Analytical Geometry and Linear Algebra 1

Cross product
Dot product

Questions from the class

No questions for today

Lab 1, Task 3: answers

Basis in $\mathbb{R}^2$

A **set** of vectors is a *basis* of $\mathbb{R}^2$ if it spans $\mathbb{R}^2$ and this set is **linearly independent.**

Check for each case if the following set of vectors is a basis or not. Explain your answer.

1. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ —
2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ —
4. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$
5. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ —
6. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —

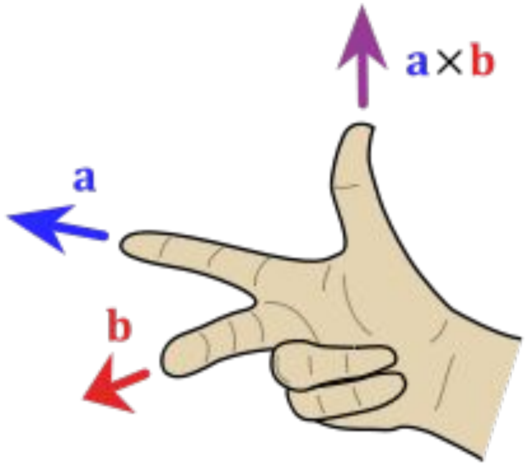
Lab objectives, 1st part

1. What does cross product mean?
2. How to calculate it?
3. What the properties of cross product, how to use it?

Cross product: Definition

a × **b** is defined as a vector **c** that is perpendicular (orthogonal) to both **a** and **b**, with:

- *direction* given by the right-hand rule
- *magnitude equal to the area* of the parallelogram that the vectors span




Video by: Eugene Khutoryansky
Narrator and dialogue editor: Kira Vincent

Cross product: where it can be used?

1. Physics: angular velocity, torque
2. Find a vector, which are perpendicular to the plane
3. Find a square of parallelogram

How to calculate it?

2 Approaches for full vector:

1. Classical one
2. Using skew-symmetric matrix

1 Approach for only magnitude:

1. Geometrical representation

Classical one

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\mathbf{X}) = a * d - b * c$$

Skew-symmetric matrix

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = a \times b \Rightarrow c = \hat{a} b$$

vectors $\Rightarrow$ matrices

$a \times \Rightarrow \hat{a}$: a skew-symmetric matrix

$$c = \hat{a} b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

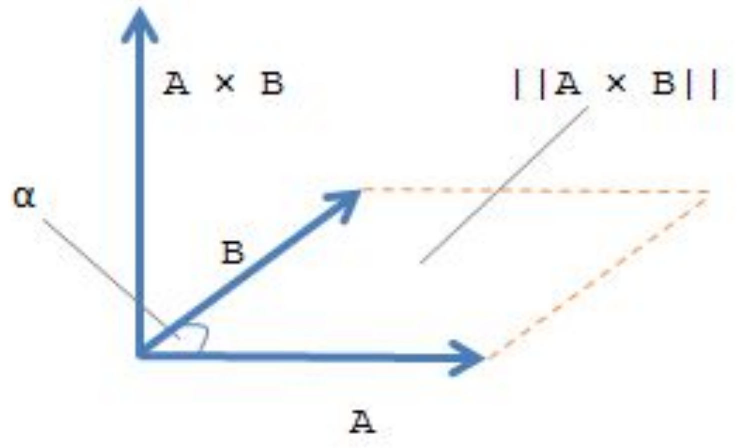
$$\boxed{c = \hat{a} b}$$

Geometrical representation

$||A \times B|| = ||A||\ ||B|| \sin\alpha$

$||A \times B||$ – area

$||A||$ – length of the vector



Case study

Calculate cross product between **a** and **b**
a =[-2;-2;10]
b=[-4;1,10]

Classic

$$(-2,-2,10)\times(-4,1,10) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 10 \\ -4 & 1 & 10 \end{pmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -2 & 10 \\ 1 & 10 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 10 \\ -4 & 10 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -2 \\ -4 & 1 \end{vmatrix}$$

$$= -30\mathbf{i} - 20\mathbf{j} - 10\mathbf{k}$$

$$= (-30,-20,-10)$$

Skew-symmetric

$$[\vec{a}\times]\vec{b} = \begin{bmatrix} 0 & -10 & -2 \\ 10 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix}$$

Cross product properties

1. $\overline{a} \times \overline{b} = -(\overline{b} \times \overline{a});$
2. $\overline{a} \times (\overline{b} + \overline{c}) = \overline{a} \times \overline{b} + \overline{a} \times \overline{c};$
3. $(\overline{a} + \overline{b}) \times \overline{c} = \overline{a} \times \overline{c} + \overline{b} \times \overline{c};$
4. $\overline{\lambda a} \times \overline{b} = \overline{a} \times \overline{\lambda b} = \lambda \cdot (\overline{a} \times \overline{b});$
5. $\overline{a} \times \overline{a} = \overline{0};$
6. $\overline{a} \times \overline{b} = \overline{0} \quad \Leftrightarrow \quad \overline{a} \parallel \overline{b}$

Task 1

Find cross product, if

VAR1 $\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}; \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$

VAR3 $\vec{a} = \begin{bmatrix} -9 \\ 3 \\ -6 \end{bmatrix}; \vec{b} = \begin{bmatrix} 3 \\ 5 \\ -8 \end{bmatrix}$

VAR2 $\vec{a} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}; \vec{b} = \begin{bmatrix} 8 \\ 8 \\ -5 \end{bmatrix}$

VAR4 $\vec{a} = \begin{bmatrix} 8 \\ 3 \\ -9 \end{bmatrix}; \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -6 \end{bmatrix}$

Task 2

2. Simplify the expressions:

(a) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$;

(b) $(3\mathbf{a} - \mathbf{b} - \frac{1}{3}\mathbf{c}) \times (2\mathbf{a} + \frac{3}{2}\mathbf{b} - 3\mathbf{c})$.

```
a = sym('a',[3 1]);
b = sym('b',[3 1]);
simplify(cross(a+b,a-b))
```

ans =

$$\begin{pmatrix} 2\,a_3\,b_2 - 2\,a_2\,b_3 \\ 2\,a_1\,b_3 - 2\,a_3\,b_1 \\ 2\,a_2\,b_1 - 2\,a_1\,b_2 \end{pmatrix}$$

```
>> 2 * cross(b,a)

ans =

 2*a3*b2 - 2*a2*b3
 2*a1*b3 - 2*a3*b1
 2*a2*b1 - 2*a1*b2
```

Task 3

A triangle is constructed on vectors $\mathbf{a}(2; 4; -1)$ and $\mathbf{b}(-2; 1; 1)$.

(a) Find the area of this triangle.

(b) Find the altitudes of this triangle.

Lab objectives, 2nd part

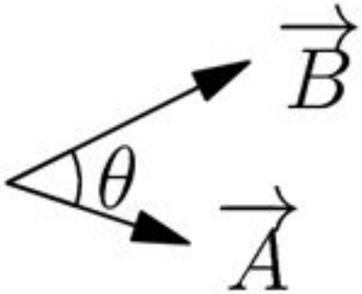
1. What does dot product mean?
2. How to calculate it?
3. How to use it?

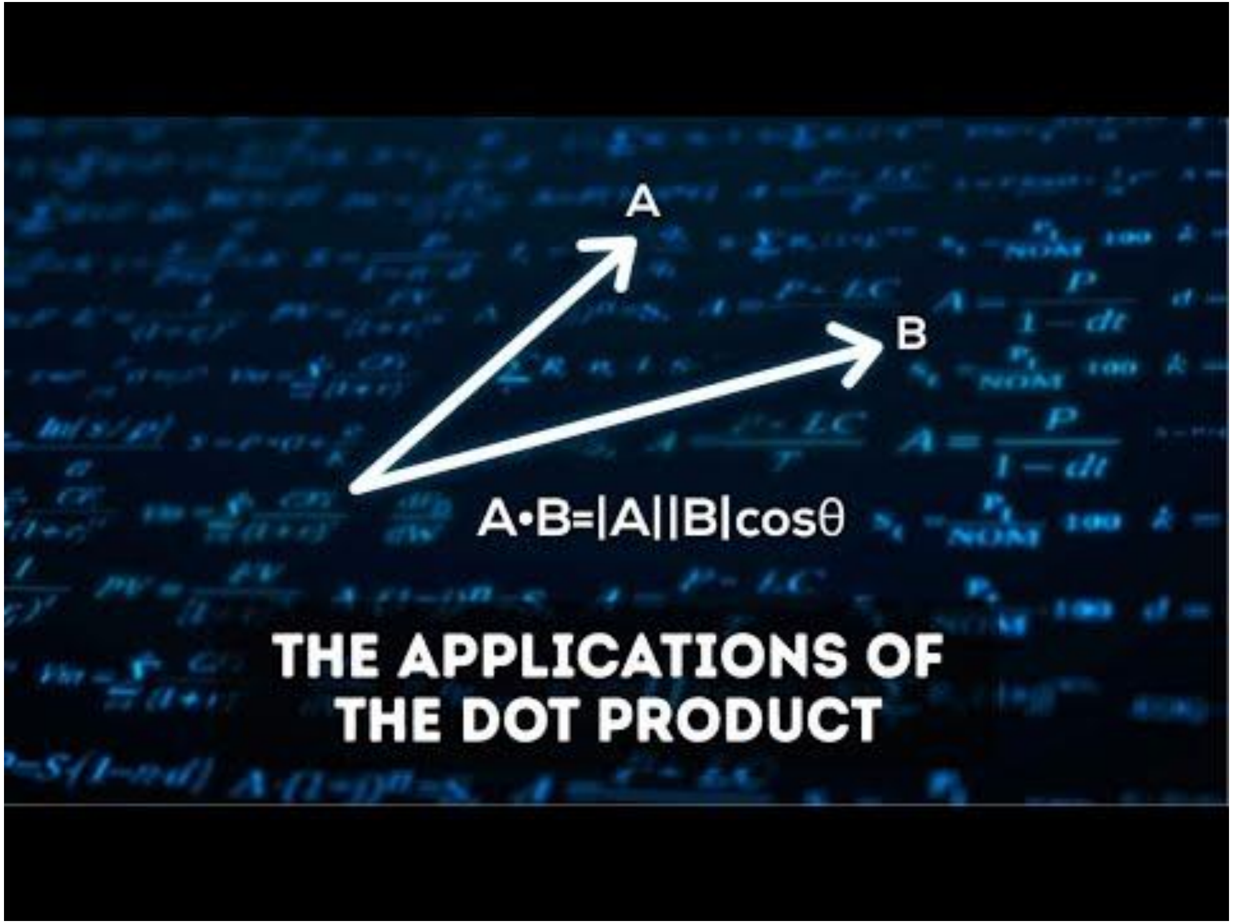
Dot product: Definition

Definition: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

This is a scalar.

Geometrically $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos(\theta)$




A
B
A•B=|A||B|cosθ
THE APPLICATIONS OF
THE DOT PRODUCT

Task 4 and 5

1. Find $|\mathbf{a}|^2 - 2\sqrt{3}\mathbf{a} \cdot \mathbf{b} - 7|\mathbf{b}|^2$ given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 1$, $\angle(\mathbf{a}, \mathbf{b}) = 150°$.

2. Find the angle[1] between $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$.

Task 6

All three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have length of 3 and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. Find $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$

4) $\vec{c} = -\vec{a} - \vec{b}$ $\qquad |\vec{a}| = 3,\ |\vec{b}| = 3$

$$9 = |\vec{c}|^2 = (\vec{a}+\vec{b})^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = 18 + 2\vec{a}\vec{b}$$

$$\vec{a}\vec{b} = -\frac{9}{2}$$

$$\vec{a}\vec{b} + (\vec{a}+\vec{b})\vec{c} = \vec{a}\vec{b} - (\vec{a}+\vec{b})^2$$

$\cancel{ab} + bc + ca \qquad -\frac{9}{2} + bc + ca$

b

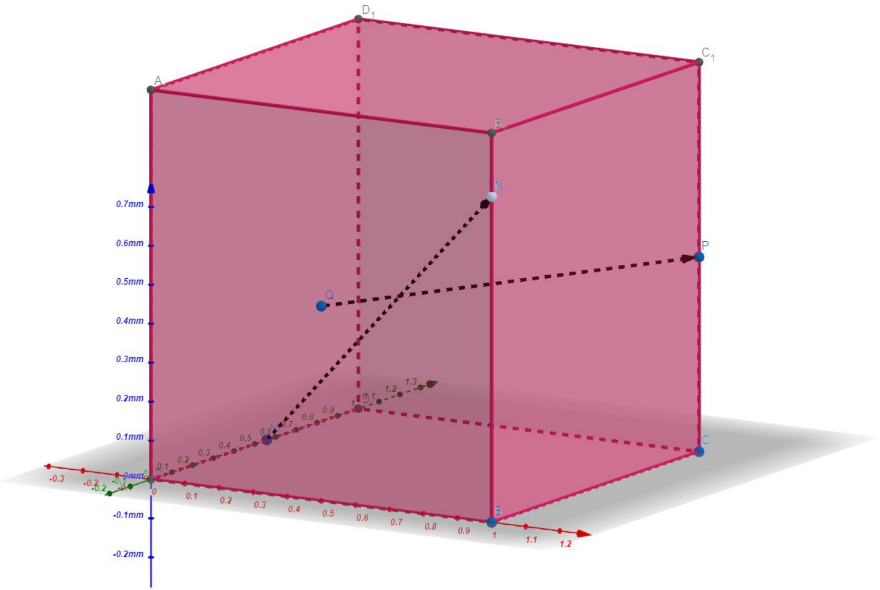
$ac + bc$


DichЪ project
super low-fidelity prototype

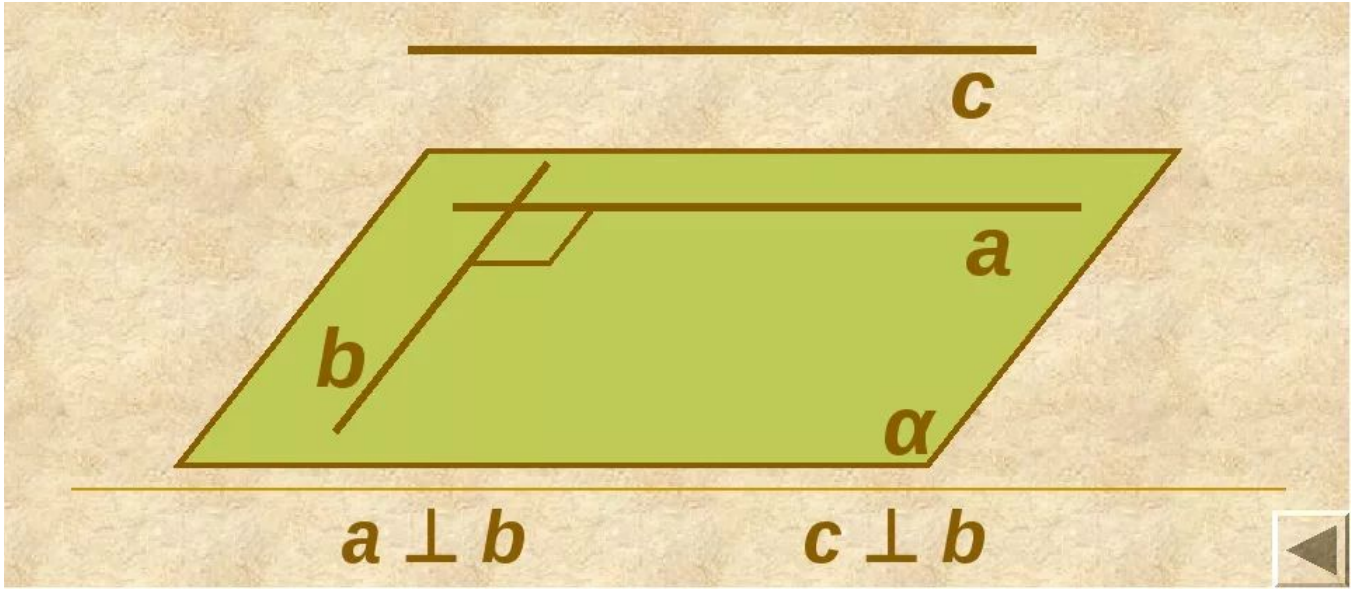
Task 7 (1)

5. The edges of cube $ABCDA_1B_1C_1D_1$ have length of 1. P is a midpoint of CC_1, and Q is a center of face AA_1B_1B. Points M and N belong to lines AD and A_1B_1 respectively, and at that MN intersects with PQ and is perpendicular to it. Find MN.

Task 7 Geogebra (2)



Task 7 (3)



It is needed to make 2 equations: 1 for perpendicularity, 2nd - intersection

Task 8

7. There are two vectors on some basis $\mathbf{a}=\begin{bmatrix} x \\ 1-x \end{bmatrix}$, $\mathbf{b}=\begin{bmatrix} x^2-2x \\ x^2-2x+1 \end{bmatrix}$. It is needed to find x, when:

   (a) vectors are collinear;

   (b) they have the same direction.

Task 8

Condition of vectors collinearity

Two vectors are collinear, if any of these conditions done:

Condition of vectors collinearity 1. Two vectors $\overline{a}$ and $\overline{b}$ are collinear if there exists a number n such that

$$\overline{a} = n \cdot \overline{b}$$

Condition of vectors collinearity 2. Two **vectors are collinear** if relations of their coordinates are equal.

N.B. Condition 2 is not valid if one of the components of the vector is zero.

Condition of vectors collinearity 3. Two **vectors are collinear** if their cross product is equal to the zero vector.

N.B. Condition 3 applies only to three-dimensional (spatial) problems.

Task 9

8. There are two vectors $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$. $\mathbf{c}$ length equal to 1. The vector is perpendicular to $\mathbf{a}$. The angle between $\mathbf{b}$ and $\mathbf{c}$ is $arccos(\sqrt{\frac{2}{27}})$. Find the coordinates of $\mathbf{c}$. How many solutions the task have?

Best extra extra materials

1) RUS - Матпрофи

2) ENG - OnlineMSchool - not only explanation, but also online calculators

3) ENG Videos - 3Blue1Brown

4) ENG Lectures - Gilbert Strang - AGLA2 based on the course

Deserve “A” grade!

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Room 105 (Underground robotics lab)