Analytical Geometry and Linear Algebra I, Lab 7

Plane

Line in Space

Questions from the class

No questions for today

Objectives

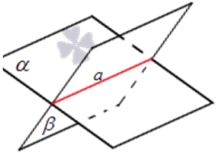
- To see the structure of all formulas, which were covered
- To understand how to transform one from to another and vise-versa
- How to apply the knowledge

Type of elements

Line in plane

Plane

Line in space



Plane

Formulas

1. ***General*** $Ax + By + Cz + D = 0$, where
   $D = -Ax_0 - By_0 - Cz_0$

2. ***Point-normal*** $(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$, where $\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$,

   $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{r_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

3. ***In segments*** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$,
   where $a = -\frac{D}{A}, b = -\frac{D}{B}, c = -\frac{D}{C}$ are points of intersection with the corresponded axes

4. ***Through three points***
   $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$, where
   $x_{1,2,3}, y_{1,2,3}, z_{1,2,3}$ are some particular coordinates of points on the plane

5. ***Parametric***
   $\vec{r} = \vec{r_0} + \alpha\vec{u} + \beta\vec{v} = \begin{cases} x = x_0 + \alpha u_x + \beta v_x \\ y = y_0 + \alpha u_y + \beta v_y \\ z = z_0 + \alpha u_z + \beta v_z \end{cases}$,
   where $\vec{r_0}$ – some point on a plane,
   $\vec{u}, \vec{v}$ – direction vectors, α, β are parameters

Plane

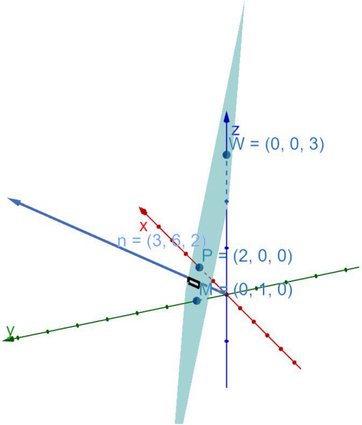
Task 0

1. Write down all forms of the line
2. Draw this plane

$$\frac{x}{2} + \frac{y}{1} + \frac{z}{3} = 1$$

Plane

Task 0, Answer



③ $\frac{x}{2} + \frac{y}{1} + \frac{z}{3} = 1$ it means we have 3 points $P = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $M = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $W = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

④ $\begin{vmatrix} x-2 & y & z \\ 0-2 & 1-0 & 0-0 \\ 0-2 & 0-0 & 3-0 \end{vmatrix} = 0 \Rightarrow 3x + 6y + 2z - 6 = 0$

④ → ①

A, B, C, D

② $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = 0$

⑤ $r_0 = M$; $\overline{MP} = (2; -1; 0)$, $\overline{MW} = (0; -1; 3)$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

Task 1

Find the equation of the plane passing through the point $(2, -3, 4)$ and parallel to the plane $2x - 5y - 7z + 15 = 0$. And draw the picture.

Task 1

Answer

The equation of the plane parallel to $2x - 5y - 7z + 15 = 0$ is $2x - 5y - 7z + k = 0$. This plane passes through the point $(2, -3, 4)$.

$$\therefore 4 + 15 - 28 + k = 0 \text{ or } k = 9$$

Hence, the equation of the required plane is $2x - 5y - 7z + 9 = 0$.

Line in space

Formulas

1. ***Passing through two points*** $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$ where $x_{1,2}, y_{1,2}, z_{1,2}$ are some particular coordinates of points on the line
2. ***Canonical*** $\dfrac{x-x_0}{a_x}=\dfrac{y-y_0}{a_y}=\dfrac{z-z_0}{a_z}$, where x_0, y_0, z_0 is a point on the line and a_x, a_y, a_z – direction vector coefficients on a basis
3. ***Parametric*** $\vec{r}=\vec{r_0}+\tau\vec{a}=\begin{cases} x=x_0+\tau a_x \\ y=y_0+\tau a_y \\ z=z_0+\tau a_z \end{cases}$, where τ is parameter, which can be received from canonical form $\left(\dfrac{x-x_0}{a_x}=\dfrac{y-y_0}{a_y}=\dfrac{z-z_0}{a_z}=\tau\right)$
4. ***General*** $\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$, where $D_{1,2}=-A_{1,2}x_0-B_{1,2}y_0-C_{1,2}z_0$
5. ***Point - 2 normal lines*** $\begin{cases} (\vec{r}-\vec{r_0})\cdot\vec{n_1}=0 \\ (\vec{r}-\vec{r_0})\cdot\vec{n_2}=0 \end{cases}$, where $\vec{n}=\begin{bmatrix} A \\ B \\ C \end{bmatrix}, \vec{r}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{r_0}=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$
6. ***Point - direction vector*** $\vec{a}\times(\vec{r}-\vec{r_0})=0$, where $\vec{a}=\vec{n_1}\times\vec{n_2}$

Line in space

Task 0

1. Write down all forms of the line
2. Draw this line

$$\begin{cases} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{cases}$$

Line in space

Task 0, Answer

(4) $\begin{cases} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{cases}$

$n_1 = \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix}$ $n_2 = \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix}$

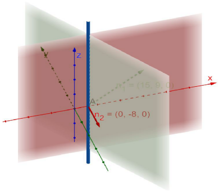
Let's solve this system

$\begin{cases} 15x = 45 - 0z \\ y = 0 - 0z \end{cases}$, we have more var then eq.; transform z to λ $\Rightarrow$

$\Rightarrow \begin{cases} x = 3 \\ y = 0 \\ z = \lambda \end{cases}$; let's choose our v_0 as $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1$

(5) $\begin{cases} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix} = 0 \\ \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix} = 0 \end{cases}$



(6) $\bar{a} = n_1 \times n_2 = \begin{bmatrix} 0 \\ 0 \\ -120 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ -120 \end{bmatrix} \times \begin{bmatrix} x - 3 \\ y - 0 \\ z - 1 \end{bmatrix} = 0$

(2) $\dfrac{x-3}{0} = \dfrac{y-0}{0} = \dfrac{z-1}{-120}$

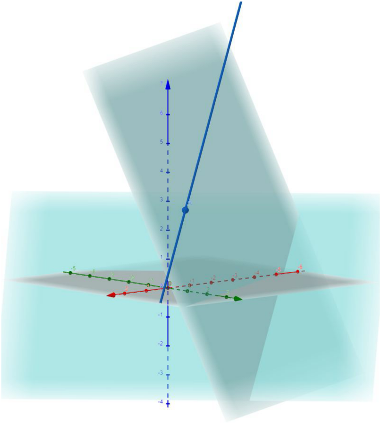
(3) $\begin{cases} x = 3 \\ y = 0 \\ z \quad 1 - 120\lambda \end{cases}$

(1) $v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 0 \\ 200 \end{bmatrix}$ $\dfrac{x-3}{3-3} = \dfrac{y-0}{0-0} = \dfrac{z-1}{200-1}$

Task 2

Find the equations of the line passing through the point $(1, 2, 3)$ and perpendicular to the planes $x - 2y - z + 5 = 0$ and $x + y + 3z + 6 = 0$. And draw the picture.

perpendicular to the planes — it cannot be physically (typo). It should be either "Perpendicualr to the normals of planes", or "parallel to the planes".



Task 2

Answer

Any plane passing through (2, 2, 4) is $a(x - 2) + b(y - 2) + c(z - 4) = 0$.

This plane is perpendicular to the planes $2x - 2y - 4z - 3 = 0$ and $3x + y + 6z - 4 = 0$.

$$\text{(i.e.)} \quad 2a - 2b - 4c = 0$$
$$3a + b + 6c = 0$$

$$\therefore \frac{a}{-12+4} = \frac{b}{-12-12} = \frac{c}{2+6} \quad \text{or} \quad \frac{a}{-8} = \frac{b}{-24} = \frac{c}{+8}$$

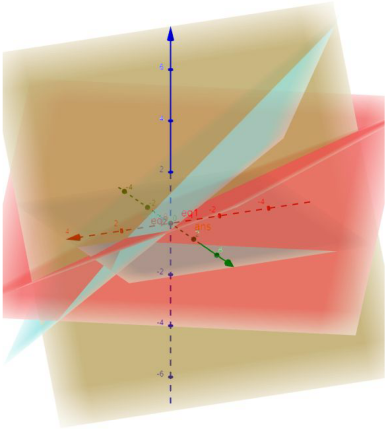
Therefore, the direction ratios of the normal to the required plane are 1, 3, –1.

Therefore, the equation of the plane is $(x - 2) + 0 + (z - 4) = 0$ (i.e.) $(x - 2) + 3(y - 2) - (z - 4) = 0$.

$$x + 3y - z - 4 = 0$$

Task 3

Find the equation of the plane which passes through the intersection of the planes $2x + 3y + 10z - 8 = 0$, $2x - 3y + 7z - 2 = 0$ and is perpendicular to the plane $3x - 2y + 4z - 5 = 0$. And draw the picture.



Task 3

Answer

The equation of any plane passing through the intersection of the planes $2x + 3y + 10z - 8 = 0$ and $2x - 3y + 7z - 2 = 0$ is $2x + 3y + 10z - 8 + \lambda(2x - 3y + 7z - 2) = 0$.

The direction ratios of the normal to this plane are $2 + 2\lambda$, $3 - 3\lambda$, $10 + 7\lambda$. The direction ratios of the plane $3x - 2y + 4z - 5 = 0$ are $3, -2, 4$. Since these two planes are perpendicular, $3(2 + 2\lambda) - 2(3 - 3\lambda) + 4(10 + 7\lambda) = 0$.

$$6 + 6\lambda - 6 + 6\lambda + 40 + 28\lambda = 0 \text{ or } 40\lambda = -40 \text{ or } \lambda = -1$$

Therefore, the required plane is $2x + 3y + 10z - 8 - (2x - 3y + 7z - \lambda) = 0$.

$$\therefore\ 6y + 3z - 6 = 0 \text{ or } 2y + z - 2 = 0$$


WHY
SLAVS
WEAR ADIDAS

Formulas of distances (Umnov 107)

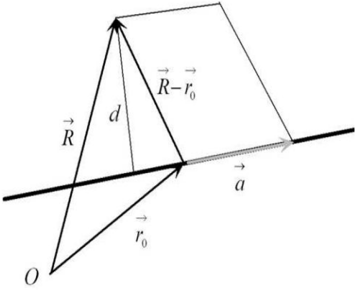
Point to line

Point to line $d = \dfrac{|(\vec{R} - \vec{r_0}) \times \vec{a}|}{|\vec{a}|}$, where

$\vec{R}$ - is a **point**,

a – directed vector of a line,

r_0 – point on this line



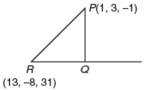
Task 4

Find the perpendicular distance from the point $(1, 3, -1)$ to the line $\dfrac{x-13}{5} = \dfrac{y+8}{-8} = \dfrac{z-31}{1}$. And draw the picture.

Task 4

Answer

$$\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}=r$$



Any point on this line are ($5r + 13$, $-8r - 8$, $r + 31$). Draw *PQ* perpendicular to the plane. The direction ratios of the line are ($5r + 12$, $-8r - 11$, $r + 32$).

Since the line *PQ* is perpendicular to *QR*, we have

$$5(5r+12)-8(-8r-11)+1(r+32)=0$$
$$25r+60+64r+88+r+32=0$$
$$\text{or } 90r+180=0 \quad \Rightarrow r=-2$$

Q is the point (3, 8, 29) and *P* is (1, 3, −1)

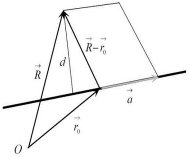
$$\therefore PQ^2=(3-1)^2+(8-3)^2+(29+1)^2=4+25+900=929$$
$$\therefore PQ^2=929 \text{ units}$$

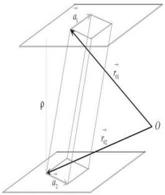
Formulas of distances (Umnov 107)

Line to line

Line to line

- *if collinear* $d = \dfrac{|(\vec{r}_2 - \vec{r}_1) \times \vec{a}_1|}{|\vec{a}_1|}$, where
  $\vec{r_{1,2}}$ - are points of lines and a_1 – directed vector of one line
- *if skew* $d = \dfrac{|(\vec{r}_2 - \vec{r}_1, \vec{a}_1, \vec{a}_2)|}{|\vec{a}_1 \times \vec{a}_2|}$, where
  $\vec{r_{1,2}}$ - are points of lines and a_1 – direction vector of one line

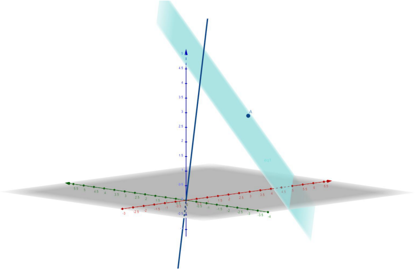




Task 5

Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

And draw the picture.



Task 5

Answer

The equations of the line through (1, −2, 3) and parallel to the line

$\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ are $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r$. Any point on this line is $(2r + 1, 3r - 2, -6r + 3)$. If this point lies on the plane $x - y + z = 5$ then $(2r + 1) - (3r - 2) + (-6r + 3) = 5$.

(i.e.) $-7r + 1 = 0$ or $r=\frac{1}{7}$.

Therefore, the point P is $\left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right)$.

Therefore, the distance between the points A (1, −2, 3) and

$\left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right)$ is $AP^2=\left(1-\frac{9}{7}\right)^2+\left(-2+\frac{11}{7}\right)^2+\left(3-\frac{15}{7}\right)^2=\frac{4}{49}+\frac{9}{49}+\frac{36}{49}=1$.

Task 6

Draw the following three cases and find how much solutions does the systems have.

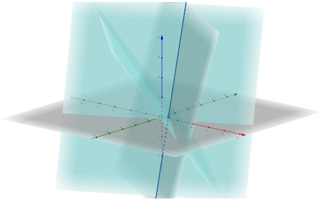
1. $$\begin{cases} 2x + 3y - z = 1, \\ x - 2y + z = 2, \\ x + y + z = 1 \end{cases}$$

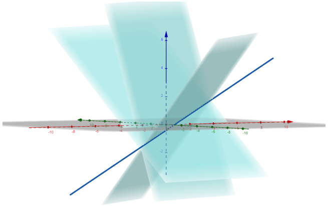
2. $$\begin{cases} 2x - 3y + z = 3, \\ x - 2y + 2z = 2, \\ x - y - z = 1 \end{cases}$$

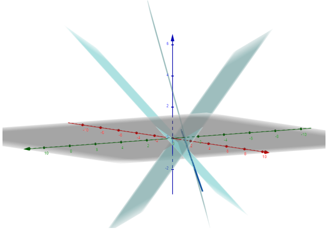
3. $$\begin{cases} 2x - 3y + z = 3, \\ x - 2y + 2z = 0, \\ x - y - z = 1 \end{cases}$$

Task 6

Answer







1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 2/9 \end{array}\right]$

Point

2. $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Plane

3. $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 6 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right]$

No exists

Task 7

Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may be perpendicular to one of the lines given by $a_1x^2 + 2h_1xy + by^2 = 0$. And draw the picture.

Task 7

Answer

Let $y = mx$ be a line of $ax^2 + 2hxy + by^2 = 0$. Then

$$ax^2 + 2hmx^2 + bm^2x^2 = 0 \Rightarrow a + 2hm + bm^2 = 0 \qquad (3.31)$$

Then $y = -\dfrac{1}{m}$ is a line of $a_1x^2 + 2hx_1y_1 + b_1y^2 = 0$.

Hence,

$$a_1x^2 - 2h\frac{mx_1^2}{m} + \frac{h_1}{m^2}x^2 = 0 \text{ or } a_1m^2 - 2h_1m + b_1 = 0 \qquad (3.32)$$

From (3.31) and (3.32), we get $\dfrac{m^2}{2hb_1 + 2h_1a} = \dfrac{m}{aa_1 - bb_1} = \dfrac{1}{-(2bh_1 + a_1h)}$.

Hence, the required condition is $(aa_1 - bb_1)^2 + 4(ha_1 + h_1b)(bh_1 + a_1h) = 0$.

Reference material

- Line in space (OnlineMschool)
- Gauss elimination online (OnlineMSchool)

Deserve “A” grade!

– Oleg Bulichev

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@Lupasic

Room 105 (Underground robotics lab)