



# Analytical Geometry and Linear Algebra II, Lab 7

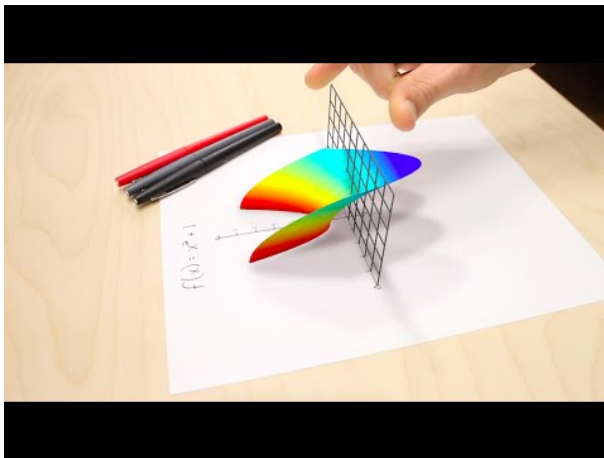
Complex numbers

Complex matrices

Hermitian and Unitary Matrices

# Complex numbers

Video



# Complex numbers

## Forms

**Rectangular form:**  $z = x + iy$ ,  $i^2 = -1$

$\text{Re}(z) = x$  - real part,  $\text{Im}(z) = y$  - imaginary part

Example:  $z = 5 + i6$

**Polar form:**  $z = r \cos(\phi) + i r \sin(\phi)$ , where

$\phi = \text{atan2}(\text{Im}(z), \text{Re}(z))$ ;

$r = |z| = \sqrt{x^2 + y^2}$  - magnitude

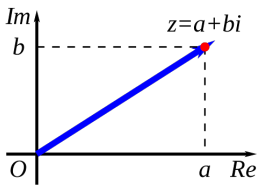
Example:  $z = 8 \cos(24) + i \sin(24)$

**Exponential form:**  $z = re^{i\phi}$

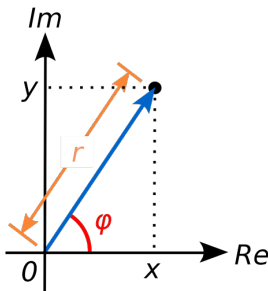
Example:  $z = 6e^{i2.5}$

**Euler formula:** transformation from exp. to polar

$$e^{i\phi} = \cos(\phi) + i \sin(\phi)$$



Rectangular  
form



Polar or  
Exponential  
form

# Complex numbers

## Operations



**General Idea:** you should work with *Im* and *Re* part separately (you cannot sum or multiply them)

- *Summarization and Subtraction* -  $(x_1 \pm iy_1) + (x_2 \pm iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- *Multiplication* -  $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
- *Division* -  $\frac{(x_2 + iy_2)}{(x_1 + iy_1)} = \frac{(x_1x_2 + y_1y_2)}{x_1^2 + y_1^2} + i\frac{(y_1x_2 - x_1y_2)}{x_1^2 + y_1^2}$

# Complex numbers

## Complex conjugate

Complex conjugate of complex number  $z = x + iy$  -  $\bar{z} = x - iy$ .

Geometrically it's reflection of  $z$  about Re axis.

### Properties:

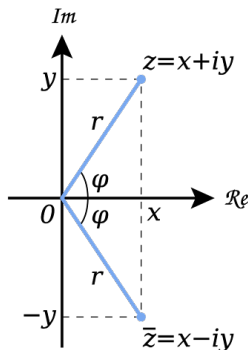
$\operatorname{Re}(\bar{z}) = \operatorname{Re}(z)$  and  $|\bar{z}| = |z|$ ;

$\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$  and  $\arg \bar{z} \equiv -\arg z \pmod{2\pi}$

$z\bar{z} = x^2 + y^2 = |z|^2$  - absolute square

### Operations

- Summarization and Subtraction -  $\overline{z \pm w} = \bar{z} \pm \bar{w}$
- Multiplication -  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- Division -  $\overline{z/w} = \bar{z}/\bar{w}$



Complex Conjugate

# Complex numbers

*Applications (1)*



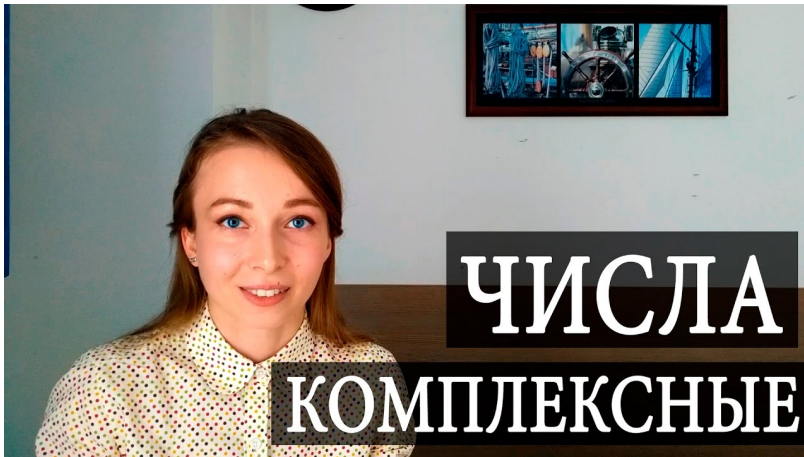
## Typical reason

Most of the time for simplicity your calculations, except Quantum computing and some other topics.

1. *Electromagnetism*: electric and magnetic fields as complex vector fields to describe electromagnetic waves. However, this is just a computational trick.
2. *Aerodynamics*: calculating wing shapes using Joukowski transform.
3. *Geodesy*: Google Maps (from Geoid to 2d map).
4. *Quantum Computing*

# Complex numbers

*Applications (2): Video*



# Complex numbers



*Applications(3): more about Quantum Computing (Links)*

- Representing Qubit States as complex numbers
- Why do we have to use complex numbers in Quantum computing (Nature paper)
- Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit)

*If you are interested in the topic, you can ask **Stas Protasov**.*





## Task 1

1 For  $z = \frac{1+i}{\sqrt{2}}$ :

- Compute  $z^2$
- Find  $r$
- Find  $\phi$
- Find  $z$  in exponential form

- 2
- Find the 8 solutions to the equation  $z^8 = 1$
  - Plot those 8 solutions in the complex plane

3 For  $z = -1 + i\frac{1}{2}$

- Find complex conjugate ( $\bar{z}$ )
- Find  $z\bar{z}$
- Find  $z + \bar{z}$
- Plot each result in the complex plane

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$
- $i^i$
- Show each result in the complex plane



## Task 1

1 For  $z = \frac{1+i}{\sqrt{2}}$ :

- Compute  $z^2$

- Find  $r$
- Find  $\phi$
- Find  $z$  in exponential form

## Answer

- $z^2 = i$
- $r = 1$
- $\phi = 45^\circ$
- $z = 1e^{i\frac{\pi}{4}}$



## Task 1

- 2 – Find the 8 solutions to the equation  $z^8 = 1$
- Plot those 8 solutions in the complex plane

### Answer

Solution (rus):  $\pm 1; \pm j; \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2};$



## Task 1

3 For  $z = -1 + i\frac{1}{2}$

- Find complex conjugate ( $\bar{z}$ )
- Find  $z\bar{z}$

- Find  $z + \bar{z}$

- Plot each result in the complex plane

### Answer

- $\bar{z} = -1 - i\frac{1}{2}$
- $z\bar{z} = 1.25$
- $z + \bar{z} = -2$



## Task 1

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$

- $i^i$
- Show each result in the complex plane

### Answer

- $e^{i\frac{\pi}{2}} = i$
- $e^{i\pi} = -1$
- $i^i = e^{\ln(i)} = e^{i \ln(i)}$ ;  $\begin{cases} i = e^{i\frac{\pi}{2}} - \text{from 1-st bullet} \\ \ln(i) = \ln(e^{i\frac{\pi}{2}}) \end{cases} \rightarrow i\frac{\pi}{2} = \ln(i); e^{i \ln(i)} = e^{-\frac{\pi}{2}} = 0.208$

# Complex Matrices

## Common and Special



Many concepts have a new name, but old meaning:

- $A^T \rightarrow A^H$ ;  $A^H = \bar{A}^T$ ,  $H$  - conjugate  $T$ ; Exp:  $\begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$
- $Q \rightarrow U$ ;  $U = Q$ , where  $U$  - Unitary matrix

$\mathbf{R}^n$ ( $n$ real components)	$\leftrightarrow$	$\mathbf{C}^n$ ( $n$ complex components)
length: $\ x\ ^2 = x_1^2 + \dots + x_n^2$	$\leftrightarrow$	length: $\ x\ ^2 =  x_1 ^2 + \dots +  x_n ^2$
transpose: $A_{ij}^T = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^H = \bar{A}_{ji}$
$(AB)^T = B^T A^T$	$\leftrightarrow$	$(AB)^H = B^H A^H$
inner product: $x^T y = x_1 y_1 + \dots + x_n y_n$	$\leftrightarrow$	inner product: $x^H y = \bar{x}_1 y_1 + \dots + \bar{x}_n y_n$
$(Ax)^T y = x^T (A^T y)$	$\leftrightarrow$	$(Ax)^H y = x^H (A^H y)$
orthogonality: $x^T y = 0$	$\leftrightarrow$	orthogonality: $x^H y = 0$
symmetric matrices: $A^T = A$	$\leftrightarrow$	Hermitian matrices: $A^H = A$
$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$ (real $\Lambda$ )	$\leftrightarrow$	$A = U \Lambda U^{-1} = U \Lambda U^H$ (real $\Lambda$ )
skew-symmetric $K^T = -K$	$\leftrightarrow$	skew-Hermitian $K^H = -K$
orthogonal $Q^T Q = I$ or $Q^T = Q^{-1}$	$\leftrightarrow$	unitary $U^H U = I$ or $U^H = U^{-1}$
$(Qx)^T (Qy) = x^T y$ and $\ Qx\  = \ x\ $	$\leftrightarrow$	$(Ux)^H (Uy) = x^H y$ and $\ Ux\  = \ x\ $

## Task 2



Compute  $A^H A$  and  $AA^H$ . Those are both \_\_\_\_\_ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

## Task 2



Compute  $A^H A$  and  $AA^H$ . Those are both \_\_\_\_\_ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

Answer

$$A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix} \text{ and } AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ are Hermitian matrices.}$$



## Task 3



Solve  $Az = \mathbf{0}$  to find a vector  $z$  in the nullspace of  $A$  in Problem 2. Show that  $z$  is orthogonal to the columns of  $A^H$ . Show that  $z$  is *not* orthogonal to the columns of  $A^T$ . ***The good row space is no longer  $C(A^T)$ . Now it is  $C(A^H)$ .***



## Task 3

Solve  $Az = \mathbf{0}$  to find a vector  $z$  in the nullspace of  $A$  in Problem 2. Show that  $z$  is orthogonal to the columns of  $A^H$ . Show that  $z$  is *not* orthogonal to the columns of  $A^T$ . *The good row space is no longer  $C(A^T)$ . Now it is  $C(A^H)$ .*

## Answer

$z = \text{multiple of } (1+i, 1+i, -2)$ ;  $Az = \mathbf{0}$  gives  $z^H A^H = \mathbf{0}^H$  so  $z$  (not  $\bar{z}$ !) is orthogonal to all columns of  $A^H$  (using complex inner product  $z^H$  times columns of  $A^H$ ).

## Task 4



If  $A + iB$  is Hermitian ( $A$  and  $B$  are real) show that  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$  is symmetric.

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If  $A + iB$  is Hermitian ( $A$  and  $B$  are real) show that  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$  is symmetric.

### Answer

We are given  $A + iB = (A + iB)^H = A^T - iB^T$ . Then  $A = A^T$  and  $B = -B^T$ . So that  $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$  is symmetric.

## Reference material



- Lecture 26
- Complex Numbers Part Imaginary, but Really Simple
- *"Linear Algebra and Applications"*, pdf pages 322–335  
Complex numbers and matrices
- *"Introduction to Linear Algebra"*, pdf pages 504–519  
Complex numbers and matrices
- Complex Numbers calculator

# Deserve "A" grade!

– Oleg Bulichev

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📍 @Lupasic

🏢 Room 105 (Underground robotics lab)