# Linear Algebra. Retake exam.

1. If a 4x4 matrix has a $det(A)=1/2$, find $det(2A)$, $det(-A)$, $det(A^2)$, and $det(A^{-1})$. (**10** points)

   $det(2A) = 2^3 = 8$

   $det(-A) = 1/2$

   $det(A^2) = 1/4$

   $det(A^{-1}) = 2$

2. Consider matrix: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}$. Find the symmetric factorization $A = LDL^T$, Find $A^{-1}$ (**10** points)

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 25 & -11 & 4 \\ -11 & 11 & -2 \\ 4 & -2 & 1 \end{bmatrix}$$

3. Find a parabola that best fits to the following points:

   (-1, 2), (0, 0), (1, -3), (2, - 5). (**10** points)

$$y(x) = a + bx + cx^2$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} -6 \\ -15 \\ -21 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3/10 \\ -12/5 \\ 0 \end{bmatrix}$$

$$y(x) = -\frac{3}{10} - \frac{12}{5} x$$

4. Let $S_1 = \{x, y, z : x - 5y + 8z = 11\}$ and $S_2 = \{x, y, z : x - y = 1\}$.

   $\vec{v} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$ – lies at the line of intersection of the planes $S_1$ and $S_2$. Find $\vec{v}$. (**10** points).

$$\begin{cases} y = x - 1 \\ z = \frac{1}{8}(11 + 5y - x) = \frac{3}{4} + \frac{1}{2} x \end{cases} \Rightarrow \vec{v} = x \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 3/4 \end{bmatrix}$$

5. Prove that for any square matrix $A$ ($n$x$n$) with eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ the multiplication: $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ produces the zero matrix? (**12** points).

$$(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I) = S(\Lambda-\lambda_1 I)S^{-1}S(\Lambda-\lambda_2 I)S^{-1}\cdots S(\Lambda-\lambda_n I)S^{-1} =$$

$$= S(\Lambda-\lambda_1 I)(\Lambda-\lambda_2 I)\cdots(\Lambda-\lambda_n I)S^{-1} =$$

$$= S\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \lambda_2-\lambda_1 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n-\lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1-\lambda_2 & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n-\lambda_2 \end{bmatrix}\cdots\begin{bmatrix} \lambda_1-\lambda_n & \cdots & \cdots & 0 \\ \vdots & \lambda_2-\lambda_n & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}S^{-1} =$$

$$= S\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}S^{-1} = [0]$$

6. Find $A^{10}$ for the matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. (**10** points).

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \Rightarrow$$

$$A^{10} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^{10} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 7324219 & 7324218 \\ 2441406 & 2441407 \end{bmatrix}$$

7. Find eigenvector of the circulant matrix $C$ for the eigenvalue $\lambda = c_1 + c_2 + c_3 + c_4$. (**12** points)

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} -c_2-c_3-c_4 & c_2 & c_3 & c_4 \\ c_4 & -c_2-c_3-c_4 & c_2 & c_3 \\ c_3 & c_4 & -c_2-c_3-c_4 & c_2 \\ c_2 & c_3 & c_4 & -c_2-c_3-c_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

8. Solve the differential equation, (**10** points):

$$\frac{d\vec{u}}{dt}=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}\vec{u}(t),\quad \vec{u}(0)=\begin{bmatrix}3\\ 2\end{bmatrix}$$

$$\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\Rightarrow$$

$$\vec{u}(t)=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{3t} & 0\\ 0 & e^{2t}\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}=\begin{bmatrix}e^{2t}+2e^{3t}\\ e^{2t}+e^{3t}\end{bmatrix}$$

What happens to $\vec{u}(t)$ as $t\to\infty$ (**2** points).

$$\vec{u}(t)=\begin{bmatrix}e^{2t}+2e^{3t}\\ e^{2t}+e^{3t}\end{bmatrix}\xrightarrow{t\to\infty}\infty$$

9. Apply the Gram-Schmidt process to: $x_1=\begin{bmatrix}0\\ 0\\ 2\\ 0\end{bmatrix}$, $x_2=\begin{bmatrix}-1\\ 0\\ 1\\ 0\end{bmatrix}$, $x_3=\begin{bmatrix}2\\ 1\\ 0\\ 0\end{bmatrix}$, $x_4=\begin{bmatrix}1\\ 0\\ 0\\ -1\end{bmatrix}$ and write the result in form $A=QR$ (**12** points)

$$A=\begin{bmatrix}0 & -1 & 2 & 1\\ 0 & 0 & 1 & 0\\ 2 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 1 & -2 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=QR$$

10. Find the SVD and the pseudoinverse of the matrix: $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$ (**10** points).

$$AA^T=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},\text{ with eigenvalues }\lambda_1=2\text{ and }\lambda_2=1$$

$$A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}=U\Sigma V^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & 1\end{bmatrix}$$

$$\text{Pseudoinverse }\; A^+=V\Sigma U^T=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\\ \frac{1}{2} & 0\end{bmatrix}$$