



# Analytical Geometry and Linear Algebra I, Lab 12

Affine Transformation

Bijection, injection and surjection

# Affine Transformation

Video: *formal definition*



## Affine Transformations

- \* Combines linear transformations, and Translations
- \* Properties
  - \* Origin does not necessarily map to origin



$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

# Affine Transformation

*Formal definition*

**Classical representation:**

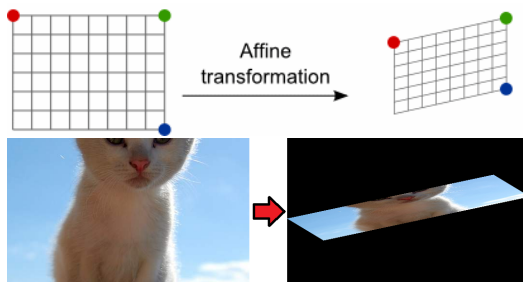
$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

**System of equations representation:**

$$\begin{cases} x^* = ax + by + x_0 \\ y^* = cx + dy + y_0 \end{cases}$$

**Homogeneous representation:**

$$\begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



[Link to code + extra tasks](#)



# Affine Transformation

## *Properties*

- *Collinearity between points*: three or more points which lie on the same line (called collinear points) continue to be collinear after the transformation.
- *Parallelism*: two or more lines which are parallel, continue to be parallel after the transformation.
- *Convexity of sets*: a convex set continues to be convex after the transformation. Moreover, the extreme points of the original set are mapped to the extreme points of the transformed set.
- *Ratios of lengths* of parallel line segments are the same after the transformation.

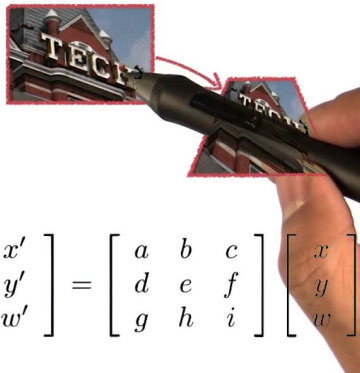
# Projective Transformation

Video: *formal definition*



## Projective Transformations

- \* Combination of Affine transformations, and Projective warps
- \* Properties:
  - \* Origin does not necessarily map to origin
  - \* Lines map to lines
  - \* Parallel lines do not necessarily remain parallel



$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

## Task 1



Linear transformation of a real axis is given by  $f(x) = ax + b$ . (a) Find all fixed points of this transformation. (b) Find the transformation that is inverse for  $f$ .



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Linear transformation of a real axis is given by  $f(x) = ax + b$ . (a) Find all fixed points of this transformation. (b) Find the transformation that is inverse for  $f$ .

### Answer

(a) If  $a \neq 1$  then there is one fixed point  $x = \frac{b}{1-a}$ ; if  $a = 1$  and  $b = 0$  then all points are fixed; if  $a = 1$  and  $b \neq 0$  then there are no fixed points. (b) It exists only if  $a \neq 0$ :

$$f(x^*) = \frac{x^* - b}{a}.$$

## Task 2



An affine transformation is given by  $x^* = 3x + 2y - 6$ ,  $y^* = 4x - 3y + 1$ . Find the images of (a) point  $M(-1; 5)$ ; (b) line  $2x + 3y = 7$ .





## Task 2

An affine transformation is given by  $x^* = 3x + 2y - 6$ ,  $y^* = 4x - 3y + 1$ . Find the images of (a) point  $M(-1; 5)$ ; (b) line  $2x + 3y = 7$ .

### Answer

(a)  $(1; -18)$

(b)  $18x - 5y - 6 = 0$ .

## Task 3



Two linear transformations of a real axis  $f$  and  $g$  are given by  $f(x) = ax + b$ ,  $g(x) = cx + d$ . Find compositions of transformations  $fg$  and  $gf$ . What are the necessary and sufficient conditions for  $fg$  to be equal to  $gf$ ?



## Task 3

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### Answer

$$(fg)(x) = acx + ad + b; (gf)(x) = acx + bc + d; fg = gf \Leftrightarrow d(a - 1) = b(c - 1).$$

# Bijection, injection and surjection

## Injective and surjective (bijective)

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto x.$$

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+ : x \mapsto x^2, \text{ and thus also its inverse}$$

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+ : x \mapsto \sqrt{x}.$$

## Injective and non-surjective

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto e^x.$$

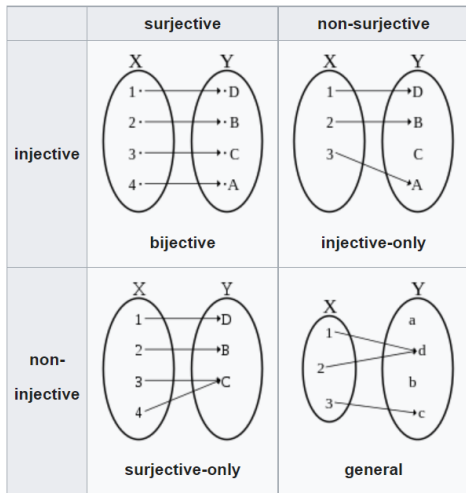
## Non-injective and surjective

$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^3 - x.$$

$$\mathbb{R} \rightarrow [-1, 1] : x \mapsto \sin(x).$$

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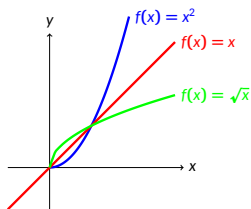
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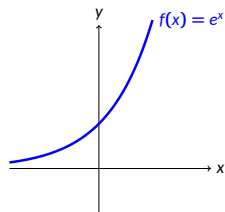
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## Non-injective and non-surjective

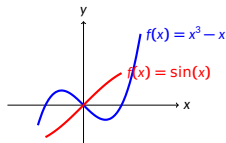
$$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto \sin(x).$$



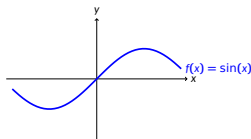
Bijjective



Injective-only



Surjective-only



General

## Task 4



Transformation of a plane is given by  $x^* = x^2 - y^2$ ,  $y^* = 2xy$ . Is this transformation an (a) injection; (b) surjection; (c) bijection?

## Task 4



### Answer

- (a): Not Injection. We can take 2 pairs  $(a, b)$ ,  $(-a, -b)$ , which provides the same result
- (b): Surjection. Need to proof that it is a continuous function ()
- (c): Not Bijection. Because of (a)

# Affine Transformation

*In Computer Vision (CV)*

$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

is equivalent to the following

$$\vec{y} = A\vec{x} + \vec{b}.$$

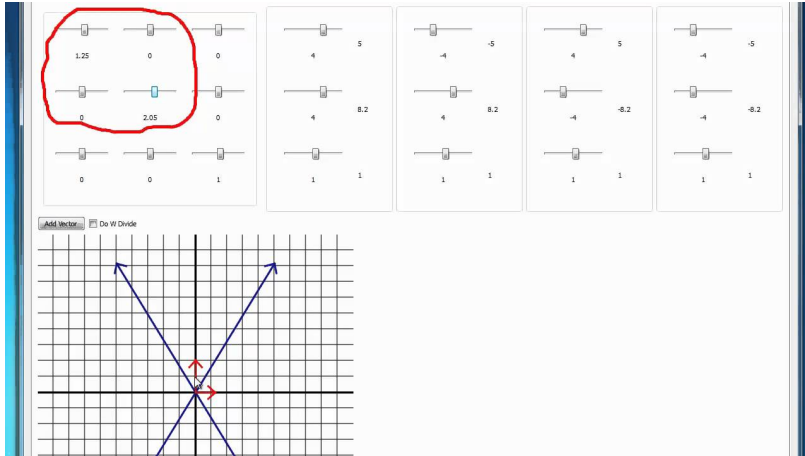
|        |  |   |
|--------|--|---|
| Scale  | $\begin{bmatrix} c_x = 2 & 0 & 0 \\ 0 & c_y = 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$                                  |   |
| Rotate | $\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | where $\theta = \frac{\pi}{6} = 30^\circ$ |
| Shear  | $\begin{bmatrix} 1 & c_x = 0.5 & 0 \\ c_y = 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$                                |   |

| Transformation name                    | Affine matrix  | Example |
|--|--|---------|
| Identity (transform to original image) | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  |         |
| Reflection                             | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |         |



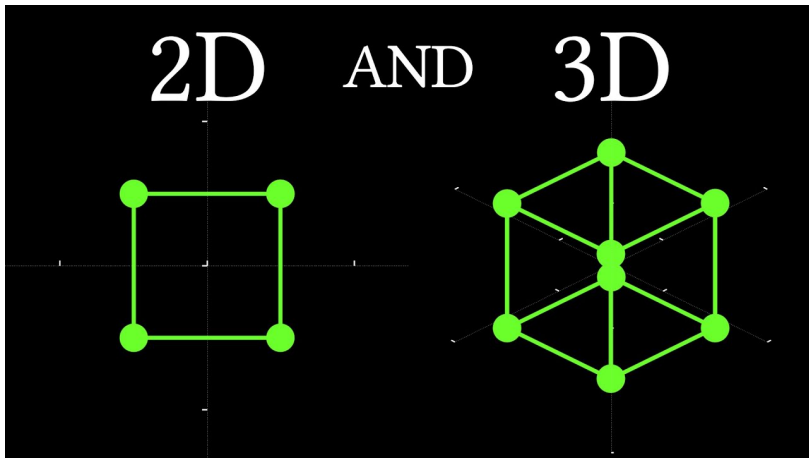
# Affine Transformation

*Video: intuition besides numbers 2D*



# Affine Transformation

*Video*



# Affine Transformation

## *Applications*

- Generation dataset for machine learning
- Camera calibration [What Is Camera Calibration?](#)
- Making panorama



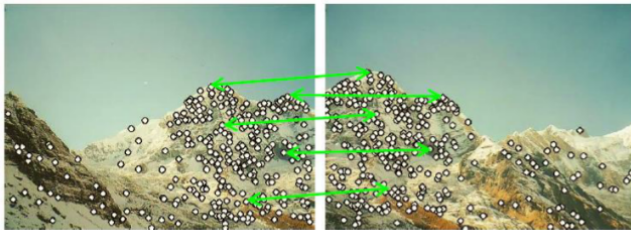
# Affine Transformation

*Application in CV (1)*



## Point Features: how to build a panorama?

- Detect feature points in both images
- Find corresponding pairs



# Affine Transformation

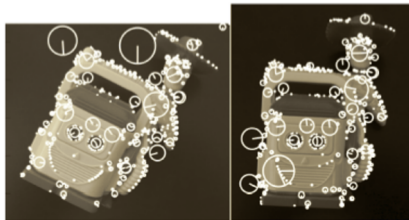
*Application in CV (2)*

## SIFT features [Lowe et al., IJCV 2004]

- **SIFT** = Scale Invariant Feature Transform  
an approach for detecting and describing regions of interest in an image
- SIFT features are reasonably **invariant** to changes in:  
rotation, scaling, changes in viewpoint, illumination
- SIFT **detector** uses **DoG kernel**, SIFT **descriptor** is based on **gradient orientations**
- Very powerful in capturing + describing **distinctive** structure, but also **computationally demanding**

**Main SIFT stages:**

1. Extract keypoints + scale
2. Assign keypoint orientation
3. Generate keypoint descriptor



## Task 5

Find the image of an arbitrary point  $M$  which has position vector  $\mathbf{r}$  by the following transformations:

- (a) homothety with center  $M_0(\mathbf{r}_0)$  and ratio  $\lambda \neq 0$ ;
- (b) reflection across point  $M_0(\mathbf{r}_0)$ ;
- (c) translation by vector  $\mathbf{a}$ ;
- (d) orthogonal projection onto the line  $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$ ;
- (e) reflection across the line  $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$ ;
- (f) dilation of factor  $\lambda > 0$  from the line  $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$ .



## Task 5

### Answer

$$(a) \mathbf{r}^* = \mathbf{r}_0 + \lambda(\mathbf{r} - \mathbf{r}_0);$$

$$(b) \mathbf{r}^* = -\mathbf{r} + 2\mathbf{r}_0;$$

$$(c) \mathbf{r}^* = \mathbf{r} + \mathbf{a};$$

$$(d) \mathbf{r}^* = \mathbf{r}_0 + \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a};$$

$$(e) \mathbf{r}^* = 2\mathbf{r}_0 - \mathbf{r} + 2 \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a};$$

$$(f) \mathbf{r}^* = \lambda \mathbf{r} + (1 - \lambda) \mathbf{r}_0 + (1 - \lambda) \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a}.$$



## Task 6

Find formulas for the following affine transformations:

(a) orthogonal projection onto line  $x - 3y + 1 = 0$ ;

(b) reflection across line  $3x + 4y - 1 = 0$ ;

(c) dilation from line  $x + y - 2 = 0$  of factor  $\frac{1}{3}$ ;

(d) dilation from line  $2x - y + 5 = 0$  of factor 2.





## Task 6

### Answer

$$(a) x^* = \frac{9x + 3y - 1}{10}, y^* = \frac{3x + y + 3}{10};$$

$$(b) x^* = \frac{7x - 24y + 6}{25}, y^* = -24x - 7y + 825;$$

$$(c) x^* = \frac{2x - y + 2}{3}, y^* = \frac{-x + 2y + 2}{3};$$

$$(d) x^* = \frac{9x - 2y + 10}{5}, y^* = \frac{-2x + 6y - 5}{5}.$$



## Task 7

Find formulas for an affine mapping that transforms

(a) points  $A(\frac{3}{7}; 1)$ ,  $B(1; \frac{1}{4})$ ,  $C(2; -1)$  into points  $A^*(-4; 2)$ ,  $B^*(-1; 6)$ ,  $C^*(4; 13)$  respectively;

(b) points  $A(0; 0)$ ,  $B(-1; 2)$ ,  $C(1; -2)$  into points  $A^*(-1; -1)$ ,  $B^*(0; 0)$ ,  $C^*(1; 1)$  respectively;

(c) points  $A(2; 0)$ ,  $B(3; -1)$ ,  $C(4; -2)$  into points  $A^*(2; 1)$ ,  $B^*(-2; -1)$ ,  $C^*(-6; -3)$  respectively;

(d) points  $A(-2; 0)$ ,  $B(2; -1)$ ,  $C(0; 4)$  into points  $A^*(-2; 1)$ ,  $B^*(2; 1)$ ,  $C^*(0; 1)$  respectively.

## Task 7



### Answer

- (a)  $x^* = -4y$ ,  $y^* = 7x - 1$ ;
- (b) no solutions;
- (c)  $x^* = px + (p + 4)y + 2 - 2p$ ,  $y^* = qx + (q + 2)y + 1 - 2q$ , where  $p$  and  $q$  are any real numbers;
- (d) no solutions (there exists a linear transformation that is not affine).

## Task 8



Find all invariant lines of an affine transformation given by

(a)  $x^* = y, y^* = 1 - x;$

(b)  $x^* = 2x + y - 3, y^* = -3x - y;$

(c)  $x^* = 5x + 3y + 1, y^* = -3x - y.$



## Task 8

Find all invariant lines of an affine transformation given by

(a)  $x^* = y, y^* = 1 - x$ ;

(b)  $x^* = 2x + y - 3, y^* = -3x - y$ ;

(c)  $x^* = 5x + 3y + 1, y^* = -3x - y$ .

### Answer

(a) no solutions;

(b)  $x + y - 3 = 0, 2x - y + p = 0$ , where  $p$  can be any real number;

(c)  $x + y + 1 = 0$ .

# Reference material



- [Bijection, injection and surjection \(wiki\)](#)
- [Affine transformation \(wiki\)](#)
- [The Math behind \(most\) 3D games - Perspective Projection - YouTube](#)
- [Affine transformations | Brilliant Math & Science Wiki](#)
- [OpenCV: Affine Transformations](#)

# Deserve "A" grade!

– Oleg Bulichev

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📍 @Lupasic

🏢 Room 105 (Underground robotics lab)