Analytical Geometry and Linear Algebra II, Lab 1

The geometry of linear equation

Gaussian elimination

Matrix notation and matrix multiplication

Task 1

Warm up

Rewrite the following system in the matrix form:

$$\begin{cases} 3x + 4y - 2z = 1 \\ 3y - 2z + x = -2 \\ 5x - 7z - 2y = 3 \end{cases}$$

Task 2

Describe geometrically (line, plane, or whole space) all linear combinations of:

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$
3. $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Task 2

Describe geometrically (line, plane, or whole space) all linear combinations of:

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ *Ans*: line
2. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$
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Task 3

Draw $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $v + w$, $v - w$ in a single xy plane

Task 4

Explain, why the system is singular?

$$\begin{cases} u+v+w=2 \\ u+2v+3w=1 \\ v+2w=0 \end{cases}$$

What value should replace that last zero on the right side, to allow the equations to have solutions, and what is one of the solutions?

What does it mean, singular?

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Example 2. Singular (incurable)

$$\begin{cases} u+v+w= \\ 2u+2v+5w= \\ 4u+4v+8w= \end{cases} \Rightarrow \begin{cases} u+v+w= \\ 3w= \\ 4w= \end{cases}$$

There is no exchange of equations that can avoid zero in the second pivot position. The equations themselves may be solvable or unsolvable. If the last two equations are $3w=6$ and $4w=7$, there is no solution. If those two equations happen to be consistent as in $3w=6$ and $4w=8$ then this singular case has an infinity of solutions. We know that $w=2$, but the first equation cannot decide both u and v.

What does it mean, pivot point?

Definition

The pivot or pivot element is the element of a matrix, or an array, which is selected first on a particular step by an algorithm (e.g. Gaussian elimination, simplex algorithm, etc.), to do certain calculations.

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$$\begin{cases} u + v + w = 2 \\ u + 2v + 3w = 1 \\ v + 2w = 0 \end{cases}$$

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Task 5

1. Choose a coefficient b that makes this system singular.
2. Then choose a right-hand side g that makes it solvable.
3. Find two solutions in that singular case.

$$\begin{cases} 2x + by = 16 \\ 4x + 8y = g \end{cases}$$

Task 6

Give 3x3 examples (not just the zero matrix):

1. a diagonal matrix: $a_{ij} = 0$, if $i \neq j$;
2. a symmetric matrix: $a_{ij} = a_{ji}$ for all i and j;
3. an upper triangular matrix: $a_{ij} = 0$, if $i > j$;
4. a skew-symmetric matrix: $a_{ij} = -a_{ji}$ for all i and j.

Task 7

Obtain a Reduced Row Echelon Form (rref) of

$$\begin{cases} 2u + 3v + 0w = 0 \\ 4u + 5v + w = 3 \\ 2u - 1v - 3w = 5 \end{cases}$$

The difference between REF and RREF

Gaussian elimination vs Gauss-Jordan elimination

Gaussian elimination refers to the process until it has reached its upper triangular.

Gauss-Jordan elimination is the algorithm for converting a matrix into RREF.

The **Row Echelon Form** is *not unique*

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{\text{add row 2 to row 1}} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 7 \end{bmatrix}.$$

Every matrix has a *unique* **Reduced Row Echelon Form**

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{\text{subtract } 3 \times \text{(row 2) from row 1}} \begin{bmatrix} 1 & 0 & -22 \\ 0 & 1 & 7 \end{bmatrix}.$$

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Obtain a Reduced Row Echelon Form (rref) of

$$\begin{cases} 2u + 3v + 0w = 0 \\ 4u + 5v + w = 3 \\ 2u - 1v - 3w = 5 \end{cases}$$

$$Ans: \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Reference material

- Lectures 1 – 3

Deserve “A” grade!

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Room 105 (Underground robotics lab)