



# Gaussian elimination

*Algorithm*

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$



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$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

# Gaussian elimination

*Training task*

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases}$$



# Gaussian elimination

*Training task*

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases} \quad \text{Ans: } \begin{cases} x_1 = 4 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

# A=LU factorization

*Example of  $A=LU$*



$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_A$$

# A=LU factorization

*Example of A=LU*

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_A \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_U$$

# A=LU factorization

*Example of A=LDV*

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_A \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}}_{V \text{ or } U}$$



# A=LU factorization

*How to convert LU to LDV*



Split  $U$  into

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

## A=LU factorization

*Formal Idea*

$$A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$$

**Requirements:** No row exchanges as Gaussian elimination reduces square  $A$  to  $U$

$$A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$$

**Requirements:** No row exchanges. The pivots in  $D$  are divided out to leave 1's on the diagonal of  $U$ . If  $A$  is symmetric, then  $U$  is  $L^T$  and  $A = LDL^T$ .

# A=LU factorization

*Application: Theory*



$$Ax = b \rightarrow$$

# A=LU factorization

*Application: Theory*

$$Ax = b \rightarrow \underbrace{LU}_A x = b$$

# A=LU factorization

*Application: Theory*



$$Ax = b \rightarrow \underbrace{LU}_A x = b \rightarrow L \underbrace{c}_{Ux} = b$$

# A=LU factorization

*Application: Theory*



$$Ax = b \rightarrow \underbrace{LU}_A x = b \rightarrow L \underbrace{c}_{Ux} = b \rightarrow Ux = c \rightarrow \text{Ans: } x$$

There are two steps:

1. **Factor** (from  $A$  find its factors  $L$  and  $U$ )
2. **Solve** (from  $L$  and  $U$  and  $b$  find the solution  $x$ )

# A=LU factorization

*Application: Case Study*

## Description

Try to solve such systems using:

- Gauss-Jordan elimination
- LU factorization

Compare time consumption

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & 10 & 2 \end{bmatrix}, B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

# A=LU factorization

*Application: Case Study*

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$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \overbrace{\begin{bmatrix} -1 & 16 & 1 \\ 3 & 29 & 0 \\ -2 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 10 & 2 \\ 0 & 29 & 13 \\ 0 & 3 & -21 \\ 0 & 0 & 2 \end{bmatrix}}^{\text{Matlab}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -16 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 21 \end{bmatrix}}_U, B_2 = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{-4}{7} \end{bmatrix}, B_3 = \begin{bmatrix} \frac{73}{147} \\ \frac{49}{109} \\ \frac{147}{-185} \\ \frac{-147}{147} \end{bmatrix}$$



## Task 1



Which number  $c$  leads to zero in the second pivot position? A row exchange is needed and  $A = LU$  will not be possible. Which  $c$  produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

## Task 1



Which number  $c$  leads to zero in the second pivot position? A row exchange is needed and  $A = LU$  will not be possible. Which  $c$  produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

## Answer

$c = 2$  leads to zero in the second pivot position: exchange rows and not singular.  
 $c = 1$  leads to zero in the third pivot position. In this case the matrix is *singular*.

## Task 2



This nonsymmetric matrix will have the same  $L$  as in Problem 13:

**Find  $L$  and  $U$  for**

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on  $a, b, c, d, r, s, t$  to get  $A = LU$  with four pivots.

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Answer

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}. \text{ Need } \begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$$

## Task 3



If  $A$  and  $B$  have nonzeros in the positions marked by  $x$ , which zeros (marked by 0) stay zero in their factors  $L$  and  $U$ ?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

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## Answer

For the first matrix  $A$ ,  $L$  keeps the 3 lower zeros at the start of rows. But  $U$  may not have the upper zero where  $A_{24} = 0$ . For the second matrix  $B$ ,  $L$  keeps the bottom left zero at the start of row 4.  $U$  keeps the upper right zero at the start of column 4. One zero in  $A$  and two zeros in  $B$  are filled in.



## Reference material

- Lecture 4 MIT course
- *"Linear Algebra and Applications"*, pdf pages 46–86
- *"Introduction to Linear Algebra"*, pdf pages 42–134  
2.2 – The Idea of Elimination, 2.6 – Elimination = Factorization:  
 $A = LU$
- This lab video, 2022 year

# Preparation material for the next class



## 1. Lecture 6, Gilbert Strang

**Goal** is to understand the basics of spaces and how Null Space appeared.

## 2. Khan Academy: Null space

It contains a good case study how to calculate Null space.

## 3. Matrix Algebra for Engineers: Null Space

Another nice example how to find  $N(A)$ .

## 4. "Linear Algebra and Applications", pdf pages 96–106

What does partial and full solutions means

## 5. The Big Picture of Linear Algebra

*Extra for now* If you want to get the global view of four subspaces

## 6. Understand the application from next few slides and make HW tasks!



# Deserve "A" grade!

– Oleg Bulichev

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🏢 Room 105 (Underground robotics lab)