



Analytical Geometry and Linear Algebra II, Lab 7

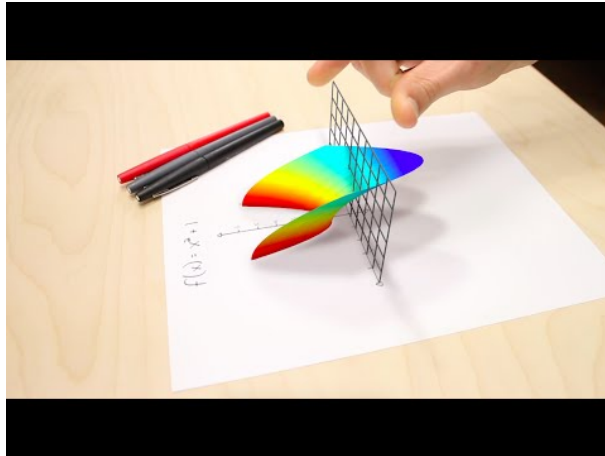
Complex numbers

Complex matrices

Hermitian and Unitary Matrices

Complex numbers

Video



Complex numbers

Forms

Rectangular form: $z = x + iy$, $i^2 = -1$

$\text{Re}(z) = x$ - real part, $\text{Im}(z) = y$ - imaginary part

Example: $z = 5 + i6$

Polar form: $z = r \cos(\phi) + i r \sin(\phi)$, where

$\phi = \text{atan2}(\text{Im}(z), \text{Re}(z))$;

$r = |z| = \sqrt{x^2 + y^2}$ - magnitude

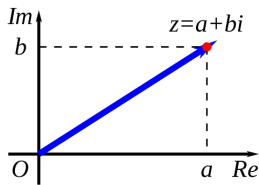
Example: $z = 8 \cos(24) + i \sin(24)$

Exponential form: $z = re^{i\phi}$

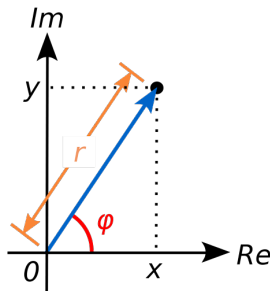
Example: $z = 6e^{i2.5}$

Euler formula: transformation from exp. to polar

$$e^{i\phi} = \cos(\phi) + i \sin(\phi)$$



Rectangular
form



Polar or
Exponential
form

Complex numbers

Operations



General Idea: you should work with *Im* and *Re* part separately (you cannot sum or multiply them)

- *Summarization and Subtraction* - $(x_1 \pm iy_1) + (x_2 \pm iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- *Multiplication* - $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
- *Division* - $\frac{(x_2 + iy_2)}{(x_1 + iy_1)} = \frac{(x_1x_2 + y_1y_2)}{x_1^2 + y_1^2} + i\frac{(y_1x_2 - x_1y_2)}{x_1^2 + y_1^2}$

Complex numbers

Complex conjugate

Complex conjugate of complex number $z = x + iy$ - $\bar{z} = x - iy$.

Geometrically it's reflection of z about Re axis.

Properties:

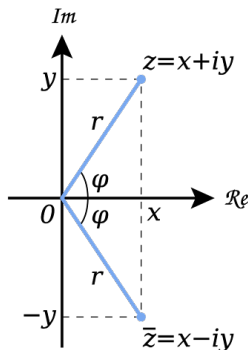
$\operatorname{Re}(\bar{z}) = \operatorname{Re}(z)$ and $|\bar{z}| = |z|$;

$\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$ and $\arg \bar{z} \equiv -\arg z \pmod{2\pi}$

$z\bar{z} = x^2 + y^2 = |z|^2$ - absolute square

Operations

- Summarization and Subtraction - $\overline{z \pm w} = \bar{z} \pm \bar{w}$
- Multiplication - $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- Division - $\overline{z/w} = \bar{z}/\bar{w}$



Complex Conjugate

Complex numbers

Applications (1)



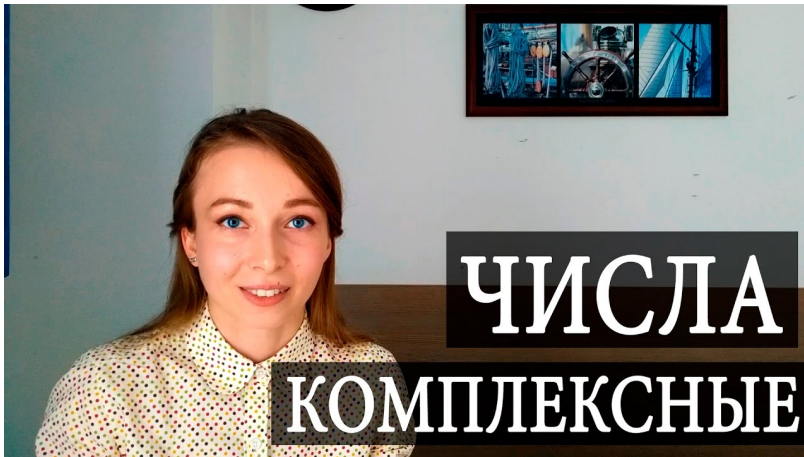
Typical reason

Most of the time for simplicity your calculations, except Quantum computing and some other topics.

1. *Electromagnetism*: electric and magnetic fields as complex vector fields to describe electromagnetic waves. However, this is just a computational trick.
2. *Aerodynamics*: calculating wing shapes using Joukowski transform.
3. *Geodesy*: Google Maps (from Geoid to 2d map).
4. *Quantum Computing*

Complex numbers

Applications (2): Video



Complex numbers



Applications(3): more about Quantum Computing (Links)

- Representing Qubit States as complex numbers
- Why do we have to use complex numbers in Quantum computing (Nature paper)
- Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit)

*If you are interested in the topic, you can ask **Stas Protasov**.*



Task 1

1 For $z = \frac{1+i}{\sqrt{2}}$:

- Compute z^2
- Find r
- Find ϕ
- Find z in exponential form

- 2
- Find the 8 solutions to the equation $z^8 = 1$
 - Plot those 8 solutions in the complex plane

3 For $z = -1 + i\frac{1}{2}$

- Find complex conjugate (\bar{z})
- Find $z\bar{z}$
- Find $z + \bar{z}$
- Plot each result in the complex plane

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$
- i^i
- Show each result in the complex plane



Task 1

1 For $z = \frac{1+i}{\sqrt{2}}$:

- Compute z^2

- Find r
- Find ϕ
- Find z in exponential form

Answer

- $z^2 = i$
- $r = 1$
- $\phi = 45^\circ$
- $z = 1e^{i\frac{\pi}{4}}$



Task 1

- 2 – Find the 8 solutions to the equation $z^8 = 1$
- Plot those 8 solutions in the complex plane

Answer

Solution (rus): $\pm 1; \pm j; \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2};$



Task 1

3 For $z = -1 + i\frac{1}{2}$

- Find complex conjugate (\bar{z})
- Find $z\bar{z}$

- Find $z + \bar{z}$

- Plot each result in the complex plane

Answer

- $\bar{z} = -1 - i\frac{1}{2}$
- $z\bar{z} = 1.25$
- $z + \bar{z} = -2$



Task 1

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$

- i^i
- Show each result in the complex plane

Answer

- $e^{i\frac{\pi}{2}} = i$
- $e^{i\pi} = -1$
- $i^i = e^{\ln(i)} = e^{i \ln(i)}$; $\begin{cases} i = e^{i\frac{\pi}{2}} - \text{from 1-st bullet} \\ \ln(i) = \ln(e^{i\frac{\pi}{2}}) \end{cases} \rightarrow i\frac{\pi}{2} = \ln(i); e^{i \ln(i)} = e^{-\frac{\pi}{2}} = 0.208$

Complex Matrices

Common and Special



Many concepts have a new name, but old meaning:

- $A^T \rightarrow A^H$; $A^H = \bar{A}^T$, H - conjugate T ; Exp: $\begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$
- $Q \rightarrow U$; $U = Q$, where U - Unitary matrix

\mathbf{R}^n (n real components)	\leftrightarrow	\mathbf{C}^n (n complex components)
length: $\ x\ ^2 = x_1^2 + \dots + x_n^2$	\leftrightarrow	length: $\ x\ ^2 = x_1 ^2 + \dots + x_n ^2$
transpose: $A_{ij}^T = A_{ji}$	\leftrightarrow	Hermitian transpose: $A_{ij}^H = \bar{A}_{ji}$
$(AB)^T = B^T A^T$	\leftrightarrow	$(AB)^H = B^H A^H$
inner product: $x^T y = x_1 y_1 + \dots + x_n y_n$	\leftrightarrow	inner product: $x^H y = \bar{x}_1 y_1 + \dots + \bar{x}_n y_n$
$(Ax)^T y = x^T (A^T y)$	\leftrightarrow	$(Ax)^H y = x^H (A^H y)$
orthogonality: $x^T y = 0$	\leftrightarrow	orthogonality: $x^H y = 0$
symmetric matrices: $A^T = A$	\leftrightarrow	Hermitian matrices: $A^H = A$
$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$ (real Λ)	\leftrightarrow	$A = U \Lambda U^{-1} = U \Lambda U^H$ (real Λ)
skew-symmetric $K^T = -K$	\leftrightarrow	skew-Hermitian $K^H = -K$
orthogonal $Q^T Q = I$ or $Q^T = Q^{-1}$	\leftrightarrow	unitary $U^H U = I$ or $U^H = U^{-1}$
$(Qx)^T (Qy) = x^T y$ and $\ Qx\ = \ x\ $	\leftrightarrow	$(Ux)^H (Uy) = x^H y$ and $\ Ux\ = \ x\ $

Task 2



Compute $A^H A$ and AA^H . Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

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$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

Answer

$$A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix} \text{ and } AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ are Hermitian matrices.}$$

Task 3



Solve $Az = \mathbf{0}$ to find a vector z in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^H . Show that z is *not* orthogonal to the columns of A^T . ***The good row space is no longer $C(A^T)$. Now it is $C(A^H)$.***



Task 3

Solve $Az = \mathbf{0}$ to find a vector z in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^H . Show that z is *not* orthogonal to the columns of A^T . *The good row space is no longer $C(A^T)$. Now it is $C(A^H)$.*

Answer

$z = \text{multiple of } (1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^H A^H = \mathbf{0}^H$ so z (not \bar{z} !) is orthogonal to all columns of A^H (using complex inner product z^H times columns of A^H).

Task 4



If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

Task 4



If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

Answer

We are given $A + iB = (A + iB)^H = A^T - iB^T$. Then $A = A^T$ and $B = -B^T$. So that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

Reference material



- Lecture 26
- "*Linear Algebra and Applications*", pdf pages 322–335
Complex numbers and matrices
- "*Introduction to Linear Algebra*", pdf pages 504–519
Complex numbers and matrices
- Complex Numbers calculator

Deserve "A" grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru

📍 @Lupasic

🏢 Room 105 (Underground robotics lab)