Analytical Geometry and Linear Algebra I, Lab 5

Test 1 Solutions

Matrix Rank

Q/A session

Questions from the class

No questions for today

Test 1, Solutions

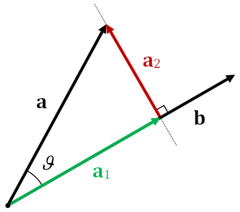
Task 1

(2 points) Decompose the vector $\mathbf{p} = (1, 2, 3)$ into components parallel and perpendicular to the vector $\mathbf{q} = (1, -2, 2)$.

Projection

Definition

The *vector projection* of a vector **a** on (or onto) a nonzero vector **b**, sometimes denoted $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$ is the orthogonal projection of **a** onto a straight line parallel to **b**.



Projection of **a** on **b** ($\mathbf{a}_1$), and rejection of **a** from **b** ($\mathbf{a}_2$)

Where it can be used:

- Maps
- Blueprints
- Fitting algorithms (Least squares)
- Reduce matrix dimension
- Reinforcement Learning (RL) fitness functions

Projection

2D Case, Classical way

Project $\vec{b}$ on $\vec{a}_1$

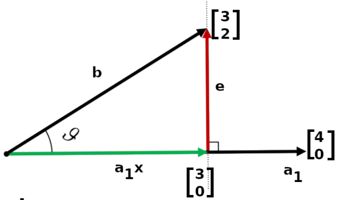
$e = b - a_1 x$, e — error b/w similar vectors

$$a_1 \cdot (b - a_1 x) = 0$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T b = a_1^T a_1 x$$

$\dfrac{a_1^T b}{a_1^T a_1} = x$ — Classical formula from school



Case study

$$\frac{a_1^T b}{a_1^T a_1} = x \Rightarrow \frac{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}} = \frac{3}{4}$$

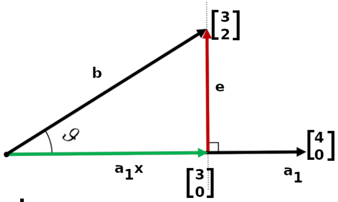
Projection $p = a_1 x = \frac{3}{4} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Projection

2D Case, Projection matrix

$$\left.\begin{array}{r} Pb = xa_1 = a_1 x \\ \dfrac{a_1^T b}{a_1^T a_1} = x \end{array}\right\} P = \frac{a_1 a_1^T}{a_1^T a_1}$$

Where P — projection matrix



Case study

$$P = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{\begin{bmatrix} 4 \\ 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \end{bmatrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p = Pb = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Projection

2D Case, Projection matrix

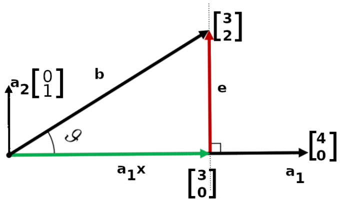
Project $\vec{b}$ on $\vec{a}_2$, which is perpendicular to $\vec{a}_1$

$$P_{d_1} = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Where $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

P_{d_1} is an error between the whole space and current projection matrix.

$$p_{d_1} = P_{d_1} b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



Projection

Case study: Reinforcement Learning fitness function

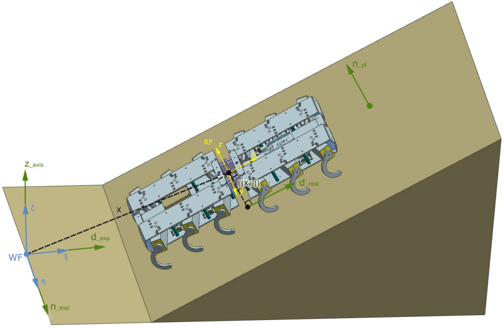
Goal

It is necessary for the robot to move in a straight line in all directions, as well as as as efficiently as possible.
The efficiency criteria are: course deviation error, max velocity and clearance.

$$F = \omega_1 X_z + \omega_2 \frac{1}{|err| + \varepsilon} + \omega_3 (P_{d_{real}} \vec{X}), \text{ where}$$

$$err = |(I - P_{d_{real}})(I - P_{n_{pl}})\vec{X}|,$$

P_* – projection matrix, ω_* – weight coeffs.



StriRus – task description

Task 1

Answer

$$\cos\alpha = \frac{pq}{\|p\|\|q\|} = \frac{1-4+6}{\sqrt{14}\cdot 3} = \frac{1}{\sqrt{14}}$$

$$\begin{cases} \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 2 \\ a & b & c \end{vmatrix} = 0 \\ a - 2b + 2c = 0 \end{cases}$$

$r(2, 8, 7)$

$q(1, -2, 2)$

$$a[4+6] + b + c[-2-2] =$$

$$= \begin{cases} 10a + b - 4c = 0 \\ a - 2b + 2c = 0 \end{cases}$$

$a = 2b - 2c$

$20b - 20c + b - 4c = 0$

$21b = 24c \qquad b = \frac{8}{7}c$

$c = 7 \qquad b = 8 \qquad a = 16 - 14 = 2$

$$p = xq + yr = x\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} x + 2y \\ -2x + 8y \\ 2x + 7y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x = 1/3$

$15y = 5$

$y = 1/3$

$$\begin{cases} \frac{1}{3} + \frac{2}{3} = 1 \\ -2/3 + \frac{8}{3} = 2 \\ 2/3 + 7/3 = 3 \end{cases} \quad V$$

$$= \frac{1}{3}q + \frac{1}{3}r$$

Task 2

1. Find the matrix product AB if $A = \begin{bmatrix} x & -2 & -1 \\ 4 & 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 1 & -3 \\ 2 & x \end{bmatrix}$

2. Find the largest possible value of determinant (AB).

Task 2

Answer

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & x \end{bmatrix} \begin{bmatrix} 5 & -1 \\ x & 2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 5+2x-15 & -1+4-5 \\ 15+7x-3x & -3+14-x \end{bmatrix} =$$

$$= \begin{bmatrix} 2x-10 & -2 \\ 15+4x & x+11 \end{bmatrix}$$

$$\det = (2x-10)(x+11) + 2(15+4x) =$$

$$= 2x^2+22x-10x-110+30+8x = 2x^2+20x+80$$

$$x_{min} = -\frac{b}{2a} = -\frac{20}{4} = -5 \qquad f_{max} = +\infty$$

$$y_{min} = 50-100+80 = 30$$

Task 3

For which values x, vectors **a** and **b** are basis of some space? Explain your answer.

$$\mathbf{a} = \begin{bmatrix} 1 - x \\ x \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 - x \\ -2 \end{bmatrix}$$

Task 3

Answer

$$\begin{vmatrix} 1-x & 1-x \\ x & -2 \end{vmatrix} = 0$$

$$-2+2x-x+x^2=0$$

$$x^2+x-2=0$$

$$x_{1,2}=\frac{-1\pm\sqrt{1+8}}{2}=\frac{-1\pm 3}{2}=$$

$$=\begin{cases} 1 \\ -2 \end{cases}$$

$$\boxed{\underline{\text{Ans}}: \; x \neq 1, \; x \neq -2}$$

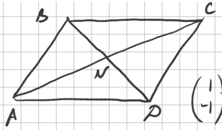
Task 4

Given a parallelogram $ABCD$. Point N is the crossing of its diagonals. The old coordinate system has origin A and the basis AB, AD.

1. Define a new coordinate system formed by the point D and two new basis vectors: DB and DC.
2. Compute the transitions matrix A from the old basis to the new basis.
3. Calculate coordinates of point N in both bases, using the transition matrix A.

Task 4

Answer



$A,\ AB,\ AD$ – old

$D,\ DB,\ DC$ – new

$$\begin{pmatrix}1\\-1\end{pmatrix} = DB = DA + AB = AB - AD$$

$$\begin{pmatrix}1\\0\end{pmatrix} = DC = AB + 0\cdot AD$$

$$\begin{pmatrix}0\\1\end{pmatrix} = AD = 0\cdot AB + AD$$

$$A = \begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}x\\y\end{pmatrix}}_{\text{old}} = \underbrace{\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x'\\y'\end{pmatrix}}_{\text{new}} + \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\text{shift}}$$

N in new $= \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix} \equiv DN = \frac{1}{2}DB + 0\,DC$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1/2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix}$$

$= \frac{1}{2}AB + \frac{1}{2}AD$ – correct

How to get out of an exam

Video

Matrix Rank

Definition

$N_r(A)$ — max number of **lineary independent** rows of matrix A.
$N_c(A)$ — max number of **lineary independent** columns of matrix A.

$$Rank(A) = N_r(A) = N_c(A)$$

The rank of the matrix is how many of the rows (columns) are «unique»: not formed out by other rows (columns).

Matrix Rank

Motivation

- Computation of the number of solutions of a system of linear equations.
- Analysis of the linear dependency of rows and columns.
- Applications in Control Theory (next year): observability and controllability.

Matrix Rank

How to find

There are 3 ways:

1. **Look at matrix** and find linear dependencies.
2. **Reduced form** (transform matrix to upper triangular form (The first part of the algorithm for finding inverse matrix).
3. **Minor method** (Метод окаймляющих миноров) *not popular in western education*.

Matrix Rank

Case Study (on whiteboard)

Calculate the rank of the following matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

Answer: 2

Matrix Rank

Task 2

Determine the ranks of the following matrices for all real values of parameter α:

1. $\begin{bmatrix} 1 & \alpha & -1 & 2 \\ 2 & -1 & \alpha & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix}$;
2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$;

Reference material

- Matrix Rank (OnlineMschool)

Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)