Essentials of Analytical Geometry and Linear Algebra I, Class #2

Innopolis University, September 2022

1 Dot Product

1. Find $|\mathbf{a}|^2 - 2\sqrt{3}\mathbf{a}\cdot\mathbf{b} - 7|\mathbf{b}|^2$ given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 1$, $\angle(\mathbf{a}, \mathbf{b}) = 150°$.

2. Find the angle[1] between $\mathbf{a}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}-5\\1\\-1\end{bmatrix}$.

3. Prove that vectors $\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ and $\mathbf{a}$ are perpendicular to each other.

4. All three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have length of 3 and $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. Find $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$

5. Find an angle between $\mathbf{a}$ and $\mathbf{b}$ if:
   (a) $\mathbf{a}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-2\\2\\-2\end{bmatrix}$;
   (b) $\mathbf{a}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}5\\1\\1\end{bmatrix}$;
   (c) $\mathbf{a}=\begin{bmatrix}3\\1\\-2\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-2\\2\\-2\end{bmatrix}$.

6. There are two vectors on some basis $\mathbf{a}=\begin{bmatrix}x\\1-x\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}x^2-2x\\x^2-2x+1\end{bmatrix}$. It is needed to find x, when:
   (a) vectors are collinear;
   (b) they have the same direction.

7. There are two vectors $\mathbf{a}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}5\\1\\1\end{bmatrix}$. Length of $\mathbf{c}$ is equal to 1 and the vector is perpendicular to $\mathbf{a}$. The angle between $\mathbf{b}$ and $\mathbf{c}$ is $arccos(\sqrt{\frac{2}{27}})$. Find the coordinates of $\mathbf{c}$. How many solutions the task have?

[1] If not stated otherwise, the coordinate system in this section is supposed to be Cartesian.

2 Cross Product

1. Find the cross product[2] of:

   (a) vectors $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$;

   (b) vectors $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -18 \\ 12 \\ -6 \end{bmatrix}$.

2. Simplify the expressions:

   (a) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$;

   (b) $(3\mathbf{a} - \mathbf{b} - \frac{1}{3}\mathbf{c}) \times (2\mathbf{a} + \frac{3}{2}\mathbf{b} - 3\mathbf{c})$.

[2] If not stated otherwise, the coordinate system in this assignment is supposed to be Cartesian.

3 How to Solve?

3.1 Dot Product

1. For $|\mathbf{a}| = 4$, $|\mathbf{b}| = 1$, $\angle(\mathbf{a}, \mathbf{b}) = 150°$

   $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\alpha$

   $|\mathbf{a}|^2 - 2\sqrt{3}\mathbf{a} \cdot \mathbf{b} - 7|\mathbf{b}|^2 = |\mathbf{4}|^2 - 2\sqrt{3}|\mathbf{a}| \cdot |\mathbf{b}| \cos 150° - 7|\mathbf{1}|^2$
   $|\mathbf{a}|^2 - 2\sqrt{3}\mathbf{a} \cdot \mathbf{b} - 7|\mathbf{b}|^2 = 16 - 2\sqrt{3} \cdot 4 \cdot 1 \cdot (-\frac{\sqrt{3}}{2}) - 7$
   $|\mathbf{a}|^2 - 2\sqrt{3}\mathbf{a} \cdot \mathbf{b} - 7|\mathbf{b}|^2 = 21$

2. For $\mathbf{a}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}-5\\1\\-1\end{bmatrix}$

   $\alpha = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|}$

   $\mathbf{a} \cdot \mathbf{b} = (1 \cdot (-5) + 1 \cdot 1 + 1 \cdot (-1))$
   $|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2}$
   $|\mathbf{b}| = \sqrt{(-5)^2 + 1^2 + (-1)^2}$
   $\alpha = \arccos \frac{-5}{9}$

3. $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ and $\mathbf{a}$ are perpendicular

   It is usually called **vector triple product**:
   $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

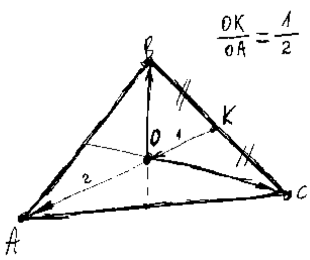
   q.e.d.

4. All three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have length of 3 and $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Find $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$

   Let us remember some properties of medians of triangle:

$\vec{OA}+\vec{OB}+\vec{OC}=0$
$2\vec{KO}+\vec{OB}+\vec{OC}=0$
$2(\vec{KC}+\vec{CO})+\vec{OB}+\vec{OC}=0$
$2\vec{KC}+2\vec{CO}+\vec{OB}+\vec{OC}=0$
$2\vec{KC}+\vec{OB}-2\vec{OC}+\vec{OC}=0$
$\vec{OC}-\vec{OC}=0$

$\blacksquare$
q.e.d



In such a way if sum of three vectors is equal to zero we may conclude that tails of vectors are vertices of triangle and their heads lie in the point which is the center of mass of triangle.

In this particular case all the three segments have the same lengths, then the triangle is an equilateral triangle. So, $\angle(\mathbf{a}, \mathbf{b}) = \angle(\mathbf{b}, \mathbf{c}) = \angle(\mathbf{c}, \mathbf{a}) = 120°$. Then,
$\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = 3 * 3 * (-\frac{1}{2}) + 3 * 3 * (-\frac{1}{2}) + 3 * 3 * (-\frac{1}{2}) = -\frac{27}{2}$

5. See task 2.

6. Find x, if:

   (a) $\mathbf{a}=\begin{bmatrix} x \\ 1-x \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} x^2-2x \\ x^2-2x+1 \end{bmatrix}$ are collinear;

   Two vectors are collinear if relations of their coordinates are equal, i.e. $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$. Note: This condition is not valid if one of the components of the vector is zero.
   In other words, two vectors are collinear if their cross product is equal to the null vector.
   $x(x^2 - 2x + 1) - (1 - x)(x^2 - 2x) = 0$
   $x^3 - 2x^2 + x - x^2 + 2x + x^3 - 2x^2 = 0$
   $2x^3 - 5x^2 + 3x = 0$
   $x(2x^2 - 5x + 3) = 0$
   Solving the equation we obtain the roots:
   $x_1 = 0$, $x_2 = \frac{3}{2}, x_3 = 1$

   (b) they have the same direction.

   If the cross product of two vectors is zero then it can be said that they are collinear.
   Two vectors are in exactly the same direction if one is a positive scalar multiple of the other.
   For $x_1 = 0$, $\mathbf{a}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $k = 1$;
   for $x_2 = \frac{3}{2}$, $\mathbf{a}=\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$, then $k = -\frac{1}{2}$;
   for $x_3 = 1$, $\mathbf{a}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, then $k = -1$.

7. Let us denote $\mathbf{c}=\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$ and write other given conditions in the sense of analytical geometry:

$$\begin{cases} x_c^2 + y_c^2 + z_c^2 = 1 \\ \arccos \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| \cdot |\mathbf{c}|} = \arccos \frac{\sqrt{2}}{\sqrt{27}} \\ \mathbf{a} \cdot \mathbf{c} = 0 \end{cases}$$

$$\begin{cases} x_c^2 + y_c^2 + z_c^2 = 1 \\ \mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| \cdot |\mathbf{c}| \frac{\sqrt{2}}{\sqrt{27}} \\ \mathbf{a} \cdot \mathbf{c} = 0 \end{cases}$$

$$\begin{cases} x_c^2 + y_c^2 + z_c^2 = 1 \\ 5x_c + y_c + z_c = \sqrt{27} \cdot 1 \cdot \frac{\sqrt{2}}{\sqrt{27}} \\ x_c - y_c + z_c = 0 \end{cases}$$

$$\begin{cases} x_c^2 + y_c^2 + z_c^2 = 1 \\ 5x_c + y_c + z_c = \sqrt{2} \\ x_c - y_c + z_c = 0 \end{cases}$$

$$\begin{cases} x_c^2 + y_c^2 + z_c^2 = 1 \\ x_c^2 = (\frac{\sqrt{2}}{6} - \frac{z_c}{3})^2 \\ y_c^2 = (x_c + z_c)^2 \end{cases}$$

Substituting 2nd and 3 rd equations to the 1st one:

$(\frac{\sqrt{2}}{6} - \frac{z_c}{3})^2 + (\frac{\sqrt{2}}{6} + \frac{2z_c}{3})^2 + z_c^2 = 1$

$\frac{2}{36} - 2\frac{\sqrt{2}}{6}\frac{z_c}{3} + \frac{z_c^2}{9} + \frac{2}{36} + 2 \cdot 2\frac{\sqrt{2}}{6}\frac{z_c}{3} + \frac{4z_c^2}{9} + z_c^2 - 1 = 0$

$\frac{14}{9}z_c^2 + \frac{\sqrt{2}}{9}z_c - \frac{8}{9} = 0$

Solving quadratic equation:

$z_{1,2} = \frac{-\sqrt{2} \mp 15\sqrt{2}}{2 \cdot 14}$

$z_1 = -\frac{4\sqrt{2}}{7}$

$z_2 = \frac{\sqrt{2}}{2}$

Then

$x_{c_1} = \frac{\sqrt{2}}{6} + \frac{4\sqrt{2}}{21} = \frac{15\sqrt{2}}{42}$

$x_{c_2} = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6} = 0$

and

$y_{c_1} = \frac{15\sqrt{2}}{42} - \frac{8\sqrt{2}}{42} = \frac{\sqrt{2}}{6}$

$y_{c_2} = \frac{\sqrt{2}}{2} + 0 = \frac{\sqrt{2}}{2}$

3.2 Cross Product

1. (a) vectors $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 \cdot b_3 - a_3 \cdot b_2 \\ a_3 \cdot b_1 - a_1 \cdot b_3 \\ a_1 \cdot b_2 - a_2 \cdot b_1 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} -2 \cdot (-3) - 1 \cdot (-5) \\ 1 \cdot 2 - 3 \cdot (-3) \\ 3 \cdot (-5) - (-2) \cdot 2 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 11 \\ 11 \\ -11 \end{bmatrix}$$

(b) vectors $\mathbf{a}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}-18\\12\\-6\end{bmatrix}$

$\mathbf{a}\times\mathbf{a}=0$
$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

$\mathbf{a}\times\mathbf{b}=6\cdot\mathbf{a}\times\mathbf{a}$
$\mathbf{a}\times\mathbf{b}=0$

2. (a) $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$

$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}$
$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2(\mathbf{a}\times\mathbf{b})$

(b) $(3\mathbf{a}-\mathbf{b}-\frac{1}{3}\mathbf{c})\times(2\mathbf{a}+\frac{3}{2}\mathbf{b}-3\mathbf{c})$.