Analytical Geometry and Linear Algebra II, Lab 3

Null space

innopolis university

How to study Null Space

Step-by-step guide

1. Lecture 6, Gilbert Strang

   Goal is to understand the basics of spaces and how Null Space appeared.

2. Khan Academy: Null space

   It contains a good case study how to calculate Null space.

3. Matrix Algebra for Engineers: Null Space

   Another nice example how to find $N(A)$.

4. *"Linear Algebra and Applications"*, *pdf pages 96–106*

   What does partial and full solutions means

5. The Big Picture of Linear Algebra

   Extra for now If you want to get the global view of four subspaces

6. Understand the application from next few slides and make HW tasks!

Null Space: Application from robotics

Video

Null Space: Application from robotics

Theory (1)

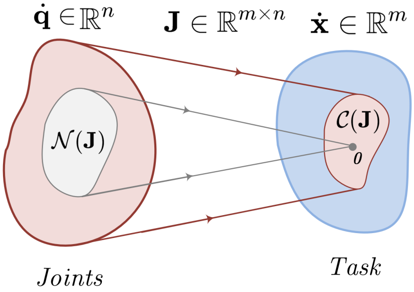

Figure 1: Click for google Collab

Let us consider differential kinematic relationship:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{1}$$

where

- $\mathbf{x} \in \mathbb{R}^m$ task space variables (for instance Cartesian coordinates)
- $\mathbf{q} \in \mathbb{R}^n$ joint space variables (positions of joints)
- $\mathbf{J} \in \mathbb{R}^{m \times n}$ manipulator Jacobian

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Theory (2)

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = \underbrace{J^{\#}\dot{r}}_{\text{a particular solution (here, the pseudoinverse) in } \mathcal{R}(J^T)} + \underbrace{\left(I - J^{\#}J\right)}_{\text{"orthogonal" projection of } \dot{q}_0 \text{ in } \mathcal{N}(J)}\dot{q}_0$$

all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

properties of projector $[I - J^{\#}J]$

- symmetric
- idempotent: $[I - J^{\#}J]^2 = [I - J^{\#}J]$
- $[I - J^{\#}J]^{\#} = [I - J^{\#}J]$
- $J^{\#}\dot{r}$ is orthogonal to $[I - J^{\#}J]\dot{q}_0$

even more in general...

$$\dot{q} = K_1\dot{r} + (I - K_2J)\dot{q}_0$$

... but with less nice properties!

K_1, K_2 generalized inverses of J $(JK_iJ = J)$

$\boxed{\text{how do we choose } \dot{q}_0\text{?}}$

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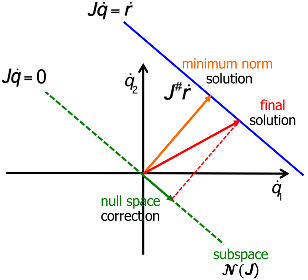
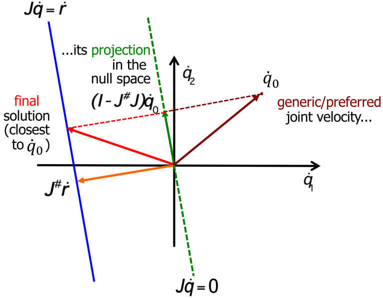
Theory (3)

in the space of velocity commands





a correction is added to the original pseudoinverse (minimum norm) solution

i) which is in the null space of the Jacobian

ii) and possibly satisfies additional criteria or objectives

Task 1

Reduce these matrices to their ordinary echelon forms U:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

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$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

Answer

(a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3

(b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free x_3 Pivot x_1, x_2

Task 2

Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

Task 2

Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

Answer

(a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I$, $R = I$.

Task 3

If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \text{ (empty 3 by 1).}$$

Task 3

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Answer

Any zero rows come after these rows: $R = [1 \quad -2 \quad -3]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

Reference material

- Robotics 2 course from Sapienza
- Gilbert Strang Book 2.1-2.2
- *"Linear Algebra and Applications", pdf pages 46–86*
- *"Introduction to Linear Algebra", pdf pages 42–134*
  2.2 – The Idea of Elimination, 2.6 – Elimination = Factorization:
  $A = LU$
- this lab video, 2022 year

Preparation material for the next class

TODO

Deserve “A” grade!

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Room 105 (Underground robotics lab)