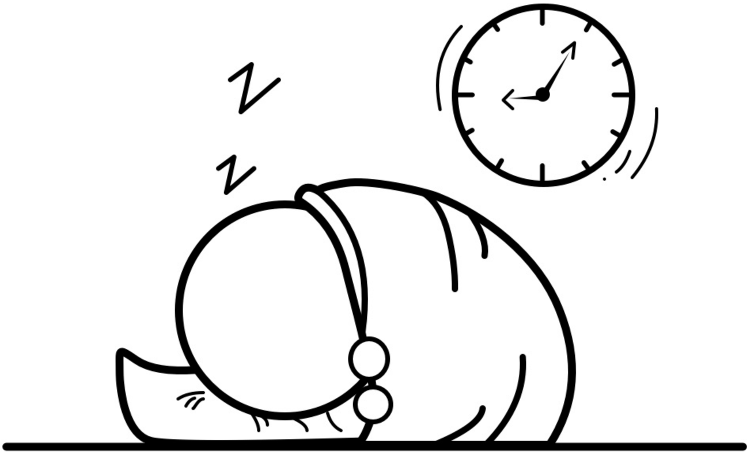
Time to Wake up

Daily Reminder

AGLA-I [f.23]

Midterm



beautiful things happen
when you do the work to reprogram
that negative voice in your head

V1

Theory. Maximum 5 points

- *Definitions, simple proofs (from lectures).*

1. (1 point) Vectors $v_1, v_2, ..., v_n$ are linearly independent if...
2. (2 points) Give a condition of coplanarity of three vectors.
3. (2 points) Give definition of a trace, $Tr(A)$, of matrix and prove linearity of trace.

1 a) $V_1 ... V_n$ are linearly independent if $\alpha_1 \bar{V}_1 + ... + \alpha_n V_n = 0$ can be solved only by $\alpha_i = 0 \ \forall i$, $\alpha_i = const$

1 b) 3 vectors $V_1(x_1 y_1 z_1)$, $V_2(x_2 y_2 z_2)$, $V_3(x_3 y_3 z_3)$ are coplanar, if they are linearly dependent, so that $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

1 c) $tr(A) = \sum_{i=1}^{n} a_{ii}$ – sum of the elements standing on the principle diagonal.

$$tr(\alpha A + \beta B) = \sum_{k=1}^{n} (\alpha A_{kk} + \beta B_{kk}) =$$

$$= \sum_{k=1}^{n} \alpha A_{kk} + \sum_{k=1}^{n} \beta B_{kk} = \alpha \sum_{k=1}^{n} A_{kk} + \beta \sum_{k=1}^{n} B_{kk} =$$

$$= \alpha\, tr(A) + \beta\, tr(B)$$

V1

1. *Vector operations / Matrices*

(a) (2 points) Find the determinant of the 3x3 matrix:

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & 4 & -2 \\ -7 & 3 & 0 \end{bmatrix}$$

(b) (4 points) Find a vector that is orthogonal to both $v_1 = (1,0,1)$ and $v_2 = (1,3,0)$ and which dot product with vector $v_3 = (1,1,0)$ equals to 8.

✓

✓

1a $A = \begin{pmatrix} 2 & 5 & -3 \\ 1 & 4 & -2 \\ -7 & 3 & 0 \end{pmatrix}$

$\det A = -7(-10+12) - 3(-4+3) =$

$= -14 + 3 = -11$

1b $v \parallel v_1 \times v_2$

$\alpha v = \alpha \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = \alpha(-3i + j + 3k) =$

$= -3\alpha \bar{i} + \alpha \bar{j} + 3\alpha \bar{k}$

$\alpha v \cdot v_3 = 8$

$(-3\alpha, \alpha, 3\alpha)(1, 1, 0) = 8$

$-3\alpha + \alpha = 8$

$-2\alpha = 8$

$\alpha = -4 \Rightarrow \alpha v = 12i - 4j - 12k$

Ans: $(12, -4, -12)$

V1

2. *Lines / Planes*

(a) (2 points) Find the angle between the planes $2x - y + z = 6$, $x + y + 2z = 3$.

(b) (4 points) What is the general equation of the plane which contains the following two parallel lines: $\frac{x+1}{6} = \frac{y-2}{7} = z$ and $\frac{x-3}{6} = \frac{y+4}{7} = z - 1$

2a

$n_1\,(2, -1, 1) \qquad \|n_1\| = \sqrt{6}$

$n_2\,(1, 1, 2) \qquad \|n_2\| = \sqrt{6}$

$$\cos \widehat{n_1, n_2} = \frac{n_1 \cdot n_2}{\|n_1\| \|n_2\|} = \frac{2 - 1 + 2}{6} = \frac{1}{2}$$

$\widehat{n_1, n_2} = \theta$

$\cos\theta = \frac{1}{2} \qquad \theta = \frac{\pi}{3}$

2b

$x + 1 = 6t \qquad y - 2 = 7t \qquad z = t$

$\underline{t = 0} \quad x = -1 \quad y = 2 \quad z = 0 \quad M_1(-1, 2, 0)$

$\underline{t = 1} \quad x = 5 \quad y = 9 \quad z = 1 \quad M_2(5, 9, 1)$

$x - 3 = 6t \qquad y + 4 = 7t \qquad z - 1 = t \qquad M(x, y, z)$

$\underline{t = 0} \quad x = 3 \quad y = -4 \quad z = 1 \quad M_3(3, -4, 1)$

$M_1M_2\,(6, 7, 1) \qquad M_1M_3\,(4, -6, 1) \qquad M_1M\,(x+1, y-2, z)$

$$\begin{vmatrix} x+1 & y-2 & z \\ 6 & 7 & 1 \\ 4 & -6 & 1 \end{vmatrix} = 0$$

$(x+1)(7+6) - (y-2)(6-4) + z(-36-28) = 0$

$13x + 13 - 2y + 4 - 64z = 0$

$13x - 2y - 64z + 17 = 0$

3. (4 points) Find the distance from the point $(1,1,-1)$ to the line of intersection of the planes $x+y+z=1$ and $2x-y-5z=1$.

3

$x+y+z=1 \qquad n_1=(1,1,1)$

$2x-y-5z=1 \qquad n_2=(2,-1,-5)$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & -5 \end{vmatrix} = i(-5+1)-j(-5-2)+k(-1-2) = $$

$$= -4i+7j-3k = \bar{a}$$

$$\|\bar{a}\| = \sqrt{16+49+9} = \sqrt{74}$$

$$\begin{cases} x+y+z=1 \\ 2x-y-5z=1 \end{cases}$$

$y = 1-z-x \qquad y = 1-z-\frac{2}{3}-\frac{4}{3}z = \frac{1}{3}-\frac{7}{3}z$

$2x-1+z+x-5z=1$

$3x = 2+4z \rightarrow x = \frac{2}{3}+\frac{4}{3}z$

A point on a line of intersection is:

$z=1 \qquad x=2 \qquad y=-2 \qquad (2\ \ -2\ \ 1)=P$

$A=(1,1,-1) \qquad AP(1,-3,2)$

line becomes $\frac{x-2}{-4} = \frac{y+2}{7} = \frac{z-1}{-3}$

Distance from a point to a line:

$$d = \frac{\|AP \times a\|}{\|a\|} = \frac{5\sqrt{3}}{\sqrt{74}}$$

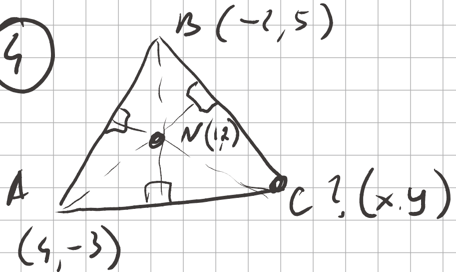
$$AP \times a = \begin{vmatrix} i & j & k \\ 1 & -3 & 2 \\ -4 & 7 & -3 \end{vmatrix} = i(9-14) - j(-3+8) + k(7-12)$$

$$= -5i-5j-5k$$

$$\|AP \times a\| = 5\sqrt{3}$$

4. (4 points) Two vertices of a triangle are $(4, -3)$ and $(-2, 5)$. If the orthocenter (intersection of altitudes) of the triangle is at $(1, 2)$, find the coordinates of the third vertex.

4



$BN\,(3, -3)$ $\quad AC\,(x-4,\ y+3)$

$AN\,(-3, 5)$ $\quad BC\,(x+2,\ y-5)$

$$\begin{cases} BN \cdot AC = 0 \\ AN \cdot BC = 0 \end{cases} \qquad \begin{cases} 3x - 12 - 3y - 9 = 0 \\ -3x - 6 + 5y - 25 = 0 \end{cases}$$

$$\begin{cases} x - y - 7 = 0 \\ -3x + 5y - 31 = 0 \end{cases} \qquad x = y + 7$$

$$-3y - 21 + 5y - 31 = 0$$

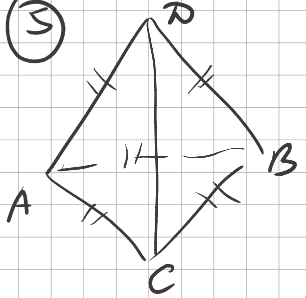
$$2y = 52$$

$y = 26$
$x = 33$

Ans $(33, 26)$

5. (5 points) In a regular tetrahedron ABCD, find the coordinates of the point M in the basis $\{D, \mathbf{DA}, \mathbf{DB}, \mathbf{DC}\}$, if the point M has coordinates $(0, 1/3, 1/3)$ in the basis $\{A, \mathbf{AD}, \mathbf{AB}, \mathbf{AC}\}$.

5



$M\left(0, \frac{1}{3}, \frac{1}{3}\right)$ new

in $(A, \overline{AD}, \overline{AB}, \overline{AC})$

$M = ?$ in

$(D, \overline{DA}, \overline{DB}, \overline{DC})$ old

$= DA$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\text{old}} = \text{Tr. M.} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\text{new}} + \text{shift}$$

given

$$\overline{AD} = -\overline{DA} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\overline{AB} = AD + DB = -DA + DB = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\overline{AC} = AD + DC = -DA + DC = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

V2

Theory. Maximum 5 points

1. *Definitions, simple proofs.*

   (a) (1 point) Let $V = \{v_1, v_2, ..., v_n\}$ be a set of vectors. Give a definition of a span $S(V)$

   (b) (2 points) What is the geometrical interpretation of the magnitude of $\mathbf{a} \times \mathbf{b}$?

   (c) (2 points) Given that BC and CB are valid, prove that $Tr(BC) = Tr(CB)$

1a The set of all possible linear combinations $w = \alpha_1 v_1 + \dots + \alpha_n v_n$, $\alpha_1 - \alpha_n$ - consts.

1b Area of the parallelogram spanned by a & b

1c $tr(BC) = \sum_{i=1}^{n} (BC)_{ii} =$

$= \sum_{i=1}^{n} \sum_{k=1}^{m} B_{ik} C_{ki} = \sum_{k=1}^{m} \sum_{i=1}^{n} C_{ki} B_{ik} =$

$= \sum_{k=1}^{m} (CB)_{kk} = tr(CB)$

V2

1. *Vector operations / Matrices*

(a) (2 points) Find the determinant of matrix

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ 4 & -3 & 5 \end{bmatrix}$$

(b) (4 points) Find a vector that is orthogonal to both $v_1 = (1, -1, 1)$ and $v_2 = (6, -3, 0)$ and which dot product with a vector $v_3 = (3, 2, 3)$ equals to 10.

1a

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ 4 & -3 & 5 \end{pmatrix}$$

$$\det A = (5-3) + 2(10+4) + 2(-6-4) = 2 + 28 - 20 = 10$$

Ans: 10

1b

$$\alpha v = \alpha \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 6 & -3 & 0 \end{vmatrix} = \alpha(3i + 6j + 3k)$$

$$(3\alpha, 6\alpha, 3\alpha)(3, 2, 3) = 10$$

$$9\alpha + 12\alpha + 9\alpha = 10$$

$$30\alpha = 10 \qquad \alpha = \frac{1}{3}$$

Ans $(1, 2, 1)$

V2

2. *Lines / Planes*

(a) (2 points) Find the angle between the planes $x + y - 4z = 8$, $4x + y - z = 8$.

(b) (4 points) What is the general equation of the plane which contains the following two parallel lines: $\frac{x-1}{5} = \frac{y+2}{3} = z$ and $\frac{x+3}{5} = \frac{y-4}{3} = z - 1$

2a

$n_1(1, 1, -4)$ $\quad \|n_1\| = \sqrt{18}$

$n_2(4, 1, -1)$ $\quad \|n_2\| = \sqrt{18}$

$$\cos\theta = \frac{n_1 n_2}{\|n_1\| \|n_2\|} = \frac{4 + 1 + 4}{18} = \frac{1}{2}$$

Ans: $\theta = \frac{\pi}{3}$

2b

$$\frac{x-1}{5} = \frac{y+2}{3} = \frac{z-0}{1}$$

$$\frac{x+3}{5} = \frac{y-4}{3} = \frac{z-1}{1}$$

$x - 1 = 5t \quad y + 2 = 3t \quad z = t$

$t = 0 \quad x = 1 \quad y = -2 \quad z = 0$

$t = 1 \quad x = 6 \quad y = 1 \quad z = 1$

$\}$ 2 points $\in \ell_1$

$x + 3 = 5t \quad y - 4 = 3t \quad z - 1 = t$

$t = 0 \quad x = -3 \quad y = 4 \quad z = 1 \quad (-3, 4, 1)$

$M_1(1, -2, 0) \quad M_2(6, 1, 1) \quad M_3(-3, 4, 1)$

$M_1M_2(5, 3, 1) \quad M_1M_3(-4, 6, 1)$

$M(x, y, z) \quad M_1M(x-1, y+2, z)$

$$\begin{vmatrix} x-1 & y+2 & z \\ 5 & 3 & 1 \\ -4 & 6 & 1 \end{vmatrix} = 0$$

$(x-1)(3-6) \bullet (y+2)(5+4) + z \cdot 42 = 0$

$-3x + 3 - 9y \bullet 18 + 42z = 0 \quad | : -3$

Ans $x + 3y - 14z + 5 = 0$

V2

3. (4 points) Find the distance from the point $(1, 1, -1)$ to the line of intersection of the planes $x + y + z = 1$ and $2x - y - 5z = 1$.

Same

(3) $x+y+z=1$ $\quad n_1=(1,1,1)$

$2x-y-5z=1$ $\quad n_2=(2,-1,-5)$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & -5 \end{vmatrix} = i(-5+1) - j(-5-2) + k(-1-2) = -4i + 7j - 3k = a$$

$\|a\| = \sqrt{16+49+9} = \sqrt{74}$

$$\begin{cases} x+y+z=1 \\ 2x-y-5z=1 \end{cases}$$

$y = 1 - z - x$ $\quad y = 1 - z - \frac{2}{3} - \frac{4}{3}z = \frac{1}{3} - \frac{7}{3}z$

$2x - 1 + z + x - 5z = 1$

$3x = 2 + 4z$ $\rightarrow$ $x = \frac{2}{3} + \frac{4}{3}z$

A point on a line of intersection is:

$z=1$ $\quad x=2$ $\quad y=-2$ $\quad (2\ \ -2\ \ 1) = P$

$A = (1,1,-1)$ $\quad AP\,(1,-3,2)$

line becomes $\dfrac{x-2}{-4} = \dfrac{y+2}{7} = \dfrac{z-1}{-3}$

Distance from a point to a line:

$$d = \frac{\|AP \times a\|}{\|a\|} = \frac{5\sqrt{3}}{\sqrt{74}}$$

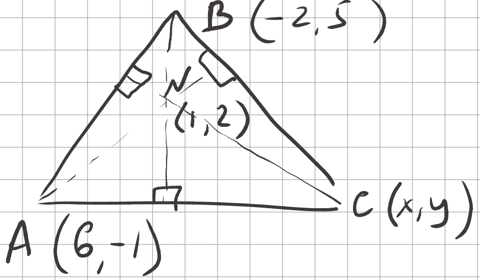
$$AP \times a = \begin{vmatrix} i & j & k \\ 1 & -3 & 2 \\ -4 & 7 & -3 \end{vmatrix} = i(9-14) - j(-3+8) + k(7-12)$$

$= -5i - 5j - 5k$

$\|AP \times a\| = 5\sqrt{3}$

VC

4. (4 points) Two vertices of a triangle are $(6,-1)$ and $(-2,5)$. If the orthocenter (intersection of altitudes) of the triangle is at $(1,2)$, find the coordinates of the third vertex.



$AC\ (x-6,\ y+1)$

$BC\ (x+2,\ y-5)$

$BN\ (3,\ -3)$

$AN\ (-5,\ 3)$

$$\begin{cases} AN \cdot BC = 0 \\ BN \cdot AC = 0 \end{cases}$$

$$\begin{cases} (-5,3)(x+2,\ y-5) = 0 \\ (3,-3)(x-6,\ y+1) = 0 \end{cases}$$

$$\begin{cases} -5x-10+3y-15 = 0 \\ 3x-18-3y-3 = 0 \end{cases}$$

$$\oplus \begin{cases} -5x+3y = 25 \\ 3x-3y = 21 \end{cases}$$

$-2x = 46$

$x = -23$

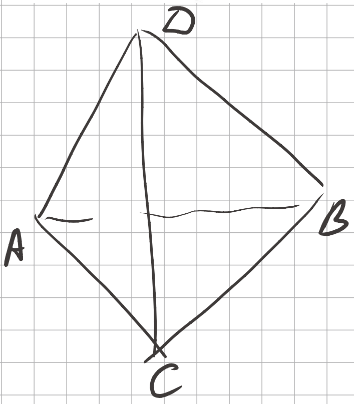
$y = -30$

$x - y = 7$

$y = x - 7 = -30$

<u>Ans</u> $(-23,-30)$

5. (5 points) In a regular tetrahedron ABCD, find the coordinates of the point M in the basis $\{A, \mathbf{AD}, \mathbf{AB}, \mathbf{AC}\}$, if the point M has coordinates $(1/3, 1/3, 1/3)$ in the basis $\{D, \mathbf{DA}, \mathbf{DB}, \mathbf{DC}\}$.



Inverse to v1

$M\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

In (D, DA, DB, DC) new

$M = ?$ In

(A, AD, AB, AC) old

$$\underset{\text{old}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \text{Tr. M.} \underset{\text{new}}{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}} + \text{shift}$$

$$\text{shift} \quad AD = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$DA = -AD = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$DB = DA + AB = -AD + AB = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$DC = DA + AC = -AD + AC = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 + 1 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \\ 1/3 \end{pmatrix}$$