# Analytical Geometry and Linear Algebra I, Lab 2

Cross product
Dot product

# Lab objectives, 1st part

1. What does cross product mean?
2. How to calculate it?
3. What properties of cross product exists? How to use them?

# Cross product
*Definition*

$\vec{a} \times \vec{b} = [\vec{a}, \vec{b}]$ is defined as a vector $\vec{c}$, that is perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$, with:

- *direction* given by the right-hand rule
- *magnitude* is equal to the area of the parallelogram, that the vectors span.
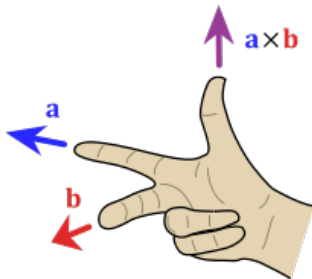
*Video*

# Cross product: shoe polishing robot
*Video*

# How to calculate it?

## Approaches for finding a full vector:

1. Classical One
2. Using skew-symmetric matrix

## Approach for finding only magnitude:

1. Geometrical representation

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\text{where,} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A = a \cdot d - b \cdot c$$

# Using Skew-symmetric matrix

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \qquad c = a \times b \implies c = \hat{a}b$$

vectors $\implies$ matrices

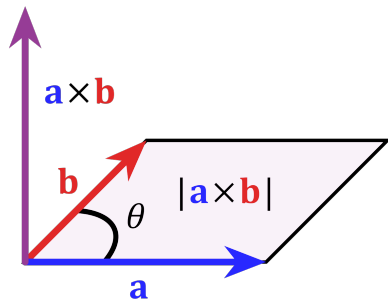$$a \times \implies \hat{a} \quad : \text{a skew-symmetric matrix}$$

$$c = \hat{a}b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \qquad \boxed{c = \hat{a}b}$$

# Geometrical representation

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \alpha$

where $|\vec{a} \times \vec{b}|$ is an area of parallelogram

*Tip:* The length or magnitude or norm of the vector a is denoted by $\|\vec{a}\|$ or, less commonly, $|\vec{a}|$, which is not to be confused with the absolute value (a scalar "norm"). *In Russia, $|\vec{a}|$ is more popular*.

*Case study*

*Task*: to find cross product between $\vec{a} = \begin{bmatrix} -2 \\ -2 \\ 10 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix}$

## Classical

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & 10 \\ -4 & 1 & 10 \end{vmatrix} =$$

$$= \vec{i} \begin{vmatrix} -2 & 10 \\ 1 & 10 \end{vmatrix} + \vec{j} \begin{vmatrix} -2 & 10 \\ -4 & 10 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & -2 \\ -4 & 1 \end{vmatrix} = \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix}$$

## Skew-symmetric

$$[\vec{a} \times] \vec{b} = \begin{bmatrix} 0 & -10 & -2 \\ 10 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} =$$

$$= \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix}$$

# Cross product properties

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
3. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
4. $\lambda \vec{a} \times \vec{b} = \vec{a} \times \lambda \vec{b} = \lambda(\vec{a} \times \vec{b})$
5. $\vec{a} \times \vec{a} = \vec{0}$
6. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

# Task 1

Find cross product between $\vec{a}$ and $\vec{b}$, if:

1. $\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$

2. $\vec{a} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 8 \\ -5 \end{bmatrix}$

3. $\vec{a} = \begin{bmatrix} -9 \\ 3 \\ -6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 5 \\ -8 \end{bmatrix}$

4. $\vec{a} = \begin{bmatrix} 8 \\ 3 \\ -9 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -6 \end{bmatrix}$

# Task 2

Simplify the expressions:

1. $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$
2. $(3\vec{a} - \vec{b} - \frac{1}{3}\vec{c}) \times (2\vec{a} + \frac{3}{2}\vec{b} - 3\vec{c})$

# Task 2

Simplify the expressions:

1. $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$
2. $(3\vec{a} - \vec{b} - \frac{1}{3}\vec{c}) \times (2\vec{a} + \frac{3}{2}\vec{b} - 3\vec{c})$

## Answer for «1»

```
a = sym('a',[3 1]);
b = sym('b',[3 1]);
simplify(cross(a+b,a-b))
```

$$
ans =
\begin{pmatrix}
2\,a_3\,b_2 - 2\,a_2\,b_3 \\
2\,a_1\,b_3 - 2\,a_3\,b_1 \\
2\,a_2\,b_1 - 2\,a_1\,b_2
\end{pmatrix}
$$

```
>> 2 * cross(b,a)

ans =

 2*a3*b2 - 2*a2*b3
 2*a1*b3 - 2*a3*b1
 2*a2*b1 - 2*a1*b2
```

# Task 3

A triangle is constructed on vectors $\vec{a} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

It is needed to

1. Find the area of the triangle.
2. Find the altitudes of this triangle.

# Lab objectives, 2nd part

1. What does dot product mean?
2. How to calculate it?
3. How to use it?

# Dot product
*Definition*

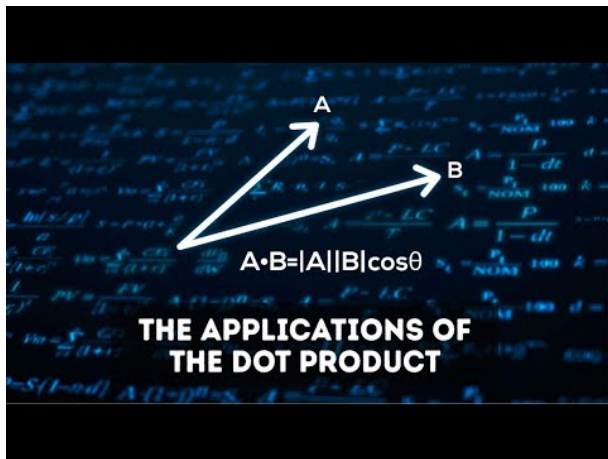The result of **dot product / inner product**
$\vec{a} \cdot \vec{b} = (\vec{a}, \vec{b}) = \vec{a}^T\vec{b} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ between $\vec{a}$ and $\vec{b}$ is a **scalar**.

Geometrically — $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \beta$ — measure of how similar two vectors are.

# Dot product
*Video*

# Tasks 4 and 5

1. Solve $|\vec{a}|^2 - 2\sqrt{3}\vec{a} \cdot \vec{b} - 7|\vec{b}|^2$, where $|\vec{a}| = 4,\ |\vec{b}| = 1,\ \angle(\vec{a}, \vec{b}) = 150°$

2. Find the angle between $\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\ \vec{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$

All three vectors $\vec{a}, \ \vec{b}, \ \vec{c}$ have length of 3 and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Find $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

# Task 6

Answer: $-\dfrac{27}{2}$



Pure math solution

# Dot and cross product possible application
*Video*

# Task 7

There are two vectors on some basis $\vec{a} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x^2 - 2x \\ x^2 - 2x + 1 \end{bmatrix}$.

It is needed to find $x$, when:

1. Vectors are collinear
2. They have the same direction

# Task 7

**Condition of vectors collinearity**

Two vectors are collinear, if any of these conditions done:

*Condition of vectors collinearity 1.* Two vectors $\overline{a}$ and $\overline{b}$ are collinear if there exists a number $n$ such that

$$\overline{a} = n \cdot \overline{b}$$

*Condition of vectors collinearity 2.* Two **vectors are collinear** if relations of their coordinates are equal.

**N.B.** Condition 2 is not valid if one of the components of the vector is zero.

*Condition of vectors collinearity 3.* Two **vectors are collinear** if their cross product is equal to the zero vector.

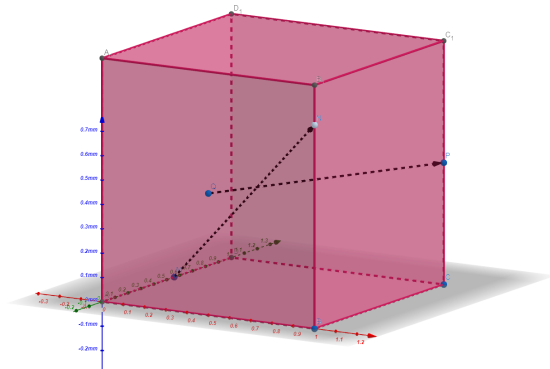**N.B.** Condition 3 applies only to three-dimensional (spatial) problems.

The edges of cube $ABCDA_1B_1C_1D_1$ have length of 1. $P$ is a midpoint of $CC_1$ and $Q$ is a center of face $AA_1B_1B$. Points $M$ and $N$ belong to lines $AD$ and $A_1B_1$ respectively, and at that $MN$ intersects with $PQ$ and is perpendicular to it. Find $MN$.
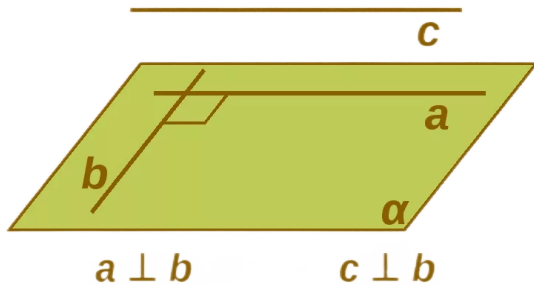
# Task 8

*Geogebra with link*

It is needed to make 2 equations: 1 for perpendicularity, 2nd — intersection. Think about amount of unknowns and possible equations.



$a \perp b \qquad\qquad c \perp b$

# Reference material

- Cross products | Chapter 10, Essence of linear algebra - YouTube
- Dot product | Chapter 9, Essence of linear algebra - YouTube
- Cross product of two vectors - OnlineMSchool
- Векторное произведение векторов - Матпрофи

# Deserve "A" grade!

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🚪 Room 105 (Underground robotics lab)