Analytical Geometry and Linear Algebra II, Lab 3

Quiz

Four Fundametal Subspaces

I am on Robotics Sirius conference

I am calling to Kholodov to make an agreement about A grade for my students

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МЕЖДУНАРОДНАЯ КОНФЕРЕНЦИЯ
ЗИМНЯЯ ШКОЛА РОБОТОТЕХНИКИ
В СИРИУСЕ 2022
СЕРТИФИКАТ
ПРИЗЁРА
КОНКУРСА СТУДЕНЧЕСКИХ НАУЧНЫХ ДОКЛАДОВ
выдан
БУЛИЧЕВУ
Олегу Викторовичу
Ректор университета

Second place

Quiz

1) Obtain P, L, U matrices from A, using $PA = LU$ factorization.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad (1) \qquad A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 12 & 1 & 5 \\ 2 & 8 & 1 & 5 \\ 0 & 2 & 2 & 3 \end{bmatrix} \quad (2)$$

2) For

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for 5th point

$Ax = [0,\ 6,\ -6]$

1. Reduce $Ax = b$ to $Ux = c$, to reach a triangular system.
2. Find the condition on b_1, b_2, b_3 to have a soultion.
3. Describe the column space of A. Find the basis of the column space.
4. Describe the nullspace of A. Declare free variables.
5. Find a particular solution and the complete solution $x_p + x_n$

Quiz

Answers (1)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} L = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 9 & 2 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1)$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 12 & 1 & 5 \\ 2 & 8 & 1 & 5 \\ 0 & 2 & 2 & 3 \end{bmatrix} L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \end{bmatrix} U = \begin{bmatrix} 3 & 12 & 1 & 5 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Quiz

Answers (2)

1. The multipliers in elimination are 2 and 3 and -1, taking $[A \;\; b]$ to $[U \;\; c]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 2 & 4 & 8 & 12 & \boldsymbol{b_2} \\ 3 & 6 & 7 & 13 & \boldsymbol{b_3} \end{bmatrix} \to \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \boldsymbol{b_3 - 3b_1} \end{array}\right] \to \left[\begin{array}{cccc|l} 1 & 2 & 3 & 5 & \boldsymbol{b_1} \\ 0 & 0 & 2 & 2 & \boldsymbol{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \boldsymbol{b_3 + b_2 - 5b_1} \end{array}\right].$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.

3. The column space of A is the plane containing all combinations of the pivot columns $(1,2,3)$ and $(3,8,7)$.

   Second description: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = b$ solvable, so b is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane (in the first description of the column space).*

Quiz

Answers (3)

4. The special solutions in N have free variables $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$:

$$\begin{array}{l} \textbf{Nullspace matrix} \\ \textbf{Special solutions to } Ax = 0 \\ \textbf{Back-substitution in } Ux = 0 \\ \textbf{Just switch signs in } Rx = 0 \end{array} \qquad N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

5. Choose $b = (0,6,-6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0,6,0)$. Back-substitute with free variables $= 0$:

$$\textbf{Particular solution to } Ax_p = (0,6,-6) \qquad x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{array}{l} \\ \text{free} \\ \\ \text{free} \end{array}$$

The complete solution to $Ax = (0,6,-6)$ is (this x_p) + (all x_n).

Reference material

- Lecture 9 and 10
- *"Linear Algebra and Applications", pdf pages 139–149*

  The application of four fundamental subspaces in CS
- Matrix Transpose and the Four Fundamental Subspaces

  Video is about how A transpose appeared
- Matrix online calculator(russian)

Problem 1

Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

Problem 1 (sol.)

Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $c_3 = 0$

Plug this in the following equation.

$$c_2 + c_3 = 0$$
$$\Rightarrow c_2 = 0$$

Plug these values in the following equation.

$$c_1 + c_2 + c_3 = 0$$
$$\Rightarrow c_1 = 0$$

Therefore, $\boxed{c_1 = c_2 = c_3 = 0}$

Problem 1 (sol.)

Now,

let $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + c_3 + 2c_4 \\ c_2 + c_3 + 3c_4 \\ c_3 + 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $c_3 + 4c_4 = 0$

$\Rightarrow c_3 = -4c_4$

Plug this value in the following equation.

$$c_2 + c_3 + 3c_4 = 0$$

$$\begin{aligned} c_2 &= -c_3 - 3c_4 \\ &= +4c_4 - 3c_4 \\ &= c_4 \end{aligned}$$

Plug this value in the following equation.

$$c_1 + c_2 + c_3 + 2c_4$$

$$\begin{aligned} c_1 &= -c_2 - c_3 - 2c_4 \\ &= -c_4 + 4c_4 - 2c_4 \end{aligned}$$

$$c_1 = c_4$$

If $c_4 = 1$, then $c_1 = 1, c_2 = 1, c_3 = -4$

$$v_1 + v_2 - 4v_3 + v_4 = 0$$

Therefore, $\boxed{v_1 + v_2 - 4v_3 + v_4 = 0}$

$\Rightarrow v_1, v_2, v_3, v_4$ are linearly dependent

Problem 2

Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

This number is the ______ of the space spanned by the v's.

Problem 2 (sol.)

That is;

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 &= 0 \\ -c_2 &= 0 \\ -c_3 &= 0 \end{aligned}$$

Therefore, v_1, v_2, v_3 are linearly independent.

Let,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

This implies;

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_4 \\ -c_2 - c_4 \\ -c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 + c_4 &= 0 \\ -c_2 - c_4 &= 0 \\ -c_4 &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} c_4 &= 0 \\ c_2 &= 0 \\ c_1 &= 0 \\ c_3 &= 0 \end{aligned}$$

Therefore, v_1, v_2, v_3, v_4 are linearly independent.

Problem 2 (sol.)

Now,

Let $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 = 0$

$$c_1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+c_4\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}+c_5\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}=0$$

$$\begin{bmatrix}c_1+c_2+c_3\\-c_1+c_4+c_5\\-c_2-c_4\\-c_3-c_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

That is;

$$\begin{aligned}c_1+c_2+c_3&=0\\-c_1+c_4+c_5&=0\\-c_2-c_4&=0\\-c_3-c_5&=0\end{aligned}$$

This implies,

$$\begin{aligned}c_3&=-c_5\\c_2&=-c_4\\c_1&=-c_2-c_3\\&=c_4+c_5\end{aligned}$$

Thus,

$$(c_4+c_5)v_1+(-c_4)v_2+(-c_5)v_3+c_4v_4+c_5v_5=0$$

Therefore v_1, v_2, v_3, v_4, v_5 are linearly dependent.

Similarly, $v_1, v_2, v_3, v_4, v_5, v_6$ are linearly dependent. Here the largest possible number is 4 of independent vectors. This number four of the space spared by v's is the dimension of the space spanned by the v's.

Therefore, This number $\boxed{\text{four}}$ of the space spared by v's

Problem 4

If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

Problem 4 (sol.)

Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$\Rightarrow c_1\left(w_2 - w_3\right) + c_2\left(w_1 - w_3\right) + c_3\left(w_1 - w_2\right) = 0$$

$$\Rightarrow \left(c_2 + c_3\right) w_1 + \left(c_1 - c_3\right) w_2 + \left(-c_1 - c_2\right) w_3 = 0$$

So,

$$\Rightarrow c_2 + c_3 = 0$$

$$c_1 - c_3 = 0$$

$$-c_1 - c_2 = 0 \qquad \left(\text{since } w_1, w_2, w_3 \text{ are linearly independent}\right)$$

But,

$$-c_1 - c_2 = 0$$

$$\Rightarrow c_1 = -c_2$$

And,

$$c_1 - c_3 = 0$$

$$\Rightarrow c_3 = c_1$$

Therefore, $c_3 = c_1 = -c_2$

So,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 v_1 - c_1 v_2 + c_1 v_3 = 0$$

Let $c_1 = 1, v_1 - v_2 + v_3 = 0,$ therefore v_1, v_2, v_3 are linear dependent

Therefore, the sum $\boxed{v_1 - v_2 + v_3 = 0}$

Problem 6

To decide whether b is in the subspace spanned by $w_1, \ldots, w_n$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$?
(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

Problem 6 (sol.)

(a) Suppose the vectors *w* be the columns of *A* and consider $w_1 = (1,1,0)$, $w_2 = (2,2,1)$, $w_3 = (0,0,2)$, and $b = (3,4,5)$.

So we have,

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

To solve for $Ax = b$, use Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 1 & 2 & 0 & | & 4 \\ 0 & 1 & 2 & | & 5 \end{bmatrix}$$

By using $R_2 \to R_2 - R_1$, we get:

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 0 & 0 & | & 1 \\ 0 & 1 & 2 & | & 5 \end{bmatrix}$$

Second row represents the equation,

$$0x_1 + 0x_2 + 0x_3 = 1$$

By solving the equation $0x_1 + 0x_2 + 0x_3 = 1$, we get

$$0 = 1$$

As we know $0 \neq 1$, therefore, $Ax = b$ has $\boxed{\text{no solution}}$ and *b* $\boxed{\text{is not in it}}$.

Problem 6 (sol.)

(b) Suppose the vectors w be the columns of A and consider $w_1=(1,2,0)$, $w_2=(2,5,1)0$,

$w_3=(0,0,2)$, and $w_4=(0,0,0)$.

We know that the system of equation $Ax=b$ has a solution if and only if the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

Let

$$A=\begin{bmatrix}1&2&0&0\\2&5&0&0\\0&0&2&0\end{bmatrix}$$

And

$$b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

To solve $Ax=b$, use Gaussian elimination.

$$\left[\begin{array}{cccc|c}1&2&0&0&b_1\\2&5&0&0&b_2\\0&0&2&0&b_3\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&0&0&0&5b_1-2b_2\\0&1&0&0&b_2-2b_1\\0&0&1&0&b_3/2\end{array}\right]$$

Therefore, $\boxed{\text{yes there is a } b \text{ in it}}$.

Problem 8

U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

Problem 8 (sol.)

Consider the matrices,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the matrix U is obtained from A by subtracting row 1 from row 3.

The objective is to find the bases for the column spaces of A and U, the bases for the row spaces of A and U and the bases for the null spaces of A and U.

Reduce the matrix A to the reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 8 (sol.)

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon forms of the matrices A and U represent the same matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second columns.

Therefore, the corresponding columns in the matrix A form a basis for the column space of A.

Hence, the basis for the column space of the matrix A is $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}}$.

The pivot positions in the matrix U are in the first and second columns. Therefore, the corresponding columns in the matrix U form a basis for the column space of U.

Therefore, the basis for the column space of the matrix U is $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}}$.

Problem 8 (sol.)

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second rows.

Therefore, the basis for the row space of the matrix A is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

The pivot positions in the reduced row echelon form of the matrix U are in the first and second rows.

Therefore, the basis for the row space of the matrix U is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

Problem 8 (sol.)

Now find the bases for the null spaces of the *A* and *U*.

From the first and second rows of the reduced row echelon form, the obtained equations are,

$x_1 - x_3 = 0$ and $x_2 + x_3 = 0$.

Here, x_3 is a free variable.

So choose $x_3 = t$, where *t* is a parameter.

Then $x_1 = t,\ x_2 = -t$.

Therefore, the vector $\mathbf{x} = (x_1, x_2, x_3)$ can be written as,

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, x_3)\\ &= (t, -t, t)\\ &= t(1, -1, 1)\end{aligned}$$

Hence, the basis for the null spaces of the matrices *A* and *U* is $\boxed{\{(1, -1, 1)\}}$.

Problem 9

Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 9 (sol.)

Consider the matrix:

$$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$$

Reduce the matrix by taking the elementary operations to form matrix U.

$$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$$

$$\overset{R_3-R_1}{:}\begin{bmatrix}1&2&0&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=U$$

$$\overset{R_1-2R_2}{:}\begin{bmatrix}1&0&-2&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

Here, columns 1, 2 are pivot columns.

Therefore, columns space of $A=\boxed{\left\{s\begin{bmatrix}1\\0\\1\end{bmatrix}+t\begin{bmatrix}2\\1\\2\end{bmatrix}/s,t\in R\right\}}$.

And $\boxed{\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}\right\}}$ is basis for column space of A.

Dimension of columns space of A, $r=\boxed{2}$.

Problem 9 (sol.)

To calculate the dimension of null space of A;

$$n-r=4-2$$
$$=2$$

The null space of $A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} / a \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Now, $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 + x_4 = 0$$
$$x_2 + x_3 = 0$$

$$x_1 = -2x_2 - x_4$$
$$x_3 = -x_2$$

The null space of A is written as below:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -x_2 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the null space of A:

Null space of $A = \left\{ s \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} / s, t \in R \right\}$

Here $\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of A.

dim Null A= dim null U = 2

Here 1, 2 columns are pivot columns of U.

Column space of $U = \boxed{\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} / s, t \in R \right\}}$.

And $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$ is a basis for U.

Dimension of columns space of $U = \boxed{2}$.

Problem 9 (sol.)

To calculate the dimension of null space of U;

$$n - r = 4 - 2$$
$$= 2$$

The null space of $U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} / a \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Now, $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 - 2x_3 + x_4 = 0$$
$$x_2 + x_3 = 0$$

$$x_1 = 2x_3 - x_4$$
$$x_2 = -x_3$$

The null space of U is written as below:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Here $\boxed{\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of U

Now, to find the transpose of matrix A.

Transpose matrix is obtained to interchange the rows and columns.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \overset{R_4 - R_1}{\vdots} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2 - 2R_1}{\vdots} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_3 - R_2}{\vdots} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 9 (sol.)

Therefore, 1, 2 columns are pivots.

Columns space of $A^T = \left\{ r\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\1\\0\end{bmatrix} / r, s \in R \right\}$

Row space of $A^T = \{ r(1,2,0,1) + s(0,1,1,0) / r, s \in R \}$

The basis for row space of $A^T = \boxed{\{(1,2,0,1),(0,1,1,0)\}}$ dimension of row space $r = \boxed{2}$.

The dimension of null space of A^T,

$$m - r = 3 - 2$$
$$= 1$$

To find null space of A^T.

$$\begin{bmatrix}1&0&1\\2&1&2\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Perform the elementary row operations.

$$\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1 + x_3 = 0$$
$$x_2 = 0$$

$$x_1 = -x_3$$
$$x_2 = 0$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-x_3\\0\\x_3\end{bmatrix}$$
$$= x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Null space of $A^T = \left\{ x_3\begin{bmatrix}-1\\0\\1\end{bmatrix} / x_3 \in R \right\}$

The row space of $A = \{ x(-1,0,1) / x \in R \}$

Dimension of row space of $A^T = \boxed{1}$.

Here $\boxed{\{(-1,0,1)\}}$ is a basis for null space of A^T.

Problem 9 (sol.)

Now, to find the transpose of matrix U.

$$U^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Null space of $U^T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} / U^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$
$$2x_1 + x_2 = 0$$
$$x_2 = 0$$

$x_1 = 0, x_2 = 0$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Null space of $U^T = \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} / x \in R \right\}$

dim Null of $U^T = 1$

1, 2 columns are independent.

Column space of $U^T = \left\{ r \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} / r, s \in R \right\}$

Basis of columns space of $U^T = \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$

Dimension of column space of $U^T = \boxed{2}$.

Here $\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of U^T, dim null $U^T = \boxed{1}$.

Deserve “A” grade!

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Room 105 (Underground robotics lab)