



# Analytical Geometry and Linear Algebra I, Visiting Lecture

Parametric form of segment line, polyline

Splines: Cubic, Bezier

Surfaces: made by extruding, sweeping, revolving



# Disclaimer

## Goal

The goal of this lecture is to get acquainted with splines, surfaces, their applications. Obtain some basic intuition.

## Constraints

1. Only really necessary proofs (others can be found in reference material).
2. I show today topics only from practical perspective (how to use it as a user and a programmer, not as a creator of new algorithms).
3. I used Matlab code (link in the references).
4. I have to make a small recap of some topics due to the reason of your misunderstanding of some concepts.



# Lecture Objectives

- To get the main benefit of parametric form.
- To have an intuition where and how splines can be used.
- To understand a relationship between splines and conic sections.
- How to make surfaces using curves.





# Computer Aided Design

## Form types

Type	Form	Example	Description
Explicit	$y = f(x)$	$y = mx + b$	Line
Implicit	$f(x, y) = 0$	$(x - a)^2 + (y - b)^2 = r^2$	Circle
Parametric	$x = \frac{g(t)}{w(t)}; y = \frac{h(t)}{w(t)}$	$x = a_0 + a_1 t; y = b_0 + b_1 t$ $x = a + r \cos t; y = b + r \sin t$	Line Circle

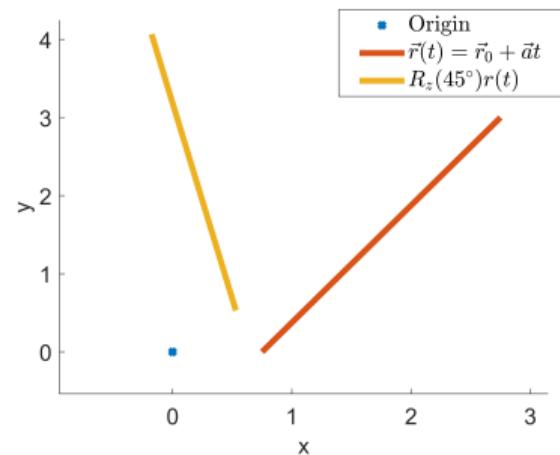


# Benefits of parametric form

## Definition

A parametric description of a curve is called such if the coordinates of the curve point are **continuous** and **unambiguous** functions of the parameter  $t$ .

- The result is a point cloud, which can be easily discretized.
- It can be easily controlled.
- We can work with our parametric curves as with coordinates (change basis, apply affine transformation).





## Segment line in parametric form

AGLA (6th lab) —  $\vec{r}(t) = \vec{p}_0 + a(t)$  or

$$\begin{cases} x = p_{0x} + a_x t \\ y = p_{0y} + a_y t \\ z = p_{0z} + a_z t \end{cases}$$

Not easy to make a segment line (we have only one clear point and a direction)

$\vec{r}(t) = \vec{p}_0(1 - t) + \vec{p}_1 t$  or

$$\begin{cases} x = p_{0x}(1 - t) + p_{1x} t \\ y = p_{0y}(1 - t) + p_{1y} t \\ z = p_{0z}(1 - t) + p_{1z} t \end{cases}$$

It is really convenient, if you know 2 points and want to make a segment line. We will meet this form a lot of times today



# Polyline (Polygonal chain)

$\vec{r}(t) = \vec{p}_i(1-w) + \vec{p}_{i+1}w$ , where  $w$  is a local parameter  
 $w = \frac{t - t_i}{t_{i+1} - t_i}$ ,  $t_i \leq t \leq t_{i+1}$

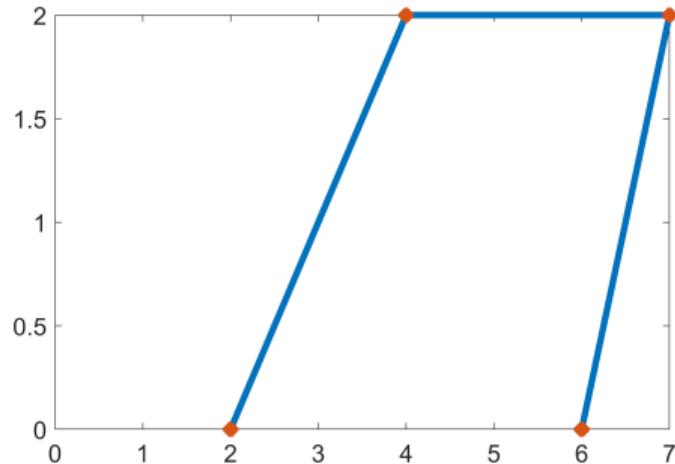
Polyline passes through given control points

$$\vec{p}_0, \vec{p}_1, \dots, \vec{p}_n. t_i \leq t_{i+1}$$

## Example

$$\vec{p}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

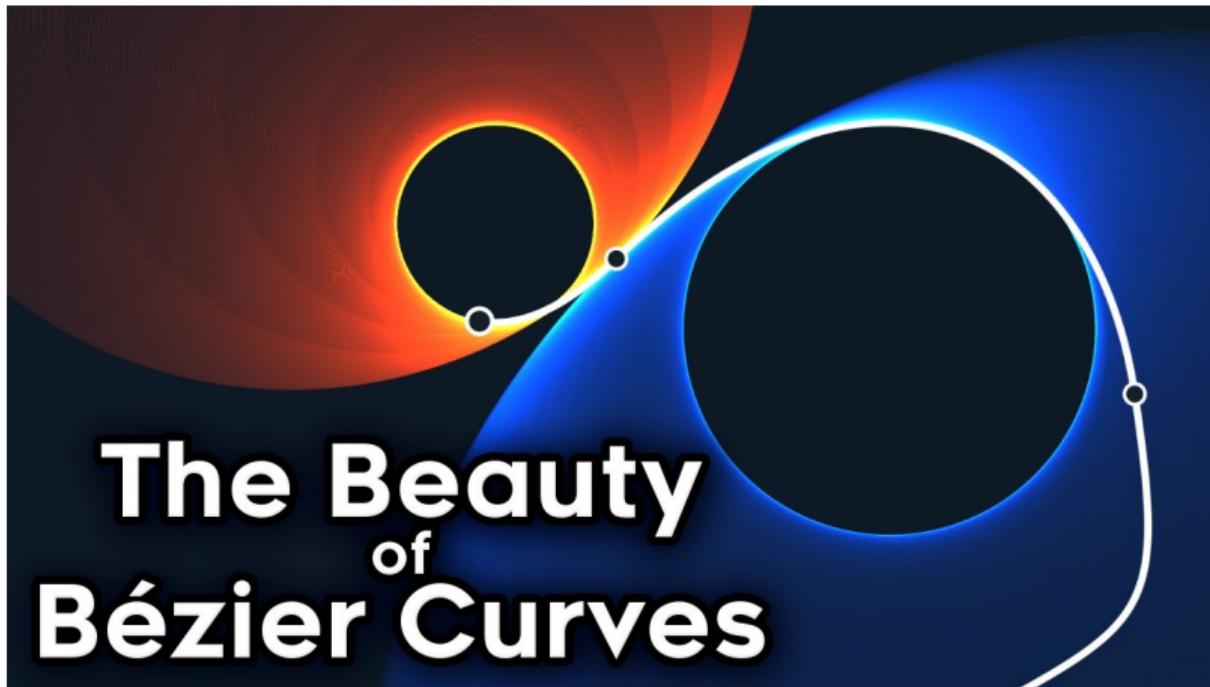
If each knot will be the same and equal to 1, then  $\vec{r}(t) = \vec{p}_i(1-t) + \vec{p}_{i+1}t, i = 0 \dots n-1$





# Intro to splines

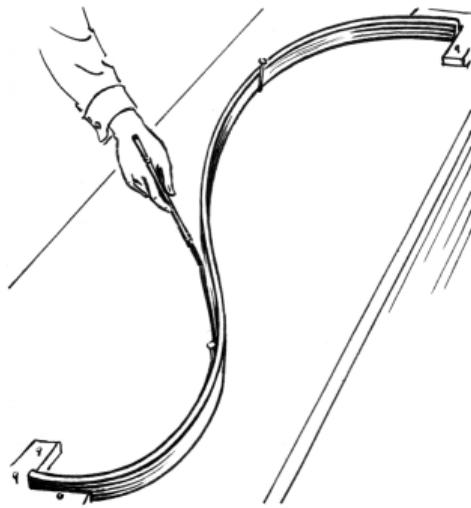
Video



# Splines

## *Informal Definition*

**Splines** (*piecewise polynomial functions*) are awesome tool to construct *smooth* and *flexible* shapes in computer graphics.



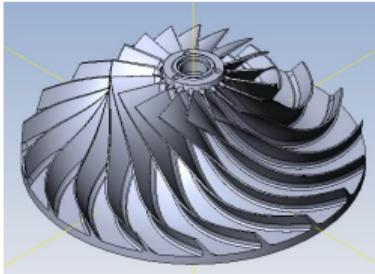
Starting 15th century, ship hull designers used splines for making a smooth shape



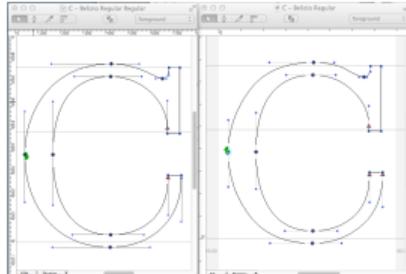
French curve (Лекало)



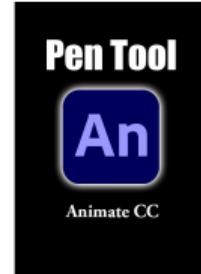
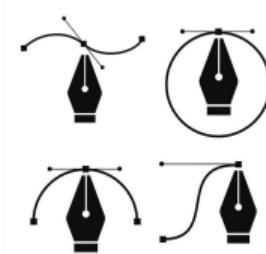
# Splines: Applications



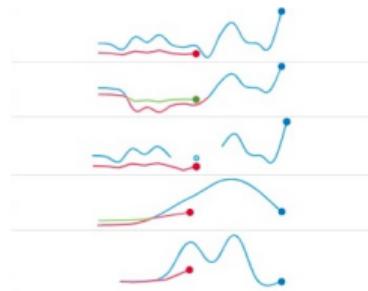
User: Car shape design, aircrafting



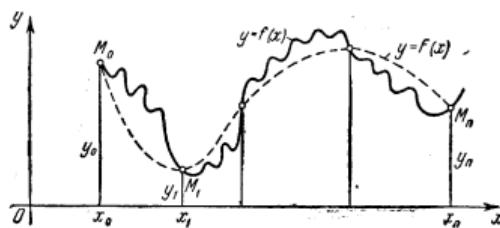
User: Make fonts



User: Pen tool in PhotoShop



Math: Interpolation — advanced data analysis



Math: Approximation — signal post processing (reduce noise)



Math: Extrapolation — revenue prediction during the covid



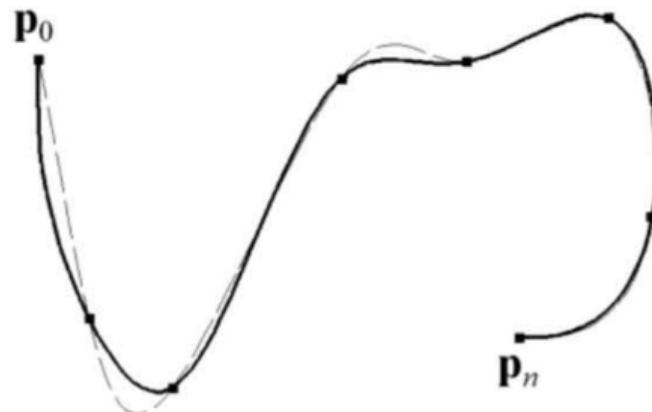
## Cubic spline (proof in (1) p. 33-35)

$$\vec{r}(t) = (1-w)\vec{p}_0 + w\vec{p}_{i+1} + ((-2w + 3w^2 - w^3)\vec{s}_i + (-w + w^3)\vec{s}_{i+1}) \frac{(t_{i+1} - t_i)^2}{6},$$

$$w = \frac{t - t_i}{t_{i+1} - t_i}, \quad t_i \leq t \leq t_{i+1}$$

$s_i$  — second derivative in control points

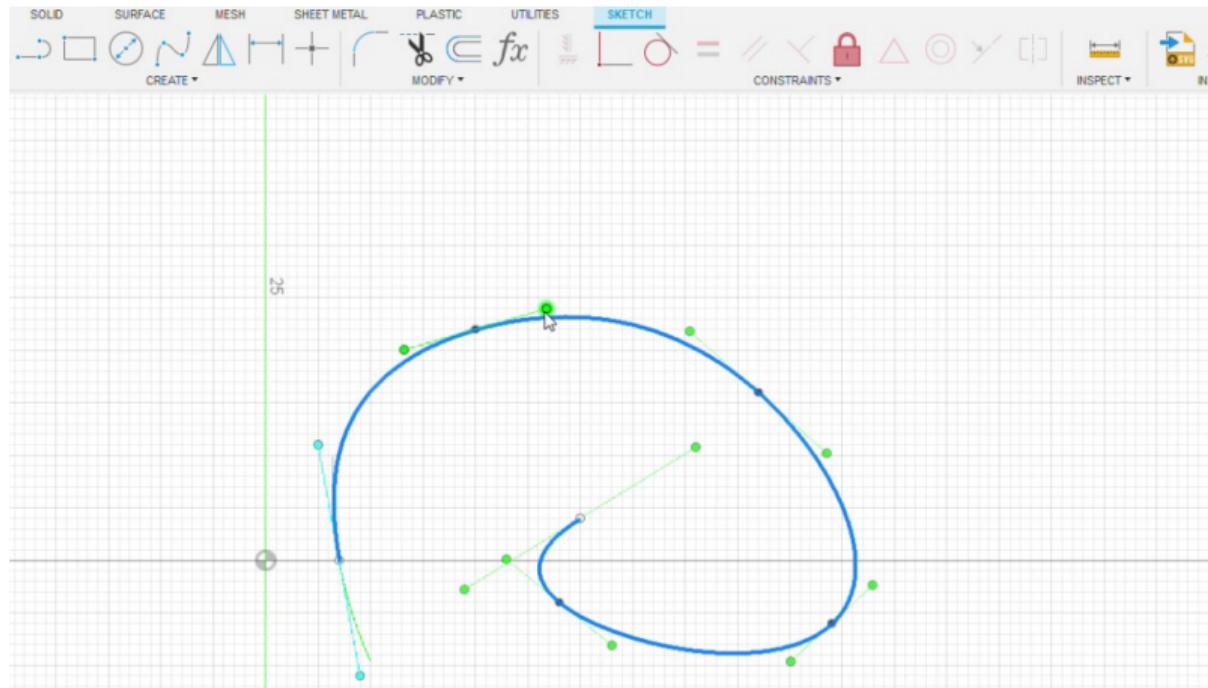
$$s_0 = s_n = 0$$





# Cubic spline in Fusion 360

Video





# Bezier spline (proof in (1) p. 38-42)

1-st order curve

$$\vec{r}(t) = (1-t)\vec{p}_0 + t\vec{p}_1$$

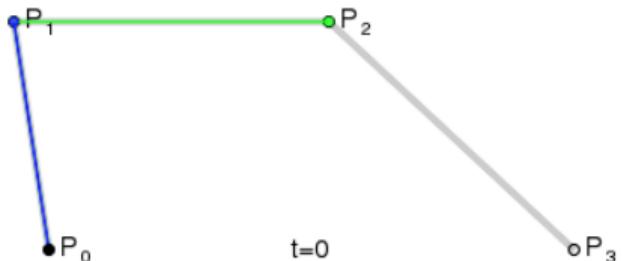
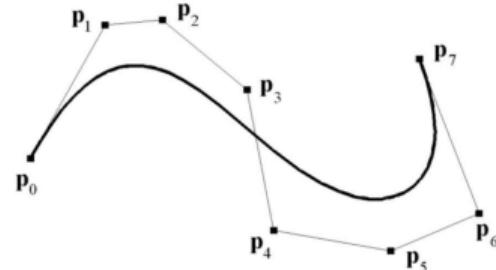
2-nd order curve

$$\vec{r}(t) = (1-t)^2\vec{p}_0 + 2t(1-t)\vec{p}_1 + t^2\vec{p}_2$$

General form

$$\vec{r}(t) = \sum_{i=0}^n B_i^n(t) \vec{p}_i$$

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$
 Bernstein polynomials

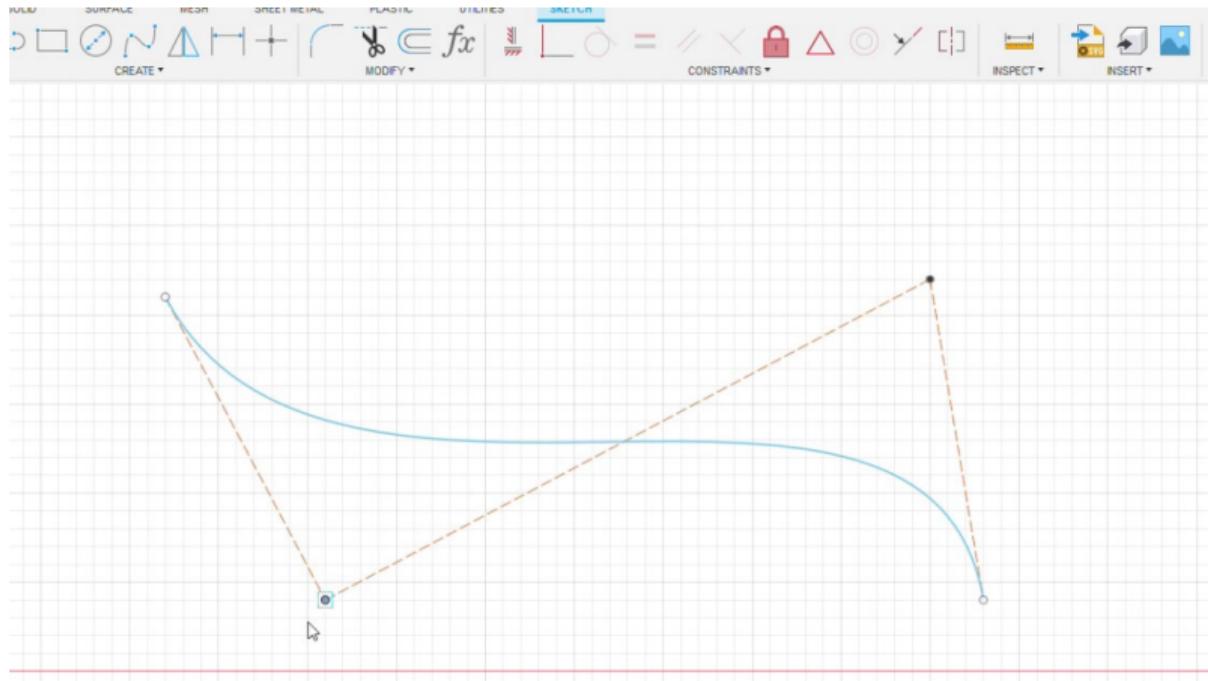


Click on the figure for a video



# Bezier spline in Fusion 360

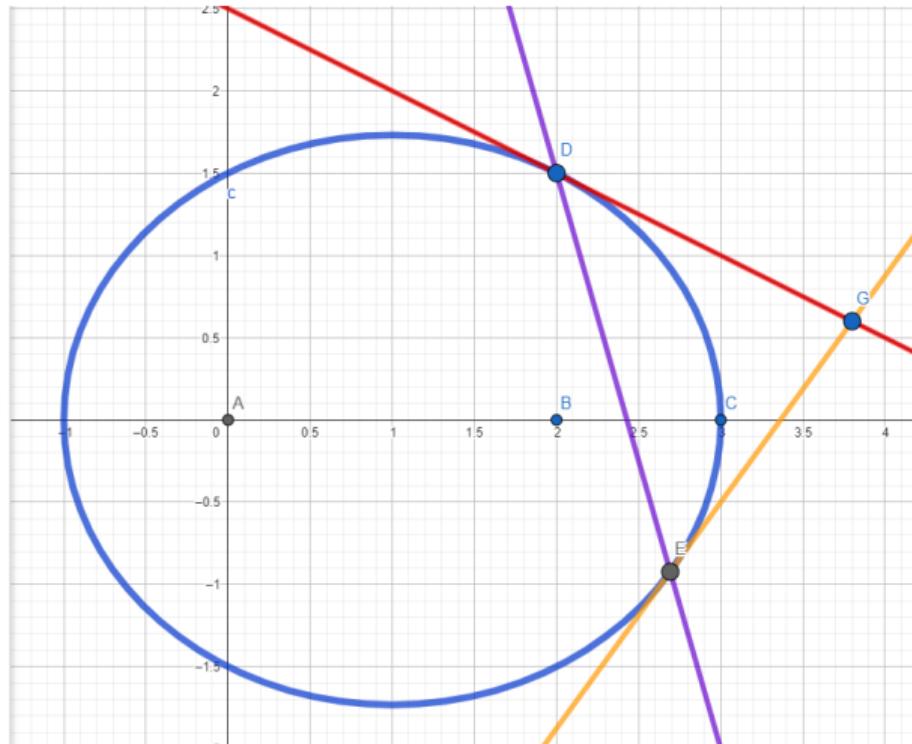
Video





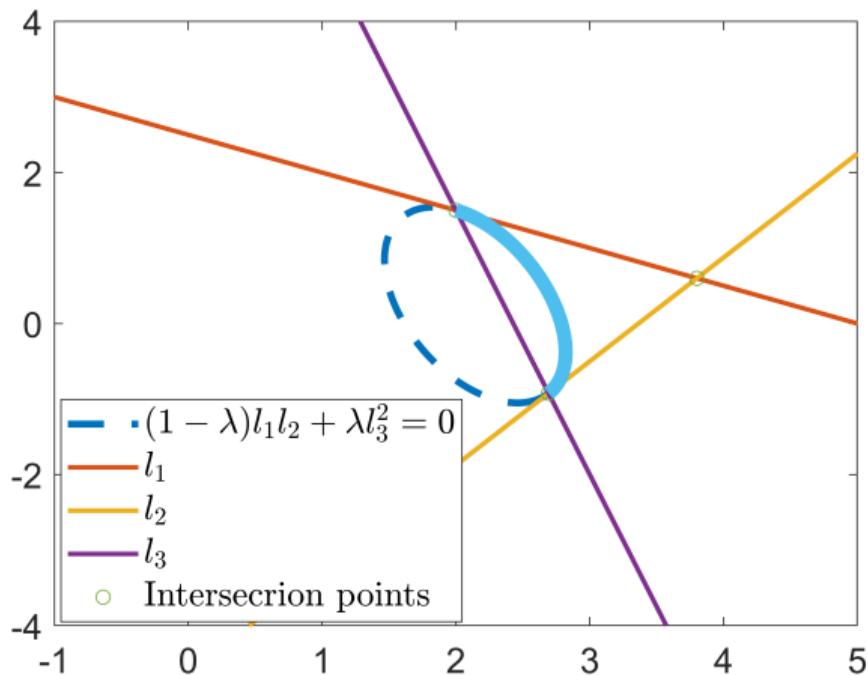
# Bezier splines & conic sections (proof in (1) p. 42-48) (1)

●	c : Ellipse(A, B, C)	⋮
	→ $3x^2 + 4y^2 - 6x = 9$	
●	D = Point(c)	⋮
	→ (2, 1.5)	◎
●	Tangent(D, c)	⋮
	→ g: $y = -0.5x + 2.5$	
●	G = Point(g)	⋮
	→ (3.8, 0.6)	◎
●	Tangent(G, c)	⋮
	→ i: $y = -0.5x + 2.5$	
●	→ j: $y = 1.375x - 4.625$	⋮
●	E = Intersect(c, j)	⋮
	→ (2.6923076923077, -0.9230769230769)	
●	f : Line(E, D)	⋮
	→ $y = -3.5x + 8.5$	
+	Input...	





## Bezier splines & conic sections (proof in (1) p. 42-48) (2)



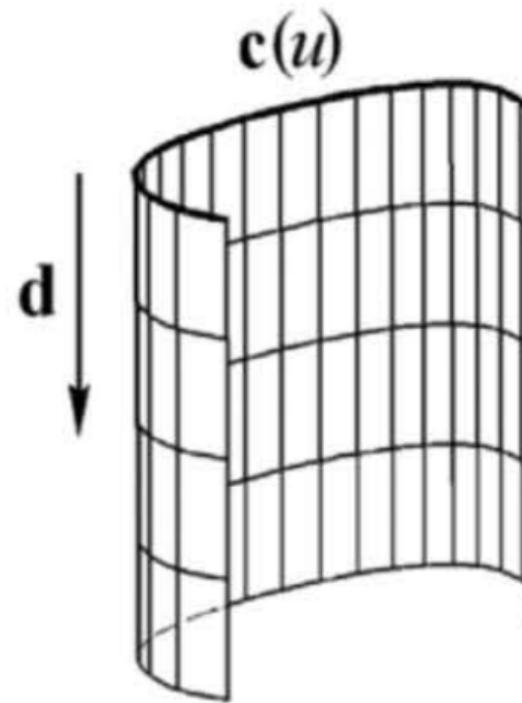


## Extrude and shear surfaces (proof in (1) p. 114-118)

When a generating curve moves along a guiding curve, the orientation of the generating curve may remain unchanged related to guiding line.

If the guiding line is a straight line — **extrude surface**, otherwise — **shear surface**

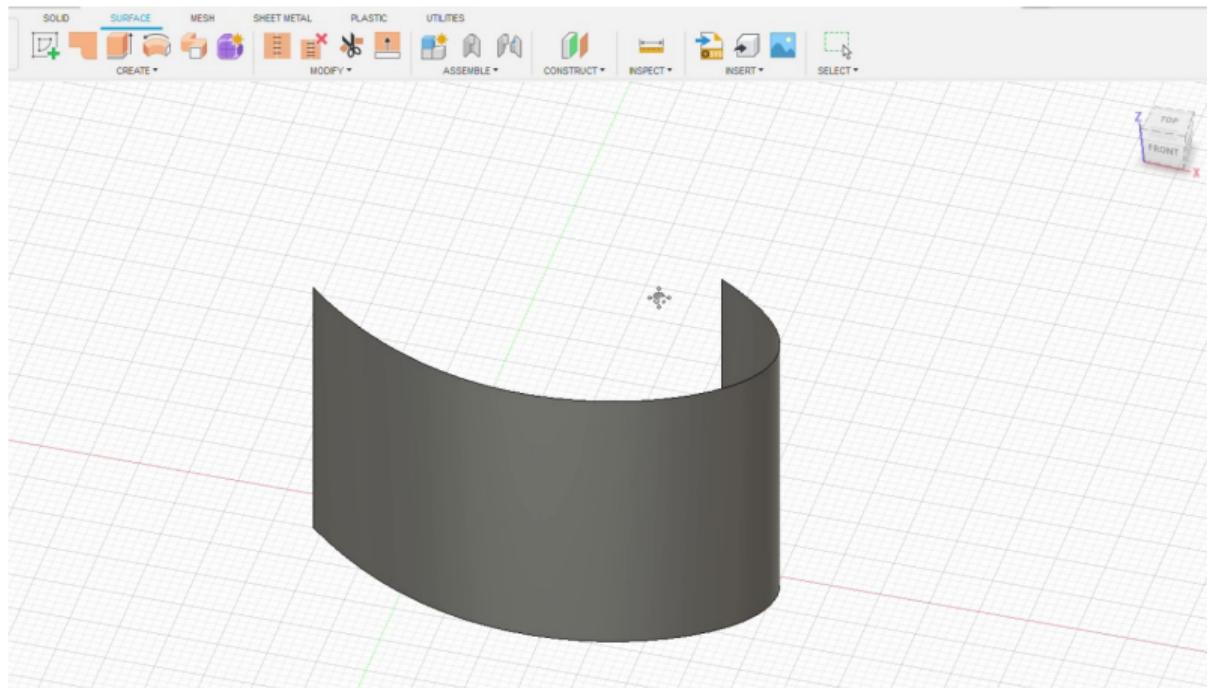
$$\vec{r}(u, v) = \vec{c}(u) + \vec{d}v$$





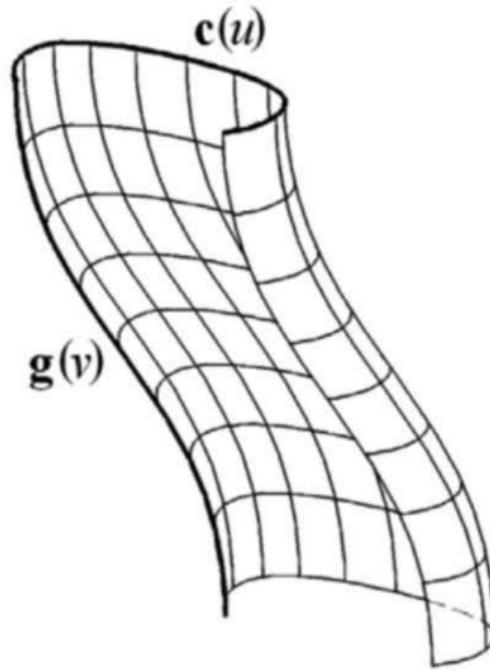
# Surface made by extruding in Fusion 360

Video



# Shear Surface

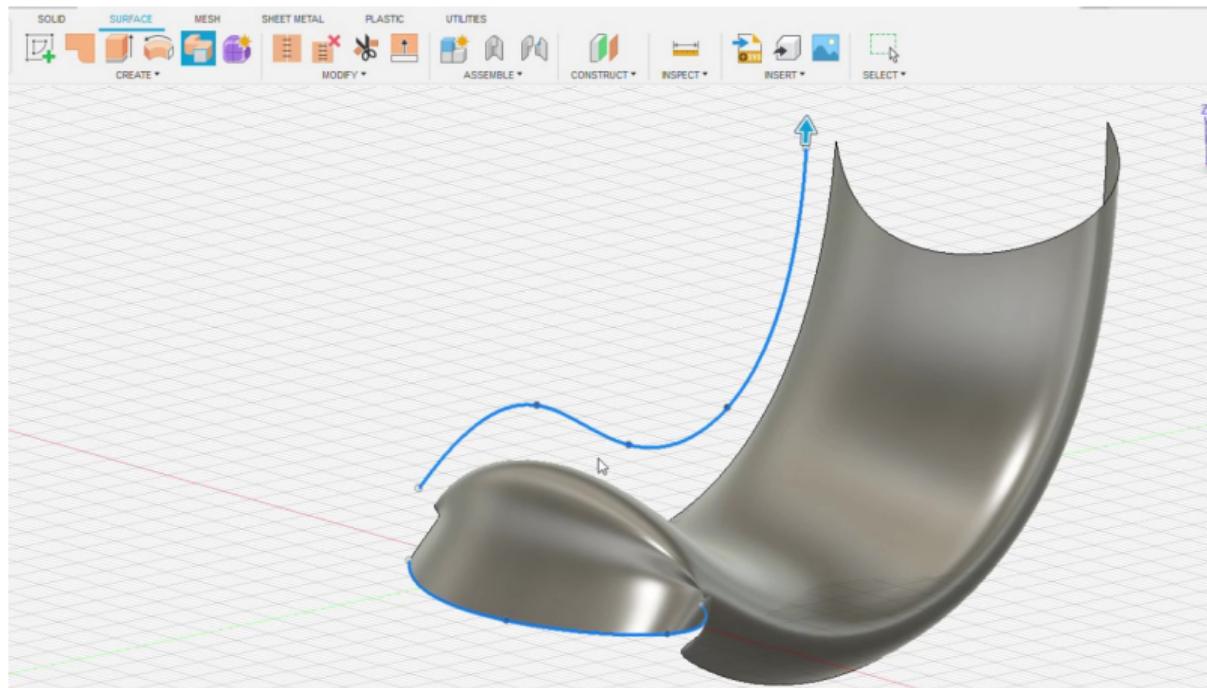
$$\vec{r}(u, v) = \vec{g}(v) + \vec{c}(u) - \vec{g}(v_{min})$$





# Surface made by shearing in Fusion 360

Video





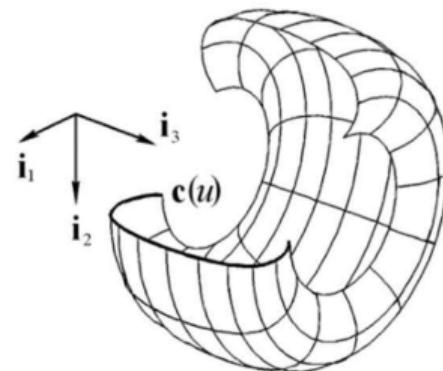
# Revolve surface

$$\vec{r}(u, v) = \vec{p} + M(v)(\vec{c}(u) - \vec{p}),$$

$$u \in [u_{min}, u_{max}], v \in [0, \alpha]$$

$$M(v) = A \begin{bmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1}, \text{ — rotation matrix}$$

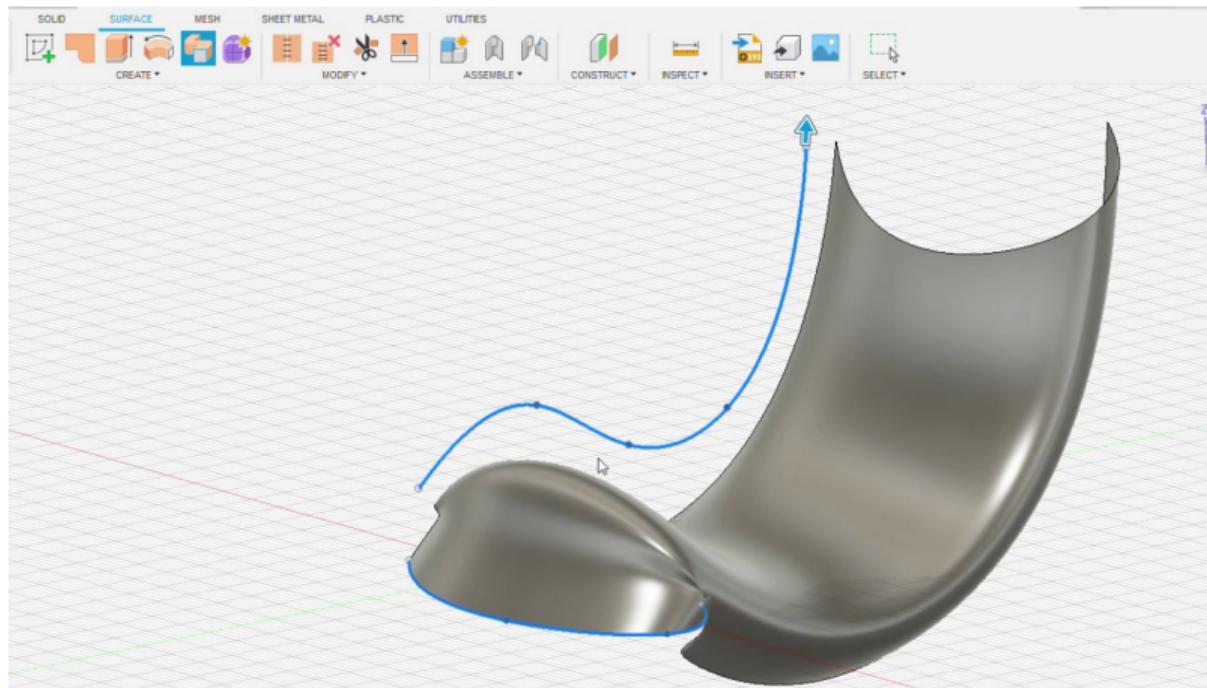
$A = [i_1 \ i_2 \ i_3]$  — transformation from local to global CS





# Surface made by revolving in Fusion 360

Video





# Summary

1. We touched the benefits of a parametric form.



# Summary

1. We touched the benefits of a parametric form.
2. We know the equation of the line and polyline.



# Summary

1. We touched the benefits of a parametric form.
2. We know the equation of the line and polyline.
3. We heard about spline applications.



# Summary

1. We touched the benefits of a parametric form.
2. We know the equation of the line and polyline.
3. We heard about spline applications.
4. We know 2 types of splines and the difference between them.



# Summary

1. We touched the benefits of a parametric form.
2. We know the equation of the line and polyline.
3. We heard about spline applications.
4. We know 2 types of splines and the difference between them.
5. We got acquainted with a proof how correlates conic sections and Bezier splines.
6. We know 3 methods of obtaining surfaces.



## Reference material

- Geometrical Modeling, Golovanov N.N. (book, rus)
- The Beauty of Bézier Curves (video, eng)
- Computer Graphics course, lectures notes 12 and 13 (Imperial College London)
- 12 Spline Curves (video, eng)
- Data Fitting: Polynomial Fitting and Splines, Part 4 (video, eng)
- Cubic spline (habr, rus)

# Deserve “A” grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru

↗ @Lupasic

🚪 Room 105 (Underground robotics lab)