LINEAR ALGEBRA. LECTURE 3

Vector spaces

Elimination can simplify, one entry at a time, the linear system $Ax = b$. Fortunately it also simplifies the theory. The basic questions of *existence* and *uniqueness* — Is there one solution, or no solution, or an infinity of solutions — are much easier to answer after elimination, we need to devote one more section to those questions, to find every solution for an m by n system. Then that circle of ideas will be complete.

We can add vectors and multiply them by numbers, which means we can discuss *linear combinations* of vectors. These combinations follow the rules of a *vector space*. One such vector space is $\mathbf{R}^2$, the set of all vectors with exactly two real number components. We depict the vector $\{a,b\}$ by drawing an arrow from the origin to the point (a, b) which is a units to the right of the origin and b units above it, and we call $\mathbf{R}^2$ the "x - y plane". Another example of a space is $\mathbf{R}^n$, the set of vectors (columns) with n real number components.

Closure

The collection of vectors with exactly two *positive* real valued components is *not* a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, -5, gives a vector that's not in the collection. We say that this collection of positive vectors is *closed* under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example, take any non-zero vector V in $\mathbf{R}^2$. Then the set of all vectors cV, where c is a real number, forms a subspace of $\mathbf{R}^2$. This collection of vectors describes a line through $\{0,0\}$ in $\mathbf{R}^2$ and is closed under addition.

Definition. A *subspace* of a vector space is a nonempty subset that satisfies the requirements for a vector space: *Linear combinations stay in the subspace*.

(i) If we add any vectors x and y in the subspace, $x+y$ is *in the subspace*.

(ii) If we multiply any vector x in the subspace by any scalar c, cx is *in the subspace*.

A line in $\mathbf{R}^2$ that does not pass through the origin is *not* a subspace of $\mathbf{R}^2$. Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of $\mathbf{R}^2$ are:

1. all of $\mathbf{R}^2$,
2. any line through {0,0} and
3. the zero vector alone (Z).

The subspaces of $\mathbf{R}^3$ are:

1. all of $\mathbf{R}^3$,
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

Column space and nullspace

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors V and W in the space and any two real numbers c and d, the vector $cV+dW$ is also in the vector space. A subspace is a vector space contained inside a vector space. A plane P containing {0,0,0} and a line L containing {0,0,0} are both subspaces of $\mathbf{R}^3$. The union $P \cup L$ of those two subspaces is generally not a subspace, because the sum of a vector in P and a vector in L is probably not contained in $P \cup L$. The intersection $S \cap T$ of two subspaces S and T is a subspace. To prove this, use the fact that both S and T are closed under linear combinations to show that their intersection is closed under linear combinations.

Column space

Given a matrix A with columns in $\mathbf{R}^3$, these columns and all their linear combinations form a subspace of $\mathbf{R}^3$. This is the *column space* $C(A)$. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ the column space of A is the plane through the origin in $\mathbf{R}^3$ containing vectors $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

Our next task will be to understand the equation $Ax = b$ in terms of subspaces and the column space of A.

Column space of *A*

The *column space* of a matrix A (mxn) *contains all linear combinations of the columns of* A. It is a subspace of R^m. With $m > n$ we have more equations than unknowns in $Ax=b$ and *usually there will be no solution*. The system will be solvable only for a very “thin” subset of all possible b’s. One way of describing this thin subset is so simple that it is easy to overlook. The system $Ax = b$ is solvable if and only if the vector b can be expressed as a *combination of the columns* of A. Then b is in the *column space*.

Solving *Ax* = *b*

Given a matrix A, for what vectors b does $Ax = b$ has a solution x?

$$\text{Let} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Then $Ax = b$ does not have a solution for every choice of b because solving $Ax = b$ is equivalent to solving four linear equations in three unknowns. If there is a solution x to $Ax = b$, then b

must be a linear combination of the columns of *A*. Only three columns cannot fill the entire four dimensional vector space – any vectors *b* cannot be expressed as linear combinations of columns of *A*.

Big question: what *b*'s allow $Ax = b$ to be solved?

A useful approach is to choose *x* and find the vector $b = Ax$ corresponding to that solution. The components of *x* are just the coefficients in a linear combination of columns of *A*. The system of linear equations $Ax = b$ is *solvable* exactly when *b* is a vector in the *column space* of *A*.

For our example matrix *A*, what can we say about the column space of *A*? Are the columns of *A independent*? In other words, does each column contribute something new to the subspace? The third column of *A* is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix *A* is a two dimensional subspace $\mathbf{R}^2$ of $\mathbf{R}^4$.

Nullspace of *A*

The *nullspace* of a matrix *A* is the collection of all solutions $x = \{x_1,x_2,x_3\}$ to the equation $Ax = 0$. The column space of the matrix in our example was a $\mathbf{R}^2$ subspace of $\mathbf{R}^4$. The nullspace of *A* is a subspace of $\mathbf{R}^3$. To see that it's a vector space, check that any sum or multiple of solutions to $Ax = 0$ is also a solution: $A(x_1+x_2) = Ax_1 + Ax_2 = 0+0 = 0$ and $A(cx)= cAx = 0$.

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the nullspace $N(A)$ consists of all multiples of $c\{1,1,-1\}=\{c,c,-c\}$; column 1 plus column 2 minus column 3 equals the zero vector. The nullspace of *A* is the line in in $\mathbf{R}^3$ of all points $x_1 = c$, $x_2 = c$, $x_3 = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides *b* and all solutions to $Ax = 0$.

The vectors *b* are in the column space and the vectors *x* are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them.

Other values of *b*

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in $\mathbf{R}^3$ that passes through the points {1,0,0} and {0,-1,1} but not {0,0,0}.

Solving $Ax = 0$: factor variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

Computing the nullspace

The *Nullspace* of a matrix A is made up of the vectors x for which $Ax = 0$. Suppose:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

(Note that the columns of this matrix A are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that A is not invertible. We don't need to use an augmented matrix because the right side (the vector b) is 0 in this computation.

The row operations used in the method of elimination don't change the solution to $Ax = b$ so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

We don't find a factor in the second column, so our next factor is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix U is in *Echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated. The *Rank* of a matrix A equals the number of factors it has. In this example, the rank of A (and of U) is 2.

Special solutions

Once we've found U we can use back-substitution to find the solutions x to the equation $Ux = 0$. In our example, columns 1 and 3 are *factor columns* containing factors, and columns 2 and 4 are *free columns*. We can assign any value to x_2 and x_4; we call these *free variables*. Suppose $x_2 = 1$ and $x_4 = 0$. Then:

$$2x_3 + 4x_4 = 0 \Rightarrow \quad x_3 = 0 \quad \text{and} \quad x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \Rightarrow \quad x_1 = -2$$

So, one solution is $\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, because the second column is just twice the first column. Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has $x_4 = 1$ and $x_2 = 0$. The nullspace of A is the collection of all linear combinations of these "special solution" vectors.

The rank r of A equals the number of factor columns, so the number of free columns is $n - r$: the number of columns (variables) minus the number of factor columns. This equals the number of special solution vectors and the dimension of the nullspace.

Reduced row echelon form

By continuing to use the method of elimination we can convert U to a matrix R in *Reduced Row Echelon Form* (RREF form), with factors equal to 1 and zeros above and below the factors.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Here I is an rxr square matrix and F is an rx$(n - r)$ free columns matrix.

If N is the *Nullspace matrix:* $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ then $RN = 0$. Here I is an $(n - r)$x$(n - r)$ square matrix and N is an mx$(n - r)$ matrix. The columns of N are the special solution.

Example: Find the *Nullspace* of a matrix A^T:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

First find the matrix U in *Echelon* (staircase) form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this example, the rank of A (and of U) is 2 again.

Once we've found U we can use back-substitution to find the solutions x to the equation $Ux = 0$. In this example, columns 1 and 2 are *factor columns* containing factors, and column 3 is *free column*. We can assign any value to x_3; we call these *free variable*. Suppose $x_3 = 1$. Then:

$$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1 \text{ and } x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -1$$

So one solution is $\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, because the third column is just the sum of the first and second columns. Any multiple of this vector is in the nullspace: $\vec{x}_N = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

By continuing to use the method of elimination we can convert U to a matrix R in *reduced row echelon form* (RREF form):

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Here I is an rxr square matrix and F is an rx$(n-r)$ one free column matrix.

Here the *nullspace matrix* $N = \begin{bmatrix} -F \\ I \end{bmatrix} = \vec{x}_N = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ and $RN = 0$. Here I is an $(n-r)$x$(n-r)$ one element matrix and N is an mx$(n-r)$ matrix-vector. The column of N is the special solution.

Solving $Ax = b$: row reduced form R

When does $Ax = b$ has solutions x, and how can we describe those solutions?

Solvability conditions on *b*

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

The third row of A is the sum of its first and second rows, so we know that if $Ax = b$ the third component of b equals the sum of its first and second components. If b does not satisfy $b3 = b1 + b2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of b must equal zero.

One way to find out whether $Ax = b$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in b. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

If $Ax=b$ has a solution, then $b3 - b2 - b1 = 0$. For example, we could choose $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $Ax = b$ is solvable exactly when b is in the column space $C(A)$. We have these two conditions on b; in fact, they are equivalent.

Complete solution

In order to find all solutions to $Ax = b$ we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $Ax = b$ is to set all free variables to zero, then solve for the factor variables. For our example matrix A, we let $x2 = x4 = 0$ to get the system of equations:

$$\begin{cases} x_1 + 2x_3 = 1 \\ 2x_3 = 3 \end{cases}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our *Particular solution* is: $\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

The general solution to $Ax = b$ is given by $\vec{x}_{complete} = \vec{x}_N + \vec{x}_p$, where $\vec{x}_N$ is a generic vector in the nullspace. To see this, we add $Ax_p = b$ to $Ax_N = 0$ and get $A(x_p + x_N) = b$ for every vector x_N in the nullspace.

We know that the nullspace of A is the collection of all combinations of the special solutions: $\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\vec{x}_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The nullspace of A is a two-dimensional subspace of $\mathbf{R}^4$, and the solutions to the equation $Ax = b$ form a plane parallel to that through $\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of factors(pivots) of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $Ax = b$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$. For any vector b in $\mathbf{R}^\mathbf{m}$ that's not a linear combination of the columns of A, there is no solution to $Ax = b$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of b to satisfy. The equation $Ax = b$ is solvable for every b. There are $n-r = n-m$ free variables, so there are $n-m$ special solutions to $Ax = 0$.

Full row and column rank

If $r = m = n$ is the number of factors of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $Ax = b$ has a unique solution for every b in $\mathbf{R}^\mathbf{m}$.

Summary

If R is in row reduced form with pivot columns first (RREF), the table below summarizes our results.

Rank(A)	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
RREF	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Number solutions to $Ax = b$	1	$1,\ b \in C(A)$ $0,\ b \notin C(A)$	Infinitely many (∞)	$\infty,\ b \in C(A)$ $0,\ b \notin C(A)$