Analytical Geometry and Linear Algebra II, Lab 7

Complex numbers

Complex matrices

Hermitian and Unitary Matrices

Innopolis University

Complex numbers

Forms

Rectangular form: $z = x + iy$,

$Re(z) = x$ – real part, $Im(z) = y$ – imaginary part

Example: $z = 5 + i6$

Polar form: $z = r\,cos(\phi]) + i\,r\,sin(\phi)$, where

$\theta = atan2(Im(z), Re(z))$;

$r = |z| = \sqrt{x^2 + y^2}$ – magnitude

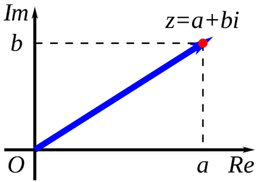
Example: $z = 8\,cos(24) + i\,sin(24))$

Exponential form: $z = re^{i\phi}$

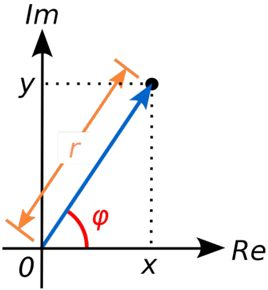
Example: $z = 6e^{i2.5}$

Euler formula: *transformation from exp. to polar*

$e^{i\phi} = cos(\phi) + i\,sin(\phi)$



Rectangular form



Polar or Exponential form

Complex numbers

Operations

General Idea: you should work with *Im* and *Re* part separately (you cannot sum or multiply them)

- *Summarization and Substraction* – $(x_1 \pm iy_1) + (x_2 \pm iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- *Multiplication* – $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1x_2 + y_1y_2)$
- *Division* – $\dfrac{(x_1 + iy_1)}{(x_2 + iy_2)} = \dfrac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\dfrac{(y_1y_2 - x_1x_2)}{x_2^2 + y_2^2}$

Complex numbers

Complex conjugate

Complex conjugate of complex number $z = x + iy$ - $\bar{z} = x - iy$.

Geometrically it's reflection of z about *Re* axis.

Properties:

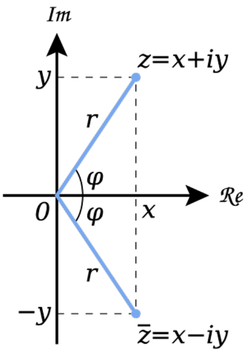
$Re(\bar{z}) = Re(z))$ and $|\bar{z}| = |z|$;

$Im(\bar{z}) = -Im(z)$ and $arg\ \bar{z} \equiv -arg\ z (mod\ 2\pi)$

$z\bar{z} = x^2 + y^2 = |z|^2$ – absolute square

Operations

- *Summarization and Substraction* – $\overline{z \pm w} = \bar{z} \pm \bar{w}$
- *Multiplication* – $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- *Division* – $\overline{z/w} = \bar{z}/\bar{w}$



Complex Conjugate

Task 1

1. For $z = \dfrac{1+i}{\sqrt{2}}$:
   - Compute z^2
   - Find r
   - Find ϕ
   - Find z in exponential form
2. - Find the 8 solutions to the equation $z^8 = 1$
   - Plot those 8 solutions in the complex plane
3. For $z = -1 + i\frac{1}{2}$
   - Find complex conjugate ($\bar{z}$)
   - Find $z\bar{z}$
   - Find $z + \bar{z}$
   - Plot each result in the complex plane
4. Find:
   - $e^{i\frac{\pi}{2}}$
   - $e^{i\pi}$
   - i^i
   - Show each result in the complex plane

Task 1

1 For $z = \dfrac{1+i}{\sqrt{2}}$:

- Compute z^2
- Find r
- Find ϕ
- Find z in exponential form

Answer

Fix Using matlab

Solution (rus)

Task 1

2 – Find the 8 solutions to the equation $z^8 = 1$

– Plot those 8 solutions in the complex plane

Answer

Fix Using matlab

Task 1

3 For $z = -1 + i\frac{1}{2}$

- Find complex conjugate $(\bar{z})$
- Find $z\bar{z}$
- Find $z + \bar{z}$
- Plot each result in the complex plane

Answer

Fix Using matlab

Task 1

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$
- i^i
- Show each result in the complex plane

Answer

Fix Using matlab

Complex Matrices

Common and Special

Many concepts have a new name, but old meaning:

- $A^T \to A^H$; $A^H = \bar{A}^T$, H – conjugate T; *Exp:* $\begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$
- $Q \to U$; $U = Q$, where U – Unitary matrix

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|x\|^2 = \lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}}y = \bar{x}_1y_1 + \cdots + \bar{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $\lvert\lambda\rvert = 1$

Task 2

Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

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Answer

$$A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix} \text{ and } AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ are Hermitian matrices.}$$

Task 3

Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $\boldsymbol{C}(A^{\mathrm{T}})$. Now it is $\boldsymbol{C}(A^{\mathrm{H}})$.

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Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $\boldsymbol{C}(A^{\mathrm{T}})$. Now it is $\boldsymbol{C}(A^{\mathrm{H}})$.

Answer

z = multiple of $(1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^{\mathrm{H}}A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so z (not $\overline{z}$!) is orthogonal to all columns of A^{H} (using complex inner product z^{H} times columns of A^{H}).

Task 4

If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is symmetric.

Task 4

If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

Answer

We are given $A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. So that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

Reference material

- Lecture 26
- *”Linear Algebra and Applications”, pdf pages 322–335*
  Complex numbers and matrices
- *”Introduction to Linear Algebra”, pdf pages 504–519*
  Complex numbers and matrices

Deserve “A” grade!

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Room 105 (Underground robotics lab)