Analytical Geometry and Linear Algebra I, Lab 3

Intro to matrices

Determinant

Scalar Triple Product

Questions from the class

No questions for today

WHAT IS THE MATRIX?

Matrix Definition

Definition. **Matrix** with size $n \times m$ is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns, which consisting of n rows and m columns.

The number of rows and columns are defined the matrix size.

Operations with matrices

Between 2 or more matrices

- Summation
- Multiplication (Order is important!)
- Cross Product
- Dot Product

No Division!

With one matrix

- Multiplication on Scalar
- Length
- Transpose
- Trace
- Determinant
- Inverse Matrix

Summation and multiplication

Case Study

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} & \text{2x3} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \text{3x2} \\ \begin{bmatrix} 4 & 5 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} = \begin{bmatrix} \overset{a_{1,1}}{2 \cdot 4 + 1 \cdot 1 + 3 \cdot 2} & \overset{a_{1,2}}{2 \cdot 5 + 1 \cdot 3 + 3 \cdot 1} \\ \overset{a_{2,1}}{4 \cdot 4 + 2 \cdot 1 + 1 \cdot 2} & \overset{a_{2,2}}{4 \cdot 5 + 2 \cdot 3 + 1 \cdot 1} \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 16 \\ 20 & 27 \end{bmatrix} \text{2x2}$$

Transpose

Case Study

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

Transpose

Why do we need it?

We know the dot product (inner product) of x and y. It is the sum of numbers $x_i y_i$. Now we have a better way to write $x \cdot y$, without using that unprofessional dot. Use matrix notation instead:

$^{\mathrm{T}}$ **is inside**	***The dot product or inner product is*** $x^{\mathrm{T}} y$	$(1 \times n)(n \times 1)$
$^{\mathrm{T}}$ **is outside**	***The rank one product or outer product is*** $x y^{\mathrm{T}}$	$(n \times 1)(1 \times n)$

$x^{\mathrm{T}} y$ is a number, $x y^{\mathrm{T}}$ is a matrix. Quantum mechanics would write those as $< x|y >$ (inner) and $|x >< y|$ (outer). I think the world is governed by linear algebra, but physics disguises it well. Here are examples where the inner product has meaning:

From mechanics	Work = (Movements) (Forces) = $x^{\mathrm{T}} f$
From circuits	Heat loss = (Voltage drops) (Currents) = $e^{\mathrm{T}} y$
From economics	Income = (Quantities) (Prices) = $q^{\mathrm{T}} p$

Trace

Definition and Case Study

Definition

The trace of an $n \times n$ square matrix A is defined as:

$\mathbf{tr(A)} = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$, where a_{ij} denotes the entry on the ith row and jth column of A.

The trace is not defined for non-square matrices.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 11 & 5 & 2 \\ 6 & 12 & -5 \end{pmatrix}$$ Then

$$\mathbf{tr(A)} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 5 + (-5) = 1$$

Intro to matrices

Task 1

Let $A = \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

1. Find $A + B$;
2. Find $2A - 3B + I$;
3. Find AB and BA (make sure that, in general, $AB \neq BA$ for matrices);
4. Find AI and IA.

Intro to matrices

Task 2

Let $A = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$:

1. Find AB and BA if they exist;
2. Find $A^T B$ and BA^T if they exist.

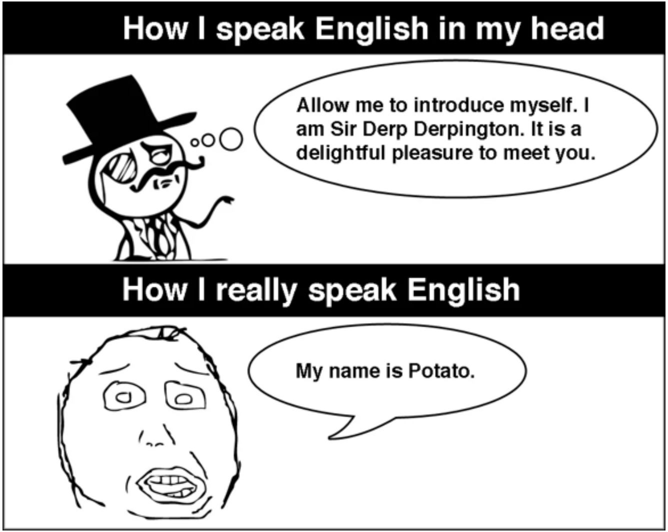
Intro to matrices

Task 3

If solution exists, what the dimension of the result matrix.

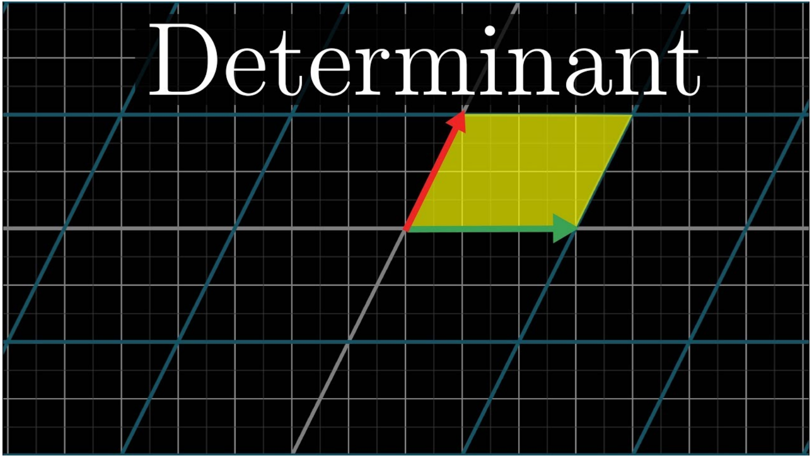
There are several matrices: $\underset{3\times 3}{A}, \underset{2\times 3}{B}, \underset{3\times 2}{C}, \underset{3\times 5}{D}, \underset{3\times 5}{D}, \underset{1\times 2}{E}, \underset{3\times 1}{K}$.

1. ABC;
2. AB^TC^T;
3. $EBAE$;
4. $AK \times KK^TB^T$.


How I speak English in my head
Allow me to introduce myself. I am Sir Derp Derpington. It is a delightful pleasure to meet you.
How I really speak English
My name is Potato.

Determinant

Video



Determinant

Where it can be used

1. Find inverse matrix (next class)
2. Find matrix rank (nex class)
3. Solve SLE using Cramer’s rule (this HW)

Determinant

How to Find (1)

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

The minor $M_{i,j}$ is defined to be the determinant of the $(n-1) \times (n-1)$-matrix that results from A by removing the i-th row and the j-th column.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant

How to Find (2)

Special case for 3x3 matrix

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Special case for 2x2 matrix

$$\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\mathbf{X}) = a * d - b * c$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

Determinant

Task 1

Find the determinants of the following matrices:

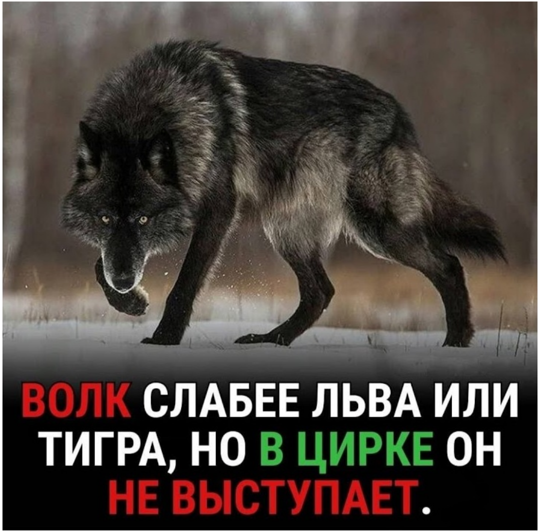
(a) $A = \begin{bmatrix} 5 & -2 \\ 1 & 6 \end{bmatrix}$; (b) $B = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 7 & 2 \\ 3 & 2 & -4 \end{bmatrix}$, (c) $C = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 0 & 2 \\ 3 & 0 & -4 \end{bmatrix}$

Determinant

Task 2

Find the matrix product AB if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & x \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ x & 2 \\ -3 & -1 \end{bmatrix}$.

Then find the largest possible value of $det(AB)$.


ВОЛК СЛАБЕЕ ЛЬВА ИЛИ ТИГРА, НО В ЦИРКЕ ОН НЕ ВЫСТУПАЕТ.


15 ЛЕТ
НИКОГДА НЕ ПРОГНУСЬ ПОД СИСТЕМУ, ВСЕГДА БУДУ СВОБОДНЫМ
ПУСТЬ ЛЬВЫ И ТИГРЫ ПЛЯШУТ ЗА ДЕНЬГИ
25 ЛЕТ
ОП, ОП
НУ А ЧЁ, НУЖНО ЖЕ КАК-ТО ЗАРАБАТЫВАТЬ

Wolf Ballet

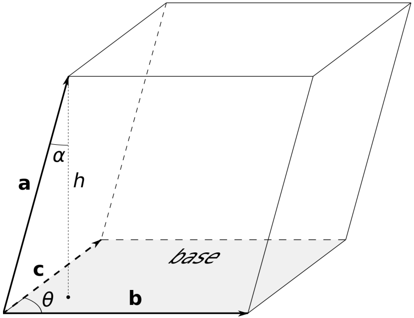
Video

Scalar Triple Product

Definition

(a, b, c) — is defined as the dot product of one of the vectors with the cross product of the other two.

Geometrically — a signed volume of the parallelepiped defined by the three vectors given



Scalar Triple Product

How to calculate

$$a \cdot (b \times c) = det(a, b, c) = det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right)$$

Scalar Triple Product

Case Study

Calcute a triple scalar prodict between $\vec{a}$, $\vec{b}$ and $\vec{c}$.

$$\vec{a} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -1 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

$$\det(A) = (-1)\begin{vmatrix} -1 & -2 \\ -2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} + (5)\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} =$$

Scalar Triple Product

Properties

Geometric interpretation. Module of scalar triple product of vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ is equal to the volume of the parallelepiped formed by these vectors:

$$V_{\text{parallelepiped}} = |\overline{a} \cdot [\overline{b} \times \overline{c}]|$$

Geometric interpretation. The volume of the pyramid formed by three vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ is equal to one-sixth of the modulus of the scalar triple product of this vectors:

$$V_{\text{pyramid}} = \frac{1}{6}|\overline{a} \cdot [\overline{b} \times \overline{c}]|$$

If the mixed product of three non-zero vectors equal to zero, these vectors are coplanar.

$\overline{a} \cdot [\overline{b} \times \overline{c}] = \overline{b} \cdot (\overline{a} \cdot \overline{c}) - \overline{c} \cdot (\overline{a} \cdot \overline{b})$

$\overline{a} \cdot [\overline{b} \times \overline{c}] = \overline{b} \cdot [\overline{c} \times \overline{a}] = \overline{c} \cdot [\overline{a} \times \overline{b}] = -\overline{a} \cdot [\overline{c} \times \overline{b}] = -\overline{b} \cdot [\overline{a} \times \overline{c}] = -\overline{c} \cdot [\overline{b} \times \overline{a}]$

$\overline{a} \cdot [\overline{b} \times \overline{c}] + \overline{b} \cdot [\overline{c} \times \overline{a}] + \overline{c} \cdot [\overline{a} \times \overline{b}] = 0$ - **Jacobi identity**.

Scalar Triple Product

Task 1

Find the scalar triple product of $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}$.

Reference material

OnlineMschool

- Matrix definition
- Matrix multiplication
- Transpose
- Determinant
- Scalar Triple Product

Deserve "A" grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)