Analytical Geometry and Linear Algebra I, Lab 9

Conic sections (2nd order curve equation):

a) Parabola

b) Ellipse

Innopolis University

Questions from the class

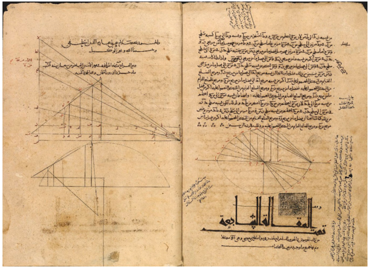
No questions for today

Questions for today

- How can I work with general form of 2nd order curve equation?
- How it relates with cone?
- What forms of equation do we have?

Why it is called <<Conic Sections>>

The greatest progress in the study of conics by the ancient Greeks is due to *Apollonius of Perga* (died c. 190 BCE), whose eight-volume **Conic Sections or Conics**. More info here.



Books 5-7 are only available in an Arabic translation (9th century)

FRANCISCI MAVROLYCI
MESSANENSIS,
EMENDATIO, ET RESTITVTIO
CONICORVM
APOLLONII PERGAEI
Nunc primùm typis excusa, ubi primi quatuor eiusdem Apollonij libri mendis, quibus fœde scatebant, expurgantur, nouisque interdum demonstrationibus illustrantur: quintus verò, sextusque liber, quorum tituli dumtaxat habebantur, maximo labore, summaque industria denuo restituuntur.
AD ILLVSTRISS.MVM SENATVM
MESSANENSEM.
MESSANAE,
Typis Hæredum Petri Breæ. MDCLIIII.
Superiorum Permissu.

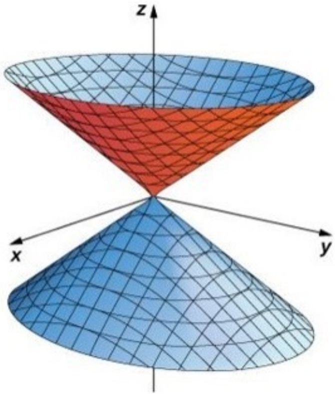
1654 edition of Conica

Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

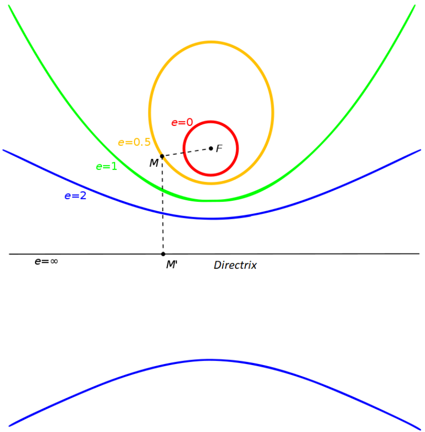


Some definitions, which can be helpful

Eccentricity, Directrix

Eccentricity is a measure of how much a conic section deviates from being circular.

It is a constant ration between distance from focal to point on the curve and from the point on the curve to **directrix**.



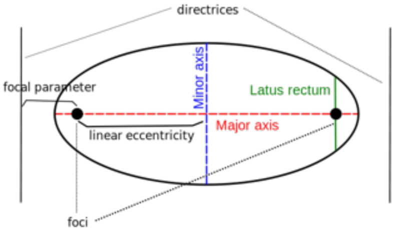
Some definitions, which can be helpful

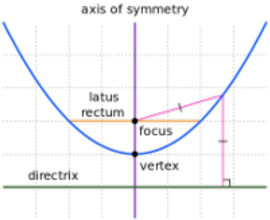
Linear eccentricity, Latus Rectrum, Focal parameter

The **linear eccentricity** is the distance between the center and the focus (or one of the two foci).

The **latus rectum** is the chord parallel to the directrix and passing through the focus (or one of the two foci).

The **focal parameter** is the distance from the focus (or one of the two foci) to the directrix.

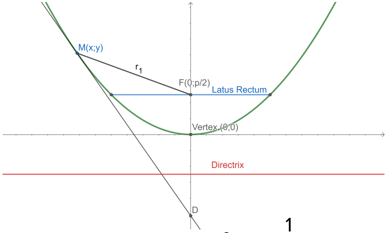




Parabola

Forms:

- **Canonical** $(x - x_{\text{shift}})^2 = 2p(y - y_{\text{shift}})$
- **General** $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where either $A = 0$ or $C = 0$, not both, if $B = 0$
- **Parametric** $\begin{cases} x = \sqrt{2}pt \\ y = pt^2 \end{cases}$



Parabola $y = x^2$, $p = \frac{1}{2}$

Properties:

- **Vertex** $\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center** Not defined
- **Eccentricity** $e = 1$
- **Linear Eccentricity** Not defined
- **Foci** $F = \begin{pmatrix} 0 \\ \frac{p}{2} \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$

- **Latus Rectum** (length of chord) $|2p|$
- **Focal parameter** p
- **Discriminant** $\mathfrak{D} = B^2 - 4AC = 0$
- **Directrix eq.** $y = -\frac{p}{2} + y_{\text{shift}}$

- **Tangent eq.** $x(x_{\tan} - x_{\text{shift}}) = p(y - y_{\tan}) + x_{\tan}(x_{\tan} - x_{\text{shift}})$
- $r = |\overline{FM}| = \sqrt{(x - \frac{p}{2})^2 + y^2}$
- ΔMFD is isosceles, where MD – tangent to M

From general to canonical form

When $B = 0$

$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0$ — General form.

Example of transformation from general to canonical form:

$$16x^2 + 25y^2 - 32x + 50y - 359 = 0 \Rightarrow$$
$$\left(16x^2 - 32x\right) + \left(25y^2 + 50y\right) - 359 = 0 \Rightarrow$$
$$16(x^2 - 2x) + 25(y^2 + 2y) = 359 \Rightarrow$$
$$16(x^2 - 2x + 1) + 25(y^2 + 2y + 1) = 350 + 16 + 25 \Rightarrow$$
$$16(x-1)^2 + 25(y+1)^2 = 400 \Rightarrow$$
$$\frac{(x-1)^2}{25} + \frac{(y+1)^2}{16} = 1$$

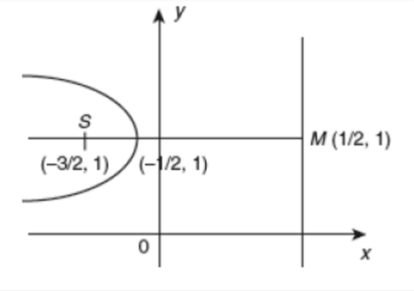
Task 1

Find the foci, latus rectum, vertices and directrices of the following parabola:

$y^2 + 4x - 2y + 3 = 0.$

Task 1

Answer



i.

$$y^2 + 4x - 2y + 3 = 0$$
$$y^2 - 2y = -4x - 3$$
$$y^2 - 2y + 1 = -4x - 3 + 1$$
$$\Rightarrow (y-1)^2 = -4\left(x + \frac{1}{2}\right)$$

Take $x + \frac{1}{2} = X$, $y - 1 = Y$. Shifting the origin to the point $\left(\frac{-1}{2}, 1\right)$ the equation of the parabola becomes $Y^2 = -4X$.

$\therefore$ Vertex is $\left(\frac{-1}{2}, 1\right)$, latus rectum is 4, focus is $\left(\frac{-3}{2}, 1\right)$ and foot of the directrix is $\left(\frac{1}{2}, 1\right)$.

The equation of the directrix is $x = \frac{1}{2}$ or $2x - 1 = 0$.

Task 2

Find the equations of the tangent and normal to the parabola $y^2 = 4(x-1)$ at $(5, 4)$.

Task 2

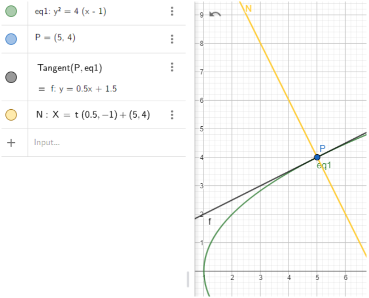
Answer

$$y^2 = 4(x - 1)$$

Differentiating with respect to x,

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\left(\frac{dy}{dx}\right)_{\text{at}(5,\,4)} = \frac{2}{4} = \frac{1}{2} = \text{Slope of the tangent at } (5, 4)$$



$\therefore$ The equation of the tangent at (5, 4) is $y - 4 = \frac{1}{2}(x - 5)$.

$2y - 8 = x - 5$ or $x - 2y + 3 = 0$. The slope of the normal at (5, 4) is -2.

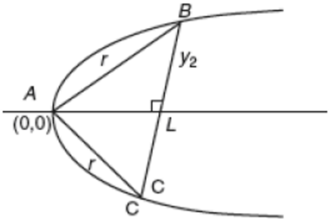
$\therefore$ The equation of normal at (5, 4) is $y - 4 = -2(x - 5)$ or $2x + y = 14$.

Task 3

An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ one of whose vertices is at the vertex of the parabola. Find its side.

Task 3

Answer



The coordinates of *B* are $B(r\cos 30°, r\sin 30°)$, $\left(\frac{\sqrt{3}}{2}r, \frac{r}{2}\right)$.

Since this point lies on the parabola $y^2 = 4ax$, then

$$\frac{r^2}{4} = 4a \cdot \frac{r}{2}\sqrt{3} \qquad \therefore r = 8a\sqrt{3}$$

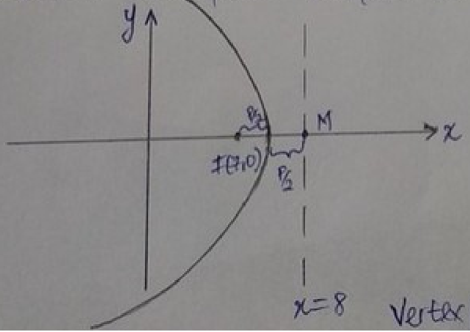
Task 4

Find the equation of a parabola that has a line $x = 8$ for a directrix and point $F(7; 0)$ for a focus.

Task 4

Answer

Find the equation of a parabola that has line $x=8$ for a directrix and point $F(7;0)$ for a focus.



$MF = p$

$\Rightarrow p = 8-7$

$\frac{p}{2} = \frac{1}{2}$

$a = 7 + \frac{p}{2}$

$= \frac{14}{2} + \frac{1}{2}$

Vertex $(a,0)$ $a = \frac{15}{2}$

$y^2 = -2px + c$

$y^2 = -2x + c$

at vertex

$0 = -2\cdot\frac{15}{2} + c$

$c = 15$

$\Rightarrow y^2 = -2x + 15$

$y^2 = -2(x - \frac{15}{2})$

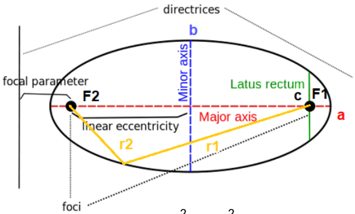
Collisions

Video

Ellipse

Forms:

- **Canonical** $\frac{(x-x_{\text{shift}})^2}{a^2} + \frac{(y-y_{\text{shift}})^2}{b^2} = 1$
- **General** $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $AC > 0$, if $B = 0$
- **Parametric** $\begin{cases} x = a\cos(\alpha) \\ y = b\sin(\alpha) \end{cases}$



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Properties:

- **Vert** $\begin{pmatrix} \pm a \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ \pm b \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Center** $(0;0) + (x_{\text{shift}}; y_{\text{shift}})$
- **Eccentricity** $0 \le e < 1$, $e = \sqrt{1 - \frac{b^2}{a^2}}$
- **Linear Eccentricity** $c = \sqrt{a^2 - b^2}$
- **Foci** $F = \begin{pmatrix} \pm(c = e\,a) \\ 0 \end{pmatrix} + \begin{pmatrix} x_{\text{shift}} \\ y_{\text{shift}} \end{pmatrix}$
- **Latus Rectum** (length of chord) $\frac{2b^2}{a}$
- **Focal parameter** $\frac{b^2}{\sqrt{a^2-b^2}}$
- **Discriminant** $\mathfrak{D} = B^2 - 4AC < 0$
- **Directrix eq.** $x = \pm\frac{a}{e} + x_{\text{shift}}$
- **Tangent eq. (w/o shift)** $\frac{x_{tangent}x}{a^2} + \frac{y_{tangent}y}{b^2} = 1$
- $r_1 + r_2 = 2a$
- $r_{1,2} = |\overline{F_{1,2}M}| = \sqrt{(x \pm c)^2 + y^2}$
- $\frac{r_1}{d_1} = e$

Task 5

Find the equation of the ellipse whose foci are $(4, 0)$ and $(-4, 0)$ and $e = 1/3$

Task 5

Answer

i. If the foci are (ae, 0) and ($-ae$, 0) then the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Here, ae = 4 and $e=\frac{1}{3}$.

$$a=\frac{4}{e}=4\times 3=12$$

$$b^2=a^2(1-e^2)=144\left(1-\frac{1}{9}\right)=144\times\frac{8}{9}=128$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{144}+\frac{y^2}{128}=1$.

Task 6

Find the eccentricity, foci and the length of the latus rectum of the ellipse $9x^2 + 4y^2 = 36$

Task 6

Answer

i. $9x^2 + 4y^2 = 36$

Dividing by 36, we get

$$\frac{9x^2}{36} + \frac{4y^2}{36} = 1$$

$$\text{(i.e.)} \quad \frac{x^2}{4} + \frac{y^2}{5} = 1$$

$$\therefore a^2 = 4, \ b^2 = 9.$$

This is an ellipse whose major axis is the *y*-axis and minor axis is the *x*-axis and centre at the origin.

$$\therefore a^2 = b^2(1 - e^2) \Rightarrow 4 = 9(1 - e^2)$$

$$\therefore 9e^2 = 5$$

Therefore, eccentricity = $e = \frac{\sqrt{5}}{3}$

Therefore, foci are $\left(0, \pm\frac{be}{1}\right)$ (i.e.) $(0, \pm\sqrt{5})$.

Therefore, latus rectum = $\frac{2a^2}{b} = 2 \times \frac{4}{3} = \frac{8}{3}$.

Task 7

The equation $25(x^2 - 6x + 9) + 16y^2 = 400$ represents an ellipse. Find the centre and foci of the ellipse. How should the axis be transformed so that the ellipse is represented by the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$?

Task 7

Answer

$25(x^2 - 6x + 9) + 16y^2 = 400$
$25(x - 3)^2 + 16y^2 = 400$

Dividing by 400, $\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1$; Take $x - 3 = X$, $y = Y$.

Then $\frac{X^2}{16} + \frac{Y^2}{25} = 1$.

The major axis of this ellipse is the Y-axis.

$$\therefore 16 = 25(1 - e^2) \Rightarrow 1 - e^2 = \frac{16}{25} \Rightarrow e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$
$$\therefore e = \frac{3}{5}.$$

Centre is (3, 0). Foci are (3, $\pm$ ae) (i.e.) $\left(3, \pm 5 \times \frac{3}{5}\right)$ (i.e.) (3, $\pm$3). Now

shift origin to the point (3, 0) and then rotate the axes through right

angles. Then the equation of the ellipse becomes $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Task 8

Find the eccentricity of an ellipse given that:

1. its major axis subtends an angle of $120°$ at the endpoints of its minor axis;
2. the segment between a focus and the farthest vertex subtends an angle of $90°$ at the endpoints of its minor axis.

Task 8

Answer

Find the eccentricity of an ellipse given that

(a) Its major axis subtends an angle $120°$ at the endpoints of its minor axis.

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

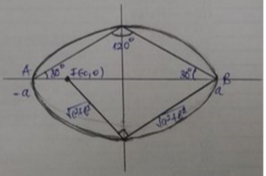
$$tg\, 30° = \frac{b}{a}$$

$$\sqrt{3}\, b = a \Rightarrow b^2 = \frac{a^2}{3}$$

$$\frac{b^2}{a^2} = \frac{1}{3}$$

$$\varepsilon = \sqrt{1 - \frac{1}{3}}$$

$$\varepsilon = \sqrt{\frac{2}{3}}$$



(b) The segment between a focus and the farthest vertex subtend an angle of $90°$ at the endpoints of its minor axis

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}$$

$$c^2 + b^2 + b^2 + a^2 = (a+c)^2$$

$$2b^2 + \cancel{a^2} + \cancel{c^2} = \cancel{a^2} + 2ac + \cancel{c^2}$$

$$2b^2 = 2ac$$

$$b^2 = ac$$

$$c^2 = a^2 - b^2$$

$$c^2 = a^2 - ac$$

$$c^2 + ac - a^2 = 0$$

$$c_{1,2} = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2}$$

$$\varepsilon > 0, \quad \varepsilon = \frac{c}{a}$$

$$\varepsilon = \frac{-1 + \sqrt{5}}{2}$$

Reference material

- Conics Section (Wiki)
- Conic sections (Khan Academy, full playlist, eng)

Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)