Analytical Geometry and Linear Algebra I, Lab 4

Matrix Rank

Test 1 Solutions

Q/A session

1. Чилл пара

Questions from the class

No questions for today

Matrix Rank

Definition

$N_r(A)$ — max number of **lineary independent** rows of matrix A.

$N_c(A)$ — max number of **lineary independent** columns of matrix A.

$$Rank(A) = N_r(A) = N_c(A)$$

The rank of the matrix is how many of the rows (columns) are «unique»: not formed out by other rows (columns).

Matrix Rank

Motivation

- Computation of the number of solutions of a system of linear equations.
- Analysis of the linear dependency of rows and columns.
- Applications in Control Theory (next year): observability and controllability.

Matrix Rank

How to find

There are 3 ways:

1. **Look at matrix** and find linear dependencies.
2. **Reduced form** (transform matrix to upper triangular form (The first part of the algorithm for finding inverse matrix).
3. **Minor method** (Метод окаймляющих миноров) *not popular in western education*.

Matrix Rank

Case Study (on whiteboard)

Calculate the rank of the following matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

Answer: 2

Matrix Rank

Task 2

Determine the ranks of the following matrices for all real values of parameter α:

1. $\begin{bmatrix} 1 & \alpha & -1 & 2 \\ 2 & -1 & \alpha & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix}$;
2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$;

How to get out of an exam

Video

Test 1, Solutions

Task 1

(2 points) For each of the following statements mark it as True or False. Justify each answer.

1. If matrix B is produced by interchanging two columns of matrix A, then $\det(B) = -\det(A)$. Explain your answer in 2×2 case.	
2. For any square matrix A there exists exactly one inverse matrix.	

Test 1, Solutions

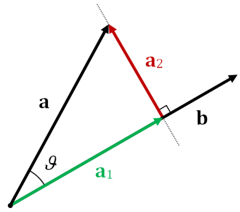
Task 2

(2 points) Decompose the vector $\mathbf{p} = (1, 2, 3)$ into components parallel and perpendicular to the vector $\mathbf{q} = (1, -2, 2)$.

Projection

Definition

The *vector projection* of a vector **a** on (or onto) a nonzero vector **b**, sometimes denoted $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$ is the orthogonal projection of **a** onto a straight line parallel to **b**.



Projection of **a** on **b** ($\mathbf{a}_1$), and rejection of **a** from **b** ($\mathbf{a}_2$)

Where it can be used:

- Maps
- Blueprints
- Fitting algorithms (Least squares)
- Reduce matrix dimension
- Reinforcement Learning (RL) fitness functions

Projection (1)

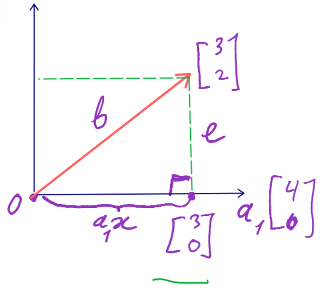
2D case Classical way
Project "b" on "a_1"

$$e = b - a_1 x$$

$$a_1 \cdot (b - a_1 x) = 0$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T b = a_1^T a_1 x$$



$$\frac{a_1^T b}{a_1^T a_1} = x$$ – classic formula from school

Particular example

$$\frac{a_1^T b}{a_1^T a_1} = x$$

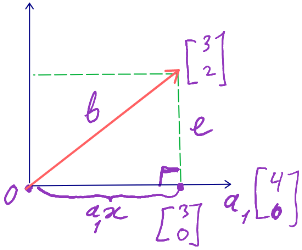
$$\Downarrow$$

$$\frac{[4\ 0]\begin{bmatrix}3\\2\end{bmatrix}}{[4\ 0]\begin{bmatrix}4\\0\end{bmatrix}} = \frac{12}{16} = \frac{3}{4}$$

projection $p = a_1 x = \frac{3}{4}\begin{bmatrix}4\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$

Projection (2)

2D case
Projection matrix
Project "b" on "a_1"



Like affine
transformation matrix

$$Pb = x a_1 = a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x \quad \Rightarrow \quad P = \frac{a_1 a_1^T}{a_1^T a_1}$$

Projection matrix

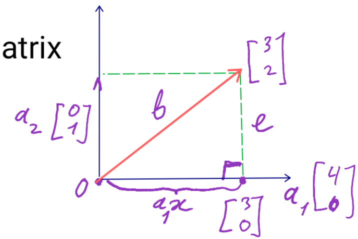
Particular example

$$P = \frac{\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}4 & 0\end{bmatrix}}{16} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$p = Pb = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

Projection (3)

3D case
Projection matrix
Project "b"
on ["a_1", "a_2"]



$A^T(b - Ax) = 0$

$A^TAx = A^Tb$

$\underbrace{A(A^TA)^{-1}A^T}_{P} b = Ax$

projection matrix

It will be Identity for full rank square matrix

Particular example

$P = [a_1\ a_2]\left(\begin{bmatrix}a_1^T\\a_2^T\end{bmatrix}[a_1\ a_2]\right)^{-1}\begin{bmatrix}a_1^T\\a_2^T\end{bmatrix}$

$= \begin{bmatrix}4&0\\0&1\end{bmatrix}\left(\begin{bmatrix}4&0\\0&1\end{bmatrix}\begin{bmatrix}4&0\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}4&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

It's correct, because we project "b" on a plane, where it lies

$p = Pb = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$

Projection (4)

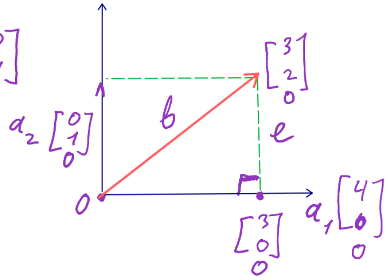
2D case
Project "b"
on "d_1" perpendicular to "a_1"

Error between whole space and current projection matrices

$P_{d_1} = I - P = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

$P = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$p_{d_1} = P_{d_1} b = \begin{bmatrix}0\\ 2\end{bmatrix}$



3D case
Project "b"
on "d_2" perpendicular
to ["a_1", "a_2"]

$P_{d_2} = I - P = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$; $p_{d_2} = P_{d_2} b = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$P = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$

Projection

Case study: Reinforcement Learning fitness function

Goal

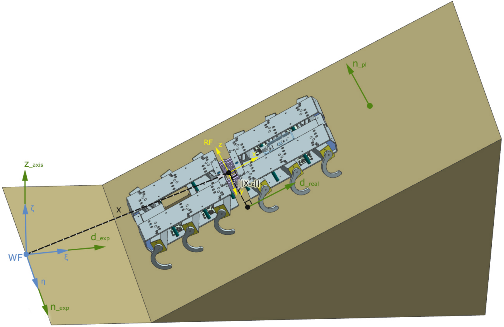
It is necessary for the robot to move in a straight line in all directions, as well as as as efficiently as possible.

The efficiency criteria are: course deviation error, max velocity and clearance.

$$F = \omega_1 X_z + \omega_2 \frac{1}{|err| + \varepsilon} + \omega_3 (P_{d_{real}} \vec{X}), \text{ where}$$

$$err = |(I - P_{d_{real}})(I - P_{n_{pl}})\vec{X}|,$$

P_* – projection matrix, ω_* – weight coeffs.



StriRus – task description

Test 1, Solutions

Task 3

1. Find the matrix product AB if $A = \begin{bmatrix} x & -2 & -1 \\ 4 & 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 1 & -3 \\ 2 & x \end{bmatrix}$
2. Find the largest possible value of determinant (AB).

Test 1, Solutions

Task 4

(3 points) Point A has coordinates (5; -1; 8) in the old coordinate system. Find its coordinates in the new coordinate system obtained from the initial one by transferring the origin to point N that has coordinates (33; -1; 2) in the old coordinate system.

Test 1, Solutions

Task 5

(3 points) Subspace S of $\mathbb{R}^3$ is formed by linear combination of vectors v_1 and v_2. Find a vector v that is orthogonal to S, if $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

2022-09-27

AGLA1

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Task 5

(3 points) Subspace S of $\mathbb{R}^3$ is formed by linear combination of vectors v_1 and v_2. Find a vector v that is orthogonal to S, if $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Обсудить когда будут линейно зависимые и линейно независимые результаты

Test 1, Solutions

Task 6

(3 points) Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1}$$

Find All **Natural** numbers ($k \in \mathbb{N}$) where: $A^k = A^{-1}$ (Also you need to check if A is invertible), Note that $A^k = \underbrace{A.A\dots A}_{k\ times}$

Reference material

- Matrix Rank (OnlineMschool)

Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)