



Analytical Geometry and Linear Algebra I, Lab 7

Plane

Line in Space

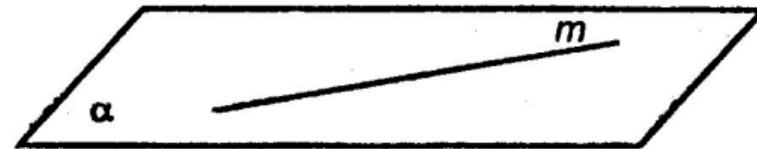


Questions from the class

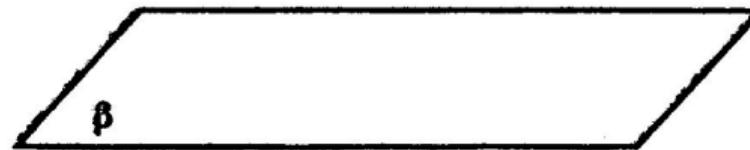
No questions for today

Type of elements

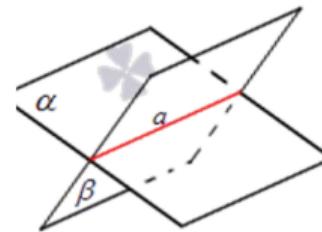
Line in plane



Plane



Line in space





Plane

Formulas

1. **General** $Ax + By + Cz + D = 0$, where

$$D = -Ax_0 - By_0 - Cz_0$$

2. **Point-normal** $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$, where $\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$,

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

3. **In segments** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$,

$$\frac{D}{A}, \frac{D}{B}, \frac{D}{C}$$

where $a = -\frac{D}{A}$, $b = -\frac{D}{B}$, $c = -\frac{D}{C}$ are points of intersection with the corresponded axes

4. **Through three points**

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0, \text{ where}$$

$x_{1,2,3}, y_{1,2,3}, z_{1,2,3}$ are some particular coordinates of points on the plane

5. **Parametric**

$$\vec{r} = \vec{r}_0 + \alpha \vec{u} + \beta \vec{v} = \begin{cases} x = x_0 + \alpha u_x + \beta v_x \\ y = y_0 + \alpha u_y + \beta v_y \\ z = z_0 + \alpha u_z + \beta v_z \end{cases}$$

where \vec{r}_0 – some point on a plane,
 \vec{u}, \vec{v} – direction vectors, α, β are parameters



Plane

Task 0

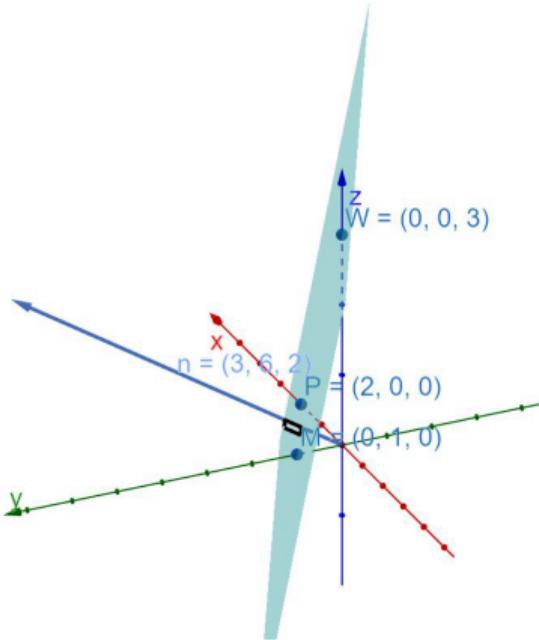
1. Write down all forms of the plane
2. Draw this plane

$$\frac{x}{2} + \frac{y}{1} + \frac{z}{3} = 1$$



Plane

Task 0, Answer



$$\text{③ } \frac{x}{2} + \frac{y}{1} + \frac{z}{3} = 1 \quad \begin{array}{l} \text{it means we have} \\ 3 \text{ points } P = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} M = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} W = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

$$\text{④ } \left| \begin{array}{ccc|c} x-2 & y & z & \\ 0-2 & 1-0 & 0-0 & 0 \\ 0-2 & 0-0 & 3-0 & \end{array} \right| = 0 \xrightarrow{\text{④} \rightarrow \text{①}} 3x + 6y + 2z - 6 = 0$$

A B C D

$$\text{② } \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = 0$$

$$\text{⑤ } r_0 = M ; \quad \overline{MP} = (2; -1; 0) \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$



Task 1

Find the equation of the plane passing through the point $(2, -3, 4)$ and parallel to the plane $2x - 5y - 7z + 15 = 0$. And draw the picture.



Task 1

Answer

The equation of the plane parallel to $2x - 5y - 7z + 15 = 0$ is $2x - 5y - 7z + k = 0$. This plane passes through the point $(2, -3, 4)$.

$$\therefore 4 + 15 - 28 + k = 0 \text{ or } k = 9$$

Hence, the equation of the required plane is $2x - 5y - 7z + 9 = 0$.



Line in space

Formulas

1. Passing through two points

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

where $x_{1,2}, y_{1,2}, z_{1,2}$ are some particular coordinates of points on the line

$$2. \text{ Canonical } \frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z},$$

where x_0, y_0, z_0 is a point on the line and a_x, a_y, a_z – direction vector coefficients on a basis

$$3. \text{ Parametric } \vec{r} = \vec{r}_0 + \tau \vec{a} = \begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y, \text{ where } \tau \text{ is} \\ z = z_0 + \tau a_z \end{cases}$$

parameter,

which can be received from canonical form

$$\left(\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z} = \tau \right)$$

$$4. \text{ General } \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}, \text{ where}$$

$$D_{1,2} = -A_{1,2}x_0 - B_{1,2}y_0 - C_{1,2}z_0$$

$$5. \text{ Point - 2 normal lines } \begin{cases} (\vec{r} - \vec{r}_0) \cdot \vec{n}_1 = 0 \\ (\vec{r} - \vec{r}_0) \cdot \vec{n}_2 = 0 \end{cases}, \text{ where}$$

$$\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$6. \text{ Point - direction vector } \vec{a} \times (\vec{r} - \vec{r}_0) = 0, \text{ where}$$

$$\vec{a} = \vec{n}_1 \times \vec{n}_2$$



Line in space

Task 0

1. Write down all forms of the line
2. Draw this line

$$\begin{cases} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{cases}$$



Line in space

Task 0, Answer

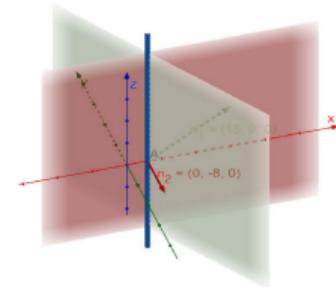
$$\textcircled{4} \quad \left\{ \begin{array}{l} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{array} \right. \\ n_1 = \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix} \quad n_2 = \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix}$$

Let's solve this system

$$\left\{ \begin{array}{l} 15x = 45 - 0z \\ y = 0 - 0z \end{array} \right. , \text{ we have more var than eq;} \\ \text{transform } z \text{ to } 2 \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} x = 3 \\ y = 0 \\ z = 2 \end{array} \right. ; \text{ let's choose our } v_0 \text{ as } \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$\textcircled{5} \quad \left[\begin{bmatrix} x & 3 \\ y & 0 \\ z & 2 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix} \right] = 0 \\ \left[\begin{bmatrix} x & 3 \\ y & 0 \\ z & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix} \right] = 0$$



$$\textcircled{6} \quad \bar{a} = n_1 \times n_2 = \begin{bmatrix} 0 \\ 0 \\ -120 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ -120 \end{bmatrix} \times \begin{bmatrix} x-3 \\ y-0 \\ z-1 \end{bmatrix} = 0$$

$$\textcircled{7} \quad \frac{x-3}{0} = \frac{y-0}{0} = \frac{z-1}{-120}$$

$$\textcircled{8} \quad \left\{ \begin{array}{l} x = 3 \\ y = 0 \\ z = 1 - 120\lambda \end{array} \right.$$

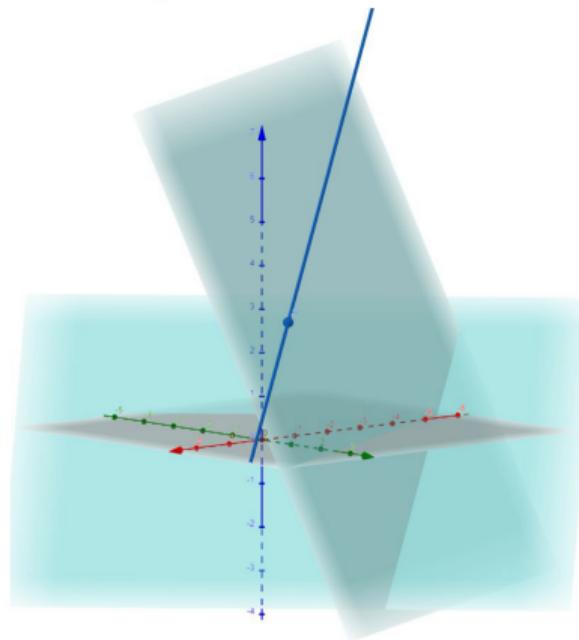
$$\textcircled{9} \quad v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 0 \\ -120 \end{bmatrix} \quad \frac{x-3}{3-3} = \frac{y-0}{0-0} = \frac{z-1}{-120-1}$$



Task 2

Find the equations of the line passing through the point $(1, 2, 3)$ and perpendicular to the planes $x - 2y - z + 5 = 0$ and $x + y + 3z + 6 = 0$. And draw the picture.

perpendicular to the planes — it cannot be physically (typo). It should be either "Perpendicular to the normals of planes", or "parallel to the planes".





Task 2

Answer

Let l, m, n be the direction ratios of the line of intersection of the planes $x - 2y - z + 5 = 0$ and $x + y + 3z + 6 = 0$.

Then $l - 2m - n = 0$
and $l + m + 3n = 0$

$$\therefore \frac{l}{-6+1} = \frac{m}{-1-3} = \frac{n}{1+2}$$

(i.e.) $\frac{l}{-5} = \frac{m}{-4} = \frac{n}{3}$.

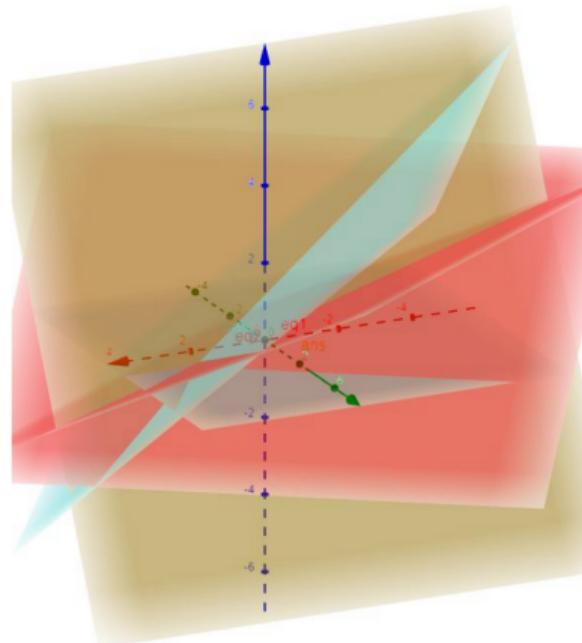
Since the line also passes through the point $(-1, 2, 3)$, its equations

is $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{-3}$.



Task 3

Find the equation of the plane which passes through the intersection of the planes $2x + 3y + 10z - 8 = 0$, $2x - 3y + 7z - 2 = 0$ and is perpendicular to the plane $3x - 2y + 4z - 5 = 0$. And draw the picture.





Task 3

Answer

The equation of any plane passing through the intersection of the planes $2x + 3y + 10z - 8 = 0$ and $2x - 3y + 7z - 2 = 0$ is $2x + 3y + 10z - 8 + \lambda(2x - 3y + 7z - 2) = 0$.

The direction ratios of the normal to this plane are $2 + 2\lambda, 3 - 3\lambda, 10 + 7\lambda$. The direction ratios of the plane $3x - 2y + 4z - 5 = 0$ are $3, -2, 4$. Since these two planes are perpendicular, $3(2 + 2\lambda) - 2(3 - 3\lambda) + 4(10 + 7\lambda) = 0$.

$$6 + 6\lambda - 6 + 6\lambda + 40 + 28\lambda = 0 \text{ or } 40\lambda = -40 \text{ or } \lambda = -1$$

Therefore, the required plane is $2x + 3y + 10z - 8 - (2x - 3y + 7z - \lambda) = 0$.

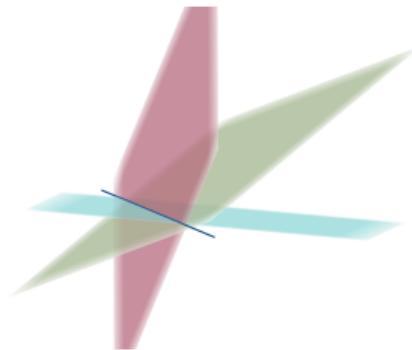
$$\therefore 6y + 3z - 6 = 0 \text{ or } 2y + z - 2 = 0$$



Pencil (sheaf) of planes (Пучок плоскостей)

$$Ax + By + Cz + D + \lambda(A_1x + B_1y + C_1z + D_1) = 0$$

Where λ — a variable, which represents a set of planes, which passes through intersection of two main planes.





Formulas of distances (Umnov 107)

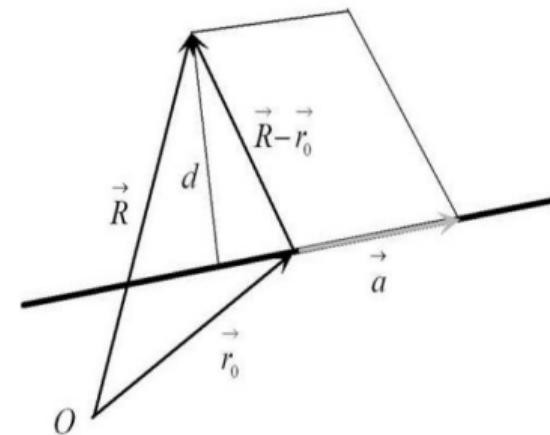
Point to line

$$\text{Point to line } d = \frac{|(\vec{R} - \vec{r}_0) \times \vec{a}|}{|\vec{a}|}, \text{ where}$$

\vec{R} - is a **point**,

\vec{a} - directed vector of a line,

\vec{r}_0 - point on this line





Task 4

Find the perpendicular distance from the point $(1, 3, -1)$ to the line

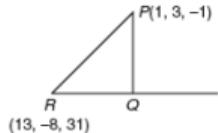
$$\frac{x-13}{5} = \frac{y+8}{-8} = \frac{z-31}{1}. \text{ And draw the picture.}$$



Task 4

Answer

$$\frac{x-13}{5} = \frac{y+8}{-8} = \frac{z-31}{1} = r$$



Any point on this line are $(5r + 13, -8r - 8, r + 31)$. Draw PQ perpendicular to the plane. The direction ratios of the line are $(5r + 12, -8r - 11, r + 32)$.

Since the line PQ is perpendicular to QR , we have

$$5(5r + 12) - 8(-8r - 11) + 1(r + 32) = 0$$

$$25r + 60 + 64r + 88 + r + 32 = 0$$

$$\text{or } 90r + 180 = 0 \Rightarrow r = -2$$

Q is the point $(3, 8, 29)$ and P is $(1, 3, -1)$

$$\therefore PQ^2 = (3-1)^2 + (8-3)^2 + (29+1)^2 = 4 + 25 + 900 = 929$$

$$\therefore PQ^2 = 929 \text{ units}$$

Formulas of distances (Umnov 107)

Line to line

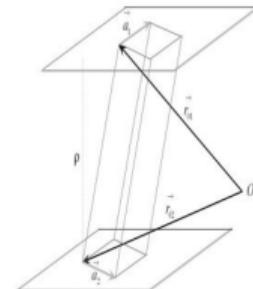
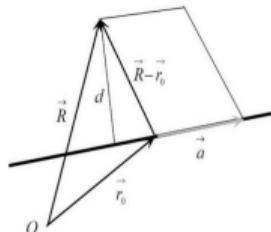
Line to line

- if collinear $d = \frac{|(\vec{r}_2 - \vec{r}_1) \times \vec{a}_1|}{|\vec{a}_1|}$, where

$\vec{r}_{1,2}$ – are points of lines and a_1 – directed vector of one line

- if skew $d = \frac{|(\vec{r}_2 - \vec{r}_1, \vec{a}_1, \vec{a}_2)|}{|\vec{a}_1 \times \vec{a}_2|}$, where

$\vec{r}_{1,2}$ – are points of lines and a_1 – direction vector of one line

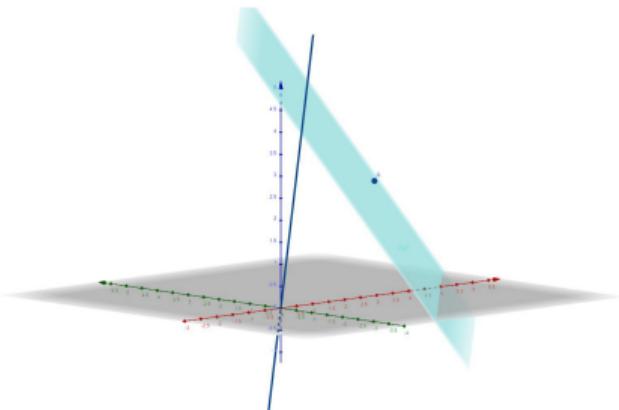




Task 5

Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

And draw the picture.





Task 5

Answer

The equations of the line through $(1, -2, 3)$ and parallel to the line

$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r$. Any point on this line is $(2r + 1, 3r - 2, -6r + 3)$.

If this point lies on the plane $x - y + z = 5$ then $(2r + 1) - (3r - 2) + (-6r + 3) = 5$.

$$\text{(i.e.) } -7r + 1 = 0 \text{ or } r = \frac{1}{7}.$$

Therefore, the point P is $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$.

Therefore, the distance between the points $A(1, -2, 3)$ and

$$\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right) \text{ is } AP^2 = \left(1 - \frac{9}{7}\right)^2 + \left(-2 + \frac{11}{7}\right)^2 + \left(3 - \frac{15}{7}\right)^2 = \frac{4}{49} + \frac{9}{49} + \frac{36}{49} = 1.$$



Task 6

Draw the following three cases and find how much solutions does the systems have.

$$1. \begin{cases} 2x + 3y - z = 1, \\ x - 2y + z = 2, \\ x + y + z = 1 \end{cases}$$

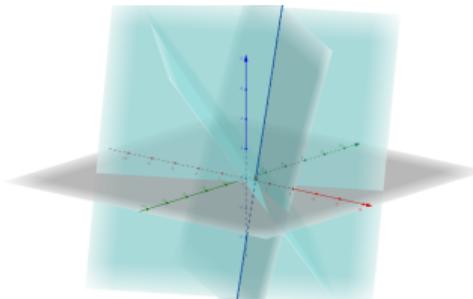
$$2. \begin{cases} 2x - 3y + z = 3, \\ x - 2y + 2z = 2, \\ x - y - z = 1 \end{cases}$$

$$3. \begin{cases} 2x - 3y + z = 3, \\ x - 2y + 2z = 0, \\ x - y - z = 1 \end{cases}$$



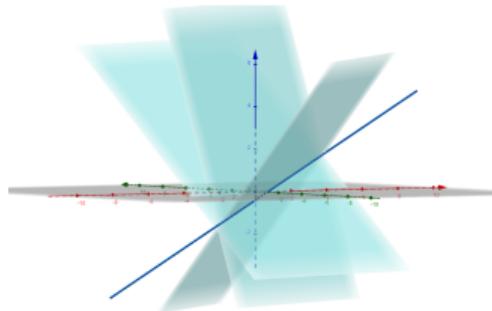
Task 6

Answer



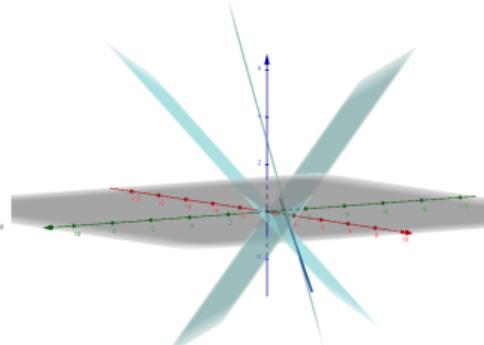
$$1. \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 2/9 \end{array} \right]$$

Point



$$2. \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Line



$$3. \left[\begin{array}{ccc|c} 1 & 0 & -4 & 6 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & -2 \end{array} \right]$$

No exists



Task 7

Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may be perpendicular to one of the lines given by $a_1x^2 + 2h_1xy + by^2 = 0$. And draw the picture.



Task 7

Answer

Let $y = mx$ be a line of $ax^2 + 2hxy + by^2 = 0$. Then

$$ax^2 + 2hmx^2 + bm^2x^2 = 0 \Rightarrow a + 2hm + bm^2 = 0 \quad (3.31)$$

Then $y = -\frac{1}{m}x$ is a line of $a_1x^2 + 2hx_1y_1 + b_1y^2 = 0$.

Hence,

$$a_1x^2 - 2h\frac{mx_1^2}{m} + \frac{h_1}{m^2}x^2 = 0 \text{ or } a_1m^2 - 2h_1m + b_1 = 0 \quad (3.32)$$

From (3.31) and (3.32), we get $\frac{m^2}{2hb_1 + 2h_1a} = \frac{m}{aa_1 - bb_1} = \frac{1}{-(2bh_1 + a_1h)}$.

Hence, the required condition is $(aa_1 - bb_1)^2 + 4(ha_1 + h_1b)(bh_1 + a_1h) = 0$.



Reference material

- Line in space (OnlineMschool)
- Gauss elimination online (OnlineMSchool)
- Pencil(geometry) definition (Wiki)
- Pencil of planes (rus)
- Pencil of planes (video, rus)
- Pencil of planes (video, eng)

Deserve “A” grade!

– Oleg Bulichev

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↗ @Lupasic

🚪 Room 105 (Underground robotics lab)