Analytical Geometry and Linear Algebra I, Lab 14

Quadric surfaces: Cone, Cylinder

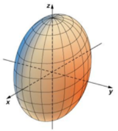
Case studies of 2nd order curve equation (ENG)

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

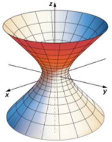


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

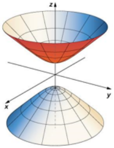


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

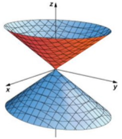


Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

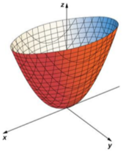


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

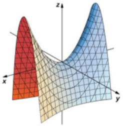


Hyperbolic Paraboloid

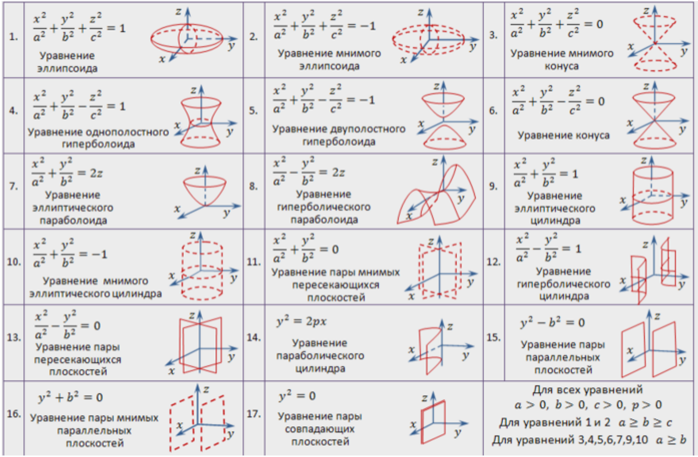
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.



Case studies of 2nd order curve equation (RUS)



1.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Уравнение эллипсоида	2.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ Уравнение мнимого эллипсоида	3.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ Уравнение мнимого конуса
4.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Уравнение однополостного гиперболоида	5.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ Уравнение двуполостного гиперболоида	6.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Уравнение конуса
7.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ Уравнение эллиптического параболоида	8.	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ Уравнение гиперболического параболоида	9.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Уравнение эллиптического цилиндра
10.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ Уравнение мнимого эллиптического цилиндра	11.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ Уравнение пары мнимых пересекающихся плоскостей	12.	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Уравнение гиперболического цилиндра
13.	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ Уравнение пары пересекающихся плоскостей	14.	$y^2=2px$ Уравнение параболического цилиндра	15.	$y^2-b^2=0$ Уравнение пары параллельных плоскостей
16.	$y^2+b^2=0$ Уравнение пары мнимых параллельных плоскостей	17.	$y^2=0$ Уравнение пары совпадающих плоскостей		Для всех уравнений $a>0,\ b>0,\ c>0,\ p>0$ Для уравнений 1 и 2 $a \ge b \ge c$ Для уравнений 3,4,5,6,7,9,10 $a \ge b$

Task 1

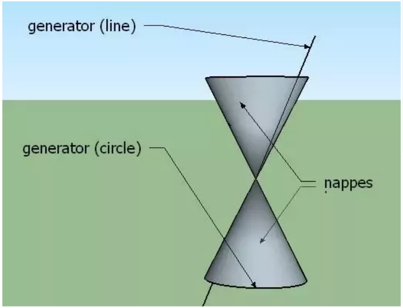
Find the equation of the cone with its vertex at $(1, 1, 1)$ and which passes through the curve $x^2 + y^2 = 4, z = 2$.

Task 1

Answer

Let V be the vertex of the cone and P be any point on the surface of the cone. Let the equations of the generator VP be

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-1}{n} \qquad (15.3)$$



This line intersects the plane $z = 2$.

$$\therefore \frac{x-1}{l}=\frac{y-1}{m}=\frac{1}{n} \quad \therefore \quad x=1+\frac{l}{n}, y=1+\frac{m}{n}$$

This point lies on the curve $x^2 + y^2 = 4$.

$$\therefore \left(1+\frac{l}{n}\right)^2+\left(1+\frac{m}{n}\right)^2=4 \quad \text{or} \quad (l+n)^2+(m+n)^2=4n^2$$

Eliminating l, m, n from (15.3) and (15.4) we get

$$\left(1+\frac{x-1}{z-1}\right)^2+\left(1+\frac{y-1}{z-1}\right)^2=4 \quad \text{or} \quad \begin{aligned}(z-1+x-1)^2+(z-1+y-1)^2&=4(z-1)^2\\ (x+z-2)^2+(y+z-2)^2&=4(z-1)^2\end{aligned}$$

The **generators** of a cone are a straight line and a closed curve (usually a circle). The line intersects the curve and passes through a point (called the apex) not in the plane of the curve.

The nappes are the two halves of the cone. Usually, we consider one such half to be a cone by itself, but both are necessary when, for example, characterising a hyperbola as a section of the cone.

Task 2

Find the equation of the cone with its vertex at the origin an which passes through the curve $ax^2 + by^2 + cz^2 - 1 = 0 = \alpha x^2 - \beta y^2 - 2z$.

Task 2

Answer

Let the equation of the generator be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r$

Any point on this line is (lr, mr, nr). This point lies on the curve

$$ax^2 + by^2 + cz^2 = 1$$

$$\alpha x^2 + \beta y^2 - 2z = 0$$

$$r^2(al^2 + bm^2 + cn^2) = 1 \qquad (15.13)$$

$$r(\alpha rl^2 + \beta rm^2 - 2n) = 0 \qquad (15.14)$$

From (15.14), $r = \frac{2n}{\alpha l^2 + \beta m^2}$ Substituting this in (15.13) we get

$$\frac{4n^2}{(\alpha l^2 + \beta m^2)^2}(al^2 + bm^2 + cn^2) = 1 \quad \text{(i.e.)} \quad 4n^2(al^2 + bm^2 + cn^2) = (\alpha l^2 + \beta m^2)^2$$

As l, m, n are proportional to x, y, z the equation of the cone is $4z^2(ax^2 + by^2 + cz^2) = (\alpha x^2 + \beta y^2)^2$.

Task 3

Find the equation of the cone of the second degree which passes through the axes.

Task 3

Answer

The cone passes through the axes. Therefore, the vertex of the cone is the origin.

The equations of the cone is a homogeneous equation of second degree in x, y and z.

$$\text{(i.e.)} \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad (15.25)$$

Given that x-axis is a generator.

Then $y = 0$, $z = 0$ must satisfy the equation (15.25)

$$\therefore a = 0$$

Since y-axis is a generator $b = 0$.
Since z-axis is a generator $c = 0$.
Hence the equation of the cone is $fyz + gzx + hxy = 0$.

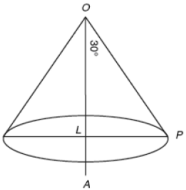
Task 4

Fin the equation of the right circular cone whose vertex is at the origin, whose axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and which has a vertical angle of 60°.

Task 4

Answer

The axis of the cone is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.



Therefore, the direction ratios of the axis of the cone are 1, 2, 3.

The direction cosines are $\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}$.

Let $P(x, y, z)$ be any point on the surface of the cone.
Let PL be perpendicular to OA.

$$\angle POL = 30°$$

$$\frac{OL}{OP} = \cos 30° \text{ or } 2OL = \sqrt{3}OP$$

Also, $$OP^2 = x^2 + y^2 + z^2$$

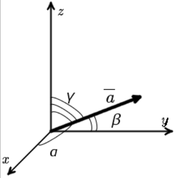
$$OL = \text{Projection of } OP \text{ on } OA$$

$$= \frac{x}{\sqrt{14}} + y\times\frac{2}{\sqrt{14}} + z\times\frac{3}{\sqrt{14}} = \frac{x+2y+3z}{\sqrt{14}}$$

$$\therefore \frac{2(x+2y+3z)}{\sqrt{14}} = \sqrt{3}\sqrt{x^2+y^2+z^2}$$

$$4(x+2y+3z)^2 = 42(x^2+y^2+z^2)$$

or $$19x^2+13y^2+3z^2-8xy-24yz-12zx=0$$



The **direction cosines** of the vector a are the cosines of angles that the vector forms with the coordinate axes.

$$\cos\alpha = \frac{a_x}{|\bar{a}|};\quad \cos\beta = \frac{a_y}{|\bar{a}|};\quad \cos\gamma = \frac{a_z}{|\bar{a}|}$$

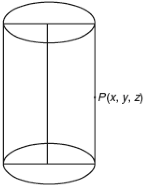
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Task 5

Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{-1} = \frac{y}{2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 9, z = 1$.

Task 5

Answer



The equations of the generator through *P* and parallel to the line

$\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ are $\frac{x-x_1}{-1}=\frac{y-y_1}{2}=\frac{z-z_1}{3}$ (16.8)

The guiding curve is $x^2+y^2=9, z=1$ (16.9)

When the generator through *P* meets the guiding curve,

$$\frac{x-x_1}{-1}=\frac{y-y_1}{2}=\frac{z-z_1}{3}$$

$$\therefore x=x_1-\frac{1-z_1}{3}=\frac{3x_1+z_1-1}{3}, y=y_1+\frac{2(1-z_1)}{3}=\frac{3y_1-2z_1+2}{3}$$

Since this point lies on the curve (16.9),

$$(3x_1+z_1-1)^2+(3y_1-2z_1+2)^2=81$$

The locus of (x_1,y_1,z_1) is $(3x+z-1)^2+(3y-2z+2)^2=81$

(i.e.) $9x^2+9y^2+5z^2+6xz-12yz-6x+12y-10z-76=0$

This is the equation of the required cylinder.

Task 6

Find the equations of the right circular cylinder of radius 3 with equations of axis as

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}.$$

Task 6

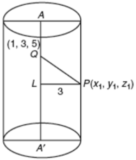
Answer

The equations of the axis are

$$\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$$

(1, 3, 5) is a point on the axis.
2, 2, –1 are the direction ratios of the axis.

$\therefore$ direction cosines are $\frac{2}{3},\frac{2}{3},\frac{-1}{3}$



Let $P\,(x_1, y_1, z_1)$ be any point on the cylinder.

$$\begin{aligned} QL &= \text{Projection of } PQ \text{ on the axis} \\ &= (x-x_1)l+(y-y_1)m+(z-z_1)n \\ &= (x_1-1)\frac{2}{3}+(y_1-3)\frac{2}{3}-(z_1-5)\frac{1}{3} \\ &= \frac{2x_1+2y_1-z_1-3}{3} \end{aligned}$$

Also, $PQ^2 = QL^2 + LP^2$

$$\text{(i.e.)}\quad (x_1-1)^2+(y_1-3)^2+(z_1-5)^2=\left(\frac{2x_1+2y_1-z_1-3}{3}\right)^2+9$$

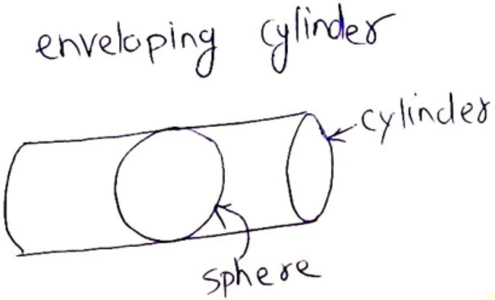
The locus of (x_1, y_1, z_1) is

$$9(x^2-2x+1+y^2-6y+9+z^2-10z+25)$$
$$=4x^2+4y^2+z^2+9+8xy-4xz-12x-4yz-12y+6z+81$$
$$\text{(i.e.)}\quad 5x^2+5y^2+8z^2-8xy+4xz+4yz-6x-42y-96z+225=0$$

This is the equation of the required cylinder.

Task 7

Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generators parallel to the line $x = y = z$.



Task 7

Answer

Let $P(x_1,\ y_1,\ z_1)$ be any point on a tangent, which is parallel to the line $x = y = z$.

Hence, the equation of the tangent lines are

$$\frac{x - x_1}{1} = \frac{y - y_1}{1} = \frac{z - z_1}{1} \qquad (1)$$

Any point on this line is $(x_1 + \tau,\ y_1 + \tau,\ z_1 + \tau)$. This point lies in this sphere.

$$x^2 + y^2 + z^2 - 2x + 4y - 1 = 0 \qquad (2)$$

$$(x_1 + \tau)^2 + (y_1 + \tau)^2 + (z_1 + \tau)^2 - 2(x_1 + \tau) + 4(y_1 + \tau) - 1 = 0 \Rightarrow$$

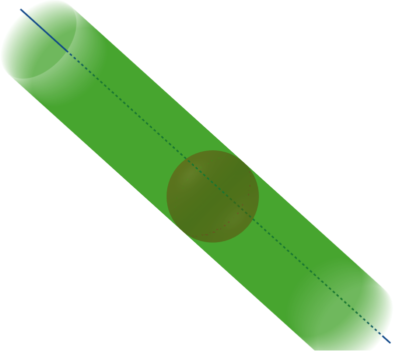
$$\Rightarrow 3\tau^2 + 2\tau(x_1 + y_1 + z_1 + 1) + (x_1^2 + y_1^2 + z_1^2 - 2x_1 + 4y_1 - 1) = 0$$

If (1) touches (2), τ is unique ($\mathscr{D} = 0$).

$$\mathscr{D} = 4(x_1 + y_1 + z_1 + 1)^2 - 12(x_1^2 + y_1^2 + z_1^2 - 2x_1 + 4y_1 - 1) = 0 \quad (3)$$

Solving (3) and change P to general form, we are obtaining the answer

$$x^2 + y^2 + 5y + z^2 - 4x - z - xy - xz - yz - 2 = 0$$



Reference material

- Cone, generatrix (OnlineMSchool)
- Direction cosines (OnlineMSchool)

Deserve “A” grade!

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