



# How I spent last weekend



Watched both seasons in 1 day  
(24 series) of "Mushoku Tensei"



RAGE and VEGs clubs cooking  
collaboration event

# Circulant Matrix



Watch first video on page 11, if you want to look at derivation and the necessity of the matrix.

**Circulant matrix** ( $N = 4$ ) is:

$$C_4 = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}, \text{ where } P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

**Properties:**

It has **eigenvectors** in the Fourier Matrix columns  $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & i^2 & 1 & (-i)^2 \\ 1 & i^3 & -1 & (-i)^3 \end{bmatrix}$

**Eigenvalues** of  $C$  can be found by Fourier trans.  $F_4 [c_0, c_1, c_2, c_3]^T = [\lambda_0, \lambda_1, \lambda_2, \lambda_3]^T$

# Circulant Matrix

## Example

**Example 2** The same ideas work for a Fourier matrix  $F$  and a circulant matrix  $C$  of any size. Two by two matrices look trivial but they are very useful. Now eigenvalues of  $P$  have  $\lambda^2 = 1$  instead of  $\lambda^4 = 1$  and the complex number  $i$  is not needed:  $\lambda = \pm 1$ .

Fourier matrix  $F$  from  
eigenvectors of  $P$  and  $C$   $F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$   $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  Circulant  $c_0 I + c_1 P$   $C = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix}$ .

The eigenvalues of  $C$  are  $c_0 + c_1$  and  $c_0 - c_1$ . Those are given by the Fourier transform  $F\mathbf{c}$  when the vector  $\mathbf{c}$  is  $(c_0, c_1)$ . This transform  $F\mathbf{c}$  gives the eigenvalues of  $C$  for any size  $n$ .



## Task 1

What are the 3 solutions to  $\lambda^3 = 1$  ? They are complex numbers  $\lambda = \cos \theta + i \sin \theta = e^{i\theta}$ . Then  $\lambda^3 = e^{3i\theta} = 1$  when the angle  $3\theta$  is 0 or  $2\pi$  or  $4\pi$ . Write the 3 by 3 Fourier matrix  $F$  with columns  $(1, \lambda, \lambda^2)$ .

Check that any 3 by 3 circulant  $C$  has eigenvectors  $(1, \lambda, \lambda^2)$   
If the diagonals of your matrix  $C$  contain  $c_0, c_1, c_2$  then its eigenvalues are in  $F\mathbf{c}$ .



# Task 1

## Answer

$\lambda^3 = 1$  has 3 roots  $\lambda = 1$  and  $e^{2\pi i/3}$  and  $e^{4\pi i/3}$ . Those are  $1, \lambda, \lambda^2$  if we take  $\lambda = e^{2\pi i/3}$ . The Fourier matrix is

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda^2 \\ 1 & \lambda^2 & \lambda^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{8\pi i/3} \end{bmatrix}.$$

A 3 by 3 circulant matrix has the form on page 425 :

$$C = \begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix} \text{ with } C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (c_0 + c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = (c_0 + c_1\lambda + c_2\lambda^2) \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \quad C \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix} = (c_0 + c_1\lambda^2 + c_2\lambda^4) \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix}.$$

Those 3 eigenvalues of  $C$  are exactly the 3 components of  $Fc = F \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ ,

# 1st order system of linear differential equations

## Algorithm

1. Write equation in  $u' = Au$  form;
2. Find eigenpairs of  $A$
3. Subtract  $\lambda$  and  $x_{\lambda_i}$  to  $u(t) = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n$
4. Find  $c_i$  using  $u(0)$  – which is vector.  $u(0) = \underbrace{u(t=0)}_{\text{from above}}$ . Solve it.
5. Put  $c_i$  to  $u(t)$ .

**Example 1** Solve  $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}$  starting from  $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ .

This is a vector equation for  $\mathbf{u}$ . It contains two scalar equations for the components  $y$  and  $z$ . They are “coupled together” because the matrix  $A$  is not diagonal:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} \quad \frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad \text{means that} \quad \frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations  $y + z$  and  $y - z$  will do it. Add and subtract equations:

$$\frac{d}{dt}(y + z) = z + y \quad \text{and} \quad \frac{d}{dt}(y - z) = -(y - z).$$

The combination  $y + z$  grows like  $e^t$ , because it has  $\lambda = 1$ . The combination  $y - z$  decays like  $e^{-t}$ , because it has  $\lambda = -1$ . Here is the point: We don’t have to juggle the original equations  $d\mathbf{u}/dt = A\mathbf{u}$ , looking for these special combinations. The eigenvectors and eigenvalues of  $A$  will do it for us.

This matrix  $A$  has eigenvalues 1 and  $-1$ . The eigenvectors  $\mathbf{x}$  are  $(1, 1)$  and  $(1, -1)$ . The pure exponential solutions  $\mathbf{u}_1$  and  $\mathbf{u}_2$  take the form  $e^{\lambda t}\mathbf{x}$  with  $\lambda_1 = 1$  and  $\lambda_2 = -1$ :

$$\mathbf{u}_1(t) = e^{\lambda_1 t} \mathbf{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2(t) = e^{\lambda_2 t} \mathbf{x}_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (4)$$

Notice: These  $\mathbf{u}$ ’s satisfy  $A\mathbf{u}_1 = \mathbf{u}_1$  and  $A\mathbf{u}_2 = -\mathbf{u}_2$ , just like  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . The factors  $e^t$  and  $e^{-t}$  change with time. Those factors give  $d\mathbf{u}_1/dt = \mathbf{u}_1 = A\mathbf{u}_1$  and  $d\mathbf{u}_2/dt = -\mathbf{u}_2 = A\mathbf{u}_2$ . **We have two solutions to  $d\mathbf{u}/dt = A\mathbf{u}$ .** To find all other solutions, **multiply those special solutions by any numbers  $C$  and  $D$  and add:**

**Complete solution**

$$\mathbf{u}(t) = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \quad (5)$$



With these two constants  $C$  and  $D$ , we can match any starting vector  $\mathbf{u}(0) = (u_1(0), u_2(0))$ . Set  $t = 0$  and  $e^0 = 1$ . Example 1 asked for the initial value to be  $\mathbf{u}(0) = (4, 2)$ :

$$\mathbf{u}(0) \text{ decides } C, D \quad C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{yields } C = 3 \quad \text{and } D = 1.$$

With  $C = 3$  and  $D = 1$  in the solution (5), the initial value problem is completely solved. The same three steps that solved  $\mathbf{u}_{k+1} = A\mathbf{u}_k$  now solve  $d\mathbf{u}/dt = A\mathbf{u}$ :

1. Write  $\mathbf{u}(0)$  as a **combination**  $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$  **of the eigenvectors of  $A$** .
2. Multiply each eigenvector  $\mathbf{x}_i$  by **its growth factor**  $e^{\lambda_i t}$ .
3. The solution is the same combination of those pure solutions  $e^{\lambda t}\mathbf{x}$ :

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} \quad \mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + \cdots + c_n e^{\lambda_n t} \mathbf{x}_n. \quad (6)$$

*Not included:* If two  $\lambda$ 's are equal, with only one eigenvector, another solution is needed. (It will be  $te^{\lambda t}\mathbf{x}$ .) Step 1 needs to diagonalize  $A = X\Lambda X^{-1}$ : a basis of  $n$  eigenvectors.

**Example 2** Solve  $d\mathbf{u}/dt = A\mathbf{u}$  knowing the eigenvalues  $\lambda = 1, 2, 3$  of  $A$ :

$$\begin{array}{l} \text{Typical example} \\ \text{Equation for } \mathbf{u} \\ \text{Initial condition } \mathbf{u}(0) \end{array} \quad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{u} \quad \text{starting from } \mathbf{u}(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}.$$

The eigenvectors are  $\mathbf{x}_1 = (1, 0, 0)$  and  $\mathbf{x}_2 = (1, 1, 0)$  and  $\mathbf{x}_3 = (1, 1, 1)$ .

**Step 1** The vector  $\mathbf{u}(0) = (9, 7, 4)$  is  $2\mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3$ . Thus  $(c_1, c_2, c_3) = (2, 3, 4)$ .

**Step 2** The factors  $e^{\lambda t}$  give exponential solutions  $e^t\mathbf{x}_1$  and  $e^{2t}\mathbf{x}_2$  and  $e^{3t}\mathbf{x}_3$ .

**Step 3** The combination that starts from  $\mathbf{u}(0)$  is  $\mathbf{u}(t) = 2e^t\mathbf{x}_1 + 3e^{2t}\mathbf{x}_2 + 4e^{3t}\mathbf{x}_3$ .

The coefficients 2, 3, 4 came from solving the linear equation  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{u}(0)$ :

$$\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} \quad \text{which is } X\mathbf{c} = \mathbf{u}(0). \quad (7)$$



## Task 2

Find two  $\lambda$ 's and  $\mathbf{x}$ 's so that  $\mathbf{u} = e^{\lambda t} \mathbf{x}$  solves

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}.$$

What combination  $\mathbf{u} = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2$  starts from  $\mathbf{u}(0) = (5, -2)$ ?



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$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}.$$

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## Answer

Eigenvalues 4 and 1 with eigenvectors  $(1, 0)$  and  $(1, -1)$  give solutions  $\mathbf{u}_1 = e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   
and  $\mathbf{u}_2 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . If  $\mathbf{u}(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , then  $\mathbf{u}(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .



## Task 3

Suppose  $P$  is the projection matrix onto the  $45^\circ$  line  $y = x$  in  $\mathbf{R}^2$ . What are its eigenvalues? If  $du/dt = -Pu$  (notice minus sign) can you find the limit of  $u(t)$  at  $t = \infty$  starting from  $u(0) = (3, 1)$ ?



## Task 3

Suppose  $P$  is the projection matrix onto the  $45^\circ$  line  $y = x$  in  $\mathbf{R}^2$ . What are its eigenvalues? If  $du/dt = -Pu$  (notice minus sign) can you find the limit of  $u(t)$  at  $t = \infty$  starting from  $u(0) = (3, 1)$ ?

### Answer

A projection matrix has eigenvalues  $\lambda = 1$  and  $\lambda = 0$ . Eigenvectors  $Px = x$  fill the subspace that  $P$  projects onto: here  $x = (1, 1)$ . Eigenvectors with  $Px = 0$  fill the perpendicular subspace: here  $x = (1, -1)$ . For the solution to  $u' = -Pu$ ,

$$u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad u(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

# Higher order differential equation

*Idea*



$$u'' + Bu' + Cu = 0 \text{ is equivalent to } \begin{bmatrix} u \\ u' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}$$

Everything else is the same as in first order system.

**The most important equation in mechanics is  $my'' + by' + ky = 0$ .** The first term is the mass  $m$  times the acceleration  $a = y''$ . This term  $ma$  balances the force  $F$  (that is *Newton's Law*). The force includes the damping  $-by'$  and the elastic force  $-ky$ , proportional to distance moved. This is a second-order equation because it contains the second derivative  $y'' = d^2y/dt^2$ . It is still linear with constant coefficients  $m, b, k$ .

In a differential equations course, the method of solution is to substitute  $y = e^{\lambda t}$ . Each derivative of  $y$  brings down a factor  $\lambda$ . We want  $y = e^{\lambda t}$  to solve the equation:

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0 \quad \text{becomes} \quad (m\lambda^2 + b\lambda + k) e^{\lambda t} = 0. \quad (8)$$

Everything depends on  $m\lambda^2 + b\lambda + k = 0$ . This equation for  $\lambda$  has two roots  $\lambda_1$  and  $\lambda_2$ . Then the equation for  $y$  has two pure solutions  $y_1 = e^{\lambda_1 t}$  and  $y_2 = e^{\lambda_2 t}$ . Their combinations  $c_1 y_1 + c_2 y_2$  give the complete solution unless  $\lambda_1 = \lambda_2$ .

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with  $y''$ ) into a *vector equation for  $y$  and  $y'$* : first derivative only. Suppose the mass is  $m = 1$ . Two equations for  $\mathbf{u} = (y, y')$  give  $d\mathbf{u}/dt = A\mathbf{u}$ :

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -ky - by' \end{aligned} \quad \text{converts to} \quad \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}. \quad (9)$$

The first equation  $dy/dt = y'$  is trivial (but true). The second is equation (8) connecting  $y''$  to  $y'$  and  $y$ . Together they connect  $\mathbf{u}'$  to  $\mathbf{u}$ . So we solve  $\mathbf{u}' = A\mathbf{u}$  by eigenvalues of  $A$ :

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

**The equation for the  $\lambda$ 's is the same as (8)! It is still  $\lambda^2 + b\lambda + k = 0$ , since  $m = 1$ .** The roots  $\lambda_1$  and  $\lambda_2$  are now *eigenvalues of  $A$* . The eigenvectors and the solution are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \quad \mathbf{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of  $\mathbf{u}(t)$  has  $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ —the same solution as before. It can't be anything else. In the second component of  $\mathbf{u}(t)$  you see the velocity  $dy/dt$ . The vector problem is completely consistent with the scalar problem. The 2 by 2 matrix  $A$  is called a *companion matrix*—a companion to the second order equation with  $y''$ .

### Example 3 *Motion around a circle with $y'' + y = 0$ and $y = \cos t$*

This is our master equation with mass  $m = 1$  and stiffness  $k = 1$  and  $d = 0$ : no damping. Substitute  $y = e^{\lambda t}$  into  $y'' + y = 0$  to reach  $\lambda^2 + 1 = 0$ . The roots are  $\lambda = i$  and  $\lambda = -i$ . Then half of  $e^{it} + e^{-it}$  gives the solution  $y = \cos t$ .

As a first-order system, the initial values  $y(0) = 1, y'(0) = 0$  go into  $\mathbf{u}(0) = (1, 0)$ :

$$\text{Use } y'' = -y \qquad \frac{d\mathbf{u}}{dt} = \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}. \qquad (10)$$

The eigenvalues of  $A$  are again the same  $\lambda = i$  and  $\lambda = -i$  (no surprise).  $A$  is anti-symmetric with eigenvectors  $\mathbf{x}_1 = (1, i)$  and  $\mathbf{x}_2 = (1, -i)$ . The combination that matches  $\mathbf{u}(0) = (1, 0)$  is  $\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ . Step 2 multiplies the  $\mathbf{x}$ 's by  $e^{it}$  and  $e^{-it}$ . Step 3 combines the pure oscillations into  $\mathbf{u}(t)$  to find  $y = \cos t$  as expected:

$$\mathbf{u}(t) = \frac{1}{2}e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}. \qquad \text{This is } \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

All good. The vector  $\mathbf{u} = (\cos t, -\sin t)$  goes around a circle (Figure 6.3). The radius is 1 because  $\cos^2 t + \sin^2 t = 1$ .



## Task 4



Solve  $y'' + 4y' + 3y = 0$  by linear algebra.

## Task 4



Solve  $y'' + 4y' + 3y = 0$  by linear algebra.

### Answer

To use linear algebra we set  $\mathbf{u} = (y, y')$ . Then the vector equation is  $\mathbf{u}' = A\mathbf{u}$ :

$$\begin{aligned} \frac{dy}{dt} &= y' \\ \frac{dy'}{dt} &= -3y - 4y' \end{aligned} \quad \text{converts to} \quad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{u}.$$

This  $A$  is a “companion matrix” and its eigenvalues are again  $-1$  and  $-3$ :

**Same quadratic**  $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$

The eigenvectors of  $A$  are  $(1, \lambda_1)$  and  $(1, \lambda_2)$ . Either way, the decay in  $y(t)$  comes from  $e^{-t}$  and  $e^{-3t}$ . With constant coefficients, calculus leads to linear algebra  $A\mathbf{x} = \lambda\mathbf{x}$ .

## Task 5

Find solution in general form  $(c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n)$ :

$$u''' + 2u'' - u' - 2u = 0, u(0) = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

## Task 5



$$u''' + 2u'' - u' - 2u = 0, u(0) = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

Answer

1. Equation:  $\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix};$

2. Eigenpairs:  $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1; x_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$

3.  $c_1 = 1, c_2 = -1, c_3 = 3;$

4. General form:  $u(t) = 1e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + (-1)e^{-1t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3e^{1t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$



## Reference material

- Eigenvectors of Circulant Matrices: Fourier Matrix
- Lecture 23, Differential Equations and  $\exp(At)$
- *"Linear Algebra and Applications"*, pdf pages 435–436  
Circulant Matrix 8.3
- *"Linear Algebra and Applications"*, pdf pages 330–348  
Systems of Differential Equations 6.3

# Deserve "A" grade!

– Oleg Bulichev

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🏢 Room 105 (Underground robotics lab)