Analytical Geometry and Linear Algebra I, Lab 12

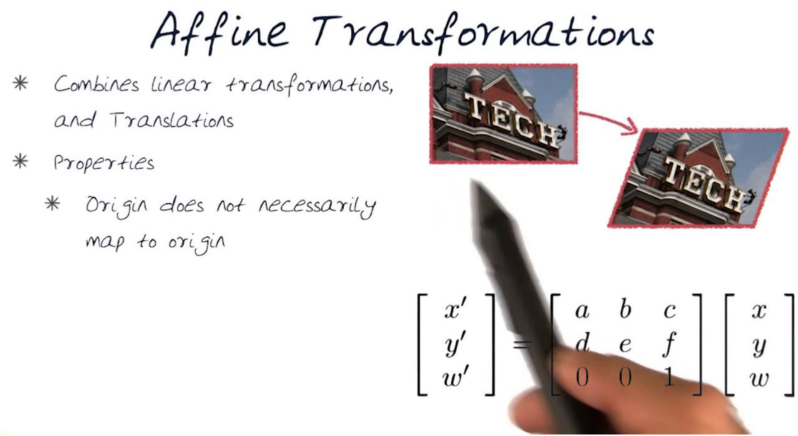
Affine Transformation

Bijection, injection and surjection

Innopolis University

Affine Transformation

Video: formal definition



Affine Transformation

Formal definition

Classical representation:

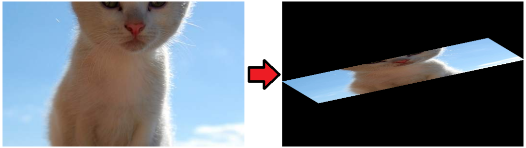
$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

System of equations representation:

$$\begin{cases} x^* = ax + by + x_0 \\ y^* = cx + dy + y_0 \end{cases}$$

Homogeneous representation:

$$\begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Link to code + extra tasks

Affine Transformation

Properties

- *Collinearity between points*: three or more points which lie on the same line (called collinear points) continue to be collinear after the transformation.
- *Parallelism*: two or more lines which are parallel, continue to be parallel after the transformation.
- *Convexity of sets*: a convex set continues to be convex after the transformation. Moreover, the extreme points of the original set are mapped to the extreme points of the transformed set.
- *Ratios of lengths* of parallel line segments are the same after the transformation.

Projective Transformation

Video: formal definition

Projective Transformations

* Combination of Affine transformations, and Projective warps
* Properties:
  * Origin does not necessarily map to origin
  * Lines map to lines
  * Parallel lines do not necessarily remain parallel

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Task 1 (Solutions are in *handwritten_solutions* folder)

Linear transformation of a real axis is given by $f(x) = ax + b$. (a) Find all fixed points of this transformation. (b) Find the transformation that is inverse for f.

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Answer

(a) If $a \neq 1$ then there is one fixed point $x = \dfrac{b}{1-a}$; if $a = 1$ and $b = 0$ then all points are fixed; if $a = 1$ and $b \neq 0$ then there are no fixed points. (b) It exists only if $a \neq 0$:

$$f(x^*) = \frac{x^* - b}{a}.$$

Task 2

An affine transformation is given by $x^* = 3x + 2y - 6$, $y^* = 4x - 3y + 1$. Find the images of (a) point $M(-1;\ 5)$; (b) line $2x + 3y = 7$.

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Answer

(a) $(1; -18)$

(b) $18x - 5y - 6 = 0$.

Task 3

Two linear transformations of a real axis f and g are given by $f(x) = ax + b$, $g(x) = cx + d$. Find compositions of transformations fg and gf. What are the necessary and sufficient conditions for fg to be equal to gf?

Task 3

Two linear transformations of a real axis f and g are given by $f(x) = ax + b$, $g(x) = cx + d$. Find compositions of transformations fg and gf. What are the necessary and sufficient conditions for fg to be equal to gf?

Answer

$(fg)(x) = acx + ad + b$; $(gf)(x) = acx + bc + d$; $fg = gf \Leftrightarrow d(a-1) = b(c-1)$.

Bijection, injection and surjection

Injective and surjective (bijective)

$\mathbb{R} \to \mathbb{R} : x \mapsto x.$

$\mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^2$, and thus also its inverse

$\mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto \sqrt{x}..$

Injective and non-surjective

$\mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{e}^x.$

Non-injective and surjective

$\mathbb{R} \to \mathbb{R} : x \mapsto x^3 - x.$

$\mathbb{R} \to [-1, 1] : x \mapsto \sin(x).$

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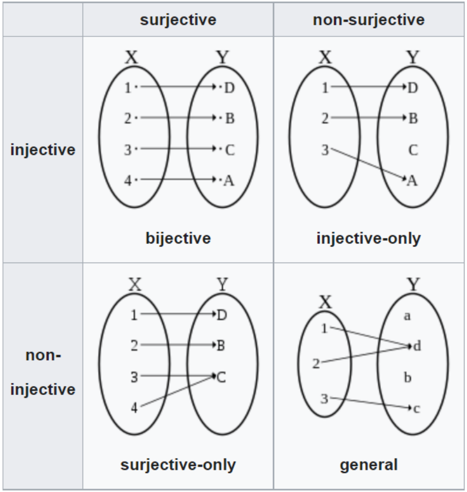

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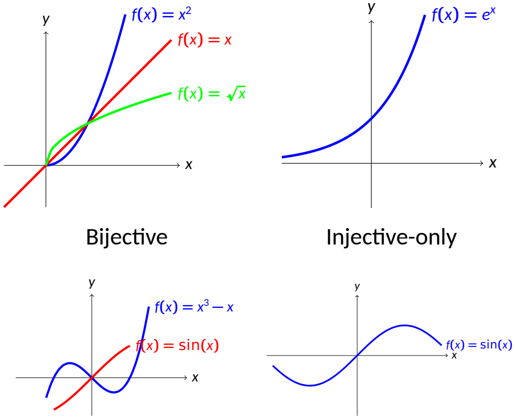


Bijective — Injective-only — Surjective-only — General

Task 4

Transformation of a plane is given by $x^* = x^2 - y^2$, $y^* = 2xy$. Is this transformation an (a) injection; (b) surjection; (c) bijection?

Task 4

Answer

(a): Not Injection. We can take 2 pairs (a, b), $(-a, -b)$, which provides the same result

(b): Surjection. Need to proof that it is a continuous function ()

(c): Not Bijection. Because of (a)

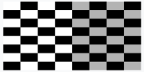
Affine Transformation

In Computer Vision (CV)

$$\begin{bmatrix} \vec{y} \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & A & & \vec{b} \\ 0 & \ldots & 0 & 1 \end{array} \right] \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

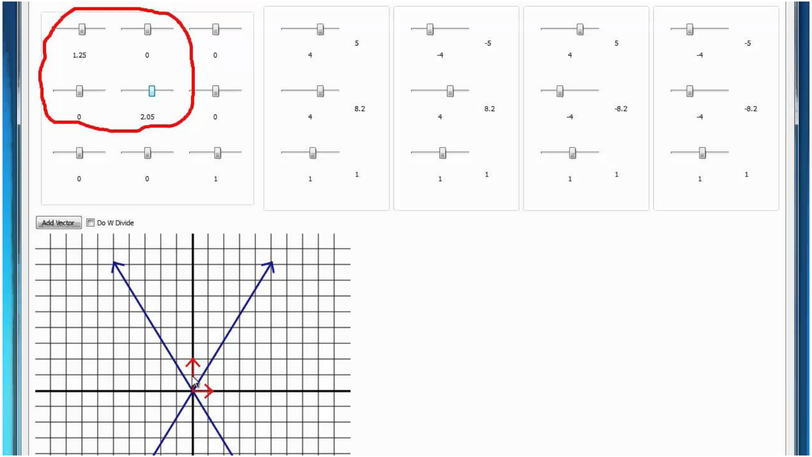
is equivalent to the following

$$\vec{y} = A\vec{x} + \vec{b}.$$

Transformation name	Affine matrix	Example
Identity (transform to original image)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
Reflection	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
Scale	$\begin{bmatrix} c_x = 2 & 0 & 0 \\ 0 & c_y = 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
Rotate	$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$	 where $\theta = \frac{\pi}{6} = 30°$
Shear	$\begin{bmatrix} 1 & c_x = 0.5 & 0 \\ c_y = 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

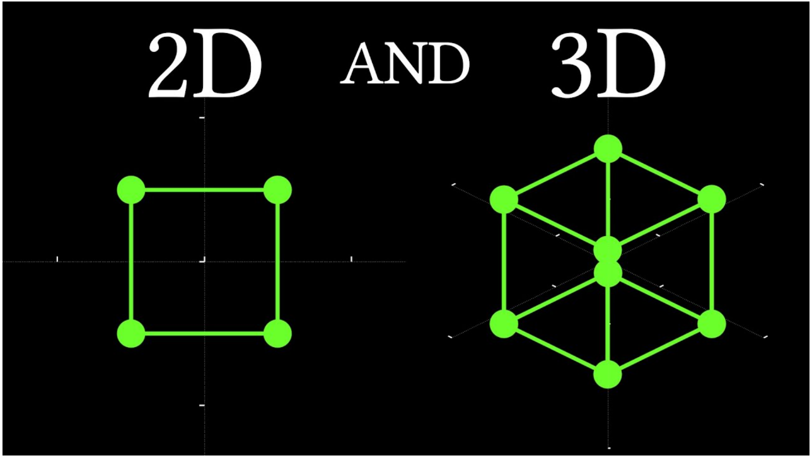
Affine Transformation

Video: intuition besides numbers 2D



Affine Transformation

Video



Affine Transformation

Applications

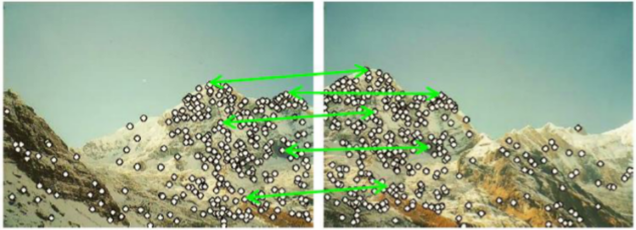
- Generation dataset for machine learning
- Camera calibration What Is Camera Calibration?
- Making panorama

Affine Transformation

Application in CV (1)

Point Features: how to build a panorama?

- Detect feature points in both images
- Find corresponding pairs



Affine Transformation

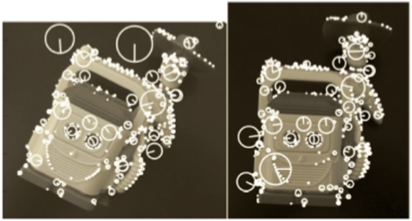
Application in CV (2)

SIFT features [Lowe et al., IJCV 2004]

- **SIFT** = Scale Invariant Feature Transform
  an approach for detecting and describing regions of interest in an image
- SIFT features are reasonably **invariant** to changes in:
  rotation, scaling, changes in viewpoint, illumination
- SIFT **detector** uses **DoG kernel**, SIFT **descriptor** is based on **gradient orientations**
- Very powerful in capturing + describing **distinctive** structure, but also **computationally demanding**

Main SIFT stages:

1. Extract keypoints + scale
2. Assign keypoint orientation
3. Generate keypoint descriptor



Task 5

Find the image of an arbitrary point M which has position vector $\mathbf{r}$ by the following transformations:

(a) homothety with center $M_0(\mathbf{r}_0)$ and ratio $\lambda \neq 0$;

(b) reflection across point $M_0(\mathbf{r}_0)$;

(c) translation by vector $\mathbf{a}$;

(d) orthogonal projection onto the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$;

(e) reflection across the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$;

(f) dilation of factor $\lambda > 0$ from the line $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t$.

Task 5

Answer

(a) $\mathbf{r}^* = \mathbf{r}_0 + \lambda(\mathbf{r} - \mathbf{r}_0)$;

(b) $\mathbf{r}^* = -\mathbf{r} + 2\mathbf{r}_0$;

(c) $\mathbf{r}^* = \mathbf{r} + \mathbf{a}$;

(d) $\mathbf{r}^* = \mathbf{r}_0 + \dfrac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$;

(e) $\mathbf{r}^* = 2\mathbf{r}_0 - \mathbf{r} + 2\dfrac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$;

(f) $\mathbf{r}^* = \lambda\mathbf{r} + (1 - \lambda)\mathbf{r}_0 + (1 - \lambda)\dfrac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$.

Task 6

Find formulas for the following affine transformations:

(a) orthogonal projection onto line $x - 3y + 1 = 0$;

(b) reflection across line $3x + 4y - 1 = 0$;

(c) dilation from line $x + y - 2 = 0$ of factor $\frac{1}{3}$;

(d) dilation from line $2x - y + 5 = 0$ of factor 2.

Task 6

Answer

(a) $x^* = \dfrac{9x + 3y - 1}{10}$, $y^* = \dfrac{3x + y + 3}{10}$;

(b) $x^* = \dfrac{7x - 24y + 6}{25}$, $y^* = -24x - 7y + 825$;

(c) $x^* = \dfrac{2x - y + 2}{3}$, $y^* = \dfrac{-x + 2y + 2}{3}$;

(d) $x^* = \dfrac{9x - 2y + 10}{5}$, $y^* = \dfrac{-2x + 6y - 5}{5}$.

Task 7

Find formulas for an affine mapping that transforms

(a) points $A(\frac{3}{7};\ 1)$, $B(1;\ \frac{1}{4})$, $C(2;-1)$ into points $A^*(-4;\ 2)$, $B^*(-1;\ 6)$, $C^*(4;\ 13)$ respectively;

(b) points $A(0;\ 0)$, $B(-1;\ 2)$, $C(1;-2)$ into points $A^*(-1;-1)$, $B^*(0;\ 0)$, $C^*(1;\ 1)$ respectively;

(c) points $A(2;\ 0)$, $B(3;-1)$, $C(4;-2)$ into points $A^*(2;\ 1)$, $B^*(-2;-1)$, $C^*(-6;-3)$ respectively;

(d) points $A(-2;\ 0)$, $B(2;-1)$, $C(0;\ 4)$ into points $A^*(-2;\ 1)$, $B^*(2;\ 1)$, $C^*(0;\ 1)$ respectively.

Task 7

Answer

(a) $x^* = -4y$, $y^* = 7x - 1$;

(b) no solutions;

(c) $x^* = px + (p+4)y + 2 - 2p$, $y^* = qx + (q+2)y + 1 - 2q$, where p and q are any real numbers;

(d) no solutions (there exists a linear transformation that is not affine).

Task 8

Find all invariant lines of an affine transformation given by

(a) $x^* = y,\ y^* = 1 - x$;

(b) $x^* = 2x + y - 3,\ y^* = -3x - y$;

(c) $x^* = 5x + 3y + 1,\ y^* = -3x - y$.

Task 8

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(b) $x^* = 2x + y - 3, y^* = -3x - y$;

(c) $x^* = 5x + 3y + 1, y^* = -3x - y$.

Answer

(a) no solutions;

(b) $x + y - 3 = 0$, $2x - y + p = 0$, where p can be any real number;

(c) $x + y + 1 = 0$.

Reference material

- Bijection, injection and surjection (wiki)
- Affine transformation (wiki)
- The Math behind (most) 3D games - Perspective Projection - YouTube
- Affine transformations | Brilliant Math & Science Wiki
- OpenCV: Affine Transformations

Deserve “A” grade!

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