# Analytical Geometry and Linear Algebra II, Lab 3

## Null space

# Disclaimer regarded to online classes

**How to work with slides**

I am giving the lecture-like class and answer on questions. All tasks you have to solve by your own at home.

# How to study Null Space

*Step-by-step guide*

1. Lecture 6, Gilbert Strang

   **Goal** is to understand the basics of spaces and how Null Space appeared.

2. Khan Academy: Null space

   It contains a good case study how to calculate Null space.

3. Matrix Algebra for Engineers: Null Space

   Another nice example how to find $N(A)$.

4. *"Linear Algebra and Applications"*, *pdf pages 96–106*

   What does partial and full solutions means

5. The Big Picture of Linear Algebra

   *Extra for now* If you want to get the global view of four subspaces

6. Understand the application from next few slides and make HW tasks!

# Null Space: Application from robotics

*Video*

# Null Space: Application from robotics

*Theory (1)*

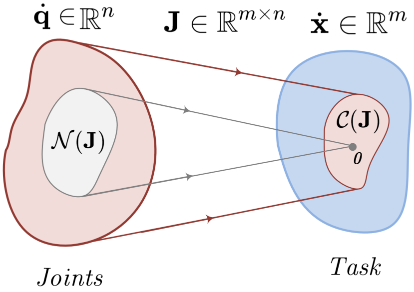



Figure 1: Click for google collab

Let us consider differential kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{1}$$

where

- $\mathbf{x} \in \mathbb{R}^m$ task space variables (for instance Cartesian coordinates)
- $\mathbf{q} \in \mathbb{R}^n$ joint space variables (positions of joints)
- $\mathbf{J} \in \mathbb{R}^{m \times n}$ mainpulator Jacobian

# Null Space: Application from robotics

*Theory (2)*

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = \underbrace{J^{\#}\dot{r}}_{\text{a particular solution (here, the pseudoinverse) in } \mathcal{R}(J^T)} + \underbrace{\left(I - J^{\#}J\right)}_{\text{"orthogonal" projection of } \dot{q}_0 \text{ in } \mathcal{N}(J)}\dot{q}_0$$

all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

a particular solution (here, the pseudoinverse) in $\mathcal{R}(J^T)$

"orthogonal" projection of $\dot{q}_0$ in $\mathcal{N}(J)$

properties of projector $[I - J^{\#}J]$

- symmetric
- idempotent: $[I - J^{\#}J]^2 = [I - J^{\#}J]$
- $[I - J^{\#}J]^{\#} = [I - J^{\#}J]$
- $J^{\#}\dot{r}$ is orthogonal to $[I - J^{\#}J]\dot{q}_0$

even more in general…

$$\dot{q} = K_1\dot{r} + (I - K_2 J)\dot{q}_0$$

$K_1, K_2$ generalized inverses of $J$ $(JK_iJ = J)$

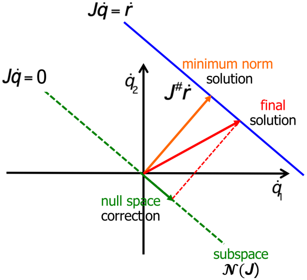
… but with less nice properties!

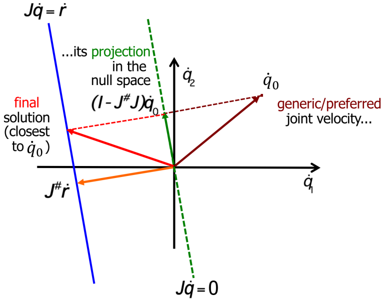
how do we choose $\dot{q}_0$?

# Null Space: Application from robotics

*Theory (3)*

in the space of velocity commands




a correction is added to the original pseudoinverse (minimum norm) solution

i) which is in the null space of the Jacobian

ii) and possibly satisfies additional criteria or objectives

# Task 1

Reduce these matrices to their ordinary echelon forms $U$:

(a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$.

Which are the free variables and which are the pivot variables?

# Task 1

Reduce these matrices to their ordinary echelon forms $U$:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

## Answer

$$\text{(a)}\quad U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables $x_2, x_4, x_5$
Pivot variables $x_1, x_3$

$$\text{(b)}\quad U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Free $x_3$
Pivot $x_1, x_2$

# Task 2

Construct 3 by 3 matrices $A$ to satisfy these requirements (if possible):

(a) $A$ has no zero entries but $U = I$.

(b) $A$ has no zero entries but $R = I$.

(c) $A$ has no zero entries but $R = U$.

(d) $A = U = 2R$.

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(c) $A$ has no zero entries but $R = U$.

(d) $A = U = 2R$.

## Answer

(a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I$, $R = I$.

# Task 3

If the special solutions to $Rx = \mathbf{0}$ are in the columns of these $N$, go backward to find the nonzero rows of the reduced matrices $R$:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \end{bmatrix} \text{ (empty 3 by 1).}$$

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## Answer

Any zero rows come after these rows: $R = [1 \quad -2 \quad -3]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

# Reference material

- Robotics 2 course from Sapienza
- Gilbert Strang Book 2.1-2.2

# Deserve “A” grade!

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