



Analytical Geometry and Linear Algebra I, Lab 4

Inverse Matrix
Change of basis

Questions from the class



No questions for today

Inverse Matrix

What is it?

Inverse matrix A^{-1} is the matrix, the product of which to original matrix A is equal to the *identity matrix* I :

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Inverse Matrix

Why do we need it?



Say we want to find matrix X , and we know matrix A and B :

$$XA = B$$

It would be nice to divide both sides by A (to get $X=B/A$), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from $1x = ab$ for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

Inverse Matrix

Properties

1. $\det(A^{-1}) = \frac{1}{\det(A)}$
2. $(AB)^{-1} = A^{-1}B^{-1}$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $(kA)^{-1} = \frac{A^{-1}}{k}$
5. $(A^{-1})^{-1} = A$

Inverse Matrix

How to find

There are 2 ways:

1. Classical approach
2. Gauss-Jordan / Reduced Row Echelon Form (RREF)



Inverse Matrix: Classical Approach

Theory

$$A_{2 \times 2}^{-1} = \frac{C^T}{\det(A)}, \text{ where } C \text{ is a matrix of cofactors.}$$

$$C_{2 \times 2} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \text{ where } C_{ij} = (-1)^{i+j} M_{ij} \text{ — (we met it on previous lab (lab 3))}$$

Inverse Matrix: Classical Approach

Case Study

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \text{ Let's find } A^{-1}.$$

1. Find a determinant (shouldn't be equal to 0, otherwise \rightarrow stop calculations).

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2$$

2. Find Cofactor matrix

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2|4| = 4$$

$$C_{12} = (-1)^{1+2}M_{12} = (-1)^3|3| = -3$$

$$C_{21} = (-1)^{2+1}M_{21} = (-1)^3|2| = -2$$

$$C_{22} = (-1)^{2+2}M_{22} = (-1)^4|1| = 1$$

$$C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

3. Transpose cofactor matrix

$$C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

4. Substitute it to the main formula

$$A^{-1} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$



Inverse Matrix: Gauss-Jordan

Core Idea for Inverse Matrices

$$(A|I) \rightarrow \dots \rightarrow (I|A^{-1})$$

1. Using sequence of *elementary row operations* to modify the matrix until the lower left-hand corner of the matrix is filled with zeros (**Row Echelon Form**/Upper Triangular Matrix). $(A|I) \rightarrow \dots \rightarrow (\triangleleft_{Upper}|B)$
2. Using *elementary row operations* transform Upper Triangular Matrix to Identity Matrix. $(\triangleleft_{Upper}|B) \rightarrow \dots \rightarrow (I|A^{-1})$

Elementary Row Operations

- Swapping two rows,
- Multiplying a row by a nonzero number,
- Adding a multiple of one row to another row. (subtraction can be achieved by multiplying one row with -1 and adding the result to another row)

Inverse Matrix: Gauss-Jordan

Case study (2×2)

$$(A|E) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{(1)} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{(2)} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{(3)} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array} \right)$$

Inverse Matrix: Gauss-Jordan

Case study (4×4)

$$\begin{aligned}(A|I) &= \left(\begin{array}{cccc|cccc} 2 & 3 & 2 & 2 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{(1)} \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right) \xrightarrow{(2)} \\ &\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{(3)} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{(4)} \\ &\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array} \right) \xrightarrow{(5)} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & -1 & -3 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array} \right) \xrightarrow{(6)} \\ &\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array} \right) = (I|A^{-1})\end{aligned}$$

Inverse Matrix

Task 1

Find inverse matrices for the following matrices:

1. $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix};$

2. $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix};$

Inverse Matrix

Task 2

Solve matrix equations:

$$1. \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix};$$

$$2. X \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix};$$

***Don't say you
love the anime***

***If you haven't
read the manga***

Spanning Vectors #1

Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

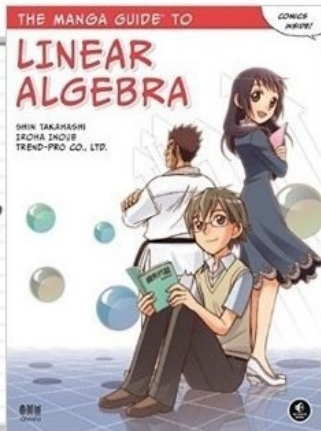
Does $\{v_1, v_2\}$ span \mathbb{R}^2 ?

Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ Can we write $x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$?

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \xrightarrow{\text{Aug}} \begin{bmatrix} 1 & 1 & a \\ 1 & -2 & b \end{bmatrix}$

$E_2 \leftarrow E_2 - E_1 \quad \begin{bmatrix} 1 & 1 & a \\ 0 & -3 & b-a \end{bmatrix} \quad E_2 \leftarrow -\frac{1}{3}E_2 \quad \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & \frac{a-b}{3} \end{bmatrix}$

$E_1 \leftarrow E_1 - E_2$



Preparation to Exam / Test

Strategy for efficient exam solving



Problem Statement

During an exam, I *spend too much time* on finding the solution

Solution

To find the right strategy for *preparation* and *behavior* during an exam.

Preparation to Exam / Test

My own guide and thoughts



I should pay attention on this parts:

1. Preparation before a test
2. Preparation in a day of a test
3. Behavior on exam

Approx time consuming:

- Preparation: 3-8 hours in overall
- Exam:
 - Find the idea how to solve a particular task — 10 sec - 2 min
 - Implement the idea — 10-20 min

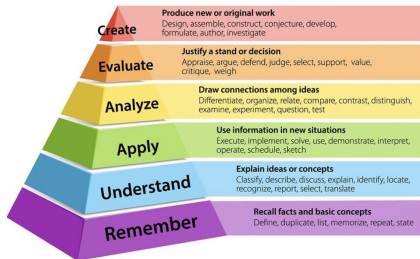
Preparation to Exam / Test

Preparation strategy

1. Understand the concept of a new topic
(Apply or Analyze in terms of Bloom's Taxonomy)
 - 1.1 Look at slides and videos
 - 1.2 Play with concept (suggest some ideas and prove it or disprove via computer or hand calculations)
2. Take a book (material) with exercises and solutions for it.
 - 2.1 Look at a task, imagine how to solve it.
 - 2.2 Check a suggestion from solutions. If you sure that your solution is also applicable — check it.

Important: For some tasks *practical skills* are crucial (find not only an idea, but implement it)!

Bloom's Taxonomy



Preparation to Exam / Test

Behavior before and on exam

Before:

Prepare your brain (skim the material) and mentality (by self hypnosis techniques)

(took from science russian book "Преодолей себя! Психическая подготовка в спорте")

During an exam:

1. Rank tasks by doing speed:

- Can be solved on-a-fly (expect max grade)
- Easy concept — tough implementation (expect that some computational mistakes can be done)
- Tough concept (cannot find the solution on-a-fly) (time consuming tasks)

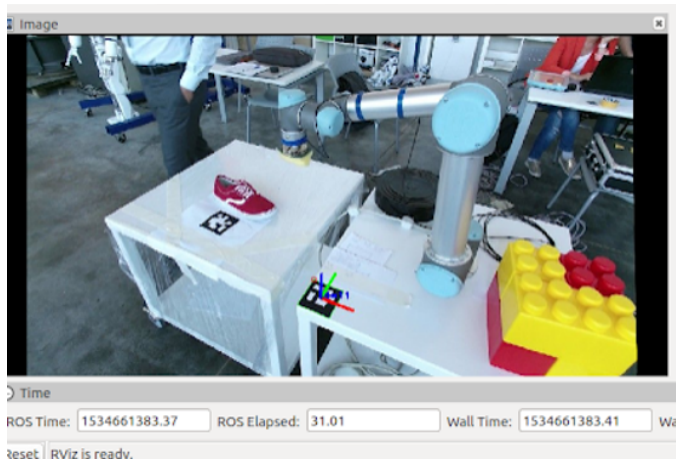
2. Solve it in such order

3. Profit! You are awesome!



Changing Basis

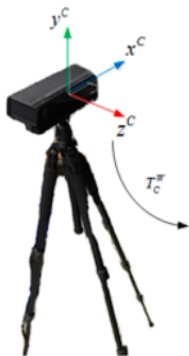
Case Study: Shoe Polishing Robot



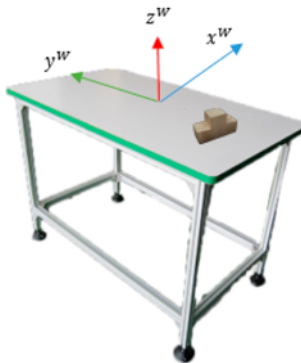
Changing Basis

Case Study: Shoe Polishing Robot, How it works

Camera coordinate system



World coordinate system



Robot coordinate system

Changing Basis

Two Bros in the World of Changing Basis



$$E' = EA \text{ and } Ex = E'x'.$$

Extended Bro Equation: $Ex = Eb + E'x'$, where

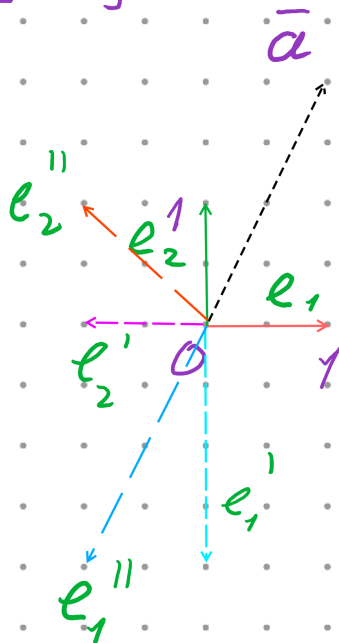
$$E = [e_1 \ e_2 \ e_3] = \begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{bmatrix}, \begin{bmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{bmatrix}, \begin{bmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{bmatrix}, E' = [e'_1 \ e'_2 \ e'_3]$$

More info in Appendix 2

Answer from figure

$$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad a' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$a' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



$$E' = EA \quad E^{-1} = E'^{-1} A^{-1}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E \rightarrow E'$$

$$1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = E' \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_2'$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = E' \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_1'$$

$$I \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} = E' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \in \rightarrow E'$$

$$E' \rightarrow E''; E' = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$$

$$2) \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = E'' \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_1''$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} a'$$

$$\begin{bmatrix} 0 & -0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$a'$$

$$\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = E'' \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_2''$$

$$\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix} = E'' = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$A \in \rightarrow A \in''$$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix} a'' \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$a''$$

$$\begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = a''$$

Changing Basis

Task 3

If vectors **a** and **b** form a basis (you should check it), it is needed to find coordinates **c** in the basis.

$$\mathbf{a} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Changing Basis

Task 3

Answer

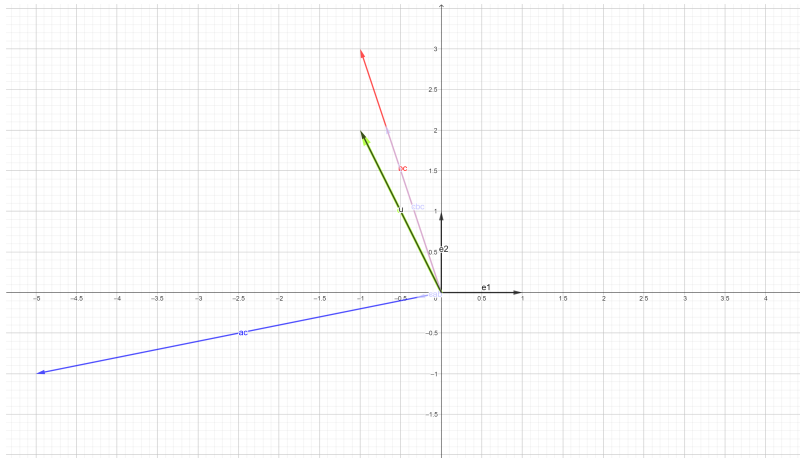
$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E' = \begin{bmatrix} -5 & -1 \\ -1 & 3 \end{bmatrix}$$

Because we are interested in coordinates, 2nd bro

$$E_{Cold} = E'x \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_{new_x} \\ c_{new_y} \end{bmatrix}$$
$$\begin{bmatrix} c_{new_x} \\ c_{new_y} \end{bmatrix} = (E')^{-1} c_{old} = \begin{bmatrix} \frac{1}{16} \\ \frac{16}{11} \\ \frac{16}{16} \end{bmatrix}$$

Changing Basis

Task 3



Changing Basis

Task 4



Two bases are given in the plane: $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}'_1, \mathbf{e}'_2$. The vectors of the second basis have coordinates $(-1; 3)$ and $(2; -7)$ in the first basis.

- (a) Compose transition matrices from the old basis to the new and vice versa.
- (b) Find the coordinates of a vector in the old basis given that it has coordinates α'_1, α'_2 in the new basis.
- (c) Find the coordinates of a vector in the new basis given that it has coordinates α_1, α_2 in the old basis.

Changing Basis

Task 4

Answer

$$E = [\vec{e}_1 \quad \vec{e}_2], \quad E' = [\vec{e}'_1 \quad \vec{e}'_2]$$

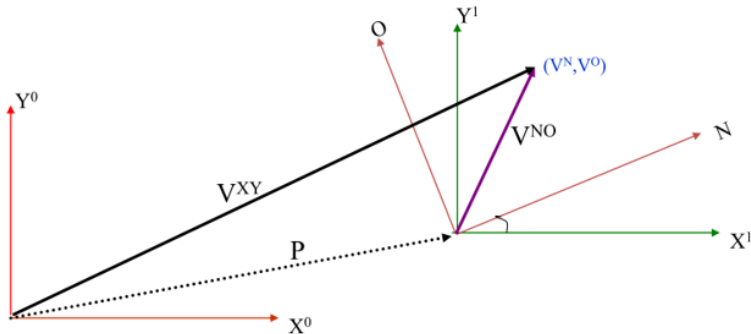
$$E \begin{bmatrix} -1 \\ 3 \end{bmatrix} = E' \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E \begin{bmatrix} 2 \\ -7 \end{bmatrix} = E' \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ because it's a second basis itself}$$

combine together

$$E \begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix} = E' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E \underbrace{\begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix}}_A = E'$$

Changing Basis

Change the coordinate frame



$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$

Changing Basis

Homogeneous representation



$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Padding with 0's and 1's

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Simplifying into a matrix form

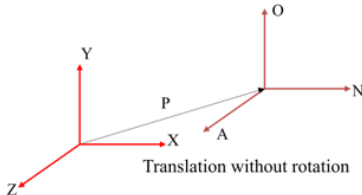
$$\mathbf{H} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous Matrix for a Translation in XY plane, followed by a Rotation around the z-axis

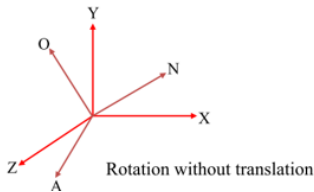
Changing Basis

Special Cases of Homogeneous matrices in 3D

H is a 4x4 matrix that can describe a translation, rotation, or both in one matrix



$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



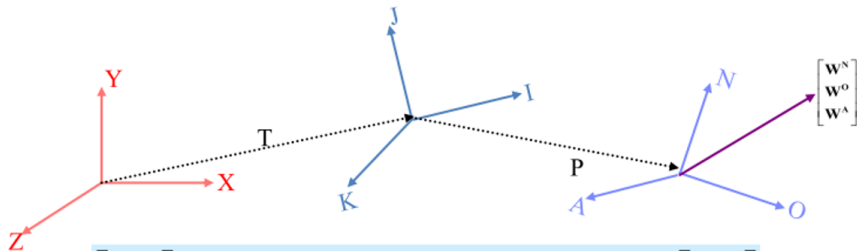
$$\mathbf{H} = \begin{bmatrix} \mathbf{n}_x & \mathbf{o}_x & \mathbf{a}_x & 0 \\ \mathbf{n}_y & \mathbf{o}_y & \mathbf{a}_y & 0 \\ \mathbf{n}_z & \mathbf{o}_z & \mathbf{a}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation part:

Could be rotation around z-axis
x-axis, y-axis or a combination of

Changing Basis

Change the coordinate frame: Case Study



$$\begin{bmatrix} \mathbf{W}^X \\ \mathbf{W}^Y \\ \mathbf{W}^Z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{i}_x & \mathbf{j}_x & \mathbf{k}_x & \mathbf{T}_x \\ \mathbf{i}_y & \mathbf{j}_y & \mathbf{k}_y & \mathbf{T}_y \\ \mathbf{i}_z & \mathbf{j}_z & \mathbf{k}_z & \mathbf{T}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{n}_i & \mathbf{o}_i & \mathbf{a}_i & \mathbf{P}_i \\ \mathbf{n}_j & \mathbf{o}_j & \mathbf{a}_j & \mathbf{P}_j \\ \mathbf{n}_k & \mathbf{o}_k & \mathbf{a}_k & \mathbf{P}_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{W}^N \\ \mathbf{W}^O \\ \mathbf{W}^A \\ 1 \end{bmatrix}$$

Changing Basis

Task 5



Let us consider two coordinate systems in the plane: $O, \mathbf{e}_1, \mathbf{e}_2$ and $O', \mathbf{e}'_1, \mathbf{e}'_2$. Point O' has coordinates $(7; -2)$ in the old coordinate system, and vectors $\mathbf{e}'_1, \mathbf{e}'_2$ can be obtained from vectors $\mathbf{e}_1, \mathbf{e}_2$ by rotating them 60° (a) clockwise; (b) counterclockwise. Find the old coordinates of a point x, y given its new coordinates x', y' .

Changing Basis

Task 5

Answer

Classical way, using Extended 2nd Bro eqn

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} + \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Or using Homogeneous matrix

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 60 & -\sin 60 & 7 \\ \sin 60 & \cos 60 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Changing Basis

Task 6

There are two bases in \mathbb{R}^3 :

$$e_1 = i, e_2 = j, e_3 = k \text{ and } e'_1 = i + j + k, e'_2 = i + j, e'_3 = i$$

Find coordinates of $x = 2i - 3j + k$ in the basis e'_1, e'_2, e'_3 .



Changing Basis

Task 7

There are 4 vectors f_1, f_2, f_3, x and the basis

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the coordinates of x in the basis (f_1, f_2, f_3) , if

$$f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, f_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Reference material



- Inverse Matrix (OnlineMschool)
- Gauss-Jordan (Wiki)
- Changing Basis (3Blue1Brown)

Deserve "A" grade!

– Oleg Bulichev

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📍 @Lupasic

🏢 Room 105 (Underground robotics lab)

1 Changing Basis and Coordinates

Suppose we have two different coordinate systems. The first (so-called “old” coordinate system¹) is given by origin O and basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$; the second (the “new” one) is given by $O', \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. Let point N have coordinates $(x_1 \ x_2 \ x_3)^T$ in the old coordinate system and coordinates $(x'_1 \ x'_2 \ x'_3)^T$ in the new one. It means that

$$\begin{aligned}\overrightarrow{ON} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3; \\ \overrightarrow{O'N} &= x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3.\end{aligned}\tag{1}$$

Let us say that we know how the coordinate systems are related with each other. That is, we can express new basis vectors via the old ones and we know the coordinates of the new origin in the old basis. In detail,

$$\begin{aligned}\mathbf{e}'_1 &= \alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3 \\ \mathbf{e}'_2 &= \alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3 \\ \mathbf{e}'_3 &= \alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3 \\ \overrightarrow{OO'} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3\end{aligned}$$

As $\overrightarrow{ON} = \overrightarrow{OO'} + \overrightarrow{O'N}$, we get

$$\begin{aligned}\overrightarrow{ON} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + \\ & x'_1(\alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3) + x'_2(\alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3) + x'_3(\alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3) = \\ & (b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3)\mathbf{e}_1 + (b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3)\mathbf{e}_2 + (b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3)\mathbf{e}_3.\end{aligned}$$

Taking (1) into account yields

$$\begin{aligned}x_1 &= b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3, \\ x_2 &= b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3, \\ x_3 &= b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3,\end{aligned}$$

or, using matrix notation,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

Thus knowing how new basis depends on the old one enables us to immediately express the old coordinates through the new ones.

Matrix $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$ is called a *transition matrix from the old basis to the new basis*. Using matrix notation, one can easily derive that basis vectors satisfy the equality

$$(\mathbf{e}'_1 \ \mathbf{e}'_2 \ \mathbf{e}'_3) = (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) A.$$

As for coordinates,

$$\mathbf{x} = \mathbf{b} + A\mathbf{x}'.$$

¹In order not to get confused we will refer to a basis, coordinates etc. without primes as to “old” ones and to those with primes as to “new” ones.