Analytical Geometry and Linear Algebra I, Lab 12

Test 2

Polar coordinates

Task 1, Var 1

Similar to lab 9 or 10, task 1

Find the equations of directrices and coordinates of focus (or foci) of the following curve:

$x^2 - 2x + y = 0$

Task 1, Var 1

Similar to lab 9 or 10, task 1

Find the equations of directrices and coordinates of focus (or foci) of the following curve:
$x^2 - 2x + y = 0$

Algorithm

1. Transform to canonical form (don't forget to change the variables, canonical form is a form, where is nothing near to x and y variables)
2. Find parameters and coordinate dependent stuff in a new basis
3. Represent $x'',\ y''$ respect to $x,\ y$ variables and substitute it in equations from previous step.
4. Highlight answers.

Task 2, Var 1

Similar to lab 9, task 4

Find the canonical equation of an ellipse (major axis is horizontal), if it is known that the eccentricity equals 0.5 and the distance from its focus to the nearest vertex is 2.

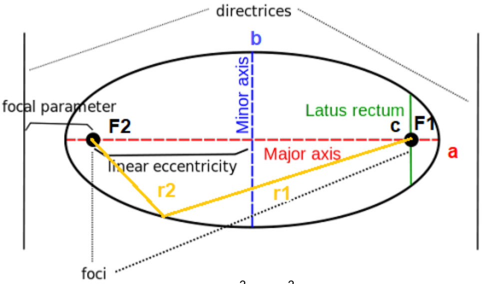
Task 2, Var 1

Similar to lab 9, task 4

Find the canonical equation of an ellipse (major axis is horizontal), if it is known that the eccentricity equals 0.5 and the distance from its focus to the nearest vertex is 2.

Algorithm

1. Refresh the properties of each parameter by the cheat sheet.
2. Draw a picture
3. Based on it, find some parameters
4. Find other parameters, knowing other stuff



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Task 3, Var 1

Similar to lab 9, task 2 or lab 10, task 5

Find the equations of the tangent and normal lines to the curve defined by the equation $x^2 - xy - y^2 + x + y = 0$ at the point with coordinates $(0.6, 1.2)$

Task 3, Var 1

Similar to lab 9, task 2 or lab 10, task 5

Find the equations of the tangent and normal lines to the curve defined by the equation $x^2 - xy - y^2 + x + y = 0$ at the point with coordinates $(0.6, 1.2)$

Algorithm

1. Task looks similar to lab 9, task 2. Or lab 10, task 5.
2. Take a derivative $\frac{dy}{dx}$ — slope (k).
3. Knowing slope and the concrete coordinate, find b. Obtain needed equation.
4. Either knowing the property $k_t k_n = -1$, find normal line. Or using the knowledge $A,\ B$ in general form is a normal vector.

Task 4, Var 1

Similar to lab 12, tasks 1, 7

In a triangle *ABC* have vertices with coordinates $A(2, 0)$; $B(-1, 2)$, $C(-1, -2)$.

1. Find the transform that maps each vertex of the triangle to the middle point of the opposite side.
2. Find the fixed point of the transform.
3. How the area of the transformed triangle relates to the area of the triangle *ABC*.

Task 4, Var 1

Similar to lab 12, tasks 1, 7

Algorithm

1. (1) (Lab 11, task 7) Draw a figure.
2. For finding affine transformation, we should have 6 equations, hence $\rightarrow$ a mapping between 3 points. Find such points by the figure.
3. Solve $\underbrace{P_{mid}}_{[\vec{K}\vec{M}\vec{N}]}\ \underbrace{inv(P_{ver})}_{[\vec{A}\vec{B}\vec{C}]} = H$. Don't forget to add 1 in last column of each vector.
4. (2) (Lab 11, task 1) $H\begin{bmatrix} x_{fix} \\ y_{fix} \end{bmatrix} = \begin{bmatrix} x_{fix} \\ y_{fix} \end{bmatrix}$. Find $x_{fix},\ y_{fix}$
5. (3) Find determinant.

Polar coordinates

Straight Line

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \end{cases}, \text{ where} \qquad (1)$$

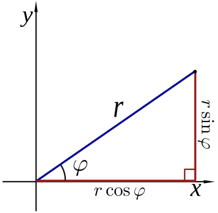
$$r = \sqrt{x^2 + y^2}; \qquad (2)$$

The general equation in polar coordinates ((1) to $Ax + By + C = 0$).

$$A\cos(\phi) + B\sin(\phi) + \frac{C}{r} = 0 \qquad (3)$$

Eqn. of the line joining the 2 points:

$$\frac{1}{r}\sin(\phi_2 - \phi_1) = \frac{1}{r_1}\sin(\phi_2 - \phi) + \frac{1}{r_2}\sin(\phi - \phi_1)$$



Polar equation of the straight line perpendicular to (3)

$$A\cos(\phi + \frac{\pi}{2}) + B\sin(\phi + \frac{\pi}{2}) = -k\frac{C}{r}$$

Polar equation of the straight line parallel to (3):

$$A\cos(\phi) + B\sin(\phi) = -k\frac{C}{r}$$

Task 1

Find the equation of the line joining the points $\begin{bmatrix} 2 \\ \frac{\pi}{3} \end{bmatrix}$ and $\begin{bmatrix} 3 \\ \frac{\pi}{6} \end{bmatrix}$. It should deduce that this line also passes through the point $\begin{bmatrix} \frac{6}{3\sqrt{3}-2} \\ \frac{\pi}{2} \end{bmatrix}$.

Task 1

Answer

The equation of the line joining the points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_2-\theta_1)=\frac{1}{r_1}\sin(\theta_2-\theta)+\frac{1}{r_2}\sin(\theta-\theta_1).$$

Therefore, the equation of the line joining the points $\left(2,\frac{\pi}{3}\right)$ and $\left(3,\frac{\pi}{6}\right)$ is

$$\frac{1}{r}\sin\left(\frac{\pi}{6}-\frac{\pi}{3}\right)=\frac{1}{2}\sin\left(\frac{\pi}{6}-\theta\right)+\frac{1}{3}\sin\left(\theta-\frac{\pi}{3}\right).$$

$$-\frac{1}{r}\sin\left(\frac{\pi}{6}\right)=\frac{-3\sin\left(\theta-\frac{\pi}{6}\right)+2\sin\left(\theta-\frac{\pi}{3}\right)}{6}$$

when $\theta=\frac{\pi}{2}$

$$\therefore \frac{1}{r}=\frac{-3\cos\frac{\pi}{6}+2\cos\frac{\pi}{3}}{-6\sin\frac{\pi}{6}}=\frac{-3\frac{\sqrt{3}}{2}+2\frac{1}{2}}{-6\times\frac{1}{2}}=\frac{3\sqrt{3}-2}{6}$$

$$r=\frac{6}{3\sqrt{3}-2}$$

Hence, the point $\left(\frac{6}{3\sqrt{3}-2},\frac{\pi}{2}\right)$ lies on the straight line.

Task 2

Find the equation of the line perpendicular to $\frac{l}{r} = \cos(\theta - \alpha) + e\ \cos(\theta)$ and passing through the point $\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix}$.

Task 2

Answer

$$\frac{r_1 \sin(\theta_1 - \alpha) + e\ \sin(\theta_1)}{r} = \sin(\theta - \alpha) + e\ \sin(\theta)$$

Task 3

Show that the feet of the perpendiculars from the origin on the sides of the triangle formed by the points with vectorial angles $\alpha,\ \beta,\ \gamma$ and which lie on the circle $r = 2a\cos(\theta)$ lie on the straight line $2a\ \cos(\alpha)\cos(\beta)\cos(\gamma) = r\ \cos(\pi - \alpha - \beta - \gamma)$.

Task 3

Answer

The equation of the circle is $r = 2a \cos \theta$.
Let the vectorial angles of P, Q, R be α, β, γ respectively.
The equations of the chord PQ, QR and RP are

$$2a\cos\alpha\cos\beta = r\cos(\theta - \overline{\alpha+\beta})$$
$$2a\cos\beta\cos\gamma = r\cos(\theta - \overline{\beta+\gamma})$$
$$2a\cos\gamma\cos\alpha = r\cos(\theta - \overline{\gamma+\alpha})$$

Let L, M and N be the feet of the perpendiculars from O on the PQ, QR and RP

Then from the above equations, we infer that the coordinates of L, M and N are

$$(2a\cos\alpha\cos\beta, \alpha+\beta)$$
$$(2a\cos\beta\cos\gamma, \beta+\gamma)$$
$$(2a\cos\gamma\cos\alpha, \gamma+\alpha)$$

These three points satisfy the equation

$$2a \cos\alpha \cos\beta \cos\gamma = r \cos(\theta - \alpha - \beta - \gamma)$$

Hence L, M and N lies on the above line.

Polar coordinates

Conic Sections

Equation of all types of conic section curves, when a conic is inclined at an angle α (if not inclined $\rightarrow \alpha = 0$)

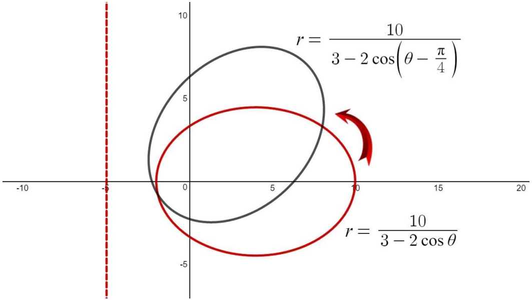
$$\frac{l}{r} = 1 + ecc\cos(\theta - \alpha),$$ where l is semi-latus rectum

Polar equation of the directrix:

$$\frac{l}{r} = ecc\cos(\theta)$$

Equation of tangent is given as

$$1 + ecc\cos(\theta) + \cos(\theta - \alpha) = \frac{l}{r}$$



Task 4

A focal chord SP of an ellipse is inclined at an angle α to the major axis. Prove that the perpendicular from the focus on the tangent at P makes with the axis an angle

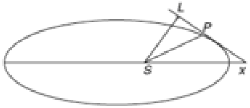
$$\arctan\left(\frac{\sin(\alpha)}{e+\cos(\alpha)}\right)$$

Task 4

Answer

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta$$



The equation of tangent at P is

$$\frac{l}{r} = 1 + e\cos\theta + \cos(\theta - \alpha)$$

The equation of the perpendicular line to the tangent at P is

$$\frac{k}{r} = e\cos\left(\theta + \frac{\pi}{2}\right) + \cos\left(\theta + \frac{\pi}{2} - \alpha\right)$$

$$\text{(i.e.)}\quad \frac{k}{r} = -e\sin\theta - \sin(\theta - \alpha)$$

If the perpendicular passes through the focus then $k = 0$

$$-e\sin\theta - \sin(\theta - \alpha) = 0$$

$$\text{(i.e.)}\quad e\sin\theta + \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\tan\theta = \frac{\sin\alpha}{e + \cos\alpha}$$

$$\text{or}\quad \theta = \tan^{-1}\left(\frac{\sin\alpha}{e + \cos\alpha}\right)$$

Reference material

- Polar coordinates (wiki)
- Polar coordinates (video, eng)
- Conics in Polar Coordinates: Rotation (video, eng)
- Multivariable Calculus, chapter 1.2 Polar Coordinates (book, eng)

Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)