Analytical Geometry and Linear Algebra II, Lab 3

Null space

How to study Null Space

Step-by-step guide

1. Lecture 6, Gilbert Strang

   Goal is to understand the basics of spaces and how Null Space appeared.

2. Khan Academy: Null space

   It contains a good case study how to calculate Null space.

3. Matrix Algebra for Engineers: Null Space

   Another nice example how to find $N(A)$.

4. *"Linear Algebra and Applications", pdf pages 96–106*

   What does partial and full solutions means

5. The Big Picture of Linear Algebra

   Extra for now If you want to get the global view of four subspaces

6. Understand the application from next few slides and make HW tasks!

Null Space: Application from robotics

Video

Null Space: Application from robotics

Theory (1)

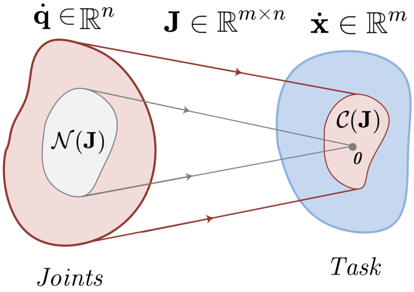


Figure 1: Click for google Collab

Let us consider differential kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{1}$$

where

- $\mathbf{x} \in \mathbb{R}^m$ task space variables (for instance Cartesian coordinates)
- $\mathbf{q} \in \mathbb{R}^n$ joint space variables (positions of joints)
- $\mathbf{J} \in \mathbb{R}^{m\times n}$ manipulator Jacobian

Null Space: Application from robotics

Theory (2)

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = \underbrace{J^{\#}\dot{r}}_{\text{a particular solution (here, the pseudoinverse) in } \mathcal{R}(J^T)} + \underbrace{\left(I - J^{\#}J\right)}_{\text{"orthogonal" projection of } \dot{q}_0 \text{ in } \mathcal{N}(J)}\dot{q}_0$$

$\left(I - J^{\#}J\right)\dot{q}_0$ ⟷ all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

a particular solution (here, the pseudoinverse) in $\mathcal{R}(J^T)$

"orthogonal" projection of $\dot{q}_0$ in $\mathcal{N}(J)$

properties of projector $[I - J^{\#}J]$

- symmetric
- idempotent: $[I - J^{\#}J]^2 = [I - J^{\#}J]$
- $[I - J^{\#}J]^{\#} = [I - J^{\#}J]$
- $J^{\#}\dot{r}$ is orthogonal to $[I - J^{\#}J]\dot{q}_0$

even more in general...

$$\dot{q} = K_1\dot{r} + (I - K_2J)\dot{q}_0$$

... but with less nice properties!

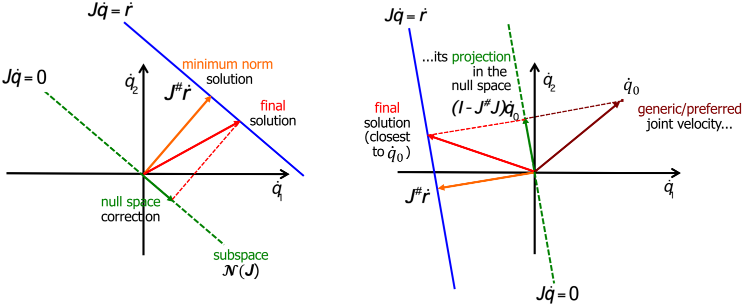
K_1, K_2 generalized inverses of J $(JK_iJ = J)$

how do we choose $\dot{q}_0$?

Null Space: Application from robotics

Theory (3)

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution
i) which is in the null space of the Jacobian
ii) and possibly satisfies additional criteria or objectives

Task 1

Reduce A and B to their triangular echelon forms U. Which variables are free?

(a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$.

Task 1

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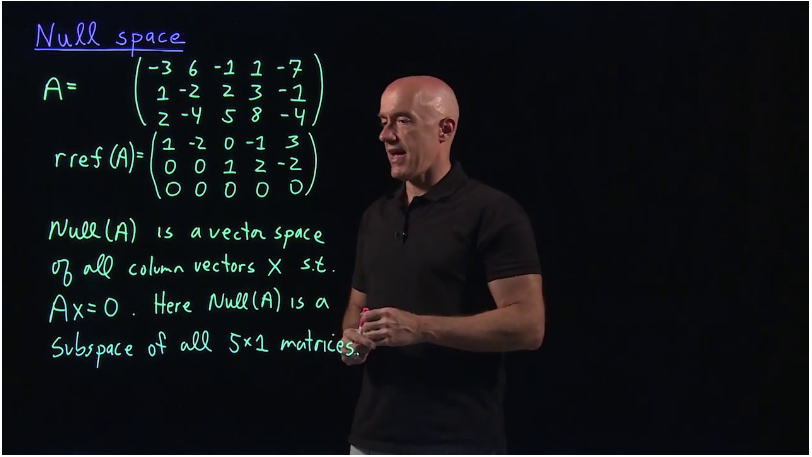
Answer

(a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3

(b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free x_3 Pivot x_1, x_2

Null space

Video



Task 1.5

Find a Nullspace of such matrices.

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Task 1.5

Answer

(a) $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Task 2

Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

Task 2

Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

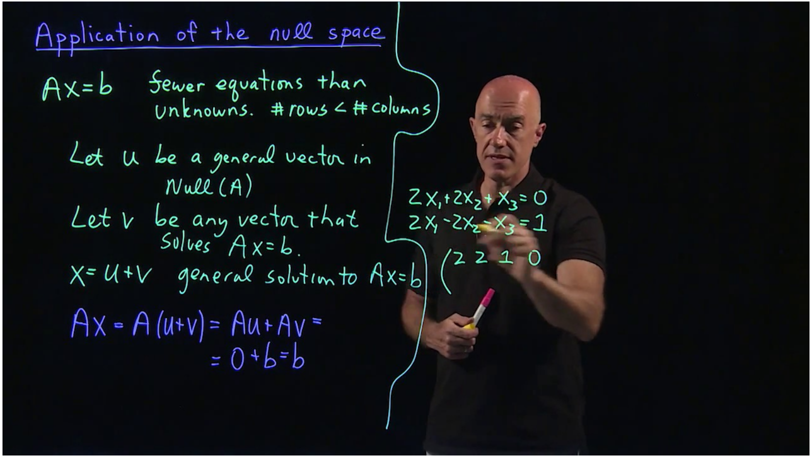
(d) $A = U = 2R$.

Answer

(a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I$, $R = I$.

Underdetermined linear system of equations

Video



Task 3

1. Reduce $[\,A \quad \boldsymbol{b}\,]$ to $[\,U \quad \boldsymbol{c}\,]$, so that $A\boldsymbol{x} = \boldsymbol{b}$ becomes a triangular system $U\boldsymbol{x} = \boldsymbol{c}$.
2. Find the condition on b_1, b_2, b_3 for $A\boldsymbol{x} = \boldsymbol{b}$ to have a solution.
3. Describe the column space of A. Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A. Which special solutions in $\mathbf{R}^4$?
5. Reduce $[\,U \quad c\,]$ to $[\,R \quad \boldsymbol{d}\,]$: Special solutions from R, particular solution from $\boldsymbol{d}$.
6. Find a particular solution and then the complete solution.

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Task 3

Answer

$$\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{s}_2 = (2, -2, 0, 1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ gives the particular solution $x_p = (4, -1, 0, 0)$.

Task 4

If the special solutions to $Rx = \mathbf{0}$ are in the columns of these nullspace matrices N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \text{ (empty 3 by 1).}$$

Task 4

If the special solutions to $Rx = \mathbf{0}$ are in the columns of these nullspace matrices N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \text{ (empty 3 by 1).}$$

Answer

Any zero rows come after these rows: $R = [1 \ \ -2 \ \ -3]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

Reference material

- Robotics 2 course from Sapienza
- *”Linear Algebra and Applications”, pdf pages 87–101*
  2.1-2.2
- *”Introduction to Linear Algebra”, pdf pages 132–152*
  3.1 – 3.2
- this lab video, 2022 year

Preparation material for the next class

- Lecture 9 and 10
- *"Linear Algebra and Applications", pdf pages 139–149*

  The application of four fundamental subspaces in CS
- Matrix Transpose and the Four Fundamental Subspaces

  Video is about how A transpose appeared

Deserve “A” grade!

– Oleg Bulichev

o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)