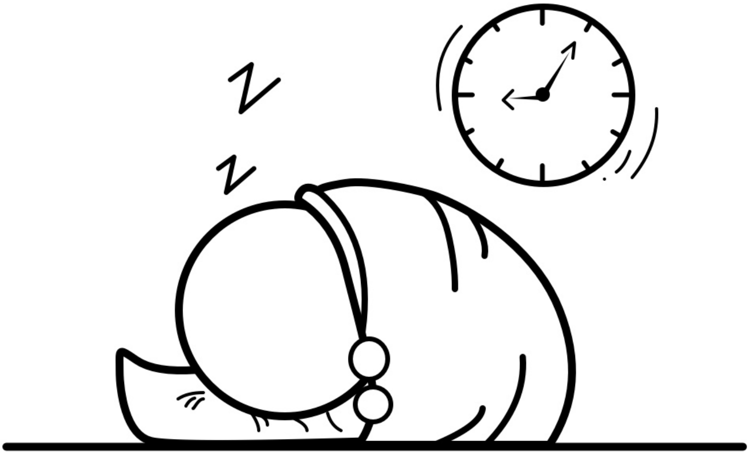
Time to Wake up

Daily Reminder



beautiful things happen
when you do the work to reprogram
that negative voice in your head

AGLA 1. TEST 1. **VARIANT 1.** 15 points, 60-75 minutes

1. (2 points) Given two vectors $\mathbf{p} = (1, 2, 3)$ and $\mathbf{q} = (1, -2, 2)$.

   (a) Decompose the vector $\mathbf{p}$ into two components that are parallel and perpendicular to the vector $\mathbf{q}$.

   (b) Find the angle between $\mathbf{p}$ and $\mathbf{q}$.

$\cos\alpha = \frac{pq}{\|p\| \|q\|} = \frac{1-4+6}{\sqrt{14}\cdot 3} = \frac{1}{\sqrt{14}}$

$\begin{cases} \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 2 \\ a & b & c \end{vmatrix} = 0 \\ a - 2b + 2c = 0 \end{cases}$

$a[4+6] + b + c[-2-2] =$

$= \begin{cases} 10a + b - 4c = 0 \\ a - 2b + 2c = 0 \end{cases}$

$a = 2b - 2c$

$20b - 20c + b - 4c = 0$

$21b = 24c \qquad b = \frac{8}{7}c$

$c = 7 \qquad b = 8 \qquad a = 16 - 14 = 2$

$r(2, 8, 7)$

$q(1, -2, 2)$

$p = xq + yr = x\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} x + 2y \\ -2x + 8y \\ 2x + 7y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$x = 1/3$

$15y = 5$

$y = 1/3$

$\begin{cases} \frac{1}{3} + \frac{2}{3} = 1 \\ -2/3 + \frac{8}{3} = 2 \\ 2/3 + 7/3 = 3 \end{cases}$ ✓

$= \frac{1}{3}q + \frac{1}{3}r$

2. (3 points)

   (a) Find the matrix product AB if $A = \begin{bmatrix} x & -2 & -1 \\ 4 & 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 1 & -3 \\ 2 & x \end{bmatrix}$

   (b) Find the largest and the smallest possible value of determinant $|AB|$.

$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & x \end{bmatrix} \begin{bmatrix} 5 & -1 \\ x & 2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 5+2x-15 & -1+4-5 \\ 15+7x-3x & -3+14-x \end{bmatrix} =$

$= \begin{bmatrix} 2x-10 & -2 \\ 15+4x & x+11 \end{bmatrix}$

$\det = (2x-10)(x+11) + 2(15+4x) =$

$= 2x^2 + 22x - 10x - 110 + 30 + 8x = 2x^2 + 20x + 80$

$x_{min} = -\frac{b}{2a} = -\frac{20}{4} = -5 \qquad y_{max} = +\infty$

$y_{min} = 50 - 100 + 80 = 30$

3. (4 points) For which values x, vectors $\mathbf{a}$ and $\mathbf{b}$ are basis of some space? Explain your answer.
$\mathbf{a} = \begin{bmatrix} 1-x \\ x \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1-x \\ -2 \end{bmatrix}$

$$\begin{vmatrix} 1-x & 1-x \\ x & -2 \end{vmatrix} = 0$$

$$-2+2x-x+x^2=0$$
$$x^2+x-2=0$$
$$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

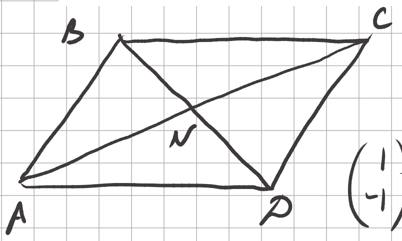
$$\boxed{\underline{\text{Ans}}: \; x \neq 1, \; x \neq -2}$$

4. (6 points) Given a parallelogram $ABCD$. Point N is the crossing of its diagonals. The old coordinate system has origin A and the basis AB, AD.

(a) Define a new coordinate system formed by the point D and two new basis vectors: DB and DC.

(b) Compute the transitions matrix A from the old basis to the new basis.

(c) Calculate coordinates of point N in both bases, using the transition matrix A.



A, AB, AD – old

D, DB, DC – new

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = DB = DA + AB = AB - AD$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = DC = AB + 0 \cdot AD$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = AD = 0 \cdot AB + AD$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\underset{\text{old}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{A}{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}} \underset{\text{new}}{\begin{pmatrix} x' \\ y' \end{pmatrix}} + \underset{\text{shift}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$N \text{ in new} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad 2 \; DN = \frac{1}{2} DB + 0\, DC$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$= \frac{1}{2} AB + \frac{1}{2} AD \text{ – correct}$$

AGLA 1. TEST 1. **VARIANT 2.** 15 points, 60-75 minutes

1. (2 points) Given two vectors $\mathbf{p} = (2, 4, 6)$ and $\mathbf{q} = (1, 2, -2)$.

   (a) Decompose the vector $\mathbf{p}$ into two components that are parallel and perpendicular to the vector $\mathbf{q}$.

   (b) Find the angle between $\mathbf{p}$ and $\mathbf{q}$.

$r = (a, b, c)$

$$\begin{cases} \begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & -2 \\ a & b & c \end{vmatrix} = 0 \\ a + 2b - 2c = 0 \end{cases} \qquad \begin{cases} -10a - 10b = 0 \\ a + 2b - 2c = 0 \end{cases}$$

$b = -2a$

$a - 4a - 2c = 0$

$2c = -3a$

$c = -\frac{3}{2}a$

$a = 2 \quad b = -4 \quad c = -3$

$$p = xq + yr = x\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad \begin{cases} x + 2y = 2 \\ 2x + 4y = 4 \\ -2x - 3y = 6 \end{cases} \qquad \begin{matrix} x = -18 \\ y = 10 \end{matrix}$$

$p = -18q + 10r$

$$(b)\quad \cos\alpha = \frac{pq}{\|p\|\,\|q\|} = \frac{2 + 8 - 12}{\sqrt{56} \cdot 3} = -\frac{2}{3\sqrt{56}} = -\frac{1}{3\sqrt{14}} \qquad \alpha = \arccos\left(-\frac{1}{3\sqrt{14}}\right)$$

2. (3 points)

   (a) Find the matrix product AB if $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & x & -5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ x & -3 \end{bmatrix}$

   (b) Find the largest and the smallest possible value of determinant $|AB|$.

$$\begin{bmatrix} 2 & x & 5 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & x \\ -3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 - 3x - 5 & 2x + 2x + 10 \\ 4 - 9 + 2 & 4x + 6 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3x - 3 & 4x + 10 \\ -3 & 4x + 2 \end{bmatrix}$$

$\det = -12x^2 - 6x \cancel{-12x} - 6 \cancel{+12x} + 30 =$

$= -12x^2 - 6x + 24$

$x_{max} = \frac{-6}{2a} = \frac{6}{-24} = -\frac{1}{4}$

$y_{max} = -12 \cdot \frac{1}{16} + \frac{6}{4} + 24 = -\frac{3}{4} + \frac{6}{4} + 24 = 24\frac{3}{4}$

$y_{min} = -\infty$

3. (4 points) For which values x, vectors **a** and **b** are basis of some space? Explain your answer.

$\mathbf{a} = \begin{bmatrix} 1-x \\ x \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1-x \\ -2 \end{bmatrix}$

the ~~same~~ ?

$a = \begin{bmatrix} x-6 \\ x-4 \end{bmatrix}$ $b = \begin{bmatrix} -x \\ -1 \end{bmatrix}$

$\begin{vmatrix} x-6 & -x \\ x-4 & -1 \end{vmatrix} = -x+6+x^2-4x = x^2-5x+6$

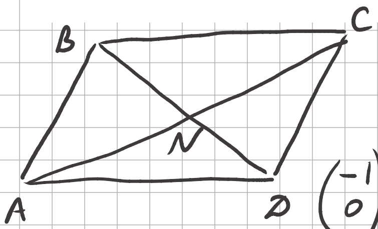
$x_{1,2} \neq \dfrac{5 \pm \sqrt{25-24}}{2} = \dfrac{5\pm 1}{2} \neq \begin{cases} 3 \\ 2 \end{cases}$

4. (6 points) Given a parallelogram $ABCD$. Point N is the crossing of its diagonals. The **old** coordinate system has origin A and the basis AN, AD.

(a) Define a **new** coordinate system formed by the point C and two **new** basis vectors: CN and CD.

(b) Compute the transitions matrix A from the old basis to the new basis.

(c) Calculate coordinates of point N in both bases, using the transition matrix A.



old : A, AN, AD

new : C, CN, CD

$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = CN = -AN + 0 \cdot AD$

$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = CD = CA + AD = -2AN + AD$

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = AC = 2AN + 0 \cdot AD$

$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$

$\underset{old}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{A}{\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}} \underset{new}{\begin{pmatrix} x' \\ y' \end{pmatrix}} + \underset{shift}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}$

N in new : $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

N in old : $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} =$

$= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = AN + 0 \cdot AD$