Analytical Geometry and Linear Algebra II, Lab 11

Symmetric matrices

Positive definite matrices and minima

How I spent last weekend

Symmetric Matrices (1)

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $q_1, \ldots, q_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{\mathbf{T}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

Symmetric Matrices (2)

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

Symmetry $\quad S = X \Lambda X^{-1} \quad$ becomes $\quad S = Q \Lambda Q^{\mathrm{T}} \quad$ with $\quad Q^{\mathrm{T}} Q = I$.

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q \Lambda Q^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

Columns $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ multiply rows $\lambda_1 \boldsymbol{q}_1^{\mathrm{T}}$ and $\lambda_2 \boldsymbol{q}_2^{\mathrm{T}}$ to produce $S = \lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{q}_2 \boldsymbol{q}_2^{\mathrm{T}}$.

Write A as $S + N$, symmetric matrix S plus skew-symmetric matrix N:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \qquad (S^T = S \text{ and } N^T = -N).$$

For any square matrix, $S = \frac{1}{2}(A + A^T)$ and $N = $ _____ add up to A.

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For any square matrix, $S = \frac{1}{2}(A + A^{\mathrm{T}})$ and $N = $ _____ add up to A.

Answer

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \begin{aligned} &= \tfrac{1}{2}(A + A^{\mathrm{T}}) + \tfrac{1}{2}(A - A^{\mathrm{T}}) \\ &= \textbf{symmetric} + \textbf{skew-symmetric}. \end{aligned}$$

Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

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Answer

$\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to

unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Task 3

True (with reason) *or false* (with example).

(a) A matrix with real eigenvalues and n real eigenvectors is symmetric.

(b) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.

(c) The inverse of an invertible symmetric matrix is symmetric.

(d) The eigenvector matrix Q of a symmetric matrix is symmetric.

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Answer

(a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) True from $A^T = Q \Lambda Q^T = A$

(c) True from $S^{-1} = Q \Lambda^{-1} Q^T$

(d) False!

Positive Definite Matrices

This section concentrates on ***symmetric matrices that have positive eigenvalues***. If symmetry makes a matrix important, this extra property (*all* $\lambda > 0$) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called ***positive definite***.

Positive Definite Matrices
Applications from ML

- Cholesky decomposition – $A = LDL^H$ (A special case of $A = LU$)
- Least squares computation reduction
- Support Vector Machine (SVM), *kernel* – Positive-definite kernel
- Representer Theorem

# Positive Definite Matrices
Five tests

Positive definite matrices are the best. How to test S for $\lambda_i > 0$?

Test **1**   Compute the **eigenvalues** of S: All eigenvalues positive

Test **2**   The **energy** $x^{\mathbf{T}} S x$ is positive for every vector $x \neq \mathbf{0}$

Test **3**   The **pivots** in elimination on S are all positive

Test 4   The upper left **determinants** of S are all positive

Test 5 $\boldsymbol{S} = \boldsymbol{A}^{\mathbf{T}} \boldsymbol{A}$ for some matrix A with independent columns
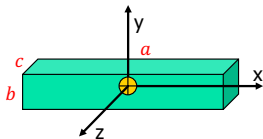
Dynamics of an actuated pendulum
a first example
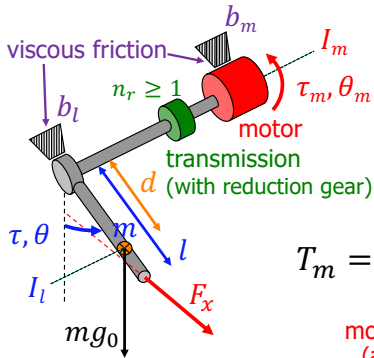
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2$$
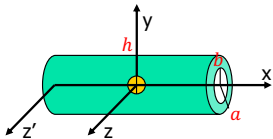
motor inertia
(around its
spinning axis)

Examples of body inertia matrices
homogeneous bodies of mass m, with axes of symmetry

parallelepiped with sides
a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$

empty cylinder with length h,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)+h^2) & \\ & & I_{zz} \end{pmatrix}$$

$I_{zz} = I_{yy}$

$$= \frac{1}{2} m \, v_c^T v_c$$

translational
kinetic energy
(point mass
at CoM)

+

rotational
kinetic energy
(of the whole body)

$$= \frac{1}{2} \omega^T I_c \, \omega$$

body inertia matrix
(around the CoM)

Test S for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Positive Definite Matrices

Solution: Test 1, 3, 4

The pivots of S are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of S are $2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**.

Solution: Test 5

A_2 comes from $S = LDL^{\mathrm{T}}$ (the symmetric version of $S = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $A_2 = L\sqrt{D}$.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = A_2^{\mathrm{T}} A_2.$$

A_2 is the Cholesky factor of S

Eigenvalues give the symmetric choice $A_3 = Q\sqrt{\Lambda}Q^{\mathrm{T}}$. This is also successful with $A_3^{\mathrm{T}} A_3 = Q\Lambda Q^{\mathrm{T}} = S$. All tests show that the $-1, 2, -1$ matrix S is positive definite.

Positive Definite Matrices

Solution: Test 2

$$\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \qquad \textbf{Rewrite with squares}$$

$$\|A_2\boldsymbol{x}\|^2 = 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}\,x_3^2 \qquad \textbf{Using } S = LDL^{\mathbf{T}}$$

$$\|A_3\boldsymbol{x}\|^2 = \lambda_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_3(\boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{x})^2 \qquad \textbf{Using } S = Q\Lambda Q^{\mathbf{T}}$$

Task 4

What is the function $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write each f as a sum of one or two squares $f = d_1(\quad)^2 + d_2(\quad)^2$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \; S \; \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Answer

$$f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2; \quad x^2 + 6xy + 9y^2 = (x + 3y)^2.$$

For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Task 5

For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Answer

S is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c - 1)^2(c + 2) > 0$.

T is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Sx > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad, \quad, \quad).$$

Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $x^T S x > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad, \quad, \quad).$$

Answer

$x^T S x$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $x^T S x$ goes *negative* for $x = (1, -10, 0)$ because the second pivot is *negative*.
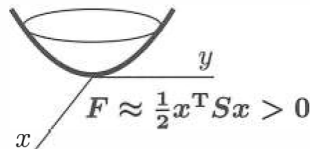
Positive Definite Matrices
Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S. It contains *four second derivatives*. **Positive d^2f/dx^2 changes to positive definite S:**

Second derivatives
$$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$

$F \approx \frac{1}{2} x^{\mathrm{T}} S x > 0$

$F(x, y)$ has a minimum if $\partial F/\partial x = \partial F/\partial y = 0$ and S is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

For $F_1(x, y) = \frac{1}{4}x^4 + x^2 y + y^2$ and $F_2(x, y) = x^3 + xy - x$ find the second derivative matrices S_1 and S_2:

Test for minimum $\quad S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$ is positive definite

S_1 is positive definite so F_1 is concave up ($=$ convex). Find the minimum point of F_1.
Find the saddle point of F_2 (look only where first derivatives are zero).

Answer

$S_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is semidefinite; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$;

$S_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite at $(0, 1)$ where first derivatives $= 0$. Then

$x = 0, y = 1$ is a saddle point of the function $f_2(x, y)$.

Reference material

- Lecture 28, Positive Definite Matrices and Minima

- 5. Positive Definite and Semidefinite Matrices

- *"Introduction to Linear Algebra", pdf pages 349–374*
  6.4 – Symmetric, 6.5 – Positive Definite matrices

- *"Linear Algebra and Applications", pdf pages 355–376*
  Positive Definite Matrices 6.1, 6.2

Deserve "A" grade!

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🚪 Room 105 (Underground robotics lab)