Linear Algebra. Test 1: Solutions

1.1   The 4x4 matrix A reduces to the identity matrix I by the following four row operations:

- E_{21}: subtract row 1 from row 2;
- E_{31}: add 2*row 1 to row 3;
- E_{32}: subtract 3*row 2 from row 3;
- E_{41}: subtract 2*row 1 from row 4;
- E_{43}: add row 3 to row 4;

Write and compute A^{-1} (3 points) and A (2 points).

$$E_{43}E_{41}E_{32}E_{31}E_{21}A = I \quad \Rightarrow \quad A^{-1} = E_{43}E_{41}E_{32}E_{31}E_{21},$$

$$A^{-1} = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\-2&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&-3&1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\2&0&1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\-1&1&0&0\\5&-3&1&0\\3&-3&1&1 \end{pmatrix}$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{43}^{-1}$$

$$A = \begin{pmatrix} 1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\-2&0&1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&3&1&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\2&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\1&1&0&0\\-2&3&1&0\\2&0&-1&1 \end{pmatrix}$$

1.2   The 4x4 matrix A reduces to the identity matrix I by the following four row operations:

- E_{21}: add 2*row 1 to row 2;
- E_{31}: subtract row 1 from row 3;
- E_{32}: subtract 3*row 2 from row 3;
- E_{41}: add row 1 to row 4;
- E_{43}: subtract 2*row 3 from row 4;

Write and compute A^{-1} (3 points) and A (2 points).

$$E_{43}E_{41}E_{32}E_{31}E_{21}A = I \quad \Rightarrow \quad A^{-1} = E_{43}E_{41}E_{32}E_{31}E_{21},$$

$$A^{-1} = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-2&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&-3&1&0\\0&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\-1&0&1&0\\0&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\2&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\2&1&0&0\\-7&-3&1&0\\15&6&-2&1 \end{pmatrix}$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{43}^{-1}$$

$$A = \begin{pmatrix} 1&0&0&0\\-2&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&3&1&0\\0&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&0&1 \end{pmatrix}\cdot\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&2&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\-2&1&0&0\\1&3&1&0\\-1&0&2&1 \end{pmatrix}$$

2.1 Consider matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Find the factorization $A = LDU$ (3 points)
Find A inverse (2 points).

$$A = LDU = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\dfrac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -\dfrac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

2.2 Consider matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Find the factorization $A = LDU$ (3 points)
Find A inverse (2 points).

$$A = LDU = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & \dfrac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -\dfrac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \dfrac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -5 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -3 \end{bmatrix}$$

3.1 Consider matrix A

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Reduce the matrix A to its ordinary Echelon form U (1 point).

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a solution for each free variable and describe every solution to $Ax = 0$ (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \quad \vec{x}_{N1} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{x}_{N2} = c_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

For which nontrivial right-hand sides (find a condition on b_1, b_3, b_3, b_4) $Ax = b$ is solvable (1 point)?

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} \Rightarrow$$

Provide an example of vector $b \neq 0$ that makes this system solvable (1 point).

$$\Rightarrow b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$$

Find the complete solution for $Ax = b$ using b from previous part (3.d) (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_{N1} + \vec{x}_{N2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

3.2 Consider matrix A

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 1 & 4 & 3 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Reduce the matrix A to its ordinary Echelon form U (1 point).

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a solution for each free variable and describe every solution to $Ax = 0$ (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \vec{x}_{N1} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{x}_{N2} = c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

For which nontrivial right-hand sides (find a condition on b_1, b_3, b_3, b_4) $Ax = b$ is solvable (1 point)?

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_2 - b_1 \\ b_2 - b_1 \end{bmatrix} \Rightarrow$$

Provide an example of vector $b \ne 0$ that makes this system solvable (1 point).

$$\Rightarrow b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Find the complete solution for $Ax = b$ using b from previous part (3.d) (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_{N1} + \vec{x}_{N2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$