



# How to study Null Space

## Step-by-step guide

### 1. Lecture 6, Gilbert Strang

**Goal** is to understand the basics of spaces and how Null Space appeared.

### 2. Khan Academy: Null space

It contains a good case study how to calculate Null space.

### 3. Matrix Algebra for Engineers: Null Space

Another nice example how to find  $N(A)$ .

### 4. "Linear Algebra and Applications", pdf pages 96–106

What does partial and full solutions means

### 5. The Big Picture of Linear Algebra

*Extra for now* If you want to get the global view of four subspaces

### 6. Understand the application from next few slides and make HW tasks!



# Null Space: Application from robotics

*Video*



# Null Space: Application from robotics

## Theory (1)

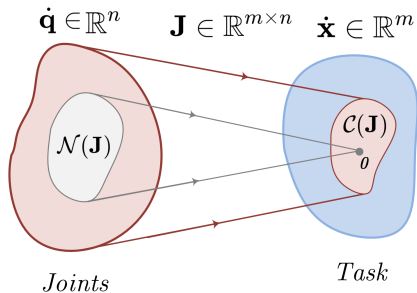


Figure 1: Click for google Collab

Let us consider differential kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \quad (1)$$

where

- $\mathbf{x} \in \mathbb{R}^m$  task space variables (for instance Cartesian coordinates)
- $\mathbf{q} \in \mathbb{R}^n$  joint space variables (positions of joints)
- $\mathbf{J} \in \mathbb{R}^{m \times n}$  manipulator Jacobian

# Null Space: Application from robotics

## Theory (2)

general solution of  $J\dot{q} = \dot{r}$

$$\dot{q} = J^\# \dot{r} + (I - J^\# J) \dot{q}_0$$

$J^\# \dot{r}$  is a particular solution (here, the pseudoinverse) in  $\mathcal{R}(J^T)$

$(I - J^\# J) \dot{q}_0$  is "orthogonal" projection of  $\dot{q}_0$  in  $\mathcal{N}(J)$

all solutions of the associated homogeneous equation  $J\dot{q} = 0$  (self-motions)

properties of projector  $[I - J^\# J]$

- symmetric
- idempotent:  $[I - J^\# J]^2 = [I - J^\# J]$
- $[I - J^\# J]^\# = [I - J^\# J]$
- $J^\# \dot{r}$  is orthogonal to  $[I - J^\# J] \dot{q}_0$

even more in general...

$$\dot{q} = K_1 \dot{r} + (I - K_2 J) \dot{q}_0$$

$K_1, K_2$  generalized inverses of  $J$  ( $JK_i J = J$ )

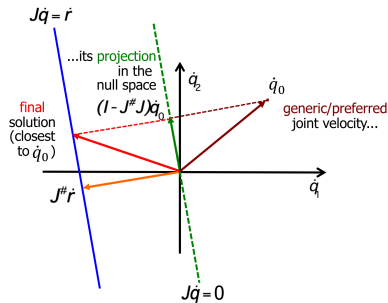
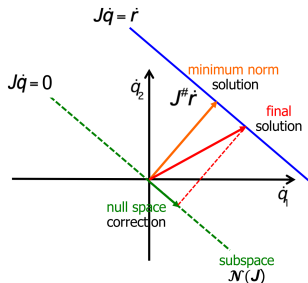
... but with less nice properties!

how do we choose  $\dot{q}_0$ ?

# Null Space: Application from robotics

## Theory (3)

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution

- i) which is in the **null space** of the Jacobian
- ii) and possibly satisfies **additional criteria** or objectives

## Task 1



Reduce  $A$  and  $B$  to their triangular echelon forms  $U$ . Which variables are free?

$$(a) \quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad (b) \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

## Task 1



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## Answer

$$(a) \quad U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Free variables } x_2, x_4, x_5 \\ \text{Pivot variables } x_1, x_3 \end{array} \quad (b) \quad U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Free } x_3 \\ \text{Pivot } x_1, x_2 \end{array}$$



## Task 2



Construct 3 by 3 matrices  $A$  to satisfy these requirements (if possible):

- (a)  $A$  has no zero entries but  $U = I$ .
- (b)  $A$  has no zero entries but  $R = I$ .
- (c)  $A$  has no zero entries but  $R = U$ .
- (d)  $A = U = 2R$ .

## Task 2



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- (a)  $A$  has no zero entries but  $U = I$ .
- (b)  $A$  has no zero entries but  $R = I$ .
- (c)  $A$  has no zero entries but  $R = U$ .
- (d)  $A = U = 2R$ .

## Answer

- (a) Impossible row 1    (b)  $A =$  invertible    (c)  $A =$  all ones    (d)  $A = 2I, R = I$ .

## Task 3



If the special solutions to  $R\mathbf{x} = \mathbf{0}$  are in the columns of these nullspace matrices  $N$ , go backward to find the nonzero rows of the reduced matrices  $R$ :

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} \quad (\text{empty } 3 \text{ by } 1).$$



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## Answer

Any zero rows come after these rows:  $R = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$ ,  $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,  $R = I$ .

## Reference material



- Robotics 2 course from Sapienza
- "*Linear Algebra and Applications*", pdf pages 87–101  
2.1–2.2
- "*Introduction to Linear Algebra*", pdf pages 132–152  
3.1 – 3.2
- this lab video, 2022 year



## Preparation material for the next class

- Lecture 9 and 10
- "*Linear Algebra and Applications*", pdf pages 139–149  
The application of four fundamental subspaces in CS
- Matrix Transpose and the Four Fundamental Subspaces  
Video is about how  $A$  transpose appeared
- Matrix online calculator(russian)

# Deserve "A" grade!

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🏢 Room 105 (Underground robotics lab)