



# Analytical Geometry and Linear Algebra I, Lab 9

Conic sections (2nd order curve equation):

- a) Parabola
- b) Ellipse

## Questions from the class



*No questions for today*

# Questions for today

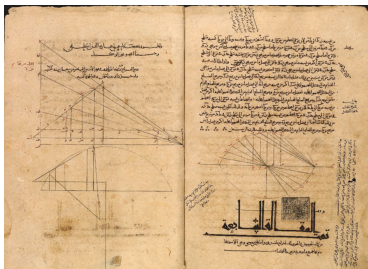


- How can I work with general form of 2nd order curve equation?
- How it relates with cone?
- What forms of equation do we have?



# Why it is called <<Conic Sections>>

The greatest progress in the study of conics by the ancient Greeks is due to *Apollonius of Perga* (died c. 190 BCE), whose eight-volume **Conic Sections or Conics**. More info [here](#).



1654 edition of Conica

Books 5-7 are only available in an Arabic translation (9th century)

# Elliptic Cone



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

## Traces

In plane  $z = p$ : an ellipse

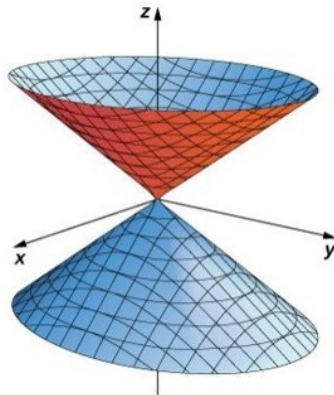
In plane  $y = q$ : a hyperbola

In plane  $x = r$ : a hyperbola

In the  $xz$  - plane: a pair of lines that intersect at the origin

In the  $yz$  - plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

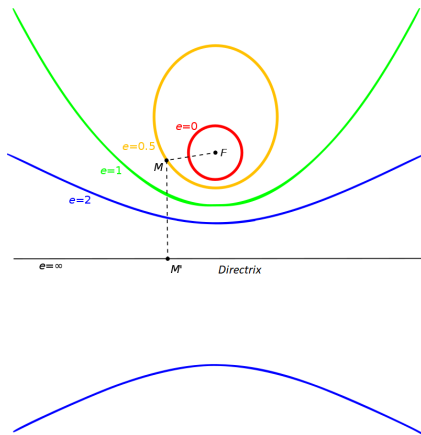


# Some definitions, which can be helpful

## *Eccentricity, Directrix*

**Eccentricity** is a measure of how much a conic section deviates from being circular.

It is a constant ration between distance from focal to point on the curve and from the point on the curve to **directrix**.



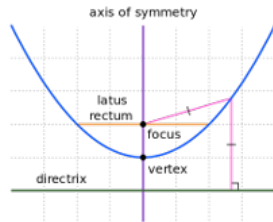
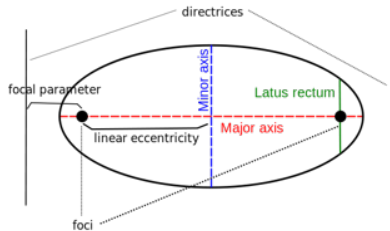
# Some definitions, which can be helpful

*Linear eccentricity, Latus Rectum, Focal parameter*

The **linear eccentricity** is the distance between the center and the focus (or one of the two foci).

The **latus rectum** is the chord parallel to the directrix and passing through the focus (or one of the two foci).

The **focal parameter** is the distance from the focus (or one of the two foci) to the directrix.



# Parabola

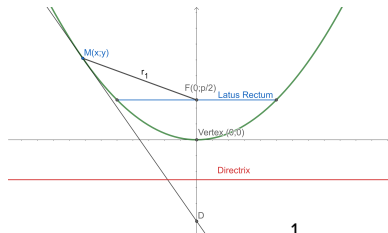
Forms:

- **Canonical (should be w/o shift)**  $(x - x_{\text{shift}})^2 = 2p(y - y_{\text{shift}})$
- **General**  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , where either  $A = 0$  or  $C = 0$ , not both, if  $B = 0$
- **Parametric**  $\begin{cases} x = \sqrt{2}pt \\ y = pt^2 \end{cases}$

Properties:

- **Vertex**  $\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center** Not defined
- **Eccentricity**  $e = 1$
- **Linear Eccentricity** Not defined
- **Foci**  $F = \begin{pmatrix} 0 \\ \frac{p}{2} \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$

- **Latus Rectum** (length of chord)  $|2p|$
- **Focal parameter**  $p$
- **Discriminant**  $\mathfrak{D} = B^2 - 4AC = 0$
- **Directrix eq.**  $y = -\frac{p}{2} + y_{\text{shift}}$



Parabola  $y = x^2$ ,  $p = \frac{1}{2}$

- **Tangent eq.**  $x(x_{\text{tan}} - x_{\text{shift}}) = p(y - y_{\text{tan}}) + x_{\text{tan}}(x_{\text{tan}} - x_{\text{shift}})$
- $r = |FM| = \sqrt{(y - \frac{p}{2})^2 + x^2}$
- $\triangle MFD$  is isosceles, where  $MD$  - tangent to  $M$



# Some words about definitions



## General equation

On my slides — there are no 2 near to components  $B$ ,  $D$  etc. I put it inside of parameters. All future formulas took it into account. It's OK, to use lecturer's notation with 2 in formulas.

## Focal parameter

In different books or sources, there are different formulas and definitions.

**My slides.** *It is the distance from the focus to the directrix.* [Wiki](#), [Calc](#).

**Lecture slides.** *It is a semi latus rectum.* [OnlineMathSchool](#), Lectures, [Rus Wiki](#).

*It's a good practice to define definitions before using them.*

## From general to canonical form

When  $B = 0$

$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0$  — General form.

Example of transformation from general to canonical form:

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0 \Rightarrow$$

$$(16x^2 - 32x) + (25y^2 + 50y) - 359 = 0 \Rightarrow$$

$$16(x^2 - 2x) + 25(y^2 + 2y) = 359 \Rightarrow$$

$$16(x^2 - 2x + 1) + 25(y^2 + 2y + 1) = 350 + 16 + 25 \Rightarrow$$

$$16(x - 1)^2 + 25(y + 1)^2 = 400 \Rightarrow$$

$$\frac{(x - 1)^2}{25} + \frac{(y + 1)^2}{16} = 1$$

## Task 1

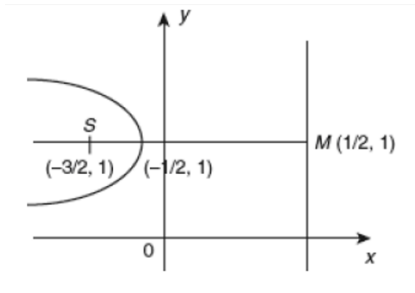


Find the foci, latus rectum, vertices and directrices of the following parabola:

$$y^2 + 4x - 2y + 3 = 0.$$

# Task 1

## Answer



i.

$$y^2 + 4x - 2y + 3 = 0$$

$$y^2 - 2y = -4x - 3$$

$$y^2 - 2y + 1 = -4x - 3 + 1$$

$$\Rightarrow (y-1)^2 = -4\left(x + \frac{1}{2}\right)$$

Take  $x + \frac{1}{2} = X$ ,  $y - 1 = Y$ . Shifting the origin to the point  $\left(-\frac{1}{2}, 1\right)$  the equation of the parabola becomes  $Y^2 = -4X$ .

$\therefore$  Vertex is  $\left(-\frac{1}{2}, 1\right)$ , latus rectum is 4, focus is  $\left(-\frac{3}{2}, 1\right)$  and foot of the directrix is  $\left(\frac{1}{2}, 1\right)$ .

The equation of the directrix is  $x = \frac{1}{2}$  or  $2x - 1 = 0$ .

# Tangent line to parabola/ellipse/hyperbola



There are 3 methods how to do it:

1. Use a formula for canonical form, but you should correctly shift the line.
2. Take a derivative of the equation. It works, because in this case we will find a slope.
3. Derive a formula, which will includes shifting (I did it only for parabola).



# Tangent line to parabola/ellipse/hyperbola

## Case study

$$\begin{aligned}
 &\underline{x x_{can} = p (y + y_{can})} \quad | \quad "2" \quad x, y(x); (x-4)^2 - 6(y-2) = 0 = f(x) \\
 &(x-4)^2 = 6(y-2) \text{ at } (10, 8) \quad | \quad \text{derivative} \\
 &"1" \text{ a) Transform to canon} \quad | \quad \frac{d}{dx} (x-4)^2 - 6 \left( \frac{d}{dx} y + \frac{d}{dx} (-2) \right) = 0 \\
 &\quad x'^2 = 2 \cdot 3 y' \text{ at } (10-4, 8-2) \quad | \quad 2x - 8 - 6y' = 0 \Rightarrow y' = \frac{x-4}{3} = k \\
 &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{put } (10, 8) \rightarrow \\
 &\text{b) use eqn.} \quad | \quad k = \frac{6}{3} = 2; \quad 8 = 2 \cdot 10 + b \Rightarrow b = -12 \\
 &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \underline{y = 2x - 12} \\
 &\text{c) take 2 points, shift, find} \quad | \quad \begin{cases} -4 = k \cdot 4 + b \\ 2 = k \cdot 7 + b \end{cases} \Rightarrow \begin{cases} k = 2 \\ b = -12 \end{cases} \Rightarrow \underline{y = 2x - 12}
 \end{aligned}$$

## Task 2



Find the equations of the tangent and normal to the parabola  $y^2 = 4(x - 1)$  at  $(5, 4)$ .

## Task 2

### Answer

$$y^2 = 4(x - 1)$$

Differentiating with respect to  $x$ ,

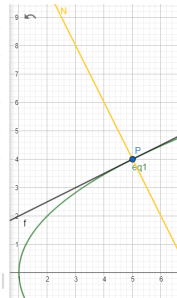
$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$
$$\left( \frac{dy}{dx} \right)_{at(5,4)} = \frac{2}{4} = \frac{1}{2} = \text{Slope of the tangent at } (5, 4)$$

$\therefore$  The equation of the tangent at  $(5, 4)$  is  $y - 4 = \frac{1}{2}(x - 5)$ .

$2y - 8 = x - 5$  or  $x - 2y + 3 = 0$ . The slope of the normal at  $(5, 4)$  is  $-2$ .

$\therefore$  The equation of normal at  $(5, 4)$  is  $y - 4 = -2(x - 5)$  or  $2x + y = 14$ .

●	eq1: $y^2 = 4(x - 1)$	::
●	$P = (5, 4)$	::
●	Tangent(P, eq1)	::
	= f: $y = 0.5x + 1.5$	
●	N: $X = t(0.5, -1) + (5, 4)$	::
+	Input...	





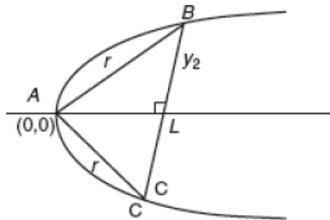
## Task 3



An equilateral triangle is inscribed in the parabola  $y^2 = 4ax$  one of whose vertices is at the vertex of the parabola. Find its side.

## Task 3

### Answer



The coordinates of  $B$  are  $B(r \cos 30^\circ, r \sin 30^\circ)$ ,  $\left(\frac{\sqrt{3}}{2}r, \frac{r}{2}\right)$ .

Since this point lies on the parabola  $y^2 = 4ax$ , then

$$\frac{r^2}{4} = 4a \cdot \frac{r}{2} \sqrt{3} \quad \therefore r = 8a\sqrt{3}$$

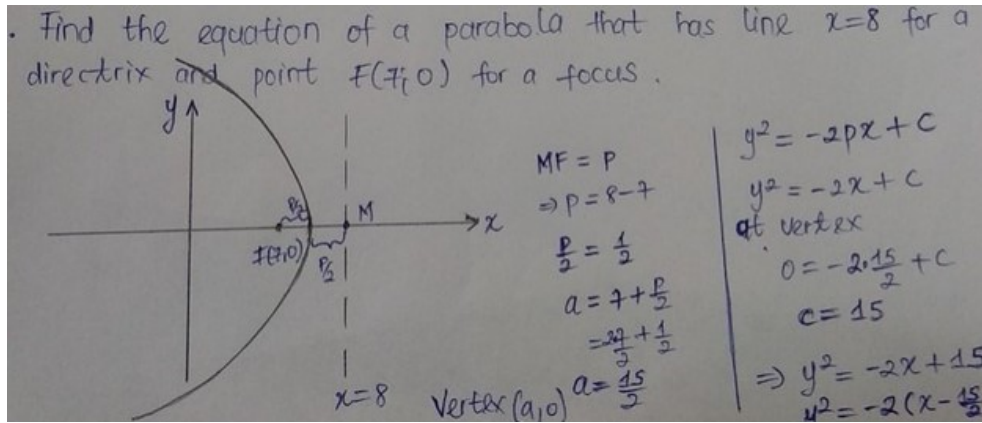
## Task 4



Find the equation of a parabola that has a line  $x = 8$  for a directrix and point  $F(7; 0)$  for a focus.

## Task 4

### Answer



# Collisions

Video



# Ellipse

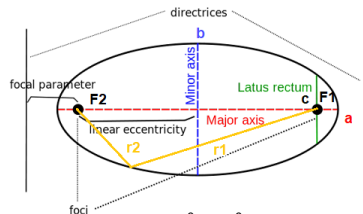
Forms:

- **Canonical (should be w/o shift)**  $\frac{(x-x_{\text{shift}})^2}{a^2} + \frac{(y-y_{\text{shift}})^2}{b^2} = 1$
- **General**  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , where  $AC > 0$ , if  $B = 0$
- **Parametric**  $\begin{cases} x = a \cos(\alpha) \\ y = b \sin(\alpha) \end{cases}$

Properties:

- **Vert**  $\begin{pmatrix} \pm a \\ 0 \end{pmatrix}$  &  $\begin{pmatrix} 0 \\ \pm b \end{pmatrix}$  +  $\begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center**  $(0; 0) + (x_{\text{shift}}; y_{\text{shift}})$
- **Eccentricity**  $0 \leq e < 1$ ,  
 $e = \sqrt{1 - \frac{b^2}{a^2}}$
- **Linear Eccentricity**  
 $c = \sqrt{a^2 - b^2}$

- **Foci**  
 $F = \begin{pmatrix} \pm(c = e a) \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Latus Rectum** (length of chord)  
 $\frac{2b^2}{a}$
- **Focal parameter**  $\frac{b^2}{\sqrt{a^2 - b^2}}$
- **Discriminant**  $\mathfrak{D} = B^2 - 4AC < 0$



$$\text{Ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- **Directrix eq.**  $x = \pm \frac{a}{e} + x_{\text{shift}}$
- **Tangent eq. (w/o shift)**  
 $\frac{x_{\text{tangent}}x}{a^2} + \frac{y_{\text{tangent}}y}{b^2} = 1$
- $r_1 + r_2 = 2a$
- $r_{1,2} = |F_{1,2}M| = \sqrt{(x \pm c)^2 + y^2}$
- $\frac{r_1}{d_1} = e$

## Task 5



Find the equation of the ellipse whose foci are  $(4, 0)$  and  $(-4, 0)$  and  $e = 1/3$



## Task 5

### Answer

i. If the foci are  $(ae, 0)$  and  $(-ae, 0)$  then the equation of the ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

Here,  $ae = 4$  and  $e = \frac{1}{3}$ .

$$a = \frac{4}{e} = 4 \times 3 = 12$$

$$b^2 = a^2(1 - e^2) = 144 \left(1 - \frac{1}{9}\right) = 144 \times \frac{8}{9} = 128$$

$\therefore$  The equation of the ellipse is  $\frac{x^2}{144} + \frac{y^2}{128} = 1$ .



## Task 6



Find the eccentricity, foci and the length of the latus rectum of the ellipse  $9x^2 + 4y^2 = 36$



## Task 6

### Answer

i.  $9x^2 + 4y^2 = 36$

Dividing by 36, we get

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1$$

$$\text{(i.e.) } \frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\therefore a^2 = 4, \quad b^2 = 9.$$

This is an ellipse whose major axis is the  $y$ -axis and minor axis is the  $x$ -axis and centre at the origin.

$$\therefore a^2 = b^2(1 - e^2) \Rightarrow 4 = 9(1 - e^2)$$

$$\therefore 9e^2 = 5$$

$$\text{Therefore, eccentricity} = e = \frac{\sqrt{5}}{3}$$

$$\text{Therefore, foci are } \left(0, \pm \frac{be}{1}\right) \text{ (i.e.) } (0, \pm \sqrt{5}).$$

$$\text{Therefore, latus rectum} = \frac{2a^2}{b} = 2 \times \frac{4}{3} = \frac{8}{3}.$$



## Task 7

The equation  $25(x^2 - 6x + 9) + 16y^2 = 400$  represents an ellipse. Find the centre and foci of the ellipse. How should the axis be transformed so that the ellipse is represented by the equation  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ?



## Task 7

### Answer

$$25(x^2 - 6x + 9) + 16y^2 = 400$$

$$25(x - 3)^2 + 16y^2 = 400$$

Dividing by 400,  $\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1$ ; Take  $x - 3 = X$ ,  $y = Y$ .

$$\text{Then } \frac{X^2}{16} + \frac{Y^2}{25} = 1.$$

The major axis of this ellipse is the Y-axis.

$$\begin{aligned}\therefore 16 &= 25(1 - e^2) \Rightarrow 1 - e^2 = \frac{16}{25} \Rightarrow e^2 = 1 - \frac{16}{25} = \frac{9}{25} \\ \therefore e &= \frac{3}{5}.\end{aligned}$$

Centre is  $(3, 0)$ . Foci are  $(3, \pm ae)$  (i.e.)  $\left(3, \pm 5 \times \frac{3}{5}\right)$  (i.e.)  $(3, \pm 3)$ . Now

shift origin to the point  $(3, 0)$  and then rotate the axes through right

angles. Then the equation of the ellipse becomes  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ .



## Task 8

Find the eccentricity of an ellipse given that:

1. its major axis subtends an angle of  $120^\circ$  at the endpoints of its minor axis;
2. the segment between a focus and the farthest vertex subtends an angle of  $90^\circ$  at the endpoints of its minor axis.

# Task 8

## Answer

Find the eccentricity of an ellipse given that  
(a) Its major axis subtends an angle  $120^\circ$  at the endpoints of its minor axis.

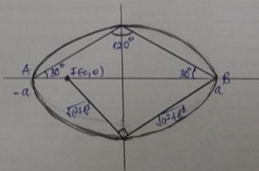
$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$\tan 30^\circ = \frac{b}{a}$$

$$\sqrt{3}b = a \Rightarrow b^2 = \frac{a^2}{3}$$

$$e = \sqrt{1 - \frac{1}{3}} \quad \frac{b^2}{a^2} = \frac{1}{3}$$

$$e = \sqrt{\frac{2}{3}}$$



(b) The segment between a focus and the farthest vertex subtend an angle of  $90^\circ$  at the endpoints of its minor axis

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}$$

$$c^2 = a^2 - b^2$$

$$c^2 = a^2 - ac$$

$$c^2 + ac - a^2 = 0$$

$$c_{1,2} = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2}$$

$$c^2 + b^2 + b^2 + a^2 = (a+c)^2$$

$$2b^2 + a^2 + c^2 = a^2 + 2ac + c^2$$

$$2b^2 = 2ac$$

$$b^2 = ac$$

$$e > 0, \quad e = \frac{c}{a}$$

$$e = \frac{-1 + \sqrt{5}}{2}$$

## Reference material



- [Conics Section \(Wiki\)](#)
- [Conic sections \(Khan Academy, full playlist, eng\)](#)

# Deserve "A" grade!

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📍 @Lupasic

🏢 Room 105 (Underground robotics lab)