Analytical Geometry and Linear Algebra II, Lab 8

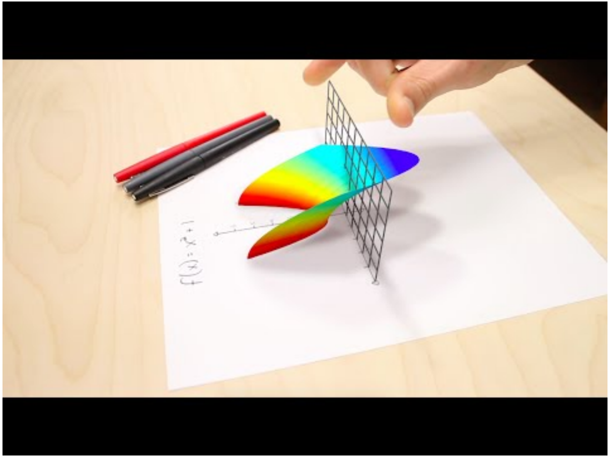
Complex numbers

Complex matrices

Hermitian and Unitary Matrices

Complex numbers

Video



Complex numbers

Forms

Rectangular form: $z = x + iy,\ i^2 = -1$

$Re(z) = x$ – real part, $Im(z) = y$ – imaginary part

Example: $z = 5 + i6$

Polar form: $z = r\,cos(\phi) + i\,r\,sin(\phi)$, where

$\phi = atan2(Im(z), Re(z))$;

$r = |z| = \sqrt{x^2 + y^2}$ – magnitude

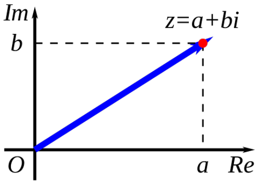
Example: $z = 8\,cos(24) + i\,sin(24))$

Exponential form: $z = re^{i\phi}$

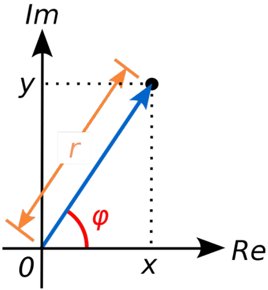
Example: $z = 6e^{i2.5}$

Euler formula: *transformation from exp. to polar*

$e^{i\phi} = cos(\phi) + i\,sin(\phi)$



Rectangular form



Polar or Exponential form

Complex numbers

Operations

General Idea: you should work with *Im* and *Re* part separately (you cannot sum or multiply them)

- *Summarization and Subtraction* – $(x_1 \pm iy_1) + (x_2 \pm iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- *Multiplication* – $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
- *Division* – $\dfrac{(x_2 + iy_2)}{(x_1 + iy_1)} = \dfrac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\dfrac{(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$

Complex numbers

Complex conjugate

Complex conjugate of complex number $z = x + iy$ – $\bar{z} = x - iy$.

Geometrically it's reflection of z about *Re* axis.

Properties:

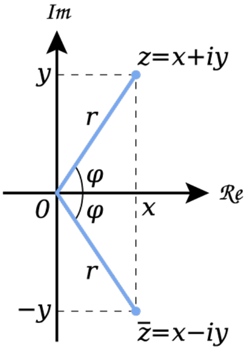
$Re(\bar{z}) = Re(z))$ and $|\bar{z}| = |z|$;

$Im(\bar{z}) = -Im(z)$ and $arg\ \bar{z} \equiv -arg\ z (mod\ 2\pi)$

$z\bar{z} = x^2 + y^2 = |z|^2$ – absolute square

Operations

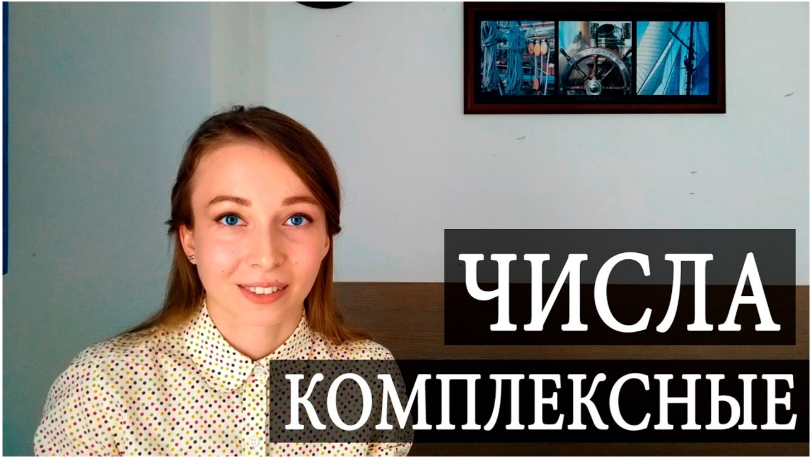
- *Summarization and Subtraction* – $\overline{z \pm w} = \bar{z} \pm \bar{w}$
- *Multiplication* – $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
- *Division* – $\overline{z/w} = \bar{z}/\bar{w}$



Complex Conjugate

Complex numbers

Applications: Video



Complex numbers

Applications: more about Quantum Computing (Links)

- Representing Qubit States as complex numbers
- Why do we have to use complex numbers in Quantum computing (Nature paper)
- Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit)

*If you are interested in the topic, you can ask **Stas Protasov**.*

Complex numbers

Applications: Summary

Typical reason

Most of the time for simplicity your calculations, except Quantum computing and some other topics.

1. *Electromagnetism*: electric and magnetic fields as complex vector fields to describe electromagnetic waves. However, this is just a computational trick.
2. *Aerodynamics*: calculating wing shapes using Joukowsky transform.
3. *Geodesy*: Google Maps (from Geoid to 2d map).
4. *Quantum Computing*

Task 1

1. For $z = \dfrac{1+i}{\sqrt{2}}$:
   - Compute z^2
   - Find r
   - Find ϕ
   - Find z in exponential form
2. - Find the 8 solutions to the equation $z^8 = 1$
   - Plot those 8 solutions in the complex plane
3. For $z = -1 + i\frac{1}{2}$
   - Find complex conjugate $(\bar{z})$
   - Find $z\bar{z}$
   - Find $z + \bar{z}$
   - Plot each result in the complex plane
4. Find:
   - $e^{i\frac{\pi}{2}}$
   - $e^{i\pi}$
   - i^i
   - Show each result in the complex plane

Task 1

1. For $z = \dfrac{1+i}{\sqrt{2}}$:
   - Compute z^2
   - Find r
   - Find ϕ
   - Find z in exponential form

Answer

- $z^2 = i$
- $r = 1$
- $\phi = 45°$
- $z = 1e^{i\frac{\pi}{4}}$

Task 1

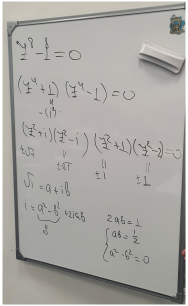
2 – Find the 8 solutions to the equation $z^8 = 1$

– Plot those 8 solutions in the complex plane

Answer

Solution (rus):

$\pm 1;\ \pm i;\ \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2};\ -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2};$



One of possible solutions

Task 1

3 For $z = -1 + i\frac{1}{2}$

- Find complex conjugate ($\bar{z}$)
- Find $z\bar{z}$
- Find $z + \bar{z}$
- Plot each result in the complex plane

Answer

- $\bar{z} = -1 - i\frac{1}{2}$
- $z\bar{z} = 1.25$
- $z + \bar{z} = -2$

Task 1

4 Find:

- $e^{i\frac{\pi}{2}}$
- $e^{i\pi}$
- i^i
- Show each result in the complex plane

Answer

- $e^{i\frac{\pi}{2}} = i$
- $e^{i\pi} = -1$
- $i^i = e^{ln(i^i)} = e^{i\ ln(i)}; \begin{cases} i = e^{i\frac{\pi}{2}} - \textit{from 1-st bullet} \\ ln(i) = ln(e^{i\frac{\pi}{2}}) \end{cases} \rightarrow i\frac{\pi}{2} = ln(i);\ e^{i\ ln(i)} = e^{-\frac{\pi}{2}} = 0.208$

Complex Matrices

Common and Special

Many concepts have a new name, but old meaning:

- $A^T \rightarrow A^H$; $A^H = \bar{A}^T$, H – conjugate T; *Exp:* $\begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$
- $Q \rightarrow U$; $U = Q$, where U – Unitary matrix

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\\|x\\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\\|x\\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}}y = \bar{x}_1y_1 + \cdots + \bar{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\\|Qx\\| = \\|x\\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\\|Ux\\| = \\|x\\|$

Task 2

Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both ______ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

Task 2

Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

Answer

$$A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix} \text{ and } AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ are Hermitian matrices.}$$

Task 3

Solve $A\boldsymbol{z} = \mathbf{0}$ to find a vector $\boldsymbol{z}$ in the nullspace of A in Problem 2. Show that $\boldsymbol{z}$ is orthogonal to the columns of A^{H}. Show that $\boldsymbol{z}$ is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $C(A^{\mathrm{T}})$. **Now it is** $\boldsymbol{C(A^{\mathrm{H}})}$.

Task 3

Solve $A\boldsymbol{z} = \mathbf{0}$ to find a vector $\boldsymbol{z}$ in the nullspace of A in Problem 2. Show that $\boldsymbol{z}$ is orthogonal to the columns of A^{H}. Show that $\boldsymbol{z}$ is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer $C(A^{\mathrm{T}})$*. Now it is $C(A^{\mathrm{H}})$.**

Answer

$\boldsymbol{z} =$ multiple of $(1+i, 1+i, -2)$; $A\boldsymbol{z} = \mathbf{0}$ gives $\boldsymbol{z}^{\mathrm{H}} A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times columns of A^{H}).

Task 4

If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is symmetric.

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If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is symmetric.

Answer

We are given $A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. So that $\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$ is symmetric.

Reference material

- Lecture 26
- Complex Numbers Part Imaginary, but Really Simple
- *"Linear Algebra and Applications", pdf pages 322–335*
  Complex numbers and matrices
- *"Introduction to Linear Algebra", pdf pages 504–519*
  Complex numbers and matrices
- Complex Numbers calculator

Deserve “A” grade!

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Room 105 (Underground robotics lab)