



Analytical Geometry and Linear Algebra I, Lab 9

Conic sections (2nd order curve equation):

- a) Parabola
- b) Ellipse

Questions from the class



No questions for today

Questions for today

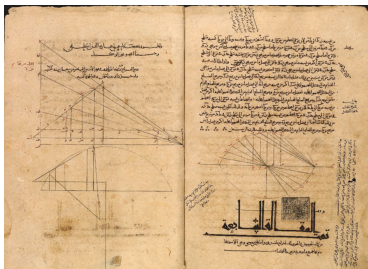


- How can I work with general form of 2nd order curve equation?
- How it relates with cone?
- What forms of equation do we have?



Why it is called <<Conic Sections>>

The greatest progress in the study of conics by the ancient Greeks is due to *Apollonius of Perga* (died c. 190 BCE), whose eight-volume **Conic Sections** or **Conics**. More info [here](#).



1654 edition of Conica

Books 5-7 are only available in an Arabic translation (9th century)

Elliptic Cone



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces

In plane $z = p$: an ellipse

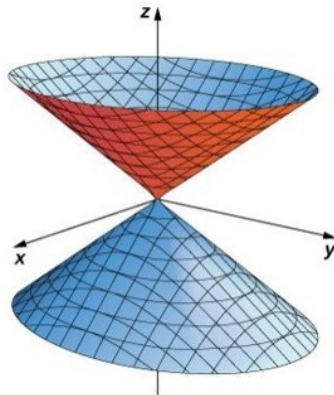
In plane $y = q$: a hyperbola

In plane $x = r$: a hyperbola

In the xz - plane: a pair of lines that intersect at the origin

In the yz - plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

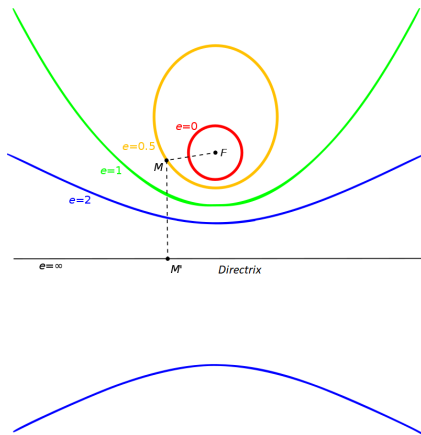


Some definitions, which can be helpful

Eccentricity, Directrix

Eccentricity is a measure of how much a conic section deviates from being circular.

It is a constant ration between distance from focal to point on the curve and from the point on the curve to **directrix**.



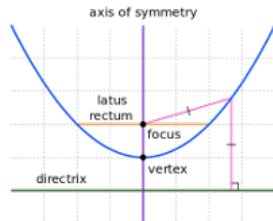
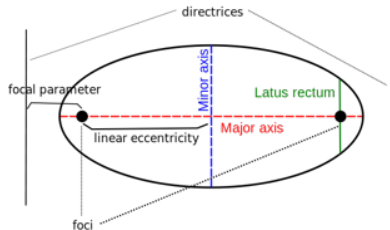
Some definitions, which can be helpful

Linear eccentricity, Latus Rectum, Focal parameter

The **linear eccentricity** is the distance between the center and the focus (or one of the two foci).

The **latus rectum** is the chord parallel to the directrix and passing through the focus (or one of the two foci).

The **focal parameter** is the distance from the focus (or one of the two foci) to the directrix.



Parabola

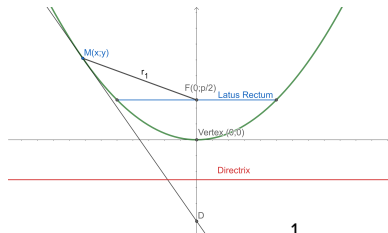
Forms:

- **Canonical** $(x - x_{\text{shift}})^2 = p(y - y_{\text{shift}})$
- **General** $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where either $A = 0$ or $C = 0$, not both, if $B = 0$
- **Parametric** $\begin{cases} x = \sqrt{2}pt \\ y = pt^2 \end{cases}$

Properties:

- **Vertex** $\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center** Not defined
- **Eccentricity** $e = 1$
- **Linear Eccentricity** Not defined
- **Foci** $F = \begin{pmatrix} 0 \\ \frac{p}{2} \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$

- **Latus Rectum** (length of chord) $|2p|$
- **Focal parameter** p
- **Discriminant** $\mathfrak{D} = B^2 - 4AC = 0$
- **Directrix eq.** $y = -\frac{p}{2} + y_{\text{shift}}$



Parabola $y = x^2$, $p = \frac{1}{2}$

- **Tangent eq.** $x(x_{\text{tan}} - x_{\text{shift}}) = p(y - y_{\text{tan}}) + x_{\text{tan}}(x_{\text{tan}} - x_{\text{shift}})$
- $r = |FM| = \sqrt{(x - \frac{p}{2})^2 + y^2}$
- $\triangle MFD$ is isosceles, where MD - tangent to M

From general to canonical form

When $B = 0$

$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0$ — General form.

Example of transformation from general to canonical form:

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0 \Rightarrow$$

$$(16x^2 - 32x) + (25y^2 + 50y) - 359 = 0 \Rightarrow$$

$$16(x^2 - 2x) + 25(y^2 + 2y) = 359 \Rightarrow$$

$$16(x^2 - 2x + 1) + 25(y^2 + 2y + 1) = 350 + 16 + 25 \Rightarrow$$

$$16(x - 1)^2 + 25(y + 1)^2 = 400 \Rightarrow$$

$$\frac{(x - 1)^2}{25} + \frac{(y + 1)^2}{16} = 1$$

Task 1

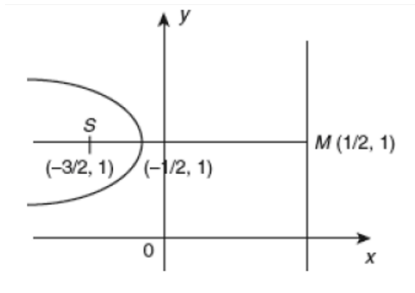


Find the foci, latus rectum, vertices and directrices of the following parabola:

$$y^2 + 4x - 2y + 3 = 0.$$

Task 1

Answer



i.

$$y^2 + 4x - 2y + 3 = 0$$

$$y^2 - 2y = -4x - 3$$

$$y^2 - 2y + 1 = -4x - 3 + 1$$

$$\Rightarrow (y-1)^2 = -4\left(x + \frac{1}{2}\right)$$

Take $x + \frac{1}{2} = X$, $y - 1 = Y$. Shifting the origin to the point $\left(-\frac{1}{2}, 1\right)$ the equation of the parabola becomes $Y^2 = -4X$.

\therefore Vertex is $\left(-\frac{1}{2}, 1\right)$, latus rectum is 4, focus is $\left(-\frac{3}{2}, 1\right)$ and foot of the directrix is $\left(\frac{1}{2}, 1\right)$.

The equation of the directrix is $x = \frac{1}{2}$ or $2x - 1 = 0$.

Task 2



Find the equations of the tangent and normal to the parabola $y^2 = 4(x - 1)$ at $(5, 4)$.

Task 2

Answer

$$y^2 = 4(x - 1)$$

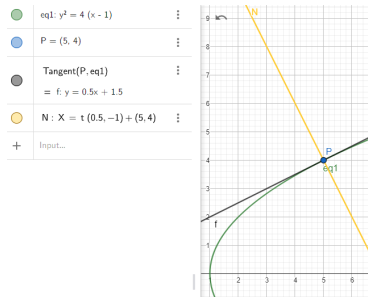
Differentiating with respect to x ,

$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$
$$\left(\frac{dy}{dx} \right)_{at(5, 4)} = \frac{2}{4} = \frac{1}{2} = \text{Slope of the tangent at } (5, 4)$$

\therefore The equation of the tangent at $(5, 4)$ is $y - 4 = \frac{1}{2}(x - 5)$.

$2y - 8 = x - 5$ or $x - 2y + 3 = 0$. The slope of the normal at $(5, 4)$ is -2 .

\therefore The equation of normal at $(5, 4)$ is $y - 4 = -2(x - 5)$ or $2x + y = 14$.



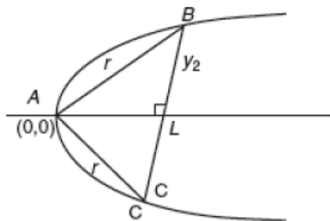
Task 3



An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ one of whose vertices is at the vertex of the parabola. Find its side.

Task 3

Answer



The coordinates of B are $B(r \cos 30^\circ, r \sin 30^\circ)$, $\left(\frac{\sqrt{3}}{2}r, \frac{r}{2}\right)$.

Since this point lies on the parabola $y^2 = 4ax$, then

$$\frac{r^2}{4} = 4a \cdot \frac{r}{2} \sqrt{3} \quad \therefore r = 8a\sqrt{3}$$

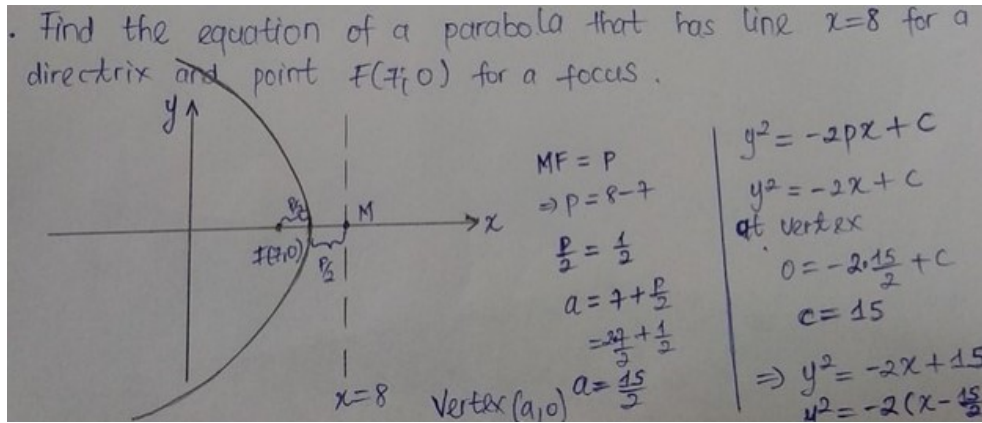
Task 4



Find the equation of a parabola that has a line $x = 8$ for a directrix and point $F(7; 0)$ for a focus.

Task 4

Answer



Collisions

Video



Ellipse

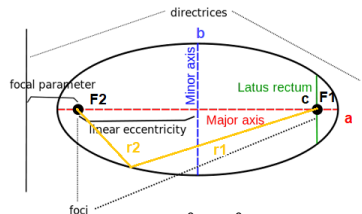
Forms:

- **Canonical** $\frac{(x-x_{\text{shift}})^2}{a^2} + \frac{(y-y_{\text{shift}})^2}{b^2} = 1$
- **General** $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $AC > 0$, if $B = 0$
- **Parametric** $\begin{cases} x = a \cos(\alpha) \\ y = b \sin(\alpha) \end{cases}$

Properties:

- **Vert** $\begin{pmatrix} \pm a \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ \pm b \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center** $(0; 0) + (x_{\text{shift}}; y_{\text{shift}})$
- **Eccentricity** $0 \leq e < 1$,
 $e = \sqrt{1 - \frac{b^2}{a^2}}$
- **Linear Eccentricity**
 $c = \sqrt{a^2 - b^2}$

- **Foci**
 $F = \begin{pmatrix} \pm(c = e a) \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Latus Rectum** (length of chord)
 $\frac{2b^2}{a}$
- **Focal parameter** $\frac{b^2}{\sqrt{a^2 - b^2}}$
- **Discriminant** $\mathfrak{D} = B^2 - 4AC < 0$



$$\text{Ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- **Directrix eq.** $x = \pm \frac{a}{e} + x_{\text{shift}}$
- **Tangent eq. (w/o shift)**
 $\frac{x_{\text{tangent}}x}{a^2} + \frac{y_{\text{tangent}}y}{b^2} = 1$
- $r_1 + r_2 = 2a$
- $r_{1,2} = |F_{1,2}M| = \sqrt{(x \pm c)^2 + y^2}$
- $\frac{r_1}{d_1} = e$

Task 5



Find the equation of the ellipse whose foci are $(4, 0)$ and $(-4, 0)$ and $e = 1/3$



Task 5

Answer

i. If the foci are $(ae, 0)$ and $(-ae, 0)$ then the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Here, $ae = 4$ and $e = \frac{1}{3}$.

$$a = \frac{4}{e} = 4 \times 3 = 12$$

$$b^2 = a^2(1 - e^2) = 144 \left(1 - \frac{1}{9}\right) = 144 \times \frac{8}{9} = 128$$

\therefore The equation of the ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$.

Task 6



Find the eccentricity, foci and the length of the latus rectum of the ellipse $9x^2 + 4y^2 = 36$



Task 6

Answer

i. $9x^2 + 4y^2 = 36$

Dividing by 36, we get

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1$$

$$\text{(i.e.) } \frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\therefore a^2 = 4, \quad b^2 = 9.$$

This is an ellipse whose major axis is the y -axis and minor axis is the x -axis and centre at the origin.

$$\therefore a^2 = b^2(1 - e^2) \Rightarrow 4 = 9(1 - e^2)$$

$$\therefore 9e^2 = 5$$

$$\text{Therefore, eccentricity} = e = \frac{\sqrt{5}}{3}$$

$$\text{Therefore, foci are } \left(0, \pm \frac{be}{1}\right) \text{ (i.e.) } (0, \pm \sqrt{5}).$$

$$\text{Therefore, latus rectum} = \frac{2a^2}{b} = 2 \times \frac{4}{3} = \frac{8}{3}.$$



Task 7

The equation $25(x^2 - 6x + 9) + 16y^2 = 400$ represents an ellipse. Find the centre and foci of the ellipse. How should the axis be transformed so that the ellipse is represented by the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$?



Task 7

Answer

$$25(x^2 - 6x + 9) + 16y^2 = 400$$

$$25(x - 3)^2 + 16y^2 = 400$$

Dividing by 400, $\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1$; Take $x - 3 = X$, $y = Y$.

$$\text{Then } \frac{X^2}{16} + \frac{Y^2}{25} = 1.$$

The major axis of this ellipse is the Y-axis.

$$\begin{aligned}\therefore 16 &= 25(1 - e^2) \Rightarrow 1 - e^2 = \frac{16}{25} \Rightarrow e^2 = 1 - \frac{16}{25} = \frac{9}{25} \\ \therefore e &= \frac{3}{5}.\end{aligned}$$

Centre is $(3, 0)$. Foci are $(3, \pm ae)$ (i.e.) $\left(3, \pm 5 \times \frac{3}{5}\right)$ (i.e.) $(3, \pm 3)$. Now

shift origin to the point $(3, 0)$ and then rotate the axes through right

angles. Then the equation of the ellipse becomes $\frac{x^2}{25} + \frac{y^2}{16} = 1$.



Task 8

Find the eccentricity of an ellipse given that:

1. its major axis subtends an angle of 120° at the endpoints of its minor axis;
2. the segment between a focus and the farthest vertex subtends an angle of 90° at the endpoints of its minor axis.

Task 8

Answer

Find the eccentricity of an ellipse given that
(a) Its major axis subtends an angle 120° at the endpoints of its minor axis.

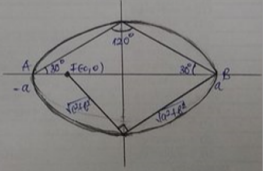
$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$\tan 30^\circ = \frac{b}{a}$$

$$\sqrt{3}b = a \Rightarrow b^2 = \frac{a^2}{3}$$

$$e = \sqrt{1 - \frac{1}{3}} \quad \frac{b^2}{a^2} = \frac{1}{3}$$

$$e = \sqrt{\frac{2}{3}}$$



(b) The segment between a focus and the farthest vertex subtend an angle of 90° at the endpoints of its minor axis

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}$$

$$c^2 = a^2 - b^2$$

$$c^2 = a^2 - ac$$

$$c^2 + ac - a^2 = 0$$

$$c_{1,2} = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2}$$

$$e > 0, \quad e = \frac{c}{a}$$

$$e = \frac{-1 + \sqrt{5}}{2}$$

$$c^2 + b^2 + b^2 + a^2 = (a+c)^2$$

$$2b^2 + a^2 + c^2 = a^2 + 2ac + c^2$$

$$2b^2 = 2ac$$

$$b^2 = ac$$

Reference material



- [Conics Section \(Wiki\)](#)
- [Conic sections \(Khan Academy, full playlist, eng\)](#)

Deserve "A" grade!

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📍 @Lupasic

🏠 Room 105 (Underground robotics lab)