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Modern Robotics

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Robird

PortWings

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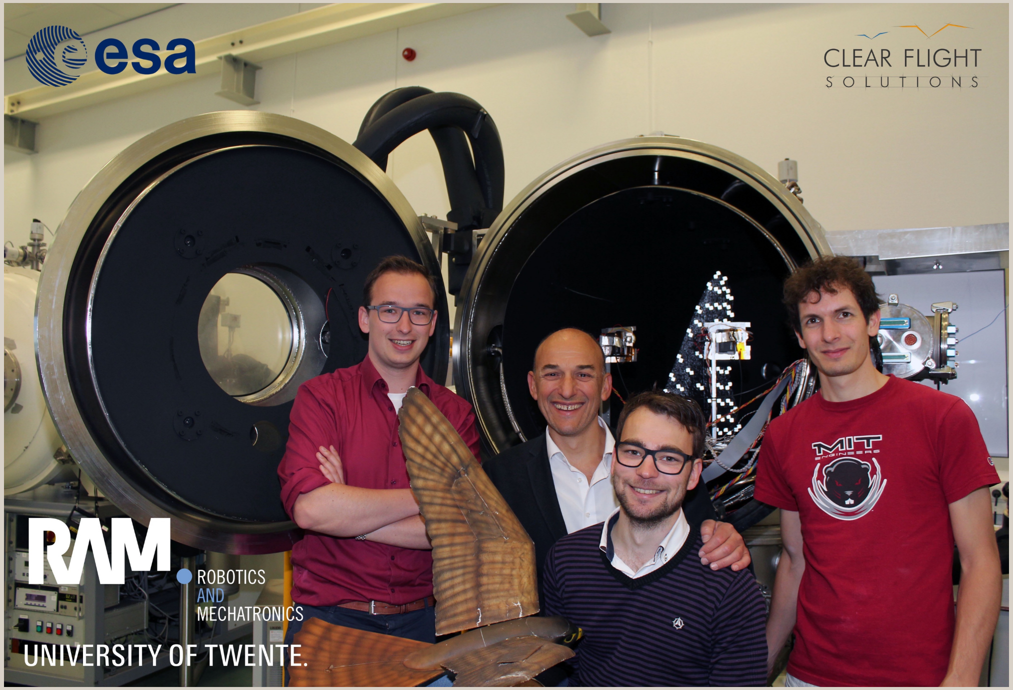
Lecturers

- **Stefano** Stramigioli, UT
- **Geert** Folkertsma, UT



Video Connection

https://bluejeans.com/207790103

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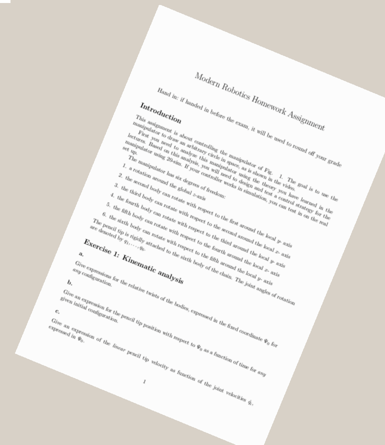
General Information

- **Questions**: In the 15 academic minutes at the beginning of each lesson hour.
- **Exam**: Written with THEORY (no book) and PRACTICE (with everything).
- **Problems Sets** available on webpage
- **Practical** Assignments in the lab (from June)


Modern Robotics Homework Assignment

Introduction

Exercise 1: Kinematic analysis

General Information

- **Main Contact via email**: Geert
- **Video Lectures** (iTunes-U+others) available
- **All infos via Blackboard:**
- **Background**:
  - Linear Algebra
  - Basic Geometry
  - Basic Mechanics

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What is this course about?

- Geometry of rigid mechanisms
- Kinematics of Serial Mechanism
- Dynamics of Serial Mechanism
- Position Control
- Interaction Control

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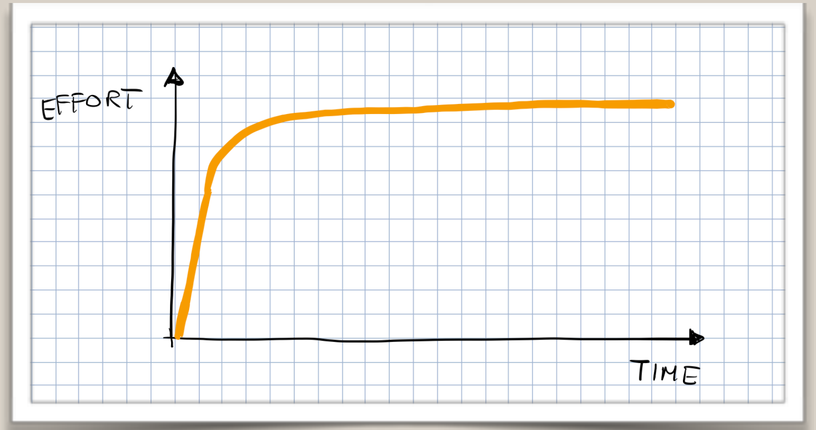
Why “Modern” Robotics

- **Different** than >95% of the Robotics course around
- Uses **geometrical powerful tools**
- To **learn the new tools** will be tough, but then **multibody dynamics will become as simple as 1D**!!
- The learning curve is



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Lecture 1

Introduction Euclidian Spaces

How can we model space ?

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Contents

- Mathematical Background
- Modeling The Euclidean Spaces
- Coordinate changes and motions
- SE(3)
- Conclusions

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Mathematical Background

Goal

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To give a basic and INTUITIVE background of the tools which will be used for the course and the understanding of basic concepts of physics and robotics.

General Remarks

- Stop me at ANY time if something is not clear!!
- The treatment will NOT be mathematically precise but yields to convey intuition
- Hopefully you will appreciate concepts which are very often discarded in basic courses, but that they are ESSENTIAL in engineering sciences (i.e.tensors).
- Managing abstractions and models is a must for academic people!

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- More meant as a reference, do not concentrate on the details!!!
- Most of the material is not ALL essential for understanding, but it usually HELPs in focusing the robotics concepts and understand them deeply

Contents

- Vectors and Co-Vectors
- Higher Order Tensors
- Linear maps, Quadratic Forms..
- Intuition of Differential Geometric Concepts
- Manifolds, (co-)tangent spaces
- Groups
- Lie-Groups

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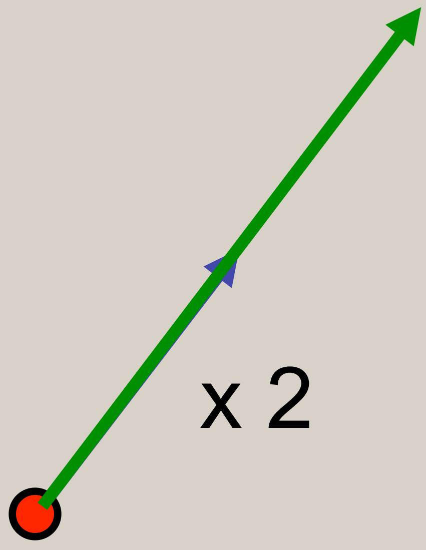
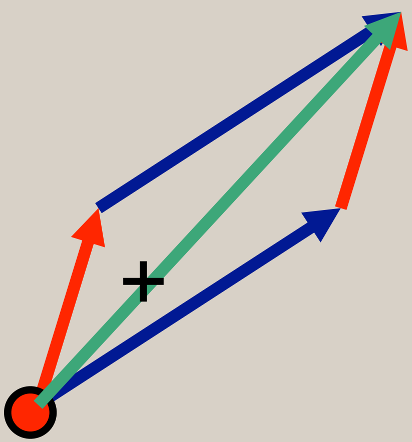
Vectors and Co-vectors

Vectors Spaces

Product for a scalar

Sum





Origin of the vector space

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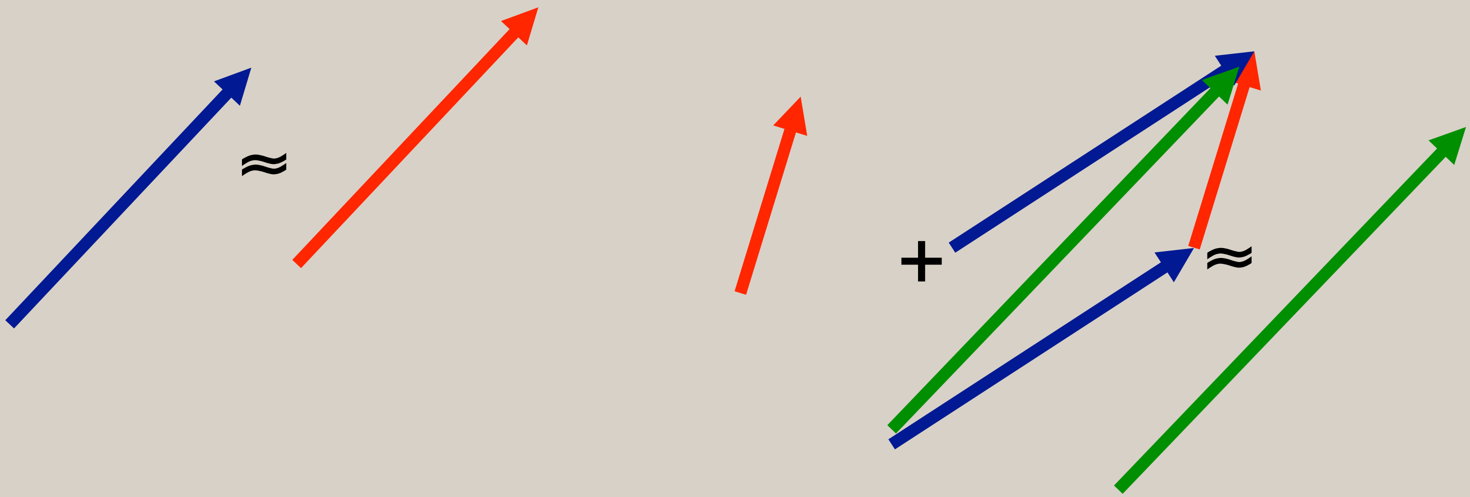
Vector Spaces

- A vector Space has an origin
- Has a scalar product
- Has a vector sum
- It does NOT in general have an inner product: we CANNOT say if two vectors are orthogonal or we CANNOT ”absolutely” measure its length.
- Vectors are more than arrays of numbers!!!!!

Free Vectors

- We may want to work with vectors which are NOT bound to a origin, but free to move around by **parallel transport** which can be defined independently from the path in **non-curved spaces**: free vectors



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Vector Space (finite dimensional)

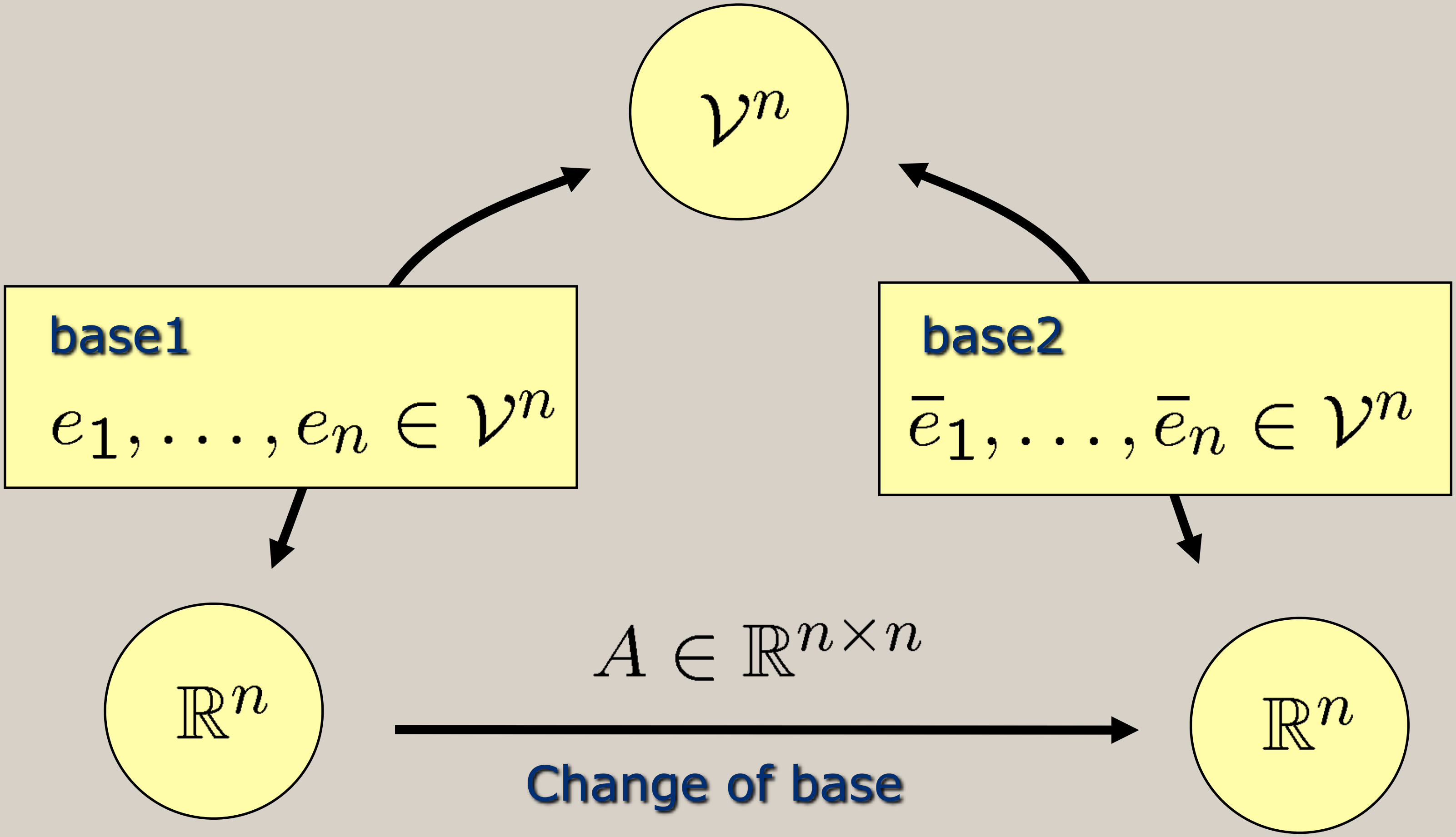
- A real vector space $\mathcal{V}^n$ is characterized by
  - An origin $o \in \mathcal{V}^n$
  - Elements $v \in \mathcal{V}^n$ can be scaled by any real number α and still belong to the vector space:

$$\alpha v \in \mathcal{V}^n$$

  - You can take a linear combination of elements and get again an element in the vector space

$$v, w \in \mathcal{V}^n, \alpha, \beta \in \mathbb{R} \longrightarrow \alpha v + \beta w \in \mathcal{V}^n$$

From a Vector Space $\mathcal{V}^n$ to $\mathbb{R}^n$



Co-vectors

Consider the set of LINEAR operators:

$$L : \mathcal{V}^n \to \mathbb{R}$$

Once a base has been chosen, it can be seen that numerically they are represented by an n-dim row vector:

$$(L(e_1), \ldots, L(e_n))$$

• =

Changing coordinates

- The vector space of this linear operators, called dual space is indicated with $\mathcal{V}^*$.
- Consider the velocity of a point mass as an element of a 3D vector space $v \in \mathcal{V}^3$
- A force F applied to this mass will transfer power to the mass. The value of power is a scalar INDEPENDENT of the coordinate choice.

Changing coordinates

Coord 1 $F^T v = \bar{F}^T \bar{v}$ Coord 2

Consider the change of base such that $v = \boxed{A}\bar{v}$

$$F^T A \bar{v} = \bar{F}^T \bar{v} \qquad \forall \bar{v}$$

Different!!

$$\bar{F} = A^T F \quad \Rightarrow \quad F = \boxed{A^{-T}}\bar{F}$$

Co-vectors

- Co-vectors are also represented by numerical arrays once a base is chosen, but they are different than vectors since they transform diferently !!

- A **velocity** is a **vector**
- A **force** is a **co-vector** NOT a vector !!
- A force is a linear operator from velocity to power

Examples

- Vectors
  - Velocities
  - Twists

- Co-vectors
  - Forces
  - Wrenches

Tensors

Tensors

A tensor of type $\binom{p}{q}$
is defined as a multi-linear operator of the following form:

$$L : (\underbrace{\mathcal{V} \times \ldots \mathcal{V}}_{q\text{-times}} \times \underbrace{\mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{p\text{-times}}) \to \mathbb{R}$$

Order 1(=p+q) tensors are

$\binom{1}{0}$ Vector

$\binom{0}{1}$ Co-vector

Matrices and 2nd order tensors

What does a matrix represent ?

4 options:

- Map $\mathcal{V} \times \mathcal{V}^* \longrightarrow \mathbb{R}$ (linear map)
- Map $\mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ (quadratic form)
- Map $\mathcal{V}^* \times \mathcal{V} \longrightarrow \mathbb{R}$ (linear map on dual space)
- Map $\mathcal{V}^* \times \mathcal{V}^* \longrightarrow \mathbb{R}$ (quadratic form on d.s.)

Linear Map

$$\mathcal{V} \times \mathcal{V}^* \longrightarrow \mathbb{R}$$

We are used to think of a linear map as

$$\mathcal{V} \rightarrow \mathcal{V}$$

But $\mathcal{V}$ can be seen as a l.m. map $\mathcal{V}^* \longrightarrow \mathbb{R}$

$$w = Av, \qquad w, v \in \mathcal{V}$$

$$F^T w \in \mathbb{R}, \qquad F \in \mathcal{V}^*$$

$$\mathcal{V} \times \mathcal{V}^* \rightarrow \mathbb{R} \quad ; \quad (v, F) \mapsto F^T A v$$

Linear Map

$$v_1 = Bv_2$$

$$\bar{v}_1 = \bar{B}\bar{v}_2$$

Change of coord of vectors $v_i = A\bar{v}_i$

$$A\bar{v}_1 = BA\bar{v}_2$$

$$\bar{v}_1 = A^{-1}BA\bar{v}_2$$

$$\bar{v}_1 = \bar{B}\bar{v}_2$$

$$\bar{B} = A^{-1}BA$$

Quadratic Form $\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$
(i.e. Kinetic energy)

$$v_1^T M v_2$$

$$\bar{v}_1^T \bar{M} \bar{v}_2$$

Change of coord of vectors $v_i = A\bar{v}_i$

$$\bar{v}_1^T A^T M A \bar{v}_2$$

$$\bar{v}_1^T \bar{M} \bar{v}_2$$

$$\bar{M} = A^T M A$$

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Does it make sense to take eigen-values of a matrix ?

vector $\lambda v = A v$ vector

$A : \mathcal{V} \rightarrow \mathcal{V}$ Linear operator!!

If A had been a quadratic form like an inertia matrix, it would not mean anything to take eigenvalues: other coordinates would give different values!!

Conclusions

- Vectors transform differently than co-vectors changing coordinates
- Linear maps transform differently than quadratic forms changing coordinates
- Always think about the kind of objects you are working with (Einstein notation !!)
- Be aware of nonsense like eigenvalues of an inertia matrix!

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Examples

- (1 1) Tensors
  - Mechanisms transformation matrix
- (0 2) Tensors
  - Inertia matrices, Inertial elipsoids, point mass, stiffness matrices at equilibrium, symplectic structure, any metric (!!)
- (2 0) Tensors
  - Poisson structure

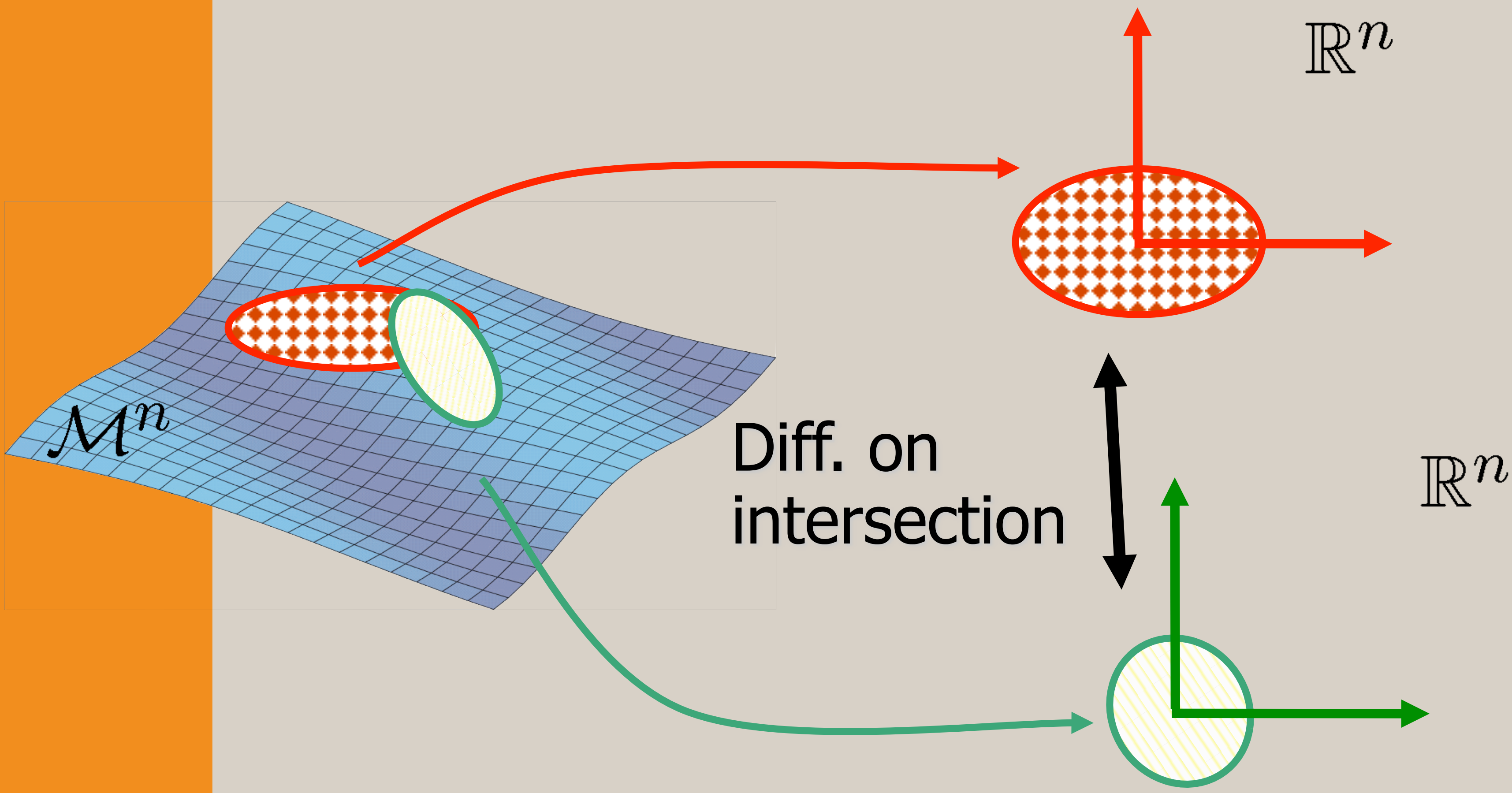
Remarks

- Frames are an artifact: physics is not based on coordinates
- Be aware of `user metric choices' !!
  - Manipulability index $\sqrt{\det(J M J^T)}$
  - Pseudo-inverses $J^+ = J^T (J M J^T)^{-1}$
  - Hybrid Position-Force Control
    - No metric in se(3) !!

....

Manifolds

Manifold



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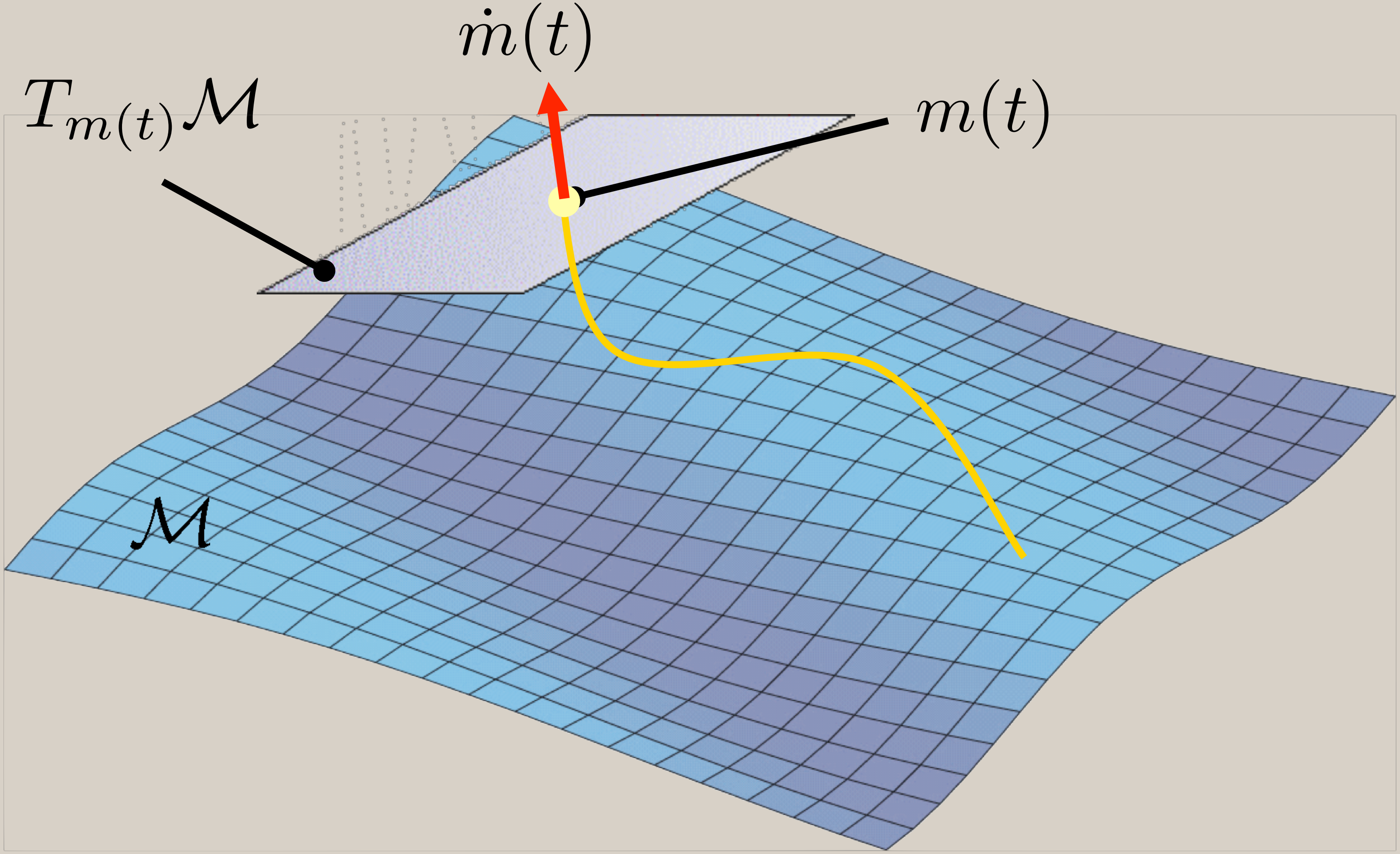
Tangent Spaces



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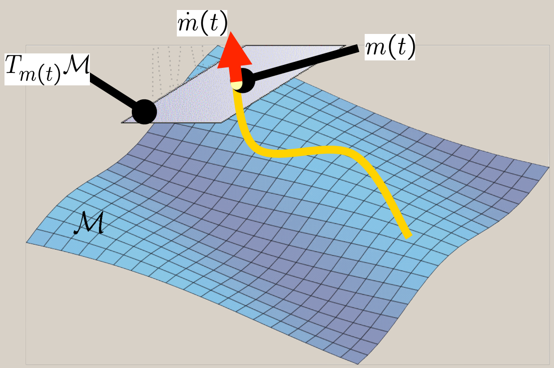
Co-Tangent spaces

The vector space of co-vectors at each configuration m is indicated with

$$T_m^*\mathcal{M}$$



$$v \in T_m\mathcal{M} \rightarrow (m, v) \in T\mathcal{M}$$

$$\langle F | v \rangle \in \mathbb{R}$$

$$F \in T_m^*\mathcal{M} \rightarrow (m, F) \in T^*\mathcal{M}$$

$$F^T v \in \mathbb{R}$$

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Examples

- Configuration space of a manipulator
- Relative configurations of bodies
- Line segment
- Earth surface
- Space
- …

Groups

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Group

A group is a set $\mathcal{M}$ and an operation $\bullet$ for which

Associativity

$$m_1, m_2, m_3 \in \mathcal{M}$$
$$\Rightarrow (m_1 \bullet m_2) \bullet m_3 = m_1 \bullet (m_2 \bullet m_3) \in \mathcal{M}$$

Identity

$$\exists I \in \mathcal{M} \quad \text{s.t.} \quad \forall m \quad m \bullet I = I \bullet m = m$$

Inverse

$$\forall m \in \mathcal{M} \qquad \exists m^{-1} \in \mathcal{M} \quad \text{s.t.} \quad m m^{-1} = I$$

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Examples

- Set of nonsingular matrices with matrix product operation
- Flow of a differential equation
- Object motions
- Quaternions
- …

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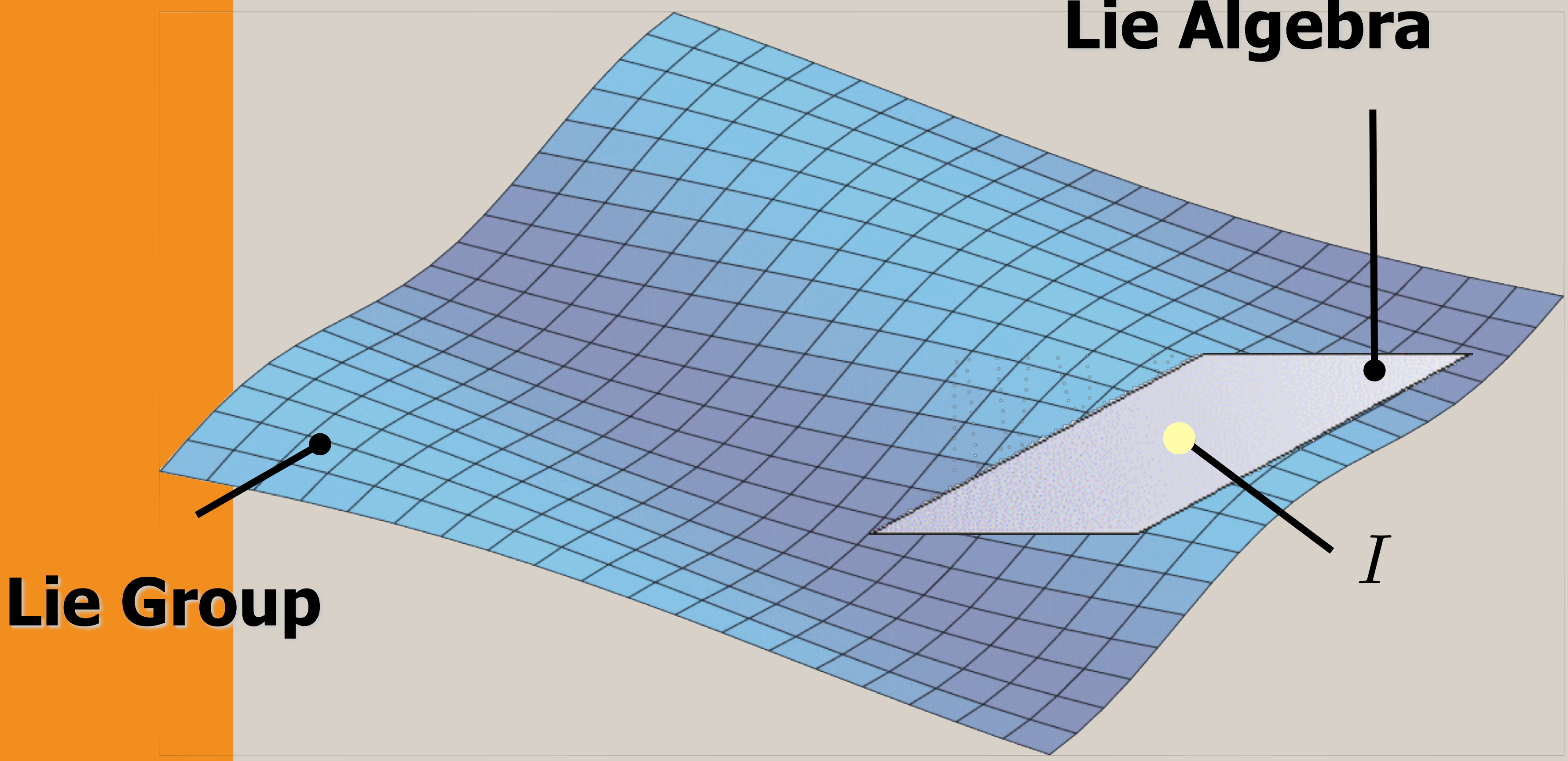
Lie Groups

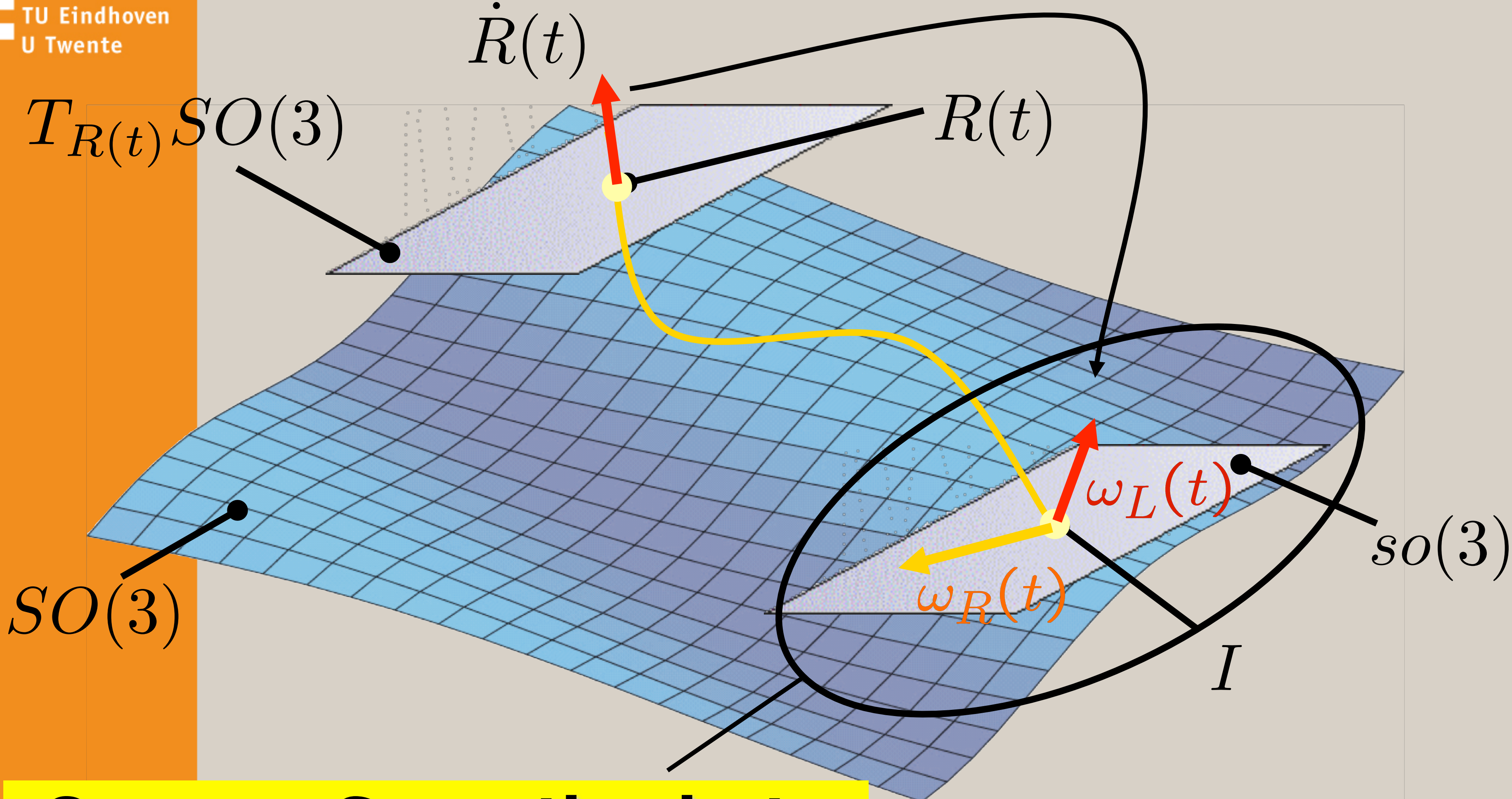
What is a Lie-Group

- A Lie group is a ”manifold group”
  - It is smooth
  - It is a group
- The tangent space in the identity is an algebra (has a skew symmetric operation) called Lie algebra

Lie Group



Lie Groups



Common Space thanks to Lie group structure

Examples

- Space of Rotation matrices
- Space of Homogeneous matrices
- Space of “Abstract” rotations
- Space of “Abstract” motions
- Unit Quaternions
- …

Euclidean Space and Motions

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Euclidean Space

- A Euclidean Space is a Space of Free vectors plus an Inner Product which can tell us about:

  - Orthogonally of Vectors
  - Absolute length of vectors

We indicate this space of FREE VECTORS as

$$\boxed{\mathcal{E}_*}$$

Euclidian Space

- An Euclidian space is a vector space with an extra structure called Inner Product: given 2 vectors it returns a real number.

$$\langle , \rangle : \mathcal{E}_* \times \mathcal{E}_* \to \mathbb{R} \; ; \; (v, w) \mapsto \langle v, w \rangle$$

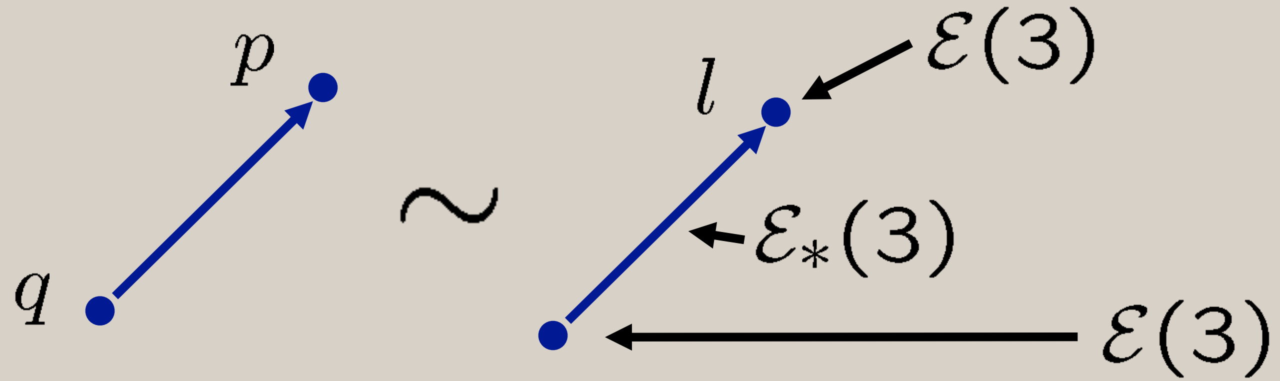
$\longmapsto$ **3.13**

(just as an example)

Euclidian Spaces and motions

- An 3-dimensional Euclidean space $\mathcal{E}(3)$ is characterized by a inner product
- We can consider the relative position of

$p, q \in \mathcal{E}(3)$ as a vector $(p - q)$:



The set of these free-vectors will be indicateds with $\mathcal{E}_*(3)$

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Notation and concepts

Inner Product

$$\langle , \rangle : \mathcal{E}_* \times \mathcal{E}_* \to \mathbb{R} \; ; \; (v, w) \mapsto \langle v, w \rangle$$

Norm (distance)

$$|| \cdot || : \mathcal{E}_* \to \mathbb{R} \; ; \; v \mapsto \sqrt{\langle v, v \rangle}$$

Orthogonality of $v, w \in \mathcal{E}_*$

$$v \perp w \Leftrightarrow \langle v, w \rangle = 0$$

Notation and concepts (cont.)

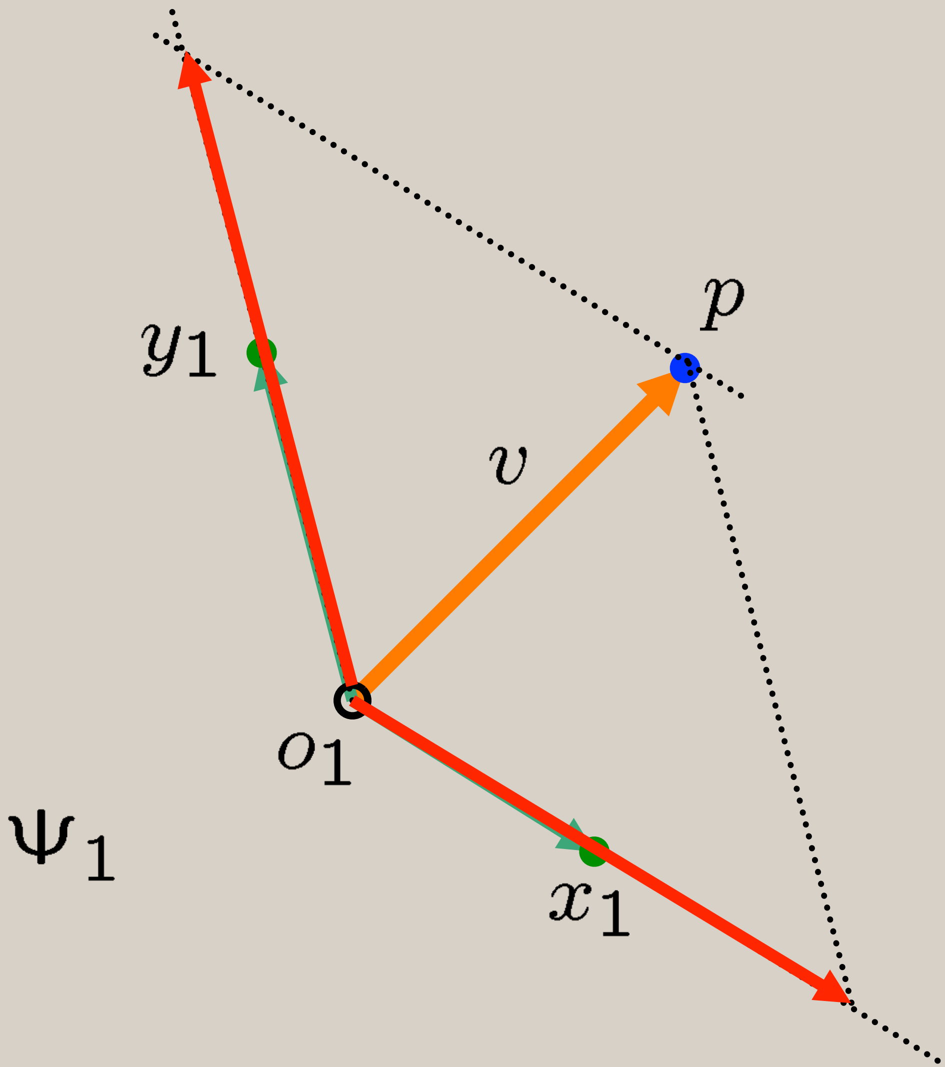
Angles between vectors $v, w \in \mathcal{E}_*$

$$\cos v \angle w := \frac{\langle v, w \rangle}{||v|| \cdot ||w||}$$

Conclusion

All concepts defined using only the inner product.

Coordinate Systems



Notation

- Physical Points (NOT NUMBERS)

$$p, p_i, q, \dots$$

No super-scipts

- Points/Vectors expressed in frame j : they are arrays of numbers (coordinate)

$$p^j, p_i^j, q^j, \dots$$

Coordinate Systems

- A coordinate system is composed of an origin and 3 linear independent free vectors:

$$\Psi_o := (o, e_1, e_2, e_3) \in \mathcal{E}(3) \times \mathcal{E}_*(3) \times \mathcal{E}_*(3) \times \mathcal{E}_*(3)$$

Coordinate Systems (cont.)

- A Coordinate System is ortho-normal iff

$$||e_i|| = 1 \quad \forall i \quad \text{(unit vectors)}$$

$$\langle e_i, e_j \rangle = 0 \quad \forall i \neq j \quad \text{(orthogonality)}$$

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Coordinates

Coordinates are real numbers:

For $p \in \mathcal{E}$ (points)

$$x_i = \langle (p - o), e_i \rangle \in \mathbb{R} \quad \forall i$$

For $v \in \mathcal{E}_*$ (vectors)

$$x_i = \langle v, e_i \rangle \in \mathbb{R} \quad \forall i$$

We will use often the following notation:

$$\hat{x} := e_1,\ \hat{y} := e_2,\ \hat{z} := e_3$$

Coordinate Mappings

$$\Psi_i = (o_i, \hat{x}_i, \hat{y}_i, \hat{z}_i)$$

$$\psi_i : \mathcal{E}(3) \rightarrow \mathbb{R}^3 \; ; \; p \mapsto \begin{pmatrix} \langle (p - o_i), \hat{x}_i \rangle \\ \langle (p - o_i), \hat{y}_i \rangle \\ \langle (p - o_i), \hat{z}_i \rangle \end{pmatrix}$$

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Change of coordinates

A change of coordinates from Ψ_2 to Ψ_1 can be expressed with:

$$p^2 = (\psi_2 \, o \, \psi_1^{-1})(p^1)$$

Which is a mapping like

$$\mathbb{R}^3 \xrightarrow{\psi_1^{-1}} \mathcal{E}(3) \xrightarrow{\psi_2} \mathbb{R}^3 \; ; \; p^1 \mapsto p \mapsto p^2$$

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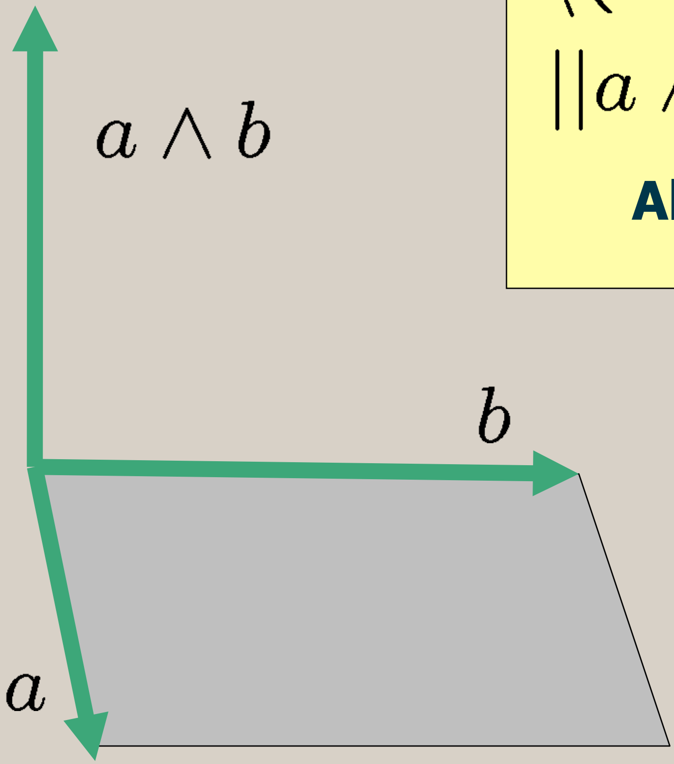
Vector Product



$$\langle (a \wedge b), a \rangle = 0$$
$$\langle (a \wedge b), b \rangle = 0$$
$$||a \wedge b|| = ||a|| \; ||b|| \sin a \angle b$$

All based on Inner Product

Orientation

- The vector product $\wedge$ operation often used in mechanics, is NOT as often thought an extra structure/operation defined, but it is only a consequence of the Lie group structure of motions:

$$\wedge : \mathcal{E}_*(3) \times \mathcal{E}_*(3) \rightarrow \mathcal{E}_*(3); (v, w) \mapsto v \wedge w$$

Tilde Operator (Notation)

We will often use the following notation

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow y := \tilde{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

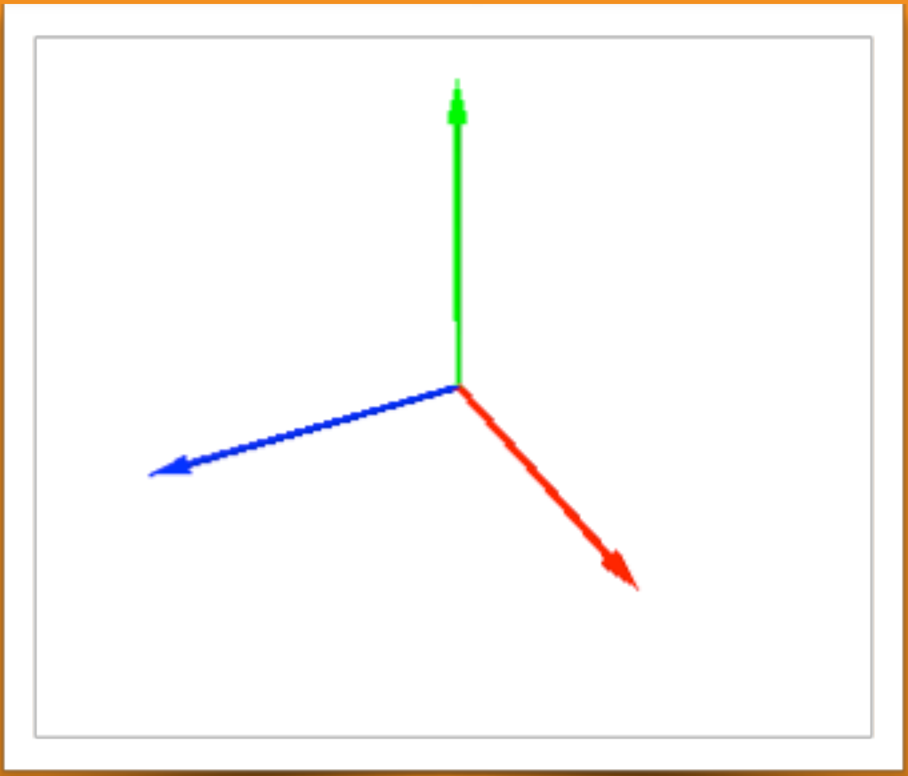
$$\Downarrow$$

$$x \wedge y = \tilde{x} y$$

In 20-SIM

Y=Skew(X)

Rotations



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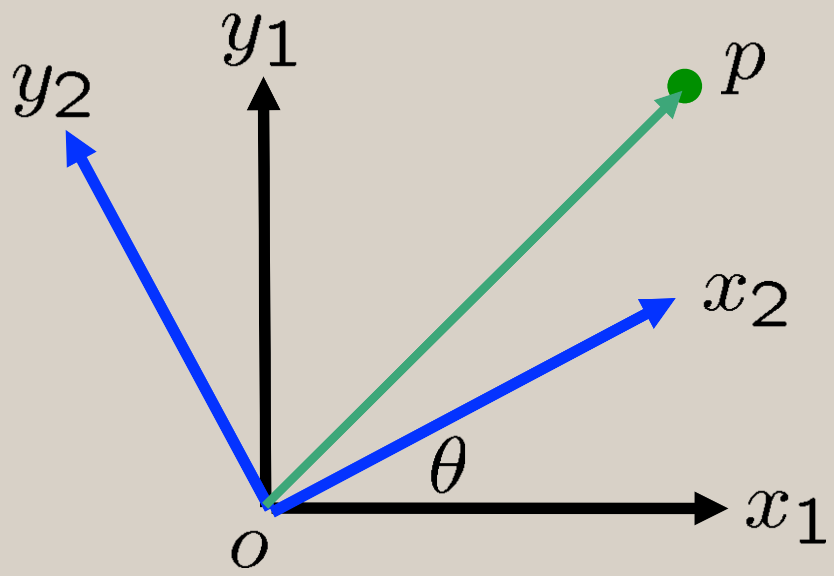
Approach

- We will first introduce change of coordinates for rotated frames
  - 2D
  - 3D
- Relate changes of coordinates to motion of objects

Rotations in 2D

Consider the following coordinates systems



What is the relation of the coordinates of p in Ψ_1 and in Ψ_2 ?

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We have that

$$p^1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \psi_1(p) = \begin{pmatrix} \langle (p - o_1), \hat{x}_1 \rangle \\ \langle (p - o_1), \hat{y}_1 \rangle \end{pmatrix}$$

Or equivalently $(p - o_1) = x_1 \hat{x}_1 + y_1 \hat{y}_1$

With $x_1, y_1 \in \mathbb{R}$ $\quad \hat{x}_1, \hat{y}_1 \in \mathcal{E}_*(2)$

Similarly, for $p^2 \in \mathbb{R}^2$ we have

$$p^2 := \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle (p - o_2), \hat{x}_2 \rangle \\ \langle (p - o_2), \hat{y}_2 \rangle \end{pmatrix}$$

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$$p^2 := \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle (p - o_2), \hat{x}_2 \rangle \\ \langle (p - o_2), \hat{y}_2 \rangle \end{pmatrix}$$

and since $o_1 = o_2$ using the expression of $(p - o_1)$ in Ψ_1, we have that

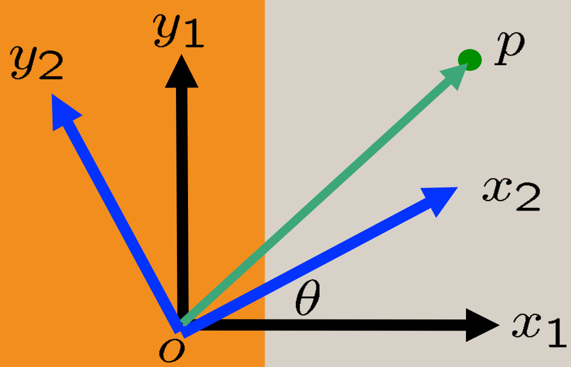
$$= \begin{pmatrix} \langle x_1 \hat{x}_1 + y_1 \hat{y}_1, \hat{x}_2 \rangle \\ \langle x_1 \hat{x}_1 + y_1 \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} = \begin{pmatrix} x_1 \langle \hat{x}_1, \hat{x}_2 \rangle + y_1 \langle \hat{y}_1, \hat{x}_2 \rangle \\ x_1 \langle \hat{x}_1, \hat{y}_2 \rangle + y_1 \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix}$$

Conclusion

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Rotation Matrix

$$p^2 = R_1^2 p^1$$



$$\ldots = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Remarks

Due to the fact we are considering right-handed orthonormal frames:

1. $\det(R_1^2)=1$
2. $R_2^1 = (R_1^2)^{-1} = (R_1^2)^T$
3. The columns and row vectors of R_1^2 have length 1 and they are orthogonal

3D Rotation

In 3D, if the coordinates we use are all right handed, we have for a general rotation around the origin:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle & \langle \hat{z}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle & \langle \hat{z}_1, \hat{y}_2 \rangle \\ \langle \hat{x}_1, \hat{z}_2 \rangle & \langle \hat{y}_1, \hat{z}_2 \rangle & \langle \hat{z}_1, \hat{z}_2 \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Basis of Ψ_2 **espressed in** Ψ_1

Basis of Ψ_1 **espressed in** Ψ_2

Example

- A rotation around the y axis would be

$$R_1^2 = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Special Orthonormal Group

- A square matrix $R \in \mathbb{R}^{3\times 3}$ such that $R^{-1} = R^T$ is called orthonormal. The group of orthonormal matrices with determinant 1 is called Special Orthonormal group of $\mathbb{R}^3$ and indicated as:

$$SO(3) := \{R \in \mathbb{R}^{3\times 3}\ ;\ R^{-1} = R^T, \det R = 1\}$$

Theorem

If $R(t) \in SO(3)$ is a differentiable function of time,

$$\dot{R}R^T \text{and } R^T\dot{R}$$

are Skew-Symmetric and belonging to $so(3)$:

$$so(3) := \{\tilde{\omega} \in \mathbb{R}^{3\times 3}\ ;\ -\tilde{\omega} = \tilde{\omega}^T\}$$

Angular velocities

This implies that

$$\exists \omega_1, \omega_2 \in \mathbb{R}^3$$

such that:

Remember:

$$\tilde{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

$$\boxed{\tilde{\omega}_1 = \dot{R} R^T}$$

$$\boxed{\tilde{\omega}_2 = R^T \dot{R}}$$

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$so(3)$ is a Lie algebra

- The linear combination of skew-symmetric matrices is still skew-symmetric
- To each $\Omega \in so(3)$ matrix we can associate a vector $\omega \in \mathbb{R}^3$ such that $\tilde{\omega} = \Omega$

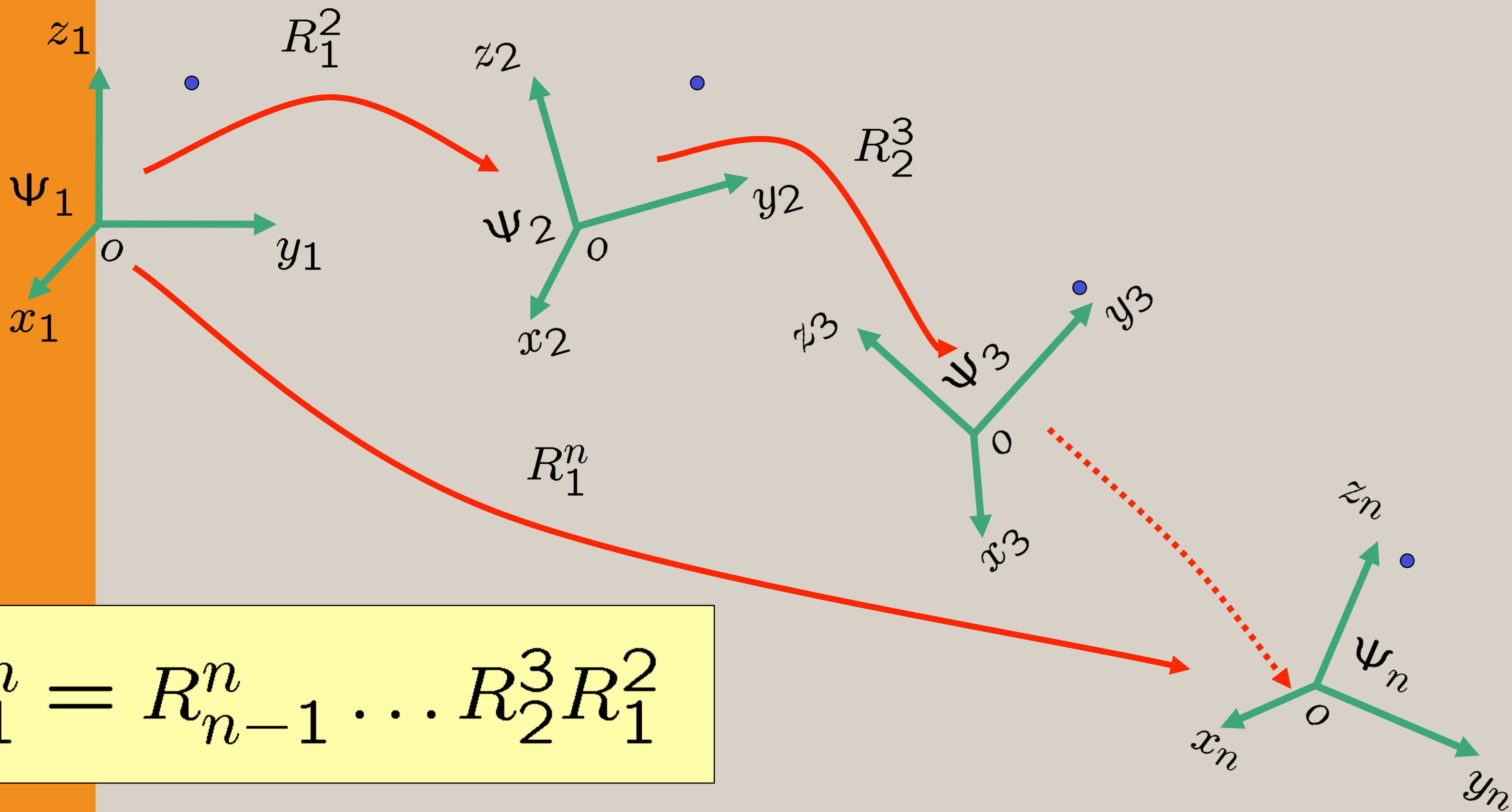
... It is a vector space

$$\Omega_1, \Omega_2 \in so(3) \Rightarrow$$

$$[\Omega_1, \Omega_2] := \Omega_1\Omega_2 - \Omega_2\Omega_1 \in so(3)$$

- It is a Lie Algebra !!

Chain Rule



$$R_1^n = R_{n-1}^n \dots R_2^3 R_1^2$$

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SO(3) is a Group

It is a Group because

- Associativity

$$R_1, R_2, R_3 \in SO(3) \Rightarrow (R_1 R_2) R_3 = R_1 (R_2 R_3)$$

- Identity

$$\exists I \in SO(3) \quad \text{s.t.} \quad \forall R \quad RI = IR = R$$

- Inverse

$$R \in SO(3) \Rightarrow R^{-1} \in SO(3), RR^{-1} = I$$

It is a Lie Group (group AND manifold)

$$(R,\dot R) \in TSO(3), \qquad \dot R \in T_R SO(3)$$

$$(L_{R^{-1}})_*(R,\dot R) = (R^{-1}R, R^{-1}\dot R) = (I,\tilde\omega_L)$$

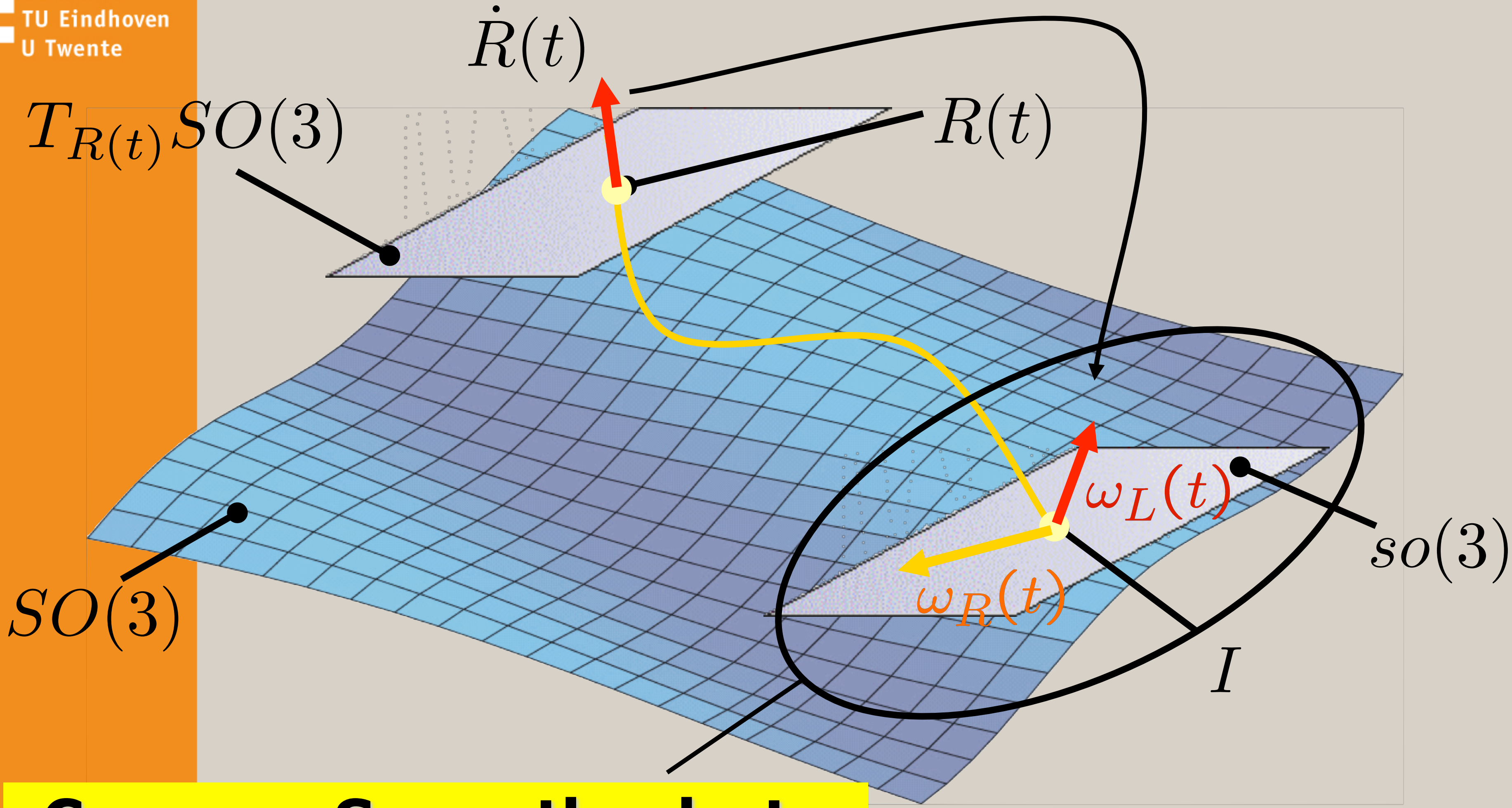
$$\tilde\omega_L \in T_I SO(3) =: so(3)$$

$$(R_{R^{-1}})_*(R,\dot R) = (RR^{-1}, \dot R R^{-1}) = (I,\tilde\omega_R)$$

$$\tilde\omega_R \in T_I SO(3) =: so(3)$$

$$[,] \qquad [\tilde\omega_x,\tilde\omega_y] := \tilde\omega_x\tilde\omega_y - \tilde\omega_y\tilde\omega_x \in so(3)$$

Lie Groups



Common Space thanks to Lie group structure

Why is the Lie group structure so important?

- It makes it possible to talk about motion ω without knowing the pose R of the object!
- We can get rid of any dependency from configuration: ESSENTIAL to talk about interconnection throw Power Ports of bodies with different poses.

Dual Space

- For any finite dimensional vector space we can define the space of linear operators from that space to

τ

co-vector

$$\omega = P \in \mathbb{R}$$

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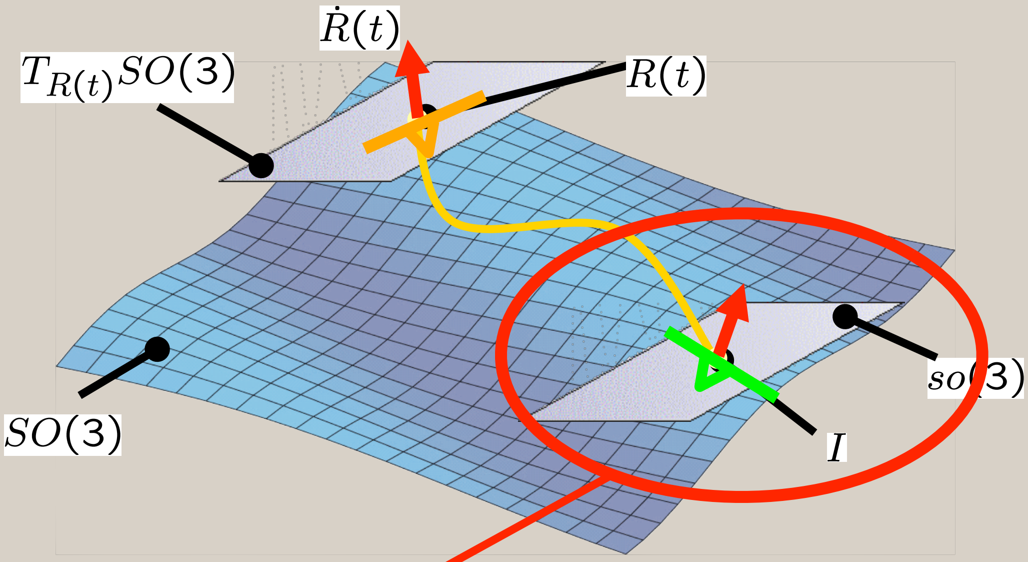
In our case we have

V	V^*
$so(3)$	$so^*(3)$
$T_R SO(3)$	$T_R^* SO(3)$



Configuration Independent Port !

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Other Rotations Representations

- Generalized angles:
  - Euler, Bryan etc. (singularity positions and NOT geometric
- Quaternions:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \rightarrow$$
$$Q = (q_0, q) = \left(\cos(\frac{\theta}{2}), \omega \sin(\frac{\theta}{2})\right)$$

Other Rotations Representations

- $SU(2)$ matrix Lie group of unitary matrices of determinant +1

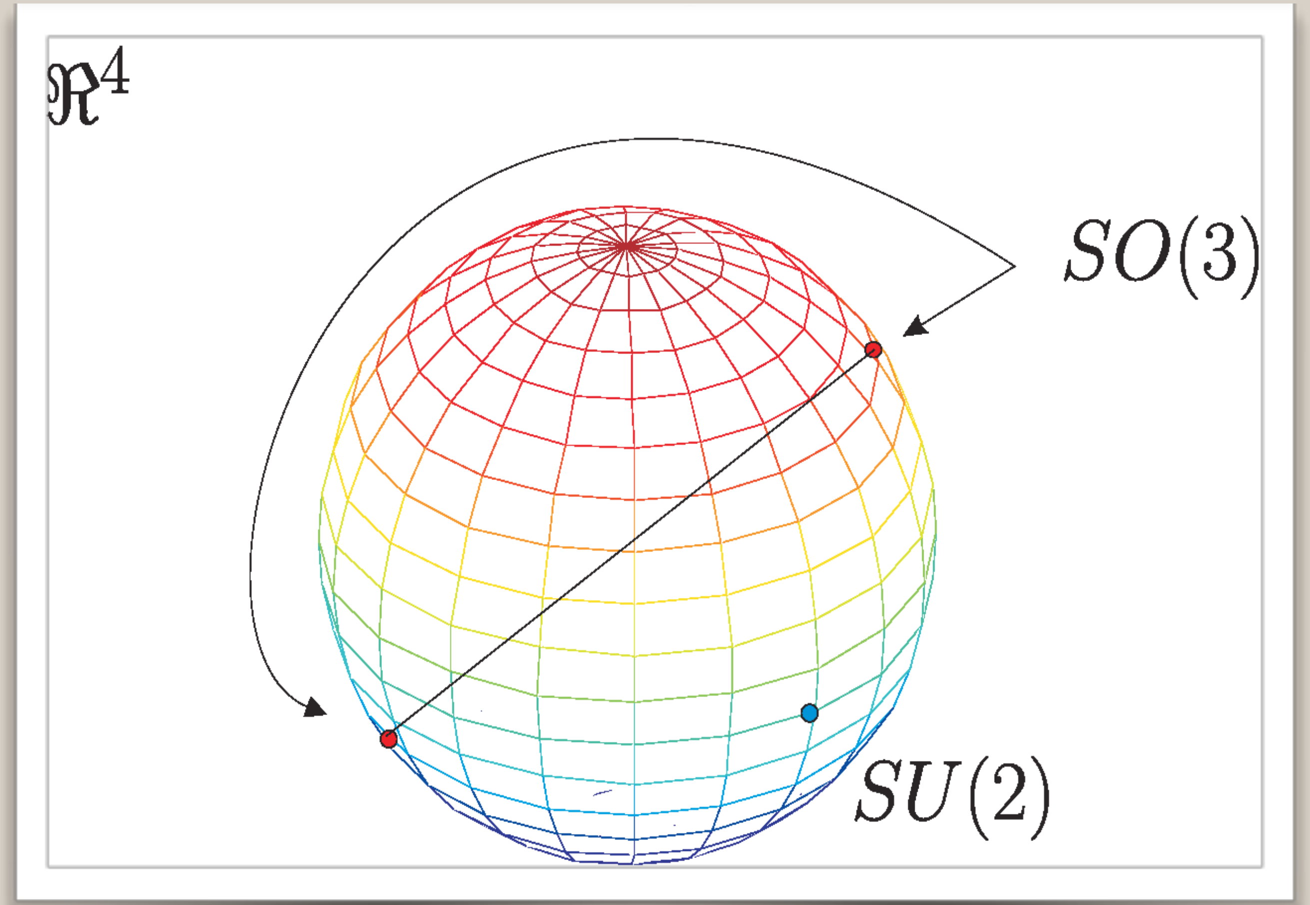
$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \qquad a^2+b^2+c^2+d^2=1$$

Unit Quaternions and $SU(2)$ are identifiable with points of a 3-sphere

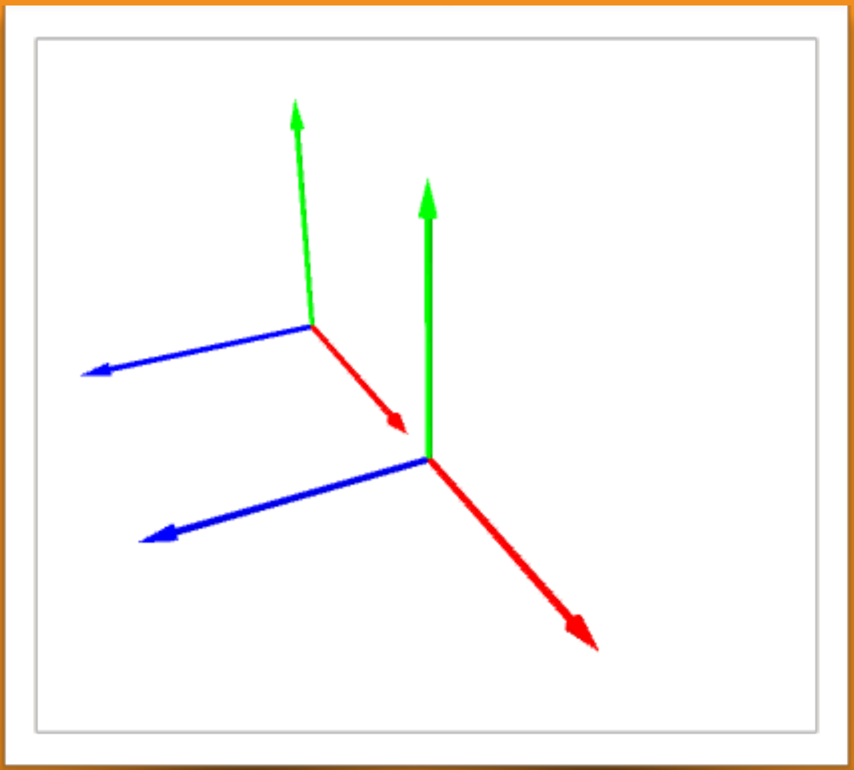
Double Covering of $SO(3)$

- The quaternions and $SU(2)$ double cover $SO(3)$
- Quaternions and $SU(2)$ are simply connected
- $SO(3)$ is connected but NOT simply connected: see the Dirac Belt Trick !
- $SO(3)$ is one of the few examples of a NON simply connected manifolds WITHOUT holes!

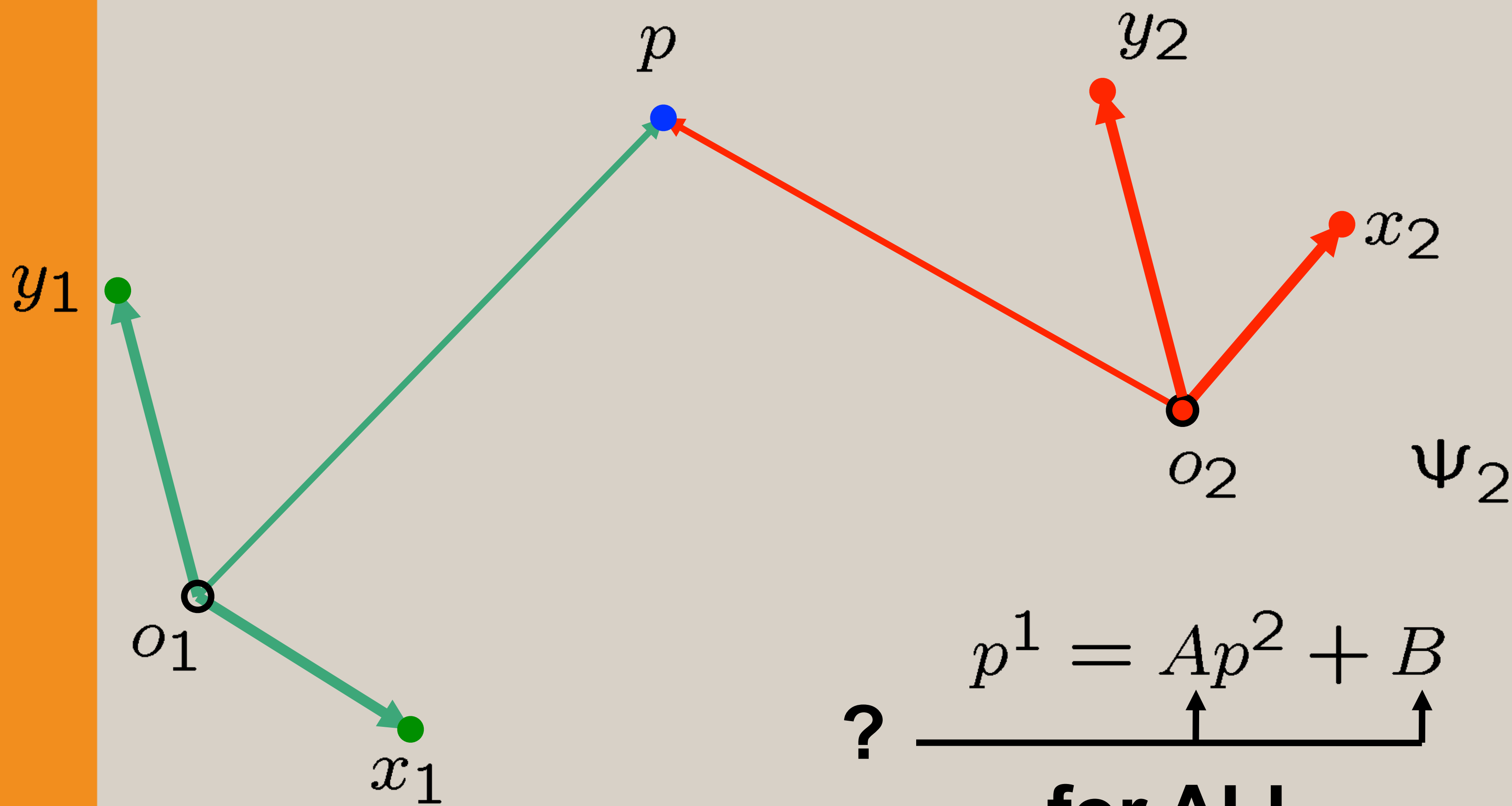
About the topology of rotations



General Motions



Coordinate Changes 2D



$$p^1 = Ap^2 + B$$

?

for ALL p

Calculation of B

If we choose

$$p = 0_2 \Rightarrow p^2 = 0_2^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$p^1 = \cancel{Ap^2} + B \Rightarrow \boxed{0_2^1 = B}$$

B is equal to the origin of Ψ_2 expressed in frame Ψ_1

Calculation of A

$$p^1 = Ap^2 + B \quad \Rightarrow \quad (p^1 - B) = Ap^2$$

If we choose $p = x_2 \quad \Rightarrow \quad p^2 = x_2^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(p^1 - B) = Ap^2 \quad \Rightarrow \quad (x_2^1 - B) = \boxed{A_1}$$

where $A = [\boxed{A_1} \ \ A_2]$

first column second column

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Similarly...

$$p^1 = Ap^2 + B \quad \Rightarrow \quad (p^1 - B) = Ap^2$$

If we choose $p = y_2 \quad \Rightarrow \quad p^2 = y_2^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(p^1 - B) = Ap^2 \quad \Rightarrow \quad (y_2^1 - B) = \boxed{A_2}$$

where $A = [A_1 \ \boxed{A_2}]$

first column

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Ortho-normal Change of Coordinates

$$p^1 = Ap^2 + B$$

$$p^j = R_i^j p^i + o_i^j$$

$$R_i^j \in SO(3)$$

$$o_i^j \in \mathbb{R}^3$$

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General Motions

$$\mathbb{R}^3 \xrightarrow{\psi_i^{-1}} \mathcal{E} \xrightarrow{\psi_j} \mathbb{R}^3$$

It can be seen that in general, for right handed frames

$$\psi_j \; o \; \psi_i^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \; ; \; p^i \mapsto R_i^j p^i + o_i^j$$

where $R_i^j \in SO(3)$ $o_i^j \in \mathbb{R}^3$

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Homogeneous Matrices

- Due to the group structure of $SO(3)$ it is easy to compose changes of coordinates in rotations

$$R_i^k = R_m^k R_s^m \dots R_l^q R_i^l$$

- Can we do the same for general motions ?

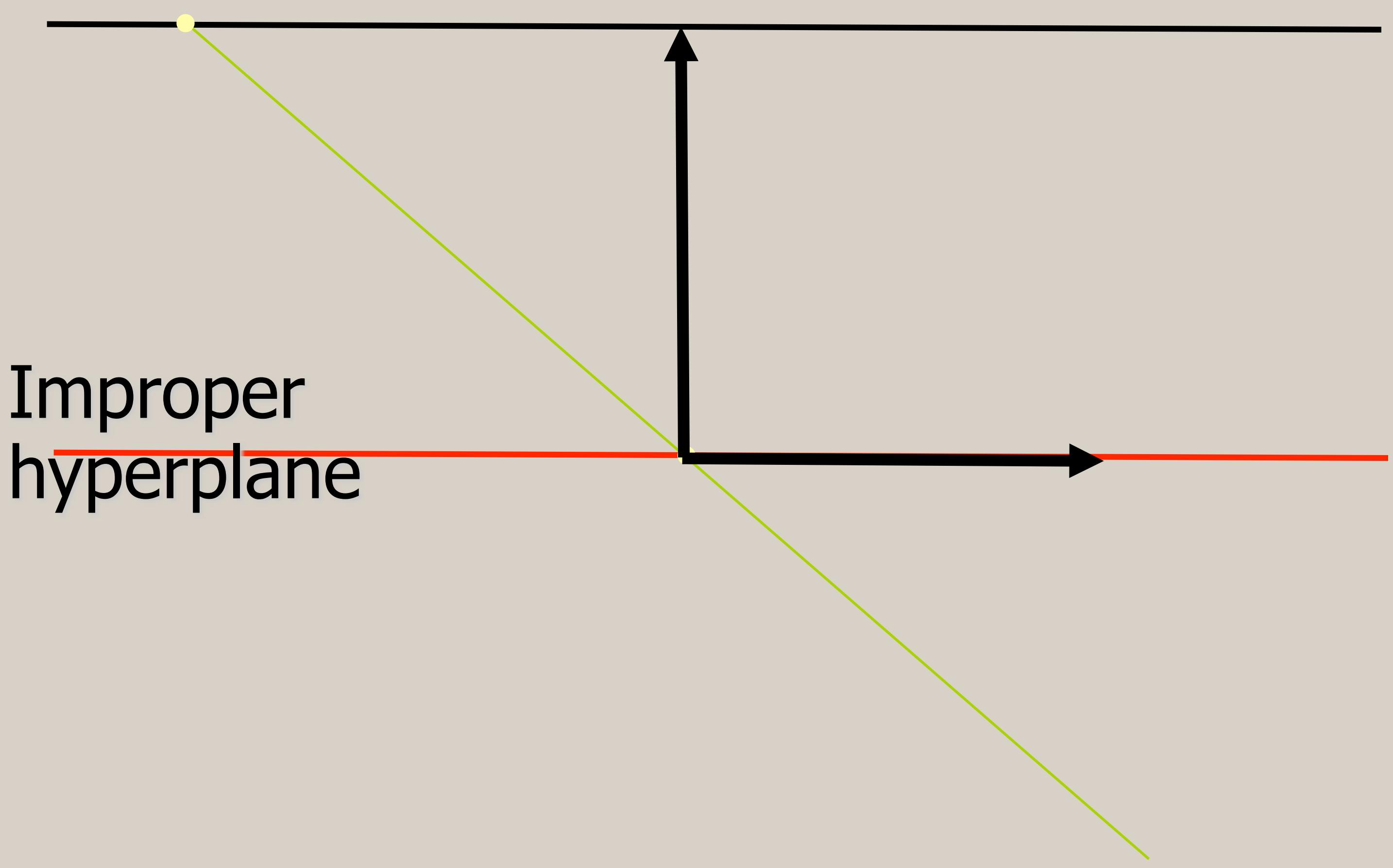
$$p := \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \to P := \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

$$H_i^j$$

$$p^i \mapsto R_i^j p^i + o_i^j$$

$$P^i \mapsto \begin{pmatrix} R_i^j & o_i^j \\ 0 & 1 \end{pmatrix} P^i$$

Projective space



Special Cases

Pure Translations $R_1^2 = I$

$$H_1^2 := \begin{pmatrix} I & o_1^2 \\ 0_3^T & 1 \end{pmatrix}$$

Pure Rotations $o_1^2 = 0$

$$H_1^2 := \begin{pmatrix} R_1^2 & 0_3 \\ 0_3^T & 1 \end{pmatrix}$$

Inverse change of coordinates

$$H_2^1 = (H_1^2)^{-1} = \begin{pmatrix} (R_1^2)^T & -(R_1^2)^T o_1^2 \\ 0_3^T & 1 \end{pmatrix}$$

Only Tanspositions !!!

$SE(3)$

$$P := \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \qquad H_i^j := \begin{pmatrix} R_i^j & o_i^j \\ 0 & 1 \end{pmatrix}$$

$$H_i^k = H_m^k H_s^m \dots H_l^q H_i^l$$

$$SE(3) := \left\{ \begin{pmatrix} R & o \\ 0_3 & 1 \end{pmatrix} \; s.t. \; R \in SO(3), o \in \mathbb{R}^3 \right\}$$

SE(3) is a Group

It is a Group because

- Associativity

$$H_1, H_2, H_3 \in SE(3) \Rightarrow (H_1 H_2) H_3 = H_1 (H_2 H_3)$$

- Identity

$$I \in SE(3)$$

- Inverse

$$H \in SE(3) \Rightarrow H^{-1} \in SE(3), HH^{-1} = I$$

Theorem

If $H(t) \in SE(3)$ is a differentiable function of time

$$\dot{H}H^{-1} \qquad \text{and} \qquad H^{-1}\dot{H}$$

belong to $se(3)$ where

$$se(3) := \left\{ \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \text{ s.t. } \tilde{\omega} \in so(3), v \in \mathbb{R}^3 \right\}$$

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Tilde operator

In 20-SIM

$\tilde{T}$=Tilde(T)

$$T := \begin{pmatrix} \omega \\ v \end{pmatrix} \Rightarrow \tilde{T} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix}$$

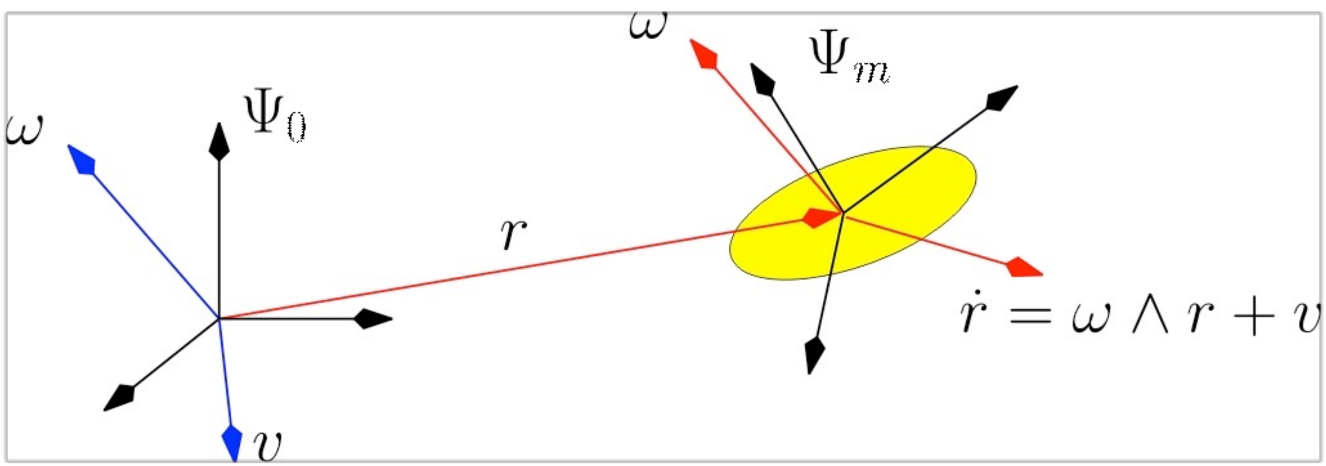
$$so(3) := \{\tilde{\omega} \in \mathbb{R}^{3\times 3} \text{ s.t. } -\tilde{\omega} = \tilde{\omega}^T\}$$

Elements of se(3): Twists

The following are vector and matrix coordinate notations for twists:

$$T := \begin{pmatrix} \omega \\ v \end{pmatrix} \Rightarrow \tilde{T} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in se(3)$$

The following $(\omega, \dot{r})$ are often called twists too, but they are no geometrical entities !



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$SO(3)$ is a Lie Group (group AND manifold)

$$(R, \dot{R}) \in TSO(3), \qquad \dot{R} \in T_R SO(3)$$

$$(L_{R^{-1}})_*(R, \dot{R}) = (R^{-1}R, R^{-1}\dot{R}) = (I, \tilde{\omega}_L)$$
$$\tilde{\omega}_L \in T_I SO(3) =: so(3)$$

$$(R_{R^{-1}})_*(R, \dot{R}) = (RR^{-1}, \dot{R}R^{-1}) = (I, \tilde{\omega}_R)$$
$$\tilde{\omega}_R \in T_I SO(3) =: so(3)$$

$$[,] \qquad [\tilde{\omega}_x, \tilde{\omega}_y] := \tilde{\omega}_x\tilde{\omega}_y - \tilde{\omega}_y\tilde{\omega}_x \in so(3)$$

$SE(3)$ is a Lie Group (group AND manifold)

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$$(H, \dot{H}) \in TSE(3), \quad \dot{H} \in T_H SE(3)$$

$$(L_{H^{-1}})_*(H, \dot{H}) = (H^{-1}H, H^{-1}\dot{H}) = (I, \tilde{T}_L)$$
$$\tilde{T}_L \in T_I SE(3) =: se(3)$$

$$(R_{H^{-1}})_*(H, \dot{H}) = (HH^{-1}, \dot{H}H^{-1}) = (I, \tilde{T}_R)$$
$$\tilde{T}_R \in T_I SE(3) =: se(3)$$

$$[,] \qquad [\tilde{T}_1, \tilde{T}_2] := \tilde{T}_1\tilde{T}_2 - \tilde{T}_2\tilde{T}_1 \in se(3)$$

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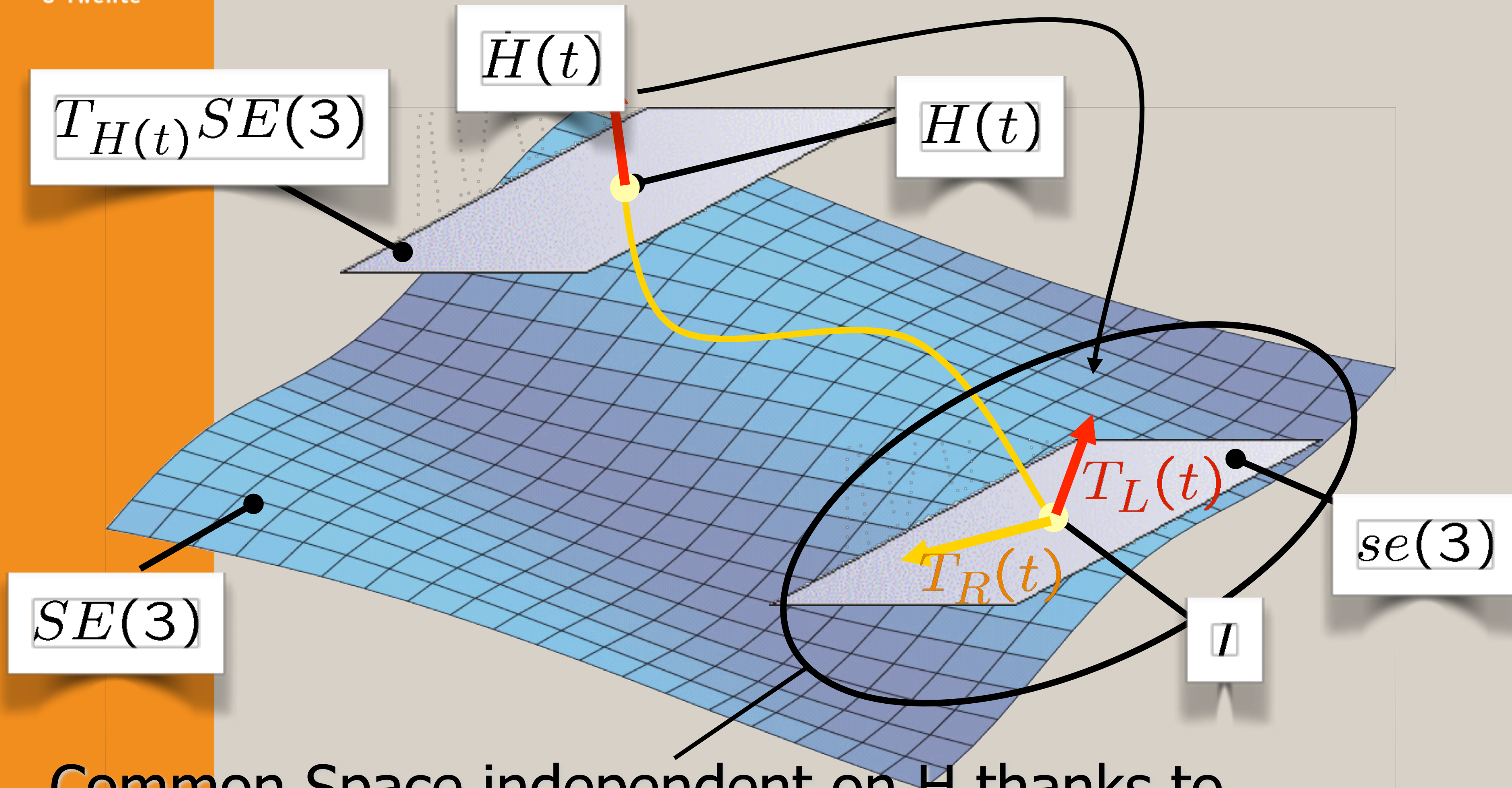
Commutator

$$[\tilde{T}_1, \tilde{T}_2] := \tilde{T}_1\tilde{T}_2 - \tilde{T}_2\tilde{T}_1 \in se(3)$$

$$\tilde{T}_i := \begin{pmatrix} \tilde{\omega}_i & v_i \\ 0 & 0 \end{pmatrix}$$

$$[\tilde{T}_1, \tilde{T}_2] = \begin{pmatrix} [\tilde{\omega}_1, \tilde{\omega}_2] & \tilde{\omega}_1 v_2 - \tilde{\omega}_2 v_1 \\ 0 & 0 \end{pmatrix}$$

Lie Groups



Common Space independent on H thanks to
Lie group structure

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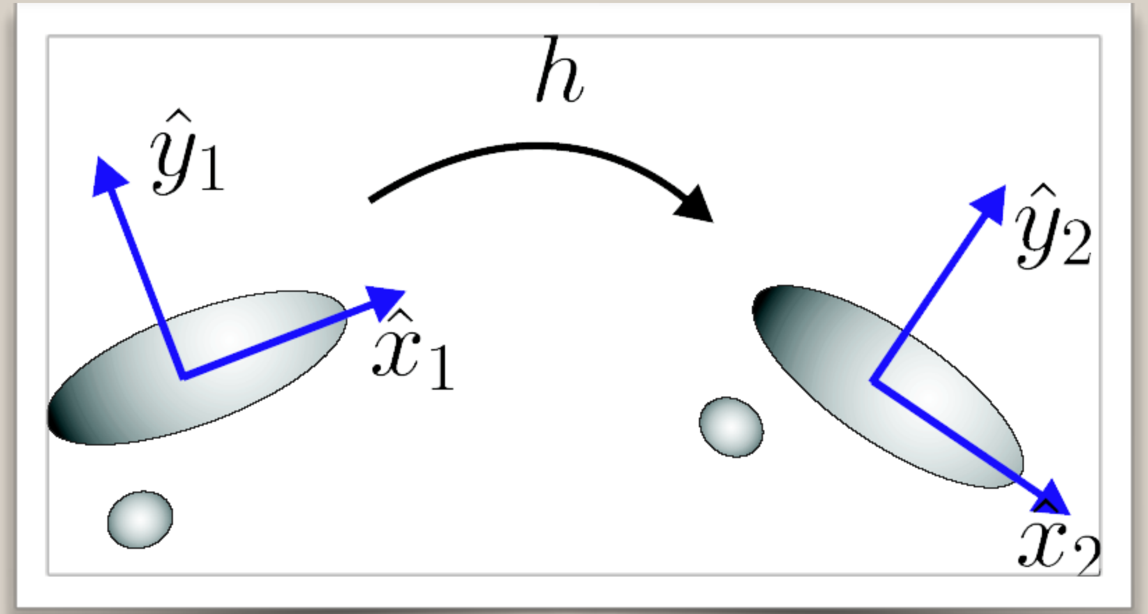
Changes of Coordinates versus Motions

Motions versus Coordinate Changes

A rigid motion is a map like:

$$h : \mathcal{E} \to \mathcal{E} \; ; \; p \mapsto h(p)$$



Features of motions

$$h : \mathcal{E} \to \mathcal{E} \; ; \; p \mapsto h(p)$$

must be both:

Isometry: it does not change the distance between points.

Orientation preserving: it does not change the orientation: swap the z axis.

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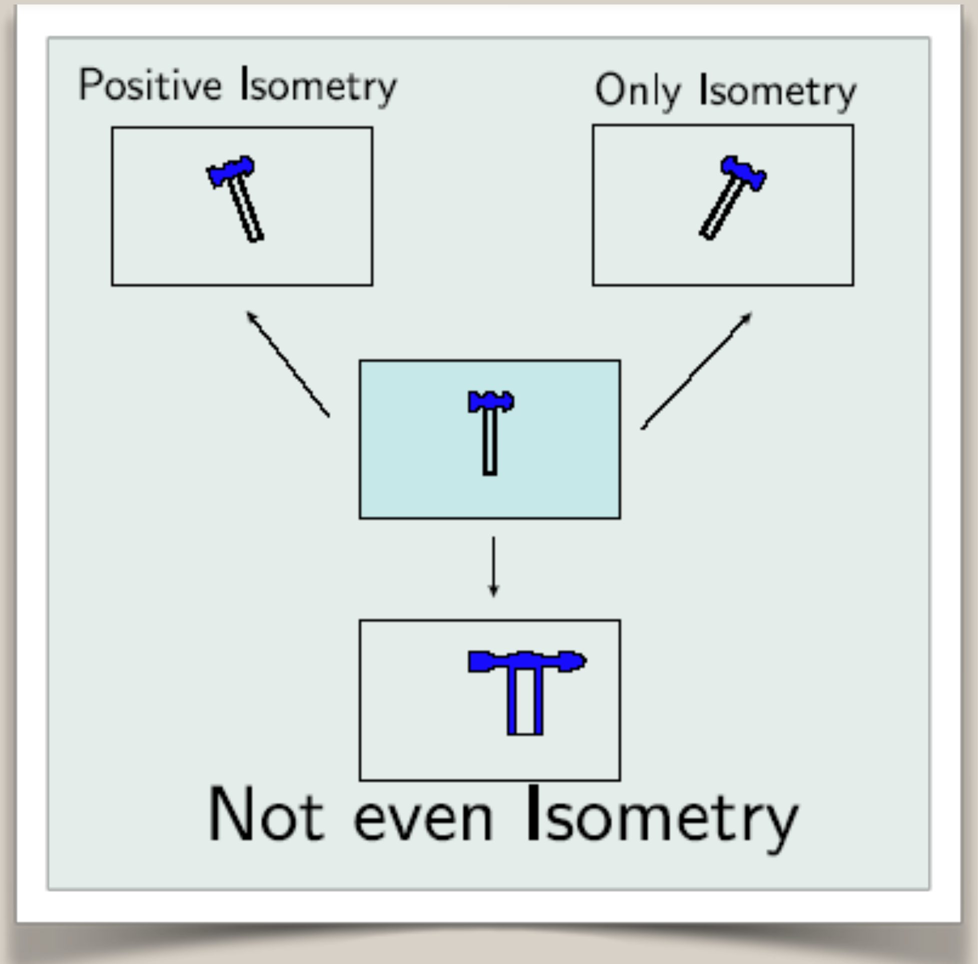
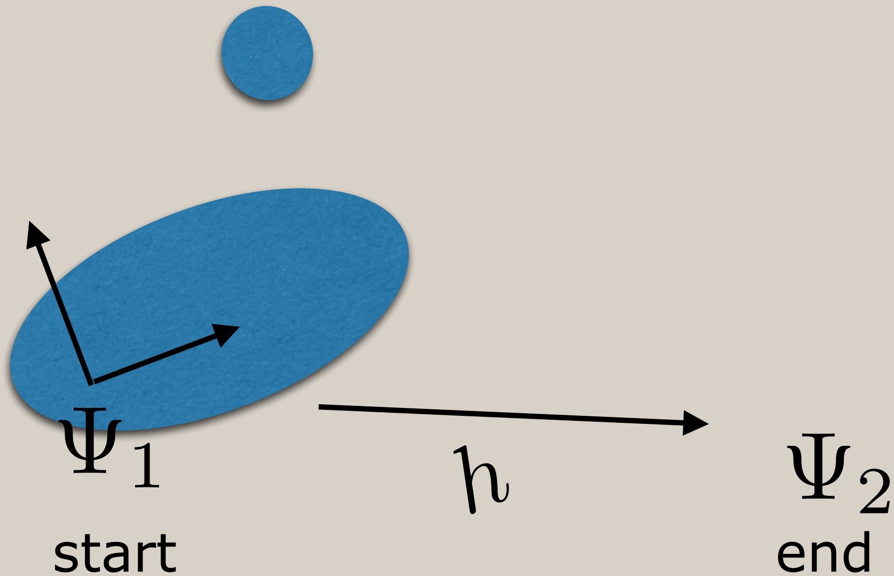
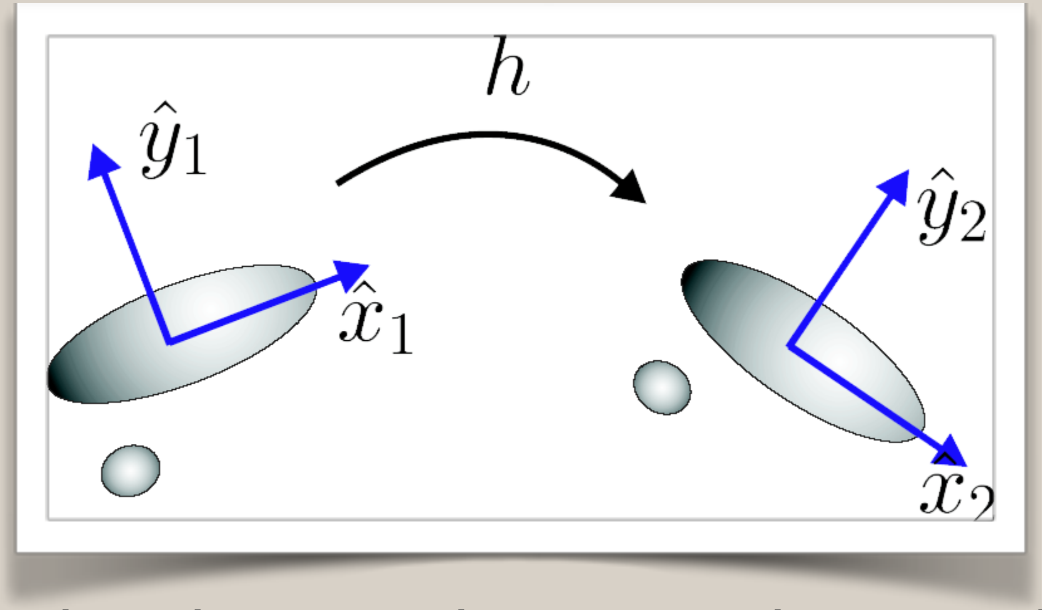
Examples





We can relate the motion h to the change of coordinates as we have learned it

Connection with coordinate changes



we want that the moved points in the moved coordinate system Ψ_2 have the same coordinates than the original points in the original coordinate system Ψ_1 :

$$\psi_2 \; o \; h \; o \; \psi_1^{-1} = I$$

But

$$\psi_2 \circ h \circ \psi_1^{-1} = I$$

is the same as:

$$\underbrace{\psi_2 \circ \psi_1^{-1}}_{H_1^2} \circ \underbrace{\psi_1 \circ h \circ \psi_1^{-1}}_{H_2^1} = I$$

The representation of the motion from Ψ_1 to Ψ_2 in the coordinate Ψ_1 ($\psi_1 \circ h \circ \psi_1^{-1}$) is the INVERSE of the coordinate change from Ψ_1 to Ψ_2 ($\psi_2 \circ \psi_1^{-1}$).

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Conclusions

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Conclusions

- A general change of coordinates in $\mathcal{E}(3)$ from Ψ_i to Ψ_j can be expressed with a matrix of the form

$$H_i^j = \begin{pmatrix} R_i^j & o_i^j \\ O_3^T & 1 \end{pmatrix} \in SE(3)$$

Where $R_i^j \in SO(3)$ and $p_i^j \in \mathbb{R}^3$

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Conclusions (cont.)

- For a rigid motion

$$h : \mathcal{E} \to \mathcal{E};\ p \mapsto q$$

- Which maps Ψ_i to Ψ_j we have that

$$q^i = H^i_j p^i \qquad q^j = H^i_j p^j$$

- Note that

$$H^i_j = (H^j_i)^{-1}.$$