

Affine Transformation

Formal definition

Classical representation:

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

System of equations representation:

$$\begin{cases} x^* = ax + by + x_0 \\ y^* = cx + dy + y_0 \end{cases}$$

Homogeneous representation:

$$\begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine Transformation

Video: *formal definition*



Affine Transformations

- * Combines linear transformations, and Translations
- * Properties
 - * Origin does not necessarily map to origin



$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Affine Transformation

Properties



- *Collinearity between points*: three or more points which lie on the same line (called collinear points) continue to be collinear after the transformation.
- *Parallelism*: two or more lines which are parallel, continue to be parallel after the transformation.
- *Convexity of sets*: a convex set continues to be convex after the transformation. Moreover, the extreme points of the original set are mapped to the extreme points of the transformed set.
- *Ratios of lengths of parallel line segments* are the same after the transformation.

Task 1



Linear transformation of a real axis is given by $f(x) = ax + b$. (a) Find all fixed points of this transformation. (b) Find the transformation that is inverse for f .



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Answer

(a) If $a \neq 1$ then there is one fixed point $x = \frac{b}{1-a}$; if $a = 1$ and $b = 0$ then all points are fixed; if $a = 1$ and $b \neq 0$ then there are no fixed points. (b) It exists only if $a \neq 0$:

$$f^{-1}(y) = \frac{y-b}{a}.$$

Task 2



An affine transformation is given by $x^* = 3x + 2y - 6$, $y^* = 4x - 3y + 1$. Find the images of
(a) point $M(-1; 5)$; (b) line $2x + 3y = 7$.

Task 2



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(a) point $M(-1; 5)$; (b) line $2x + 3y = 7$.

Answer

(a) $(1; -18)$

(b) $18x - 5y - 6 = 0$.

Task 3



Two linear transformations of a real axis f and g are given by $f(x) = ax + b$, $g(x) = cx + d$. Find compositions of transformations fg and gf . What are the necessary and sufficient conditions for fg to be equal to gf ?

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Answer

$$(fg)(x) = acx + ad + b; (gf)(x) = acx + bc + d; fg = gf \Leftrightarrow d(a - 1) = b(c - 1).$$

Bijection, injection and surjection

Injective and surjective (bijective)

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto x.$$

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+ : x \mapsto x^2, \text{ and thus also its inverse}$$

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+ : x \mapsto \sqrt{x}.$$

Injective and non-surjective

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto e^x.$$

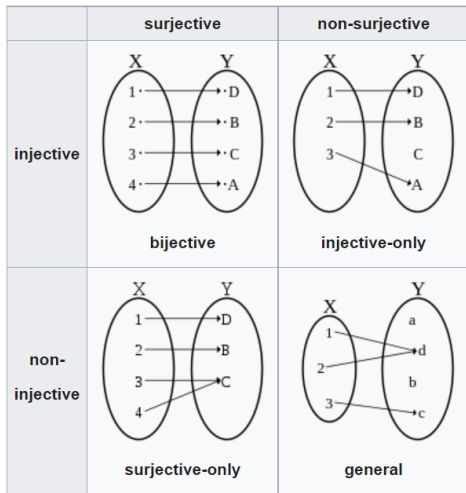
Non-injective and surjective

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto (x-1)x(x+1) = x^3 - x.$$

$$\mathbb{R} \rightarrow [-1, 1] : x \mapsto \sin(x).$$

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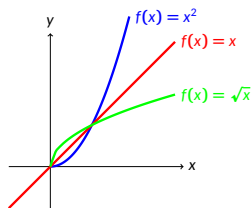
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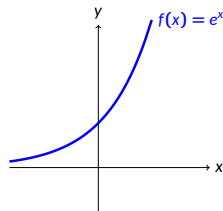
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Non-injective and non-surjective

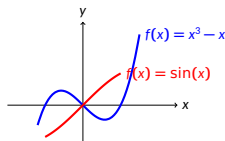
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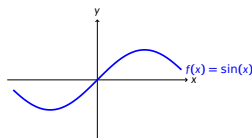
Bijjective



Injective-only



Surjective-only



General

Task 4



Transformation of a plane is given by $x^* = x^2 - y^2$, $y^* = 2xy$. Is this transformation an (a) injection; (b) surjection; (c) bijection? (d) Find the preimage of point $(x^*; y^*)$ by this transformation.

Task 4



Answer

(a): Not Injection

(b): Is surjection

(c): Not Bijection

Affine Transformation

In Computer Vision (CV)

$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

is equivalent to the following

$$\vec{y} = A\vec{x} + \vec{b}$$

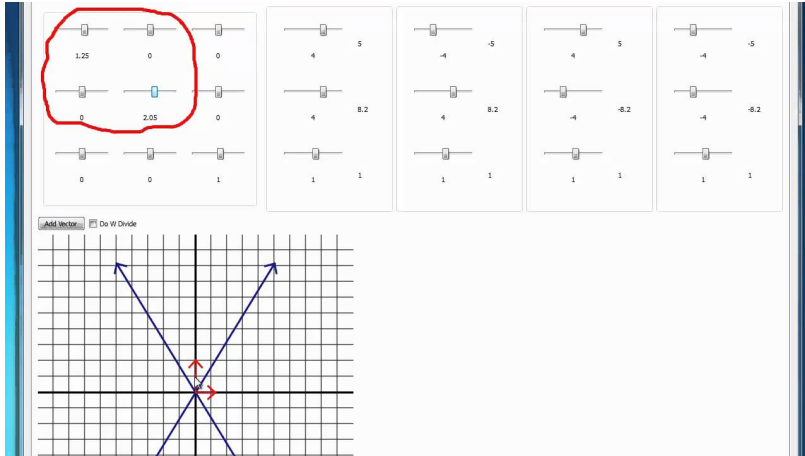
Scale	$\begin{bmatrix} c_x = 2 & 0 & 0 \\ 0 & c_y = 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
Rotate	$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$	where $\theta = \frac{\pi}{6} = 30^\circ$
Shear	$\begin{bmatrix} 1 & c_x = 0.5 & 0 \\ c_y = 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

Transformation name	Affine matrix	Example
Identity (transform to original image)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
Reflection	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	



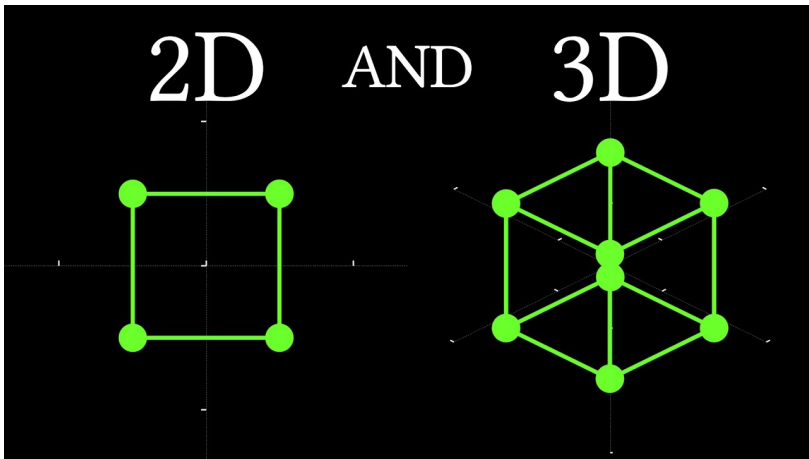
Affine Transformation

Video: intuition besides numbers 2D



Affine Transformation

Video



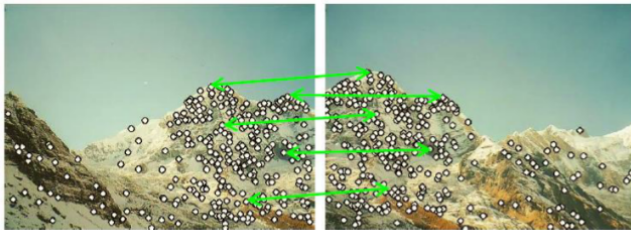
Affine Transformation

Application in CV (1)



Point Features: how to build a panorama?

- Detect feature points in both images
- Find corresponding pairs



Affine Transformation

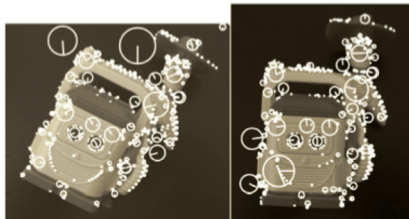
Application in CV (2)

SIFT features [Lowe et al., IJCV 2004]

- **SIFT** = Scale Invariant Feature Transform
an approach for detecting and describing regions of interest in an image
- SIFT features are reasonably **invariant** to changes in:
rotation, scaling, changes in viewpoint, illumination
- SIFT **detector** uses **DoG kernel**, SIFT **descriptor** is based on **gradient orientations**
- Very powerful in capturing + describing **distinctive** structure, but also **computationally demanding**

Main SIFT stages:

1. Extract keypoints + scale
2. Assign keypoint orientation
3. Generate keypoint descriptor

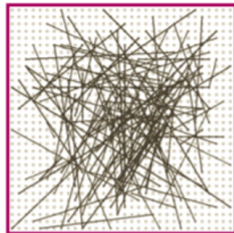


Affine Transformation

Application in CV (3)

BRIEF detector [Calonder et. al, ECCV 2010]

- BRIEF : Binary Robust Independent Elementary Features
- Goal: high speed (in description and matching)
- Binary descriptor formation:
 - Smooth image
 - for each detected keypoint (e.g. FAST),
 - sample all intensity pairs (I_1, I_2) (typically 256 pairs) according to pattern around the keypoint
 - for each pair p
 - if $I_1 < I_2$ then set bit p of descriptor to 1
 - else set bit p of descriptor to 0
- Not scale/rotation invariant (extensions exist...)
- Allows very fast Hamming Distance matching: counting the no. different bits in the descriptors tested



Pattern for intensity pair samples – generated randomly

Task 5



Find the image of an arbitrary point M which has position vector \mathbf{r} by the following transformations:

- (a) homothety with center $M_0(\mathbf{r}_0)$ and ratio $\lambda \neq 0$;
- (b) reflection across point $M_0(\mathbf{r}_0)$;
- (c) translation by vector \mathbf{a} ;
- (d) orthogonal projection onto the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$;
- (e) reflection across the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$;
- (f) dilation of factor $\lambda > 0$ from the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$.



Task 5

Answer

$$(a) \mathbf{r}^* = \mathbf{r}_0 + \lambda(\mathbf{r} - \mathbf{r}_0);$$

$$(b) \mathbf{r}^* = -\mathbf{r} + 2\mathbf{r}_0;$$

$$(c) \mathbf{r}^* = \mathbf{r} + \mathbf{a};$$

$$(d) \mathbf{r}^* = \mathbf{r}_0 + \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a};$$

$$(e) \mathbf{r}^* = 2\mathbf{r}_0 - \mathbf{r} + 2 \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a};$$

$$(f) \mathbf{r}^* = \lambda \mathbf{r} + (1 - \lambda) \mathbf{r}_0 + (1 - \lambda) \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a}.$$



Task 6

Find formulas for the following affine transformations:

(a) orthogonal projection onto line $x - 3y + 1 = 0$;

(b) reflection across line $3x + 4y - 1 = 0$;

(c) dilation from line $x + y - 2 = 0$ of factor $\frac{1}{3}$;

(d) dilation from line $2x - y + 5 = 0$ of factor 2.



Task 6

Answer

$$(a) x^* = \frac{9x + 3y - 1}{10}, y^* = \frac{3x + y + 3}{10};$$

$$(b) x^* = \frac{7x - 24y + 6}{25}, y^* = -24x - 7y + 825;$$

$$(c) x^* = \frac{2x - y + 2}{3}, y^* = \frac{-x + 2y + 2}{3};$$

$$(d) x^* = \frac{9x - 2y + 10}{5}, y^* = \frac{-2x + 6y - 5}{5}.$$



Task 7

Find formulas for an affine mapping that transforms

(a) points $A(\frac{3}{7}; 1)$, $B(1; \frac{1}{4})$, $C(2; -1)$ into points $A^*(-4; 2)$, $B^*(-1; 6)$, $C^*(4; 13)$ respectively;

(b) points $A(0; 0)$, $B(-1; 2)$, $C(1; -2)$ into points $A^*(-1; -1)$, $B^*(0; 0)$, $C^*(1; 1)$ respectively;

(c) points $A(2; 0)$, $B(3; -1)$, $C(4; -2)$ into points $A^*(2; 1)$, $B^*(-2; -1)$, $C^*(-6; -3)$ respectively;

(d) points $A(-2; 0)$, $B(2; -1)$, $C(0; 4)$ into points $A^*(-2; 1)$, $B^*(2; 1)$, $C^*(0; 1)$ respectively.

Task 7



Answer

(a) $x^* = -4y$, $y^* = 7x - 1$;

(b) no solutions;

(c) $x^* = px + (p + 4)y + 2 - 2p$, $y^* = qx + (q + 2)y + 1 - 2q$, where p and q are any real numbers;

(d) no solutions (there exists a linear transformation that is not affine).

Task 8



Find all invariant lines of an affine transformation given by

(a) $x^* = y, y^* = 1 - x;$

(b) $x^* = 2x + y - 3, y^* = -3x - y;$

(c) $x^* = 5x + 3y + 1, y^* = -3x - y.$

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(b) $x^* = 2x + y - 3, y^* = -3x - y$;

(c) $x^* = 5x + 3y + 1, y^* = -3x - y$.

Answer

(a) no solutions;

(b) $x + y - 3 = 0, 2x - y + p = 0$, where p can be any real number;

(c) $x + y + 1 = 0$.

Reference material



- [Bijection, injection and surjection \(wiki\)](#)
- [Affine transformation \(wiki\)](#)

Deserve "A" grade!

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📍 @Lupasic

🏢 Room 105 (Underground robotics lab)