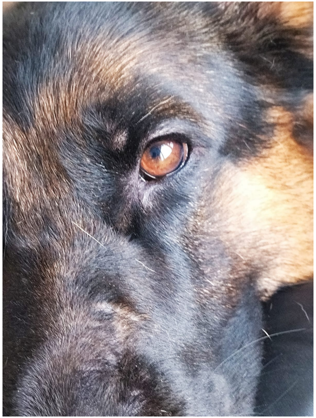
Analytical Geometry and Linear Algebra II, Lab 10

Symmetric matrices

Positive definite matrices and minima

Innopolis University

How I spent last weekend



Symmetric Matrices (1)

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q\Lambda Q^{\mathrm{T}}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

Symmetric Matrices (2)

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

Symmetry $\quad S = X\Lambda X^{-1}$ becomes $S = Q\Lambda Q^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$.

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

Columns q_1 and q_2 multiply rows $\lambda_1 q_1^{\mathrm{T}}$ and $\lambda_2 q_2^{\mathrm{T}}$ to produce $S = \lambda_1 q_1 q_1^{\mathrm{T}} + \lambda_2 q_2 q_2^{\mathrm{T}}$.

Task 1

Write A as $S + N$, symmetric matrix S plus skew-symmetric matrix N:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \qquad (S^{\mathrm{T}} = S \text{ and } N^{\mathrm{T}} = -N).$$

For any square matrix, $S = \frac{1}{2}(A + A^{\mathrm{T}})$ and $N =$ ____ add up to A.

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Answer

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \begin{aligned} &= \tfrac{1}{2}(A + A^{\mathrm{T}}) + \tfrac{1}{2}(A - A^{\mathrm{T}}) \\ &= \textbf{symmetric} + \textbf{skew-symmetric}. \end{aligned}$$

Task 2

Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

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Answer

$\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Task 3

True (with reason) ***or false*** (with example).

(a) A matrix with real eigenvalues and n real eigenvectors is symmetric.

(b) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.

(c) The inverse of an invertible symmetric matrix is symmetric.

(d) The eigenvector matrix Q of a symmetric matrix is symmetric.

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Answer

(a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$

(c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$

(d) False!

Positive Definite Matrices

Positive Definite Matrices

Applications from ML

- Cholesky decomposition – $A = LL^H$ (A special case of $A = LU$)
- Least squares computation reduction
- Support Vector Machine (SVM), *kernel* – Positive-definite kernel
- Representer Theorem

Positive Definite Matrices

Five tests

Positive definite matrices are the best. How to test S for $\lambda_i > 0$?

Test **1**	Compute the **eigenvalues** of S: All eigenvalues positive
Test **2**	The **energy** $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$ is positive for every vector $\boldsymbol{x} \neq \mathbf{0}$
Test **3**	The **pivots** in elimination on S are all positive
Test **4**	The upper left **determinants** of S are all positive
Test **5**	$\boldsymbol{S} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ for some matrix A with independent columns

Positive Definite Matrices

Applications from Robotics

- Matrix form of Lagrange

Positive Definite Matrices

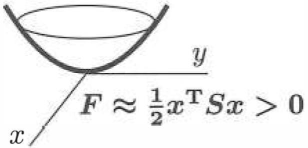
Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S. It contains *four second derivatives*. **Positive d^2f/dx^2 changes to positive definite S**:

Second derivatives

$$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$



$F(x, y)$ has a minimum if $\partial F/\partial x = \partial F/\partial y = 0$ and S is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

Reference material

- Lecture 28, Positive Definite Matrices and Minima
- *"Introduction to Linear Algebra", pdf pages 349–374*

  6.4 – Symmetric, 6.5 – Positive Definite matrices
- *"Linear Algebra and Applications", pdf pages 355–376*

  Positive Definite Matrices 6.1, 6.2

Deserve “A” grade!

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Room 105 (Underground robotics lab)