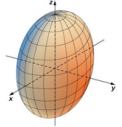
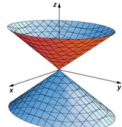
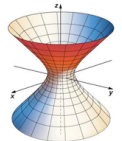
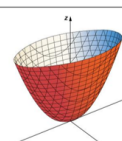
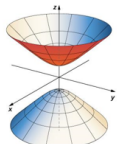
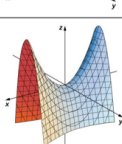


Case studies of 2nd order curve equation (ENG)



<p>Ellipsoid</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: an ellipse In plane $x = r$: an ellipse</p> <p>If $a = b = c$, then this surface is a sphere.</p>		<p>Elliptic Cone</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the xz-plane: a pair of lines that intersect at the origin In the yz-plane: a pair of lines that intersect at the origin</p> <p>The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.</p>	
<p>Hyperboloid of One Sheet</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola</p> <p>In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.</p>		<p>Elliptic Paraboloid</p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a parabola In plane $x = r$: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	
<p>Hyperboloid of Two Sheets</p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse or the empty set (no trace) In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola</p> <p>In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.</p>		<p>Hyperbolic Paraboloid</p> $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p><i>Traces</i> In plane $z = p$: a hyperbola In plane $y = q$: a parabola In plane $x = r$: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	

Case studies of 2nd order curve equation (RUS)



1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Уравнение эллипсоида		2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ Уравнение мнимого эллипсоида		3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ Уравнение мнимого конуса	
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Уравнение однополостного гиперболоида		5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ Уравнение двуполостного гиперболоида		6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ Уравнение конуса	
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ Уравнение эллиптического параболоида		8. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ Уравнение гиперболического параболоида		9. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Уравнение эллиптического цилиндра	
10. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ Уравнение мнимого эллиптического цилиндра		11. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ Уравнение пары мнимых пересекающихся плоскостей		12. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Уравнение гиперболического цилиндра	
13. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ Уравнение пары пересекающихся плоскостей		14. $y^2 = 2px$ Уравнение параболического цилиндра		15. $y^2 - b^2 = 0$ Уравнение пары параллельных плоскостей	
16. $y^2 + b^2 = 0$ Уравнение пары мнимых параллельных плоскостей		17. $y^2 = 0$ Уравнение пары совпадающих плоскостей		<p>Для всех уравнений $a > 0, b > 0, c > 0, p > 0$</p> <p>Для уравнений 1 и 2 $a \geq b \geq c$</p> <p>Для уравнений 3, 4, 5, 6, 7, 9, 10 $a \geq b$</p>	

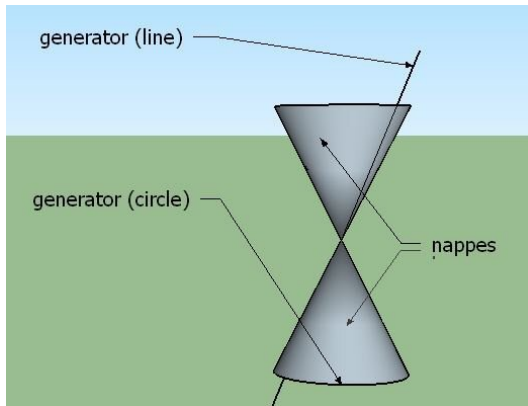
Generator

Definition

A line all points of which lie on a quadric is called a **generator of the quadric**. An elliptic cone is a quadric surface which is generated by a straight line which passes through a fixed point and which intersect an ellipse.

Then, the equation of generator is:

$$\frac{x - x_v}{l} = \frac{y - y_v}{m} = \frac{z - z_v}{n}$$



Task 1



Find the equation of the cone with its vertex at $(1, 1, 1)$ and which passes through the curve $x^2 + y^2 = 4, z = 2$.



Task 1

Answer

Let's find a generator of the cone, when $V(1, 1, 1)$. Also, let's consider that our line intersects the plane $z = 2$:

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{1}{n}, \Rightarrow x = \frac{l}{n} + 1, y = \frac{m}{n} + 1 \quad (1)$$

This point lies on the curve $x^2 + y^2 = 4$, hence - put (1) into the circle

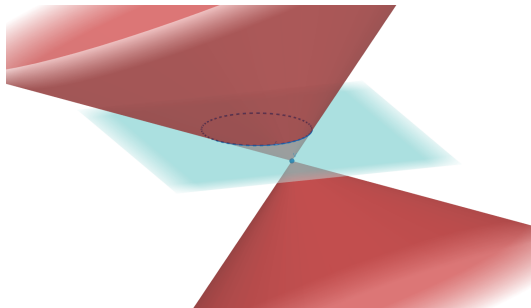
$$\left(\frac{l}{n} + 1\right)^2 + \left(\frac{m}{n} + 1\right)^2 = 4 \quad (2)$$

Our goal to eliminate l, m, n from equation above.

$$\frac{l}{n} = \frac{x-1}{z-1}, \frac{m}{n} = \frac{y-1}{z-1} \rightarrow (2)$$

$$\left(\frac{x-1}{z-1} + 1\right)^2 + \left(\frac{y-1}{z-1} + 1\right)^2 = 4 \text{ after simplification } \Rightarrow$$

$$\Rightarrow x^2 + y^2 - 2z^2 + 2xz + 2yz - 4x - 4y + 4 = 0$$



Task 2



Find the equation of the cone with its vertex at the origin, which passes through the curve $ax^2 + by^2 + cz^2 - 1 = 0 = \alpha x^2 - \beta y^2 - 2z$.



Task 2

Answer

This task is an upgraded version on the 1-st task.
 Right now, we don't know neither the center of a given curve, nor the conic section passed through the center. Let's find it

$$ax^2 + by^2 + cz^2 - 1 - \alpha x^2 + \beta y^2 + 2z = 0 \Rightarrow$$

$$\Rightarrow (a - \alpha)x^2 + (b + \beta)y^2 + c(z + \frac{1}{c})^2 = \frac{c + 1}{c}$$

The center of the curve is $(0, 0, -\frac{1}{c})$. Let's find a conic section in a plane, perpendicular to z axis (just simplification, can choose any).

$$(a - \alpha)x^2 + (b + \beta)y^2 = \frac{c + 1}{c} \quad (3)$$

Now, the task is the same as the first one. We know the vertex (origin) and a curve, which should be passed through.

$$\frac{x - 0}{l} = \frac{y - 0}{m} = \frac{-\frac{1}{c} - 0}{n}, \Rightarrow x = -\frac{l}{cn}, y = -\frac{m}{cn}$$

Put x, y into (3):

$$(a - \alpha) \frac{l^2}{c^2 n^2} + (b + \beta) \frac{m^2}{c^2 n^2} = \frac{c + 1}{c} \quad (4)$$

Based on equation of generator we will receive

$$\frac{l}{n} = \frac{x}{z}; \frac{m}{n} = \frac{y}{z}. \text{ Put it into (4)}$$

$$(a - \alpha) \frac{x^2}{z^2} + (b + \beta) \frac{y^2}{z^2} = c^2 + c \Rightarrow$$

$$(a - \alpha)x^2 + (b + \beta)y^2 - (c^2 + c)z^2 = 0$$

Task 3



Find the equation of the cone, which passes through the axes.

Task 3

Answer



The cone passes through the axes. Therefore, the vertex of the cone is the origin.

The equations of the cone is a homogeneous equation of second degree in x , y and z .

$$\text{(i.e.) } ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (15.25)$$

Given that x -axis is a generator.

Then $y = 0$, $z = 0$ must satisfy the [equation \(15.25\)](#)

$$\therefore a = 0$$

Since y -axis is a generator $b = 0$.

Since z -axis is a generator $c = 0$.

Hence the equation of the cone is $fyz + gzx + hxy = 0$.

Task 4



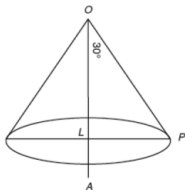
Find the equation of the right circular cone whose vertex is at the origin, whose axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which has a vertical angle of 60° .



Task 4

Answer

The axis of the cone is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.



Therefore, the direction ratios of the axis of the cone are 1, 2, 3.

The direction cosines are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

Let $P(x, y, z)$ be any point on the surface of the cone.
Let PL be perpendicular to OA .

$$\angle POL = 30^\circ$$

$$\frac{OL}{OP} = \cos 30^\circ \text{ or } 2OL = \sqrt{3}OP$$

Also,

$$OP^2 = x^2 + y^2 + z^2$$

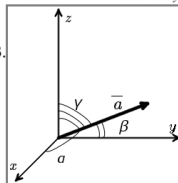
OL = Projection of OP on OA

$$= \frac{x}{\sqrt{14}} + y \times \frac{2}{\sqrt{14}} + z \times \frac{3}{\sqrt{14}} = \frac{x + 2y + 3z}{\sqrt{14}}$$

$$\therefore \frac{2(x + 2y + 3z)}{\sqrt{14}} = \sqrt{3} \sqrt{x^2 + y^2 + z^2}$$

$$4(x + 2y + 3z)^2 = 42(x^2 + y^2 + z^2)$$

$$\text{or } 19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$$



The **direction cosines** of the vector a are the cosines of angles that the vector forms with the coordinate axes.

$$\cos \alpha = \frac{a_x}{|a|}; \quad \cos \beta = \frac{a_y}{|a|}; \quad \cos \gamma = \frac{a_z}{|a|}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Task 5

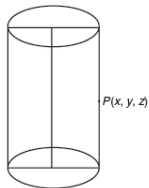


Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{-1} = \frac{y}{2} = \frac{z}{3}$ and whose directional curve is $x^2 + y^2 = 9, z = 1$.



Task 5

Answer



When the generator through P meets the guiding curve,

$$\frac{x-x_1}{-1} = \frac{y-y_1}{2} = \frac{z-z_1}{3}$$

$$\therefore x = x_1 - \frac{1-z_1}{3} = \frac{3x_1+z_1-1}{3}, y = y_1 + \frac{2(1-z_1)}{3} = \frac{3y_1-2z_1+2}{3}$$

Since this point lies on the curve (16.9),

$$(3x_1 + z_1 - 1)^2 + (3y_1 - 2z_1 + 2)^2 = 81$$

The locus of (x_1, y_1, z_1) is $(3x + z - 1)^2 + (3y - 2z + 2)^2 = 81$

$$\text{(i.e.) } 9x^2 + 9y^2 + 5z^2 + 6xz - 12yz - 6x + 12y - 10z - 76 = 0$$

This is the equation of the required cylinder.

The equations of the generator through P and parallel to the line

$$\frac{x}{-1} = \frac{y}{2} = \frac{z}{3} \text{ are } \frac{x-x_1}{-1} = \frac{y-y_1}{2} = \frac{z-z_1}{3} \quad (16.8)$$

The guiding curve is $x^2 + y^2 = 9, z = 1 \quad (16.9)$

Task 6



Find the equations of the right circular cylinder of radius 3 with equations of axis as

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}.$$



Task 6

Answer

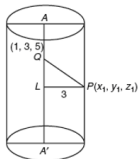
The equations of the axis are

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$$

(1, 3, 5) is a point on the axis.

2, 2, -1 are the direction ratios of the axis.

\therefore direction cosines are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$



Let $P(x_1, y_1, z_1)$ be any point on the cylinder. This is the equation of the required cylinder.

QL = Projection of PQ on the axis

$$\begin{aligned} &= (x-x_1)l + (y-y_1)m + (z-z_1)n \\ &= (x_1-1)\frac{2}{3} + (y_1-3)\frac{2}{3} - (z_1-5)\frac{1}{3} \\ &= \frac{2x_1 + 2y_1 - z_1 - 3}{3} \end{aligned}$$

Also, $PQ^2 = QL^2 + LP^2$

$$\text{(i.e.) } (x_1-1)^2 + (y_1-3)^2 + (z_1-5)^2 = \left(\frac{2x_1 + 2y_1 - z_1 - 3}{3} \right)^2 + 9$$

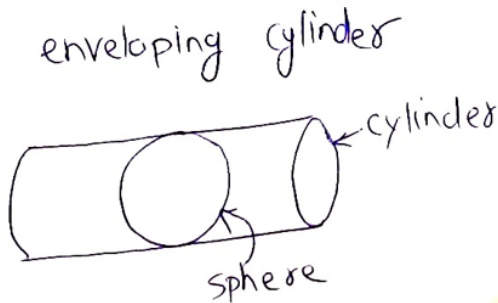
The locus of (x_1, y_1, z_1) is

$$\begin{aligned} &9(x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 10z + 25) \\ &= 4x^2 + 4y^2 + z^2 + 9 + 8xy - 4xz - 12x - 4yz - 12y + 6z + 81 \\ \text{(i.e.) } &5x^2 + 5y^2 + 8z^2 - 8xy + 4xz + 4yz - 6x - 42y - 96z + 225 = 0 \end{aligned}$$

Task 7



Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generators parallel to the line $x = y = z$.



Task 7

Answer

Let $P(x_1, y_1, z_1)$ be any point on a tangent, which is parallel to the line $x = y = z$.

Hence, the equation of the tangent lines are

$$\frac{x - x_1}{1} = \frac{y - y_1}{1} = \frac{z - z_1}{1} \quad (5)$$

Any point on this line is $(x_1 + \tau, y_1 + \tau, z_1 + \tau)$. This point lies in this sphere.

$$x^2 + y^2 + z^2 - 2x + 4y - 1 = 0 \quad (6)$$

$$(x_1 + \tau)^2 + (y_1 + \tau)^2 + (z_1 + \tau)^2 - 2(x_1 + \tau) + 4(y_1 + \tau) - 1 = 0 \Rightarrow$$

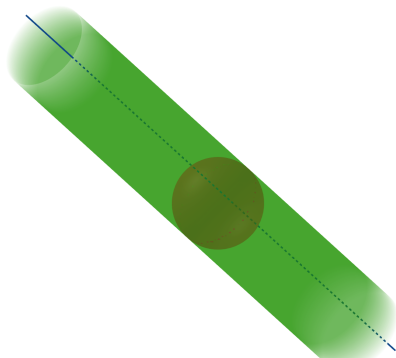
$$\Rightarrow 3\tau^2 + 2\tau(x_1 + y_1 + z_1 + 1) + (x_1^2 + y_1^2 + z_1^2 - 2x_1 + 4y_1 - 1) = 0$$

If (5) touches (6), τ is unique ($\mathcal{D} = 0$).

$$\mathcal{D} = 4(x_1 + y_1 + z_1 + 1)^2 - 12(x_1^2 + y_1^2 + z_1^2 - 2x_1 + 4y_1 - 1) = 0 \quad (7)$$

Solving (7) and change P to general form, we are obtaining the answer

$$x^2 + y^2 + 5y + z^2 - 4x - z - xy - xz - yz - 2 = 0$$



Reference material



- Cone, generatrix (OnlineMSchool)
- Direction cosines (OnlineMSchool)

Deserve "A" grade!

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📍 @Lupasic

🏠 Room 105 (Underground robotics lab)