Linear Algebra. Test 2. Solutions

1. Find matrix e^A (4 points) and $det(e^A)$ (1 points):

$$1.1)\quad A = \begin{bmatrix} -3 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$e^A = Se^{\Lambda}S^{-1} = \frac{1}{20}\begin{bmatrix} 16 & 1 & 0 \\ -4 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-4} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 4 & 16 & 0 \\ -5 & -15 & 1 \end{bmatrix} =$$

$$= \frac{1}{20}\begin{bmatrix} 16e^{-4} + 4e & -16e^{-4} + 16e & 0 \\ -4e^{-4} + 4e & 4e^{-4} + 16e & 0 \\ e^{-4} + 4e - 5 & -e^{-4} + 16e - 15 & 20 \end{bmatrix}$$

$$\det(e^A) = \det(S)\det(e^{\Lambda})\det(S^{-1}) = \det(e^{\Lambda}) = \det\begin{bmatrix} e^{-4} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{-3}$$

$$1.2)\quad A = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$e^A = Se^{\Lambda}S^{-1} = \frac{1}{12}\begin{bmatrix} 9 & 1 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-3} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 3 & 9 & 0 \\ -4 & -8 & 12 \end{bmatrix} =$$

$$= \frac{1}{12}\begin{bmatrix} 9e^{-3} + 3e & -9e^{-3} + 9e & 0 \\ -3e^{-3} + 3e & 3e^{-3} + 9e & 0 \\ e^{-3} + 3e - 4 & e^{-3} + 9e - 8 & 12 \end{bmatrix}$$

$$\det(e^A) = \det(S)\det(e^{\Lambda})\det(S^{-1}) = \det(e^{\Lambda}) = \det\begin{bmatrix} e^{-3} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{-2}$$

2. Write down the first order equation system for the following differential equation and solve it (4 points):

$$2.1)\ \begin{cases} y''' + 3y'' - 4y' = 0 \\ y''(0) = 2, \quad y'(0) = 0, \quad y(0) = -1 \end{cases}$$

Is the solution of this system will be stable? (1 points)

IF we introduce the vector $\vec{u}(t) = \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}$ the ODE becomes:

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}, \quad \vec{u}(0) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \Rightarrow$$

$$\vec{u}(t) = Se^{\Lambda t}S^{-1}\vec{u}(0) = \frac{1}{20} \begin{bmatrix} 16e^{-4t} + 4e^{t} & -16e^{-4t} + 16e^{t} & 0 \\ -4e^{-4t} + 4e^{t} & 4e^{-4t} + 16e^{t} & 0 \\ e^{-4t} + 4e^{t} - 5 & -e^{-4t} + 16e^{t} - 15 & 20 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \Rightarrow$$

$$y(t) = \frac{1}{10}(e^{-4t} + 4e^{t} - 15) \quad \Rightarrow \quad \lim_{t\to\infty} y(t) = \infty$$

$$2.2)\ \begin{cases} y''' + 2y'' - 3y' = 0 \\ y''(0) = 3, \quad y'(0) = 0, \quad y(0) = 1 \end{cases}$$

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}, \quad \vec{u}(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\vec{u}(t) = Se^{\Lambda t}S^{-1}\vec{u}(0) = \frac{1}{12} \begin{bmatrix} 9e^{-3t} + 3e^{t} & -9e^{-3t} + 9e^{t} & 0 \\ -3e^{-3t} + 3e^{t} & (3e^{-3t} + 9e^{t}) & 0 \\ e^{-3t} + 3e^{t} - 4 & -e^{-3t} + 9e^{t} - 8 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$y(t) = \frac{e^{-3t}}{4} + \frac{3e^{t}}{4} \quad \Rightarrow \quad \lim_{t\to\infty} y(t) = \infty$$

3. For what number a is the matrix A positive definite?

$$3.1)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a+1 & -2 \\ 2 & -2 & a-2 \end{bmatrix}$$

$$1 > 0,\ \det\begin{vmatrix} 1 & 1 \\ 1 & a+1 \end{vmatrix} = a > 0,\ \det\begin{vmatrix} 1 & 1 & 2 \\ 1 & a+1 & -2 \\ 2 & -2 & a-2 \end{vmatrix} = a^2 - 6a - 16 > 0$$

$$\begin{cases} a > 0 \\ a^2 - 6a - 16 > 0 \end{cases} \Rightarrow\ a > 8$$

$$3.2)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a-1 & 2 \\ 2 & 2 & a+1 \end{bmatrix}$$

$$1 > 0,\ \det\begin{vmatrix} 1 & 1 \\ 1 & a-1 \end{vmatrix} = a - 2 > 0,\ \det\begin{vmatrix} 1 & 1 & 2 \\ 1 & a-1 & 2 \\ 2 & 2 & a+1 \end{vmatrix} = a^2 - 5a + 6 > 0$$

$$\begin{cases} a - 2 > 0 \\ a^2 - 5a + 6 > 0 \end{cases} \Rightarrow\ a > 3$$