All these steps use the λ's and the $\boldsymbol{x}$'s. This section solves the constant coefficient problems that turn into linear algebra. It clarifies these simplest but most important differential equations—whose solution is completely based on growth factors $e^{\lambda t}$.

Second Order Equations

The most important equation in mechanics is $\boldsymbol{my'' + by' + ky = 0}$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (that is *Newton's Law*). The force includes the damping $-by'$ and the elastic force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative of y brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$\boldsymbol{m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0} \quad \text{becomes} \quad \boldsymbol{(m\lambda^2 + b\lambda + k)}\, e^{\lambda t} = \mathbf{0}. \tag{8}$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a *vector equation for y and y'*: first derivative only. Suppose the mass is $m = 1$. Two equations for $\boldsymbol{u} = (y,\ y')$ give $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} \boldsymbol{dy/dt} &= \boldsymbol{y'} \\ \boldsymbol{dy'/dt} &= \boldsymbol{-ky - by'} \end{aligned} \quad \text{converts to} \quad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \boldsymbol{-k} & \boldsymbol{-b} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{9}$$

The first equation $dy/dt = y'$ is trivial (but true). The second is equation (8) connecting y'' to y' and y. Together they connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve $\boldsymbol{u}' = A\boldsymbol{u}$ by eigenvalues of A:

$$\boxed{A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix}} \quad \text{has determinant} \quad \boxed{\lambda^2 + b\lambda + k = 0.}$$

The equation for the λ's **is the same as** (8)! **It is still $\boldsymbol{\lambda^2 + b\lambda + k = 0}$, since** $m = 1$. The roots λ_1 and λ_2 are now *eigenvalues of* A. The eigenvectors and the solution are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of $\boldsymbol{u}(t)$ has $y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$—the same solution as before. It can't be anything else. In the second component of $\boldsymbol{u}(t)$ you see the velocity dy/dt. The vector problem is completely consistent with the scalar problem. The 2 by 2 matrix A is called a *companion matrix*—a companion to the second order equation with y''.