

Disclaimer regarded to online classes



How to work with slides

I am giving the lecture-like class and answer on questions. All tasks you have to solve by your own at home.

How to study Null Space

Step-by-step guide



1. [Lecture 6, Gilbert Strang](#)

Goal is to understand the basics of spaces and how Null Space appeared.

2. [Khan Academy: Null space](#)

It contains a good case study how to calculate Null space.

3. [Matrix Algebra for Engineers: Null Space](#)

Another nice example how to find $N(A)$.

4. *"Linear Algebra and Applications", pdf pages 96–106*

What does partial and full solutions means

5. Understand the application from next few slides.

6. Make HW tasks ... profit!

Null Space: Application from robotics

Video



Null Space: Application from robotics

Theory (1)

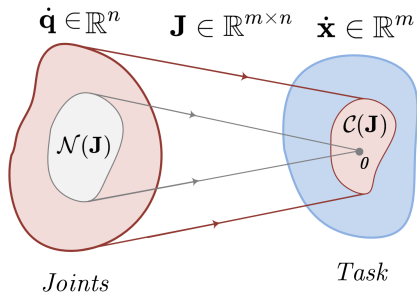


Figure 1: Click for google collab

Let us consider differential kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \quad (1)$$

where

- $\mathbf{x} \in \mathbb{R}^m$ task space variables (for instance Cartesian coordinates)
- $\mathbf{q} \in \mathbb{R}^n$ joint space variables (positions of joints)
- $\mathbf{J} \in \mathbb{R}^{m \times n}$ manipulator Jacobian

Null Space: Application from robotics

Theory (2)

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = J^\# \dot{r} + (I - J^\# J) \dot{q}_0$$

$J^\# \dot{r}$ is a particular solution (here, the pseudoinverse) in $\mathcal{R}(J^T)$

$(I - J^\# J) \dot{q}_0$ is the "orthogonal" projection of \dot{q}_0 in $\mathcal{N}(J)$

all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

properties of projector $[I - J^\# J]$

- symmetric
- idempotent: $[I - J^\# J]^2 = [I - J^\# J]$
- $[I - J^\# J]^\# = [I - J^\# J]$
- $J^\# \dot{r}$ is orthogonal to $[I - J^\# J] \dot{q}_0$

even more in general...

$$\dot{q} = K_1 \dot{r} + (I - K_2 J) \dot{q}_0$$

K_1, K_2 generalized inverses of J

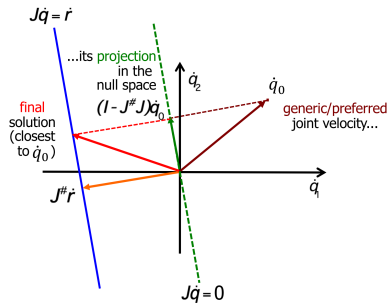
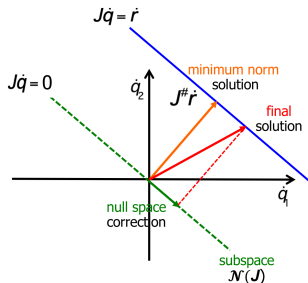
... but with less nice properties! ($JK_i J = J$)

how do we choose \dot{q}_0 ?

Null Space: Application from robotics

Theory (3)

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution

- i) which is in the **null space** of the Jacobian
- ii) and possibly satisfies **additional criteria** or objectives

Task 1



Reduce these matrices to their ordinary echelon forms U :

$$(a) \quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad (b) \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

Task 1



Reduce these matrices to their ordinary echelon forms U :

$$(a) \quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad (b) \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

Answer

$$(a) \quad U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Free variables } x_2, x_4, x_5 \\ \text{Pivot variables } x_1, x_3 \end{array} \quad (b) \quad U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Free } x_3 \\ \text{Pivot } x_1, x_2 \end{array}$$

Task 2



Construct 3 by 3 matrices A to satisfy these requirements (if possible):

- (a) A has no zero entries but $U = I$.
- (b) A has no zero entries but $R = I$.
- (c) A has no zero entries but $R = U$.
- (d) $A = U = 2R$.

Task 2



Construct 3 by 3 matrices A to satisfy these requirements (if possible):

- (a) A has no zero entries but $U = I$.
- (b) A has no zero entries but $R = I$.
- (c) A has no zero entries but $R = U$.
- (d) $A = U = 2R$.

Answer

- (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

Task 3



If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N , go backward to find the nonzero rows of the reduced matrices R :

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \\ \\ \end{bmatrix} \quad (\text{empty } 3 \text{ by } 1).$$

Task 3



If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N , go backward to find the nonzero rows of the reduced matrices R :

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \\ \\ \end{bmatrix} \quad (\text{empty } 3 \text{ by } 1).$$

Answer

Any zero rows come after these rows: $R = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

Reference material



- Robotics 2 course from Sapienza
- Gilbert Strang Book 2.1-2.2

Deserve "A" grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru

📍 @Lupasic

🏠 Room 105 (Underground robotics lab)