Analytical Geometry and Linear Algebra II, Lab 9

Eigenvalues and Eigenvectors

Diagonalization of a Matrix

Difference equation

How I spent last weekend

Hiking Club KFU ”Tochka” event

Where it can be used

- Machine learning (transform data in more suitable usage)
- Make some calculations easier (matrix^{1}00 – piece of cake)
- Predict the behavior of linear systems (physics, biology, etc)
- Design the controller for our system
- Estimate the complexity of calculations
- ...

Definition

In linear algebra, an **eigenvector or characteristic vector** of a linear transformation is a non-zero *vector that changes by only a scalar factor when that linear transformation is applied to it.* Wiki

$A\mathbf{x} = \lambda\mathbf{x}$, where

x – eigenvector (should be non-zero),

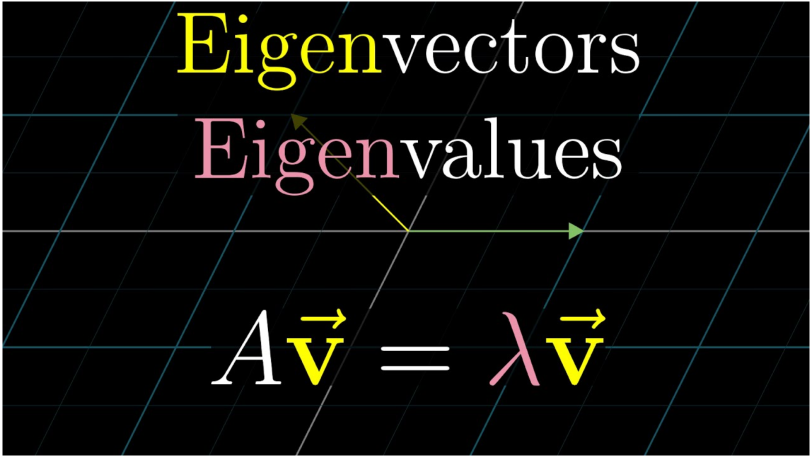
λ – eigenvalue,

A – *square* matrix.

For $n \times n$ matrix – max amount of λ is a number of n .

EigenValues concept

Video



Calculation (1)

Classical approach (max 3x3)

There are 2 steps:

1. Find λ (eigenvalue) — $det(A-\lambda I)=0$
   - 2×2 matrix: $det(A-\lambda I)=\lambda^2-trace(A)\lambda+det(A)=0$, where $trace(A)$ – sum of diag values of A;
   - 3×3 matrix: $det(A-\lambda I)=\lambda^3-trace(A)\lambda^2-\frac{1}{2}(trace(A^2)-trace(A)^2)\lambda-det(A)=0$
2. Find $\mathbf{x}$ for each λ — $(A-\lambda_i I)\mathbf{x}=0$

Case study, 2×2 matrix: $A=\begin{bmatrix}4 & 3\\ -2 & -3\end{bmatrix}$

1. $trace(A)=4+(-3)=1$,
   $det(A)=4(-3)-3(-2)=-6$, hence
   $\lambda^2-\lambda-6=(\lambda-3)(\lambda+2)$, $\rightarrow$
   $\rightarrow\lambda_1=3,\ \lambda_2=-2$
2. 2.1 $A-3I=\begin{bmatrix}1 & 3\\ -2 & -6\end{bmatrix}$; $x_{\lambda=3}=\begin{bmatrix}3\\ -1\end{bmatrix}$

   2.2 $A+2I=\begin{bmatrix}6 & 3\\ -2 & -1\end{bmatrix}$; $x_{\lambda=2}=\begin{bmatrix}1\\ -2\end{bmatrix}$

Task 1

Find the eigenvalues and eigenvectors:

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

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Find the eigenvalues and eigenvectors:

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Answer

1. $\lambda_1 = -5,\ \lambda_2 = 9$

   $x_{\lambda=-5} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, x_{\lambda=9} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$
2. $\lambda_1 = 3 + 1i,\ \lambda_2 = 3 - 1i$

   $x_{\lambda=3+1i} = \begin{bmatrix} 3 + 1i \\ 0 \end{bmatrix}, x_{\lambda=3-1i} = \begin{bmatrix} 0 \\ 3 - 1i \end{bmatrix}$

Calculation (2)

Real life approach (Iterative algorithms)

Due to the reason that computers appeared recently, eigenpairs weren't used broadwide.

Nowadays, it can be found easily by iteration method, which implemented in most programming languages.

Method	Applies to	Produces	Cost per step	Convergence
Lanczos algorithm	Hermitian	m largest/smallest eigenpairs		
Power iteration	general	eigenpair with largest value	$O(n^2)$	linear
Inverse iteration	general	eigenpair with value closest to μ		linear
Rayleigh quotient iteration	Hermitian	any eigenpair		cubic
Preconditioned inverse iteration[11] or LOBPCG algorithm	positive-definite real symmetric	eigenpair with value closest to μ		
Bisection method	real symmetric tridiagonal	any eigenvalue		linear
Laguerre iteration	real symmetric tridiagonal	any eigenvalue		cubic[12]
QR algorithm	Hessenberg	all eigenvalues	$O(n^2)$	cubic
		all eigenpairs	$6n^3 + O(n^2)$	
Jacobi eigenvalue algorithm	real symmetric	all eigenvalues	$O(n^3)$	quadratic
Divide-and-conquer	Hermitian tridiagonal	all eigenvalues	$O(n^2)$	
		all eigenpairs	$(\frac{4}{3})n^3 + O(n^2)$	

Eigenvector and eigenvalue iterative algorithms Wiki

Eigenpair properties and features

- $\sum \lambda = trace(A)$
- $A_{new} = A_{old} + a\lambda_{old}, \rightarrow$ eigenvectors won't change, $\lambda_{new} = \lambda_{old} + a$
- If matrix is triangular – the eigenvalues are on the main diagonal
- If matrix is symmetric – λ is *definitely* real
- If matrix is not symmetric – λ *can* contain imaginary part
- $Ax = \lambda x \rightarrow A^2x = Ax\ (left\ mult) \rightarrow A^2x = Ax(\lambda\ is\ const) = \lambda^2 x$

Diagonalization

Key idea / **Follow each eigenvector separately** / $\boldsymbol{n}$ **simple problems**

Eigenvector matrix $\boldsymbol{X}$

Assume independent $\boldsymbol{x}$'s

Then X is invertible

$$AX = A\begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix}$$

$$\boxed{\begin{aligned} \boldsymbol{AX} &= \boldsymbol{X\Lambda} \\ \boldsymbol{X^{-1}AX} &= \boldsymbol{\Lambda} \\ \boldsymbol{A} &= \boldsymbol{X\Lambda X^{-1}} \end{aligned}}$$

$$\begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Diagonalization properties

Some matrices are not diagonalizable

They don't have n independent vectors

$$A = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} \text{ has } \lambda = \mathbf{3} \text{ and } \mathbf{3}$$

That A has double eigenvalue, single eigenvector

Only one $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Task 2

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

- Find eigenpairs;
- Write down A in diagonal from;
- Draw several vectors: one, which are parallel to an eigenvector, other – not.
- Multiply chosen vectors on A, draw the new ones.

Task 2

Answer

$Ax = \lambda x$; $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$D = 16 - 4 \cdot 1 \cdot 3 = 4$

$\lambda_{1,2} = \frac{4 \pm \sqrt{4}}{2} \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 3 \end{matrix}$

(1) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$, for instance $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \to x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

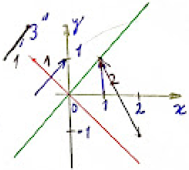
„2" $S^{-1} = S^T$ if orthonormal, here not appl.

$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = S^{-1}AS$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$

„3"



2 $\cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

It's scale + shear

Task 3

True or false: If the columns of X (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) X is invertible (d) X is diagonalizable.

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(c) X is invertible (d) X is diagonalizable.

Answer

(a) False: We are not given the λ's (b) True (c) True (d) False: For this we would need the eigenvectors of X

Task 4

If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix X is triangular, then X^{-1} is triangular. Prove that A is also triangular.

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If the eigenvectors of A are the columns of I, then A is a ______ matrix. If the eigenvector matrix X is triangular, then X^{-1} is triangular. Prove that A is also triangular.

Answer

With $X = I$, $A = X\Lambda X^{-1} = \Lambda$ is a diagonal matrix. If X is triangular, then X^{-1} is triangular, so $X\Lambda X^{-1}$ is also triangular.

A^k

A^k becomes easy $\quad A^k = (X\Lambda X^{-1})\,(X\Lambda X^{-1})\cdots(X\Lambda X^{-1})$

Same eigenvectors in X $\quad \boxed{\boldsymbol{A^k = X\Lambda^k X^{-1}}}$ $\quad \boldsymbol{\Lambda^k} = (\text{eigenvalues})^{\boldsymbol{k}}$

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}^{\mathbf{4}} = X\Lambda^{\mathbf{4}}X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{1^4} & 0 \\ 0 & \mathbf{3^4} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 81 \\ 0 & 81 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{80} \\ \mathbf{0} & \mathbf{81} \end{bmatrix}$$

Question : **When does $\boldsymbol{A^k} \to$ zero matrix ?**

Answer : $\boxed{\textbf{All } |\boldsymbol{\lambda_i}| < \mathbf{1}}$

Task 5

$A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____ . Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

Task 5

$A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____ . Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

Answer

$A^k = X\Lambda^k X^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; A_1 is a Markov matrix so $\lambda_{\max} = 1$ and $A_1^k \to A_1^\infty$, A_2 has $\lambda = .6 \pm .3$ so $A_2^k \to 0$.

Applications (1)

Computer Vision

Applications (2)

Machine learning + optimization

Applications (3)

Predict the behavior of linear systems

Applications (4)

Fast Calculations

Task 6

(Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Task 6

(Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Answer

Two problems: The nullspace and column space can overlap, so x could be in both.

There may not be r independent eigenvectors in the column space.

Reference material

- Lecture 21, Eigenvalues and Eigenvectors
- Lecture 22, Diagonalization and Powers of A
- *"Linear Algebra and Applications", pdf pages 270–306*
  Eigenvalues and Eigenvectors 5.1–5.3
- *"Introduction to Linear Algebra", pdf pages 299–329*
  Eigenvalues and Eigenvectors 6.1–6.2
- The eigenvalue problem | Lectures 32 – 38
  Video from Matrix Algebra for Engineers course

Deserve “A” grade!

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Room 105 (Underground robotics lab)