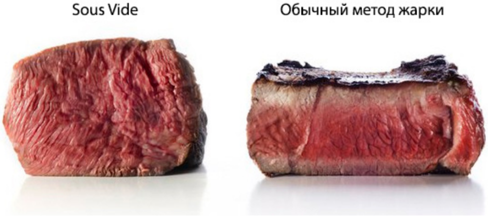
Analytical Geometry and Linear Algebra II, Lab 5

Projection

Application (Least Squares)

How I spent last weekend

(a) Cross-country skiing

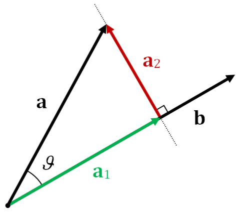


(b) Sous vide

Projection

Definition

The *vector projection* of a vector **a** on (or onto) a nonzero vector **b**, sometimes denoted $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$ is the orthogonal projection of **a** onto a straight line parallel to **b**.



Projection of **a** on **b** ($\mathbf{a}_1$), and rejection of **a** from **b** ($\mathbf{a}_2$)

Where it can be used:

- Maps
- Blueprints
- Fitting algorithms (Least squares)
- Reduce matrix dimention
- Reinforecement Learning (RL) fitness functions

Projection (1)

2D case
Classical way
Project "b" on "a_1"

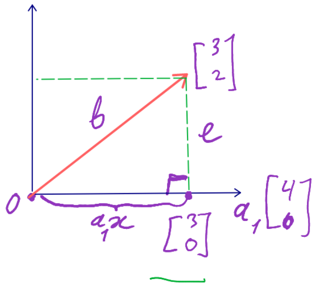
$$e = b - a_1 x$$

$$a_1 \cdot (b - a_1 x) = 0$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T b = a_1^T a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x$$ — classic formula from school



Particular example

$$\frac{a_1^T b}{a_1^T a_1} = x$$

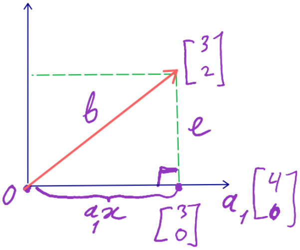
$$\Downarrow$$

$$\frac{\begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}} = \frac{12}{16} = \frac{3}{4}$$

projection $p = a_1 x = \frac{3}{4}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Projection (2)

2D case
Projection matrix
Project "b" on "a_1"



Like affine
transformation matrix

$$P b = x a_1 = a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x \quad \Rightarrow \quad P = \frac{a_1 a_1^T}{a_1^T a_1}$$

Projection matrix

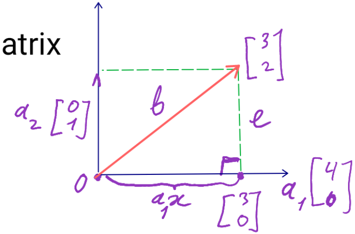
Particular example

$$P = \frac{\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}4 & 0\end{bmatrix}}{16} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$p = P b = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

Projection (3)

3D case
Projection matrix
Project "b"
on ["a_1","a_2"]



$$A^T(b - Ax) = 0$$

$$A^TAx = A^Tb$$

It will be Indentity for
full rank square matrix

$$\underbrace{A(A^TA)^{-1}A^T}_{P}\,b = Ax$$

projection matrix

Particular example

$$P = \begin{bmatrix}a_1 & a_2\end{bmatrix}\left(\begin{bmatrix}a_1^T\\a_2^T\end{bmatrix}\begin{bmatrix}a_1 & a_2\end{bmatrix}\right)^{-1}\begin{bmatrix}a_1^T\\a_2^T\end{bmatrix}$$

$$= \begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}\left(\begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}4 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

It's correct, because we project b
on a plane, where it lies

$$p = Pb = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

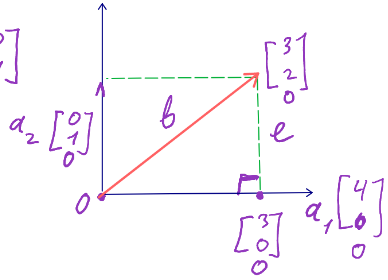
Projection (4)

2D case
Project "b"
on "d_1" perpendicular to "a_1"

$$P_{d_1} = I - P = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p_{d_1} = P_{d_1} b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$



3D case
Project "b"
on "d_2" perpendicular to
["a_1", "a_2"]

$$P_{d_2} = I - P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad p_{d_2} = P_{d_2} b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Projection

Case study: Reinforcement Learning fitness function

Goal

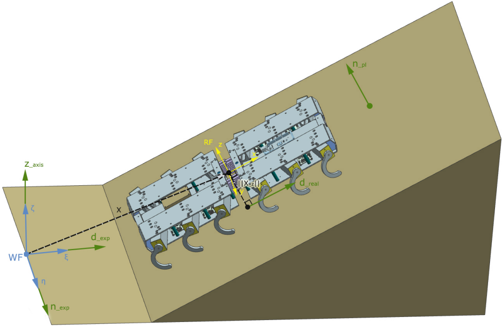
It is necessary for the robot to move in a straight line in all directions, as well as as as efficiently as possible.

The efficienty criteria are: course deviation error, max velocity and robot safety.

$$F = \omega_1 X_z + \omega_2 \frac{1}{|err| + \varepsilon} + \omega_3 (P_{d_{real}} \vec{X}), \text{ where}$$

$$err = |(I - P_{d_{real}})(I - P_{n_{pl}})\vec{X}|,$$

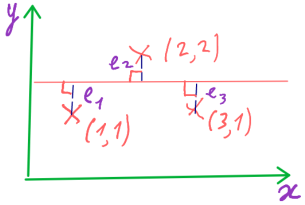
P_* – projection matrix, ω_* – weight coeffs.



StriRus – task description

Least squares (1)

Task description: a guy draw the line on a floor. We want to know the equation of this line. We send a robot, which follows the line and obtain a dataset from GPS.



$\|e\|^2 = \|Ax - y\|^2 \to \min$

Equation of the line is y=kx+t. Hence, there are 2 unknown variables. Be we obtained 3 equations. Our system is overdetermined (more equations (points), than variables).

We have to reformulate our task.
Let's fit all points on the line, where the distance between points and our resulted line will be minimized.

There are two ways of thinking:
1) Calculus way (derivatives) $\to \frac{dE^2}{dx} = 0$
2) LA way (projections) $\to A^TAx = A^Ty$

Least squares (2)

Case study:
$y = kx + t$ $\Rightarrow$ $\begin{cases} t + k\cdot 1 = 1 \\ t + k\cdot 2 = 2 \\ t + k\cdot 3 = 1 \end{cases}$ $\Rightarrow$ $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} t \\ k \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{y}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \end{bmatrix}$$

Lets find errors (subs t and k to $Ax = y$) $\begin{cases} 4/3 + 0\cdot 1 = 1 \\ 4/3 + 0\cdot 2 = 2 \\ 4/3 + 0\cdot 3 = 1 \end{cases}$ $\Rightarrow e = \begin{cases} -1/3 \\ 2/3 \\ -1/3 \end{cases}$

Task 1

4.2 A Project the vector $\boldsymbol{b} = (3, 4, 4)$ onto the line through $\boldsymbol{a} = (2, 2, 1)$ and then onto the plane that also contains $\boldsymbol{a}^* = (1, 0, 0)$. Check that the first error vector $\boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

Find the 3 by 3 projection matrix P onto that plane of $\boldsymbol{a}$ and $\boldsymbol{a}^*$. Find a vector whose projection onto the plane is the zero vector.

Task 1

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Find the 3 by 3 projection matrix P onto that plane of $\boldsymbol{a}$ and $\boldsymbol{a}^*$. Find a vector whose projection onto the plane is the zero vector.

Answer

Solution The projection of $\boldsymbol{b} = (3,4,4)$ onto the line through $\boldsymbol{a} = (2,2,1)$ is $\boldsymbol{p} = 2\boldsymbol{a}$:

Onto a line $$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = \frac{18}{9}(2,2,1) = (4,4,2).$$

The error vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1,0,2)$ is perpendicular to $\boldsymbol{a}$. So $\boldsymbol{p}$ is correct.

The plane of $\boldsymbol{a} = (2,2,1)$ and $\boldsymbol{a}^* = (1,0,0)$ is the column space of $A = [\boldsymbol{a}\ \boldsymbol{a}^*]$:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \quad (A^{\mathrm{T}}A)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

Then $\boldsymbol{p}^* = P\boldsymbol{b} = (3, 4.8, 2.4)$. The error $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^* = (0, -.8, 1.6)$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{a}^*$. This $\boldsymbol{e}^*$ is in the nullspace of P and its projection is zero! Note $P^2 = P$.

Task 2

11 Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

Task 2

11 Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad\text{and}\quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

Answer

(a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2, 3, 0)$, $\boldsymbol{e} = (0, 0, 4)$, $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$ (b) $\boldsymbol{p} = (4, 4, 6)$, $\boldsymbol{e} = \mathbf{0}$.

Task 3

(Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Task 3

(Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Answer

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \boldsymbol{p} = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

Task 4

(a) If P is the 2 by 2 projection matrix onto the line through $(1, 1)$, then $I - P$ is the projection matrix onto ______ .

(b) If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto ______ .

Task 4

(a) If P is the 2 by 2 projection matrix onto the line through $(1, 1)$, then $I - P$ is the projection matrix onto ______ .

(b) If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto ______ .

Answer

(a) $I - P$ is the projection matrix onto $(1, -1)$ in the perpendicular direction to $(1, 1)$
(b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1, 1, 1)$.

Task 5

(a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B = P_C A P_R$. Your answer B should be a little surprising—can you explain it?

Task 5

(a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B = P_C A P_R$. Your answer B should be a little surprising—can you explain it?

Answer

(a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1, 2, 2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ (columns of A project to themselves) and $A P_R = A$. Then $P_C A P_R = A$!

Reference material

- Lecture 15 and 16
- *"Linear Algebra and Applications", pdf pages 181–204*

  Projections onto lines and Least squares
- The Least-Squares Problem

  Video from Matrix Algebra for Engineers course

Deserve “A” grade!

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Room 105 (Underground robotics lab)