

INTRODUCTION TO LINEAR ALGEBRA

Third Edition

MANUAL FOR INSTRUCTORS

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Solutions to Exercises

Problem Set 1.1, page 6

- 1 Line through $(1, 1, 1)$; plane; same plane!
- 3 $\mathbf{v} = (2, 2)$ and $\mathbf{w} = (1, -1)$.
- 4 $3\mathbf{v} + \mathbf{w} = (7, 5)$ and $\mathbf{v} - 3\mathbf{w} = (-1, -5)$ and $c\mathbf{v} + d\mathbf{w} = (2c + d, c + 2d)$.
- 5 $\mathbf{u} + \mathbf{v} = (-2, 3, 1)$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = (0, 0, 0)$ and $2\mathbf{u} + 2\mathbf{v} + \mathbf{w} = (\text{add first answers}) = (-2, 3, 1)$.
- 6 The components of every $c\mathbf{v} + d\mathbf{w}$ add to zero. Choose $c = 4$ and $d = 10$ to get $(4, 2, -6)$.
- 8 The other diagonal is $\mathbf{v} - \mathbf{w}$ (or else $\mathbf{w} - \mathbf{v}$). Adding diagonals gives $2\mathbf{v}$ (or $2\mathbf{w}$).
- 9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.
- 10 $\mathbf{i} + \mathbf{j}$ is the diagonal of the base.
- 11 Five more corners $(0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The centers of the six faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.
- 12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional sides and 24 two-dimensional faces and 32 one-dimensional edges. See Worked Example 2.4 A.
- 13 sum = zero vector; sum = $-4:00$ vector; $1:00$ is 60° from horizontal $= (\cos \frac{\pi}{3}, \sin \frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.
- 14 Sum $= 12\mathbf{j}$ since $\mathbf{j} = (0, 1)$ is added to every vector.
- 15 The point $\frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ is three-fourths of the way to \mathbf{v} starting from \mathbf{w} . The vector $\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ is halfway to $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$, and the vector $\mathbf{v} + \mathbf{w}$ is $2\mathbf{u}$ (the far corner of the parallelogram).
- 16 All combinations with $c + d = 1$ are on the line through \mathbf{v} and \mathbf{w} . The point $\mathbf{V} = -\mathbf{v} + 2\mathbf{w}$ is on that line beyond \mathbf{w} .
- 17 The vectors $c\mathbf{v} + c\mathbf{w}$ fill out the line passing through $(0, 0)$ and $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$. It continues beyond $\mathbf{v} + \mathbf{w}$ and $(0, 0)$. With $c \geq 0$, half this line is removed and the “ray” starts at $(0, 0)$.
- 18 The combinations with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ fill the parallelogram with sides \mathbf{v} and \mathbf{w} .
- 19 With $c \geq 0$ and $d \geq 0$ we get the “cone” or “wedge” between \mathbf{v} and \mathbf{w} .
- 20 (a) $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ is the center of the triangle between \mathbf{u}, \mathbf{v} and \mathbf{w} ; $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ is the center of the edge between \mathbf{u} and \mathbf{w} (b) To fill in the triangle keep $c \geq 0, d \geq 0, e \geq 0$, and $c + d + e = 1$.

- 21 The sum is $(\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{u} - \mathbf{w}) = \text{zero vector}$.
- 22 The vector $\frac{1}{2}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.
- 23 All vectors are combinations of \mathbf{u} , \mathbf{v} , and \mathbf{w} .
- 24 Vectors $c\mathbf{v}$ are in both planes.
- 25 (a) Choose $\mathbf{u} = \mathbf{v} = \mathbf{w} = \text{any nonzero vector}$ (b) Choose \mathbf{u} and \mathbf{v} in different directions, and \mathbf{w} to be a combination like $\mathbf{u} + \mathbf{v}$.
- 26 The solution is $c = 2$ and $d = 4$. Then $2(1, 2) + 4(3, 1) = (14, 8)$.
- 27 The combinations of $(1, 0, 0)$ and $(0, 1, 0)$ fill the xy plane in xyz space.
- 28 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (divide by bd) the ratios a/b and c/d are equal!

Problem Set 1.2, page 17

- 1 $\mathbf{u} \cdot \mathbf{v} = 1.4$, $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{w} = 24 = \mathbf{w} \cdot \mathbf{v}$.
- 2 $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 5 = \|\mathbf{w}\|$. Then $1.4 < (1)(5)$ and $24 < (5)(5)$.
- 3 Unit vectors $\mathbf{v}/\|\mathbf{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\mathbf{w}/\|\mathbf{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{24}{25}$. The vectors \mathbf{w} , \mathbf{u} , $-\mathbf{w}$ make $0^\circ, 90^\circ, 180^\circ$ angles with \mathbf{w} .
- 4 $\mathbf{u}_1 = \mathbf{v}/\|\mathbf{v}\| = \frac{1}{\sqrt{10}}(3, 1)$ and $\mathbf{u}_2 = \mathbf{w}/\|\mathbf{w}\| = \frac{1}{3}(2, 1, 2)$. $\mathbf{U}_1 = \frac{1}{\sqrt{10}}(1, -3)$ or $\frac{1}{\sqrt{10}}(-1, 3)$. \mathbf{U}_2 could be $\frac{1}{\sqrt{5}}(1, -2, 0)$.
- 5 (a) $\mathbf{v} \cdot (-\mathbf{v}) = -1$ (b) $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} = 1 + () - () - 1 = 0$
so $\theta = 90^\circ$ (c) $(\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 4\mathbf{w} \cdot \mathbf{w} = -3$
- 6 (a) $\cos \theta = \frac{1}{(2)(1)}$ so $\theta = 60^\circ$ or $\frac{\pi}{3}$ radians (b) $\cos \theta = 0$ so $\theta = 90^\circ$ or $\frac{\pi}{2}$ radians
(c) $\cos \theta = \frac{-1+3}{(2)(2)} = \frac{1}{2}$ so $\theta = 60^\circ$ or $\frac{\pi}{3}$ (d) $\cos \theta = -1/\sqrt{2}$ so $\theta = 135^\circ$ or $\frac{3\pi}{4}$.
- 7 All vectors $\mathbf{w} = (c, 2c)$; all vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*; all vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.
- 8 (a) False (b) True: $\mathbf{u} \cdot (c\mathbf{v} + d\mathbf{w}) = c\mathbf{u} \cdot \mathbf{v} + d\mathbf{u} \cdot \mathbf{w} = 0$ (c) True
- 9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = 0$.
- 10 Slopes $\frac{2}{1}$ and $-\frac{1}{2}$ multiply to give -1 : perpendicular.
- 11 $\mathbf{v} \cdot \mathbf{w} < 0$ means angle $> 90^\circ$; this is half of the plane.
- 12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = 0$ if $c = \mathbf{v} \cdot \mathbf{w} / \mathbf{v} \cdot \mathbf{v}$.
- 13 $\mathbf{v} = (1, 0, -1)$, $\mathbf{w} = (0, 1, 0)$.
- 14 $\mathbf{u} = (1, -1, 0, 0)$, $\mathbf{v} = (0, 0, 1, -1)$, $\mathbf{w} = (1, 1, -1, -1)$.
- 15 $\frac{1}{2}(x + y) = 5$; $\cos \theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = .8$.
- 16 $\|\mathbf{v}\|^2 = 9$ so $\|\mathbf{v}\| = 3$; $\mathbf{u} = \frac{1}{3}\mathbf{v}$; $\mathbf{w} = (1, -1, 0, \dots, 0)$.
- 17 $\cos \alpha = 1/\sqrt{2}$, $\cos \beta = 0$, $\cos \gamma = -1/\sqrt{2}$, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (v_1^2 + v_2^2 + v_3^2)/\|\mathbf{v}\|^2 = 1$.

- 18** $\|v\|^2 = 4^2 + 2^2 = 20$, $\|w\|^2 = (-1)^2 + 2^2 = 5$, $\|(3, 4)\|^2 = 25 = 20 + 5$.
- 19** $v - w = (5, 0)$ also has $(\text{length})^2 = 25$. Choose $v = (1, 1)$ and $w = (0, 1)$ which are not perpendicular; $(\text{length of } v)^2 + (\text{length of } w)^2 = 1^2 + 1^2 + 1^2$ but $(\text{length of } v - w)^2 = 1$.
- 20** $(v+w) \cdot (v+w) = (v+w) \cdot v + (v+w) \cdot w = v \cdot (v+w) + w \cdot (v+w) = v \cdot v + v \cdot w + w \cdot v + w \cdot w = v \cdot v + 2v \cdot w + w \cdot w$. Notice $v \cdot w = w \cdot v$!
- 21** $2v \cdot w \leq 2\|v\|\|w\|$ leads to $\|v+w\|^2 = v \cdot v + 2v \cdot w + w \cdot w \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$.
- 22** Compare $v \cdot v + w \cdot w$ with $(v-w) \cdot (v-w)$ to find that $-2v \cdot w = 0$. Divide by -2 .
- 23** $\cos \beta = w_1/\|w\|$ and $\sin \beta = w_2/\|w\|$. Then $\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha = v_1 w_1 / \|v\|\|w\| + v_2 w_2 / \|v\|\|w\| = v \cdot w / \|v\|\|w\|$.
- 24** We know that $(v-w) \cdot (v-w) = v \cdot v - 2v \cdot w + w \cdot w$. The Law of Cosines writes $\|v\|\|w\| \cos \theta$ for $v \cdot w$. When $\theta < 90^\circ$ this is positive and $v \cdot v + w \cdot w$ is larger than $\|v-w\|^2$.
- 25** (a) $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \geq 0$.
- 26** Example 6 gives $|u_1||U_1| \leq \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \leq \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \leq (.6)(.8) + (.8)(.6) \leq \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$.
- 27** The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos \theta|^2 = x^2/(x^2 + y^2) \leq 1$.
- 28** Try $v = (1, 2, -3)$ and $w = (-3, 1, 2)$ with $\cos \theta = \frac{-7}{14}$ and $\theta = 120^\circ$. Write $v \cdot w = xz + yz + xy$ as $\frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x+y+z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2)$, so $v \cdot w / \|v\|\|w\| = -\frac{1}{2}$.
- 29** The length $\|v-w\|$ is between 2 and 8. The dot product $v \cdot w$ is between -15 and 15 .
- 30** The vectors $w = (x, y)$ with $v \cdot w = x + 2y = 5$ lie on a line in the xy plane. The shortest w is $(1, 2)$ in the direction of v .
- 31** Three vectors in the plane could make angles $> 90^\circ$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this (360° total angle). How many can do this in \mathbf{R}^3 or \mathbf{R}^n ?

Problem Set 1.3

- 1** $(x, y, z) = (2, 0, 0)$ and $(0, 6, 0)$; $n = (3, 1, -1)$; dot product $(3, 1, -1) \cdot (2, -6, 0) = 0$.
- 2** $4x - y - 2z = 1$ is parallel to every plane $4x - y - 2z = d$ and perpendicular to $n = (4, -1, -2)$.
- 3** (a) True (assuming $n \neq 0$) (b) False (c) True.
- 4** (a) $x + 5y + 2z = 14$ (b) $x + 5y + 2z = 30$ (c) $y = 0$.
- 5** The plane changes to the symmetric plane on the other side of the origin.
- 6** $x - y - z = 0$.
- 7** $x + 4y = 0$; $x + 4y = 14$.
- 8** $u = (2, 0, 0)$, $v = (0, 2, 0)$, $w = (0, 0, 2)$. Need $c + d + e = 1$.

- 9** $x + 4y + z + 2t = 8$.
- 10** $x - 4y + 2z = 0$.
- 11** We choose $\mathbf{v}_0 = (6, 0, 0)$ and then in-plane vectors $(3, 1, 0)$ and $(1, 0, 1)$. The points on the plane are $\mathbf{v}_0 + y(3, 1, 0) + z(1, 0, 1)$.
- 12** $\mathbf{v}_0 = (0, 0, 0)$; all vectors in the plane are combinations $y(-2, 1, 0) + z(\frac{1}{2}, 0, 1)$.
- 13** $\mathbf{v}_0 = (0, 0, 0)$; all solutions are combinations $y(-1, 1, 0) + z(-1, 0, 1)$.
- 14** Particular point $(9, 0)$; solution $(3, 1)$; points are $(9, 0) + y(3, 1) = (3y + 9, y)$.
- 15** $\mathbf{v}_0 = (24, 0, 0, 0)$; solutions $(-2, 1, 0, 0)$ and $(-3, 0, 1, 0)$ and $(-4, 0, 0, 1)$. Combine to get $(24 - 2y - 3z - 4t, y, z, t)$.
- 16** Choose $\mathbf{v}_0 = (0, 6, 0)$ with two zero components. Then set components to 1 to choose $(1, 0, 0)$ and $(0, -3/2, 1)$. Combinations are $(x, 6 - \frac{3}{2}z, z)$.
- 17** Now $|d|/\|\mathbf{n}\| = 12/\sqrt{56} = 12/2\sqrt{14} = 6/\sqrt{14}$. Same answer because same plane.
- 18** (a) $|d|/\|\mathbf{n}\| = 18/3 = 6$ and $\mathbf{v} = (4, 4, 2)$ (b) $|d|/\|\mathbf{n}\| = 0$ and $\mathbf{v} = \mathbf{0}$
 (c) $|d|/\|\mathbf{n}\| = 6/\sqrt{2}$ and $\mathbf{v} = 3\mathbf{n} = (3, 0, -3)$.
- 19** (a) Shortest distance is along perpendicular to line (b) Need $t + 4t = 25$ or $t = 5$
 (c) The distance to $(5, -10)$ is $\sqrt{125}$.
- 20** (a) $\mathbf{n} = (a, b)$ (b) $t = c/(a^2 + b^2)$ (c) This distance to $t\mathbf{n} = (ca, cb)/(a^2 + b^2)$ is $|c|/\sqrt{a^2 + b^2}$.
- 21** Substitute $x = 1 + t, y = 2t, z = 5 - 2t$ to find $(1 + t) + 2(2t) - 2(5 - 2t) = 27$ or $-9 + 9t = 27$ or $t = 4$. Then $\|t\mathbf{n}\| = 12$.
- 22** Shortest distance in the direction of \mathbf{n} ; $\mathbf{w} + t\mathbf{n}$ lies on the plane when $\mathbf{n} \cdot \mathbf{w} + t\mathbf{n} \cdot \mathbf{n} = d$ or $t = (d - \mathbf{n} \cdot \mathbf{w})/\mathbf{n} \cdot \mathbf{n}$. The distance is $|d - \mathbf{n} \cdot \mathbf{w}|/\|\mathbf{n}\|$ (which is $|d|/\|\mathbf{n}\|$ when $\mathbf{w} = \mathbf{0}$).
- 23** The vectors $(1, 2, 3)$ and $(1, -1, -1)$ are perpendicular to the line. Set $x = 0$ to find $y = -16$ and $z = 14$. Set $y = 0$ to find $x = 9/2$ and $z = 5/2$. These particular points are $(0, -16, 14)$ and $(9/2, 0, 5/2)$.
- 24** (a) $\mathbf{n} = (1, 1, 1, -1)$ (b) $|d|/\|\mathbf{n}\| = \frac{1}{2}$ (c) $d\mathbf{n}/\mathbf{n} \cdot \mathbf{n} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4})$
 (d) $\mathbf{v}_0 = (1, 0, 0, 0)$ (e) $(-1, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)$
 (f) all points $(1 - y - z + t, y, z, t)$.
- 25** $\mathbf{n} = (1, 1, 1)$ or any nonzero (c, c, c) .
- 26** $\cos \theta = (0, 1, 1) \cdot (1, 0, 1)/\sqrt{2}\sqrt{2} = \frac{1}{2}$ so $\theta = 60^\circ$.

Problem Set 2.1, page 29

- 1** The planes $x = 2$ and $y = 3$ and $z = 4$ are perpendicular to the x, y, z axes.
- 2** The vectors are $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ and $\mathbf{b} = (2, 3, 4) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

- 3** The planes are the same: $2y = 6$ is $y = 3$, and $3z = 12$ is $z = 4$. The solution is the same intersection point. The columns are changed; but same combination $\hat{\mathbf{x}} = \mathbf{x}$.
- 4** The solution is not changed; the second plane and row 2 of the matrix and all columns of the matrix are changed.
- 5** If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ give the point $(5, 1, 0)$. Halfway between is $(3, 0, 1)$.
- 6** If x, y, z satisfy the first two equations they also satisfy the third equation. The line \mathbf{L} of solutions contains $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\mathbf{u} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$ and all combinations $c\mathbf{v} + d\mathbf{w}$ with $c + d = 1$.
- 7** Equation 1 + equation 2 - equation 3 is now $0 = -4$. Solution impossible.
- 8** Column 3 = Column 1; solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\mathbf{b} = (4, 6, c)$ needs $c = 10$ for solvability.
- 9** Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\mathbf{x} = (3, 3, 3, 2)$ is $\mathbf{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$.
- 10** $A\mathbf{x} = (18, 5, 0)$, $A\mathbf{x} = (3, 4, 5, 5)$.
- 11** Nine multiplications for $A\mathbf{x} = (18, 5, 0)$.
- 12** $(14, 22)$ and $(0, 0)$ and $(9, 7)$.
- 13** (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.
- 14** (a) \mathbf{x} has n components, $A\mathbf{x}$ has m components (b) Planes in n -dimensional space, but the columns are in m -dimensional space.
- 15** $2x + 3y + z + 5t = 8$ is $A\mathbf{x} = \mathbf{b}$ with the 1 by 4 matrix $A = \begin{bmatrix} 2 & 3 & 1 & 5 \end{bmatrix}$. The solutions \mathbf{x} fill a 3D "plane" in 4 dimensions.
- 16** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- 17** $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.
- 18** $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z) .
- 19** $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- 20** $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\mathbf{v} = (3, 4, 8)$, $E^{-1}E\mathbf{v} = (3, 4, 5)$.
- 21** $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P_1\mathbf{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $P_2P_1\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

22 $R = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$

23 The dot product $[1 \ 4 \ 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

24 $A = [1 \ 2 \ ; \ 3 \ 4]$ and $\mathbf{x} = [5 \ -2]'$ and $\mathbf{b} = [1 \ 7]'$. $\mathbf{r} = \mathbf{b} - A * \mathbf{x}$ prints as zero.

25 $A * \mathbf{v} = [3 \ 4 \ 5]'$ and $\mathbf{v}' * \mathbf{v} = 50$; $\mathbf{v} * A$ gives an error message.

26 $\text{ones}(4, 4) * \text{ones}(4, 1) = [4 \ 4 \ 4 \ 4]'$; $B * \mathbf{w} = [10 \ 10 \ 10 \ 10]'$.

27 The row picture has two lines meeting at $(4, 2)$. The column picture has $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$.

28 The row picture shows 2 planes in 3-dimensional space. The column picture is in 2-dimensional space. The solutions normally lie on a *line*.

29 The row picture shows four *lines*. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

30 $\mathbf{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components always add to 1. They are always positive.

31 $\mathbf{u}_7, \mathbf{v}_7, \mathbf{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

32 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \text{steady state } \mathbf{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

34 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1, 1, 1) = (15, 15, 15)$;
 $M_4(1, 1, 1, 1) = (34, 34, 34, 34)$ because the numbers 1 to 16 add to 136 which is $4(34)$.

Problem Set 2.2, page 40

- 1 Multiply by $l = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$.
- 2 $y = -1$ and then $x = 2$. Multiplying the right side by 4 will multiply (x, y) by 4 to give the solution $(x, y) = (8, -4)$.
- 3 Subtract $-\frac{1}{2}$ times equation 1 (or add $\frac{1}{2}$ times equation 1). The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.
- 4 Subtract $l = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.
- 5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$.

- 6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 2 \cdot 16 = 32$ makes the system solvable. The lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.
- 7 If $a = 2$ elimination must fail. The equations have no solution. If $a = 0$ elimination stops for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.
- 8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.
- 9 $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$. Then there will be infinitely many solutions.
- 10 The equation $y = 1$ comes from elimination. Then $x = 4$ and $5x - 4y = c = 16$.
- 11 $2x + 3y + z = 8$ $x = 2$
 $y + 3z = 4$ gives $y = 1$ If a zero is at the start of row 2 or 3,
 $8z = 8$ $z = 1$ that avoids a row operation.
- 12 $2x - 3y = 3$ $2x - 3y = 3$ $x = 3$ Subtract $2 \times$ row 1 from row 2
 $y + z = 1$ gives $y + z = 1$ and $y = 1$ Subtract $1 \times$ row 1 from row 3
 $2y - 3z = 2$ $-5z = 0$ $z = 0$ Subtract $2 \times$ row 2 from row 3
- 13 Subtract 2 times row 1 from row 2 to reach $(d - 10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system is singular; third pivot is missing.
- 14 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.
- 15 $0x + 0y + 2z = 4$ $0x + 3y + 4z = 4$
(a) $x + 2y + 2z = 5$ (b) $x + 2y + 2z = 5$
 $0x + 3y + 4z = 6$ $0x + 3y + 4z = 6$
(exchange 1 and 2, then 2 and 3) (rows 1 and 3 are not consistent)
- 16 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 1 = column 2 there is no *second* pivot.
- 17 $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has infinitely many solutions.
- 18 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular — no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.
- 19 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.
- 20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1 + 2 =$ row 3 on the left side but not the right side: for example $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 1$. No parallel planes but still no solution.
- 21 Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$. Solution $t = 4$, $z = -3$, $y = 2$, $x = -1$.
- 22 The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

23 The fifth pivot is $\frac{6}{5}$. The n th pivot is $\frac{(n+1)}{n}$.

24 $A = \begin{bmatrix} 1 & 1 & 1 \\ a & a+1 & a+1 \\ b & b+c & b+c+3 \end{bmatrix}$ for any a, b, c leads to $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

25 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$.

26 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

27 Solvable for $s = 10$ (add equations); $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$. $A = [1 \ 1 \ 0 \ 0; \ 1 \ 0 \ 1 \ 0; \ 0 \ 0 \ 1 \ 1; \ 0 \ 1 \ 0 \ 1]$ and $U = [1 \ 1 \ 0 \ 0; \ 0 \ -1 \ 1 \ 0; \ 0 \ 0 \ 1 \ 1; \ 0 \ 0 \ 0 \ 0]$.

28 Elimination leaves the diagonal matrix $\text{diag}(3, 2, 1)$. Then $x = 1, y = 1, z = 4$.

29 $A(2, :) = A(2, :) - 3 * A(1, :)$ Subtracts 3 times row 1 from row 2.

30 The average pivots for `rand(3)` *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite! *With* row exchanges in MATLAB's `lu` code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for `randn` with normal instead of uniform probability distribution).

Problem Set 2.3, page 50

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\mathbf{b} = (1, -5, -35)$ but $E_{21}E_{32}\mathbf{b} = (1, -5, 0)$. Then row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} E_{21}, & E_{31}, & E_{32} \\ M = E_{32} & E_{31} & E_{21} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

4 Elimination on column 4: $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. Then back substitution in $U\mathbf{x} = (1, -4, 10)$ gives $z = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\mathbf{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31} , add 7 times row 1 to row 3. The matrix is $R_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$.

8 The same R_{31} from Problem 7 is changed to I . Thus $E_{31}R_{31} = R_{31}E_{31} = I$.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, E must act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

11 $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

12 $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB (b) E could add row 2 to row 3 to give nonzeros.

14 E_{21} has $l_{21} = -\frac{1}{2}$, E_{32} has $l_{32} = -\frac{2}{3}$, E_{43} has $l_{43} = -\frac{3}{4}$. Otherwise the E 's match the identity matrix.

15 $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix}$. $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

16 (a) $X - 2Y = 0$ and $X + Y = 33$; $X=22$, $Y=11$ (b) $2m + c = 5$ and $3m + c = 7$; $m = 2$, $c = 1$.

$a + b + c = 4$ $a = 2$

17 $a + 2b + 4c = 8$ gives $b = 1$.

$a + 3b + 9c = 14$ $c = 1$

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I, (-P)^2 = I, I^2 = I, (-I)^2 = I$ (and many more).

20 (a) Each column is E times a column of B (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ rows are multiples of $[1 \ 2 \ 4]$.

21 No. $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $x_2 - x_1$ (d) $(A\mathbf{x})_1 = \sum a_{1j}x_j$.

23 $E(EA)$ subtracts 4 times row 1 from row 2. AE subtracts 2 times **column** 2 of A from **column** 1.

- 24** $[A \ b] = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 17 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & -5 & 15 \end{bmatrix}$; $2x_1 + 3x_2 = 1$ $x_1 = 5$
 $-5x_2 = 15$ $x_2 = -3$.
- 25** The last equation becomes $0 = 3$. Change the original 6 to 3. Then row 1 + row 2 = row 3.
- 26** (a) Add two extra columns; $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -7 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.
- 27** (a) No solution if $d = 0$ and $c \neq 0$ (b) Infinitely many solutions if $d = 0$ and $c = 0$. No effect from a and b .
- 28** $A = AI = A(BC) = (AB)C = IC = C$.
- 29** Given positive integers with $ad - bc = 1$. Certainly $c < a$ and $b < d$ would be impossible. Also $c > a$ and $b > d$ would be impossible with integers. This leaves row 1 < row 2 OR row 2 < row 1. An example is $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Multiply by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then multiply twice by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This shows that $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- 30** $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ and eventually $M = \text{"inverse of Pascal"} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$
 reduces Pascal to I .

Problem Set 2.4, page 59

- 1** $BA = 3I$ is 5 by 5 $AB = 5I$ is 3 by 3 $ABD = 5D$ is 3 by 1. ABD : No $A(B + C)$: No.
- 2** (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B)
 (d) (Row 1 of C) D (column 1 of E).
- 3** $AB + AC = A(B + C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$.
- 4** $A(BC) = (AB)C = \text{zero matrix}$
- 5** $A^n = \begin{bmatrix} 1 & bn \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.
- 6** $(A + B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.
- 7** (a) True (b) False (c) True (d) False.
- 8** Rows of DA are 3·(row 1 of A) and 5·(row 2 of A). Both rows of EA are row 2 of A . Columns of AD are 3·(column 1 of A) and 5·(column 2 of A). Columns of AE are zero and column 1 of A + column 2 of A .
- 9** $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not $F(EA)$ because multiplication is not commutative.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0, \dots$

12 $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $b = c = 0$. Then $AC = CA$ gives $a = d$: $A = aI$.

13 $(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$.

14 (a) True (b) False (c) True (d) False (take $B = 0$).

15 (a) mn (every entry) (b) mnp (c) n^3 (this is n^2 dot products).

16 By linearity $(AB)c$ agrees with $A(Bc)$. Also for all other columns of C .

17 (a) Use only column 2 of B (b) Use only row 2 of A (c)-(d) Use row 2 of first A .

18 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$.

19 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix.

20 (a) a_{11} (b) $l_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

21 $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^4 = 0$; then $A\mathbf{v} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}, A^2\mathbf{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix},$
 $A^3\mathbf{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}, A^4\mathbf{v} = \mathbf{0}.$

22 $A = A^2 = A^3 = \dots$ but $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$ and $(AB)^2 = 0$.

23 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$
 $DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED.$

24 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$; $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but $A^3 = 0$.

25 $A_1^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}, A_2^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A_3^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}.$

$$26 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}.$$

$$27 \text{ (a) (Row 3 of } A) \cdot (\text{column 1 of } B) = (\text{Row 3 of } A) \cdot (\text{column 2 of } B) = 0$$

$$\text{(b) } \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

$$28 A \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} \right] ; \left[\text{---} \right] B ; \left[\text{---} \right] \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} \right] ; \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} \right] \left[\text{---} \right]$$

$$29 A\mathbf{x} = \left[\begin{array}{c} | \\ | \\ | \end{array} \right] \begin{bmatrix} \underline{x_1} \\ \underline{x_2} \\ \underline{x_3} \end{bmatrix} = x_1(\text{column 1}) + x_2(\text{column 2}) + \cdots.$$

$$30 E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \text{ then } EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

$$31 \text{ In Problem 30, } \mathbf{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}, D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}, D - \mathbf{cb}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

$$32 \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ Bx + Ay \end{bmatrix} \begin{array}{l} \text{real part} \\ \text{imaginary part.} \end{array}$$

$$33 A \text{ times } X \text{ will be the identity matrix } I.$$

$$34 \text{ The solution for } \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \text{ is } 3\mathbf{x}_1 + 5\mathbf{x}_2 + 8\mathbf{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}; A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ to produce } x_1, x_2, x_3.$$

$$35 S = D - CA^{-1}B \text{ is the Schur complement: block version of } d - (cb/a).$$

$$36 \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \text{ agrees with } \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} \text{ when } b=c \text{ and } a=d.$$

$$37 \text{ If } A \text{ is "northwest" and } B \text{ is "southeast" then } AB \text{ is upper triangular and } BA \text{ is lower triangular. One reason: Row } i \text{ of } A \text{ can have } n-i+1 \text{ nonzeros, with zeros after that. Column } j \text{ of } B \text{ has } j \text{ nonzeros, with zeros above that. If } i > j \text{ then } (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B) = 0. \text{ So } AB \text{ is upper triangular.}$$

Similarly BA is lower triangular. Problem 2.7.40 asks about inverses and transposes and permutations of a northwest A and a southeast B .

$$38 A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix}, A^3 \text{ with } A^2$$

gives diameter 3.

39 $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ need also A^4
so diameter 4.

Problem Set 2.5, Page 72

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 $P^{-1} = P$; $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} is (transpose) of P .

3 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -2 \end{bmatrix}$, $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and
any $\begin{bmatrix} x & t \\ y & z \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix}^{-1}$.

4 $x + 2y = 1$, $3x + 6y = 0$: impossible.

5 $U = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$.

6 (a) Multiply $AB = AC$ by A^{-1} to find $B = C$

(b) B and C can be any matrices $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

7 (a) In $A\mathbf{x} = (1, 0, 0)$, equation 1 + equation 2 – equation 3 is $0 = 1$ (b) The right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\mathbf{x} = (1, 1, -1)$ solves $A\mathbf{x} = \mathbf{0}$ (b) Elimination keeps columns 1+2 = column 3. When columns 1 and 2 end in zeros so does column 3: no third pivot.

9 If you exchange rows 1 and 2 of A , you exchange columns 1 and 2 of A^{-1} .

10 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block).

11 (a) $A = I$, $B = -I$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

12 $C = AB$ gives $C^{-1} = B^{-1}A^{-1}$ so $A^{-1} = BC^{-1}$.

13 $M^{-1} = C^{-1}B^{-1}A^{-1}$ so $B^{-1} = CM^{-1}A$.

14 $B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

15 If A has a column of zeros, so does BA . So $BA = I$ is impossible. There is no A^{-1} .

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$. The inverse of one matrix is the other divided by $ad-bc$.

17 $\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E$; $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} = L = E^{-1}$
after reversing the order and changing -1 to $+1$.

18 $A^2B = I$ can be written as $A(AB) = I$. Therefore A^{-1} is AB .

19 The $(1, 1)$ entry requires $4a - 3b = 1$; the $(1, 2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b - a = 0$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

20 $A * \text{ones}(4, 1)$ is the zero vector so A cannot be invertible.

21 6 of the 16 are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [I \ A^{-1}]$;
 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 & 3 \\ 0 & 1 & 3 & -1 \end{bmatrix} = [I \ A^{-1}]$.

23 $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow$
 $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \rightarrow$
 $\begin{bmatrix} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{bmatrix}.$

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$

25 $A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so B^{-1} does not exist.

26 $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Then $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$
to reach I . Here $D^{-1}E_{12}E_{21} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} = A^{-1}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 $\begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix} = [I \ A^{-1}]$.

29 (a) True (AB has a row of zeros) (b) False (matrix of all 1's) (c) True (inverse of A^{-1} is A) (d) True (inverse of A^2 is $(A^{-1})^2$).

30 Not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

31 Elimination produces the pivots a and $a - b$ and $a - b$. $A^{-1} = \frac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

32 $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The 5 by 5 A^{-1} also has 1's on the diagonal and superdiagonal.

33 $\mathbf{x} = (2, 2, 2, 1)$.

34 $\mathbf{x} = (1, 1, \dots, 1)$ has $P\mathbf{x} = Q\mathbf{x}$ so $(P - Q)\mathbf{x} = \mathbf{0}$.

35 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

36 If $AC = CA$, multiply left and right by A^{-1} to find $CA^{-1} = A^{-1}C$. If also $BC = CB$, then (using the associative law!!), $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$.

37 A can be invertible but B is always singular. Each row of B will add to zero, from $0 + 1 + 2 - 3$, so the vector $\mathbf{x} = (1, 1, 1, 1)$ will give $B\mathbf{x} = \mathbf{0}$. I thought A would be invertible as long as you put the 3's on its main diagonal, but that's wrong:

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \mathbf{0} \quad \text{but} \quad A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \text{is invertible}$$

38 $AD = \text{pascal}(4, 1)$ is its own inverse.

39 `hilb(6)` is not the exact Hilbert matrix because fractions are rounded off.

40 The three Pascal matrices have $S = LU = LL^T$ and then $\text{inv}(S) = \text{inv}(L^T)\text{inv}(L)$. Note that the triangular L is `abs(pascal(n, 1))` in MATLAB.

41 For $A\mathbf{x} = \mathbf{b}$ with $A = \text{ones}(4, 4)$ = singular matrix and $\mathbf{b} = \text{ones}(4, 1)$ in its column space, MATLAB will pick the shortest solution $\mathbf{x} = (1, 1, 1, 1)/4$. Any vector in the nullspace of A could be added to this particular solution.

42 If $AC = I$ for square matrices then $C = A^{-1}$ (it is proved in **2I** that $CA = I$ will also be true). The same will be true for C^* . But a square matrix has only one inverse so $C = C^*$.

$$\begin{aligned}
\mathbf{43} \quad MM^{-1} &= (I_n - UV)(I_n + U(I_m - VU)^{-1}V) \\
&= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V \\
&= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n \quad (\text{formulas } \mathbf{1}, \mathbf{2}, \mathbf{4} \text{ are similar})
\end{aligned}$$

Problem Set 2.6, page 84

$$\mathbf{1} \quad \ell_{21} = 1; L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ times } U\mathbf{x} = \mathbf{c} \text{ is } A\mathbf{x} = \mathbf{b}: \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

$\mathbf{2} \quad \ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse the steps to recover $x + 3y + 6z = 11$ from $U\mathbf{x} = \mathbf{c}$:
 1 times $(x + y + z = 5) + 2$ times $(y + 2z = 2) + 1$ times $(z = 2)$ gives $x + 3y + 6z = 11$.

$$\mathbf{3} \quad L\mathbf{c} = \mathbf{b} \text{ is } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \quad U\mathbf{x} = \mathbf{c} \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$\mathbf{4} \quad L\mathbf{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}. \quad U\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}.$$

$$\mathbf{5} \quad EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U; \quad A = LU = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} U.$$

$$\mathbf{6} \quad \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U. \text{ Then } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U = E_{21}^{-1}E_{32}^{-1}U = LU.$$

$$\mathbf{7} \quad E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

$$\text{This is } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U. \text{ Then } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} U = LU.$$

$$\mathbf{8} \quad E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ -c & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -b & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -a & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ ac-b & -c & 1 \end{bmatrix}. \quad \text{This is}$$

$$L^{-1} = A^{-1}.$$

$$\mathbf{9} \quad \text{2 by 2: } d = 0 \text{ not allowed; } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ f & h & \\ i & & \end{bmatrix} \quad \begin{array}{l} d = 1, e = 1, \text{ then } l = 1 \\ f = 0 \text{ is not allowed} \\ \text{no pivot in row 2} \end{array}$$

$\mathbf{10} \quad c = 2$ leads to zero in the second pivot position: exchange rows and the matrix will be OK.

$c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

11 $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$ has $L = I$ and $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$; $A = LU$ has $U = A$ (pivots on the diagonal);

$A = LDU$ has $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ with 1's on the diagonal.

12 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; notice U is L^T

$A = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T.$

13 $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$. Need $\begin{matrix} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{matrix}$

14 $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

15 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\mathbf{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

Check that $A = LU = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}$ times \mathbf{x} is $\mathbf{b} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$.

16 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ gives $\mathbf{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ gives $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

17 (a) L goes to I (b) I goes to L^{-1} (c) LU goes to U .

18 (a) Multiply $LDU = L_1 D_1 U_1$ by inverses to get $L_1^{-1} L D = D_1 U_1 U^{-1}$. The left side is lower triangular, the right side is upper triangular \Rightarrow both sides are diagonal.

(b) Since L, U, L_1, U_1 have diagonals of 1's we get $D = D_1$. Then $L_1^{-1} L$ is I and $U_1 U^{-1}$ is I .

19 $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU$; $\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = (\text{same } L) \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} (\text{same } U).$

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (so 1 operation to find the multiplier and 1 to find the new pivot!). $T = \text{bidiagonal } L \text{ times } U$:

$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Reverse steps by $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- 21 For A , L has the 3 lower zeros but U may not have the upper zero. For B , L has the bottom left zero and U has the upper right zero. One zero in A and two zeros in B are filled in.

$$22 \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & & 1 \end{bmatrix} \begin{bmatrix} * & * & * \\ 0 & & \\ 0 & & \end{bmatrix} \quad (*\text{'s are all known after the first pivot is used}).$$

$$23 \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L. \text{ Then } A = UL \text{ with } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$24 \begin{bmatrix} 1 & 1 & 0 & 0 & 5 \\ 2 & 1 & 1 & 0 & 8 \\ 0 & 1 & 3 & 2 & 8 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}. \text{ Solve } \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \text{ for } x_2 = 3$$

and $x_3 = 1$ in the middle. Then $x_1 = 2$ backward and $x_4 = 1$ forward.

- 25 The 2 by 2 upper submatrix B has the first two pivots 2, 7. Reason: Elimination on A starts in the upper left corner with elimination on B .

- 26 The first three pivots for M are still 2, 7, 6. To be sure that 9 is the fourth pivot, put zeros in the rest of row 4 and column 4.

$$27 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 3 & 6 \\ & & & 1 & 4 \\ & & & & 1 \end{bmatrix} \quad \begin{array}{l} \text{Pascal's triangle in } L \text{ and } U. \\ \text{MATLAB's lu code will wreck} \\ \text{the pattern. chol does no row} \\ \text{exchanges for symmetric} \\ \text{matrices with positive pivots.} \end{array}$$

- 28 $c = 6$ and also $c = 7$ will make LU impossible ($c = 6$ needs a row exchange).

- 32 $\text{inv}(A) * \mathbf{b}$ should take 3 times as long as $A \setminus \mathbf{b}$ (n^3 for A^{-1} vs $n^3/3$ multiplications for LU).

- 34 The upper triangular part $\text{triu}(A)$ should be about three times faster to invert.

- 35 Each new *right side* costs only n^2 steps compared to $n^3/3$ for full elimination $A \setminus \mathbf{b}$.

- 36 This L comes from the $-1, 2, -1$ tridiagonal $A = LDL^T$. (Row i of L) \cdot (Column j of L^{-1}) = $\left(\frac{1-i}{i}\right) \left(\frac{j}{i-1}\right) + (1) \left(\frac{j}{i}\right) = 0$ for $i > j$ so $LL^{-1} = I$. Then L^{-1} leads to $A^{-1} = (L^{-1})^T D^{-1} L^{-1}$.
The $-1, 2, -1$ matrix has inverse $A_{ij}^{-1} = j(n-i+1)/(n+1)$ for $i \geq j$ (reverse for $i \leq j$).

Problem Set 2.7, page 95

$$1 \quad A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}, \quad (A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}; \quad A^T = A \text{ and } A^{-1} = \frac{1}{c^2} \begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^T.$$

- 2 In case $AB = BA$, transpose both sides: A^T commutes with B^T .

$$3 \quad ((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T.$$

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. But the diagonal entries of $A^T A$ are dot products of columns of A with themselves. If $A^T A = 0$, zero dot products \Rightarrow zero columns $\Rightarrow A =$ zero matrix.

5 (a) $\mathbf{x}^T A \mathbf{y} = a_{22} = 5$ (b) $\mathbf{x}^T A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ (c) $A \mathbf{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

6 $M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$; $M^T = M$ needs $A^T = A, B^T = C, D^T = D$.

7 (a) False (needs $A = A^T$) (b) False (c) True (d) False.

8 The 1 in column 1 has n choices; then the 1 in column 2 has $n - 1$ choices; \dots ($n!$ choices overall).

9 $P_1 P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \neq P_2 P_1$.

10 (3, 1, 2, 4), (2, 3, 1, 4) keep 4 in position; 6 more keeping 1 or 2 or 3 in position; (2, 1, 4, 3) and (3, 4, 1, 2) exchanging 2 pairs.

11 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$;

No AP is lower triangular (this is a column exchange); $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

12 $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T P^T P \mathbf{y} = \mathbf{x}^T \mathbf{y}$ because $P^T P = I$; In general $P\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot P^T \mathbf{y} \neq \mathbf{x} \cdot P\mathbf{y}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

13 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose; $\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same P had $\hat{P}^4 = \hat{P}$.

14 There are $n!$ permutation matrices of order n . Eventually two powers of P must be the same:

$P^r = P^s$ and $P^{r-s} = I$. Certainly $r - s \leq n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

15 (a) $P^T(\text{row } 4) = \text{row } 1$ (b) $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^T$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ moves all rows.

16 $A^2 - B^2$ and ABA are symmetric if A and B are symmetric.

17 (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

- 18** (a) $5 + 4 + 3 + 2 + 1 = 15$ independent entries if $A = A^T$ (b) L has 10 and D has 5: total 15 in LDL^T (c) Zero diagonal if $A^T = -A$, leaving $4 + 3 + 2 + 1 = 10$.

- 19** (a) The transpose of $R^T A R$ is $R^T A^T R^{TT} = R^T A R = n$ by n (b) $(R^T R)_{jj} = (\text{column } j \text{ of } R) \cdot (\text{column } j \text{ of } R) = \text{length squared of column } j$.

20
$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

21 Lower right 2 by 2 matrix is $\begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix}, \begin{bmatrix} d - b^2 & e - bc \\ e - bc & f - c^2 \end{bmatrix}.$

22
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}; \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$$

23 $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$ and $L = U = I$; exchanges rows 1-2 then rows 2-3.

24
$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 8 & \\ -2/3 & & \end{bmatrix}.$$
 If we wait to exchange, then

$$A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

25 $\text{abs}(A(1,1)) = 0$ and $\text{abs}(A(2,1)) > \text{tol}$; $A \rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $P \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; no more elimination

so $L = I$ and $U = \text{new } A$. $\text{abs}(A(1,1)) = 0$ and $\text{abs}(A(2,1)) > \text{tol}$; $A \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 5 & 6 \end{bmatrix}$ and

$P \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{matrix} \text{abs}(A(2,2)) = 0 \\ \text{abs}(A(3,2)) > \text{tol} \end{matrix}; A \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}, L = I, P \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$

26 $\text{abs}(A(1,1)) = 0$ so find $\text{abs}(A(2,1)) > \text{tol}$; exchange rows to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}$ and $P =$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; eliminate to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, same P ; $\text{abs}(A(2,2)) > \text{tol}$

so eliminate to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \text{final } U$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$

- 27** No solution

28 $L_1 = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}$ shows the elimination steps as actually done (L is affected by P).

29 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: show that an exchange always reverses that count! Then 3 or 5 exchanges will leave that count odd.

30 $E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$ and $E_{21}AE_{21}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ is still symmetric; $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -4 & 1 \end{bmatrix}$

and $E_{32}E_{21}AE_{21}^TE_{32}^T = D$. Elimination from both sides gives the symmetric LDL^T directly.

31 Total currents are $A^T \mathbf{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

Either way $(A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (A^T \mathbf{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$.

32 Inputs $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$; $A^T \mathbf{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck
1 plane

33 $A\mathbf{x} \cdot \mathbf{y}$ is the *cost* of inputs while $\mathbf{x} \cdot A^T \mathbf{y}$ is the *value* of outputs.

34 $P^3 = I$ so three rotations for 360° ; P rotates around $(1, 1, 1)$ by 120° .

35 $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = EH$

36 $L(U^T)^{-1} =$ triangular times triangular. The transpose of $U^T D U$ is $U^T D^T U^{TT} = U^T D U$ again.

37 These are groups: Lower triangular with diagonal 1's, diagonal invertible D , permutations P , orthogonal matrices with $Q^T = Q^{-1}$.

38 $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ (I don't know any rules for constructions like this)

39 Reordering the rows and/or columns of $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ will move the entry \mathbf{a} .

40 Certainly B^T is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L , so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1} . So B^{-1} is *southeast*. Northwest times southeast is upper triangular! $B = PL$ and $C = PU$ give $BC = (PLP)U =$ upper times upper.

41 The i, j entry of PAP is the $n - i + 1, n - j + 1$ entry of A . The main diagonal reverses order.

Problem Set 3.1, Page 107

- 1 $\mathbf{x} + \mathbf{y} \neq \mathbf{y} + \mathbf{x}$ and $\mathbf{x} + (\mathbf{y} + \mathbf{z}) \neq (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ and $(c_1 + c_2)\mathbf{x} \neq c_1\mathbf{x} + c_2\mathbf{x}$.
- 2 The only broken rule is 1 times \mathbf{x} equals \mathbf{x} .
- 3 (a) $c\mathbf{x}$ may not be in our set: not closed under scalar multiplication. Also no $\mathbf{0}$ and no $-\mathbf{x}$
 (b) $c(\mathbf{x} + \mathbf{y})$ is the usual $(xy)^c$, while $c\mathbf{x} + c\mathbf{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ they equal 8. This is $3(2 + 1)!!$. The zero vector is the number 1.
- 4 The zero vector in \mathbf{M} is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace containing A consists of all matrices cA .
- 5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) All matrices whose main diagonal is all zero.
- 6 $h(x) = 3f(x) - 4g(x) = 3x^2 - 20x$.
- 7 Rule 8 is broken: If $c\mathbf{f}(x)$ is defined to be the usual $f(cx)$ then $(c_1 + c_2)\mathbf{f} = f((c_1 + c_2)x)$ is different from $c_1\mathbf{f} + c_2\mathbf{f} = \text{usual } f(c_1x) + f(c_2x)$.
- 8 If $(\mathbf{f} + \mathbf{g})(x)$ is the usual $f(g(x))$ then $(\mathbf{g} + \mathbf{f})x$ is $g(f(x))$ which is different. In Rule 2 both sides are $f(g(h(x)))$. Rule 4 is broken because there might be no inverse function $f^{-1}(x)$ such that $f(f^{-1}(x)) = x$. If the inverse function exists it will be the vector $-\mathbf{f}$.
- 9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
 (b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.
- 10 Only (a) (d) (e) are subspaces.
- 11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.
- 12 The sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane.
- 13 \mathbf{P}_0 has the equation $x + y - 2z = 0$; $(2, 0, 1)$ and $(0, 2, 1)$ and their sum $(2, 2, 2)$ are in \mathbf{P}_0 .
- 14 (a) The subspaces of \mathbf{R}^2 are \mathbf{R}^2 itself, lines through $(0, 0)$, and $(0, 0)$ itself (b) The subspaces of \mathbf{R}^4 are \mathbf{R}^4 itself, three-dimensional planes $\mathbf{n} \cdot \mathbf{v} = 0$, two-dimensional subspaces ($\mathbf{n}_1 \cdot \mathbf{v} = 0$ and $\mathbf{n}_2 \cdot \mathbf{v} = 0$), one-dimensional lines through $(0, 0, 0, 0)$, and $(0, 0, 0, 0)$ alone.
- 15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$ (b) The plane and line probably intersect in the point $(0, 0, 0)$ (c) Suppose \mathbf{x} is in $\mathbf{S} \cap \mathbf{T}$ and \mathbf{y} is in $\mathbf{S} \cap \mathbf{T}$. Both vectors are in both subspaces, so $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ are in both subspaces.
- 16 The smallest subspace containing \mathbf{P} and \mathbf{L} is either \mathbf{P} or \mathbf{R}^3 .
- 17 (a) The zero matrix is not invertible (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular.
- 18 (a) True (b) True (c) False.
- 19 The column space of A is the x axis = all vectors $(x, 0, 0)$. The column space of B is the xy plane = all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

- 20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.
- 21 A combination of the columns of C is also a combination of the columns of A (same column space; B has a different column space).
- 22 (a) Every \mathbf{b} (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.
- 23 The extra column \mathbf{b} enlarges the column space unless \mathbf{b} is *already in* the column space of A :
 $[A \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (larger column space) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (\mathbf{b} already in column space)
 (no solution to $A\mathbf{x} = \mathbf{b}$) ($A\mathbf{x} = \mathbf{b}$ has a solution)
- 24 The column space of AB is contained in (possibly equal to) the column space of A . If $B = 0$ and $A \neq 0$ then $AB = 0$ has a smaller column space than A .
- 25 The solution to $A\mathbf{z} = \mathbf{b} + \mathbf{b}^*$ is $\mathbf{z} = \mathbf{x} + \mathbf{y}$. If \mathbf{b} and \mathbf{b}^* are in the column space so is $\mathbf{b} + \mathbf{b}^*$.
- 26 The column space of any invertible 5 by 5 matrix is \mathbf{R}^5 . The equation $A\mathbf{x} = \mathbf{b}$ is always solvable (by $\mathbf{x} = A^{-1}\mathbf{b}$) so every \mathbf{b} is in the column space.
- 27 (a) False (b) True (c) True (d) False.
- 28 $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ (columns on 1 line).
- 29 Every \mathbf{b} is in the column space so that space is \mathbf{R}^9 .

Problem Set 3.2, Page 118

- 1 (a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables x_2, x_4, x_5
 Pivot variables x_1, x_3 (b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free x_3
 Pivot x_1, x_2
- 2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$
 (b) Free variable x_3 : solution $(1, -1, 1)$.
- 3 The complete solutions are $(-2x_2, x_2, -2x_4 - 3x_5, x_4, x_5)$ and $(2x_3, -x_3, x_3)$.
 The nullspace contains only $\mathbf{0}$ when there are no free variables.
- 4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A .
- 5 $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$.
- 6 (a) Special solutions $(3, 1, 0)$ and $(5, 0, 1)$ (b) $(3, 1, 0)$. Total count of pivot and free is n .
- 7 (a) Nullspace of A is the plane $-x + 3y + 5z = 0$; it contains all vectors $(3y + 5z, y, z)$
 (b) The *line* through $(3, 1, 0)$ has equations $-x + 3y + 5z = 0$ and $-2x + 6y + 7z = 0$.
- 8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [1]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- 9 (a) False (b) True (c) True (only n columns) (d) True (only m rows).

10 (a) Impossible above diagonal (b) $A = \text{invertible} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $A = 2I, U = 2I, R = I.$

11 $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12 $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

13 If column 4 is all zero then x_4 is a *free* variable. Its special solution is $(0, 0, 0, 1, 0)$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

15 There are $n - r$ special solutions. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when $r = n$. The column space is \mathbf{R}^m when $r = m$.

16 The nullspace contains only $\mathbf{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is \mathbf{R}^5 , because we can solve $A\mathbf{x} = \mathbf{b}$ and every \mathbf{b} is in the column space.

17 $A = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$; y and z are free; special solutions $(3, 1, 0)$ and $(1, 0, 1)$.

18 Fill in 12 then 3 then 1.

19 If $LU\mathbf{x} = \mathbf{0}$, multiply by L^{-1} to find $U\mathbf{x} = \mathbf{0}$. Then U and LU have the same nullspace.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\mathbf{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of \mathbf{s} (a line in \mathbf{R}^5).

21 Free variables x_3, x_4 : $A = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$

22 $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$

23 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}.$

24 This construction is impossible: 2 pivot columns, 2 free variables, only 3 columns.

25 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$

26 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$

27 If nullspace = column space (r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

28 If A times every column of B is zero, the column space of B is contained in the nullspace of

$$A: A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

29 R is most likely to be I ; R is most likely to be I with fourth row of zeros.

30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\text{rref}(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31 Three pivots (4 columns and 1 special solution); $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

32 Any zero rows come after these rows: $R = [1 \ -2 \ -3]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R 's!

34 One reason: A and $-A$ have the same nullspace (and also the same column space).

Problem Set 3.3, page 128

1 (a) and (c) are correct; (d) is false because R might happen to have 1's in nonpivot columns.

2 $R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $r = 1$; $R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $r = 2$; $R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $r = 1$

3 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = [R_A \ R_A]$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero row in the upper

R moves all the way to the bottom.

4 If all pivot variables come last then $R = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

5 I think this is true.

6 A and A^T have the same rank r . But *pivcol* (the column number) is 2 for A and 1 for A^T :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7 The special solutions are the columns of $N = \begin{bmatrix} -2 & -3 \\ -4 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$.

$$\mathbf{8} \quad A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}, \quad M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}.$$

9 If A has rank 1, the column space is a *line* in \mathbf{R}^m . The nullspace is a *plane* in \mathbf{R}^n (given by one equation). The column space of A^T is a *line* in \mathbf{R}^n .

10 $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (1, 2, 2)$; $\mathbf{u} = (2, -1)$, $\mathbf{v} = (1, 1, 3, 2)$.

11 A rank one matrix has one pivot. The second row of U is zero.

$$\mathbf{12} \quad S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad S = [1] \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^T is also r , and the example matrix A has rank 2:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \quad S^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

15 $\text{Rank}(AB) = 1$; $\text{rank}(AM) = 1$ except $AM = 0$ if $c = -1/2$.

16 $(\mathbf{u}\mathbf{v}^T)(\mathbf{w}\mathbf{z}^T) = \mathbf{u}(\mathbf{v}^T\mathbf{w})\mathbf{z}^T$ has rank one unless $\mathbf{v}^T\mathbf{w} = 0$.

17 (a) By matrix multiplication, each column of AB is A times the corresponding column of B .

So a combination of columns of B turns into a combination of columns of AB .

(b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank stays the same for $A_1 = I$ and it drops to zero for $A_2 = 0$ or $A_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$.

18 If we know that $\text{rank}(B^T A^T) \leq \text{rank}(A^T)$, then since rank stays the same for transposes, we have $\text{rank}(AB) \leq \text{rank}(A)$.

19 We are given $AB = I$ which has rank n . Then $\text{rank}(AB) \leq \text{rank}(A)$ forces $\text{rank}(A) = n$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad AB = I \quad \text{and} \quad BA \neq I.$$

21 (a) A and B will both have the same nullspace and row space as R (same R for both matrices).

(b) A equals an *invertible* matrix times B , when they share the same R . A key fact!

$$\mathbf{22} \quad A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \quad (\text{nonzero rows of } R).$$

$$23 \quad A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}.$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 8 \end{bmatrix}.$$

24 The m by n matrix Z has r ones at the start of its main diagonal. Otherwise Z is all zeros.

25 $Y = Z$ because the form is decided by the rank which is the same for A and A^T .

$$26 \quad \text{If } c = 1, R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ has } x_2, x_3, x_4 \text{ free. If } c \neq 1, R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ has } x_3, x_4 \text{ free.}$$

$$\text{Special solutions in } N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (c = 1) \text{ and } N = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} (c \neq 1)$$

$$\text{If } c = 1, R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } x_1 \text{ free; if } c = 2, R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \text{ and } x_2 \text{ free; } R = I \text{ if } c \neq 1, 2$$

$$\text{Special solutions in } N = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (c = 1) \text{ or } N = \begin{bmatrix} 2 \\ 1 \end{bmatrix} (c = 2) \text{ or } N = 2 \text{ by } 0 \text{ empty matrix.}$$

$$27 \quad N = \begin{bmatrix} I \\ -I \end{bmatrix}; \quad N = \begin{bmatrix} I \\ -I \end{bmatrix}; \quad N = \text{empty.}$$

Problem Set 3.4, page 136

$$1 \quad \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - 2\mathbf{b}_1 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$ which is the plane $b_3 + b_2 - 2b_1 = 0$ (!); the nullspace contains all combinations of $\mathbf{s}_1 = (-1, -1, 1, 0)$ and $\mathbf{s}_2 = (2, -2, 0, 1)$; $\mathbf{x}_{complete} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$;

$$[R \quad \mathbf{d}] = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ gives the particular solution } \mathbf{x}_p = (4, -1, 0, 0).$$

$$2 \quad \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - 3\mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - 2\mathbf{b}_1 \end{bmatrix} \quad \text{Then } [R \quad \mathbf{d}] = \begin{bmatrix} 1 & 1/2 & 3/2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A\mathbf{x} = \mathbf{b}$$

has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the column space is the line through $(2, 6, 4)$

which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\mathbf{s}_1 = (-1/2, 1, 0)$ and $\mathbf{s}_2 = (-3/2, 0, 1)$; particular solution $\mathbf{x}_p = (5, 0, 0)$ and complete solution $\mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$.

$$3 \quad \mathbf{x}_{complete} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

$$4 \quad \mathbf{x}_{complete} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

$$5 \quad \text{Solvable if } 2b_1 + b_2 = b_3. \text{ Then } \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$6 \quad (a) \text{ Solvable if } b_2 = 2b_1 \text{ and } 3b_1 - 3b_3 + b_4 = 0. \text{ Then } \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} \text{ (no free variables)}$$

$$(b) \text{ Solvable if } b_2 = 2b_1 \text{ and } 3b_1 - 3b_3 + b_4 = 0. \text{ Then } \mathbf{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

$$7 \quad \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix} \rightarrow \begin{array}{l} \text{row 3} - 2(\text{row 2}) + 4(\text{row 1}) \\ \text{is the zero row} \\ [0 \ 0 \ 0 \ b_3 - 2b_2 + 4b_1] \end{array}$$

8 (a) Every \mathbf{b} is in the column space: *independent rows*. (b) Need $b_3 = 2b_2$. Row $3 - 2$ row $2 = 0$.

$$9 \quad L[\mathbf{U} \quad \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = [\mathbf{A} \quad \mathbf{b}];$$

$\mathbf{x}_p = (-9, 0, 3, 0)$ so $-9(1, 2, 3) + 3(3, 8, 7) = (0, 6, -6)$ is exactly $\mathbf{A}\mathbf{x}_p = \mathbf{b}$.

$$10 \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

11 A 1 by 3 system has at least two free variables.

12 (a) $\mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{0}$ solve $\mathbf{A}\mathbf{x} = \mathbf{0}$ (b) $2\mathbf{x}_1 - 2\mathbf{x}_2$ solves $\mathbf{A}\mathbf{x} = \mathbf{0}$; $2\mathbf{x}_1 - \mathbf{x}_2$ solves $\mathbf{A}\mathbf{x} = \mathbf{b}$.

13 (a) The particular solution \mathbf{x}_p is always multiplied by 1 (b) Any solution can be the particular solution (c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
(d) The “homogeneous” solution in the nullspace is $\mathbf{x}_n = \mathbf{0}$ when \mathbf{A} is invertible.

14 If column 5 has no pivot, x_5 is a free variable. The zero vector *is not* the only solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$. If $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of \mathbf{U} has no pivot, that is a *zero row*. $\mathbf{U}\mathbf{x} = \mathbf{c}$ is solvable only if $c_3 = 0$. $\mathbf{A}\mathbf{x} = \mathbf{b}$ *might not* be solvable, because \mathbf{U} may have other zero rows.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is \mathbf{R}^3 . An example is $A = [I \ F]$ for any 3 by 2 matrix F .

17 The largest rank is 4. There is a pivot in every *column*. The solution is *unique*. The nullspace contains only the *zero vector*. An example is $A = [I \ G]$ for any 4 by 2 matrix G .

18 Rank = 3; rank = 3 unless $q = 2$ (then rank = 2).

19 All ranks = 2.

$$\mathbf{20} \quad A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}.$$

$$\mathbf{21} \quad (\text{a}) \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad (\text{b}) \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

22 If $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$ then we can add $\mathbf{x}_1 - \mathbf{x}_2$ to any solution of $A\mathbf{x} = \mathbf{B}$. But there will be *no* solution to $A\mathbf{x} = \mathbf{B}$ if \mathbf{B} is not in the column space.

23 For A , $q = 3$ gives rank 1, every other q gives rank 2. For B , $q = 6$ gives rank 1, every other q gives rank 2.

$$\mathbf{24} \quad (\text{a}) \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\text{b}) \quad [1 \ 1] \quad (\text{c}) \quad [0] \text{ or any } r < m, r < n \quad (\text{d}) \text{ Invertible.}$$

$$\mathbf{25} \quad (\text{a}) \quad r < m, \text{ always } r \leq n \quad (\text{b}) \quad r = m, r < n \quad (\text{c}) \quad r < m, r = n \quad (\text{d}) \quad r = m = n.$$

$$\mathbf{26} \quad R = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = I.$$

27 R has n pivots equal to 1. Zeros above and below pivots make $R = I$.

$$\mathbf{28} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

The pivot columns contain I so -1 and 2 go into \mathbf{x}_p .

$$\mathbf{29} \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} : \text{no solution because of row 3.}$$

$$\mathbf{30} \quad \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}; \quad \mathbf{x}_p = \begin{bmatrix} 4 \\ -3 \\ 0 \\ -2 \end{bmatrix} \text{ and } \mathbf{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{31} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}; \quad B \text{ cannot exist since 2 equations in 3 unknowns cannot have a unique solution.}$$

$$32 \quad A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \text{ and then no solution.}$$

$$36 \quad A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Problem Set 3.5, page 150

$$1 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \text{ gives } c_3 = c_2 = c_1 = 0. \text{ But } \mathbf{v}_1 + \mathbf{v}_2 - 4\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \text{ (dependent).}$$

2 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent. All six vectors are on the plane $(1, 1, 1) \cdot \mathbf{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 = $\mathbf{0}$; if $d = 0$ then $b(\text{column 1}) - a(\text{column 2}) = \mathbf{0}$; if $f = 0$ then all columns end in zero (all are perpendicular to $(0, 0, 1)$, all in the xy plane, must be dependent).

$$4 \quad U\mathbf{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } z = 0 \text{ then } y = 0 \text{ then } x = 0.$$

$$5 \quad (a) \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix} : \text{invertible} \Rightarrow \text{independent columns}$$

$$(b) \quad \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}; A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ columns add to } \mathbf{0}.$$

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A .

7 The sum $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ because $(\mathbf{w}_2 - \mathbf{w}_3) - (\mathbf{w}_1 - \mathbf{w}_3) + (\mathbf{w}_1 - \mathbf{w}_2) = \mathbf{0}$.

8 If $c_1(\mathbf{w}_2 + \mathbf{w}_3) + c_2(\mathbf{w}_1 + \mathbf{w}_3) + c_3(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\mathbf{w}_1 + (c_1 + c_3)\mathbf{w}_2 + (c_1 + c_2)\mathbf{w}_3 = \mathbf{0}$. Since the \mathbf{w} 's are independent this requires $c_2 + c_3 = 0, c_1 + c_3 = 0, c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ gives zero.

9 (a) The four vectors are the columns of a 3 by 4 matrix A . There is a nonzero solution to $A\mathbf{x} = \mathbf{0}$ because there is at least one free variable (b) dependent if $[\mathbf{v}_1 \ \mathbf{v}_2]$ has rank 0 or 1 (c) $0\mathbf{v}_1 + 3(0, 0, 0) = \mathbf{0}$.

10 The plane is the nullspace of $A = [1 \ 2 \ -3 \ -1]$. Three free variables give three solutions $(x, y, z, t) = (2, -1, 0, 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$.

11 (a) Line in \mathbf{R}^3 (b) Plane in \mathbf{R}^3 (c) Plane in \mathbf{R}^3 (d) All of \mathbf{R}^3 .

- 12 \mathbf{b} is in the column space when there is a solution to $A\mathbf{x} = \mathbf{b}$; \mathbf{c} is in the row space when there is a solution to $A^T\mathbf{y} = \mathbf{c}$. *False*. The zero vector is always in the row space.
- 13 All dimensions are 2. The row spaces of A and U are the same.
- 14 The dimension of \mathbf{S} is (a) zero when $\mathbf{x} = \mathbf{0}$ (b) one when $\mathbf{x} = (1, 1, 1, 1)$ (c) three when $\mathbf{x} = (1, 1, -1, -1)$ because all rearrangements of this \mathbf{x} are perpendicular to $(1, 1, 1, 1)$ (d) four when the \mathbf{x} 's are not equal and don't add to zero. **No \mathbf{x} gives $\dim \mathbf{S} = 2$.**
- 15 $\mathbf{v} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) + \frac{1}{2}(\mathbf{v} - \mathbf{w})$ and $\mathbf{w} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \frac{1}{2}(\mathbf{v} - \mathbf{w})$. The two pairs *span* the same space. They are a basis when \mathbf{v} and \mathbf{w} are *independent*.
- 16 The n independent vectors span a space of dimension n . They are a *basis* for that space. If they are the columns of A then m is *not less* than n ($m \geq n$).
- 17 These bases are not unique! (a) $(1, 1, 1, 1)$ (b) $(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)$ (c) $(1, -1, -1, 0), (1, -1, 0, -1)$ (d) $(1, 0)(0, 1); (-1, 0, 1, 0, 0), (0, -1, 0, 1, 0), (-1, 0, 0, 0, 1)$.
- 18 Any bases for \mathbf{R}^2 ; (row 1 and row 2) or (row 1 and row 1 + row 2).
- 19 (a) The 6 vectors *might not* span \mathbf{R}^4 (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.
- 20 Independent columns \Rightarrow rank n . Columns span $\mathbf{R}^m \Rightarrow$ rank m . Columns are basis for $\mathbf{R}^m \Rightarrow$ rank $= m = n$.
- 21 One basis is $(2, 1, 0), (-3, 0, 1)$. The vector $(2, 1, 0)$ is a basis for the intersection with the xy plane. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.
- 22 (a) The only solution is $\mathbf{x} = \mathbf{0}$ because *the columns are independent* (b) $A\mathbf{x} = \mathbf{b}$ is solvable because *the columns span \mathbf{R}^5* .
- 23 (a) True (b) False because the basis vectors may not be in \mathbf{S} .
- 24 Columns 1 and 2 are bases for the (different) column spaces; rows 1 and 2 are bases for the (equal) row spaces; $(1, -1, 1)$ is a basis for the (equal) nullspaces.
- 25 (a) False for $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (b) False (c) True: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) False, columns may be dependent.
- 26 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.
- 27 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
- (c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ are a basis for all $A = -A^T$.
- 28 $I, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; echelon matrices do not form a subspace; they *span* the upper triangular matrices (not every U is echelon).

$$29 \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

$$30 \quad - \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = 0$$

31 (a) All 3 by 3 matrices (b) Upper triangular matrices (c) All multiples cI .

$$32 \quad \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

33 (a) $y(x) = \text{constant } C$ (b) $y(x) = 3x$ (c) $y(x) = 3x + C = \mathbf{y}_p + \mathbf{y}_n$.

34 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

35 (a) $y(x) = e^{2x}$ (b) $y = x$ (one basis vector in each case).

36 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 Basis $1, x, x^2, x^3$; basis $x - 1, x^2 - 1, x^3 - 1$.

38 Basis for \mathbf{S} : $(1, 0, -1, 0), (0, 1, 0, 0), (1, 0, 0, -1)$; basis for \mathbf{T} : $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T}$ has dimension 1.

39 See Solution 30 for $I =$ combination of five other P 's. Check the $(1, 1)$ entry, then $(3, 2)$, then $(3, 3)$, then $(1, 2)$ to show that those five P 's are independent.

Four conditions on the 9 entries make all row sums and column sums equal: row sum 1 = row sum 2 = row sum 3 = column sum 1 = column sum 2 (= column sum 3 is automatic).

40 The subspace of matrices that have $AS = SA$ has dimension *three*.

41 (a) No, don't span (b) No, dependent (c) Yes, a basis (d) No, dependent

42 If the 5 by 5 matrix $[A \ \mathbf{b}]$ is invertible, \mathbf{b} is not a combination of the columns of A . If $[A \ \mathbf{b}]$ is singular, and the 4 columns of A are independent, \mathbf{b} is a combination of those columns.

Problem Set 3.6, page 161

1 (a) Row and column space dimensions = 5, nullspace dimension = 4, left nullspace dimension = 2 sum = 16 = $m + n$ (b) Column space is \mathbf{R}^3 ; left nullspace contains only $\mathbf{0}$.

2 A : Row space $(1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space $(1, 2)$; left nullspace $(-2, 1)$. B : Row space $(1, 2, 4)$ and $(2, 5, 8)$; column space $(1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty.

3 Row space $(0, 1, 2, 3, 4)$ and $(0, 0, 0, 1, 2)$; column space $(1, 1, 0)$ and $(3, 4, 1)$; nullspace basis $(1, 0, 0, 0, 0), (0, 2, -1, 0, 0), (0, 2, 0, -2, 1)$; left nullspace $(1, -1, 1)$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$
 (e) Impossible: Row space = column space requires $m = n$. Then $m - r = n - r$.

5 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}.$

- 6 A : Row space $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; column space $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; left nullspace $(0, 1, 0)$. B : Row space (1) , column space $(1, 4, 5)$, nullspace: empty basis, left nullspace $(-4, 1, 0)$ and $(-5, 0, 1)$.
- 7 Invertible A : row space basis = column space basis = $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis and left nullspace basis are empty. Matrix B : row space basis $(1, 0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$ and $(0, 0, 1, 0, 0, 1)$; column space basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis $(-1, 0, 0, 1, 0, 0)$ and $(0, -1, 0, 0, 1, 0)$ and $(0, 0, -1, 0, 0, 1)$; left nullspace basis is empty.
- 8 Row space dimensions 3, 3, 0; column space dimensions 3, 3, 0; nullspace dimensions 2, 3, 2; left nullspace dimensions 0, 2, 3.
- 9 (a) Same row space and nullspace. Therefore rank (dimension of row space) is the same
(b) Same column space and left nullspace. Same rank (dimension of column space).
- 10 Most likely rank = 3, nullspace and left nullspace contain only $(0, 0, 0)$. When the matrix is 3 by 5: Most likely rank = 3 and dimension of nullspace is 2.
- 11 (a) No solution means that $r < m$. Always $r \leq n$. Can't compare m and n
(b) If $m - r > 0$, the left nullspace contains a nonzero vector.
- 12 $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}; r + (n - r) = n = 3$ but $2 + 2$ is 4.
- 13 (a) False (b) True (c) False (choose A and B same size and invertible).
- 14 Row space basis $(1, 2, 3, 4)$, $(0, 1, 2, 3)$, $(0, 0, 1, 2)$; nullspace basis $(0, 1, -2, 1)$; column space basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; left nullspace has empty basis.
- 15 Row space and nullspace stay the same; $(2, 1, 3, 4)$ is in the new column space.
- 16 If $A\mathbf{v} = \mathbf{0}$ and \mathbf{v} is a row of A then $\mathbf{v} \cdot \mathbf{v} = 0$.
- 17 Row space = yz plane; column space = xy plane; nullspace = x axis; left nullspace = z axis.
For $I + A$: Row space = column space = \mathbf{R}^3 , nullspaces contain only zero vector.
- 18 Row $3 - 2$ row $2 +$ row $1 =$ zero row so the vectors $c(1, -2, 1)$ are in the left nullspace. The same vectors happen to be in the nullspace.
- 19 Elimination leads to $0 = b_3 - b_2 - b_1$ so $(-1, -1, 1)$ is in the left nullspace. Elimination leads to $b_3 - 2b_1 = 0$ and $b_4 + b_2 - 4b_1 = 0$, so $(-2, 0, 1, 0)$ and $(-4, 1, 0, 1)$ are in the left nullspace.
- 20 (a) All combinations of $(-1, 2, 0, 0)$ and $(-\frac{1}{4}, 0, -3, 1)$ (b) One (c) $(1, 2, 3)$, $(0, 1, 4)$.
- 21 (a) \mathbf{u} and \mathbf{w} (b) \mathbf{v} and \mathbf{z} (c) rank < 2 if \mathbf{u} and \mathbf{w} are dependent or \mathbf{v} and \mathbf{z} are dependent (d) The rank of $\mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$ is 2.

22 $\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix}.$

- 23 Row space basis $(3, 0, 3)$, $(1, 1, 2)$; column space basis $(1, 4, 2)$, $(2, 5, 7)$; rank is only 2.
- 24 $A^T \mathbf{y} = \mathbf{d}$ puts \mathbf{d} in the *row space* of A ; unique solution if the *left nullspace* (nullspace of A^T) contains only $\mathbf{y} = \mathbf{0}$.
- 25 (a) True (same rank) (b) False $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (c) False (A can be invertible and also unsymmetric) (d) True.
- 26 The rows of $AB = C$ are combinations of the rows of B . So $\text{rank } C \leq \text{rank } B$. Also $\text{rank } C \leq \text{rank } A$. (The columns of C are combinations of the columns of A).
- 27 Choose $d = bc/a$. Then the row space has basis (a, b) and the nullspace has basis $(-b, a)$.
- 28 Both ranks are 2; if $p \neq 0$, rows 1 and 2 are a basis for the row space. $N(B^T)$ has six vectors with 1 and -1 separated by a zero; $N(C^T)$ has $(-1, 0, 0, 0, 0, 0, 1)$ and $(0, -1, 0, 0, 0, 0, 1)$ and columns 3, 4, 5, 6 of I ; $N(C)$ is a challenge.
- 29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$ (not unique).

Problem Set 4.1, page 171

- 1 Both nullspace vectors are orthogonal to the row space vector in \mathbf{R}^3 . Column space is perpendicular to the nullspace of A^T in \mathbf{R}^2 .
- 2 The nullspace is \mathbf{Z} (only zero vector) so $\mathbf{x}_n = \mathbf{0}$. and row space = \mathbf{R}^2 . Plane \perp line in \mathbf{R}^3 .
- 3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $C(A)$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $N(A^T)$ is impossible: not perpendicular (d) This asks for $A^2 = 0$; take $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$
(e) $(1, 1, 1)$ will be in the nullspace and row space; no such matrix.
- 4 If $AB = 0$, the columns of B are in the *nullspace* of A . The rows of A are in the *left nullspace* of B . If $\text{rank} = 2$, all four subspaces would have dimension 2 which is impossible for 3 by 3.
- 5 (a) If $A\mathbf{x} = \mathbf{b}$ has a solution and $A^T \mathbf{y} = \mathbf{0}$, then \mathbf{y} is perpendicular to \mathbf{b} . $(A\mathbf{x})^T \mathbf{y} = \mathbf{b}^T \mathbf{y} = 0$.
(b) \mathbf{c} is in the row space, \mathbf{x} is in the nullspace: $\mathbf{c}^T \mathbf{x} = \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{0} = 0$.
- 6 Multiply the equations by $y_1 = 1, y_2 = 1, y_3 = -1$. They add to $0 = 1$ so no solution: $\mathbf{y} = (1, 1, -1)$ is in the left nullspace. Can't have $0 = (\mathbf{y}^T A)\mathbf{x} = \mathbf{y}^T \mathbf{b} = 1$.
- 7 Multiply by $\mathbf{y} = (1, 1, -1)$, then $x_1 - x_2 = 1$ plus $x_2 - x_3 = 1$ minus $x_1 - x_3 = 1$ is $0 = 1$.
- 8 $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$, where \mathbf{x}_r is in the row space and \mathbf{x}_n is in the nullspace. Then $A\mathbf{x}_n = \mathbf{0}$ and $A\mathbf{x} = A\mathbf{x}_r + A\mathbf{x}_n = A\mathbf{x}_r$. All vectors $A\mathbf{x}$ are combinations of the columns of A .
- 9 $A\mathbf{x}$ is always in the *column space* of A . If $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x}$ is also in the nullspace of A^T . Perpendicular to itself, so $A\mathbf{x} = \mathbf{0}$.
- 10 (a) For a symmetric matrix the column space and row space are the same (b) \mathbf{x} is in the nullspace and \mathbf{z} is in the column space = row space: so these "eigenvectors" have $\mathbf{x}^T \mathbf{z} = 0$.

- 11 The nullspace of A is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$.
- 12 \mathbf{x} splits into $\mathbf{x}_r + \mathbf{x}_n = (1, -1) + (1, 1) = (2, 0)$.
- 13 $V^T W = \text{zero matrix}$ makes each basis vector for V orthogonal to each basis vector for W . Then every \mathbf{v} in V is orthogonal to every \mathbf{w} in W (they are combinations of the basis vectors).
- 14 $A\mathbf{x} = B\hat{\mathbf{x}}$ means that $[A \ B] \begin{bmatrix} \mathbf{x} \\ -\hat{\mathbf{x}} \end{bmatrix} = \mathbf{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $\mathbf{x} = (3, 1)$ and $\hat{\mathbf{x}} = (1, 0)$ and $A\mathbf{x} = B\hat{\mathbf{x}} = (5, 6, 5)$ is in both column spaces. Two planes in \mathbf{R}^3 must intersect in a line at least!
- 15 A p -dimensional and a q -dimensional subspace of \mathbf{R}^n share at least a line if $p + q > n$.
- 16 $A^T \mathbf{y} = \mathbf{0} \Rightarrow (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = 0$. Then $\mathbf{y} \perp A\mathbf{x}$ and $N(A^T) \perp C(A)$.
- 17 If S is the subspace of \mathbf{R}^3 containing only the zero vector, then S^\perp is \mathbf{R}^3 . If S is spanned by $(1, 1, 1)$, then S^\perp is spanned by $(1, -1, 0)$ and $(1, 0, -1)$. If S is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, then S^\perp is spanned by $(0, 1, 0)$.
- 18 S^\perp is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore S^\perp is a *subspace* even if S is not.
- 19 L^\perp is the 2-dimensional subspace (a plane) in \mathbf{R}^3 perpendicular to L . Then $(L^\perp)^\perp$ is a 1-dimensional subspace (a line) perpendicular to L^\perp . In fact $(L^\perp)^\perp$ is L .
- 20 If V is the whole space \mathbf{R}^4 , then V^\perp contains only the *zero vector*. Then $(V^\perp)^\perp = \mathbf{R}^4 = V$.
- 21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $S^\perp = \text{nullspace of } A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.
- 22 $(1, 1, 1, 1)$ is a basis for P^\perp . $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has the plane P as its nullspace.
- 23 \mathbf{x} in V^\perp is perpendicular to any vector in V . Since V contains all the vectors in S , \mathbf{x} is also perpendicular to any vector in S . So every \mathbf{x} in V^\perp is also in S^\perp .
- 24 Column 1 of A^{-1} is orthogonal to the space spanned by the 2nd, 3rd, \dots , n th rows of A .
- 25 If the columns of A are unit vectors, all mutually perpendicular, then $A^T A = I$.
- 26 $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, $A^T A = 9I$ is *diagonal*: $(A^T A)_{ij} = (\text{column } i \text{ of } A) \cdot (\text{column } j \text{ of } A)$.
- 27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are parallel. They are the same line if $b_2 = 2b_1$. In that case (b_1, b_2) is perpendicular to $(-2, 1)$. The nullspace is the line $3x + y = 0$. One particular vector in the nullspace is $(-1, 3)$.
- 28 (a) $(1, -1, 0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect!
 (b) Need *three* orthogonal vectors to span the whole orthogonal complement.
 (c) Lines can meet without being orthogonal.
- 29 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$; \mathbf{v} can *not* be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and $\mathbf{v}^T \mathbf{v} \neq 0$.

- 30** When $AB = 0$, the column space of B is contained in the nullspace of A . So $\text{rank}(B) \leq 4 - \text{rank}(A) = (\text{dimension of the nullspace } A)$.
- 31** $\text{null}(N')$ produces a basis for the *row space* of A (perpendicular to $N(A)$).

Problem Set 4.2, page 181

- 1** (a) $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 5/3$; $\mathbf{p} = (5/3, 5/3, 5/3)$; $\mathbf{e} = (-2/3, 1/3, 1/3)$
 (b) $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = -1$; $\mathbf{p} = (1, 3, 1)$; $\mathbf{e} = (0, 0, 0)$.
- 2** (a) $\mathbf{p} = (\cos \theta, 0)$ (b) $\mathbf{p} = (0, 0)$ since $\mathbf{a}^T \mathbf{b} = 0$.
- 3** $P_1 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1 \mathbf{b} = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ and $P_1^2 = P_1$. $P_2 = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2 \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.
- 4** $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $P_1 P_2 \neq 0$ and $P_1 + P_2$ is not a projection matrix.
- 5** $P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. $P_1 P_2 = \text{zero matrix}$ because $\mathbf{a}_1 \perp \mathbf{a}_2$.
- 6** $\mathbf{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $\mathbf{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $\mathbf{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. Then $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = (1, 0, 0) = \mathbf{b}$.
- 7** $P_1 + P_2 + P_3 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I$.
- 8** $\mathbf{p}_1 = (1, 0)$ and $\mathbf{p}_2 = (0.6, 1.2)$. Then $\mathbf{p}_1 + \mathbf{p}_2 \neq \mathbf{b}$.
- 9** Since A is invertible, $P = A(A^T A)^{-1} A^T = A A^{-1} (A^T)^{-1} A^T = I$: project onto all of \mathbf{R}^2 .
- 10** $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, $P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$, $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_1 P_2 \mathbf{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$. No, $P_1 P_2 \neq (P_1 P_2)^2$.
- 11** (a) $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (2, 3, 0)$ and $\mathbf{e} = (0, 0, 4)$ (b) $\mathbf{p} = (4, 4, 6)$ and $\mathbf{e} = (0, 0, 0)$.
- 12** $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection on xy plane. $P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- 13** $\mathbf{p} = (1, 2, 3, 0)$. $P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- 14** The projection of this \mathbf{b} onto the column space of A is \mathbf{b} itself, but P is not necessarily I .
 $P = \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$ and $\mathbf{p} = (0, 2, 4)$.
- 15** The column space of $2A$ is the same as the column space of A . $\hat{\mathbf{x}}$ for $2A$ is *half* of $\hat{\mathbf{x}}$ for A .

- 16 $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. Therefore \mathbf{b} is in the plane. Projection shows $P\mathbf{b} = \mathbf{b}$.
- 17 $P^2 = P$ and therefore $(I - P)^2 = (I - P)(I - P) = I - PI - IP + P^2 = I - P$. When P projects onto the column space of A then $I - P$ projects onto the *left nullspace* of A .
- 18 (a) $I - P$ is the projection matrix onto $(1, -1)$ in the perpendicular direction to $(1, 1)$
 (b) $I - P$ is the projection matrix onto the plane $x + y + z = 0$ perpendicular to $(1, 1, 1)$.
- 19 For any choice, say $(1, 1, 0)$ and $(2, 0, 1)$, the matrix P is
$$\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}.$$
- 20 $\mathbf{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \mathbf{e}\mathbf{e}^T/\mathbf{e}^T\mathbf{e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $P = I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}.$
- 21 $(A(A^T A)^{-1}A^T)^2 = A(A^T A)^{-1}(A^T A)(A^T A)^{-1}A^T = A(A^T A)^{-1}A^T$. Therefore $P^2 = P$. $P\mathbf{b}$ is always in the column space (where P projects). Therefore its projection $P(P\mathbf{b})$ is $P\mathbf{b}$.
- 22 $P^T = (A(A^T A)^{-1}A^T)^T = A((A^T A)^{-1})^T A^T = A(A^T A)^{-1}A^T = P$. ($A^T A$ is symmetric.)
- 23 If A is invertible then its column space is all of \mathbf{R}^n . So $P = I$ and $\mathbf{e} = \mathbf{0}$.
- 24 The nullspace of A^T is *orthogonal* to the column space $\mathbf{C}(A)$. So if $A^T \mathbf{b} = \mathbf{0}$, the projection of \mathbf{b} onto $\mathbf{C}(A)$ should be $\mathbf{p} = \mathbf{0}$. Check $P\mathbf{b} = A(A^T A)^{-1}A^T \mathbf{b} = A(A^T A)^{-1}\mathbf{0} = \mathbf{0}$.
- 25 The column space of P will be \mathcal{S} (n -dimensional). Then $r = \text{dimension of column space} = n$.
- 26 A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.
- 27 $A\mathbf{x}$ is in the nullspace of A^T . But $A\mathbf{x}$ is always in the column space of A . To be in both of those perpendicular spaces, $A\mathbf{x}$ must be zero. So A and $A^T A$ have the *same nullspace*.
- 28 $P^2 = P = P^T$ give $P^T P = P$. Then the $(2, 2)$ entry of P equals the $(2, 2)$ entry of $P^T P$ which is the length squared of column 2.
- 29 Set $A = B^T$. Then A has independent columns. By 4G, $A^T A = BB^T$ is invertible.
- 30 (a) The column space is the line through $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T \mathbf{a}} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$. We can't use $(A^T A)^{-1}$ because A has dependent columns. (b) The row space is the line through $\mathbf{v} = (1, 2, 2)$ and $P_R = \mathbf{v}\mathbf{v}^T/\mathbf{v}^T \mathbf{v}$. Always $P_C A = A$ and $A P_R = A$ and then $P_C A P_R = A$!

Problem Set 4.3, page 192

$$1 \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \text{ give } A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \text{ and } A^T \mathbf{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b} \text{ gives } \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \mathbf{p} = A \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \text{ and } \mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}. \quad E = \|\mathbf{e}\|^2 = 44.$$

- 2 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Change the right side to $\mathbf{p} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$; $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ exactly solves $A\hat{\mathbf{x}} = \mathbf{b}$.
- 3 $\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (1, 5, 13, 17)$. $\mathbf{e} = (-1, 3, -5, 3)$. \mathbf{e} is indeed perpendicular to both columns of A . The shortest distance $\|\mathbf{e}\|$ is $\sqrt{44}$.
- 4 $E = (C + 0D)^2 + (C + 1D - 8)^2 + (C + 3D - 8)^2 + (C + 4D - 20)^2$. Then $\partial E / \partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E / \partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$. These normal equations are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.
- 5 $E = (C - 0)^2 + (C - 8)^2 + (C - 8)^2 + (C - 20)^2$. $A^T = [1 \ 1 \ 1 \ 1]$, $A^T A = [4]$ and $A^T \mathbf{b} = [36]$ and $(A^T A)^{-1} A^T \mathbf{b} = 9 = \text{best height } C$. Errors $\mathbf{e} = (-9, -1, -1, 11)$.
- 6 $\hat{\mathbf{x}} = \mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a} = 9$ and projection $\mathbf{p} = (9, 9, 9, 9)$; $\mathbf{e}^T \mathbf{a} = (-9, -1, -1, 11)^T (1, 1, 1, 1) = 0$ and $\|\mathbf{e}\| = \sqrt{204}$.
- 7 $A = [0 \ 1 \ 3 \ 4]^T$, $A^T A = [26]$ and $A^T \mathbf{b} = [112]$. Best $D = 112/26 = 56/13$.
- 8 $\hat{\mathbf{x}} = 56/13$, $\mathbf{p} = (56/13)(0, 1, 3, 4)$. $C = 9$, $D = 56/13$ don't match $(C, D) = (1, 4)$; the columns of A were not perpendicular so we can't project separately to find $C = 1$ and $D = 4$.
- 9 Closest parabola: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^T A \hat{\mathbf{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.
- 10 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Then $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$. Exact cubic so $\mathbf{p} = \mathbf{b}$, $\mathbf{e} = \mathbf{0}$.
- 11 (a) The best line is $x = 1 + 4t$, which goes through the center point $(\hat{t}, \hat{\mathbf{b}}) = (2, 9)$
 (b) From the first equation: $C \cdot m + D \cdot \sum_{i=1}^m t_i = \sum_{i=1}^m b_i$. Divide by m to get $C + D\hat{t} = \hat{\mathbf{b}}$.
- 12 (a) $\mathbf{a}^T \mathbf{a} = m$, $\mathbf{a}^T \mathbf{b} = b_1 + \dots + b_m$. Therefore $\hat{\mathbf{x}}$ is the mean of the b 's (b) $\mathbf{e} = \mathbf{b} - \hat{\mathbf{x}} \mathbf{a}$.
- $\|\mathbf{e}\|^2 = \sum_{i=1}^m (b_i - \hat{\mathbf{x}})^2$ (c) $\mathbf{p} = (3, 3, 3)$, $\mathbf{e} = (-2, -1, 3)$, $\mathbf{p}^T \mathbf{e} = 0$. $P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- 13 $(A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x}) = \hat{\mathbf{x}} - \mathbf{x}$. Errors $\mathbf{b} - A\mathbf{x} = (\pm 1, \pm 1, \pm 1)$ add to $\mathbf{0}$, so the $\hat{\mathbf{x}} - \mathbf{x}$ add to $\mathbf{0}$.
- 14 $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T = (A^T A)^{-1} A^T (\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T A (A^T A)^{-1}$. Average $(\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T = \sigma^2 I$ gives the *covariance matrix* $(A^T A)^{-1} A^T \sigma^2 A (A^T A)^{-1}$ which simplifies to $\sigma^2 (A^T A)^{-1}$.
- 15 Problem 14 gives the expected error $(\hat{\mathbf{x}} - \mathbf{x})^2$ as $\sigma^2 (A^T A)^{-1} = \sigma^2 / m$. By taking m measurements, the variance drops from σ^2 to σ^2 / m .
- 16 $\frac{1}{10} b_{10} + \frac{9}{10} \hat{\mathbf{x}}_9 = \frac{1}{10} (b_1 + \dots + b_{10})$.

17 $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 $\mathbf{p} = A\hat{\mathbf{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\mathbf{b} - \mathbf{p} = (2, -6, 4)$.

19 If $\mathbf{b} = \mathbf{e}$ then \mathbf{b} is perpendicular to the column space of A . Projection $\mathbf{p} = \mathbf{0}$.

20 If $\mathbf{b} = A\hat{\mathbf{x}} = (5, 13, 17)$ then error $\mathbf{e} = \mathbf{0}$ since \mathbf{b} is in the column space of A .

21 \mathbf{e} is in $\mathbf{N}(A^T)$; \mathbf{p} is in $\mathbf{C}(A)$; $\hat{\mathbf{x}}$ is in $\mathbf{C}(A^T)$; $\mathbf{N}(A) = \{\mathbf{0}\} =$ zero vector.

22 The least squares equation is $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. Solution: $C = 1$, $D = -1$.

23 The square of the distance between points on two lines is $E = (y - x)^2 + (3y - x)^2 + (1 + x)^2$.

Set $\frac{1}{2}\partial E/\partial x = -(y - x) - (3y - x) + (x + 1) = 0$ and $\frac{1}{2}\partial E/\partial y = (y - x) + 3(3y - x) = 0$.

The solution is $x = -5/7, y = -2/7; E = 2/7$, and the minimal distance is $\sqrt{2/7}$.

24 \mathbf{e} is orthogonal to \mathbf{p} ; $\|\mathbf{e}\|^2 = \mathbf{e}^T(\mathbf{b} - \mathbf{p}) = \mathbf{e}^T\mathbf{b} = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{p}$.

25 The derivatives of $\|A\mathbf{x} - \mathbf{b}\|^2$ are zero when $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$.

26 Direct approach to 3 points on a line: *Equal slopes* $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$.

Linear algebra approach: If \mathbf{y} is orthogonal to the columns $(1, 1, 1)$ and (t_1, t_2, t_3) and \mathbf{b} is in the column space then $\mathbf{y}^T \mathbf{b} = 0$. This $\mathbf{y} = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$ is in the left nullspace. Then $\mathbf{y}^T \mathbf{b} = 0$ is the same equal slopes condition written as $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$.

27 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A^T \mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$. At

$x, y = 0, 0$ the best plane $2 - x - \frac{3}{2}y$ has height $C = 2$ which is the average of 0, 1, 3, 4.

Problem Set 4.4, page 203

1 (a) *Independent* (b) *Independent and orthogonal* (c) *Independent and orthonormal*.

For orthonormal, (a) becomes $(1, 0), (0, 1)$ and (b) is $(.6, .8), (.8, -.6)$.

2 $\mathbf{q}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. $\mathbf{q}_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. $Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $Q Q^T = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$.

3 (a) $A^T A = 16I$ (b) $A^T A$ is diagonal with entries 1, 4, 9.

4 (a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Q Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(1, 0)$ and $(0, 0)$ are *orthogonal*, not *independent*

(c) $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

5 *Orthogonal* vectors are $(1, -1, 0)$ and $(1, 1, -1)$. *Orthonormal* are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

- 6 If Q_1 and Q_2 are orthogonal matrices then $(Q_1 Q_2)^T Q_1 Q_2 = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$ which means that $Q_1 Q_2$ is orthogonal also.
- 7 The least squares solution to $Q^T Q \hat{\mathbf{x}} = Q^T \mathbf{b}$ is $\hat{\mathbf{x}} = Q^T \mathbf{b}$. This is $\mathbf{0}$ if $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- 8 If \mathbf{q}_1 and \mathbf{q}_2 are *orthonormal* vectors in \mathbf{R}^5 then $(\mathbf{q}_1^T \mathbf{b})\mathbf{q}_1 + (\mathbf{q}_2^T \mathbf{b})\mathbf{q}_2$ is closest to \mathbf{b} .
- 9 (a) $P = QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(QQ^T)(QQ^T) = Q(Q^T Q)Q^T = QQ^T$.
- 10 (a) If $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ are *orthonormal* then the dot product of \mathbf{q}_1 with $c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2 + c_3 \mathbf{q}_3 = \mathbf{0}$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$ independent (b) $Q\mathbf{x} = \mathbf{0} \Rightarrow Q^T Q\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$.
- 11 (a) Two *orthonormal* vectors are $\frac{1}{10}(1, 3, 4, 5, 7)$ and $\frac{1}{10}(7, -3, -4, 5, -1)$ (b) The closest vector in the plane is the *projection* $QQ^T(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$.
- 12 (a) $\mathbf{a}_1^T \mathbf{b} = \mathbf{a}_1^T (x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3) = x_1(\mathbf{a}_1^T \mathbf{a}_1) = x_1$
 (b) $\mathbf{a}_1^T \mathbf{b} = \mathbf{a}_1^T (x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3) = x_1(\mathbf{a}_1^T \mathbf{a}_1)$. Therefore $x_1 = \mathbf{a}_1^T \mathbf{b} / \mathbf{a}_1^T \mathbf{a}_1$
 (c) x_1 is the first component of A^{-1} times \mathbf{b} .
- 13 The multiple to subtract is $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}$. Then $\mathbf{B} = \mathbf{b} - \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = (4, 0) - 2 \cdot (1, 1) = (2, -2)$.
- 14 $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2] \begin{bmatrix} \|\mathbf{a}\| & \mathbf{q}_1^T \mathbf{b} \\ 0 & \|\mathbf{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR$.
- 15 (a) $\mathbf{q}_1 = \frac{1}{3}(1, 2, -2)$, $\mathbf{q}_2 = \frac{1}{3}(2, 1, 2)$, $\mathbf{q}_3 = \frac{1}{3}(2, -2, -1)$ (b) The nullspace of A^T contains \mathbf{q}_3 (c) $\hat{\mathbf{x}} = (A^T A)^{-1} A^T (1, 2, 7) = (1, 2)$.
- 16 The projection $\mathbf{p} = (\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a} = 14\mathbf{a} / 49 = 2\mathbf{a} / 7$ is closest to \mathbf{b} ; $\mathbf{q}_1 = \mathbf{a} / \|\mathbf{a}\| = \mathbf{a} / 7$ is $(4, 5, 2, 2) / 7$. $\mathbf{B} = \mathbf{b} - \mathbf{p} = (-1, 4, -4, -4) / 7$ has $\|\mathbf{B}\| = 1$ so $\mathbf{q}_2 = \mathbf{B}$.
- 17 $\mathbf{p} = (\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}) \mathbf{a} = (3, 3, 3)$ and $\mathbf{e} = (-2, 0, 2)$. $\mathbf{q}_1 = (1, 1, 1) / \sqrt{3}$ and $\mathbf{q}_2 = (-1, 0, 1) / \sqrt{2}$.
- 18 $\mathbf{A} = \mathbf{a} = (1, -1, 0, 0)$; $\mathbf{B} = \mathbf{b} - \mathbf{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\mathbf{C} = \mathbf{c} - \mathbf{p}_A - \mathbf{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.
- 19 If $A = QR$ then $A^T A = R^T R =$ lower times upper triangular. Pivots of $A^T A$ are 3 and 8.
- 20 (a) True (b) True. $Q\mathbf{x} = x_1 \mathbf{q}_1 + x_2 \mathbf{q}_2$. $\|Q\mathbf{x}\|^2 = x_1^2 + x_2^2$ because $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$.
- 21 The orthonormal vectors are $\mathbf{q}_1 = (1, 1, 1, 1) / 2$ and $\mathbf{q}_2 = (-5, -1, 1, 5) / \sqrt{52}$. Then $\mathbf{b} = (-4, -3, 3, 0)$ projects to $\mathbf{p} = (-7, -3, -1, 3) / 2$. Check that $\mathbf{b} - \mathbf{p} = (-1, -3, 7, -3) / 2$ is orthogonal to both \mathbf{q}_1 and \mathbf{q}_2 .
- 22 $\mathbf{A} = (1, 1, 2)$, $\mathbf{B} = (1, -1, 0)$, $\mathbf{C} = (-1, -1, 1)$. Not yet orthonormal.
- 23 $\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.
- 24 (a) One basis for this subspace is $\mathbf{v}_1 = (1, -1, 0, 0)$, $\mathbf{v}_2 = (1, 0, -1, 0)$, $\mathbf{v}_3 = (1, 0, 0, 1)$
 (b) $(1, 1, 1, -1)$ (c) $\mathbf{b}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ and $\mathbf{b}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$.

25 $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$. Singular $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$.
The Gram-Schmidt process breaks down when A is singular and $ad - bc = 0$.

26 $(q_2^T C^*)q_2 = \frac{B^T c}{B^T B} B$ because $q_2 = \frac{B}{\|B\|}$ and the extra q_1 in C^* is orthogonal to q_2 .

27 When a and b are not orthogonal, the projections onto these lines do not add to the projection onto their plane.

28 $q_1 = \frac{1}{3}(2, 2, -1)$, $q_2 = \frac{1}{3}(2, -1, 2)$, $q_3 = \frac{1}{3}(1, -2, -2)$.

29 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The columns of the wavelet matrix W are orthonormal. Then $W^{-1} = W^T$. See Section 7.3 for more about wavelets.

31 (a) $c = \frac{1}{2}$ (b) Change all signs in rows 2, 3, 4; then in columns 2, 3, 4.

32 $p_1 = \frac{1}{2}(-1, 1, 1, 1)$ and $p_2 = (0, 0, 1, 1)$.

33 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

34 (a) $Qu = (I - 2uu^T)u = u - 2uu^Tu$. This is $-u$, provided that u^Tu equals 1

(b) $Qv = (I - 2uu^T)v = v - 2uu^Tv = v$, provided that $u^Tv = 0$.

35 No solution

36 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal, 0 elsewhere.

Problem Set 5.1, page 213

- 1 $\det(2A) = 8$ and $\det(-A) = (-1)^4 \det A = \frac{1}{2}$ and $\det(A^2) = \frac{1}{4}$ and $\det(A^{-1}) = 2$.
- 2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$.
- 3 (a) False: 2 by 2 I (b) True (c) False: 2 by 2 I (d) False (but trace = 0).
- 4 Exchange rows 1 and 3. Exchange rows 1 and 4, then 2 and 3.
- 5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. The determinants are 1, 1, -1, -1 repeating, so $|J_{101}| = 1$.
- 6 Multiply the zero row by t . The determinant is multiplied by t but the matrix is the same $\Rightarrow \det = 0$.
- 7 $\det(Q) = 1$ for rotation, $\det(Q) = -1$ for reflection ($1 - 2\sin^2\theta - 2\cos^2\theta = -1$).
- 8 $Q^T Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so can't blow up. Same for Q^{-1} .
- 9 $\det A = 1$, $\det B = 2$, $\det C = 0$.
- 10 If the entries in every row add to zero, then $(1, 1, \dots, 1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).
- 11 $CD = -DC \Rightarrow |CD| = (-1)^n |DC|$ and *not* $-|DC|$. If n is even we can have $|CD| \neq 0$.

$$12 \quad \det(A^{-1}) = \det \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}.$$

13 Pivots 1, 1, 1 give $\det = 1$; pivots 1, -2, -3/2 give $\det = 3$.

14 $\det(A) = 24$ and $\det(A) = 5$.

15 $\det = 0$ and $\det = 1 - 2t^2 + t^4 = (1 - t^2)^2$.

16 A singular rank one matrix has $\det = 0$; Also $\det K = 0$.

17 Any 3 by 3 skew-symmetric K has $\det(K^T) = \det(-K) = (-1)^3 \det(K)$. This is $-\det(K)$.

But also $\det(K^T) = \det(K)$, so we must have $\det(K) = 0$.

$$18 \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$$

19 $\det(U) = 6$, $\det(U^{-1}) = \frac{1}{6}$, $\det(U^2) = 36$, $\det(U) = ad$, $\det(U^2) = a^2 d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

$$20 \quad \det \begin{bmatrix} a-Lc & b-Ld \\ c-la & d-lb \end{bmatrix} = (ad-bc)(1-Ll).$$

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

22 $\det(A) = 3$, $\det(A^{-1}) = \frac{1}{3}$, $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. Then $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. *Note to instructor:* If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify 1 and 3 as the eigenvalues.

$$23 \quad \det(A) = 10, \quad A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}, \quad \det(A^2) = 100, \quad A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}, \quad \det(A^{-1}) = \frac{1}{10}.$$

$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = 2$ or $\lambda = 5$.

24 $\det(L) = 1$, $\det(U) = -6$, $\det(A) = -6$, $\det(U^{-1}L^{-1}) = -\frac{1}{6}$, and $\det(U^{-1}L^{-1}A) = 1$.

25 Row 2 = 2 times row 1 so $\det A = 0$.

26 Row 3 - row 2 = row 2 - row 1 so A is singular.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b-a)(c-b)$.

28 (a) *True*: $\det(AB) = \det(A)\det(B) = 0$ (b) *False*: may exchange rows
(c) *False*: $A = 2I$ and $B = I$ (d) *True*: $\det(AB) = \det(A)\det(B) = \det(BA)$.

29 A is rectangular so $\det(A^T A) \neq (\det A^T)(\det A)$: these are not defined.

$$30 \quad \begin{bmatrix} \partial f / \partial a & \partial f / \partial c \\ \partial f / \partial b & \partial f / \partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

31 The Hilbert determinants are 1, .08, 4.6×10^{-4} , 1.6×10^{-7} , 3.7×10^{-12} , 5.4×10^{-18} , 4.8×10^{-25} , 2.7×10^{-33} , 9.7×10^{-43} , 2.2×10^{-53} . Pivots are ratios of determinants so 10th pivot is near 10^{-10} .

32 Typical determinants of $\text{rand}(n)$ are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$). Using $\text{randn}(n)$ with normal bell-shaped probabilities these are $10^{31}, 10^{78}, 10^{186}$, $\text{Inf} \geq 2^{1024}$. MATLAB computes $1.9999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

33 $n=5$; $p=(n-1)^2$; $A0=\text{ones}(n)$; $\text{maxdet}=0$; for $k=0:2^p-1$
 $A_{\text{sub}}=\text{rem}(\text{floor}(k.*2.^{-(p+1:0)}),2)$; $A=A0$; $A(2:n,2:n)=1-2*\text{reshape}(A_{\text{sub}},n-1,n-1)$;
 if $\text{abs}(\det(A))>\text{maxdet}$, $\text{maxdet}=\text{abs}(\det(A))$; $\text{maxA}=A$; end end

Output: $\text{maxA} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix}$ $\text{maxdet} = 48$.

34 Reduce B to [row 3 : row 2; row 1]. Then $\det B = -6$.

Problem Set 5.2, page 225

- 1** $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$, rows are independent; $\det B = 0$, rows are dependent; $\det C = -1$, independent rows.
- 2** $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = (-2)(0)$, dependent.
- 3** Each of the 6 terms in $\det A$ is zero; the rank is at most 2; column 2 has no pivot.
- 4** (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one choice will be zero.
- 5** $a_{11}a_{23}a_{32}a_{44}$ gives -1 , $a_{14}a_{23}a_{32}a_{41}$ gives $+1$ so $\det A = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 48$.
- 6** Four zeros in a row guarantee $\det = 0$; $A = I$ has 12 zeros.
- 7** (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms are certainly zero.
- 8** $5!/2 = 60$ permutation matrices have $\det = +1$. Put row 5 of I at the top (4 exchanges).
- 9** Some term $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ is not zero! Move rows 1, 2, \dots , n into rows $\alpha, \beta, \dots, \omega$. Then these nonzero a 's will be on the main diagonal.
- 10** To get $+1$ for the even permutations the matrix needs an *even* number of -1 's. For the odd P 's the matrix needs an *odd* number of -1 's. So six 1 's and $\det = 6$ are impossible: $\max(\det) = 4$.
- 11** $\det(I + P_{\text{even}}) = 16$ or 4 or 0 (16 comes from $I + I$).
- 12** $C = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix}$. $C = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$.
- 13** $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^T$.
- 14** $|B_4| = 2 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ & -1 & -1 \end{bmatrix} = 2|B_3| - \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 2|B_3| - |B_2|$.
- 15** (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = -1$.

- 16** Must choose 1's from column 2 then column 1, column 4 then column 3, and so on. Therefore n must be even to have $\det A_n \neq 0$. The number of row exchanges is $\frac{1}{2}n$ so $C_n = (-1)^{n/2}$.
- 17** The 1, 1 cofactor is E_{n-1} . The 1, 2 cofactor has a single 1 in its first column, with cofactor E_{n-2} . Signs give $E_n = E_{n-1} - E_{n-2}$. Then 1, 0, -1, -1, 0, 1 repeats by sixes; $E_{100} = -1$.
- 18** The 1, 1 cofactor is F_{n-1} . The 1, 2 cofactor has a 1 in column 1, with cofactor F_{n-2} . Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so Fibonacci).
- 19** $|B_n| = |A_n| - |A_{n-1}| = (n+1) - n = 1$.
- 20** Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c , so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3 . Any Vandermonde matrix $V_{ij} = (c_i)^{j-1}$ has $\det V = \text{product of all } (c_l - c_k) \text{ for } l > k$.
- 21** $G_2 = -1, G_3 = 2, G_4 = -3$, and $G_n = (-1)^{n-1}(n-1)$ = (product of the n eigenvalues!)
- 22** $S_1 = 3, S_2 = 8, S_3 = 21$. The rule looks like every second number in Fibonacci's sequence ... 3, 5, 8, 13, 21, 34, 55, ... so the guess is $S_4 = 55$. Following the solution to Problem 32 with 3's instead of 2's confirms $S_4 = 81 + 1 - 9 - 9 - 9 = 55$.
- 23** The problem asks us to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using the Fibonacci rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = F_{2n} + (F_{2n} - F_{2n-2}) + F_{2n} = 3F_{2n} - F_{2n-2}.$$

- 24** Changing 3 to 2 in the corner reduces the determinant F_{2n+2} by 1 times the cofactor of that corner entry. This cofactor is the determinant of S_{n-1} (one size smaller) which is F_{2n} . Therefore changing 3 to 2 changes the determinant to $F_{2n+2} - F_{2n}$ which is F_{2n+1} .
- 25** (a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves a pair of entries from A times a pair from D leading to $(\det A)(\det D)$
- (b) and (c) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
- 26** (a) All L 's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, -\frac{1}{3}$.
- 27** Problem 25 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A| \text{ times } |D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.
- 28** If A is a row and B is a column then $\det M = \det AB = \text{dot product of } A \text{ and } B$. If A is a column and B is a row then AB has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$).
- 29** (a) $\det A = a_{11}A_{11} + \dots + a_{1n}A_{1n}$. The derivative with respect to a_{11} is the cofactor A_{11} .
- 30** Row 1 - 2 row 2 + row 3 = 0 so the matrix is singular.
- 31** There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+(1, 1)(2, 2)(3, 3)(4, 4) + (1, 2)(2, 1)(3, 4)(4, 3) - (1, 2)(2, 1)(3, 3)(4, 4) - (1, 1)(2, 2)(3, 4)(4, 3) - (1, 1)(2, 3)(3, 2)(4, 4)$. Total $1 + 1 - 1 - 1 - 1 = -1$.
- 32** The 5 products in solution 31 change to $16 + 1 - 4 - 4 - 4$ since A has 2's and -1's:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2).$$

- 33** $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take 4, 1, 2, 3 into 1, 2, 3, 4; $\det(P^2) = (\det P)(\det P) = +1$ so $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is *not right*.
- 34** With $a_{11} = 1$, the $-1, 2, -1$ matrix has $\det = 1$ and inverse $(A^{-1})_{ij} = n + 1 - \max(i, j)$.
- 35** With $a_{11} = 2$, the $-1, 2, -1$ matrix has $\det = n + 1$ and $(n + 1)(A^{-1})_{ij} = i(n - j + 1)$ for $i \leq j$ and symmetrically $(n + 1)(A^{-1})_{ij} = j(n - i + 1)$ for $i \geq j$.
- 36** Subtracting 1 from the n, n entry subtracts its cofactor C_{nn} from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

Problem Set 5.3, page 240

- 1** (a) $\det A = 3$, $\det B_1 = -6$, $\det B_2 = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$
 (b) $|A| = 4$, $|B_1| = 3$, $|B_2| = -2$, $|B_3| = 1$. Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.
- 2** (a) $y = -c/(ad - bc)$ (b) $y = (fg - id)/D$.
- 3** (a) $x_1 = 3/0$ and $x_2 = -2/0$: no solution (b) $x_1 = 0/0$ and $x_2 = 0/0$: *undetermined*.
- 4** (a) $x_1 = \det[\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3]/\det A$, if $\det A \neq 0$ (b) The determinant is linear in column 1 so we get $x_1|\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3| + x_2|\mathbf{a}_2 \ \mathbf{a}_2 \ \mathbf{a}_3| + x_3|\mathbf{a}_3 \ \mathbf{a}_2 \ \mathbf{a}_3|$. The last two determinants are zero.
- 5** If the first column in A is also the right side \mathbf{b} then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.
- 6** (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{4}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. The inverse of a symmetric matrix is symmetric.
- 7** If all cofactors = 0 then A^{-1} would be the zero matrix if it existed; cannot exist. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.
- 8** $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Therefore $\det A = 3$. Cofactor of 100 is 0.
- 9** If we know the cofactors and $\det A = 1$ then $C^T = A^{-1}$ and $\det A^{-1} = 1$. The inverse of A^{-1} is A , so A is the cofactor matrix for C .
- 10** Take the determinant of $AC^T = (\det A)I$. The left side gives $\det AC^T = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.
- 11** We find $\det A = (\det C)^{\frac{1}{n-1}}$ with $n = 4$. Then $\det A^{-1}$ is $1/\det A$. Construct A^{-1} using the cofactors. Invert to find A .
- 12** The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1} .

- 13 Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A = 1$ or -1 .

14 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \frac{1}{5}C^T$.

- 15 (a) $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}, C_{31} = C_{13}, C_{32} = C_{23}$ make S^{-1} symmetric.

- 16 For $n = 5$ the matrix C contains 25 cofactors and each 4 by 4 cofactor contains 24 terms and each term needs 3 multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.

- 17 (a) Area $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$ (b) 5 (c) 5.

18 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces = length of cross product $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ is $6\sqrt{2}$.

19 (a) Area $\frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) 5 + new triangle area $\frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

20 $\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ because the transpose has the same determinant. See #23.

- 21 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$.)

- 22 The maximum volume is $L_1 L_2 L_3 L_4$ reached when the four edges are orthogonal in \mathbf{R}^4 . With entries 1 and -1 all lengths are $\sqrt{1+1+1+1} = 2$. The maximum determinant is $2^4 = 16$, achieved by Hadamard above. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

- 23 TO SEND IN EMAIL

24 $A^T A = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{bmatrix} [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \begin{bmatrix} \mathbf{a}^T \mathbf{a} & 0 & 0 \\ 0 & \mathbf{b}^T \mathbf{b} & 0 \\ 0 & 0 & \mathbf{c}^T \mathbf{c} \end{bmatrix}$ has $\det A^T A = (\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|)^2$
 $\det A = \pm \|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|$

25 The box has height 4. The volume is $4 = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$; $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $(\mathbf{k} \cdot \mathbf{w}) = 4$.

- 26 The n -dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n(n-1)$ -dimensional faces. Those are coefficients of $(2+x)^n$ in Worked Example 2.4A. The cube whose edges are the rows of $2I$ has volume 2^n .

- 27 The pyramid has volume $1/6$. The 4-dimensional pyramid has volume $1/24 = 1/4!$.

- 28 $J = r$. The columns are orthogonal and their lengths are 1 and r .

29 $J = \begin{vmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & 0 \end{vmatrix} = \rho^2 \sin \varphi$, needed for triple integrals inside spheres.

30 $\begin{vmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ (-\sin \theta)/r & (\cos \theta)/r \end{vmatrix} = \frac{1}{r}$.

- 31 The triangle with corners $(0,0)$, $(6,0)$, $(1,4)$ has area 24. Rotated by $\theta = 60^\circ$ the area is *unchanged*. The determinant of the rotation matrix is $J = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = 1$.

- 32 Base area 10, height 2, volume 20.

33 $V = \det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20.$

34 $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = u_1 \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - u_2 \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + u_3 \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$

35 $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$: *Cyclic = even* permutation of $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

36 $S = (2, 1, -1)$. The area is $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners could be $(0, 0, 0)$, $(0, 0, 2)$, $(1, 2, 2)$, $(1, 1, 0)$. The volume of the tilted box is $|det| = 1$.

37 If $(1, 1, 0)$, $(1, 2, 1)$, (x, y, z) are in a plane the volume is $\det \begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0.$

38 $\det \begin{bmatrix} x & y & z \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 0 = 7x - 5y + z$; plane contains the two vectors.

39 Doubling each row multiplies the volume by 2^n . Then $2 \det A = \det(2A)$ only if $n = 1$.

Problem Set 6.1, page 253

- 1 A and A^2 and A^∞ all have the same eigenvectors. The eigenvalues are 1 and 0.5 for A , 1 and 0.25 for A^2 , 1 and 0 for A^∞ . Therefore A^2 is halfway between A and A^∞ .

Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (it is still a Markov matrix with eigenvalue 1, and the trace is now $0.2 + 0.3$ —so the other eigenvalue is -0.5).

Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

- 2 $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $\mathbf{x}_1 = (-2, 1)$ and $\mathbf{x}_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to 0 and 6.
- 3 A has $\lambda_1 = 4$ and $\lambda_2 = -1$ (check trace and determinant) with $\mathbf{x}_1 = (1, 2)$ and $\mathbf{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors as A , with eigenvalues $1/\lambda_1 = 1/4$ and $1/\lambda_2 = -1$.
- 4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace and determinant) with $\mathbf{x}_1 = (3, -2)$ and $\mathbf{x}_2 = (1, 1)$. A^2 has the same eigenvectors as A , with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.
- 5 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. $A + B$ has $\lambda_1 = 1$, $\lambda_2 = 3$. Eigenvalues of $A + B$ are *not equal* to eigenvalues of A plus eigenvalues of B .
- 6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = \frac{1}{2}(3 \pm \sqrt{5})$. Eigenvalues of AB are *not equal* to eigenvalues of A times eigenvalues of B . Eigenvalues of AB and BA **are equal**.
- 7 The eigenvalues of U are the *pivots*. The eigenvalues of L are all 1's. The eigenvalues of A are not the same as the pivots.
- 8 (a) Multiply $A\mathbf{x}$ to see $\lambda\mathbf{x}$ which reveals λ (b) Solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find \mathbf{x} .

- 9 (a) Multiply by A : $A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$ gives $A^2\mathbf{x} = \lambda^2\mathbf{x}$ (b) Multiply by A^{-1} : $A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x} = \lambda A^{-1}\mathbf{x}$ gives $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ (c) Add $I\mathbf{x} = \mathbf{x}$: $(A + I)\mathbf{x} = (\lambda + 1)\mathbf{x}$.
- 10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\mathbf{x}_1 = (1, 2)$ and $\mathbf{x}_2 = (1, -1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞ .
- 11 $M = (A - \lambda_2 I)(A - \lambda_1 I) =$ zero matrix so the columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$. This "Cayley-Hamilton Theorem" $M = 0$ in Problem 6.2.35 has a short proof: by Problem 9 $= M$ has eigenvalues $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$. Same $\mathbf{x}_1, \mathbf{x}_2$.
- 12 P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0), (2, -1, 0), (0, 0, 1)$. Add the first and last vectors: $(1, 2, 1)$ also has $\lambda = 1$. $P^{100} = P$ so P^{100} gives the same answers.
- 13 (a) $P\mathbf{u} = (\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(\mathbf{u}^T\mathbf{u}) = \mathbf{u}$ so $\lambda = 1$ (b) $P\mathbf{v} = (\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{u}(\mathbf{u}^T\mathbf{v}) = \mathbf{0}$ so $\lambda = 0$
(c) $\mathbf{x}_1 = (-1, 1, 0, 0), \mathbf{x}_2 = (-3, 0, 1, 0), \mathbf{x}_3 = (-5, 0, 0, 1)$ are eigenvectors with $\lambda = 0$.
- 14 The eigenvectors are $\mathbf{x}_1 = (1, i)$ and $\mathbf{x}_2 = (1, -i)$.
- 15 $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.
- 16 Set $\lambda = 0$ to find $\det A = (\lambda_1)(\lambda_2) \cdots (\lambda_n)$.
- 17 If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$. Always $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a - d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{\quad})$. Their sum is $a + d$.
- 18 $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.
- 19 (a) $\text{rank} = 2$ (b) $\det(B^T B) = 0$ (d) eigenvalues of $(B + I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{3}$.
- 20 $A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$ has trace 11 and determinant 28.
- 21 $a = 0, b = 9, c = 0$ multiply $1, \lambda, \lambda^2$ in $\det(A - \lambda I) = 9\lambda - \lambda^3$: $A =$ companion matrix.
- 22 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^T$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$: different eigenvectors.
- 23 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).
- 24 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always $A^2 =$ zero matrix if $\lambda = 0, 0$ (Cayley-Hamilton 6.2.35).
- 25 $\lambda = 0, 0, 6$ with $\mathbf{x}_1 = (0, -2, 1), \mathbf{x}_2 = (1, -2, 0), \mathbf{x}_3 = (1, 2, 1)$.
- 26 $A\mathbf{x} = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n$ equals $B\mathbf{x} = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n$ for all \mathbf{x} . So $A = B$.
- 27 $\lambda = 1, 2, 5, 7$.
- 28 $\text{rank}(A) = 1$ with $\lambda = 0, 0, 0, 4$; $\text{rank}(C) = 2$ with $\lambda = 0, 0, 2, 2$.
- 29 B has $\lambda = -1, -1, -1, 3$ so $\det B = -3$. The 5 by 5 matrix A has $\lambda = 0, 0, 0, 0, 5$ and $B = A - I$ has $\lambda = -1, -1, -1, -1, 4$.
- 30 $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; $\lambda(C) = 0, 0, 6$.
- 31 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix} = (a + b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = d - b$ to produce trace = $a + d$.

- 32** Eigenvector $(1, 3, 4)$ for A with $\lambda = 11$ and eigenvector $(3, 1, 4)$ for PAP .
- 33** (a) \mathbf{u} is a basis for the nullspace, \mathbf{v} and \mathbf{w} give a basis for the column space
 (b) $\mathbf{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\mathbf{u}$ from the nullspace
 (c) If $A\mathbf{x} = \mathbf{u}$ had a solution, \mathbf{u} would be in the column space, giving dimension 3.
- 34** With $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$, the determinant is $\lambda_1\lambda_2 = 1$ and the trace is $\lambda_1 + \lambda_2 = -1$:
- $$e^{2\pi i/3} + e^{-2\pi i/3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} + \cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} = -1. \text{ Also } \lambda_1^3 = \lambda_2^3 = 1.$$
- $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ has this trace -1 and determinant 1 . Then $A^3 = I$ and every $(M^{-1}AM)^3 = I$.
 Choosing $\lambda_1 = \lambda_2 = 1$ leads to I or else to a matrix like $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ that has $A^3 \neq I$.
- 35** $\det(P - \lambda I) = 0$ gives the equation $\lambda^3 = 1$. This reflects the fact that $P^3 = I$. The solutions of $\lambda^3 = 1$ are $\lambda = 1$ (real) and $\lambda = e^{2\pi i/3}, \lambda = e^{-2\pi i/3}$ (complex conjugates). The real eigenvector $\mathbf{x}_1 = (1, 1, 1)$ is not changed by the permutation P . The complex eigenvectors are $\mathbf{x}_2 = (1, e^{-2\pi i/3}, e^{-4\pi i/3})$ and $\mathbf{x}_3 = (1, e^{2\pi i/3}, e^{4\pi i/3}) = \overline{\mathbf{x}_2}$.

Problem Set 6.2, page 266

- 1** $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$
- 2** If $A = SAS^{-1}$ then $A^3 = S\Lambda^3S^{-1}$ and $A^{-1} = S\Lambda^{-1}S^{-1}$.
- 3** $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$
- 4** If $A = SAS^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S . $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.
- 5** (a) False: don't know λ 's (b) True (c) True (d) False: need eigenvectors of S !
- 6** A is a diagonal matrix. If S is triangular, then S^{-1} is triangular, so $S\Lambda S^{-1}$ is also triangular.
- 7** The columns of S are nonzero multiples of $(2, 1)$ and $(0, 1)$ in either order. Same for A^{-1} .
- 8** $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for any a and b .
- 9** $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}; F_{20} = 6765.$
- 10** (a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1, \lambda_2 = -\frac{1}{2}$ with $\mathbf{x}_1 = (1, 1), \mathbf{x}_2 = (1, -2)$
- (b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$
- (c) $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = A^k \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$

$$11 \quad A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

$$S\Lambda^k S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} - \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}.$$

12 The equation for the λ 's is $\lambda^2 - \lambda - 1 = 0$ or $\lambda^2 = \lambda + 1$. Multiply by λ^k to get $\lambda^{k+2} = \lambda^{k+1} + \lambda^k$.

13 Direct computation gives L_0, \dots, L_{10} as 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123. My calculator gives $\lambda_1^{10} = (1.618\dots)^{10} = 122.991\dots$

14 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

15 (a) True (b) False (c) False (might have 2 or 3 independent eigenvectors).

16 (a) False: don't know λ (b) True: missing an eigenvector (c) True.

$$17 \quad A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix} \text{ (or other), } A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}, A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}; \text{ only eigenvectors are } (c, -c).$$

18 The rank of $A - 3I$ is one. Changing any entry except $a_{12} = 1$ makes A diagonalizable.

19 $S\Lambda^k S^{-1}$ approaches zero if and only if every $|\lambda| < 1$; $B^k \rightarrow 0$.

$$20 \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad \Lambda^k \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } S\Lambda^k S^{-1} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}; \text{ steady state.}$$

$$21 \quad \Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}, S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}; B^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, B^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}, B^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} =$$

sum of those two.

$$22 \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } A^k = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$23 \quad B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}.$$

24 $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This works when A is *diagonalizable*.

25 $\text{trace } AB = (aq + bs) + (cr + dt) = (qa + rc) + (sb + td) = \text{trace } BA$. Proof for diagonalizable case: the trace of $S\Lambda S^{-1}$ is the trace of $(\Lambda S^{-1})S = \Lambda$ which is *the sum of the λ 's*.

$$26 \quad AB - BA = I: \text{ impossible since } \text{trace } AB - \text{trace } BA = \text{zero} \pm \text{trace } I. \quad E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

$$27 \quad \text{If } A = S\Lambda S^{-1} \text{ then } B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}.$$

28 The A 's form a subspace since cA and $A_1 + A_2$ have the same S . When $S = I$ the A 's give the subspace of diagonal matrices. Dimension 4.

29 If A has columns $\mathbf{x}_1, \dots, \mathbf{x}_n$ then $A^2 = A$ means every $A\mathbf{x}_i = \mathbf{x}_i$. All vectors in the column space are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$. Dimensions of those spaces add to n by the Fundamental Theorem so A is diagonalizable (n independent eigenvectors).

30 Two problems: The nullspace and column space can overlap, so \mathbf{x} could be in both. There may not be r independent eigenvectors in the column space.

- 31** $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. \sqrt{B} would have $\lambda = \sqrt{9}$ and $\lambda = \sqrt{-1}$ so its trace is not real. Note $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, and real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- 32** $A^T = A$ gives $\mathbf{x}^T AB\mathbf{x} = (A\mathbf{x})^T(B\mathbf{x}) \leq \|A\mathbf{x}\|\|B\mathbf{x}\|$ by the Schwarz inequality. $B^T = -B$ gives $-\mathbf{x}^T BA\mathbf{x} = (B\mathbf{x})^T A\mathbf{x} \leq \|A\mathbf{x}\|\|B\mathbf{x}\|$. Add these to get Heisenberg when $AB - BA = I$.
- 33** The factorizations of A and B into $S\Lambda S^{-1}$ are the same. So $A = B$.
- 34** $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Diagonal matrices always give $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$. Then $AB = BA$ from $S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1 \Lambda_2 S^{-1} = S\Lambda_2 \Lambda_1 S^{-1} = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = BA$.
- 35** If $A = S\Lambda S^{-1}$ then the product $(A - \lambda_1 I) \cdots (A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I)S^{-1}$. The factor $\Lambda - \lambda_j I$ is zero in row j . *The product is zero in all rows = zero matrix.*
- 36** $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = \text{zero matrix}$ confirms Cayley-Hamilton.
- 37** $(A - aI)(A - dI) = \begin{bmatrix} \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{d} - \mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{a} - \mathbf{d} & \mathbf{b} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$
- 38** (a) The eigenvectors for $\lambda = 0$ always span the nullspace (b) The eigenvectors for $\lambda \neq 0$ span the column space if there are r independent eigenvectors: then algebraic multiplicity = geometric multiplicity for each nonzero λ .
- 39** The eigenvalues 2, -1, 0 and their eigenvectors are in Λ and S . Then $A^k = S\Lambda^k S^{-1}$ is
- $$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2^k & & \\ & (-1)^k & \\ & & 0^k \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 1 & -1 \end{bmatrix} = \frac{2^k}{6} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$
- Check $k = 1$! The (2, 2) entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Harder to find the eleven 4-step paths that start and end at node 1.
- Notice the column times row multiplication above. Since $A = A^T$ the eigenvectors in the columns of S are orthogonal. They are in the rows of S^{-1} divided by their length squared.
- 40** B has the same eigenvectors (1, 0) and (0, 1) as A , so B is also diagonal. The 4 equations $AB - BA = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} - \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ have coefficient matrix with rank 2.
- 41** $AB = BA$ always has the solution $B = A$. (In case $A = 0$ every B is a solution.)
- 42** B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1; C has $\lambda = (1 \pm \sqrt{3}i)/2 = \exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1 . Then $C^3 = -I$ and $C^{1024} = -C$.

Problem Set 6.3, page 279

1 $\mathbf{u}_1 = e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\mathbf{u}(0) = (5, -2)$, then $\mathbf{u}(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- 2 $z(t) = -2e^t$; then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.
- 3 $\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$. Then $\lambda = \frac{1}{2}(5 \pm \sqrt{41})$.
- 4 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5$, $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$,
 $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches 20/10; e^{5t} dominates.
- 5 $d(v+w)/dt = dv/dt + dw/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = -2$ with $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $v(1) = 20 + 10e^{-2}$,
 $w(1) = 20 - 10e^{-2}$.
- 6 $\lambda_1 = 0$ and $\lambda_2 = 2$. Now $v(t) = 20 + 10e^{2t} \rightarrow \infty$ as $t \rightarrow \infty$.
- 7 $e^{At} = I + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{zeros} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.
- 8 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with only one independent eigenvector $(1, 3)$.
- 9 $my'' + by' + ky = 0$ is $\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$.
- 10 When A is skew-symmetric, $\|\mathbf{u}(t)\| = \|e^{At}\mathbf{u}(0)\| = \|\mathbf{u}(0)\|$. So e^{At} is an *orthogonal* matrix.
- 11 (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Then $\mathbf{u}(t) = \frac{1}{2}e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.
- 12 $y(t) = \cos t$ starts at $y(0) = 1$ and $y'(0) = 0$.
- 13 $\mathbf{u}_p = A^{-1}\mathbf{b} = 4$ and $u(t) = ce^{2t} + 4$; $\mathbf{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{u}(t) = c_1e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.
- 14 Substituting $\mathbf{u} = e^{ct}\mathbf{v}$ gives $ce^{ct}\mathbf{v} = Ae^{ct}\mathbf{v} - e^{ct}\mathbf{b}$ or $(A - cI)\mathbf{v} = \mathbf{b}$ or $\mathbf{v} = (A - cI)^{-1}\mathbf{b} =$
particular solution. If c is an eigenvalue then $A - cI$ is not invertible.
- 15 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. In each case e^{At} blows up.
- 16 $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \dots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots) = Ae^{At}$.
- 17 $e^{Bt} = I + Bt = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$. Derivative $= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = B$.
- 18 The solution at time $t+T$ is also $e^{A(t+T)}\mathbf{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.
- 19 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$; $e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$.
- 20 If $A^2 = A$ then $e^{At} = I + At + \frac{1}{2}At^2 + \frac{1}{6}At^3 + \dots = I + (e^t - 1)A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^t - 1 & e^t - 1 \\ 0 & 0 \end{bmatrix}$.
- 21 $e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$, $e^B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $e^Ae^B \neq e^Be^A = \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix} \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$.
- 22 $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$, then $e^{At} = \begin{bmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{bmatrix}$.

- 23** $A^2 = A$ so $A^3 = A$ and by Problem 20 $e^{At} = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}$.
- 24** (a) The inverse of e^{At} is e^{-At} (b) If $A\mathbf{x} = \lambda\mathbf{x}$ then $e^{At}\mathbf{x} = e^{\lambda t}\mathbf{x}$ and $e^{\lambda t} \neq 0$.
- 25** $x(t) = e^{4t}$ and $y(t) = -e^{4t}$ is a growing solution. The correct matrix for the exchanged unknown $\mathbf{u} = (y, x)$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ and it *does* have the same eigenvalues as the original matrix.

Problem Set 6.4, page 290

- 1** $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \text{symmetric} + \text{skew-symmetric}.$
- 2** $(A^T C A)^T = A^T C^T (A^T)^T = A^T C A$. When A is 6 by 3, C is 6 by 6 and $A^T C A$ is 3 by 3.
- 3** $\lambda = 0, 2, -1$ with unit eigenvectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$.
- 4** $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$
- 5** $Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$
- 6** $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or $\begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix}$ or exchange columns.
- 7** (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ 's
(c) $\text{trace} = \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues.
- 8** If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3Q^T = 0$ gives $\Lambda = 0$ and the only symmetric possibility is $A = Q0Q^T = \text{zero matrix}.$
- 9** If λ is complex then $\bar{\lambda}$ is also an eigenvalue ($A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$). Always $\lambda + \bar{\lambda}$ is real. The trace is real so the third eigenvalue must be real.
- 10** If \mathbf{x} is not real then $\lambda = \mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x}$ is *not* necessarily real. Can't assume real eigenvectors!
- 11** $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}; \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$
- 12** $[\mathbf{x}_1 \ \mathbf{x}_2]$ is an orthogonal matrix so $P_1 + P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = [\mathbf{x}_1 \ \mathbf{x}_2] \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} = I;$
 $P_1 P_2 = \mathbf{x}_1 (\mathbf{x}_1^T \mathbf{x}_2) \mathbf{x}_2^T = 0$. Second proof: $P_1 P_2 = P_1 (I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.
- 13** $\lambda = ib$ and $-ib$; $A = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ has $\det(A - \lambda I) = -\lambda^3 - 25\lambda = 0$ and $\lambda = 0, 5i, -5i$.

- 14** Skew-symmetric and orthogonal; $\lambda = i, i, -i, -i$ to have trace zero.
- 15** A has $\lambda = 0, 0$ and only one independent eigenvector $\mathbf{x} = (i, 1)$.
- 16** (a) If $A\mathbf{z} = \lambda\mathbf{y}$ and $A^T\mathbf{y} = \lambda\mathbf{z}$ then $B[\mathbf{y}; -\mathbf{z}] = [-A\mathbf{z}; A^T\mathbf{y}] = -\lambda[\mathbf{y}; -\mathbf{z}]$. So $-\lambda$ is also an eigenvalue of B . (b) $A^T A\mathbf{z} = A^T(\lambda\mathbf{y}) = \lambda^2\mathbf{z}$. The eigenvalues of $A^T A$ are ≥ 0 . (c) $\lambda = -1, -1, 1, 1$; $\mathbf{x}_1 = (1, 0, -1, 0)$, $\mathbf{x}_2 = (0, 1, 0, -1)$, $\mathbf{x}_3 = (1, 0, 1, 0)$, $\mathbf{x}_4 = (0, 1, 0, 1)$.
- 17** The eigenvalues of B are $0, \sqrt{2}, -\sqrt{2}$ with $\mathbf{x}_1 = (1, -1, 0)$, $\mathbf{x}_2 = (1, 1, \sqrt{2})$, $\mathbf{x}_3 = (1, 1, -\sqrt{2})$.
- 18** \mathbf{y} is in the nullspace of A and \mathbf{x} is in the column space. $A = A^T$ has column space = row space, and this is perpendicular to the nullspace. Then $\mathbf{y}^T \mathbf{x} = 0$. If $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \beta\mathbf{y}$ then shift by β : $(A - \beta I)\mathbf{x} = (\lambda - \beta)\mathbf{x}$ and $(A - \beta I)\mathbf{y} = \mathbf{0}$ and again $\mathbf{x} \perp \mathbf{y}$.
- 19** B has eigenvectors in $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1+d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; independent but not perpendicular.
- 20** $\lambda = -5$ and 5 have the same signs as the pivots -3 and $25/3$.
- 21** (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True (c) True. $A^{-1} = Q\Lambda^{-1}Q^T$ is also symmetric (d) False.
- 22** If $A^T = -A$ then $A^T A = AA^T = -A^2$. If A is orthogonal then $A^T A = AA^T = I$. $A = \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ is normal only if $a = d$. Then $\mathbf{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.
- 23** A and A^T have the same λ 's but the *order* of the \mathbf{x} 's can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\mathbf{x}_1 = (1, i)$ for A but $\mathbf{x}_1 = (1, -i)$ for A^T .
- 24** A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. $QR, SAS^{-1}, Q\Lambda Q^T$ possible for A ; $S\Lambda S^{-1}$ and $Q\Lambda Q^T$ possible for B .
- 25** Symmetry gives $Q\Lambda Q^T$ when $b = 1$; repeated λ and no S when $b = -1$; singular if $b = 0$.
- 26** Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so every $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \text{reflection}.$
- 27** Eigenvectors $(1, 0)$ and $(1, 1)$ give a 45° angle even with A^T very close to A .
- 28** The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.
- 29** We get good eigenvectors for the "symmetric part" $\frac{1}{2}(P + P^T)$ which MATLAB would recognize as symmetric. But the projection matrix $P = A(A^T A)^{-1}A^T$ = product of 3 matrices is not recognized as exactly symmetric.

Problem Set 6.5, page 302

- 1** A_4 has two positive eigenvalues because $a = 1$ and $ac - b^2 = 1$; $\mathbf{x}^T A_1 \mathbf{x}$ is zero for $\mathbf{x} = (1, -1)$ and $\mathbf{x}^T A_1 \mathbf{x} < 0$ for $\mathbf{x} = (6, -5)$.

- 2 Positive definite $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ for $-3 < b < 3$ $\begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^T$;
Positive definite $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ for $c > 8$ $\begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$.
- 3 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $f(x, y) = x^2 + 6xy + 9y^2 = (x + 3y)^2$.
- 4 $x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2$ is negative at $x = 2, y = -1$.
- 5 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ produces $f(x, y) = [x \ y] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2xy$. A has $\lambda = 1$ and -1 .
- 6 $\mathbf{x}^T A^T A \mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = 0$ only if $A\mathbf{x} = \mathbf{0}$. Since A has independent columns this only happens when $\mathbf{x} = \mathbf{0}$.
- 7 $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $A^T A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular.
- 8 $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots outside squares, and L inside.
- 9 $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$, rank $A = 1$, eigenvalues are 24, 0, 0, $\det A = 0$.
- 10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- 11 $|A_1| = 2, |A_2| = 6, |A_3| = 30$. The pivots are $2/1, 6/2, 30/6$.
- 12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, c^3 + 2 - 3c > 0$. B is never positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).
- 13 $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ has $a + c > 2b$ but $ac < b^2$, so not positive definite.
- 14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\mathbf{x}^T A^{-1} \mathbf{x} = (A^{-1} \mathbf{x})^T A (A^{-1} \mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- 15 Since $\mathbf{x}^T A \mathbf{x} > 0$ and $\mathbf{x}^T B \mathbf{x} > 0$ we have $\mathbf{x}^T (A + B) \mathbf{x} = \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T B \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Then $A + B$ is a positive definite matrix.
- 16 $\mathbf{x}^T A \mathbf{x}$ is not positive when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal.
- 17 If a_{jj} were smaller than all the eigenvalues, $A - a_{jj}I$ would have *positive* eigenvalues (so positive definite). But $A - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 16.
- 18 If $A\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{x}^T A \mathbf{x} = \lambda\mathbf{x}^T \mathbf{x}$. If A is positive definite this leads to $\lambda = \mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x} > 0$ (ratio of positive numbers).
- 19 All cross terms are $\mathbf{x}_i^T \mathbf{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors.
- 20 (a) The determinant is positive, all $\lambda > 0$ (b) All projection matrices except I are singular
(c) The diagonal entries of D are its eigenvalues (d) $-I$ has $\det = 1$ when n is even.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ (check determinants).

22 $R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

23 $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = \frac{1}{3}$ and $b = \frac{1}{4}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $a = 1/\sqrt{\lambda_1} = \sqrt{2}$ and $b = \sqrt{2/3}$.

25 $A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$.

26 $C = L\sqrt{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D .

27 $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$; $2x^2 + 8xy + 10y^2 = 2(x + 2y)^2 + 2y^2$.

28 $\det A = 10$; $\lambda = 2$ and 5 ; $\mathbf{x}_1 = (\cos \theta, \sin \theta)$, $\mathbf{x}_2 = (-\sin \theta, \cos \theta)$; the λ 's are positive.

29 $A_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is positive definite if $x \neq 0$; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$;
 $A_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite and $(0, 1)$ is a saddle point.

30 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$).

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a trough staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At} , matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n .

Problem Set 6.6, page 310

1 $C = (MN)^{-1}A(MN)$ so if B is similar to A and C is similar to B , then A is similar to C .

2 $B = (FG^{-1})^{-1}A(FG^{-1})$. If C is similar to A and also to B then A is similar to B .

3 $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives $B = M^{-1}AM$.

4 A has no repeated λ so it can be diagonalized: $S^{-1}AS = \Lambda$ makes A similar to Λ .

5 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by itself and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by itself.

6 Eight families of similar matrices: 6 matrices have $\lambda = 0$, 1; 3 matrices have $\lambda = 1$, 1 and 3 have $\lambda = 0$, 0 (two families each!); one has $\lambda = 1$, -1 ; one has $\lambda = 2$, 0; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$.

7 (a) $(M^{-1}AM)(M^{-1}\mathbf{x}) = M^{-1}(A\mathbf{x}) = M^{-1}\mathbf{0} = \mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues 0, 0.

9 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, every $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$, $J^3 = \begin{bmatrix} c^3 & 3c^2 \\ 0 & c^3 \end{bmatrix}$, $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$, $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

11 $w(t) = (w(0) + tx(0) + \frac{1}{2}t^2y(0) + \frac{1}{6}t^3z(0))e^{5t}$.

12 If $M^{-1}JM = K$ then $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$ and M is not invertible.

13 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^T$ in each block (2) M_0 has those blocks M_i on its block diagonal to get $M_0^{-1}JM_0 = J^T$. (3) $A^T = (M^{-1})^T J^T M^T$ is $(M^{-1})^T M_0^{-1}JM_0M^T = (MM_0M^T)^{-1}A(MM_0M^T)$, and A^T is similar to A .

14 Every matrix MJM^{-1} will be similar to J .

15 $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda IM) = \det(M^{-1}(A - \lambda I)M) = \det(A - \lambda I)$.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$; $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is similar to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. I is not similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

17 (a) True: One has $\lambda = 0$, the other doesn't (b) False. Diagonalize a nonsymmetric matrix and Λ is symmetric (c) False: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (d) True: All eigenvalues of $A + I$ are increased by 1, so different from the eigenvalues of A .

18 $AB = B^{-1}(BA)B$ so AB is similar to BA . Also $AB\mathbf{x} = \lambda\mathbf{x}$ leads to $BA(B\mathbf{x}) = \lambda(B\mathbf{x})$.

19 Diagonals 6 by 6 and 4 by 4; AB has all the same eigenvalues as BA plus 6 - 4 zeros.

20 (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$ (b) A may not be similar to $B = -A$ (but it could be!) (c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because $\lambda_1 \neq \lambda_2$
(d) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable (e) PAP^T is similar to A .

21 J^2 has three 1's down the *second* superdiagonal, and two independent eigenvectors for $\lambda = 0$.

Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & & \\ & J_2 & \\ & & \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note to professors: You could list all 3 by 3 and 4 by 4 Jordan J 's:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \text{ with } 3, 2, 1 \text{ eigenvectors; } \text{diag}(a, b, c, d) \text{ and}$$

$$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix} \text{ with } 4, 3, 2, 1 \text{ eigenvectors.}$$

Problem Set 6.7, page 318

- 1 $A^T A = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$.
- 2 (a) $AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$.
 (b) $A\mathbf{v}_1 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix} = \begin{bmatrix} \sqrt{17} \\ 2\sqrt{17} \end{bmatrix} = \sqrt{85} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \sigma_1 \mathbf{u}_1$.
- 3 $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for the column space, $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$ for the row space, $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ for the nullspace, $\mathbf{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$ for the left nullspace.
- 4 $A^T A = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \frac{3+\sqrt{5}}{2}$ and $\sigma_2^2 = \frac{3-\sqrt{5}}{2}$.

Since $A = A^T$ the eigenvectors of $A^T A$ are the same as for A . Since $\lambda_2 = \frac{1-\sqrt{5}}{2}$ is *negative*, $\sigma_1 = \lambda_1$ but $\sigma_2 = -\lambda_2$. The eigenvectors are the same as in Section 6.2 for A , except for the effect of this minus sign: $\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} \lambda_1/\sqrt{1+\lambda_1^2} \\ 1/\sqrt{1+\lambda_1^2} \end{bmatrix}$ and $\mathbf{u}_2 = -\mathbf{v}_2 = \begin{bmatrix} \lambda_2/\sqrt{1+\lambda_2^2} \\ 1/\sqrt{1+\lambda_2^2} \end{bmatrix}$.

- 6 A proof that *eigshow* finds the SVD for 2 by 2 matrices. Starting at the orthogonal pair $\mathbf{V}_1 = (1, 0)$, $\mathbf{V}_2 = (0, 1)$ the demo finds $A\mathbf{V}_1$ and $A\mathbf{V}_2$ at angle θ . After a 90° turn by the mouse to $\mathbf{V}_2, -\mathbf{V}_1$ the demo finds $A\mathbf{V}_2$ and $-A\mathbf{V}_1$ at angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\mathbf{v}_1, \mathbf{v}_2$ must have produced $A\mathbf{v}_1$ and $A\mathbf{v}_2$ at angle $\theta = \pi/2$. Those are the orthogonal directions for \mathbf{u}_1 and \mathbf{u}_2 .

$$7 \quad AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ has } \sigma_1^2 = 3 \text{ with } \mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \sigma_2^2 = 1 \text{ with } \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}. \quad A^T A =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ has } \sigma_1^2 = 3 \text{ with } \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \sigma_2^2 = 1 \text{ with } \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}; \text{ and } \mathbf{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

$$\text{Then } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]^T.$$

- 8 $A = UV^T$ since all $\sigma_j = 1$.

- 9 $A = 12UV^T$.
- 10 $A = W\Sigma W^T$ is the same as $A = U\Sigma V^T$.
- 11 Multiply $U\Sigma V^T$ using columns (of U) times rows (of ΣV^T).
- 12 Since $A^T = A$ we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. But λ_2 is negative, so $\sigma_1 = 3$ and $\sigma_2 = 2$. The unit eigenvectors of A are the same $\mathbf{u}_1 = \mathbf{v}_1$ as for $A^T A = AA^T$ and $\mathbf{u}_2 = -\mathbf{v}_2$ (notice sign change because $\sigma_2 = -\lambda_2$).
- 13 Suppose the SVD of R is $R = U\Sigma V^T$. Then multiply by Q . So the SVD of this A is $(QU)\Sigma V^T$.
- 14 The smallest change in A is to set its smallest singular value σ_2 to zero.
- 15 (a) If A changes to $4A$, multiply Σ by 4. (b) $A^T = V\Sigma^T U^T$. And if A^{-1} exists, it is square and equal to $(V^T)^{-1}\Sigma^{-1}U^{-1}$.
- 16 The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A + I)^T(A + I)$.
- 17 This simulates the random walk used by *Google* on billions of sites to solve $A\mathbf{p} = \mathbf{p}$. It is like the power method of 9.3 except that it follows the links in one “walk” where the vectors $\mathbf{p}_k = A^k \mathbf{p}_0$ averages over all walks.

Problem Set 7.1, page 325

- 1 With $\mathbf{w} = \mathbf{0}$ linearity gives $T(\mathbf{v} + \mathbf{0}) = T(\mathbf{v}) + T(\mathbf{0})$. Thus $T(\mathbf{0}) = \mathbf{0}$. With $c = -1$ linearity gives $T(-\mathbf{0}) = -T(\mathbf{0})$. Thus $T(\mathbf{0}) = \mathbf{0}$.
- 2 $T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w})$; add $eT(\mathbf{u})$.
- 3 (d) is not linear.
- 4 (a) $S(T(\mathbf{v})) = \mathbf{v}$ (b) $S(T(\mathbf{v}_1) + T(\mathbf{v}_2)) = S(T(\mathbf{v}_1)) + S(T(\mathbf{v}_2))$.
- 5 Choose $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (-1, 0)$. Then $T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{v} + \mathbf{w}$ but $T(\mathbf{v} + \mathbf{w}) = (0, 0)$.
- 6 (b) and (c) are linear (d) satisfies $T(c\mathbf{v}) = cT(\mathbf{v})$.
- 7 (a) $T(T(\mathbf{v})) = \mathbf{v}$ (b) $T(T(\mathbf{v})) = \mathbf{v} + (2, 2)$ (c) $T(T(\mathbf{v})) = -\mathbf{v}$ (d) $T(T(\mathbf{v})) = T(\mathbf{v})$.
- 8 (a) Range \mathbf{R}^2 , kernel $\{\mathbf{0}\}$ (b) Range \mathbf{R}^2 , kernel $\{(0, 0, v_3)\}$ (c) Range $\{\mathbf{0}\}$, kernel \mathbf{R}^2 (d) Range = multiples of $(1, 1)$, kernel = multiples of $(1, -1)$.
- 9 $T(T(\mathbf{v})) = (v_3, v_1, v_2)$; $T^3(\mathbf{v}) = \mathbf{v}$; $T^{100}(\mathbf{v}) = T(\mathbf{v})$.
- 10 (a) $T(1, 0) = \mathbf{0}$ (b) $(0, 0, 1)$ is not in the range (c) $T(0, 1) = \mathbf{0}$.
- 11 $\mathbf{V} = \mathbf{R}^n$, $\mathbf{W} = \mathbf{R}^m$; the outputs fill the column space; \mathbf{v} is in the kernel if $A\mathbf{v} = \mathbf{0}$.
- 12 $T(\mathbf{v}) = (4, 4); (2, 2); (2, 2)$; if $\mathbf{v} = (a, b) = b(1, 1) + \frac{a-b}{2}(2, 0)$ then $T(\mathbf{v}) = b(2, 2) + (0, 0)$.
- 13 Associative gives $A(M_1 + M_2) = AM_1 + AM_2$. Distributive over c 's gives $A(cM) = c(AM)$.
- 14 A is invertible. Multiply $AM = 0$ and $AM = B$ by A^{-1} to get $M = 0$ and $M = A^{-1}B$.
- 15 A is not invertible. $AM = I$ is impossible. $A \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

16 No matrix A gives $A \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: The matrix space has dimension 4.

Linear transformations come from 4 by 4 matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

18 $T(I) = 0$ but $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$; these fill the range. $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ in the kernel.

19 If $\mathbf{v} \neq 0$ is a column of B and $\mathbf{u}^T \neq 0$ is a row of A , choose $M = \mathbf{u}\mathbf{v}^T$.

20 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

21 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical.

23 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ (b) $A = 3I$ (c) $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

24 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0, 0)$.)

25 Rotate the house by 180° and shift one unit to the right.

27 This emphasizes that circles are transformed to ellipses (figure in Section 6.7).

30 Squeezed by 10 in y direction; flattened onto 45° line; rotated by 45° and stretched by $\sqrt{2}$; flipped over and “skewed” so squares become parallelograms.

Problem Set 7.2, page 337

1 $S\mathbf{v}_1 = S\mathbf{v}_2 = \mathbf{0}$, $S\mathbf{v}_3 = 2\mathbf{v}_1$, $S\mathbf{v}_4 = 6\mathbf{v}_2$; $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 All functions $\mathbf{v}(x) = a + bx$; all vectors $(a, b, 0, 0)$.

3 $A^2 = B$ when $T^2 = S$ and output basis = input basis.

4 Third derivative has 6 in the $(1, 4)$ position; fourth derivative of cubic is zero.

5 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

6 $T(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 2\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3$; A times $(1, 1, 1)$ gives $(2, 1, 2)$.

7 $\mathbf{v} = c(\mathbf{v}_2 - \mathbf{v}_3)$ gives $T(\mathbf{v}) = \mathbf{0}$; nullspace is $(0, c, -c)$; solutions are $(1, 0, 0) + \text{any } (0, c, -c)$.

8 $(1, 0, 0)$ is not in the column space; \mathbf{w}_1 is not in the range.

9 We don't know $T(\mathbf{w})$ unless the \mathbf{w} 's are the same as the \mathbf{v} 's. In that case the matrix is A^2 .

10 Rank = 2 = dimension of the range of T .

- 11** $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; for output $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose input $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.
- 12** $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ so $T^{-1}(\mathbf{w}_1) = \mathbf{v}_1 - \mathbf{v}_2$, $T^{-1}(\mathbf{w}_2) = \mathbf{v}_2 - \mathbf{v}_3$, $T^{-1}(\mathbf{w}_3) = \mathbf{v}_3$; the only solution to $T(\mathbf{v}) = \mathbf{0}$ is $\mathbf{v} = \mathbf{0}$.
- 13** (c) is wrong because \mathbf{w}_1 is not generally in the input space.
- 14** (a) $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_1$ (b) $T(\mathbf{v}_1) = \mathbf{v}_1$, $T(\mathbf{v}_2) = \mathbf{0}$ (c) If $T^2 = I$ and $T^2 = T$ then $T = I$.
- 15** (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
- 16** (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ (c) $ad = bc$.
- 17** $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.
- 18** Permutation matrix; positive diagonal matrix.
- 19** $(a, b) = (\cos \theta, -\sin \theta)$. Minus sign from $Q^{-1} = Q^T$.
- 20** $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$; $(a, b) = (5, -4)$ = first column of M^{-1} .
- 21** $\mathbf{w}_2(x) = 1 - x^2$; $\mathbf{w}_3(x) = \frac{1}{2}(x^2 - x)$; $\mathbf{y} = 4\mathbf{w}_1 + 5\mathbf{w}_2 + 6\mathbf{w}_3$.
- 22** \mathbf{w} 's to \mathbf{v} 's: $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$. \mathbf{v} 's to \mathbf{w} 's: inverse matrix = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.
- 23** $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$; Vandermonde determinant = $(b-a)(c-a)(c-b)$; a, b, c must be distinct.
- 24** The matrix M with these nine entries must be invertible.
- 25** $\mathbf{a}_2 = r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2$ gives \mathbf{a}_2 as a combination of the \mathbf{q} 's. So the change of basis matrix is R .
- 26** Row 2 of A is $l_{21}(\text{row 1 of } U) + l_{22}(\text{row 2 of } U)$. The change of basis matrix is always *invertible*.
- 27** The matrix is Λ .
- 28** If T is not invertible then $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ will not be a basis. Then we couldn't choose $\mathbf{w}_i = T(\mathbf{v}_i)$.
- 29** (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
- 30** $T(x, y) = (x, -y)$ and then $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

31 $S(T(\mathbf{v})) = (-1, 2)$ but $S(\mathbf{v}) = (-2, 1)$ and $T(S(\mathbf{v})) = (1, -2)$.

32 $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ rotates by $2(\theta - \alpha)$.

33 False, because the \mathbf{v} 's might not be linearly independent.

Problem Set 7.3, page 345

1 Multiply by $W^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Then $\mathbf{e} = \frac{1}{4}\mathbf{w}_1 + \frac{1}{4}\mathbf{w}_2 + \frac{1}{2}\mathbf{w}_3$ and $\mathbf{v} = \mathbf{w}_3 + \mathbf{w}_4$.

2 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

3 The wavelet basis is $(1, 1, 1, 1, 1, 1, 1, 1)$ and the long wavelet and two medium wavelets $(1, 1, -1, -1, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1, 1, -1, -1)$ and 4 short wavelets with a single pair 1, -1.

4 $W_2^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $W_1^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

5 The Hadamard matrix H has orthogonal columns of length 2. So the inverse is $H^T/4 = H/4$.

6 If $V\mathbf{b} = W\mathbf{c}$ then $\mathbf{b} = V^{-1}W\mathbf{c}$. The change of basis matrix is $V^{-1}W$.

7 The transpose of $WW^{-1} = I$ is $(W^{-1})^T W^T = I$. So the matrix W^T (which has the \mathbf{w} 's in its rows) is the inverse to the matrix that has the \mathbf{w}^* 's in its columns.

Problem Set 7.4, page 353

1 $A^T A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

2 $AA^T = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$ has $\lambda = 50$ and 0, $\mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

3 Orthonormal bases: \mathbf{v}_1 for row space, \mathbf{v}_2 for nullspace, \mathbf{u}_1 for column space, \mathbf{u}_2 for $N(A^T)$.

4 The matrices with those four subspaces are multiples cA .

5 $A = QH = \frac{1}{\sqrt{50}} \begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}} \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

6 $A^+ = V \begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^T = \frac{1}{50} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+ A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $A^T A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$ has $\lambda = 18$ and 2 , $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\sigma_1 = \sqrt{18}$ and $\sigma_2 = \sqrt{2}$.

8 $AA^T = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$ has $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

9 $[\sigma_1 \mathbf{u}_1 \quad \sigma_2 \mathbf{u}_2] \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$. In general this is $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$.

10 $Q = UV^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $K = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

11 A^+ is A^{-1} because A is invertible.

12 $A^T A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\lambda = 25, 0, 0$ and $\mathbf{v}_1 = \begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
 $AA^T = [25]$ and $\sigma_1 = 5$.

13 $A = [1] \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} V^T$ and $A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} [1] = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $AA^+ = [1]$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

14 Zero matrix; $\Sigma = 0$; $A^+ = 0$ is 3 by 2.

15 If $\det A = 0$ then $\text{rank}(A) < n$; thus $\text{rank}(A^+) < n$ and $\det A^+ = 0$.

16 A must be symmetric and positive definite.

17 (a) $A^T A$ is singular (b) $A^T A \mathbf{x}^+ = A^T \mathbf{b}$ (c) $(I - AA^+)$ projects onto $\mathcal{N}(A^T)$.

18 \mathbf{x}^+ in the row space of A is perpendicular to $\hat{\mathbf{x}} - \mathbf{x}^+$ in the nullspace of $A^T A = \text{nullspace of } A$. The right triangle has $c^2 = a^2 + b^2$.

19 $AA^+ \mathbf{p} = \mathbf{p}$, $AA^+ \mathbf{e} = \mathbf{0}$, $A^+ A \mathbf{x}_r = \mathbf{x}_r$, $A^+ A \mathbf{x}_n = \mathbf{0}$.

20 $A^+ = \frac{1}{5} [.6 \quad .8] = [.12 \quad .16]$ and $A^+ A = [1]$ and $AA^+ = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$.

21 L is determined by ℓ_{21} . Each eigenvector in S is determined by one number. The counts are 1 + 3 for LU , 1 + 2 + 1 for LDU , 1 + 3 for QR , 1 + 2 + 1 for $U\Sigma V^T$, 2 + 2 + 0 for $S\Lambda S^{-1}$.

22 The counts are 1 + 2 + 0 because A is *symmetric*.

23 Column times row multiplication gives $A = U\Sigma V^T = \sum \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ and also $A^+ = V\Sigma^+ U^T = \sum \sigma_i^{-1} \mathbf{v}_i \mathbf{u}_i^T$. Multiplying $A^+ A$ and using orthogonality of each \mathbf{u}_i to all other \mathbf{u}_j leaves the projection matrix $A^+ A$: $A^+ A = \sum 1 \mathbf{v}_i \mathbf{v}_i^T$. Similarly $AA^+ = \sum 1 \mathbf{u}_i \mathbf{u}_i^T$ from $VV^T = I$.

24 The columns of \hat{U} are a basis for the column space of A . So are the first r columns of U . Those r columns must have the form $\hat{U} M_1$ for some r by r invertible matrix M_1 . Similarly the columns of \hat{V} and the first r columns of V are bases for the row space of A . So $V = \hat{V} M_2$.

Keep only the r by r invertible corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^T$ has the required form $A = \hat{U} M_1 \Sigma_r M_2^T \hat{V}^T$ with an invertible $M = M_1 \Sigma_r M_2^T$ in the middle.

25 $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \sigma \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$. That block matrix connects to $A^T A$ and AA^T .

Problem Set 8.1, page ???

- 1 $\det A_0^T C_0 A_0$ is by direct calculation. Set $c_4 = 0$ to find $\det A_1^T C_1 A_1 = c_1 c_2 c_3$.
- 2 $(A_1^T C_1 A_1)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_1^{-1} + c_2^{-1} + c_3^{-1} & c_2^{-1} + c_3^{-1} & c_3^{-1} \\ c_2^{-1} + c_3^{-1} & c_2^{-1} + c_3^{-1} & c_3^{-1} \\ c_3^{-1} & c_3^{-1} & c_3^{-1} \end{bmatrix}$
- 3 The rows of the free-free matrix in equation (9) add to $[0 \ 0 \ 0]$ so the right side needs $f_1 + f_2 + f_3 = 0$. For $\mathbf{f} = (-1, 0, 1)$ elimination gives $c_2 u_1 - c_2 u_2 = -1$, $c_3 u_2 - c_3 u_3 = -1$, and $0 = 0$. Then $\mathbf{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $\mathbf{u}_{\text{nullspace}} = (1, 1, 1)$.
- 4 $\int -\frac{d}{dx} \left(c(x) \frac{du}{dx} \right) dx = \left[c(0) \frac{du}{dx}(0) - c(1) \frac{du}{dx}(1) \right] = 0$ so we need $\int f(x) dx = 0$.
- 5 $-\frac{dy}{dx} = f(x)$ gives $y(x) = C - \int_0^x f(t) dt$. Then $y(1) = 0$ gives $C = \int_0^1 f(t) dt$ and $y(x) = \int_x^1 f(t) dt$. If $f(x) = 1$ then $y(x) = 1 - x$.
- 6 Multiply $A_1^T C_1 A_1$ as columns of A_1^T times c 's times rows of A_1 . The first "element matrix" $c_1 E_1 = [1 \ 0 \ 0]^T c_1 [1 \ 0 \ 0]$ has c_1 in the top left corner.
- 7 For 5 springs and 4 masses, the 5 by 4 A has all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \text{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^T C A$, symmetric tridiagonal with $K_{ii} = c_i + c_{i+1}$ and $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix and $K(2, 3, 3, 2) = (1, 1, 1, 1)$.
- 8 The solution to $-u'' = 1$ with $u(0) = u(1) = 0$ is $u(x) = \frac{1}{2}(x - x^2)$. At $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ this $u(x)$ equals $\mathbf{u} = 2, 3, 3, 2$ (discrete solution in Problem 7) times $(\Delta x)^2 = 1/25$.
- 9 $-u'' = mg$ has complete solution $u(x) = A + Bx - \frac{1}{2}mgx^2$. From $u(0) = 0$ we get $A = 0$. From $u'(1) = 0$ we get $B = mg$. Then $u(x) = \frac{1}{2}mg(2x - x^2)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is not proportional to the discrete \mathbf{u} at the meshpoints.
- 10 The graphs of 100 points are "discrete parabolas" starting at $(0, 0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.
- 11 Forward vs. backward differences for du/dx have a big effect on the discrete \mathbf{u} , because that term has the large coefficient 10 (and with 100 or 1000 we would have a real boundary layer = near discontinuity at $x = 1$). The computed values are $\mathbf{u} = 0, .01, .03, .04, .05, .06, .07, .11, 0$ versus $\mathbf{u} = 0, .12, .24, .36, .46, .54, .55, .43, 0$. The MATLAB code is $E = \text{diag}(\text{ones}(6, 1), 1)$; $K = 64 * (2 * \text{eye}(7) - E - E')$; $D = 80 * (E - \text{eye}(7))$; $(K + D) \setminus \text{ones}(7, 1)$, $(K - D') \setminus \text{ones}(7, 1)$.

Problem Set 8.2, page 366

- 1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.
- 2 $A^T \mathbf{y} = \mathbf{0}$ for $\mathbf{y} = (1, -1, 1)$; current = 1 along edge 1, edge 3, back on edge 2 (full loop).

3 $U = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; tree from edges 1 and 2.

4 $A\mathbf{x} = \mathbf{b}$ is solvable for $\mathbf{b} = (1, 1, 0)$ and not solvable for $\mathbf{b} = (1, 0, 0)$; \mathbf{b} must be orthogonal to $\mathbf{y} = (1, -1, 1)$; $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^T \mathbf{y} = \mathbf{f}$ is solvable for $\mathbf{f} = (1, -1, 0)$ and not solvable for $\mathbf{f} = (1, 0, 0)$; \mathbf{f} must be orthogonal to $(1, 1, 1)$ in the nullspace.

6 $A^T A \mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \mathbf{f}$ produces $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials 1, -1, 0 and currents $-A\mathbf{x} = 2, 1, -1$; \mathbf{f} sends 3 units into node 1 and out from node 2.

7 $A^T \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\mathbf{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\mathbf{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

9 Elimination on $A\mathbf{x} = \mathbf{b}$ always leads to $\mathbf{y}^T \mathbf{b} = 0$ which is $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (\mathbf{y} 's from Problem 8 in the left nullspace). This is Kirchhoff's Voltage Law around the loops.

10 $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the matrix that keeps edges 1, 2, 4; other trees from 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 $A^T A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$ diagonal entry = number of edges into the node
off-diagonal entry = -1 if nodes are connected.

12 (1) The nullspace and rank of $A^T A$ and A are always the same (2) $A^T A$ is always positive semidefinite because $\mathbf{x}^T A^T A \mathbf{x} = \|A\mathbf{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1, 1, 1, 1)$ is in the nullspace (3) Real eigenvalues all ≥ 0 because positive semidefinite.

13 $A^T C A \mathbf{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives potentials $\mathbf{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ (grounded $x_4 = 0$ and solved 3 equations); $\mathbf{y} = -CA\mathbf{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$.

14 $A^T C A \mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (c, c, c, c)$; then \mathbf{f} must be orthogonal to \mathbf{x} .

- 15** $n - m + 1 = 7 - 7 + 1 = 1$ loop.
- 16** $5 - 7 + 3 = 1$; $5 - 8 + 4 = 1$.
- 17** (a) 8 independent columns (b) \mathbf{f} must be orthogonal to the nullspace so $f_1 + \dots + f_9 = 0$
 (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.
- 18** Complete graph has $5 + 4 + 3 + 2 + 1 = 15$ edges; tree has 5 edges.

Problem Set 8.3, page 373

- 1** $\lambda = 1$ and .75; $(A - I)\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = (.6, .4)$.
- 2** $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$;
 A^k approaches $\begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.
- 3** $\lambda = 1$ and .8, $\mathbf{x} = (1, 0)$; $\lambda = 1$ and $-.8$, $\mathbf{x} = (\frac{5}{9}, \frac{4}{9})$; $\lambda = 1, \frac{1}{4}$, and $\frac{1}{4}$, $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- 4** A^T always has the eigenvector $(1, 1, \dots, 1)$ for $\lambda = 1$.
- 5** The steady state is $(0, 0, 1) = \text{all dead}$.
- 6** If $A\mathbf{x} = \lambda\mathbf{x}$, add components on both sides to find $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.
- 7** $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; A^{16} has the same factors except now $(.5)^{16}$.
- 8** $(.5)^k \rightarrow 0$ gives $A^k \rightarrow A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $-\frac{2}{3} \leq a \leq 1$.
- 9** $\mathbf{u}_1 = (0, 0, 1, 0)$; $\mathbf{u}_2 = (0, 1, 0, 0)$; $\mathbf{u}_3 = (1, 0, 0, 0)$; $\mathbf{u}_4 = \mathbf{u}_0$. The eigenvalues 1, i , -1 , $-i$ are all on the unit circle. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue.
- 10** M^2 is still nonnegative; $[1 \ \dots \ 1]M = [1 \ \dots \ 1]$ so multiply by M to find
 $[1 \ \dots \ 1]M^2 = [1 \ \dots \ 1] \Rightarrow$ columns of M^2 add to 1.
- 11** $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\mathbf{x}_1 = (b, 1 - a)$.
- 12** Last row .2, .3, .5 makes $A = A^T$; rows also add to 1 so $(1, \dots, 1)$ is also an eigenvector of A .
- 13** B has $\lambda = 0$ and $-.5$ with $\mathbf{x}_1 = (.3, .2)$ and $\mathbf{x}_2 = (-1, 1)$; $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} \mathbf{x}_1 = c_1 \mathbf{x}_1$.
- 14** Each column of $B = A - I$ adds to zero. Then $\lambda_1 = 0$ and $e^{0t} = 1$.
- 15** The eigenvector is $\mathbf{x} = (1, 1, 1)$ and $A\mathbf{x} = (.9, .9, .9)$.
- 16** $(I - A)(I + A + A^2 + \dots) = I + A + A^2 + \dots - (A + A^2 + A^3 + \dots) = I$. This says that
 $I + A + A^2 + \dots$ is $(I - A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$ and the
 series adds to $\begin{bmatrix} 1 + \frac{1}{2} + \dots & \frac{1}{2} + \frac{1}{4} + \dots \\ 1 + \frac{1}{2} + \dots & 1 + \frac{1}{2} + \dots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I - A)^{-1}$.

- 17** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix}$ have $\lambda_{\max} < 1$.
- 18** $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.
- 19** $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).
- 20** No, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

Problem Set 8.4, page 382

- 1** Feasible set = line segment from (6, 0) to (0, 3); minimum cost at (6, 0), maximum at (0, 3).
- 2** Feasible set is 4-sided with corners (0, 0), (6, 0), (2, 2), (0, 6). Minimize $2x - y$ at (6, 0).
- 3** Only two corners (4, 0, 0) and (0, 2, 0); choose x_1 very negative, $x_2 = 0$, and $x_3 = x_1 - 4$.
- 4** From (0, 0, 2) move to $\mathbf{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\mathbf{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$ so $r = 1$ (more expensive).
- 5** Cost = 20 at start (4, 0, 0); keeping $x_1 + x_2 + 2x_3 = 4$ move to (3, 1, 0) with cost 18 and $r = -2$; or move to (2, 0, 1) with cost 17 and $r = -3$. Choose x_3 as entering variable and move to (0, 0, 2) with cost 14. Another step to reach (0, 4, 0) with minimum cost 12.
- 6** $\mathbf{c} = [3 \ 5 \ 7]$ has minimum cost 12 by the Ph.D. since $\mathbf{x} = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \leq 3$, $y \leq 5$, $y \leq 7$. Maximum = 12.

Problem Set 8.5, page 387

- 1** $\int_0^{2\pi} \cos(j+k)x \, dx = \left[\frac{\sin(j+k)x}{j+k} \right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos(j-k)x \, dx = 0$ (in the denominator notice $j - k \neq 0$). If $j = k$ then $\int_0^{2\pi} \cos^2 jx \, dx = \pi$.
- 2** $\int_{-1}^1 (1)(x) \, dx = 0$, $\int_{-1}^1 (1)(x^2 - \frac{1}{3}) \, dx = 0$, $\int_{-1}^1 (x)(x^2 - \frac{1}{3}) \, dx = 0$. Then $2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$.
- 3** $\mathbf{w} = (2, -1, 0, 0, \dots)$ has $\|\mathbf{w}\| = \sqrt{5}$.
- 4** $\int_{-1}^1 (1)(x^3 - cx) \, dx = 0$ and $\int_{-1}^1 (x^2 - \frac{1}{3})(x^3 - cx) \, dx = 0$ for all c (integral of an odd function). Choose c so that $\int_{-1}^1 x(x^3 - cx) \, dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^1 = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.
- 5** The integrals lead to $a_1 = 0$, $b_1 = 4/\pi$, $b_2 = 0$.
- 6** From equation (3) the a_k are zero and $b_k = 4/\pi k$. The square wave has $\|f\|^2 = 2\pi$. Then equation (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots)$ so this infinite series equals $\pi^2/8$.
- 8** $\|\mathbf{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$ so $\|\mathbf{v}\| = \sqrt{2}$; $\|\mathbf{v}\|^2 = 1 + a^2 + a^4 + \dots = 1/(1 - a^2)$ so $\|\mathbf{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x) \, dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

- 9 (a) $f(x) = \frac{1}{2} + \frac{1}{2}$ (square wave) so a 's are $\frac{1}{2}, 0, 0, \dots$, and b 's are $2/\pi, 0, -2/3\pi, 0, 2/5\pi, \dots$
 (b) $a_0 = \int_0^{2\pi} x dx / 2\pi = \pi$, other $a_k = 0$, $b_k = -2/k$.
- 10 The integral from $-\pi$ to π or from 0 to 2π or from any a to $a+2\pi$ is over one complete period of the function. If $f(x)$ is odd (and periodic) then $\int_0^{2\pi} f(x) dx = \int_0^\pi f(x) dx + \int_{-\pi}^0 f(x) dx$ and those integrals cancel.
- 11 $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$.
- 12 $\frac{d}{dx} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}.$
- 13 $dy/dx = \cos x$ has $y = y_p + y_n = \sin x + C$.

Problem Set 8.6, page 392

- 1 (x, y, z) has homogeneous coordinates $(x, y, z, 1)$ and also (cx, cy, cz, c) for any nonzero c .
- 2 For an affine transformation we need T (origin). Then $(x, y, z, 1) \rightarrow xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k}) + T(\mathbf{0})$.
- 3 $TT_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix}$ is translation along $(1, 6, 8)$.
- 4 $S = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ & & & 1 \end{bmatrix}$, $ST = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ 1 & 4 & 3 & 1 \end{bmatrix}$, $TS = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ c & 4c & 3c & 1 \end{bmatrix}$, use vTS .
- 5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square.
- 6 $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ 2 & 2 & 4 & 1 \end{bmatrix}.$
- 9 $\mathbf{n} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has $\|\mathbf{n}\| = 1$ and $P = I - \mathbf{n}\mathbf{n}^T = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}.$

10 Choose $(0, 0, 3)$ on the plane and multiply $T_-PT_+ = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3, 3, 3)$ projects to $\frac{1}{3}(-1, -1, 4)$ and $(3, 3, 3, 1)$ projects to $(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 1)$.

12 A parallelogram (or a line segment).

13 The projection of a cube is a hexagon.

14 $(3, 3, 3)(I - 2\mathbf{nn}^T) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = (-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3})$.

15 $(3, 3, 3, 1) \rightarrow (3, 3, 0, 1) \rightarrow (-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1) \rightarrow (-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1)$.

16 $\mathbf{v} = (x, y, z, 0)$ ending in 0; add a **vector** to a point.

17 Rescaled by $1/c$ because (x, y, z, c) is the same point as $(x/c, y/c, z/c, 1)$.

Problem Set 9.1, page 402

1 Without exchange, pivots .001 and 1000; with exchange, pivots 1 and -1 . When the pivot is

larger than the entries below it, $l_{ij} = \text{entry/pivot}$ has $|l_{ij}| \leq 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

2 $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$.

3 $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/16 \\ 13/12 \\ 47/60 \end{bmatrix} = \begin{bmatrix} 1.833 \\ 1.083 \\ 0.783 \end{bmatrix}$ compared with $A \begin{bmatrix} 0 \\ 6 \\ -3.6 \end{bmatrix} = \begin{bmatrix} 1.80 \\ 1.10 \\ 0.78 \end{bmatrix}$. $\|\Delta \mathbf{b}\| < .04$ but $\|\Delta \mathbf{x}\| > 6$.

4 The largest $\|\mathbf{x}\| = \|A^{-1}\mathbf{b}\|$ is $1/\lambda_{\min}$; the largest error is $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of \mathbf{x} (already known from below) and divide by the pivot. Total for n rows is less than wn .

6 L , U , and R need $\frac{1}{2}n^2$ multiplications to solve a linear system. Q needs n^2 to multiply the right side by $Q^{-1} = Q^T$. So QR takes 1.5 times longer than LU to reach \mathbf{x} .

7 On column j of I , back substitution needs $\frac{1}{2}j^2$ multiplications (only the j by j upper left block is involved). Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3)$.

$$\begin{aligned}
& \mathbf{8} \quad \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U \text{ with } P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}; A \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \\
& \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \text{ with } P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}.
\end{aligned}$$

9 The cofactors are $C_{13} = C_{31} = C_{24} = C_{42} = 1$ and $C_{14} = C_{41} = -1$.

10 With 16-digit floating point arithmetic the errors $\|\mathbf{x} - \mathbf{y}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

$$\mathbf{11} \quad \cos \theta = 1/\sqrt{10}, \quad \sin \theta = -3/\sqrt{10}, \quad R = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}.$$

12 Eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of Q : either

$$\begin{aligned}
Q &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } QAQ^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix} \text{ or} \\
Q &= \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \text{ and } QAQ^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}.
\end{aligned}$$

13 Changes in rows i and j ; changes also in columns i and j .

14 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos \theta$, the entries 1 and $\pm \tan \theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR .

Problem Set 9.2, page 408

- 1** $\|A\| = 2, c = 2/.5 = 4; \|A\| = 3, c = 3/1 = 3; \|A\| = 2 + \sqrt{2}, c = (2 + \sqrt{2})/(2 - \sqrt{2}) = 5.83$.
- 2** $\|A\| = 2, c = 1; \|A\| = \sqrt{2}, c = \text{infinite (singular matrix)}; \|A\| = \sqrt{2}, c = 1$.
- 3** For the first inequality replace \mathbf{x} by $B\mathbf{x}$ in $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$; the second inequality is just $\|B\mathbf{x}\| \leq \|B\|\|\mathbf{x}\|$. Then $\|AB\| = \max(\|AB\mathbf{x}\|/\|\mathbf{x}\|) \leq \|A\|\|B\|$.
- 4** Choose $B = A^{-1}$ and compute $\|I\| = 1$. Then $1 \leq \|A\|\|A^{-1}\| = c(A)$.
- 5** If $\lambda_{\max} = \lambda_{\min} = 1$ then all $\lambda_i = 1$ and $A = SIS^{-1} = I$. The only matrices with $\|A\| = \|A^{-1}\| = 1$ are *orthogonal matrices*.
- 6** $\|A\| \leq \|Q\|\|R\| = \|R\|$ and in reverse $\|R\| \leq \|Q^{-1}\|\|A\| = \|A\|$.
- 7** The triangle inequality gives $\|A\mathbf{x} + B\mathbf{x}\| \leq \|A\mathbf{x}\| + \|B\mathbf{x}\|$. Divide by $\|\mathbf{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \leq \|A\| + \|B\|$.
- 8** If $A\mathbf{x} = \lambda\mathbf{x}$ then $\|A\mathbf{x}\|/\|\mathbf{x}\| = |\lambda|$ for that particular vector \mathbf{x} . When we maximize the ratio over all vectors we get $\|A\| \geq |\lambda|$.
- 9** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\rho(A) = 0$ and $\rho(B) = 0$ but $\rho(A + B) = 1$; also $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has $\rho(AB) = 1$; thus $\rho(A)$ is not a norm.

- 10** The condition number of A^{-1} is $\|A^{-1}\| \|(A^{-1})^{-1}\| = c(A)$. Since $A^T A$ and AA^T have the same nonzero eigenvalues, A and A^T have the same norm.
- 11** $c(A) = (1.00005 + \sqrt{(1.00005)^2 - .0001}) / (1.00005 - \sqrt{\quad})$.
- 12** $\det(2A)$ is not $2 \det A$; $\det(A+B)$ is not always less than $\det A + \det B$; taking $|\det A|$ does not help. The only reasonable property is $\det AB = (\det A)(\det B)$. The condition number should not change when A is multiplied by 10.
- 13** The residual $\mathbf{b} - A\mathbf{y} = (10^{-7}, 0)$ is much smaller than $\mathbf{b} - A\mathbf{z} = (.0013, .0016)$. But \mathbf{z} is much closer to the solution than \mathbf{y} .
- 14** $\det A = 10^{-6}$ so $A^{-1} = \begin{bmatrix} 659,000 & -563,000 \\ -913,000 & 780,000 \end{bmatrix}$. Then $\|A\| > 1$, $\|A^{-1}\| > 10^6$, $c > 10^6$.
- 15** $\|\mathbf{x}\| = \sqrt{5}$, $\|\mathbf{x}\|_1 = 5$, $\|\mathbf{x}\|_\infty = 1$; $\|\mathbf{x}\| = 1$, $\|\mathbf{x}\|_1 = 2$, $\|\mathbf{x}\|_\infty = .7$.
- 16** $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $x_1^2 + \cdots + x_n^2 + 2|x_1||x_2| + \cdots = \|\mathbf{x}\|_1^2$. Certainly $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$ so $\|\mathbf{x}\| \leq \sqrt{n} \|\mathbf{x}\|_\infty$. Choose $\mathbf{y} = (\text{sign } x_1, \text{sign } x_2, \dots, \text{sign } x_n)$ to get $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_1$. By Schwarz this is at most $\|\mathbf{x}\| \|\mathbf{y}\| = \sqrt{n} \|\mathbf{x}\|$. Choose $\mathbf{x} = (1, 1, \dots, 1)$ for maximum ratios \sqrt{n} .
- 17** The largest component $|(x+y)_i| = \|\mathbf{x} + \mathbf{y}\|_\infty$ is not larger than $|x_i| + |y_i| \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$. The sum of absolute values $|(x+y)_i|$ is not larger than the sum of $|x_i| + |y_i|$. Therefore $\|\mathbf{x} + \mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$.
- 18** $|x_1| + 2|x_2|$ is a norm; $\min |x_i|$ is not a norm; $\|\mathbf{x}\| + \|\mathbf{x}\|_\infty$ is a norm; $\|A\mathbf{x}\|$ is a norm provided A is invertible (otherwise a nonzero vector has norm zero; for rectangular A we require independent columns).

Problem Set 9.3, page 417

- 1** $S = I$ and $T = I - A$ and $S^{-1}T = I - A$.
- 2** If $A\mathbf{x} = \lambda\mathbf{x}$ then $(I - A)\mathbf{x} = (1 - \lambda)\mathbf{x}$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.
- 3** This matrix A has $I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which has $|\lambda| = 2$.
- 4** Always $\|AB\| \leq \|A\| \|B\|$. Choose $A = B$ to find $\|B^2\| \leq \|B\|^2$. Then choose $A = B^2$ to find $\|B^3\| \leq \|B^2\| \|B\| \leq \|B\|^3$. Continue (or use induction). Since $\|B\| \geq \max |\lambda(B)|$ it is no surprise that $\|B\| < 1$ gives convergence.
- 5** $A\mathbf{x} = \mathbf{0}$ gives $(S - T)\mathbf{x} = \mathbf{0}$. Then $S\mathbf{x} = T\mathbf{x}$ and $S^{-1}T\mathbf{x} = \mathbf{x}$. Then $\lambda = 1$ means that the errors do not approach zero.
- 6** Jacobi has $S^{-1}T = \frac{1}{3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$.
- 7** Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9} = (|\lambda|_{\max} \text{ for Jacobi})^2$.

- 8 Jacobi has $S^{-1}T = \begin{bmatrix} a & \\ & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}$ with $|\lambda| = |bc/ad|^{1/2}$. Gauss-Seidel has $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & -bc/ad \end{bmatrix}$ with $|\lambda| = |bc/ad|$.
- 9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07.
- 11 If the iteration gives all $x_i^{\text{new}} = x_i^{\text{old}}$ then the quantity in parentheses is zero, which means $A\mathbf{x} = \mathbf{b}$. For Jacobi change the whole right side to x^{old} .
- 13 $\mathbf{u}_k/\lambda_1^k = c_1\mathbf{x}_1 + c_2\mathbf{x}_2(\lambda_2/\lambda_1)^k + \cdots + c_n\mathbf{x}_n(\lambda_n/\lambda_1)^k \rightarrow c_1\mathbf{x}_1$ if all ratios $|\lambda_i/\lambda_1| < 1$. The largest ratio controls, when k is large. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $|\lambda_2| = |\lambda_1|$ and no convergence.
- 14 The eigenvectors of A and also A^{-1} are $\mathbf{x}_1 = (.75, .25)$ and $\mathbf{x}_2 = (1, -1)$. The inverse power method converges to a multiple of \mathbf{x}_2 .
- 15 The j th component of $A\mathbf{x}_1$ is $2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms, using $\sin(a+b) = \sin a \cos b + \cos a \sin b$, combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. The eigenvalue is $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.
- 16 $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$ is converging to the eigenvector direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_{\text{max}} = 3$.
- 17 $A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{9} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{u}_3 = \frac{1}{27} \begin{bmatrix} 14 \\ 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- 18 $R = Q^T A = \begin{bmatrix} 1 & \cos\theta \sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1 + \sin^2\theta) & -\sin^3\theta \\ -\sin^3\theta & -\cos\theta \sin^2\theta \end{bmatrix}$.
- 19 If A is orthogonal then $Q = A$ and $R = I$. Therefore $A_1 = RQ = A$ again.
- 20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues from A to A_1 .
- 21 Multiply $A\mathbf{q}_j = b_{j-1}\mathbf{q}_{j-1} + a_j\mathbf{q}_j + b_j\mathbf{q}_{j+1}$ by \mathbf{q}_j^T to find $\mathbf{q}_j^T A\mathbf{q}_j = a_j$ (because the \mathbf{q} 's are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. Its entries are the a 's and b 's.
- 22 Theoretically the \mathbf{q} 's are orthonormal. In reality this algorithm is not very stable. We must stop every few steps to reorthogonalize.
- 23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^T AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A . If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.
- 24 The proof of $|\lambda| < 1$ when every absolute row sum < 1 uses $|\sum a_{ij}x_j| \leq \sum |a_{ij}||x_j| < |x_i|$. (Note $|x_i| \geq |x_j|$.) The Gershgorin circle theorem (very useful) is proved after its statement.

- 25** The maximum row sums give all $|\lambda| \leq .9$ and $|\lambda| \leq 3$. The circles around diagonal entries give tighter bounds. The circle $|\lambda - .2| \leq .7$ contains the other circles $|\lambda - .3| \leq .5$ and $|\lambda - .1| \leq .6$ and all three eigenvalues. The circle $|\lambda - 2| \leq 2$ contains the circle $|\lambda - 2| \leq 1$ and all three eigenvalues $2 + \sqrt{2}$, 2 , and $2 - \sqrt{2}$.
- 26** The circles $|\lambda - a_{ii}| \leq r_i$ don't include $\lambda = 0$ (so A is invertible!) when $a_{ii} > r_i$.
- 27** From the last line of code, \mathbf{q}_2 is in the direction of $\mathbf{v} = A\mathbf{q}_1 - h_{11}\mathbf{q}_1 = A\mathbf{q}_1 - (\mathbf{q}_1^T A \mathbf{q}_1)\mathbf{q}_1$. The dot product with \mathbf{q}_1 is zero. This is Gram-Schmidt with $A\mathbf{q}_1$ as the second input vector.
- 28** $\mathbf{r}_1 = \mathbf{b} - \alpha_1 A\mathbf{b} = \mathbf{b} - (\mathbf{b}^T \mathbf{b} / \mathbf{b}^T A\mathbf{b}) A\mathbf{b}$ is orthogonal to $\mathbf{r}_0 = \mathbf{b}$: the residuals $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ are orthogonal at each step. To show that \mathbf{p}_1 is orthogonal to $A\mathbf{p}_0 = A\mathbf{b}$, simplify \mathbf{p}_1 to $c\mathbf{P}_1$: $\mathbf{P}_1 = \|A\mathbf{b}\|^2 \mathbf{b} - (\mathbf{b}^T A\mathbf{b}) A\mathbf{b}$ and $c = \mathbf{b}^T \mathbf{b} / (\mathbf{b}^T A\mathbf{b})^2$. Certainly $(A\mathbf{b})^T \mathbf{P}_1 = 0$ because $A^T = A$. (That simplification put α_1 into $\mathbf{p}_1 = \mathbf{b} - \alpha_1 A\mathbf{b} + (\mathbf{b}^T \mathbf{b} - 2\alpha_1 \mathbf{b}^T A\mathbf{b} + \alpha_1^2 \|A\mathbf{b}\|^2) \mathbf{b} / \mathbf{b}^T \mathbf{b}$. For a good discussion see *Numerical Linear Algebra* by Trefethen and Bau.)

Problem Set 10.1, page 427

- 1** Sums 4 , $-2 + 2i$, $2 \cos \theta$; products 5 , $-2i$, 1 .
- 2** In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.
- 3** Absolute values $r = 10$, 100 , $\frac{1}{10}$, 100 ; angles θ , 2θ , $-\theta$, -2θ .
- 4** $|z \times w| = 6$, $|z + w| \leq 5$, $|z/w| = \frac{2}{3}$, $|z - w| \leq 5$.
- 5** $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, i , $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.
- 6** $1/z$ has absolute value $1/r$ and angle $-\theta$; $\frac{1}{r}e^{-i\theta}$ times $re^{i\theta} = 1$.
- 7**
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ bc + ad \end{bmatrix} \quad \begin{array}{l} \text{real part} \\ \text{imaginary part} \end{array}$$
- 8**
$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$
- 9** $2 + i$; $(2 + i)(1 + i) = 1 + 3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = (-i)^3 = i$.
- 10** $z + \bar{z}$ is real; $z - \bar{z}$ is pure imaginary; $z\bar{z}$ is positive; z/\bar{z} has absolute value 1 .
- 11** If $a_{ij} = i - j$ then $\det(A - \lambda I) = -\lambda^3 - 6\lambda = 0$ gives $\lambda = 0$, $\sqrt{6}i$, $-\sqrt{6}i$ (the conjugate of $\sqrt{6}i$).
- 12** (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ 's can be real and different.
- 13** Complex λ 's when $(a + d)^2 < 4(ad - bc)$; write $(a + d)^2 - 4(ad - bc)$ as $(a - d)^2 + 4bc$ which is positive when $bc > 0$.
- 14** $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1, 1, 1, 1)$ and $(1, -1, 1, -1)$ and $(1, i, -1, -i)$ and $(1, -i, -1, i)$ = columns of Fourier matrix.
- 15** $\det(P_6 - \lambda I) = \lambda^6 - 1 = 0$ when $\lambda = 1, w, w^2, w^3, w^4, w^5$ with $w = e^{2\pi i/6}$ as in Figure 10.3.
- 16** The block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

- 17 (a) $2e^{i\pi/3}, 4e^{2i\pi/3}$ (b) $e^{2i\theta}, e^{4i\theta}$
 (c) $73^{3\pi i/2}, 49e^{3\pi i} (= -49), \sqrt{50}e^{-\pi i/4}, 50e^{-\pi i/2}$.
- 18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.
- 19 $a + ib = 1, i, -1, -i, \pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$.
- 20 $1, e^{2\pi i/3}, e^{4\pi i/3}; -1, e^{\pi i/3}, e^{-\pi i/3}; 1$.
- 21 $\cos 3\theta = \operatorname{Re}(\cos \theta + i \sin \theta)^3 = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$; $\sin 3\theta = \operatorname{Im}(\cos \theta + i \sin \theta)^3 = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$.
- 22 If $\bar{z} = 1/z$ then $|z|^2 = 1$ and z is any point $e^{i\theta}$ on the unit circle.
- 23 (a) e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e = 1$ (c) There are infinitely many candidates $i^e = e^{i(\pi/2 + 2\pi n)e}$.
- 24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 436

- 1 $\|u\| = \sqrt{9} = 3, \|v\| = \sqrt{3}, u^H v = 3i + 2, v^H u = -3i + 2$ (conjugate of $u^H v$).
- 2 $A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are Hermitian matrices.
- 3 $z = \text{multiple of } (1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^H A^H = \mathbf{0}^H$ so z (not \bar{z} !) is orthogonal to all columns of A^H (using complex inner product z^H times column).
- 4 The four fundamental subspaces are $C(A), N(A), C(A^H), N(A^H)$.
- 5 (a) $(A^H A)^H = A^H A^{HH} = A^H A$ again (b) If $A^H A z = \mathbf{0}$ then $(z^H A^H)(Az) = 0$. This is $\|Az\|^2 = 0$ so $Az = \mathbf{0}$. The nullspaces of A and $A^H A$ are the *same*. $A^H A$ is invertible when $N(A) = \{\mathbf{0}\}$.
- 6 (a) False: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue if $A = A^H$ (c) False.
- 7 cA is still Hermitian for real c ; $(iA)^H = -iA^H = -iA$ is skew-Hermitian.
- 8 Orthogonal, invertible, unitary, factorizable into QR .
- 9 $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P^3 = I, P^{100} = P^{99}P = P; \lambda = \text{cube roots of } 1 = 1, e^{2\pi i/3}, e^{4\pi i/3}$.
- 10 $(1, 1, 1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore unitary.
- 11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2+5P+4P^2$ has $\lambda = 2+5+4 = 11, 2+5e^{2\pi i/3}+4e^{4\pi i/3}, 2+5e^{4\pi i/3}+4e^{8\pi i/3}$.

- 12** If $U^H U = I$ then $U^{-1}(U^H)^{-1} = U^{-1}(U^{-1})^H = I$ so U^{-1} is also unitary.
Also $(UV)^H(UV) = V^H U^H UV = V^H V = I$ so UV is unitary.
- 13** The determinant is the product of the eigenvalues (all real).
- 14** $(\mathbf{z}^H A^H)(A\mathbf{z}) = \|A\mathbf{z}\|^2$ is positive unless $A\mathbf{z} = \mathbf{0}$; with independent columns this means $\mathbf{z} = \mathbf{0}$;
so $A^H A$ is positive definite.
- 15** $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}.$
- 16** $K = (iA^T \text{ in Problem 15}) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix} \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix};$
 λ 's are imaginary.
- 17** $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.
- 18** $V = \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6 + 2\sqrt{3}$ has $|\lambda| = 1$.
 $V = V^H$ gives real λ , trace zero gives $\lambda = 1, -1$.
- 19** The \mathbf{v} 's are columns of a unitary matrix U . Then $\mathbf{z} = UU^H \mathbf{z} =$ (multiply by columns)
 $= \mathbf{v}_1(\mathbf{v}_1^H \mathbf{z}) + \cdots + \mathbf{v}_n(\mathbf{v}_n^H \mathbf{z})$.
- 20** Don't multiply e^{-ix} times e^{ix} ; conjugate the first, then $\int_0^{2\pi} e^{2ix} dx = [e^{2ix}/2i]_0^{2\pi} = 0$.
- 21** $\mathbf{z} = (1, i, -2)$ completes an orthogonal basis for \mathbf{C}^3 .
- 22** $R + iS = (R + iS)^H = R^T - iS^T$; R is symmetric but S is skew-symmetric.
- 23** \mathbf{C}^n has dimension n ; the columns of any unitary matrix are a basis: $(i, 0, \dots, 0), \dots,$
 $(0, \dots, 0, i)$
- 24** $[1]$ and $[-1]$; any $[e^{i\theta}]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\bar{z} \\ -z & e^{i\phi}\bar{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$.
- 25** Eigenvalues of A^H are complex conjugates of eigenvalues of A : $\det(A - \lambda I) = 0$ gives $\det(A^H - \bar{\lambda}I) = 0$.
- 26** $(I - 2\mathbf{u}\mathbf{u}^H)^H = I - 2\mathbf{u}\mathbf{u}^H$; $(I - 2\mathbf{u}\mathbf{u}^H)^2 = I - 4\mathbf{u}\mathbf{u}^H + 4\mathbf{u}(\mathbf{u}^H \mathbf{u})\mathbf{u}^H = I$; the matrix $\mathbf{u}\mathbf{u}^H$
projects onto the line through \mathbf{u} .
- 27** Unitary means $U^H U = I$ or $(A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A) = I$. Then
 $A^T A + B^T B = I$ and $A^T B - B^T A = 0$ which makes the block matrix orthogonal.
- 28** We are given $A + iB = (A + iB)^H = A^T - iB^T$. Then $A = A^T$ and $B = -B^T$.
- 29** $AA^{-1} = I$ gives $(A^{-1})^H A^H = I$. Therefore $(A^{-1})^H = (A^H)^{-1} = A^{-1}$ and A^{-1} is Hermitian.
- 30** $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}.$

Problem Set 10.3, page 444

1 Equation (3) is correct using $i^2 = -1$ in the last two rows and three columns.

$$2 \quad F^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & & \\ & 1 & i^2 & \\ & & 1 & 1 \\ & & & 1 & i^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} = \frac{1}{4} F^H.$$

$$3 \quad F = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & i^2 & \\ & & 1 & 1 \\ & & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & i & & -i \end{bmatrix}.$$

$$4 \quad D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix} \text{ and } F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

5 $F^{-1}\mathbf{w} = \mathbf{v}$ and $F^{-1}\mathbf{v} = \frac{1}{4}\mathbf{w}$.

$$6 \quad (F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \text{ and } (F_4)^4 = 16I.$$

$$7 \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = F\mathbf{c}; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

8 $\mathbf{c} \rightarrow (1, 1, 1, 1, 0, 0, 0, 0) \rightarrow (4, 0, 0, 0, 0, 0, 0, 0) \rightarrow (4, 0, 0, 0, 4, 0, 0, 0)$ which is $F_8\mathbf{c}$. The second vector becomes $(0, 0, 0, 0, 1, 1, 1, 1) \rightarrow (0, 0, 0, 0, 4, 0, 0, 0) \rightarrow (4, 0, 0, 0, -4, 0, 0, 0)$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and \sqrt{w} is a 128th root of 1.

10 For every integer n , the n th roots of 1 add to zero.

11 The eigenvalues of P are 1, i , $i^2 = -1$, and $i^3 = -i$.

$$12 \quad \Lambda = \text{diag}(1, i, i^2, i^3); \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^T \text{ lead to } \lambda^3 - 1 = 0.$$

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1i + c_2i^2 + c_3i^3$; E contains the four eigenvalues of C .

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n \log_2 n$ multiplications each by the **FFT**. Total much less than the ordinary n^2 .

16 $(c_0 + c_2) + (c_1 + c_3)$; then $(c_0 - c_2) + i(c_1 - c_3)$; then $(c_0 + c_2) - (c_1 + c_3)$; then $(c_0 - c_2) - i(c_1 - c_3)$. These steps are the **FFT**!