Essentials of Analytical Geometry and Linear Algebra I, Class #3

Innopolis University, September 2022

1 Operations with Matrices

1.1 Introduction to matices

1. Let $A = \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

   (a) Find $A + B$;

   (b) Find $2A - 3B + I$;

   (c) Find AB and BA (make sure that, in general, $AB \neq BA$ for matrices);

   (d) Find AI and IA.

2. Let $A = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$:

   (a) Find AB and BA if they exist;

   (b) Find $A^T B$ and BA^T if they exist.

3. If solution exists, what the dimension of the result matrix. There are several matrices: $\underset{3\times3}{A}, \underset{2\times3}{B}, \underset{3\times2}{C}, \underset{3\times5}{D}, \underset{3\times5}{D}, \underset{1\times2}{E}, \underset{3\times1}{K}$.

   (a) ABC;

   (b) AB^TC^T;

   (c) $EBAE$;

   (d) $K^T \times K^TCE^T$.

1.2 Determinants

1. Find the determinants of the following matrices:

   (a) $A = \begin{bmatrix} 5 & -2 \\ 1 & 6 \end{bmatrix}$; (b) $B = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 7 & 2 \\ 3 & 2 & -4 \end{bmatrix}$.

2. A triangle is constructed on vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Find the area of this triangle.

   (b) Find the altitudes of this triangle.

3. Find the matrix product AB if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & x \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ x & 2 \\ -3 & -1 \end{bmatrix}$.

   Then find the largest possible value of $det(AB)$.

4. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three pairwise non-collinear vectors. Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$ if and only if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

2 Scalar Triple Product

1. Find the scalar triple product of $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}$.

2. Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are not coplanar. Find all values of θ such that vectors $\mathbf{a} + 2\mathbf{b} + \theta\mathbf{c}$, $4\mathbf{a} + 5\mathbf{b} + 6\mathbf{c}$, $7\mathbf{a} + 8\mathbf{b} + \theta^2\mathbf{c}$ are coplanar.

3 Changing Basis and Coordinates

1. Two bases are given in the plane: $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}'_1$, $\mathbf{e}'_2$. The vectors of the second basis have coordinates $(-1; 3)$ and $(2; -7)$ in the second basis.

   (a) Compose transition matrices from the old basis to the new and vice versa.

   (b) Find the coordinates of a vector in the old basis given that it has coordinates α'_1, α'_2 in the new basis.

   (c) Find the coordinates of a vector in the new basis given that it has coordinates α_1, α_2 in the old basis.

2. Let us consider two coordinate systems in the plane: O, $\mathbf{e}_1$, $\mathbf{e}_2$ and O', $\mathbf{e}'_1$, $\mathbf{e}'_2$. Point O' has coordinates $(7; -2)$ in the old coordinate system, and vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$ can be obtained from vectors $\mathbf{e}_1$, $\mathbf{e}_2$ by rotating them $60°$ (a) clockwise; (b) counterclockwise. Find the old coordinates of a point x, y given its new coordinates x', y'.