Problems from

Sec. 2.3

Book: Linear Algebra and Its Application – 4ed

Problem 1

Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

Problem 1 (sol.)

Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$\Rightarrow c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 + c_2 + c_3\\c_2 + c_3\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

i.e. $c_3 = 0$

Plug this in the following equation.

$$c_2 + c_3 = 0$$
$$\Rightarrow c_2 = 0$$

Plug these values in the following equation.

$$c_1 + c_2 + c_3 = 0$$
$$\Rightarrow c_1 = 0$$

Therefore, $\boxed{c_1 = c_2 = c_3 = 0}$

Problem 1 (sol.)

Now,

let $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$$\Rightarrow c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} + c_4\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 + c_2 + c_3 + 2c_4\\ c_2 + c_3 + 3c_4\\ c_3 + 4c_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

i.e. $c_3 + 4c_4 = 0$

$\Rightarrow c_3 = -4c_4$

Plug this value in the following equation.

$$\begin{aligned} c_2 + c_3 + 3c_4 &= 0 \\ c_2 &= -c_3 - 3c_4 \\ &= +4c_4 - 3c_4 \\ &= c_4 \end{aligned}$$

Plug this value in the following equation.

$$\begin{aligned} &c_1 + c_2 + c_3 + 2c_4 \\ &c_1 = -c_2 - c_3 - 2c_4 \\ &\quad = -c_4 + 4c_4 - 2c_4 \\ &c_1 = c_4 \end{aligned}$$

If $c_4 = 1$, then $c_1 = 1, c_2 = 1, c_3 = -4$

$v_1 + v_2 - 4v_3 + v_4 = 0$

Therefore, $\boxed{v_1 + v_2 - 4v_3 + v_4 = 0}$

$\Rightarrow v_1, v_2, v_3, v_4$ are linearly dependent

Problem 2

Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

This number is the ______ of the space spanned by the v's.

Problem 2 (sol.)

Let,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

This implies;

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_4 \\ -c_2 - c_4 \\ -c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 + c_4 &= 0 \\ -c_2 - c_4 &= 0 \\ -c_4 &= 0 \end{aligned}$$

Thus,

$$c_4 = 0$$
$$c_2 = 0$$
$$c_1 = 0$$
$$c_3 = 0$$

Therefore, v_1, v_2, v_3, v_4 are linearly independent.

That is;

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 &= 0 \\ -c_2 &= 0 \\ -c_3 &= 0 \end{aligned}$$

Therefore, v_1, v_2, v_3 are linearly independent.

Problem 2 (sol.)

Now,

Let $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 = 0$

$$c_1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+c_4\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}+c_5\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}=0$$

$$\begin{bmatrix}c_1+c_2+c_3\\-c_1+c_4+c_5\\-c_2-c_4\\-c_3-c_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

That is;

$$\begin{aligned} c_1+c_2+c_3&=0\\ -c_1+c_4+c_5&=0\\ -c_2-c_4&=0\\ -c_3-c_5&=0 \end{aligned}$$

This implies,

$$\begin{aligned} c_3&=-c_5\\ c_2&=-c_4\\ c_1&=-c_2-c_3\\ &=c_4+c_5 \end{aligned}$$

Thus,

$$(c_4+c_5)v_1+(-c_4)v_2+(-c_5)v_3+c_4v_4+c_5v_5=0$$

Therefore v_1, v_2, v_3, v_4, v_5 are linearly dependent.

Similarly, $v_1, v_2, v_3, v_4, v_5, v_6$ are linearly dependent. Here the largest possible number is 4 of independent vectors. This number four of the space spared by v's is the dimension of the space spanned by the v's.

Therefore, This number $\boxed{\text{four}}$ of the space spared by $v's$

Problem 3

Choose three independent columns of U.
Do the same for A.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

Problem 3 (sol.)

Consider the matrix, $U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The three independent columns are $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$.

The columns are base for the columns space $= \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} / a, b, c \in R \right\}$

Now,

$$A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$$

Apply $R_4 \to R_4 - 2R_1$

$$= \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= U$

Problem 4

If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

Problem 4 (sol.)

Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$\Rightarrow c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$$

$$\Rightarrow (c_2 + c_3)w_1 + (c_1 - c_3)w_2 + (-c_1 - c_2)w_3 = 0$$

So,

$\Rightarrow c_2 + c_3 = 0$

$c_1 - c_3 = 0$

$-c_1 - c_2 = 0 \qquad (\text{since } w_1, w_2, w_3 \text{ are linearly independent})$

But,

$-c_1 - c_2 = 0$

$\Rightarrow c_1 = -c_2$

And,

$c_1 - c_3 = 0$

$\Rightarrow c_3 = c_1$

Therefore, $c_3 = c_1 = -c_2$

So,

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$c_1v_1 - c_1v_2 + c_1v_3 = 0$

Let $c_1 = 1, v_1 - v_2 + v_3 = 0,\quad \text{therefore } v_1, v_2, v_3 \text{ are linear dependent}$

Therefore, the sum $\boxed{v_1 - v_2 + v_3 = 0}$

Problem 5

Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 5 (sol.)

(a)

Consider the matrix

$$A=\begin{bmatrix}1&1&0\\1&3&1\\3&1&-1\end{bmatrix}$$

Let,

$$v_1=\begin{bmatrix}1\\1\\3\end{bmatrix}$$

$$v_2=\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$v_3=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Let $c_1v_1+c_2v_2+c_3v_3=0$

$$c_1\begin{bmatrix}1\\1\\3\end{bmatrix}+c_2\begin{bmatrix}1\\3\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

That is;

$$c_1+c_2=0$$
$$c_1+3c_2+c_3=0$$
$$3c_1+c_2-c_3=0$$

From the above system of equations,

$$(3c_1+c_2-c_3)+(c_1+3c_2+c_3)=0$$
$$4c_1+4c_2=0$$
$$c_1=-c_2$$

Now,

$$c_3=-c_1-3c_2$$
$$(\text{Since}\, c_1=-c_2)$$
$$=c_2-3c_2$$
$$=-2c_2$$

Therefore $-c_2v_1+c_2v_2-2c_2v_3=0$

If $c_2=1, -v_1+v_2-2v_3=0$

Therefore v_1,v_2,v_3 $\boxed{\text{are linearly dependent.}}$

Problem 5 (sol.)

If $c_1 v_1 + c_2 v_2 = 0$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is;

$c_1 + c_2 = 0$ ...... (1)

$c_1 + 3c_2 = 0$ ...... (2)

$3c_1 + c_2 = 0$ ...... (3)

Apply $(3\times(1)) - (2)$;

This implies;

$$2c_1 = 0$$
$$c_1 = 0$$

Plug this value in equation (1)

So, $c_2 = 0$

Therefore, v_1, v_2 are linearly independent and $\{v_1 \ \ , v_2\}$ spans columns space.

Therefore dimension of column space $\boxed{A = 2}$.

Problem 5 (sol.)

(b)

Consider the matrix; $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Clearly,

$$v_1 + 2v_3 = v_2$$

$$v_1 - v_2 + 2v_3 = 0$$

Therefore v_1, v_2, v_3 are dependent and $c_1 v_1 + c_2 v_2 = 0$

$$c_1 + c_2 = 0$$

$$2c_2 = 0$$

Thus,

$$c_1 = 0$$

$$c_2 = 0$$

Hence v_1, v_2 are linearly independent and $\{v_1, v_2\}$ spans column space of U.

Therefore, the dimension of column space of $\boxed{U = 2}$.

Problem 5 (sol.)

(c)

Let

$$v_1 = (1,1,0)$$
$$v_2 = (1,3,1)$$
$$v_3 = (3,1,-1)$$

Now,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$c_1(1,1,0) + c_2(1,3,1) + c_3(3,1,-1) = 0$$
$$c_1 + c_2 + 3c_3 = 0$$
$$c_1 + 3c_2 + c_3 = 0$$

Solve these two equations.

So,

$c_2 = c_3$ and

$c_1 = -c_2 - 3c_3$

This implies;

$$c_2 = c_3$$
$$c_1 = -4c_3$$

So,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$-4c_3 v_1 + c_3 v_2 + c_3 v_3 = 0$$

If $c_3 = 1$, then;

$$-4v_1 + v_2 + v_3 = 0$$

Therefore, $\{v_1, v_2, v_3\}$ are independent.

But, $\{v_1, v_2\}$ are independent and $\{v_1, v_2\}$ spans row spaces row space of A.

Therefore, the dimension of row space $\boxed{A = 2}$.

Problem 5 (sol.)

(d)

The row space of U = spanned by $\{(1,1,0),(0,2,1)\}$

If $(1,3,1)=(1,1,0)+(0,2,1)$ and $(1,1,0),(1,3,1)$ are linearly independent $\{(1,1,0),(1,3,1)\}$ space the row space of U.

Hence, $\boxed{\text{row space of } U = \text{Row space of } A}$.

Problem 6

To decide whether b is in the subspace spanned by $w_1, \ldots, w_n$, let the vectors w be the columns of A and try to solve $Ax = b$. What is the result for

(a) $w_1 = (1, 1, 0)$, $w_2 = (2, 2, 1)$, $w_3 = (0, 0, 2)$, $b = (3, 4, 5)$?

(b) $w_1 = (1, 2, 0)$, $w_2 = (2, 5, 0)$, $w_3 = (0, 0, 2)$, $w_4 = (0, 0, 0)$, and any b?

Problem 6 (sol.)

(a) Suppose the vectors w be the columns of A and consider $w_1 = (1,1,0)$, $w_2 = (2,2,1)$, $w_3 = (0,0,2)$, and $b = (3,4,5)$.

So we have,

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

To solve for $Ax = b$, use Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 1 & 2 & 0 & | & 4 \\ 0 & 1 & 2 & | & 5 \end{bmatrix}$$

By using $R_2 \to R_2 - R_1$, we get:

$$\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 0 & 0 & | & 1 \\ 0 & 1 & 2 & | & 5 \end{bmatrix}$$

Second row represents the equation,

$$0x_1 + 0x_2 + 0x_3 = 1$$

By solving the equation $0x_1 + 0x_2 + 0x_3 = 1$, we get

$$0 = 1$$

As we know $0 \neq 1$, therefore, $Ax = b$ has $\boxed{\text{no solution}}$ and b $\boxed{\text{is not in it}}$.

Problem 6 (sol.)

(b) Suppose the vectors *w* be the columns of *A* and consider $\mathbf{w}_1 = (1,2,0)$, $\mathbf{w}_2 = (2,5,1)0$,

$\mathbf{w}_3 = (0,0,2)$, and $\mathbf{w}_4 = (0,0,0)$.

We know that the system of equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the vector *b* can be expressed as a combination of the columns of *A*. Then *b* is in the column space.

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

And

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

To solve $A\mathbf{x} = \mathbf{b}$, use Gaussian elimination.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & b_1 \\ 2 & 5 & 0 & 0 & b_2 \\ 0 & 0 & 2 & 0 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 1 & 0 & b_3/2 \end{array}\right]$$

Therefore, $\boxed{\textbf{yes there is a } b \textbf{ in it}}$.

Problem7

If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension ______. These vectors are a ______ for that space. If the vectors are the columns of an m by n matrix, then m is ______ than n.

Problem 7 (ans.)

If $v_1, v_2, \ldots\ldots, v_n$ are linearly independent. The space the span has dimension $\boxed{n}$. These vectors are a $\boxed{\text{basis for that space}}$. If the vectors are the columns of an m by n matrix then m $\boxed{\text{is not less than}}$. n $(m \geq n)$.

Problem 8

U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

Problem 8 (sol.)

Consider the matrices,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the matrix U is obtained from A by subtracting row 1 from row 3.

The objective is to find the bases for the column spaces of A and U, the bases for the row spaces of A and U and the bases for the null spaces of A and U.

Reduce the matrix A to the reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 8 (sol.)

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon forms of the matrices A and U represent the same matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second columns.

Therefore, the corresponding columns in the matrix A form a basis for the column space of A.

Hence, the basis for the column space of the matrix A is $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}}$.

The pivot positions in the matrix U are in the first and second columns. Therefore, the corresponding columns in the matrix U form a basis for the column space of U.

Therefore, the basis for the column space of the matrix U is $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}}$.

Problem 8 (sol.)

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second rows.

Therefore, the basis for the row space of the matrix A is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

The pivot positions in the reduced row echelon form of the matrix U are in the first and second rows.

Therefore, the basis for the row space of the matrix U is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

Problem 8 (sol.)

Now find the bases for the null spaces of the *A* and *U*.

From the first and second rows of the reduced row echelon form, the obtained equations are,

$x_1 - x_3 = 0$ and $x_2 + x_3 = 0$.

Here, x_3 is a free variable.

So choose $x_3 = t$, where *t* is a parameter.

Then $x_1 = t,\ x_2 = -t$.

Therefore, the vector $\mathbf{x} = (x_1, x_2, x_3)$ can be written as,

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, x_3)\\ &= (t, -t, t)\\ &= t(1, -1, 1)\end{aligned}$$

Hence, the basis for the null spaces of the matrices *A* and *U* is $\boxed{\{(1,-1,1)\}}$.

Problems from

Sec. 2.4

Book: Linear Algebra and Its Application – 4ed

Problem 9

Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 9 (sol.)

Consider the matrix:

$$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$$

Reduce the matrix by taking the elementary operations to form matrix U.

$$A=\begin{bmatrix}1&2&0&1\\0&1&1&0\\1&2&0&1\end{bmatrix}$$

$$\overset{R_3-R_1}{:}\begin{bmatrix}1&2&0&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=U$$

$$\overset{R_1-2R_2}{:}\begin{bmatrix}1&0&-2&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

Here, columns 1, 2 are pivot columns.

Therefore, columns space of $A=\boxed{\left\{s\begin{bmatrix}1\\0\\1\end{bmatrix}+t\begin{bmatrix}2\\1\\2\end{bmatrix}/\,s,t\in R\right\}}$.

And $\boxed{\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}\right\}}$ is basis for column space of A.

Dimension of columns space of A, $r=\boxed{2}$.

Problem 9 (sol.)

To calculate the dimension of null space of A;

$$n-r=4-2$$
$$=2$$

The null space of $A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} / a \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Now, $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 + x_4 = 0$$
$$x_2 + x_3 = 0$$
$$x_1 = -2x_2 - x_4$$
$$x_3 = -x_2$$

The null space of A is written as below:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -x_2 \\ x_4 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the null space of A:

Null space of $A = \left\{ s \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} / s,t \in R \right\}$

Here $\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of A.

dim Null A= dim null U = 2

Here 1, 2 columns are pivot columns of U.

Column space of $U = \boxed{\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} / s,t \in R \right\}}$.

And $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$ is a basis for U.

Dimension of columns space of $U = \boxed{2}$.

Problem 9 (sol.)

To calculate the dimension of null space of U;

$$\begin{aligned} n-r &= 4-2 \\ &= 2 \end{aligned}$$

The null space of $U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} / a \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Now, $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} x_1 - 2x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$x_1 = 2x_3 - x_4$$
$$x_2 = -x_3$$

The null space of U is written as below:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Here $\boxed{\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of U

Now, to find the transpose of matrix A.

Transpose matrix is obtained to interchange the rows and columns.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \overset{R_4 - R_1}{:} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2 - 2R_1}{:} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_3 - R_2}{:} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 9 (sol.)

Therefore, 1, 2 columns are pivots.

Columns space of $A^T = \left\{ r\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\1\\0\end{bmatrix} / r,s \in R \right\}$

Row space of $A^T = \{r(1,2,0,1)+s(0,1,1,0)/r,s\in R\}$

The basis for row space of $A^T = \boxed{\{(1,2,0,1),(0,1,1,0)\}}$ dimension of row space $r=\boxed{2}$.

The dimension of null space of A^T,

$$\begin{aligned} m-r &= 3-2 \\ &= 1 \end{aligned}$$

To find null space of A^T.

$$\begin{bmatrix}1&0&1\\2&1&2\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Perform the elementary row operations.

$$\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{aligned} x_1+x_3&=0\\ x_2&=0 \end{aligned}$$

$$\begin{aligned} x_1&=-x_3\\ x_2&=0 \end{aligned}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_3\\0\\x_3\end{bmatrix} = x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Null space of $A^T = \left\{ x_3\begin{bmatrix}-1\\0\\1\end{bmatrix} / x_3 \in R \right\}$

The row space of $A = \{x(-1,0,1)/x\in R\}$

Dimension of row space of $A^T = \boxed{1}$.

Here $\boxed{\{(-1,0,1)\}}$ is a basis for null space of A^T.

Problem 9 (sol.)

Now, to find the transpose of matrix U.

$$U^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Null space of $U^T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} / U^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$
$$2x_1 + x_2 = 0$$
$$x_2 = 0$$

$x_1 = 0, x_2 = 0$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Null space of $U^T = \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} / x \in R \right\}$

dim Null of $U^T = 1$

1, 2 columns are independent.

Column space of $U^T = \left\{ r \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} / r, s \in R \right\}$

Basis of columns space of $U^T = \boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$

Dimension of column space of $U^T = \boxed{2}$.

Here $\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is a basis for null space of U^T , dim null $U^T = \boxed{1}$.

Problem 10

Describe the four subspaces in three-dimensional space associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 10 (sol.)

The four subspaces in three-dimensional space associated with the matrix A are column space $C(A)$ of A, the null space $N(A)$ of A, the column space $C(A^T)$ of A^T and the null space $N(A^T)$ of A^T.

Note that the matrix A is in reduced row echelon form.

The leading 1 s' are in the second and third columns. Therefore, the corresponding columns in the matrix A form a basis for the column space of A.

So the basis for $C(A)$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

Thus, the column space of A is defined as $\boxed{C(A)=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}\middle|\, x,y\in\mathbf{R}\right\}}$.

The null space $N(A)$ of the matrix A is the solution space of the system $A\mathbf{x}=\mathbf{0}$.

Thus, the system $A\mathbf{x}=\mathbf{0}$ is equivalent to $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The equations obtained from this matrix equation are,

$x_2=0, x_3=0.$

Here, x_1 is a free variable.

So choose $x_1=t$, where t is a parameter.

Therefore, the vector $\mathbf{x}$ can be written as,

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\0\\0\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Thus, the basis for the null space of A is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

And the null space of A is defined as $N(A)=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix}\middle|\, x\in\mathbf{R}\right\}$.

Problem 10 (sol.)

The transpose of the matrix A is $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Now find the column space and the null space of the matrix A^T.

The leading 1 s' in the matrix A^T are in the first and second columns. So the corresponding columns in the matrix A^T form the basis for the column space of A^T.

Thus, the basis for the column space of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Thus, the column space of A^T is defined as $\boxed{C\left(A^T\right) = \left\{ x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \; x, y \in \mathbf{R} \right\}}$.

The null space $N\left(A^T\right)$ of the matrix A is the solution space of the system $A^T\mathbf{x} = \mathbf{0}$.

Thus, the system $A^T\mathbf{x} = \mathbf{0}$ is equivalent to $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The equations obtained from this matrix equation are,

$x_1 = 0, x_2 = 0.$

Here, x_3 is a free variable.

So choose $x_3 = t$, where t is a parameter.

Therefore, the vector $\mathbf{x}$ can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the basis for the null space of A^T is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

And the null space of A^T is defined as $\boxed{N\left(A^T\right) = \left\{ x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \; x \in \mathbf{R} \right\}}$.