1.1 (3 points) Let $S_1 = \{x, y, z : x - 2y - z = 8\}$ and $S_2 = \{x, y, z : x - y + 2z = 2\}$.

Find the line of intersection of the planes S_1 and S_2 in vector form: $\vec{v} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$

$$\begin{cases} x - 2y - z = 8 \\ x - y + 2z = 2 \end{cases} \Rightarrow \begin{cases} x = -4 - 5z \\ y = -3(2 + z) \end{cases} \Rightarrow \vec{v} = -\begin{bmatrix} 4 + 5z \\ 3(2 + z) \\ z \end{bmatrix} = -z\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$$

1.2 (3 points) Let $S_1 = \{x, y, z : x + y - 4z = 8\}$ and $S_2 = \{x, y, z : x + 2y + 2z = 2\}$..

Find the line of intersection of the planes S_1 and S_2 in vector form: $\vec{v} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$

$$\begin{cases} x + y - 4z = 8 \\ x + 2y + 2z = 2 \end{cases} \Rightarrow \begin{cases} x = 2(7 + 5z) \\ y = -6(1 + z) \end{cases} \Rightarrow \vec{v} = \begin{bmatrix} 2(7 + 5z) \\ -6(1 + z) \\ z \end{bmatrix} = z\begin{bmatrix} 10 \\ -6 \\ 1 \end{bmatrix} + \begin{bmatrix} 14 \\ -6 \\ 0 \end{bmatrix}$$

2.1 Considering the matrix, A and the vector $\vec{b}$,

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

(a) (4 points) Find the projection of b onto the column space of A.

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\left[\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}^T\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\right]^{-1}\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}^T\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

(b) (1 points) Split $\vec{b}$ into $\vec{p} + \vec{e}$, with p in the column space and e orthogonal to that space.

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}$$

(c) (1 point) Which of the four fundamental spaces of A contains e. $\vec{e} \in N(A^T)$.

2.2 Considering the matrix, A and the vector $\vec{b}$,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

(a) (4 points) Find the projection of b onto the column space of A.

$$\vec{p} = P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}\left[\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}^T\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}^T\begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(b) (1 points) Split $\vec{b}$ into $\vec{p} + \vec{e}$, with p in the column space and e orthogonal to that space.

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 3 \\ 2 \\ -4 \end{bmatrix}$$

(c) (1 point) Which of the four fundamental spaces of A contains e. $\vec{e} \in N(A^T)$.

3.1 Considering the following measurements:

t	-2	-1	1	2
b	-1	0	1	1

(a) (4 points) Find the best straight-line fit (Least squares) to the measurements,

$$A=\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}.\ \text{We get a system of equations: }\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\Rightarrow$$

$$\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\Leftrightarrow\begin{bmatrix}4&0\\0&10\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow$$

The best straight-line fit to the measurements is $y(t)=\frac{1}{4}+\frac{1}{2}t$

(b) (2 point) Find the projection matrix of vector $b=[-2,0,4,6]^T$ onto the column space of matrix A:

$$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}\left[\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}\right]^{-1}\begin{bmatrix}1&-2\\1&-1\\1&1\\1&2\end{bmatrix}^T=\frac{1}{20}\begin{bmatrix}13&9&1&-3\\9&7&3&1\\1&3&7&9\\-3&1&9&13\end{bmatrix}$$

3.2 Considering the following measurements:

t	-1	0	1	2
b	-2	0	1	1

(a) (4 points) Find the best straight-line fit (Least squares) to the measurements,

$$A=\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}.\ \text{We get a system of equations: }\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\0\\1\\1\end{bmatrix}\Rightarrow$$

$$\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}-2\\0\\1\\1\end{bmatrix}\Leftrightarrow\begin{bmatrix}4&2\\2&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\5\end{bmatrix}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{1}{10}\begin{bmatrix}1\\-7\end{bmatrix}\Rightarrow$$

The best straight-line fit to the measurements is $y(t)=-\frac{1}{2}+t$

(b) (2 point) Find the projection matrix of vector $b=[1,-1,1,-2]^T$ onto the column space of matrix A:

$$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\left[\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}^T\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\right]^{-1}\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}^T=\frac{1}{10}\begin{bmatrix}7&4&1&-2\\4&3&2&1\\1&2&3&4\\-2&1&4&7\end{bmatrix}$$

4.1 (6 points) Find the dimensions of the four fundamental subspaces associated with A, depending on the parameters α and β.

$$A = \begin{bmatrix} \alpha & 7 & 5 & 10 \\ 3 & 1 & \beta & 4 \\ 2 & 1 & 1 & 2 \\ 8 & 5 & 3 & 6 \end{bmatrix} = [\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4] \implies$$

$$\text{1) If } \alpha = 12, \text{ then } \vec{a}_1 = \vec{a}_2 + \tfrac{1}{2}\vec{a}_4 \text{ and } \operatorname{rank}(A) = 3 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 3) \\ \operatorname{Dim}[C(A^T)] = (3 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 1) \\ \operatorname{Dim} N(A^T) = (4 \times 1) \end{cases}$$

$$\text{2) If } \beta = 2, \text{ then } \vec{a}_2 = \tfrac{1}{2}\vec{a}_4 \text{ and } \operatorname{rank}(A) = 3 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 3) \\ \operatorname{Dim}[C(A^T)] = (3 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 1) \\ \operatorname{Dim} N(A^T) = (4 \times 1) \end{cases}$$

$$\text{3) If } \alpha = 12 \text{ and } \beta = 2, \text{ then } \operatorname{rank}(A) = 2 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 2) \\ \operatorname{Dim}[C(A^T)] = (2 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 2) \\ \operatorname{Dim} N(A^T) = (4 \times 2) \end{cases}$$

4.2 (6 points) Find the dimensions of the four fundamental subspaces associated with A, depending on the parameters α and β.

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ \alpha & 8 & 6 & 12 \\ 1 & 1 & \beta & 0 \\ 9 & 5 & 4 & 8 \end{bmatrix} \implies$$

$$\text{1) If } \alpha = 14, \text{ then } \vec{a}_1 = \vec{a}_2 + \tfrac{1}{2}\vec{a}_4 \text{ and } \operatorname{rank}(A) = 3 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 3) \\ \operatorname{Dim}[C(A^T)] = (3 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 1) \\ \operatorname{Dim} N(A^T) = (4 \times 1) \end{cases}$$

$$\text{2) If } \beta = 0, \text{ then } \vec{a}_3 = \tfrac{1}{2}\vec{a}_4 \text{ and } \operatorname{rank}(A) = 3 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 3) \\ \operatorname{Dim}[C(A^T)] = (3 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 1) \\ \operatorname{Dim} N(A^T) = (4 \times 1) \end{cases}$$

$$\text{3) If } \alpha = 14 \text{ and } \beta = 0, \text{ then } \operatorname{rank}(A) = 2 \text{ and } \begin{cases} \operatorname{Dim}[C(A)] = (4 \times 2) \\ \operatorname{Dim}[C(A^T)] = (2 \times 4) \\ \operatorname{Dim} N(A) = (4 \times 2) \\ \operatorname{Dim} N(A^T) = (4 \times 2) \end{cases}$$

5.1 Find an orthonormal basis for the subspace spanned by the vectors: a_1, a_2 and a_3 (4 points).

Then express $A = [a_1, a_2, a_3]$ in the form of QR (2 points).

$$a_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, a_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}.$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \qquad q_2 = \frac{a_2 - q_1^T a_2}{\|a_2 - q_1^T a_2\|} = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \qquad q_3 = \frac{a_3 - q_1^T a_3 - q_2^T a_3}{\|a_3 - q_1^T a_3 - q_2^T a_3\|} = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & -2 & 1 \\ 2 & -1 & -4 \\ -1 & 2 & -1 \end{bmatrix} = QR = \begin{bmatrix} -2 & -2 & -1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.2 Find an orthonormal basis for the subspace spanned by the vectors: a_1, a_2 and a_3 (4 points).

Then express $A = [a_1, a_2, a_3]$ in the form of QR (2 points).

$$a_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \qquad q_2 = \frac{a_2 - q_1^T a_2}{\|a_2 - q_1^T a_2\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \qquad q_3 = \frac{a_3 - q_1^T a_3 - q_2^T a_3}{\|a_3 - q_1^T a_3 - q_2^T a_3\|} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 4 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & 2 \end{bmatrix} = QR = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.1 (4 points) Prove that the matrix $A(m \times n)$ has the same nullspace as $A^T A$:

Supposing that $A^T Ax = 0$, and taking the inner product with x to show that $Ax = 0$: $x^T A^T Ax = 0$, or $||Ax||^2 = 0$, or $Ax = 0$. So, if vectors x in the nullspace of $A^T A$ are also in the nullspace of A.

6.2 (4 points) Prove that the matrix $A^T(n \times m)$ has the same nullspace as AA^T:

Supposing that $AA^T x = 0$, and taking the inner product with x to show that $A^T x = 0$: $x^T AA^T x = 0$, or $||A^T x||^2 = 0$, or $A^T x = 0$. So, if vectors x in the nullspace of AA^T are also in the nullspace of A^T.

Если студенты в 6 задаче строят доказательство в обратном порядка, а именно:

1. if $Ax = 0$ then $A^T Ax = 0$. Vectors x in the nullspace of A are also in the nullspace of $A^T A$.
2. if $A^T x = 0$ then $AA^T x = 0$. Vectors x in the nullspace of A^T are also in the nullspace of AA^T.

Тогда снижайте оценку до 2 баллов вместо 4!

7 (4 points) Prove that if the matrix $A(n \times n)$ is symmetric the dot product of the vectors $x \in \mathbb{R}^n$ and Ay is the same as the vectors Ax and $y \in \mathbb{R}^n$:

$$A = A^T \quad \Rightarrow \quad x^T Ay = (Ay)^T x = y^T A^T x = y^T Ax.$$