1. Diagonalize matrix A by finding its eigenvalues and its eigenvectors. Find A inverse.

$$A = \begin{bmatrix} 7 & -2+2i \\ -2-2i & 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{27}\begin{bmatrix} 5 & 2-2i \\ 2+2i & 7 \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & -1+i \\ -1-i & 7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{54}\begin{bmatrix} 7 & 1-i \\ 1+i & 4 \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/6 & 0 \\ 0 & 1/9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}$$

2. Show that A and B are similar $B = M^{-1}AM$ by finding matrix M:

$$A = \begin{bmatrix} 7 & -2+2i \\ -2-2i & 5 \end{bmatrix} \quad B = \begin{bmatrix} 6-i & 5-5i \\ 1+i & 6+i \end{bmatrix}$$

$$B = \begin{bmatrix} 6-i & 5-5i \\ 1+i & 6+i \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 7 & -2+2i \\ -2-2i & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & -1+i \\ -1-i & 7 \end{bmatrix} \quad B = \begin{bmatrix} 7-i & 1-2i \\ 1+i & 8+i \end{bmatrix}$$

$$B = \begin{bmatrix} 7-i & 1-2i \\ 1+i & 8+i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 8 & -1+i \\ -1-i & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

3. For which α and β quadratic form $Q(x,y,z)$ is positive definite:

$$Q(x,y,z) = \alpha x^2 + 2y^2 + z^2 - 2\beta xy + 4xz$$

$$Q(x,y,z) = \alpha x^2 + 2y^2 + z^2 - 2\beta xy + 4xz = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} \alpha & -\beta & 2 \\ -\beta & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\text{all Det}\begin{vmatrix} \alpha & -\beta & 2 \\ -\beta & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} > 0 \implies \alpha > 4,\ -\sqrt{2\alpha - 8} < \beta < \sqrt{2\alpha - 8}$$

$$Q(x,y,z) = 2\alpha x^2 + y^2 + z^2 - 2\beta xy + 4xz$$

$$Q(x,y,z) = 2\alpha x^2 + y^2 + z^2 - 2\beta xy + 4xz = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 2\alpha & -\beta & 2 \\ -\beta & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\text{all Det}\begin{vmatrix} 2\alpha & -\beta & 2 \\ -\beta & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} > 0 \implies \alpha > 2,\ -\sqrt{2\alpha - 4} < \beta < \sqrt{2\alpha - 4}$$

4. Solve the second order differential equation system:

$$\begin{cases} \dfrac{d^2x(t)}{dt^2} = 7\,x(t) - (2-2\,i)y(t) \\ \dfrac{d^2y(t)}{dt^2} = -(2+2\,i)x(t) + 5\,y(t) \\ x(0) = 1, \qquad x'(0) = 0 \\ y(0) = 1, \qquad y'(0) = 0 \end{cases}$$

We know that if eigenvalues of matrix $A = S\Lambda S^{-1}$ don't equal zero the general solution $\vec{u}(t)$ is: $\vec{u}(t) = S\cosh(\Lambda\mathrm{t})S^{-1}\vec{u}(0) + S\Lambda^{-1/2}\sinh(\Lambda\mathrm{t})S^{-1}\vec{u}'(0)$

$$A = \begin{bmatrix} 7 & -2+2i \\ -2-2i & 5 \end{bmatrix} = \tfrac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix} \Rightarrow$$

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t)1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \cosh(\sqrt{3}\mathrm{t}) & 0 \\ 0 & \cosh(3\mathrm{t}) \end{bmatrix}\begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} +$$

$$+\frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{\sinh(\sqrt{3}\mathrm{t})}{\sqrt{3}} & 0 \\ 0 & \dfrac{\sinh(3\mathrm{t})}{3} \end{bmatrix}\begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x(t) \\ y(t)1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (1+i)\cosh[3t] + (2-i)\cosh[\sqrt{3}t] \\ -i(\cosh[3t] - (1-3i)\cosh[\sqrt{3}t]) \end{bmatrix}$$

$$\begin{cases} \dfrac{d^2x(t)}{dt^2} = 8\,x(t) - (1-\ i)y(t) \\ \dfrac{d^2y(t)}{dt^2} = -(1+\ i)x(t) + 7\,y(t) \\ x(0) = 1, \qquad x'(0) = 0 \\ y(0) = 1, \qquad y'(0) = 0 \end{cases}$$

$$A = \begin{bmatrix} 8 & -1+i \\ -1-i & 7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix} \Rightarrow$$

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \cosh(\sqrt{6}\mathrm{t}) & 0 \\ 0 & \cosh(3\mathrm{t}) \end{bmatrix}\begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} +$$

$$+=$$

$$\frac{1}{6}\begin{bmatrix} 1-i & -1+i \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{\sinh(\sqrt{6}\mathrm{t})}{\sqrt{6}} & 0 \\ 0 & \dfrac{\sinh(3\mathrm{t})}{3} \end{bmatrix}\begin{bmatrix} 1+i & 2 \\ -2-2i & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{6}\begin{bmatrix} (2+2i)\cosh[3t] + (4-2i)\cosh[\sqrt{6}t] \\ -2i\cosh[3t] + (6+2i)\cosh[\sqrt{6}t] \end{bmatrix}$$

I. Using least squares method to find the coefficients a and b of the curve: $f(x) = f(x) = a\sin(x) + b\,cos(x)$, that best fits following points:

x	0	π/4	π/2
$f(x)$	-1	0	0.8

$$f(x) = a\sin(x) + b\,cos(x) \;\Rightarrow$$

$$\begin{cases} b = -1 \\ a + b = 0 \\ a = 0.8 \end{cases} \;\Rightarrow\; \begin{cases} a = 0.85 \\ b = -0.95 \end{cases}$$

$$f(x) = 0.85\sin(x) - 0.95\,cos(x)$$

$f(x) = f(x) = a\sin(x) + b\,cos(x)$, that best fits following points:

x	0	π/4	π/2
$f(x)$	-0.8	0	1

$$f(x) = a\sin(x) + b\,cos(x) \;\Rightarrow$$

$$\begin{cases} b = -0.8 \\ a + b = 0 \\ a = 1 \end{cases} \;\Rightarrow\; \begin{cases} a = 0.95 \\ b = -0.85 \end{cases}$$

$$f(x) = 0.95\sin(x) - 0.85\,cos(x)$$

5. If $A + iB$ is a unitary matrix (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix

Unitary $U^H U = I$ means $(A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B + B^T A) = I$. So, $A^T A + B^T B = I$ and $A^T B + B^T A = 0$ which makes the block matrix $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ orthogonal, because:

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}^T \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A^T A + B^T B & A^T B + B^T A \\ A^T B + B^T A & A^T A + B^T B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I.$$

If $A+iB$ is Hermitian matrix (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric matrix.

We are given $A+iB=(A+iB)^H=A^T-iB^T$. Then $A=A^T$ and $B=-B^T$. So that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

6. Prove that for any square matrix $A(n\times n)$ with eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ the multiplication: $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ produces the zero matrix:

$$(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)=S(\Lambda-\lambda_1 I)S^{-1}S(\Lambda-\lambda_2 I)S^{-1}\cdots S(\Lambda-\lambda_n I)S^{-1}=$$

$$=S(\Lambda-\lambda_1 I)(\Lambda-\lambda_2 I)\cdots(\Lambda-\lambda_n I)S^{-1}=$$

$$=S\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \lambda_2-\lambda_1 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n-\lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1-\lambda_2 & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n-\lambda_2 \end{bmatrix}\cdots\begin{bmatrix} \lambda_1-\lambda_n & \cdots & \cdots & 0 \\ \vdots & \lambda_2-\lambda_n & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}S^{-1}=$$

$$=S\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}S^{-1}=[0]$$

Prove that any real square matrix can be factored into $A=QS$, where Q is orthogonal and S is symmetric positive semidefinite.

Remember it's not enough to present $A=(UV^T)(V\Sigma V^T)=QS$ but it is also necessary to show that $Q=UV^T$ is **orthogonal**: $QQ^T=UV^TVU^T=I$ and $S=V\Sigma V^T$ is **positive semidefinite**: $\forall x\neq 0$: $x^TV\Sigma V^Tx=\left(x^TV\sqrt{\Sigma}\right)\left(\sqrt{\Sigma}V^Tx\right)=y^Ty\geq 0$ where $y=\sqrt{\Sigma}V^Tx$.