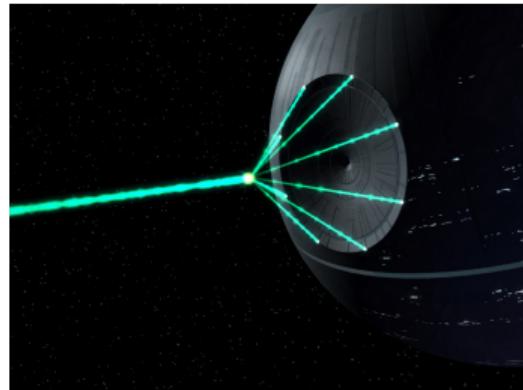


Analytical Geometry and Linear Algebra I, Lab 13

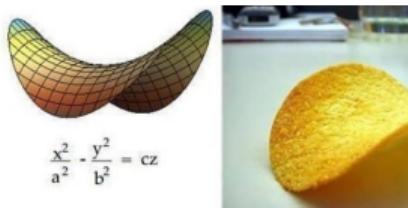
# How to determine a type of surface Sphere



# Where it can be used



Death Star — Elliptic paraboloid



Pringles — Hyperbolic paraboloid



Rugby ball — Ellipsoid



## How to determine a type of surface (Video explanation)

$$(a) 2z^2 + 3y^2 - x = 1$$

$$(b) 2x^2 + 3y^2 - z^2 = 1$$

$$(c) -3x^2 + 2y^2 - z = 0$$

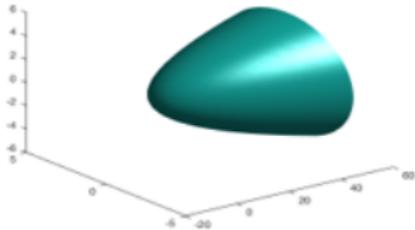
$$(d) -3x^2 + 2y^2 - z^2 = 1$$

$$(e) 2x^2 + 3y - z = 0$$

$$(f) 2x^2 + 3y^2 + 4z^2 = 24$$



# How to determine a type of surface (Video explanation)



(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid

(b)  $2x^2 + 3y^2 - z^2 = 1$

(c)  $-3x^2 + 2y^2 - z = 0$

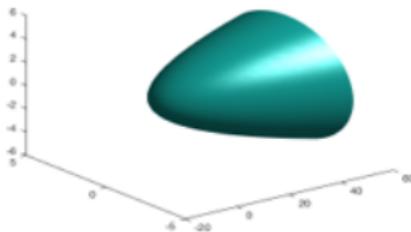
(d)  $-3x^2 + 2y^2 - z^2 = 1$

(e)  $2x^2 + 3y - z = 0$

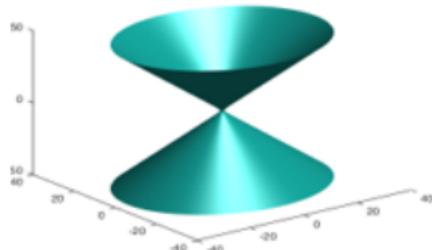
(f)  $2x^2 + 3y^2 + 4z^2 = 24$



# How to determine a type of surface (Video explanation)



(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid



(b)  $2x^2 + 3y^2 - z^2 = 1$   
Hyperboloid of one sheet

(c)  $-3x^2 + 2y^2 - z = 0$

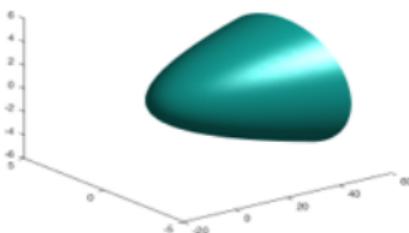
(d)  $-3x^2 + 2y^2 - z^2 = 1$

(e)  $2x^2 + 3y - z = 0$

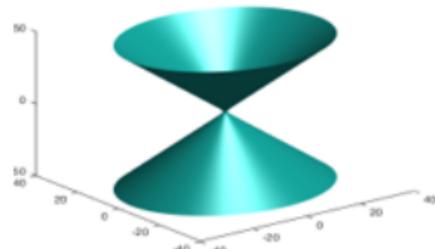
(f)  $2x^2 + 3y^2 + 4z^2 = 24$



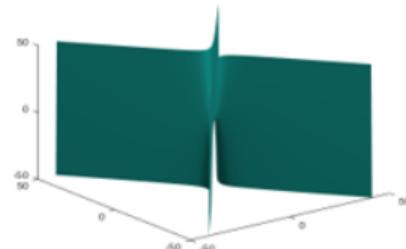
# How to determine a type of surface (Video explanation)



(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid



(b)  $2x^2 + 3y^2 - z^2 = 1$   
Hyperboloid of one sheet



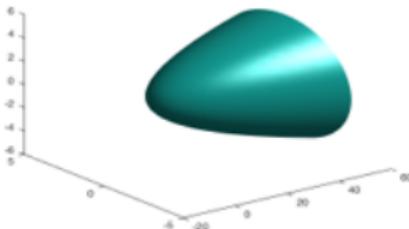
(c)  $-3x^2 + 2y^2 - z = 0$   
Hyperbolic paraboloid

(d)  $-3x^2 + 2y^2 - z^2 = 1$

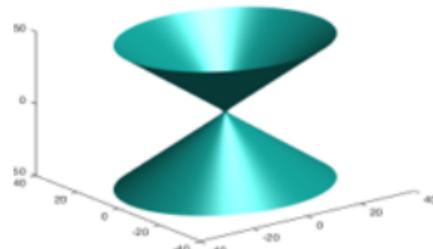
(e)  $2x^2 + 3y - z = 0$

(f)  $2x^2 + 3y^2 + 4z^2 = 24$

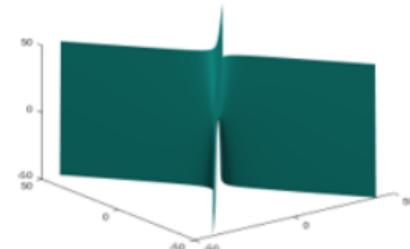
# How to determine a type of surface (Video explanation)



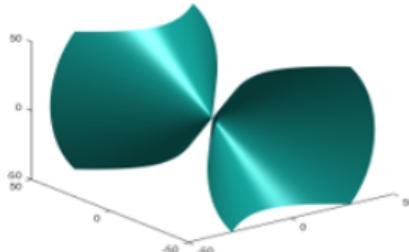
(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid



(b)  $2x^2 + 3y^2 - z^2 = 1$   
Hyperboloid of one sheet



(c)  $-3x^2 + 2y^2 - z = 0$   
Hyperbolic paraboloid

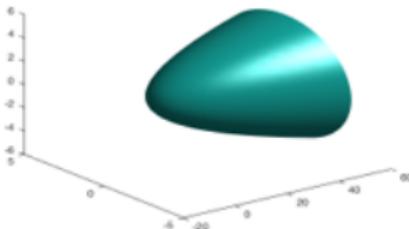


(d)  $-3x^2 + 2y^2 - z^2 = 1$   
Hyperboloid of two sheets

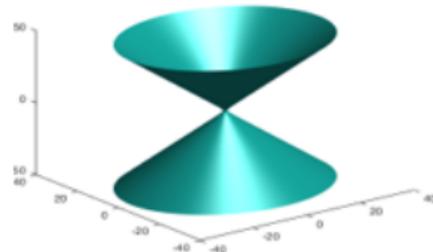
$$2x^2 + 3y - z = 0$$

$$2x^2 + 3y^2 + 4z^2 = 24$$

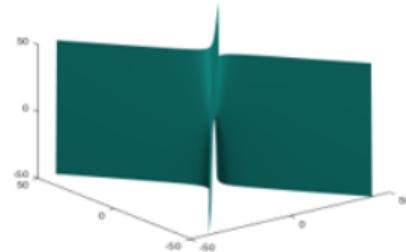
# How to determine a type of surface (Video explanation)



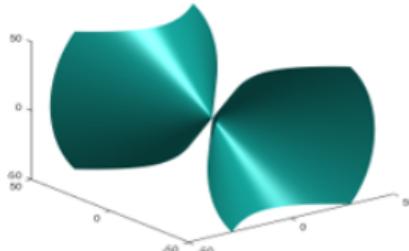
(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid



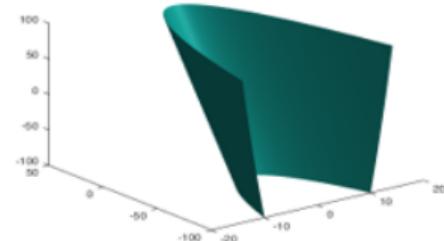
(b)  $2x^2 + 3y^2 - z^2 = 1$   
Hyperboloid of one sheet



(c)  $-3x^2 + 2y^2 - z = 0$   
Hyperbolic paraboloid



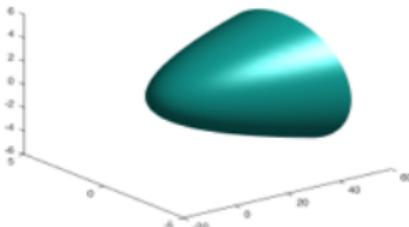
(d)  $-3x^2 + 2y^2 - z^2 = 1$   
Hyperboloid of two sheets



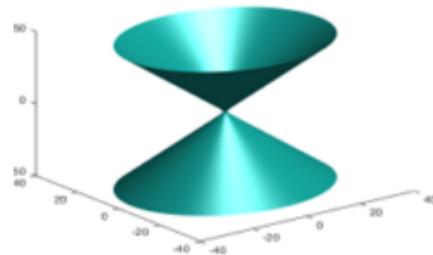
(e)  $2x^2 + 3y - z = 0$   
Parabolic cylinder

(f)  $2x^2 + 3y^2 + 4z^2 = 24$

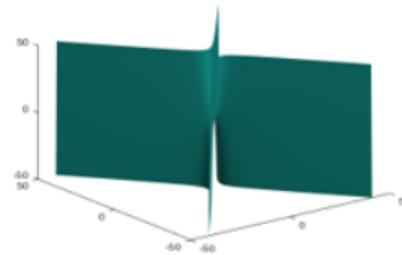
# How to determine a type of surface (Video explanation)



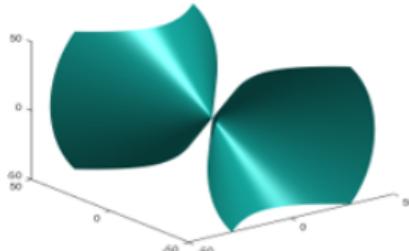
(a)  $2z^2 + 3y^2 - x = 1$   
Elliptic paraboloid



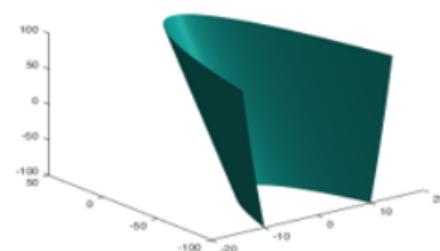
(b)  $2x^2 + 3y^2 - z^2 = 1$   
Hyperboloid of one sheet



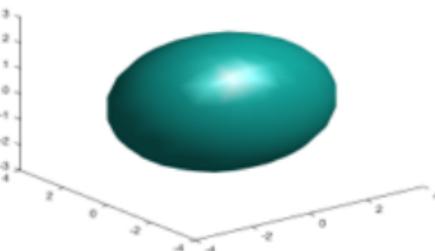
(c)  $-3x^2 + 2y^2 - z = 0$   
Hyperbolic paraboloid



(d)  $-3x^2 + 2y^2 - z^2 = 1$   
Hyperboloid of two sheets



(e)  $2x^2 + 3y - z = 0$   
Parabolic cylinder



(f)  $2x^2 + 3y^2 + 4z^2 = 24$   
Ellipsoid

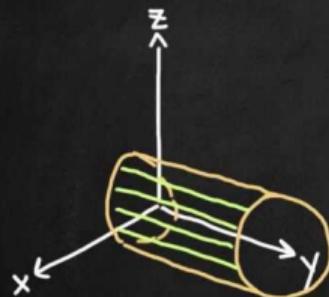


# How to sketch surfaces

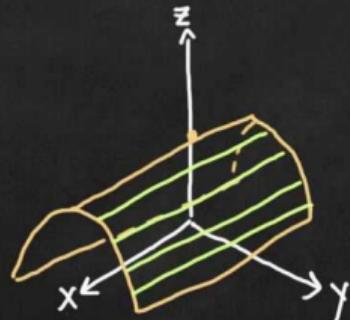
Video

sketch the surface and describe the figure

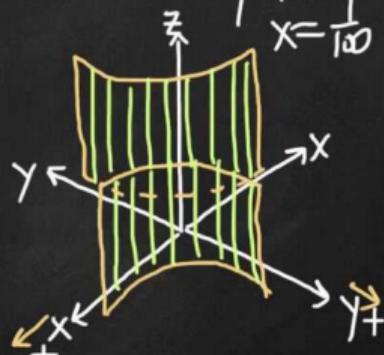
$$x^2 + z^2 = 1$$



$$z = 1 - y^2$$



$$\begin{aligned}x &= 10 & y &= \frac{1}{10} \\x &= 100 & y &= \frac{1}{100} \\xy &= 1 & y &= 100 \\y &= 100 & x &= \frac{1}{100}\end{aligned}$$



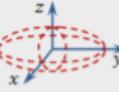
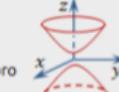
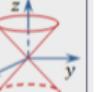
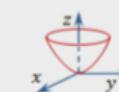
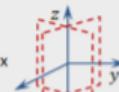
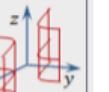
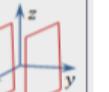
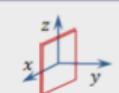


# Case studies of 2nd order curve equation (ENG)

<p><b>Ellipsoid</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: an ellipse In plane <math>y = q</math>: an ellipse In plane <math>x = r</math>: an ellipse</p> <p>If <math>a = b = c</math>, then this surface is a sphere.</p>		<p><b>Elliptic Cone</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: an ellipse In plane <math>y = q</math>: a hyperbola In plane <math>x = r</math>: a hyperbola In the <math>xz</math>-plane: a pair of lines that intersect at the origin In the <math>yz</math>-plane: a pair of lines that intersect at the origin</p> <p>The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.</p>	
<p><b>Hyperboloid of One Sheet</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: an ellipse In plane <math>y = q</math>: a hyperbola In plane <math>x = r</math>: a hyperbola</p> <p>In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.</p>		<p><b>Elliptic Paraboloid</b></p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: an ellipse In plane <math>y = q</math>: a parabola In plane <math>x = r</math>: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	
<p><b>Hyperboloid of Two Sheets</b></p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: an ellipse or the empty set (no trace) In plane <math>y = q</math>: a hyperbola In plane <math>x = r</math>: a hyperbola</p> <p>In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.</p>		<p><b>Hyperbolic Paraboloid</b></p> $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p><b>Traces</b></p> <p>In plane <math>z = p</math>: a hyperbola In plane <math>y = q</math>: a parabola In plane <math>x = r</math>: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	



# Case studies of 2nd order curve equation (RUS)

1.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Уравнение эллипсоида		2.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ Уравнение минимого эллипсоида		3.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ Уравнение минимого конуса		
4.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Уравнение однополостного гиперболоида		5.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ Уравнение двуполостного гиперболоида		6.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ Уравнение конуса		
7.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ Уравнение эллиптического параболоида		8.	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ Уравнение гиперболического параболоида		9.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Уравнение эллиптического цилиндра		
10.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ Уравнение минимого эллиптического цилиндра		11.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ Уравнение пары минимых пересекающихся плоскостей		12.	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Уравнение гиперболического цилиндра		
13.	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ Уравнение пары пересекающихся плоскостей		14.	$y^2 = 2px$ Уравнение параболического цилиндра		15.	$y^2 - b^2 = 0$ Уравнение пары параллельных плоскостей		
16.	$y^2 + b^2 = 0$ Уравнение пары минимых параллельных плоскостей		17.	$y^2 = 0$ Уравнение пары совпадающих плоскостей		Для всех уравнений $a > 0, b > 0, c > 0, p > 0$		Для уравнений 1 и 2 $a \geq b \geq c$	
						Для уравнений 3,4,5,6,7,9,10 $a \geq b$			



# Task 1

Sketch the following surfaces:

1.  $4 - 4y^2 - z^2 = x$

2.  $-x^2 - y^2 + z^2 = 1$

3.  $x^2 + y^2 - z^2 = 4$



# Task 1

Answer for  $4 - 4y^2 - z^2 = x$

$$x = 4 - 4y^2 - z^2$$

$$\Rightarrow x - 4 = -4y^2 - z^2$$

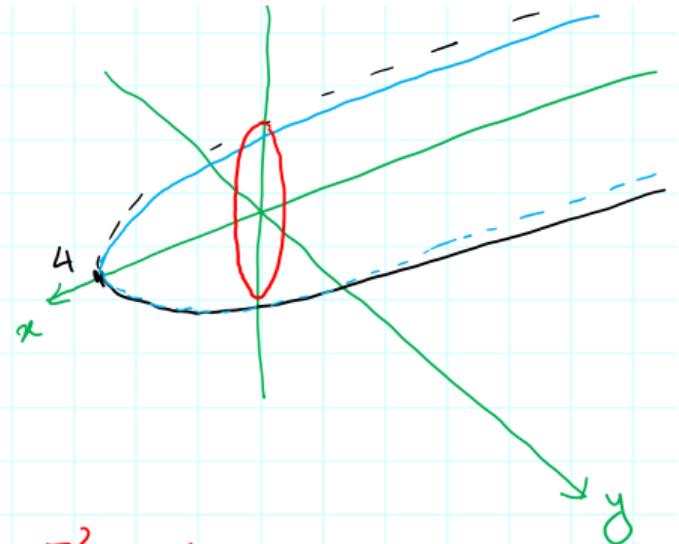
$$\Rightarrow \frac{x-4}{-4} = \frac{y^2}{1^2} + \frac{z^2}{2^2}$$

The vertex is located at  $(4, 0, 0)$

In  $xy$  plane ( $z=0$ )  $\rightarrow$  Parabola:  $(x-4) = -4y^2$

In  $xz$  plane ( $y=0$ )  $\rightarrow$  Parabola:  $(x-4) = -z^2$

In the planes  $\frac{x-4}{-4} = -c$  ( $\perp x$ )  $\rightarrow$  ellipses  $\frac{y^2}{(4c)^2} + \frac{z^2}{(8c)^2} = 1$





## Task 1

Answer for  $-x^2 - y^2 + z^2 = 1$

In xy plane ( $z=0$ ), we get nothing

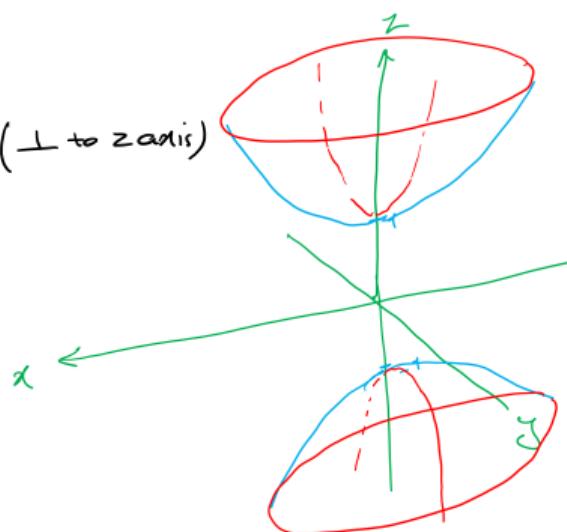
In yz plane ( $x=0$ )  $\rightarrow$  hyperbola:  $z^2 - y^2 = 1$

In xz plane ( $y=0$ )  $\rightarrow$  hyperbola:  $z^2 - x^2 = 1$

In any plane  $z=c$  &  $c \in (-\infty, -1] \cup [1, \infty)$  ( $\perp$  to z-axis)

$$\text{circle } x^2 + y^2 = (\sqrt{c^2 - 1})^2$$

vertices  $(0,0,1)$  &  $(0,0,-1)$





## Task 1

Answer for  $x^2 + y^2 - z^2 = 4$

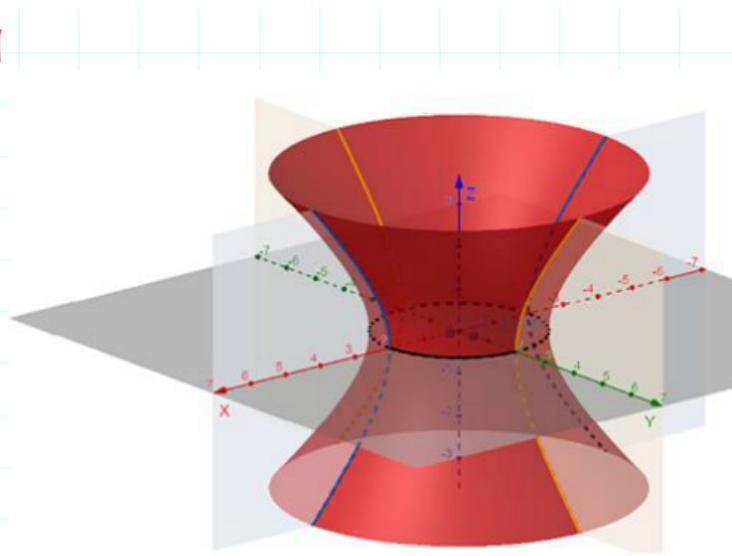
In  $yz$  plane ( $x=0$ )  $\rightarrow$  hyperbola:  $\frac{y^2}{2^2} - \frac{z^2}{2^2} = 1$

In  $xz$  plane ( $y=0$ )  $\rightarrow$  hyperbola:  $\frac{x^2}{2^2} - \frac{z^2}{2^2} = 1$

In  $xy$  plane ( $z=0$ )  $\rightarrow$  circle:  $x^2 + y^2 = 2^2$

In any plane  $\perp$  to  $Z$  axis ( $c=z$ ), circle

$$x^2 + y^2 = \left( \sqrt{4 + c^2/4} \right)^2$$





## Task 2

Find the equation of the sphere which passes through the points  $(2, 7, -4)$  and  $(4, 5, -1)$  has its centre on the line joining these two points as diameter.



## Task 2

### Answer

$$\begin{aligned}\text{Centre of the sphere} &= \left( \frac{2+4}{2}, \frac{7+5}{2}, \frac{-4+1}{2} \right) \\ &= \left( 3, 6, \frac{-5}{2} \right)\end{aligned}$$

$$\text{Radius} = \frac{1}{2} \sqrt{(2-4)^2 + (7-5)^2 + (-4+1)^2} = \frac{1}{2} \sqrt{4+4+9} = \frac{\sqrt{17}}{2}.$$

Therefore, the equation of the sphere is  $(x-3)^2 + (y-6)^2 + \left(z + \frac{5}{2}\right)^2 = \frac{17}{4}$ .

$$x^2 + y^2 + z^2 - 6x - 12y + 5z + 9 + 36 + \frac{25}{4} - \frac{17}{4} = 0$$

$$x^2 + y^2 + z^2 - 6x - 12y + 5z + 47 = 0$$



## Task 3

Find the equation of the sphere which touches the coordinate axes, whose centre lies in the positive octant and has a radius 4.



## Task 3

### Answer

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2xu + 2vy + 2wz + d = 0.$$

The equation of the  $x$ -axis is  $\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = t$ .

Any point on this line is  $(t, 0, 0)$ .

The point lies on the given sphere  $t^2 + 2ut + d = 0$ .

Since the sphere touches the  $x$ -axis the two roots of this equation are equal.

$$\therefore 4u^2 - 4d = 0 \quad \text{or} \quad u^2 = d.$$

Similarly,  $v^2 = d$  and  $w^2 = d$

The radius of the sphere is  $\sqrt{u^2 + v^2 + w^2 - d} = \sqrt{3d - d} = \sqrt{2d}$ .

$$\sqrt{2d} = 4 \quad \text{or} \quad d = 8 \quad u^2 = v^2 = w^2 = 8.$$

Since the centre lies on the  $x$ -axis,  $-u = -v = -w = 2\sqrt{2}$ .

$$(x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 + (z - 2\sqrt{2})^2 = 16$$



## Task 4

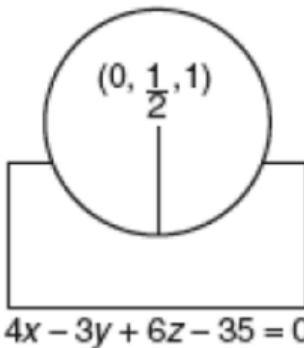
Show that the plane  $4x - 3y + 6z - 35 = 0$  is a tangent plane to the sphere  $x^2 + y^2 + z^2 - y - 2z - 14 = 0$  and find the point of contact.



## Task 4

### Answer

If the plane is a tangent plane to the sphere then the radius is equal Perpendicular distance from the centre on the plane is to the perpendicular distance from the centre on the plane.



The centre of the sphere  $x^2 + y^2 + z^2 - y - 2z - 14 = 0$  is  $\left(0, \frac{1}{2}, 1\right)$ .

$$r = \sqrt{0 + \frac{1}{4} + 1 + 14} = \sqrt{\frac{61}{4}}$$

$$\frac{\left|4(0) - 3\left(\frac{1}{2}\right) + 1(6) - 35\right|}{\sqrt{16 + 9 + 36}} = \frac{\sqrt{61}}{2}.$$

Therefore, the plane touches the sphere. The equations of the normal to the tangent plane are  $\frac{x}{4} = \frac{y - \frac{1}{2}}{-3} = \frac{z - 1}{6} = t$ .

Any point on this line is  $\left(4t, -3t + \frac{1}{2}, 6t + 1\right)$ .

If this point lies on the plane  $4x - 3y + 6z - 35 = 0$  then,

$$\begin{aligned} 4(4t) - 3\left(-3t + \frac{1}{2}\right) + 6(6t + 1) - 35 &= 0. \\ 16t + 9t + 36t - \frac{3}{2} + 6 - 35 &= 0 \quad (\text{i.e.}) \quad 61t - \frac{61}{2} = 0, \quad t = \frac{1}{2} \end{aligned}$$

Therefore, the point of contact is  $(2, -1, 4)$ .



## Task 5

Obtain the equations to the sphere through the common circle of the sphere  $x^2 + y^2 + z^2 + 2x + 2y = 0$  and the plane  $x + y + z + 4 = 0$  which intersects the plane  $x + y = 0$  in circle of radius 3 units.



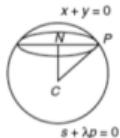
# Task 5

## Answer

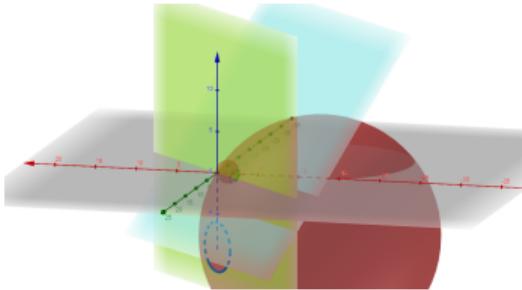
The equation of the sphere containing the given circle is  $x^2 + y^2 + z^2 + 2x + 2y + \lambda(x + y + z + 4) = 0$ .

Centre of this sphere is  $\left[-\left(1+\frac{\lambda}{2}\right), -\left(1+\frac{\lambda}{2}\right), -\frac{\lambda}{2}\right]$ .

$$\begin{aligned} \text{Radius} &= \sqrt{\left(1+\frac{\lambda}{2}\right)^2 + \left(1+\frac{\lambda}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2 - 4\lambda} \\ &= \frac{1}{2}\sqrt{3\lambda^2 - 8\lambda + 8} \end{aligned}$$



$CN$  = Perpendicular from the centre  $C$  on the plane  $x + y = 0$ .



$$= \left| \frac{-\left(1+\frac{\lambda}{2}\right) - \left(1+\frac{\lambda}{2}\right)}{\sqrt{2}} \right| = \frac{2+\lambda}{\sqrt{2}}$$

$$CP^2 = CN^2 + NP^2$$

$$\frac{1}{4}(3\lambda^2 - 8\lambda + 8) = \frac{(2+\lambda)^2}{2} + 9$$

$$3\lambda^2 - 8\lambda + 8 = 8 + 8\lambda + 2\lambda^2 + 36$$

$$(i.e.) \quad \lambda^2 - 16\lambda - 36 = 0$$

$$(\lambda + 2)(\lambda - 18) = 0$$

$$\therefore \lambda = -2 \text{ or } 18$$

Therefore, the equations of the required spheres are  $x^2 + y^2 + z^2 + 2x + 2y - 2(x + y + z + 4) = 0$  and  $x^2 + y^2 + z^2 + 2x + 2y + 18(x + y + z + 4) = 0$ .

$$(i.e.) \quad x^2 + y^2 + z^2 - 2z - 8 = 0$$

$$x^2 + y^2 + z^2 + 20x + 20y + 18z + 72 = 0$$



## Task 6

Find the equation of the sphere which touches the plane  $3x + 2y - z + 2 = 0$  at the point  $(1, -2, 1)$  and cuts orthogonally the sphere  $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$



# Task 6

## Answer

Let the equation of the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (14.54)$$

This sphere touches the plane  $3x + 2y - z + 2 = 0$  at  $(1, -2, 1)$ .

The equation of the tangent plane at  $(1, -2, 1)$  is  $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$ .

$$\text{(i.e.) } x(u+1) + y(v-2) + z(w+1) + u - 2v + w + d = 0 \quad (14.55)$$

But the tangent plane is given as

$$3x + 2y - z + 2 = 0 \quad (14.56)$$

Identifying equations (14.55) and (14.56) we get,

$$\frac{u+1}{3} = \frac{v-2}{2} = \frac{w+1}{-1} = \frac{u-2v+w+d}{2} = k \text{ (say)}$$

$$\therefore u = 3k - 1 \quad (14.57)$$

$$v = 2k + 2 \quad (14.58)$$

$$w = -k - 1 \quad (14.59)$$

$$\begin{aligned} u - 2v + w + d &= 2k \\ (3k - 1) - 2(2k + 2) + (-k - 1) + d &= 2k \\ d &= 4k + 6 \end{aligned} \quad (14.60)$$

The sphere (14.54) cuts orthogonally the sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0 \quad (14.61)$$

$$\therefore 2u(-2) + 2v(3) + d + 4 = 1$$

$$-4u + 6v + d + 4 = 1$$

$$-4(3k - 1) + 6(2k + 2) + 4k + 6 + 4 = 1$$

$$-12k + 4 + 12k + 12 = 4k + 10$$

$$16 = 4k + 10$$

$$6 = 4k \text{ or } k = \frac{3}{7}$$

$$\therefore u = 3k - 1 = \frac{7}{2}$$

$$v = 2k + 2 = 5$$

$$w = -k - 1 = -\frac{5}{2}$$

$$d = 4k + 6 = 12$$

Therefore, the equation of the sphere is  $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$ .



## Reference material

- How to sketch surfaces (Mathprofi, rus)
- Reducing a quadric surface equation to standard form (video, eng)

# Deserve “A” grade!

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↗ @Lupasic

🚪 Room 105 (Underground robotics lab)