Linear Algebra. Test 2. Solutions

1. Find $det(e^A)$ for the matrix (5 points):

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\det(e^A) = \det(S)\det(e^\Lambda)\det(S^{-1}) = \det(e^\Lambda) = \det\begin{bmatrix} e^{-2} & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^0 \end{bmatrix} = e^{-1}$$

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\det(e^A) = \det(S)\det(e^\Lambda)\det(S^{-1}) = \det(e^\Lambda) = \det\begin{bmatrix} e^{-3} & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^0 \end{bmatrix} = e^{-2}$$

2. Write down the first order equation system for the following differential equation and solve it (4 points):

$$\begin{cases} y''' + y'' - 2y' = 0 \\ y''(0) = 2, \quad y'(0) = 0, \quad y(0) = 1 \end{cases}$$

Is the solution of this system will be stable? (1 points)

IF we introduce the vector $\vec{u}(t) = \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}$ the ODE becomes:

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}, \ \vec{u}(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\vec{u}(t) = Se^{\Lambda t}S^{-1}\vec{u}(0) = \frac{1}{6}\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 0 \\ -3 & -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$y(t) = \frac{e^{-2t}}{3} + \frac{2e^t}{3} \Rightarrow \lim_{t\to\infty} y(t) = \infty$$

$$\begin{cases} y''' + 2y'' - 3y' = 0 \\ y''(0) = 3, \quad y'(0) = 0, \quad y(0) = 1 \end{cases}$$

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y''(t) \\ y'(t) \\ y(t) \end{bmatrix}, \; \vec{u}(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\vec{u}(t) = Se^{\Lambda t}S^{-1}\vec{u}(0) = \frac{1}{12}\begin{bmatrix} 9 & 1 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 3 & 9 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$y(t) = \frac{\mathrm{e}^{-3t}}{4} + \frac{3\mathrm{e}^{t}}{4} \Rightarrow \lim_{t\to\infty} y(t) = \infty$$

3. Find the SVD (4 points) and the pseudoinverse (1 points) of the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 1 & 2 \end{bmatrix}, \; A = U\Sigma V^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = U\Lambda U^T = \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} =$$

$$= \frac{1}{9}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A^T A = V\Lambda V^T = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$A^+ = V\Sigma^+ U^T = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A=\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\end{bmatrix},\ A=U\Sigma V^T=\frac{1}{3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$$

$$A^+=V\Sigma^+U^T=\frac{1}{3}\begin{bmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{bmatrix}\left[\begin{bmatrix}1/3 & 0\\ 0 & 1/3\\ 0 & 0\end{bmatrix}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}2 & -2\\ 2 & 1\\ 1 & 2\end{bmatrix}$$