Analytical Geometry and Linear Algebra I, Lab 2

Cross product

Dot product

Lab objectives, 1st part

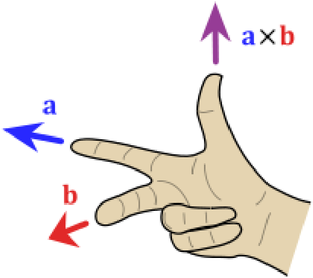
1. What does cross product mean?
2. How to calculate it?
3. What properties of cross product exists? How to use them?

Cross product

Definition

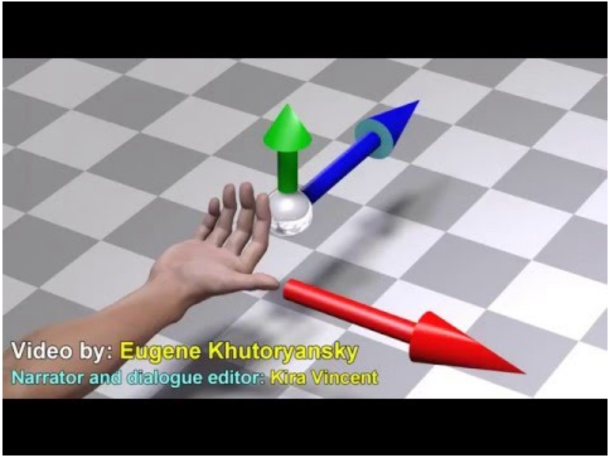
$\vec{a} \times \vec{b} = [\vec{a}, \vec{b}]$ is defined as a vector $\vec{c}$, that is perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$, with:

- *direction* given by the right-hand rule
- *magnitude* is equal to the area of the parallelogram, that the vectors span.



Cross product

Video



Cross product: shoe polishing robot

Video

How to calculate it?

Approaches for finding a full vector:

1. Classical One
2. Using skew-symmetric matrix

Approach for finding only magnitude:

1. Geometrical representation

Classical One

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\text{where, } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A = a \cdot d - b \cdot c$$

Using Skew-symmetric matrix

$$a=\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix}, b=\begin{bmatrix}b_x\\b_y\\b_z\end{bmatrix}$$

$$c = a \times b \Rightarrow c = \hat{a}b$$

vectors $\Rightarrow$ matrices

$a\times \Rightarrow \hat{a}$: a skew-symmetric matrix

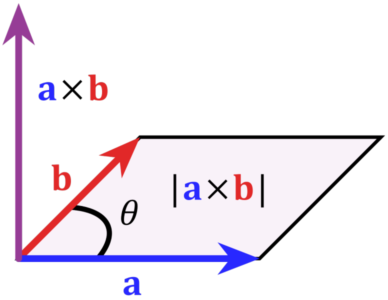
$$c=\hat{a}b=\begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}\begin{bmatrix}b_x\\b_y\\b_z\end{bmatrix}$$

$$\boxed{c=\hat{a}b}$$

Geometrical representation

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \alpha$
where $|\vec{a} \times \vec{b}|$ is an area of parallelogram

Tip: The length or magnitude or norm of the vector a is denoted by $||\vec{a}||$ or, less commonly, $|\vec{a}|$, which is not to be confused with the absolute value (a scalar "norm"). *In Russia,* $|\vec{a}|$ *is more popular.*



Cross product

Case study

Task: to find cross product between $\vec{a} = \begin{bmatrix} -2 \\ -2 \\ 10 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix}$

Classical

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & 10 \\ -4 & 1 & 10 \end{vmatrix} =$$

$$= \vec{i} \begin{vmatrix} -2 & 10 \\ 1 & 10 \end{vmatrix} + \vec{j} \begin{vmatrix} -2 & 10 \\ -4 & 10 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & -2 \\ -4 & 1 \end{vmatrix} = \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix}$$

Skew-symmetric

$$[\vec{a} \times]\vec{b} = \begin{bmatrix} 0 & -10 & -2 \\ 10 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} =$$

$$= \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix}$$

Cross product properties

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
3. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
4. $\lambda\vec{a} \times \vec{b} = \vec{a} \times \lambda\vec{b} = \lambda(\vec{a} \times \vec{b})$
5. $\vec{a} \times \vec{a} = \vec{0}$
6. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

Task 1

Find cross product between $\vec{a}$ and $\vec{b}$, if:

1. $\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$
2. $\vec{a} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 8 \\ -5 \end{bmatrix}$
3. $\vec{a} = \begin{bmatrix} -9 \\ 3 \\ -6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 5 \\ -8 \end{bmatrix}$
4. $\vec{a} = \begin{bmatrix} 8 \\ 3 \\ -9 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -6 \end{bmatrix}$

Task 2

Simplify the expressions:

1. $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$
2. $(3\vec{a}-\vec{b}-\frac{1}{3}\vec{c})\times(2\vec{a}+\frac{3}{2}\vec{b}-3\vec{c})$

Task 2

Simplify the expressions:

1. $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$
2. $(3\vec{a}-\vec{b}-\frac{1}{3}\vec{c})\times(2\vec{a}+\frac{3}{2}\vec{b}-3\vec{c})$

Answer for «1»

```
a = sym('a',[3 1]);
b = sym('b',[3 1]);
simplify(cross(a+b,a-b))
```

ans =

$$\begin{pmatrix} 2\,a_3\,b_2 - 2\,a_2\,b_3 \\ 2\,a_1\,b_3 - 2\,a_3\,b_1 \\ 2\,a_2\,b_1 - 2\,a_1\,b_2 \end{pmatrix}$$

```
>> 2 * cross(b,a)

ans =

 2*a3*b2 - 2*a2*b3
 2*a1*b3 - 2*a3*b1
 2*a2*b1 - 2*a1*b2
```

Task 3

A triangle is constructed on vectors $\vec{a} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

It is needed to

1. Find the area of the triangle.
2. Find the altitudes of this triangle.

Lab objectives, 2nd part

1. What does dot product mean?
2. How to calculate it?
3. How to use it?

Dot product

Definition

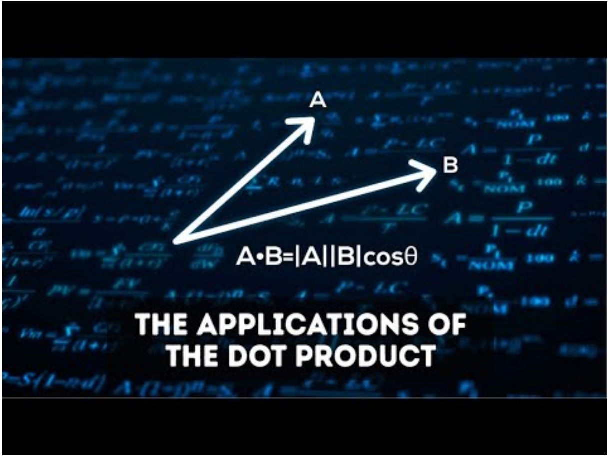
The result of **dot product / inner product**

$\vec{a} \cdot \vec{b} = (\vec{a}, \vec{b}) = \vec{a}^T\vec{b} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ between $\vec{a}$ and $\vec{b}$ is a **scalar**.

Geometrically — $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \beta$ — measure of how similar two vectors are.

Dot product

Video



Tasks 4 and 5

1. Solve $|\vec{a}|^2 - 2\sqrt{3}\vec{a} \cdot \vec{b} - 7|\vec{b}|^2$, where $|\vec{a}| = 4,\ |\vec{b}| = 1,\ \angle(\vec{a}, \vec{b}) = 150°$
2. Find the angle between $\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$

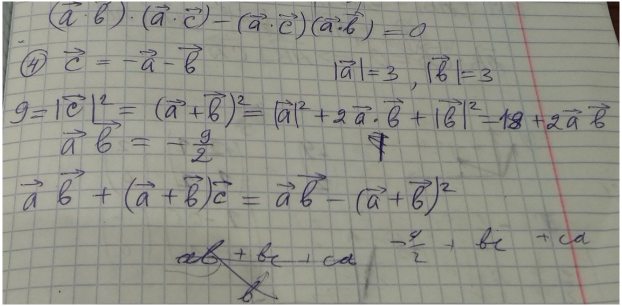
Task 6

All three vectors $\vec{a},\ \vec{b},\ \vec{c}$ have length of 3 and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Find $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$

Task 6

Answer: $-\dfrac{27}{2}$



Pure math solution

Dot and cross product possible application

Video

Task 7

There are two vectors on some basis $\vec{a} = \begin{bmatrix} x \\ 1 - x \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x^2 - 2x \\ x^2 - 2x + 1 \end{bmatrix}$.

It is needed to find x, when:

1. Vectors are collinear
2. They have the same direction

Task 7

Condition of vectors collinearity

Two vectors are collinear, if any of these conditions done:

Condition of vectors collinearity 1. Two vectors $\overline{a}$ and $\overline{b}$ are collinear if there exists a number n such that

$$\overline{a} = n \cdot \overline{b}$$

Condition of vectors collinearity 2. Two **vectors are collinear** if relations of their coordinates are equal.

N.B. Condition 2 is not valid if one of the components of the vector is zero.

Condition of vectors collinearity 3. Two **vectors are collinear** if their cross product is equal to the zero vector.

N.B. Condition 3 applies only to three-dimensional (spatial) problems.

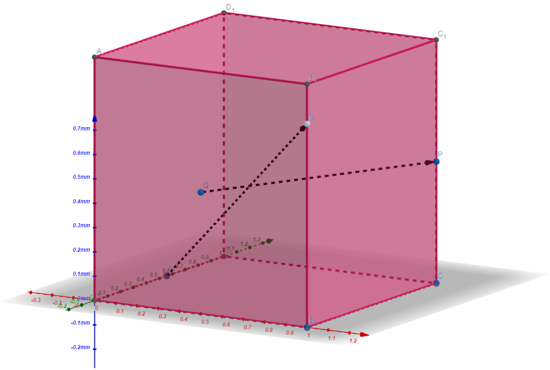
Task 8

At home

The edges of cube $ABCDA_1B_1C_1D_1$ have length of 1. P is a midpoint of CC_1 and Q is a center of face AA_1B_1B. Points M and N belong to lines AD and A_1B_1 respectively, and at that MN intersects with PQ and is perpendicular to it. Find MN.

Task 8

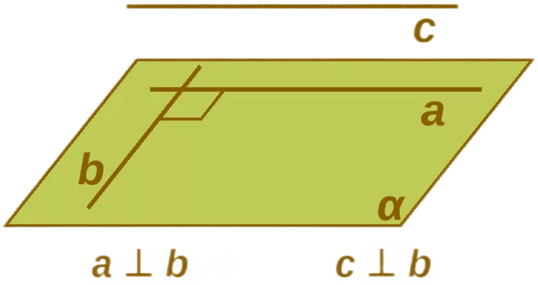
Geogebra with link



Task 8

A hint

It is needed to make 2 equations: 1 for perpendicularity, 2nd — intersection. Think about amount of unknowns and possible equations.



Reference material

- Cross products | Chapter 10, Essence of linear algebra - YouTube
- Dot product | Chapter 9, Essence of linear algebra - YouTube
- Cross product of two vectors - OnlineMSchool
- Векторное произведение векторов - Матпрофи

Deserve “A” grade!

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Room 105 (Underground robotics lab)