# Analytical Geometry and Linear Algebra I, Lab 4

Inverse Matrix

Change of basis

# Questions from the class

*No questions for today*

# Inverse Matrix

*What is it?*

**Inverse matrix** $A^{-1}$ is the matrix, the product of which to original matrix $A$ is equal to the *identity matrix* $I$:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

# Inverse Matrix

*Why do we need it?*

Say we want to find matrix X, and we know matrix A and B:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by $A^{-1}$ ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate $A^{-1}$)

# Inverse Matrix

*Properties*

1. $det(A^{-1}) = \dfrac{1}{det(A)}$
2. $(AB)^{-1} = A^{-1}B^{-1}$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $(kA)^{-1} = \dfrac{A^{-1}}{k}$
5. $(A^{-1})^{-1} = A$

# Inverse Matrix

*How to find*

There are 2 ways:

1. Classical approach
2. Gauss-Jordan / Reduced Row Echelon Form (RREF)

# Inverse Matrix: Classical Approach

*Theory*

$\underset{2\times 2}{A}^{-1} = \dfrac{C^T}{det(A)}$, where $C$ is a matrix of *cofactors*.

$\underset{2\times 2}{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, where $C_{ij} = (-1)^{i+j} M_{ij}$ — (we met it on previous lab (lab 3))

# Inverse Matrix: Classical Approach

*Case Study*

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let's find $A^{-1}$.

1. Find a determinant (shouldn't be equal to 0, otherwise $\rightarrow$ stop calculations).
   $det(A) = 1 \cdot 4 - 2 \cdot 3 = -2$
2. Find Cofactor matrix
   $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 |4| = 4$
   $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 |3| = -3$
   $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 |2| = -2$
   $C_{22} = (-1)^{2+2} M_{22} = (-1)^4 |1| = 1$

$$C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

3. Transpose cofactor matrix
   $C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$
4. Substitute it to the main formula

$$A^{-1} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

# Inverse Matrix: Gauss-Jordan

*Core Idea for Inverse Matrices*

$(A|I) \to \ldots \to (I|A^{-1})$

1. Using sequence of *elementary row operations* to modify the matrix until the lower left-hand corner of the matrix is filled with zeros (**Row Echelon Form**/Upper Trianglular Matrix). $(A|I) \to \ldots \to (\triangleleft_{Upper}|B)$
2. Using *elementary row operations* transform Upper Trianglular Matrix to Indentity Matrix. $(\triangleleft_{Upper}|B) \to \ldots \to (I|A^{-1})$

## Elementary Row Operations

- Swapping two rows,
- Multiplying a row by a nonzero number,
- Adding a multiple of one row to another row. (subtraction can be achieved by multiplying one row with -1 and adding the result to another row)

# Inverse Matrix: Gauss-Jordan

*Case study* $(2 \times 2)$

$$(A|E)=\left(\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 3 & 4 & 0 & 1\end{array}\right)\xrightarrow{(1)}\left(\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & -2 & -3 & 1\end{array}\right)\xrightarrow{(2)}\left(\begin{array}{cc|cc}1 & 0 & -2 & 1\\ 0 & -2 & -3 & 1\end{array}\right)\xrightarrow{(3)}\left(\begin{array}{cc|cc}1 & 0 & -2 & 1\\ 0 & 1 & 3/2 & -1/2\end{array}\right)$$

# Inverse Matrix: Gauss-Jordan

*Case study* (4 × 4)

$$
(A|\mathrm{I}) = \left(\begin{array}{cccc|cccc} 2 & 3 & 2 & 2 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(1)} \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{(2)}
$$

$$
\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{(3)} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{(4)}
$$

$$
\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow{(5)} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & -1 & -3 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow{(6)}
$$

$$
\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right) = (\mathrm{I}|A^{-1})
$$

# Inverse Matrix

*Task 1*

Find inverse matrices for the following matrices:

1. $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$;
2. $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$;

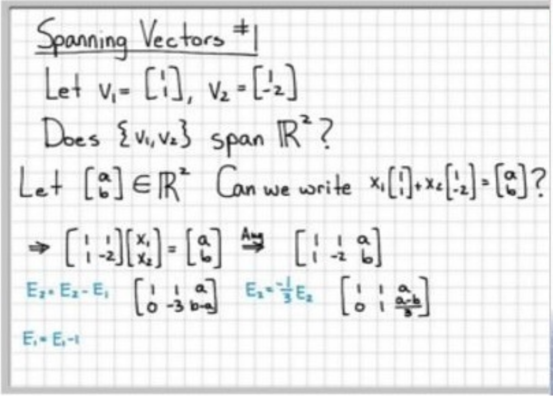
# Inverse Matrix

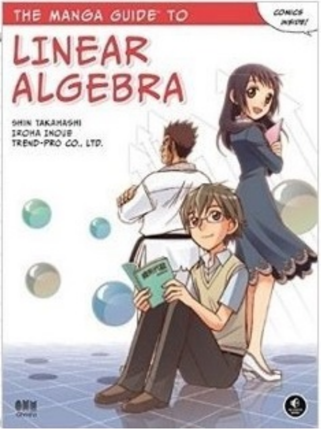
*Task 2*

Solve matrix equations:

1. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix};$
2. $X \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix};$

**Don't say you love the anime** **If you haven't read the manga**



# Preparation to Exam / Test

*Strategy for efficient exam solving*

## Problem Statement

During an exam, I *spend too much time* on finding the solution

## Solution

To find the right strategy for *preparation* and *behavior* during an exam.

# Preparation to Exam / Test

*My own guide and thoughts*

**I should pay attention on this parts:**

1. Preparation before a test
2. Preparation in a day of a test
3. Behavior on exam

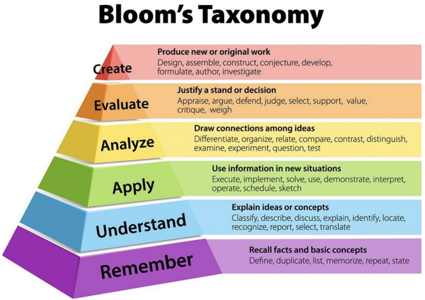
**Approx time consuming:**

- Preparation: 3-8 hours in overall
- Exam:
  - Find the idea how to solve a particular task — 10 sec - 2 min
  - Implement the idea — 10-20 min

# Preparation to Exam / Test

*Preparation strategy*

1. Understand the concept of a new topic (Apply or Analyze in terms of Bloom's Taxonomy)
   1.1 Look at slides and videos
   1.2 Play with concept (suggest some ideas and prove it or disprove via computer or hand calculations)
2. Take a book (material) with exercises and solutions for it.
   2.1 Look at a task, imagine how to solve it.
   2.2 Check a suggestion from solutions. If you sure that your solution is also applicable — check it.




Important: For some tasks *practical skills* are crucial (find not only an idea, but implement it)!

# Preparation to Exam / Test

*Behavior before and on exam*

**Before**:

Prepare your brain (skim the material) and mentality (by self hypnosis techniques) (took from science russian book ”Преодолей себя! Психическая подготовка в спорте”)

**During an exam**:

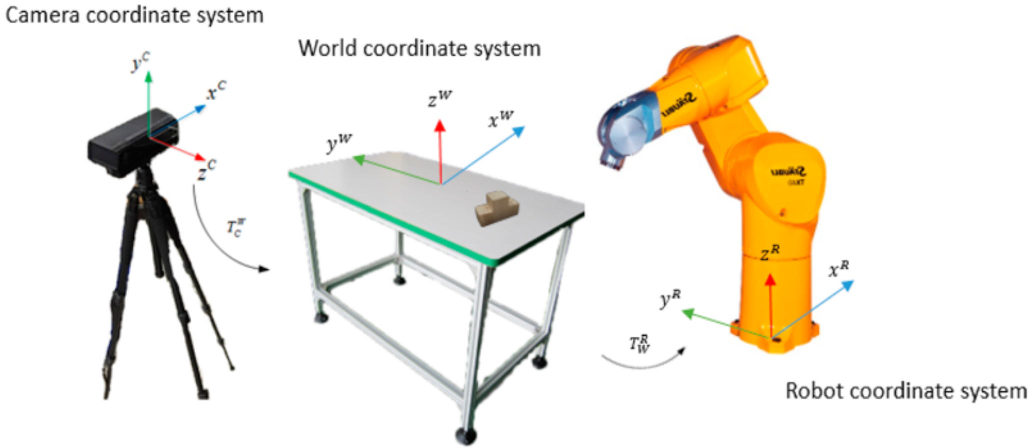
1. Rank tasks by doing speed:
   - Can be solved on-a-fly (expect max grade)
   - Easy concept — tough implementation (expect that some computational mistakes can be done)
   - Tough concept (cannot find the solution on-a-fly) (time consuming tasks)
2. Solve it in such order
3. Profit! You are awesome!

# Changing Basis

*Case Study: Shoe Polishing Robot*

# Changing Basis

*Case Study: Shoe Polishing Robot, How it works*



# Changing Basis

*Two Bros in the World of Changing Basis*

$$E' = EA \text{ and } Ex = E'x'.$$

Extended Bro Equation: $Ex = Eb + E'x'$, where

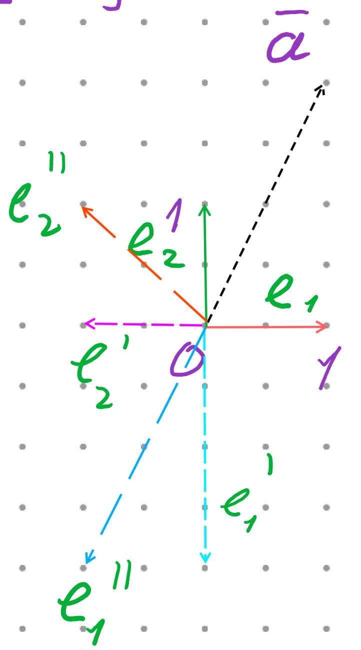
$$E = [e_1\ e_2\ e_3] = \begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{bmatrix}, \begin{bmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{bmatrix}, \begin{bmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{bmatrix}, E' = [e'_1\ e'_2\ e'_3]$$

More info in Appendix 2

Answer from figure

$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a'' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$a' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$




$E' = EA \qquad Ex = E'x'$

$x = Ax'$

$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E \to E'$

1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{e_2'} = E' \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{e_1'} = E' \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$I \underbrace{\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}}_{A_{E \to E'}} = E' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E' \to E''; \; E' = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} a'$

$\begin{bmatrix} 0 & -0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{a'}$

2) $\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{e_1''} = E'' \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}}_{e_2''} = E'' \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix}}_{A_{E' \to E''}} = E'' = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix} a'' \Rightarrow \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{a''}$

$\begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = a''$

# Changing Basis

*Task 3*

If vectors **a** and **b** form a basis (you should check it), it is needed to find coordinates **c** in the basis.

$$\mathbf{a} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

# Changing Basis

*Task 3*

## Answer

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; E' = \begin{bmatrix} -5 & -1 \\ -1 & 3 \end{bmatrix}$$

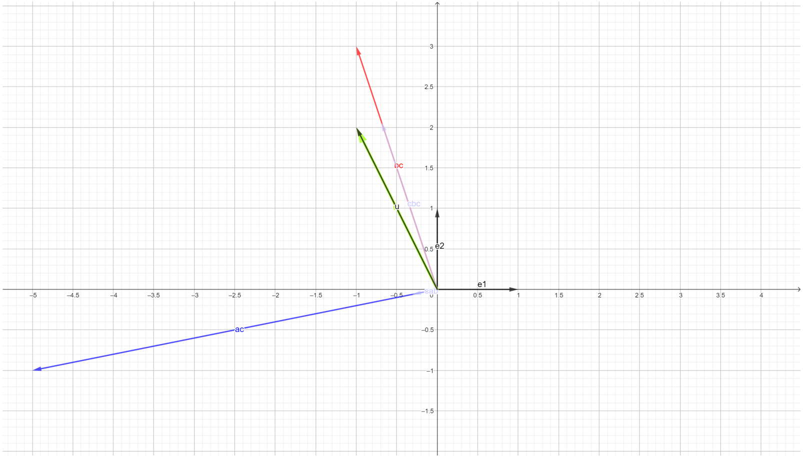
Because we are interested in coordinates, 2nd bro

$$Ec_{old} = E'x \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_{new_x} \\ c_{new_y} \end{bmatrix}$$

$$\begin{bmatrix} c_{new_x} \\ c_{new_y} \end{bmatrix} = (E')^{-1} c_{old} = \begin{bmatrix} \frac{1}{16} \\ \frac{11}{16} \end{bmatrix}$$

# Changing Basis

*Task 3*



# Changing Basis

*Task 4*

Two bases are given in the plane: $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}'_1, \mathbf{e}'_2$. The vectors of the second basis have coordinates $(-1;\ 3)$ and $(2; -7)$ in the first basis.

(a) Compose transition matrices from the old basis to the new and vice versa.

(b) Find the coordinates of a vector in the old basis given that it has coordinates $\alpha'_1, \alpha'_2$ in the new basis.

(c) Find the coordinates of a vector in the new basis given that it has coordinates $\alpha_1, \alpha_2$ in the old basis.

# Changing Basis

*Task 4*

## Answer

$$E = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix},\ E = \begin{bmatrix} \vec{e'}_1 & \vec{e'}_2 \end{bmatrix}$$

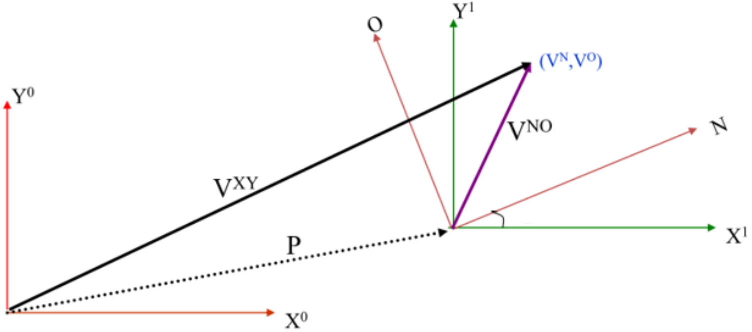
$$E\begin{bmatrix} -1 \\ 3 \end{bmatrix} = E'\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ E\begin{bmatrix} 2 \\ -7 \end{bmatrix} = E'\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$, because its a second basis itself

combine together

$$E\begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix} = E'\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E\underbrace{\begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix}}_{A} = E'$$

# Changing Basis

*Change the coordinate frame*




$$V^{XY} = \begin{bmatrix} V^X \\ V^Y \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V^N \\ V^O \end{bmatrix}$$

# Changing Basis

*Homogeneous representation*

$$V^{XY} = \begin{bmatrix} V^X \\ V^Y \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V^N \\ V^O \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} V^X \\ V^Y \\ 1 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V^N \\ V^O \\ 1 \end{bmatrix}$$

Padding with 0's and 1's

$$= \begin{bmatrix} V^X \\ V^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & P_x \\ \sin\theta & \cos\theta & P_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^N \\ V^O \\ 1 \end{bmatrix}$$

Simplifying into a matrix form

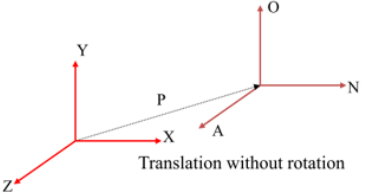
$$H = \begin{bmatrix} \cos\theta & -\sin\theta & P_x \\ \sin\theta & \cos\theta & P_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous Matrix for a Translation in XY plane, followed by a Rotation around the z-axis

# Changing Basis

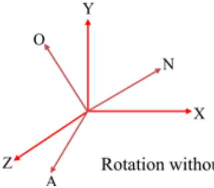
*Special Cases of Homogeneous matrices in 3D*

H is a 4x4 matrix that can describe a translation, rotation, or both in one matrix




Translation without rotation

$$H = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



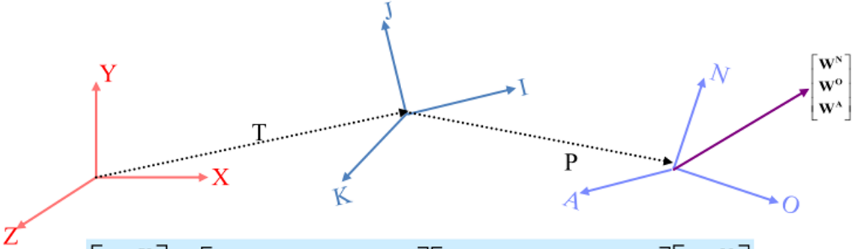

Rotation without translation

$$H = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation part:
Could be rotation around z-axis
x-axis, y-axis or a combination of

# Changing Basis

*Change the coordinate frame: Case Study*




$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & T_x \\ i_y & j_y & k_y & T_y \\ i_z & j_z & k_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$

# Changing Basis

*Task 5*

Let us consider two coordinate systems in the plane: $O, \mathbf{e}_1, \mathbf{e}_2$ and $O', \mathbf{e}_1', \mathbf{e}_2'$. Point $O'$ has coordinates $(7; -2)$ in the old coordinate system, and vectors $\mathbf{e}_1', \mathbf{e}_2'$ can be obtained from vectors $\mathbf{e}_1, \mathbf{e}_2$ by rotating them 60° (a) clockwise; (b) counterclockwise. Find the old coordinates of a point $x, y$ given its new coordinates $x', y'$.

# Changing Basis

*Task 5*

## Answer

Classical way, using Extended 2nd Bro eqn

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} + \begin{bmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Or using Homogeneous matrix

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 60 & -\sin 60 & 7 \\ \sin 60 & \cos 60 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

# Changing Basis

*Task 6*

There are to bases in $R^3$:

$e_1 = i,\ e_2 = j,\ e_3 = k$ and $e'_1 = i + j + k,\ e'_2 = i + j,\ e'_3 = i$

Find coordinates of $x = 2i - 3j + k$ in the basis $e'_1,\ e'_2,\ e'_3$.

# Changing Basis

*Task 6*

## Answer

1. We can represent $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence $E' = EA$, $A = \begin{array}{c c} & \begin{array}{ccc} e'_1 & e'_2 & e'_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$

2. $E \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = A\vec{x} \rightarrow A^{-1}E \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \vec{x} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$

# Changing Basis

*Task 7*

There are 4 vectors $f_1,\ f_2,\ f_3,\ x$ and the basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$ Find the coordinates of $x$ in the basis $(f_1,\ f_2,\ f_3)$, if

$$f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ f_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ f_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

# Reference material

- Inverse Matrix (OnlineMschool)
- Gauss-Jordan (Wiki)
- Changing Basis (3Blue1Brown)

# Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)

# 1 Changing Basis and Coordinates

Suppose we have two different coordinate systems. The first (so-called "old" coordinate system[1]) is given by origin $O$ and basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$; the second (the "new" one) is given by $O'$, $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Let point $N$ have coordinates $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ in the old coordinate system and coordinates $\begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix}^T$ in the new one. It means that

$$\begin{aligned} \overrightarrow{ON} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3; \\ \overrightarrow{O'N} &= x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3. \end{aligned} \tag{1}$$

Let us say that we know how the coordinate systems are related with each other. That is, we can express new basis vectors via the old ones and we know the coordinates of the new origin in the old basis. In detail,

$$\begin{aligned} \mathbf{e}'_1 &= \alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3 \\ \mathbf{e}'_2 &= \alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3 \\ \mathbf{e}'_3 &= \alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3 \\ \overrightarrow{OO'} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \end{aligned}$$

As $\overrightarrow{ON} = \overrightarrow{OO'} + \overrightarrow{O'N}$, we get

$$\begin{aligned} \overrightarrow{ON} = {} & b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + \\ & x'_1(\alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3) + x'_2(\alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3) + x'_3(\alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3) = \\ & (b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3)\,\mathbf{e}_1 + (b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3)\,\mathbf{e}_2 + (b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3)\,\mathbf{e}_3. \end{aligned}$$

Taking (1) into account yields

$$\begin{aligned} x_1 &= b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3, \\ x_2 &= b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3, \\ x_3 &= b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3, \end{aligned}$$

or, using matrix notation,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

Thus knowing how new basis depends on the old one enables us to immediately express the old coordinates through the new ones.

Matrix $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$ is called a *transition matrix from the old basis to the new basis.* Using matrix notation, one can easily derive that basis vectors satisfy the equality

$$\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} A.$$

As for coordinates,

$$\mathbf{x} = \mathbf{b} + A\mathbf{x}'.$$

[1]In order not to get confused we will refer to a basis, coordinates etc. without primes as to "old" ones and to those with primes as to "new" ones.