# Analytical Geometry and Linear Algebra II, Lab 2

Gaussian elimination recap
A=LU, A=LDV, PA=LU factorization

# Gaussian elimination
*Algorithm*

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

# Gaussian elimination
*Training task*

$$\begin{cases} 2x_1 + 3x_2 - x3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases}$$

$$\begin{cases} 2x_1 + 3x_2 - x3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases} \text{Ans:} \begin{cases} x_1 = 4 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

*Example of A=LU*

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A}$$

# A=LU factorization

*Example of A=LU*

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

# A=LU factorization

*Example of A=LDV*

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}}_{V \text{ or } U}$$

# A=LU factorization
*Formal Idea*

$$A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangluar } U \\ \text{pivots on the diagonal} \end{pmatrix}$$

**Requirements**: No row exchanges as Gaussian elimination reduces square $A$ to $U$

$$A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangluar } U \\ \text{1's on the diagonal} \end{pmatrix}$$

**Requirements**: No row exchanges. The pivots in $D$ are divided out to leave 1's on the diagonal of $U$. If $A$ is symmetric, then $U$ is $L^T$ and $A = LDL^T$.

*Application: Theory*

$$Ax = b \rightarrow$$

$$Ax = b \rightarrow \underbrace{LU}_{A}\ x = b$$

$$Ax = b \rightarrow \underbrace{LU}_{A}\, x = b \rightarrow L\underbrace{c}_{Ux} = b$$

# A=LU factorization
*Application: Theory*

$$Ax = b \rightarrow \underbrace{LU}_{A} x = b \rightarrow L \underbrace{c}_{Ux} = b \rightarrow Ux = c \rightarrow \text{Ans: } x$$

There are two steps:

1. **Factor** (from $A$ find its factors $L$ and $U$)

2. **Solve** (from $L$ and $U$ and $b$ find the solution $x$)

└─A=LU factorization

**A=LU factorization**
*Application: Theory*

$$Ax = b \rightarrow \underbrace{LU}_{A} \, x = b \rightarrow L \underbrace{c}_{Ux} = b \rightarrow Ux = c \rightarrow \text{Ans: } x$$

There are two steps:
1. **Factor** (from $A$ find its factors $L$ and $U$)
2. **Solve** (from $L$ and $U$ and $b$ find the solution $x$)

The best use for A=LU is when you need to solve an equation where the right-hand side depends on the previous answer

# A=LU factorization
*Application: Case Study*

## Description

Try to solve such systems using:

- Gauss-Jordan elimination
- LU factorization

Compare time consumption

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & 10 & 2 \end{bmatrix}, B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

# A=LU factorization
*Application: Case Study*

## Description

Try to solve such systems using:

- Gauss-Jordan elimination
- LU factorization

Compare time consumption

$AX = B$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & 10 & 2 \end{bmatrix}$, $B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$B_{j+1} = B_j + X_j$, for $j = 1, 2$, where $B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Matlab

$A = \begin{bmatrix} \frac{-1}{3} & \frac{16}{29} & 1 \\ \frac{-2}{3} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 10 & 2 \\ 0 & \frac{29}{3} & \frac{13}{3} \\ 0 & 0 & \frac{-21}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -16 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 21 \end{bmatrix}$, $B_2 = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{-4}{7} \end{bmatrix}$, $B_3 = \begin{bmatrix} \frac{73}{49} \\ \frac{109}{147} \\ \frac{-185}{147} \end{bmatrix}$

$L$ $\quad$ $U$ $\qquad\qquad$ $L$ $\qquad\qquad$ $U$

Which number $c$ leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which $c$ produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

Which number $c$ leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which $c$ produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

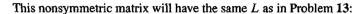$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

## Answer

$c = 2$ leads to zero in the second pivot position: exchange rows and not singular.
$c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

This nonsymmetric matrix will have the same $L$ as in Problem **13**:

**Find $L$ and $U$ for**

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on $a, b, c, d, r, s, t$ to get $A = LU$ with four pivots.

# Task 2

This nonsymmetric matrix will have the same $L$ as in Problem **13**:

**Find $L$ and $U$ for**
$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on $a, b, c, d, r, s, t$ to get $A = LU$ with four pivots.

## Answer

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}. \text{ Need } \begin{array}{l} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{array}$$

# Task 3

If $A$ and $B$ have nonzeros in the positions marked by $x$, which zeros (marked by 0) *stay zero* in their factors $L$ and $U$?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

# Task 3

If $A$ and $B$ have nonzeros in the positions marked by $x$, which zeros (marked by 0) *stay zero* in their factors $L$ and $U$?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

## Answer

For the first matrix $A$, $L$ keeps the 3 lower zeros at the start of rows. But $U$ may not have the upper zero where $A_{24} = 0$. For the second matrix $B$, $L$ keeps the bottom left zero at the start of row 4. $U$ keeps the upper right zero at the start of column 4. One zero in $A$ and two zeros in $B$ are filled in.

# Reference material

- Lecture 4 MIT course

- *"Linear Algebra and Applications", pdf pages 46–86*

- *"Introduction to Linear Algebra", pdf pages 349–374*
  2.2 – The Idea of Elimination, 2.6 – Elimination = Factorization:
  $A = LU$

- This lab video, 2022 year

# Deserve "A" grade!

– Oleg Bulichev

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🚪 Room 105 (Underground robotics lab)