**Example 1** Solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

This is a vector equation for $\boldsymbol{u}$. It contains two scalar equations for the components $y$ and $z$. They are "coupled together" because the matrix $A$ is not diagonal:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \text{ means that } \frac{d\boldsymbol{y}}{d\boldsymbol{t}} = \boldsymbol{z} \text{ and } \frac{d\boldsymbol{z}}{d\boldsymbol{t}} = \boldsymbol{y}.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it. Add and subtract equations:

$$\frac{d}{dt}(y+z) = z + y \qquad \text{and} \qquad \frac{d}{dt}(y-z) = -(y-z).$$

The combination $y + z$ grows like $e^t$, because it has $\lambda = 1$. The combination $y - z$ decays like $e^{-t}$, because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$, looking for these special combinations. The eigenvectors and eigenvalues of $A$ will do it for us.

This matrix $A$ has eigenvalues 1 and $-1$. The eigenvectors $\boldsymbol{x}$ are $(1, 1)$ and $(1, -1)$. The pure exponential solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ take the form $e^{\lambda t}\boldsymbol{x}$ with $\lambda_1 = 1$ and $\lambda_2 = -1$:

$$\boxed{\boldsymbol{u}_1(t) = e^{\lambda_1 t}\boldsymbol{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \text{and} \quad \boxed{\boldsymbol{u}_2(t) = e^{\lambda_2 t}\boldsymbol{x}_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}. \qquad (4)$$

Notice: These $\boldsymbol{u}$'s satisfy $A\boldsymbol{u}_1 = \boldsymbol{u}_1$ and $A\boldsymbol{u}_2 = -\boldsymbol{u}_2$, just like $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The factors $e^t$ and $e^{-t}$ change with time. Those factors give $d\boldsymbol{u}_1/dt = \boldsymbol{u}_1 = A\boldsymbol{u}_1$ and $d\boldsymbol{u}_2/dt = -\boldsymbol{u}_2 = A\boldsymbol{u}_2$. **We have two solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$.** To find all other solutions, **multiply those special solutions by any numbers $C$ and $D$ and add:**

**Complete solution** $$\boldsymbol{u}(t) = \boldsymbol{C}e^{\boldsymbol{t}} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} + \boldsymbol{D}e^{-\boldsymbol{t}} \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \qquad (5)$$

With these two constants $C$ and $D$, we can match any starting vector $\boldsymbol{u}(0) = (u_1(0), u_2(0))$. Set $t = 0$ and $e^0 = 1$. Example 1 asked for the initial value to be $\boldsymbol{u}(0) = (4, 2)$:

$$\boldsymbol{u}(0) \textbf{ decides } \boldsymbol{C}, \boldsymbol{D} \qquad C\begin{bmatrix}1\\1\end{bmatrix} + D\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} \quad \text{yields} \quad \boldsymbol{C = 3} \quad \text{and} \quad \boldsymbol{D = 1}.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is completely solved.

The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

> **1.** Write $\boldsymbol{u}(0)$ as a **combination** $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ **of the eigenvectors of** $\boldsymbol{A}$.
>
> **2.** Multiply each eigenvector $\boldsymbol{x}_i$ by **its growth factor** $e^{\lambda_i t}$.
>
> **3.** The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:
>
> $$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u(t) = c_1 e^{\lambda_1 t}x_1 + \cdots + c_n e^{\lambda_n t}x_n}. \tag{6}$$

*Not included*: If two $\lambda$'s are equal, with only one eigenvector, another solution is needed. (It will be $te^{\lambda t}\boldsymbol{x}$.) Step 1 needs to diagonalize $A = X\Lambda X^{-1}$: a basis of $n$ eigenvectors.

**Example 2** Solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$ knowing the eigenvalues $\lambda = 1, 2, 3$ of $A$:

**Typical example**
**Equation for $\boldsymbol{u}$**
**Initial condition $\boldsymbol{u}(0)$**

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 1\\0 & 0 & 3\end{bmatrix}\boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}(0) = \begin{bmatrix}9\\7\\4\end{bmatrix}.$$

The eigenvectors are $\boldsymbol{x}_1 = (1,0,0)$ and $\boldsymbol{x}_2 = (1,1,0)$ and $\boldsymbol{x}_3 = (1,1,1)$.

> **Step 1** The vector $\boldsymbol{u}(0) = (9,7,4)$ is $\boldsymbol{2x_1 + 3x_2 + 4x_3}$. Thus $(c_1, c_2, c_3) = (\mathbf{2, 3, 4})$.
>
> **Step 2** The factors $e^{\lambda t}$ give exponential solutions $e^t\boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.
>
> **Step 3** The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = \boldsymbol{2e^t x_1 + 3e^{2t}x_2 + 4e^{3t}x_3}$.

The coefficients $2, 3, 4$ came from solving the linear equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = \boldsymbol{u}(0)$:

$$\begin{bmatrix}\\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3\\ \\ \end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}9\\7\\4\end{bmatrix} \quad \text{which is} \quad X\boldsymbol{c} = \boldsymbol{u}(0). \tag{7}$$