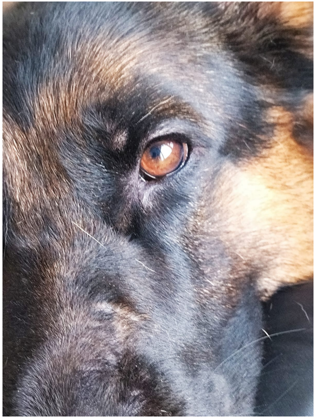
Analytical Geometry and Linear Algebra II, Lab 11

Symmetric matrices

Positive definite matrices and minima

Innopolis University

How I spent last weekend



Symmetric Matrices (1)

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q\Lambda Q^{\mathrm{T}}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

Symmetric Matrices (2)

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

Symmetry $\quad S = X\Lambda X^{-1}$ becomes $S = Q\Lambda Q^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$.

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

Columns q_1 and q_2 multiply rows $\lambda_1 q_1^{\mathrm{T}}$ and $\lambda_2 q_2^{\mathrm{T}}$ to produce $S = \lambda_1 q_1 q_1^{\mathrm{T}} + \lambda_2 q_2 q_2^{\mathrm{T}}$.

Task 1

Write A as $S+N$, symmetric matrix S plus skew-symmetric matrix N:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \qquad (S^{\mathrm{T}} = S \text{ and } N^{\mathrm{T}} = -N).$$

For any square matrix, $S = \frac{1}{2}(A + A^{\mathrm{T}})$ and $N =$ ____ add up to A.

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Answer

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \begin{aligned} &= \tfrac{1}{2}(A + A^{\mathrm{T}}) + \tfrac{1}{2}(A - A^{\mathrm{T}}) \\ &= \textbf{symmetric} + \textbf{skew-symmetric}. \end{aligned}$$

Task 2

Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -\mathbf{2} & \mathbf{6} \\ \mathbf{6} & \mathbf{7} \end{bmatrix}$. What is Λ?

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Answer

$\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Task 3

True (with reason) ***or false*** (with example).

(a) A matrix with real eigenvalues and n real eigenvectors is symmetric.

(b) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.

(c) The inverse of an invertible symmetric matrix is symmetric.

(d) The eigenvector matrix Q of a symmetric matrix is symmetric.

Task 3

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Answer

(a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$

(c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$

(d) False!

Positive Definite Matrices

This section concentrates on ***symmetric matrices that have positive eigenvalues***. If symmetry makes a matrix important, this extra property (*all* $\lambda > 0$) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called ***positive definite***.

Positive Definite Matrices

Applications from ML

- Cholesky decomposition – $A = LDL^H$ (A special case of $A = LU$)
- Least squares computation reduction
- Support Vector Machine (SVM), *kernel* – Positive-definite kernel
- Representer Theorem

Positive Definite Matrices

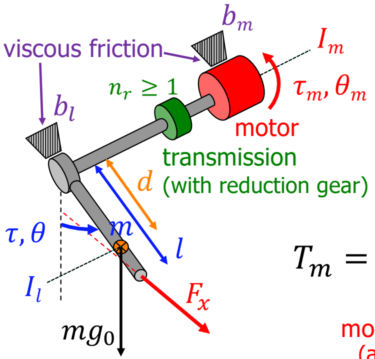
Five tests

Positive definite matrices are the best. How to test S for $\lambda_i > 0$?

Test **1**	Compute the **eigenvalues** of S : All eigenvalues positive
Test **2**	The **energy** $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x}$ is positive for every vector $\boldsymbol{x} \neq \mathbf{0}$
Test **3**	The **pivots** in elimination on S are all positive
Test **4**	The upper left **determinants** of S are all positive
Test **5**	$\boldsymbol{S} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ for some matrix A with independent columns

Dynamics of an actuated pendulum

a first example

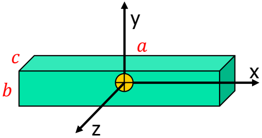


$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2$$

motor inertia
(around its
spinning axis)

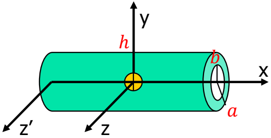
Examples of body inertia matrices

homogeneous bodies of mass m, with axes of symmetry



parallelepiped with sides
a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$



empty cylinder with length h,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)+h^2) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

Kinetic energy of a rigid body (cont)

$$\boxed{= \frac{1}{2} m\, v_c^T v_c}$$

translational
kinetic energy
(point mass
at CoM)

+

rotational
kinetic energy
(of the whole body)

$$\boxed{= \frac{1}{2} \omega^T\, I_c\, \omega}$$

body inertia matrix
(around the CoM)

Positive Definite Matrices

Example

Test S for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Positive Definite Matrices

Solution: Test 1, 3, 4

The pivots of S are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of S are $2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**.

Positive Definite Matrices

Solution: Test 5

A_2 comes from $S = LDL^{\mathrm{T}}$ (the symmetric version of $S = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $\boldsymbol{A_2 = L\sqrt{D}}$.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = A_2^{\mathrm{T}} A_2.$$

A_2 is the Cholesky factor of S

Eigenvalues give the symmetric choice $\boldsymbol{A_3 = Q\sqrt{\Lambda}Q^{\mathrm{T}}}$. This is also successful with $A_3^{\mathrm{T}} A_3 = Q\Lambda Q^{\mathrm{T}} = S$. All tests show that the $-1, 2, -1$ matrix S is positive definite.

Positive Definite Matrices

Solution: Test 2

$\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$	**Rewrite with squares**
$\lvert\lvert A_2 \boldsymbol{x} \rvert\rvert^2 = 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}\, x_3^2$	**Using $\boldsymbol{S} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$**
$\lvert\lvert A_3 \boldsymbol{x} \rvert\rvert^2 = \lambda_1 (\boldsymbol{q}_1^{\mathrm{T}} \boldsymbol{x})^2 + \lambda_2 (\boldsymbol{q}_2^{\mathrm{T}} \boldsymbol{x})^2 + \lambda_3 (\boldsymbol{q}_3^{\mathrm{T}} \boldsymbol{x})^2$	**Using $\boldsymbol{S} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{\mathrm{T}}$**

Task 4

What is the function $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write each f as a sum of one or two squares $f = d_1(\quad)^2 + d_2(\quad)^2$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Answer

$$f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2;\ \ x^2 + 6xy + 9y^2 = (x + 3y)^2.$$

Task 5

For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Task 5

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$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Answer

S is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$.

T is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

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Answer

$\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

Task 7

Without multiplying $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of S

(b) the eigenvalues of S

(c) the eigenvectors of S

(d) a reason why S is symmetric positive definite.

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(a) the determinant of S

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Answer

$\det S = (1)(10)(1) = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive. So S is positive definite.

Positive Definite Matrices

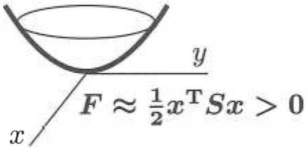
Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S. It contains *four second derivatives*. **Positive d^2f/dx^2 changes to positive definite S**:

Second derivatives

$$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$



$F(x, y)$ has a minimum if $\partial F/\partial x = \partial F/\partial y = 0$ and S is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

Task 8

For $F_1(x,y) = x^4/4 + x^2 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$, find the second-derivative matrices H_1 and H_2:

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}.$$

H_1 is positive-definite so F_1 is concave up (= convex). Find the minimum point of F_1 and the the saddle point of F_2 (look only where the first derivatives are zero).

Task 8

Answer: Subtask 1

For $F_1(x, y)$, we first solve for the stationary point

$$\frac{\partial F_1}{\partial x} = x^3 + 2x + 2xy = 0 \ (1) \quad , \frac{\partial F_1}{\partial y} = x^2 + 2y = 0 \ (2)$$

From (2), we have $y = -x^2/2$. Plug this into (1), we have $2x = 0$ and hence the only critical point is $x = y = 0$. At this point,

$$H_1 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 3x^3 + 2 + 2y & 2x \\ 2x & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is positive definite and hence $(0, 0)$ is a minimal point of $F_1(x, y)$.

Task 8

Answer: Subtask 2

For $F_2(x, y)$, we first solve for the stationary point

$$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1 = 0, \frac{\partial F_2}{\partial y} = x = 0$$

This implies that $y = 1$. At this point $(0, 1)$,

$$H_2 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of H_2 at $(0, 1)$ is the solution to $\det(H_2 - \lambda I) = \lambda^2 - 1$, which are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are with opposite signs and hence $(0, 1)$ is a saddle point of $F_2(x, y)$.

Reference material

- Lecture 28, Positive Definite Matrices and Minima
- 5. Positive Definite and Semidefinite Matrices
- *"Introduction to Linear Algebra", pdf pages 349–374*
  6.4 – Symmetric, 6.5 – Positive Definite matrices
- *"Linear Algebra and Applications", pdf pages 355–376*
  Positive Definite Matrices 6.1, 6.2

Deserve “A” grade!

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