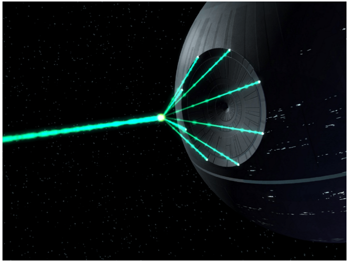
Analytical Geometry and Linear Algebra I, Lab 13

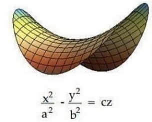
How to determine a type of surface

Sphere

Where it can be used



Death Star — Elliptic paraboloid



Pringles — Hyperbolic paraboloid

Rugby ball — Ellipsoid

How to determine a type of surface (Video explanation)

$2z^2 + 3y^2 - x = 1$

$2x^2 + 3y^2 - z^2 = 1$

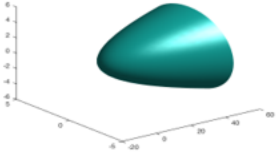
$-3x^2 + 2y^2 - z = 0$

$-3x^2 + 2y^2 - z^2 = 1$

$2x^2 + 3y - z = 0$

$2x^2 + 3y^2 + 4z^2 = 24$

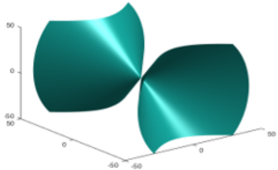
How to determine a type of surface (Video explanation)



$2z^2 + 3y^2 - x = 1$
Elliptic paraboloid

$2x^2 + 3y^2 - z^2 = 1$

$-3x^2 + 2y^2 - z = 0$



$-3x^2 + 2y^2 - z^2 = 1$

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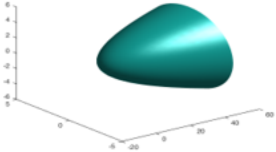
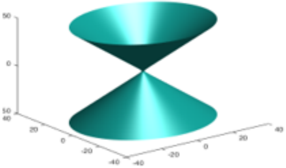
How to determine a type of surface (Video explanation)



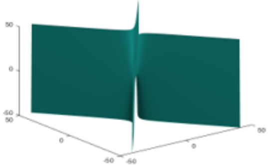
$2z^2 + 3y^2 - x = 1$

Elliptic paraboloid

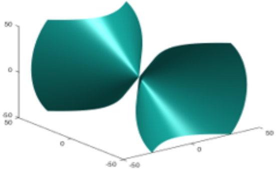


$2x^2 + 3y^2 - z^2 = 1$

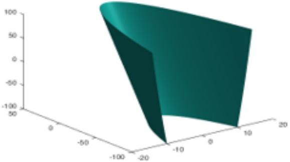
Hyperboloid of one sheet



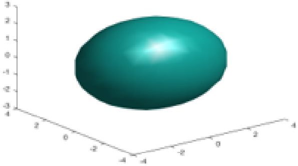
$-3x^2 + 2y^2 - z = 0$



$-3x^2 + 2y^2 - z^2 = 1$

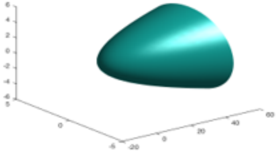


$2x^2 + 3y - z = 0$



$2x^2 + 3y^2 + 4z^2 = 24$

How to determine a type of surface (Video explanation)



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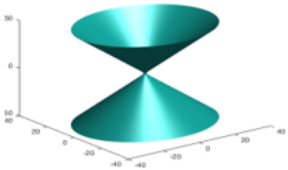
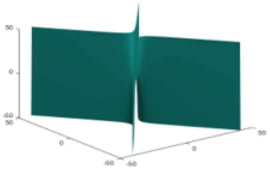
Elliptic paraboloid



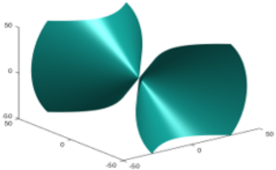
$2x^2 + 3y^2 - z^2 = 1$

Hyperboloid of one sheet

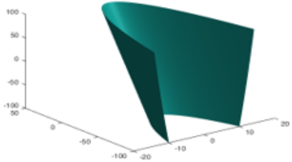


$-3x^2 + 2y^2 - z = 0$

Hyperbolic paraboloid



$-3x^2 + 2y^2 - z^2 = 1$



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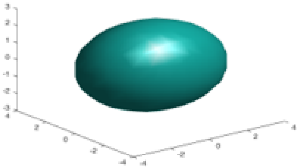

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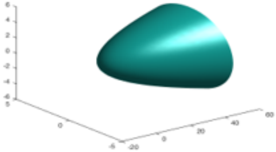
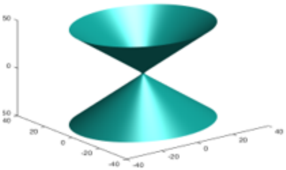
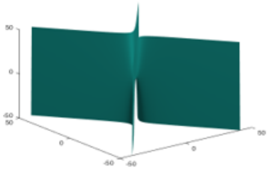
How to determine a type of surface (Video explanation)



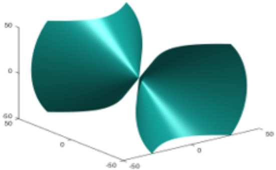
$2z^2 + 3y^2 - x = 1$
Elliptic paraboloid



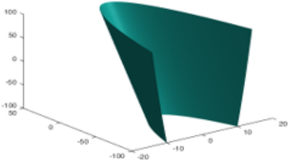
$2x^2 + 3y^2 - z^2 = 1$
Hyperboloid of one sheet



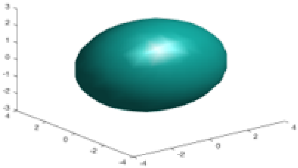
$-3x^2 + 2y^2 - z = 0$
Hyperbolic paraboloid



$-3x^2 + 2y^2 - z^2 = 1$
Hyperboloid of two sheets



$2x^2 + 3y - z = 0$



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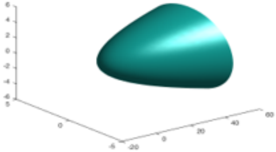

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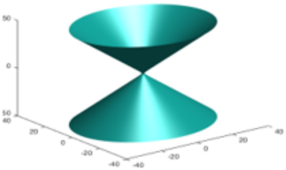
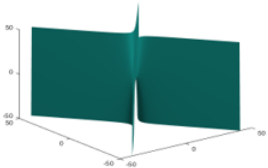
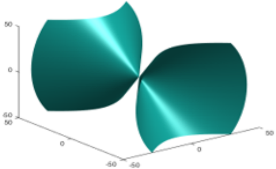


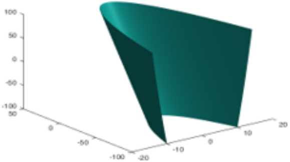
$2x^2 + 3y^2 - z^2 = 1$
Hyperboloid of one sheet



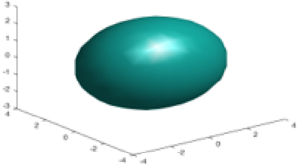
$-3x^2 + 2y^2 - z = 0$
Hyperbolic paraboloid



$-3x^2 + 2y^2 - z^2 = 1$
Hyperboloid of two sheets



$2x^2 + 3y - z = 0$
Parabolic cylinder



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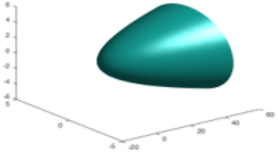
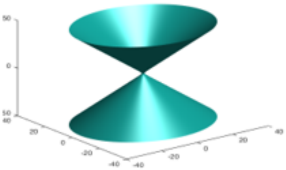
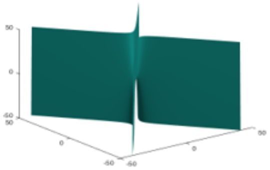
How to determine a type of surface (Video explanation)



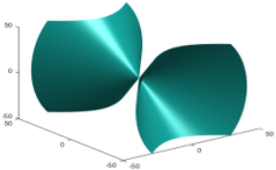
$2z^2 + 3y^2 - x = 1$
Elliptic paraboloid



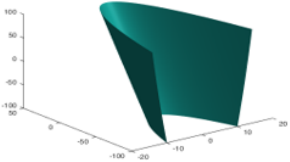
$2x^2 + 3y^2 - z^2 = 1$
Hyperboloid of one sheet



$-3x^2 + 2y^2 - z = 0$
Hyperbolic paraboloid



$-3x^2 + 2y^2 - z^2 = 1$
Hyperboloid of two sheets



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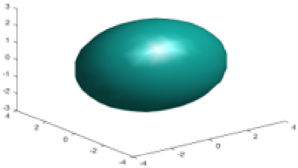
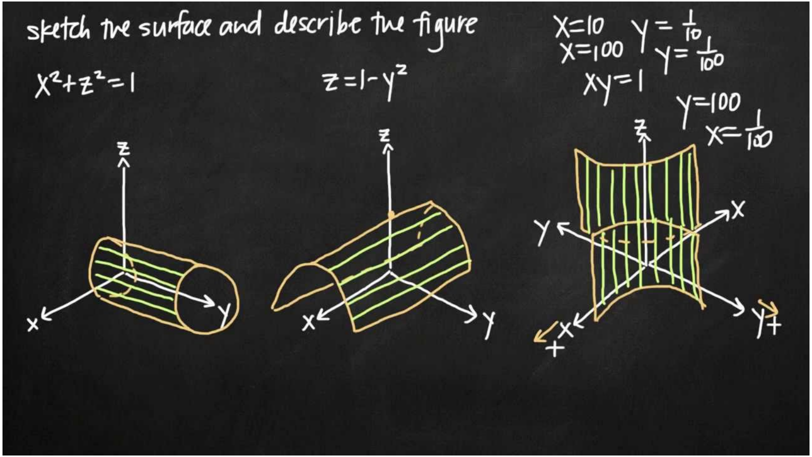


$2x^2 + 3y^2 + 4z^2 = 24$
Ellipsoid

How to sketch surfaces

Video



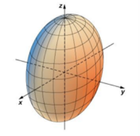
Case studies of 2nd order curve equation (ENG)

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

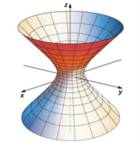


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

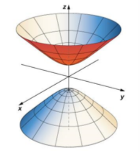


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

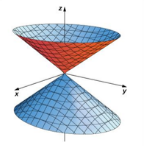


Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

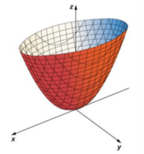


Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.

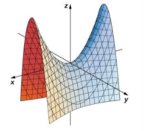


Hyperbolic Paraboloid

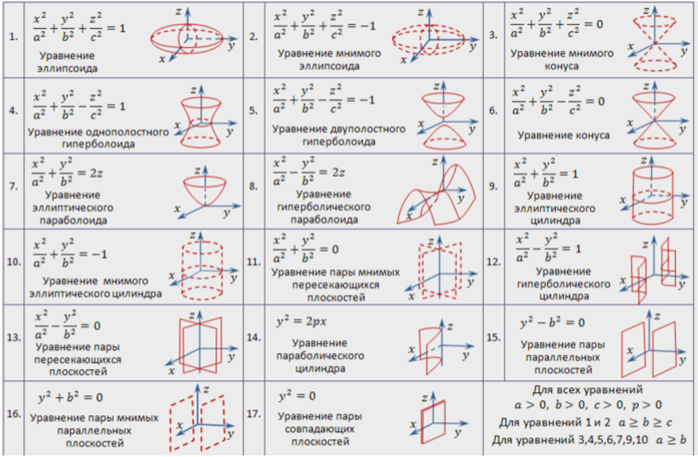
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Traces
In plane $z = p$: a hyperbola
In plane $y = q$: a parabola
In plane $x = r$: a parabola

The axis of the surface corresponds to the linear variable.



Case studies of 2nd order curve equation (RUS)



1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Уравнение эллипсоида	2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ Уравнение мнимого эллипсоида	3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ Уравнение мнимого конуса
4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Уравнение однополостного гиперболоида	5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ Уравнение двуполостного гиперболоида	6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Уравнение конуса
7. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ Уравнение эллиптического параболоида	8. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ Уравнение гиперболического параболоида	9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Уравнение эллиптического цилиндра
10. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ Уравнение мнимого эллиптического цилиндра	11. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ Уравнение пары мнимых пересекающихся плоскостей	12. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Уравнение гиперболического цилиндра
13. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ Уравнение пары пересекающихся плоскостей	14. $y^2=2px$ Уравнение параболического цилиндра	15. $y^2-b^2=0$ Уравнение пары параллельных плоскостей
16. $y^2+b^2=0$ Уравнение пары мнимых параллельных плоскостей	17. $y^2=0$ Уравнение пары совпадающих плоскостей	Для всех уравнений $a>0,\ b>0,\ c>0,\ p>0$ Для уравнений 1 и 2 $a\geq b\geq c$ Для уравнений 3,4,5,6,7,9,10 $a\geq b$

Task 1

Sketch the following surfaces:

1. $4 - 4y^2 - z^2 = x$
2. $-x^2 - y^2 + z^2 = 1$
3. $x^2 + y^2 - z^2 = 4$

Task 1

Answer for $4 - 4y^2 - z^2 = x$

$$x = 4 - 4y^2 - z^2$$

$$\Rightarrow x - 4 = -4y^2 - z^2$$

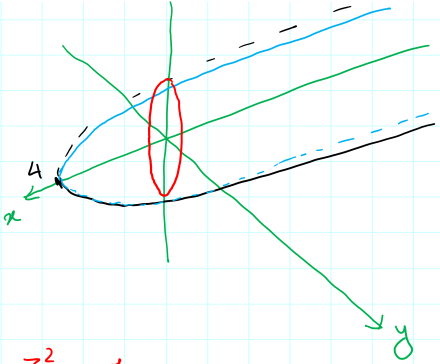
$$\Rightarrow \frac{x-4}{-4} = \frac{y^2}{1^2} + \frac{z^2}{2^2}$$

The vertex is located at $(4, 0, 0)$

In xy plane $(z=0) \longrightarrow$ Parabola: $(x-4) = -4y^2$

In xz plane $(y=0) \longrightarrow$ Parabola: $(x-4) = -z^2$

In the planes $\frac{x-4}{-4} = -c$ $(\perp x) \longrightarrow$ ellipses $\frac{y^2}{(\sqrt{c})^2} + \frac{z^2}{(2\sqrt{c})^2} = 1$



Task 1

Answer for $-x^2 - y^2 + z^2 = 1$

In xy plane $(z=0)$, we get nothing

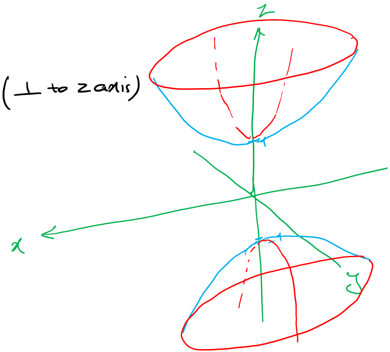
In yz plane $(x=0) \longrightarrow$ hyperbola: $z^2 - y^2 = 1$

In xz plane $(y=0) \longrightarrow$ hyperbola: $z^2 - x^2 = 1$

In any plane $z=c$ & $c \in (-\infty, -1] \cup [1, \infty)$ ($\perp$ to z axis)

circle $x^2 + y^2 = \left(\sqrt{c^2 - 1}\right)^2$

vertices $(0,0,1)$ & $(0,0,-1)$



Task 1

Answer for $x^2 + y^2 - z^2 = 4$

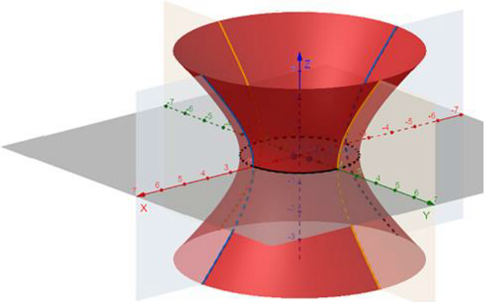
In yz plane $(x=0) \to$ hyperbola: $\frac{y^2}{2^2} - \frac{z^2}{2^2} = 1$

In xz plane $(y=0) \to$ hyperbola: $\frac{x^2}{2^2} - \frac{z^2}{2^2} = 1$

In xy plane $(z=0) \to$ circle: $x^2+y^2=2^2$

In any plane $\perp$ to Z axis $(c=z)$, circle

$$x^2+y^2=\left(\sqrt{1+c^2/4}\right)^2$$



Task 2

Find the equation of the sphere which passes through the points $(2, 7, -4)$ and $(4, 5, -1)$ has its centre on the line joining these two points as diameter.

Task 2

Answer

$$\text{Centre of the sphere} = \left(\frac{2+4}{2}, \frac{7+5}{2}, \frac{-4-1}{2}\right)$$

$$= \left(3,\ 6,\ \frac{-5}{2}\right)$$

$$\text{Radius} = \frac{1}{2}\sqrt{(2-4)^2+(7-5)^2+(-4+1)^2} = \frac{1}{2}\sqrt{4+4+9} = \frac{\sqrt{17}}{2}.$$

Therefore, the equation of the sphere is $(x-3)^2+(y-6)^2+\left(z+\frac{5}{2}\right)^2 = \frac{17}{4}$.

$$x^2+y^2+z^2-6x-12y+5z+9+36+\frac{25}{4}-\frac{17}{4}=0$$

$$x^2+y^2+z^2-6x-12y+5z+47=0$$

Task 3

Find the equation of the sphere which touches the coordinate axes, whose centre lies in the positive octant and has a radius 4.

Task 3

Answer

Let the equation of the sphere be
$x^2 + y^2 + z^2 + 2xu + 2vy + 2wz + d = 0$.

The equation of the x- axis is $\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = t$.

Any point on this line is $(t, 0, 0)$.
The point lies on the given sphere $t^2 + 2ut + d = 0$.

Since the sphere touches the x-axis the two roots of this equation are equal.

$$\therefore 4u^2 - 4d = 0 \quad \text{or} \quad u^2 = d.$$

Similarly, $v^2 = d$ and $w^2 = d$

The radius of the sphere is $\sqrt{u^2 + v^2 + w^2 - d} = \sqrt{3d - d} = \sqrt{2d}$.

$$\sqrt{2d} = 4 \text{ or } d = 8 \quad u^2 = v^2 = w^2 = 8.$$

Since the centre lies on the x-axis, $-u = -v = -w = 2\sqrt{2}$.

$$(x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 + (z - 2\sqrt{2})^2 = 16$$

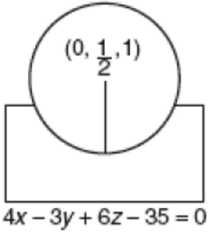
Task 4

Show that the plane $4x - 3y + 6z - 35 = 0$ is a tangent plane to the sphere $x^2 + y^2 + z^2 - y - 2z - 14 = 0$ and find the point of contact.

Task 4

Answer

If the plane is a tangent plane to the sphere then the radius is equal to the perpendicular distance from the centre on the plane.



The centre of the sphere $x^2 + y^2 + z^2 - y - 2z - 14 = 0$ is $\left(0, \frac{1}{2}, 1\right)$.

$$r = \sqrt{0 + \frac{1}{4} + 1 + 14} = \sqrt{\frac{61}{4}}$$

Perpendicular distance from the centre on the plane is

$$\frac{\left|4(0) - 3\left(\frac{1}{2}\right) + 1(6) - 35\right|}{\sqrt{16+9+36}} = \frac{\sqrt{61}}{2}.$$

Therefore, the plane touches the sphere. The equations of the normal to the tangent plane are $\frac{x}{4} = \frac{y - \frac{1}{2}}{-3} = \frac{z-1}{6} = t.$

Any point on this line is $\left(4t, -3t + \frac{1}{2}, 6t + 1\right)$.

If this point lies on the plane $4x - 3y + 6z - 35 = 0$ then,

$$4(4t) - 3\left(-3t + \frac{1}{2}\right) + 6(6t+1) - 35 = 0.$$

$$16t + 9t + 36t - \frac{3}{2} + 6 - 35 = 0 \quad \text{(i.e.)} \quad 61t - \frac{61}{2} = 0, \quad t = \frac{1}{2}$$

Therefore, the point of contact is (2, –1, 4).

Task 5

Obtain the equations to the sphere through the common circle of the sphere $x^2 + y^2 + z^2 + 2x + 2y = 0$ and the plane $x + y + z + 4 = 0$ which intersects the plane $x + y = 0$ in circle of radius 3 units.

Task 5

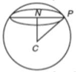
Answer

The equation of the sphere containing the given circle is $x^2 + y^2 + z^2 + 2x + 2y + \lambda(x + y + z + 4) = 0$.

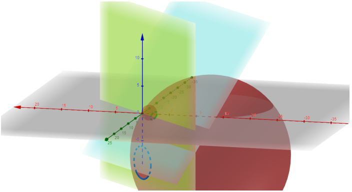
Centre of this sphere is $\left[-\left(1+\frac{\lambda}{2}\right), -\left(1+\frac{\lambda}{2}\right), -\frac{\lambda}{2}\right]$.

$$\text{Radius} = \sqrt{\left(1+\frac{\lambda}{2}\right)^2 + \left(1+\frac{\lambda}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2 - 4\lambda}$$

$$= \frac{1}{2}\sqrt{3\lambda^2 - 8\lambda + 8}$$



CN = Perpendicular from the centre C on the plane $x + y = 0$.



$$= \left|\frac{-\left(1+\frac{\lambda}{2}\right) - \left(1+\frac{\lambda}{2}\right)}{\sqrt{2}}\right| = \frac{2+\lambda}{\sqrt{2}}$$

$$CP^2 = CN^2 + NP^2$$

$$\frac{1}{4}(3\lambda^2 - 8\lambda + 8) = \frac{(2+\lambda)^2}{2} + 9$$

$$3\lambda^2 - 8\lambda + 8 = 8 + 8\lambda + 2\lambda^2 + 36$$

$$\text{(i.e.)} \quad \lambda^2 - 16\lambda - 36 = 0$$

$$(\lambda + 2)(\lambda - 18) = 0$$

$$\therefore \quad \lambda = -2 \text{ or } 18$$

Therefore, the equations of the required spheres are $x^2 + y^2 + z^2 + 2x + 2y - 2(x + y + z + 4) = 0$ and $x^2 + y^2 + z^2 + 2x + 2y + 18(x + y + z + 4) = 0$.

$$\text{(i.e.)} \quad x^2 + y^2 + z^2 - 2z - 8 = 0$$

$$x^2 + y^2 + z^2 + 20x + 20y + 18z + 72 = 0$$

Task 6

Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$

Task 6

Answer

Let the equation of the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (14.54)$$

This sphere touches the plane $3x + 2y - z + 2 = 0$ at $(1, -2, 1)$.
The equation of the tangent plane at $(1, -2, 1)$ is $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$.

$$\text{(i.e.)} \quad x(u+1) + y(v-2) + z(w+1) + u - 2v + w + d = 0 \qquad (14.55)$$

But the tangent plane is given as

$$3x + 2y - z + 2 = 0 \qquad (14.56)$$

Identifying equations (14.55) and (14.56) we get,

$$\frac{u+1}{3} = \frac{v-2}{2} = \frac{w+1}{-1} = \frac{u-2v+w+d}{2} = k(\text{say})$$

$$\therefore \quad u = 3k - 1 \qquad (14.57)$$

$$v = 2k + 2 \qquad (14.58)$$

$$w = -k - 1 \qquad (14.59)$$

$$u - 2v + w + d = 2k$$

$$(3k-1) - 2(2k+2) + (-k-1) + d = 2k$$

$$d = 4k + 6 \qquad (14.60)$$

The sphere (14.54) cuts orthogonally the sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0 \qquad (14.61)$$

$$\therefore \quad 2u(-2) + 2v(3) + d + 4 = 1$$

$$-4u + 6v + d + 4 = 1$$

$$-4(3k-1) + 6(2k+2) + 4k + 6 + 4 = 1$$

$$-12k + 4 + 12k + 12 = 4k + 10$$

$$16 = 4k + 10$$

$$6 = 4k \text{ or } k = \frac{3}{7}$$

$$\therefore u = 3k - 1 = \frac{7}{2}$$

$$v = 2k + 2 = 5$$

$$w = -k - 1 = -\frac{5}{2}$$

$$d = 4k + 6 = 12$$

Therefore, the equation of the sphere is $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$.

Reference material

- How to sketch surfaces (Mathprofi, rus)
- Reducing a quadric surface equation to standard form (video, eng)

Deserve "A" grade!

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