Analytical Geometry and Linear Algebra II, Lab 10

Circulant Matrix

Systems of linear differential equations

Innopolis University

How I spent last weekend

Watched both seasons in 1 day (24 series) of "Mushoku Tensei"

RAGE and VEGs clubs cooking collaboration event

Circulant Matrix

Watch [5] video, if you want to get how to derive this property and the necessity of it.

Circulant matrix ($N = 4$) is:

$$C_4 = c_0 I + c_1 P + c_2 P^2 + C_3 P^3 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}, \text{ where } P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Properties:

It has ***eigenvectors*** in the Fourier Matrix columns $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & i^2 & 1 & (-i)^2 \\ 1 & i^3 & -1 & (-i)^3 \end{bmatrix}$

Eigenvalues of C can be found by the Fourier transform $F_4 \bar{c} = \bar{\lambda}$

Circulant Matrix

Example

Example 2 The same ideas work for a Fourier matrix F and a circulant matrix C of any size. Two by two matrices look trivial but they are very useful. Now eigenvalues of P have $\lambda^2 = 1$ instead of $\lambda^4 = 1$ and the complex number i is not needed: $\boldsymbol{\lambda = \pm 1}$.

Fourier matrix F from eigenvectors of P and C $\quad F = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Circulant $c_0 I + c_1 P$ $\quad C = \begin{bmatrix} \boldsymbol{c_0} & \boldsymbol{c_1} \\ \boldsymbol{c_1} & \boldsymbol{c_0} \end{bmatrix}.$

The eigenvalues of C are $c_0 + c_1$ and $c_0 - c_1$. Those are given by the Fourier transform $F\boldsymbol{c}$ when the vector $\boldsymbol{c}$ is (c_0, c_1). This transform $F\boldsymbol{c}$ gives the eigenvalues of C for any size n.

Task 1

What are the 3 solutions to $\lambda^3 = 1$? They are complex numbers $\lambda = \cos\theta + i\sin\theta = e^{i\theta}$. Then $\lambda^3 = e^{3i\theta} = 1$ when the angle 3θ is 0 or 2π or 4π. Write the 3 by 3 Fourier matrix F with columns $(1, \lambda, \lambda^2)$.

Check that any 3 by 3 circulant C has eigenvectors $(1, \lambda, \lambda^2)$ from Problem 8. If the diagonals of your matrix C contain c_0, c_1, c_2 then its eigenvalues are in $F\boldsymbol{c}$.

Task 1
Answer

$\lambda^3 = 1$ has 3 roots $\lambda = 1$ and $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Those are $\mathbf{1}, \boldsymbol{\lambda}, \boldsymbol{\lambda^2}$ if we take $\lambda = e^{2\pi i/3}$. The Fourier matrix is

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda^2 \\ 1 & \lambda^2 & \lambda^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{8\pi i/3} \end{bmatrix}.$$

A 3 by 3 circulant matrix has the form on page 425 :

$$C = \begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix} \text{ with } C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (c_0 + c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = (c_0 + c_1\lambda + c_2\lambda^2) \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \qquad C \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix} = (c_0 + c_1\lambda^2 + c_2\lambda^4) \begin{bmatrix} 1 \\ \lambda^2 \\ \lambda^4 \end{bmatrix}.$$

Those 3 eigenvalues of C are exactly the 3 components of $F\boldsymbol{c} = F \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$,

Reference material

- Eigenvectors of Circulant Matrices: Fourier Matrix
- Lecture 23, Differential Equations and exp(At)
- *"Linear Algebra and Applications", pdf pages 435–436*
  Circulant Matrix 8.3
- *"Linear Algebra and Applications", pdf pages 330–348*
  Systems of Differential Equations 6.3

Deserve "A" grade!

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Room 105 (Underground robotics lab)