



Analytical Geometry and Linear Algebra II, Lab 10

Symmetric matrices

Positive definite matrices and minima

Symmetric Matrices (1)



- 1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** q_1, \dots, q_n .
- 2 Every real symmetric S can be diagonalized: $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$
- 3 The number of positive eigenvalues of S equals the number of positive pivots.

Symmetric Matrices (2)



Symmetric matrices S have orthogonal eigenvector matrices Q . Look at this again:

Symmetry $S = X\Lambda X^{-1}$ becomes $S = Q\Lambda Q^T$ with $Q^T Q = I$.

This says that every 2 by 2 symmetric matrix is **(rotation)(stretch)(rotate back)**

$$S = Q\Lambda Q^T = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}. \quad (5)$$

Columns q_1 and q_2 multiply rows $\lambda_1 q_1^T$ and $\lambda_2 q_2^T$ to produce $S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$.

Task 1



Write A as $S + N$, symmetric matrix S plus skew-symmetric matrix N :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \quad (S^T = S \text{ and } N^T = -N).$$

For any square matrix, $S = \frac{1}{2}(A + A^T)$ and $N = \underline{\hspace{2cm}}$ add up to A .



Task 1

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For any square matrix, $S = \frac{1}{2}(A + A^T)$ and $N = \text{_____}$ add up to A .

Answer

$$\begin{aligned} A &= \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \\ &= \text{symmetric} + \text{skew-symmetric}. \end{aligned}$$

Task 2



Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ ?



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Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ ?

Answer

$\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Task 3



True (with reason) *or false* (with example).

- (a) A matrix with real eigenvalues and n real eigenvectors is symmetric.
- (b) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.
- (c) The inverse of an invertible symmetric matrix is symmetric.
- (d) The eigenvector matrix Q of a symmetric matrix is symmetric.



Task 3

True (with reason) *or false* (with example).

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- (d) The eigenvector matrix Q of a symmetric matrix is symmetric.

Answer

- (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^T = Q\Lambda Q^T = A$ (d) False!
- (c) True from $S^{-1} = Q\Lambda^{-1}Q^T$

Positive Definite Matrices



This section concentrates on *symmetric matrices that have positive eigenvalues*. If symmetry makes a matrix important, this extra property (*all $\lambda > 0$*) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called *positive definite*.

Positive Definite Matrices

Applications from ML



- Cholesky decomposition - $A = LDL^H$ (A special case of $A = LU$)
- Least squares computation reduction
- Support Vector Machine (SVM), *kernel* - Positive-definite kernel
- Representer Theorem

Positive Definite Matrices

Five tests



Positive definite matrices are the best. How to test S for $\lambda_i > 0$?

Test 1 Compute the **eigenvalues** of S : All eigenvalues positive

Test 2 The **energy** $x^T S x$ is positive for every vector $x \neq 0$

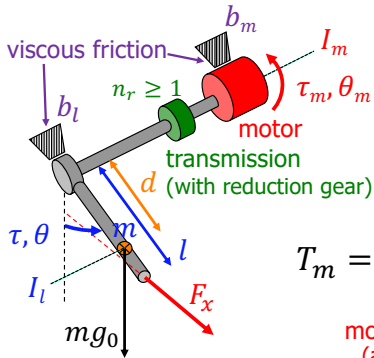
Test 3 The **pivots** in elimination on S are all positive

Test 4 The upper left **determinants** of S are all positive

Test 5 $S = A^T A$ for some matrix A with independent columns

Dynamics of an actuated pendulum

a first example



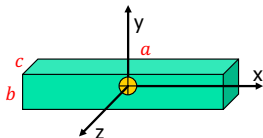
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2$$

↑
motor inertia
(around its
spinning axis)



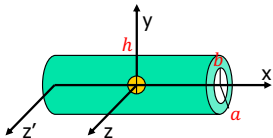
Examples of body inertia matrices

homogeneous bodies of mass m , with axes of symmetry



parallelepiped with sides
 a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2 + c^2) & & \\ & \frac{1}{12}m(a^2 + c^2) & \\ & & \frac{1}{12}m(a^2 + b^2) \end{pmatrix}$$



empty cylinder with length h ,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2 + b^2) & & \\ & \frac{1}{12}m(3(a^2 + b^2) + h^2) & \\ & & I_{zz} \end{pmatrix} \quad I_{zz} = I_{yy}$$



Kinetic energy of a rigid body (cont)

$$= \frac{1}{2} m v_c^T v_c$$

↑
translational
kinetic energy
(point mass
at CoM)

+

rotational
kinetic energy
(of the whole body)

→

$$= \frac{1}{2} \omega^T I_c \omega$$

↑
body inertia matrix
(around the CoM)

Positive Definite Matrices

Example

Test S for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Positive Definite Matrices

Solution: Test 1, 3, 4

The pivots of S are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of S are $2 - \sqrt{2}$ and $2 + \sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**.

Positive Definite Matrices

Solution: Test 5

A_2 comes from $S = LDL^T$ (the symmetric version of $S = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L . **Just put $A_2 = L\sqrt{D}$.**

$$LDL^T = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^T = A_2^T A_2.$$

A_2 is the Cholesky factor of S

Eigenvalues give the symmetric choice $A_3 = Q\sqrt{\Lambda}Q^T$. This is also successful with $A_3^T A_3 = Q\Lambda Q^T = S$. All tests show that the $-1, 2, -1$ matrix S is positive definite.

Positive Definite Matrices

Solution: Test 2



$$\mathbf{x}^T S \mathbf{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

$$\|A_2 \mathbf{x}\|^2 = 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2$$

$$\|A_3 \mathbf{x}\|^2 = \lambda_1(\mathbf{q}_1^T \mathbf{x})^2 + \lambda_2(\mathbf{q}_2^T \mathbf{x})^2 + \lambda_3(\mathbf{q}_3^T \mathbf{x})^2$$

Rewrite with squares

Using $S = LDL^T$

Using $S = Q\Lambda Q^T$



Task 4

What is the function $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write each f as a sum of one or two squares $f = d_1(\quad)^2 + d_2(\quad)^2$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$f = [x \ y] \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Task 4

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$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad f = [x \ y] \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Answer

$$f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2; \quad x^2 + 6xy + 9y^2 = (x + 3y)^2.$$

Task 5



For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$



Task 5

For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Answer

S is positive definite for $c > 1$; determinants c , $c^2 - 1$, and $(c - 1)^2(c + 2) > 0$.

T is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).



Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $\mathbf{x}^T S \mathbf{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad , \quad , \quad).$$



Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $\mathbf{x}^T S \mathbf{x} > 0$:

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Answer

$\mathbf{x}^T S \mathbf{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\mathbf{x}^T S \mathbf{x}$ goes *negative* for $\mathbf{x} = (1, -10, 0)$ because the second pivot is *negative*.

Task 7



Without multiplying $S = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, find

- (a) the determinant of S
- (b) the eigenvalues of S
- (c) the eigenvectors of S
- (d) a reason why S is symmetric positive definite.



Task 7

Without multiplying $S = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, find

- (a) the determinant of S
- (b) the eigenvalues of S
- (c) the eigenvectors of S
- (d) a reason why S is symmetric positive definite.

Answer

$\det S = (1)(10)(1) = 10$; $\lambda = 2$ and 5 ; $\mathbf{x}_1 = (\cos \theta, \sin \theta)$, $\mathbf{x}_2 = (-\sin \theta, \cos \theta)$; the λ 's are positive. So S is positive definite.

Positive Definite Matrices

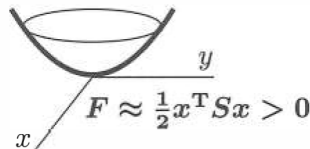
Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S . It contains *four second derivatives*. **Positive d^2f/dx^2 changes to positive definite S :**

**Second
derivatives**

$$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x\partial y \\ \partial^2 F/\partial y\partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$



$F(x, y)$ has a minimum if $\partial F/\partial x = \partial F/\partial y = 0$ and S is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

Task 8

For $F_1(x, y) = x^4/4 + x^2 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$, find the second-derivative matrices H_1 and H_2 :

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}.$$

H_1 is positive-definite so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where the first derivatives are zero).



Task 8

Answer: Subtask 1

For $F_1(x, y)$, we first solve for the stationary point

$$\frac{\partial F_1}{\partial x} = x^3 + 2x + 2xy = 0 \quad (1) \quad , \quad \frac{\partial F_1}{\partial y} = x^2 + 2y = 0 \quad (2)$$

From (2), we have $y = -x^2/2$. Plug this into (1), we have $2x = 0$ and hence the only critical point is $x = y = 0$. At this point,

$$H_1 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 3x^3 + 2 + 2y & 2x \\ 2x & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is positive definite and hence $(0, 0)$ is a minimal point of $F_1(x, y)$.



Task 8

Answer: Subtask 2

For $F_2(x, y)$, we first solve for the stationary point

$$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1 = 0, \frac{\partial F_2}{\partial y} = x = 0$$

This implies that $y = 1$. At this point $(0, 1)$,

$$H_2 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of H_2 at $(0, 1)$ is the solution to $\det(H_2 - \lambda I) = \lambda^2 - 1$, which are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are with opposite signs and hence $(0, 1)$ is a saddle point of $F_2(x, y)$.



Reference material

- Lecture 28, Positive Definite Matrices and Minima
- 5. Positive Definite and Semidefinite Matrices
- *"Introduction to Linear Algebra"*, pdf pages 349–374
6.4 – Symmetric, 6.5 – Positive Definite matrices
- *"Linear Algebra and Applications"*, pdf pages 355–376
Positive Definite Matrices 6.1, 6.2

Deserve "A" grade!

– Oleg Bulichev

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🏢 Room 105 (Underground robotics lab)