Mid Term Exam. **Innopolis University**

AG&LA II

Spring 2023

Duration: 120 Minutes **23.03.2023**

1.1 (3 points) True or False statement.

For matrix $A(m$x$n)$ with $\mathrm{rank}(A) < m$ & $\mathrm{rank}(A) < n$ is	True	False
$\mathrm{Dim}[C(A)] = \mathrm{Dim}[C(A^T)]$	+	
$C(A) = C(A^T)$		+
Dim $C(A)$ + Dim $N(A^T) = m$.	+	

1.2 (3 points) True or False statement.

For matrix $A(m$x$n)$ with $\mathrm{rank}(A) < m$ & $\mathrm{rank}(A) < n$ is	True	False
$C(A^T) \neq C(A)$	+	
$\mathrm{Dim}[C(A^T)] \neq \mathrm{Dim}[C(A)]$		+
Dim $C(A^T)$ + Dim $N(A) = n$.	+	

2.1 Considering the matrix, A and the vector $\vec{b}$,

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

(a) (4 points) Find the projection of b onto the column space of A.

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}\left[\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}\right]^{-1}\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -11 \\ -2 \\ -1 \end{bmatrix}$$

(b) (1 points) Split $\vec{b}$ into $\vec{p} + \vec{e}$, with p in the column space and e orthogonal to that space.

$$\vec{p} = \vec{b} - \vec{e} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 7 \\ -11 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 \\ -3 \\ 9 \\ 15 \end{bmatrix}$$

(c) (1 point) Which of the four fundamental spaces of A contains e. $\vec{e} \in N(A^T)$.

2.2 Considering the matrix, A and the vector $\vec{b}$,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

(a) (4 points) Find the projection of b onto the column space of A.

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}\left[\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

(b) (1 points) Split $\vec{b}$ into $\vec{p} + \vec{e}$, with p in the column space and e orthogonal to that space.

$$\vec{p} = \vec{b} - \vec{e} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 \\ 4 \\ 0 \\ -4 \end{bmatrix}$$

(c) (1 point) Which of the four fundamental spaces of A contains e. $\vec{e} \in N(A^T)$.

3.1 (4 points) Let $S_1 = \{x, y, z : x - 2y - 4z = 8\}$ and $S_2 = \{x, y, z : x - y + z = 3\}$.

$\vec{v} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$ – lies at the line of intersection of the planes S_1 and S_2. Find $\vec{v}$.

$$\begin{cases} x - 2y - 4z = 8 \\ x - y + z = 3 \end{cases} \Rightarrow \begin{cases} x = -2(3z+1) \\ y = -5(z+1) \end{cases} \Rightarrow \vec{v} = -\begin{bmatrix} 2(3z+1) \\ 5(z+1) \\ -z \end{bmatrix} = -z\begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

3.2 (4 points) Let $S_1 = \{x, y, z : x + 2y - 4z = 8\}$ and $S_2 = \{x, y, z : x + y + z = 3\}$.

$\vec{v} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$ – lies at the line of intersection of the planes S_1 and S_2. Find $\vec{v}$.

$$\begin{cases} x + 2y - 4z = 8 \\ x + y + z = 3 \end{cases} \Rightarrow \begin{cases} x = -2(3z+1) \\ y = 5(z+1) \end{cases} \Rightarrow \vec{v} = \begin{bmatrix} -2(3z+1) \\ 5(z+1) \\ z \end{bmatrix} = z\begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$$

4.1 Considering the following measurements:

t	-2	-1	1	2
b	-2	0	4	6

(a) (4 points) Find the best straight-line fit (Least squares) to the measurements,

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}. \text{ We get a system of equations: } \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 6 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} -2 \\ 0 \\ 4 \\ 6 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow$$

The best straight-line fit to the measurements is $y(t) = 2 + 2t$

(b) (2 point) Find the projection matrix of vector $b = [-2, 0, 4, 6]^T$ onto the column space of matrix A:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\left[\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}} = \frac{1}{20}\begin{bmatrix} 13 & 9 & 1 & -3 \\ 9 & 7 & 3 & 1 \\ 1 & 3 & 7 & 9 \\ -3 & 1 & 9 & 13 \end{bmatrix}$$

4.2 Considering the following measurements:

t	-1	0	1	2
b	1	-1	1	-2

(a) (4 points) Find the best straight-line fit (Least squares) to the measurements,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}. \text{ We get a system of equations: } \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ -7 \end{bmatrix} \Rightarrow$$

The best straight-line fit to the measurements is $y(t) = \frac{1}{10} - \frac{7}{10}t$

(b) (2 point) Find the projection matrix of vector $b = [1, -1, 1, -2]^T$ onto the column space of matrix A:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\left[\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}} = \frac{1}{10}\begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

5.1 (6 points) Find the dimensions of the four fundamental subspaces associated with A, depending on the parameters α and β.

$$A = \begin{bmatrix} 7 & \alpha & 5 & 10 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & \beta & 4 \\ 5 & 8 & 3 & 6 \end{bmatrix} = [\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4] \implies$$

1) If $\alpha = 12$, then $\vec{a}_2 = \vec{a}_1 + \frac{1}{2}\vec{a}_4$ and $\operatorname{rank}(A) = 3$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 3 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 1 \end{cases}$

2) If $\beta = 2$, then $\vec{a}_3 = \frac{1}{2}\vec{a}_4$ and $\operatorname{rank}(A) = 3$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 3 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 1 \end{cases}$

3) If $\alpha = 12$ and $\beta = 2$, then $\operatorname{rank}(A) = 2$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 2 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 2 \end{cases}$

5.2 (6 points) Find the dimensions of the four fundamental subspaces associated with A, depending on the parameters α and β.

$$A = \begin{bmatrix} \alpha & 6 & 8 & 12 \\ 2 & 1 & 1 & 2 \\ 1 & \beta & 1 & 0 \\ 9 & 4 & 5 & 8 \end{bmatrix} \implies$$

1) If $\alpha = 14$, then $\vec{a}_1 = \vec{a}_3 + \frac{1}{2}\vec{a}_4$ and $\operatorname{rank}(A) = 3$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 3 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 1 \end{cases}$

2) If $\beta = 0$, then $\vec{a}_2 = \frac{1}{2}\vec{a}_4$ and $\operatorname{rank}(A) = 3$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 3 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 1 \end{cases}$

3) If $\alpha = 14$ and $\beta = 0$, then $\operatorname{rank}(A) = 2$ and $\begin{cases} \operatorname{Dim}[C(A)] = \operatorname{Dim}[C(A^T)] = 2 \\ \operatorname{Dim} N(A) = \operatorname{Dim} N(A^T) = 2 \end{cases}$

6.1 Find an orthonormal basis for the subspace spanned by the vectors: a_1, a_2 and a_3 (4 points).

Then express $A = [a_1, a_2, a_3]$ in the form of QR (2 points).

$$a_1 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}, a_3 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}.$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \quad q_2 = \frac{a_2 - q_1^T a_2}{\|a_2 - q_1^T a_2\|} = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}, \quad q_3 = \frac{a_3 - q_1^T a_3 - q_2^T a_3}{\|a_3 - q_1^T a_3 - q_2^T a_3\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 & -2 \\ -1 & -1 & 2 \\ 2 & -4 & -1 \end{bmatrix} = QR = \begin{bmatrix} -2 & -1 & -2 \\ -1 & -2 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.2 Find an orthonormal basis for the subspace spanned by the vectors: a_1, a_2 and a_3 (4 points).

Then express $A = [a_1, a_2, a_3]$ in the form of QR (2 points).

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, a_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad q_2 = \frac{a_2 - q_1^T a_2}{\|a_2 - q_1^T a_2\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad q_3 = \frac{a_3 - q_1^T a_3 - q_2^T a_3}{\|a_3 - q_1^T a_3 - q_2^T a_3\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -2 & 4 & 1 \\ 2 & -1 & 2 \end{bmatrix} = QR = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7.1 (4 points) Prove that the matrix A^TA has the same nullspace as $A(m \times n)$:

Certainly, if $Ax = 0$ then $A^TAx = 0$. Vectors x in the nullspace of A are also in the nullspace of A^TA.

To go in the opposite direction, start by supposing that $A^TAx = 0$, and take the inner product with x to show that $Ax = 0$: $x^TA^TAx = 0$, or $\|Ax\|^2 = 0$, or $Ax = 0$.

7.2 (4 points) Prove that if the matrix $A(n \times n)$ is symmetric the dot product of the vectors $x \in \mathbb{R}^n$ and Ay is the same as the vectors Ax and $y \in \mathbb{R}^n$:

$$A = A^T \Rightarrow x^TAy = (Ay)^Tx = y^TA^Tx = y^TAx.$$