Analytical Geometry and Linear Algebra II, Lab 6

Orthogonality + OrthoNormality + SO(3)

Gram-Schmidt

Preparation to midterm

Naming problem

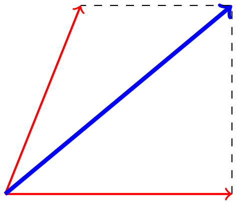
Problem statement

Orthogonality and *OrthoNormality* are vague term.

1. In Wiki they have the same meaning.
2. According to Kholodov, OrthoNormal – rectangular orthogonal matrix with normalized vectors. Orthogonal – squared normalized matrix.
3. According to Matlab documentation (skipnormalization param.) and Quora – orthogonal – rectangular matrix without normalized vectors, orthoNormal – with them.

I decide to follow ***3rd notation***.

Orthogonality + OrthoNormality



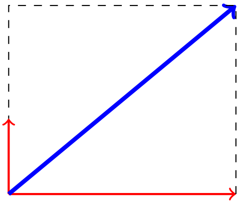
Common

$$a \cdot b \neq 0$$

$$b \cdot c \neq 0$$

$$a \cdot c \neq 0$$

$$\|a\|,\ \|b\|,\ |c\| \neq 1$$



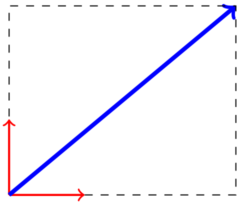
Orthogonal

$$a \cdot b = 0$$

$$b \cdot c = 0$$

$$a \cdot c = 0$$

$$\|a\|,\ \|b\|,\ |c\| \neq 1$$



OrthoNormal

$$a \cdot b = 0$$

$$b \cdot c = 0$$

$$a \cdot c = 0$$

$$\|a\|,\ \|b\|,\ |c\| = 1$$

$$det = \pm 1$$

Properties of orthonormal matrices

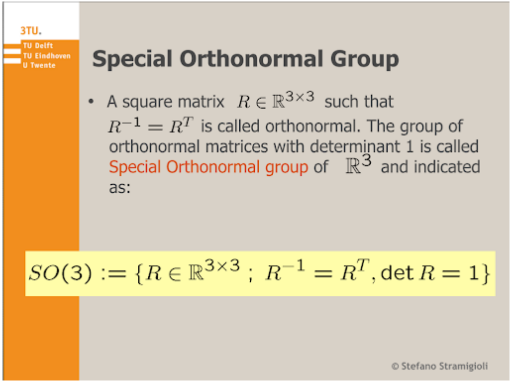
(if they are square)

$Q^T = Q^{-1}$, case study → rotation matrix

Orthonormal matrix → $Q^T Q = I$

Not normalized Q → $N^T N = Diag$

Special orthonormal group (SO(n)) group



3TU.
TU Delft
TU Eindhoven
U Twente

Special Orthonormal Group

- A square matrix $R \in \mathbb{R}^{3\times 3}$ such that $R^{-1} = R^T$ is called orthonormal. The group of orthonormal matrices with determinant 1 is called Special Orthonormal group of $\mathbb{R}^3$ and indicated as:

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} \; ; \; R^{-1} = R^T, \det R = 1\}$$

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Modern robotics course, 1 lec (math background, LA + phys + advanced stuff for self education)

Gram-Schmidt

This approach provides to **change** our common basis to orthonormal basis (our basis will have another vectors, but it represents the same (sub)space).

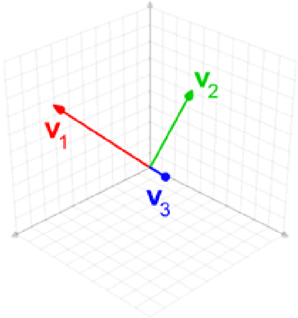
If we **had vectors in old basis before**, we should find the linear combination values again **by solving system of linear equations**.

Application: $A = QR$ Factorization. The factorization simplifies least squares solution.

Concept: The concept of orthogonalization uses Principle Component Analysis (PCA) algorithm.

Gram-Schmidt

Algorithm



Lab 6

Gram - Schmidt

```
zero = [0;0;0];
```

Define our basis

```
a = [2;1;0]
```

```
a = 3×1
     2
     1
     0
```

```
b = [0;0;4]
```

```
b = 3×1
     0
     0
     4
```

```
c = [2.5;3;0]
```

```
c = 3×1
    2.5000
    3.0000
         0
```

Current basis

```
Old_BASIS = rats([a b c])
```

```
Old_BASIS = 3×42 char array
    '       2              0             5/2      '
    '       1              0              3       '
    '       0              4              0       '
```

```
% det_old_bas = det([a b c])
```

Let's check, is it orthogonal?

```
dot(a,b)
```

```
ans = 0
```

```
dot(b,c)
```

```
ans = 0
```

```
dot(a,c)
```

```
ans = 8
```

As it can be seen, it is not (it should be equal to 0)

For checking the concepts, let's consider some vector, which will be a linear combination of our basis.

```
new_vec_el = [1;2;3];
new_vec_old_bas = new_vec_el(1)*a + new_vec_el(2)*b + new_vec_el(3)*c
```

```
new_vec_old_bas = 3×1
    9.5000
   10.0000
    8.0000
```

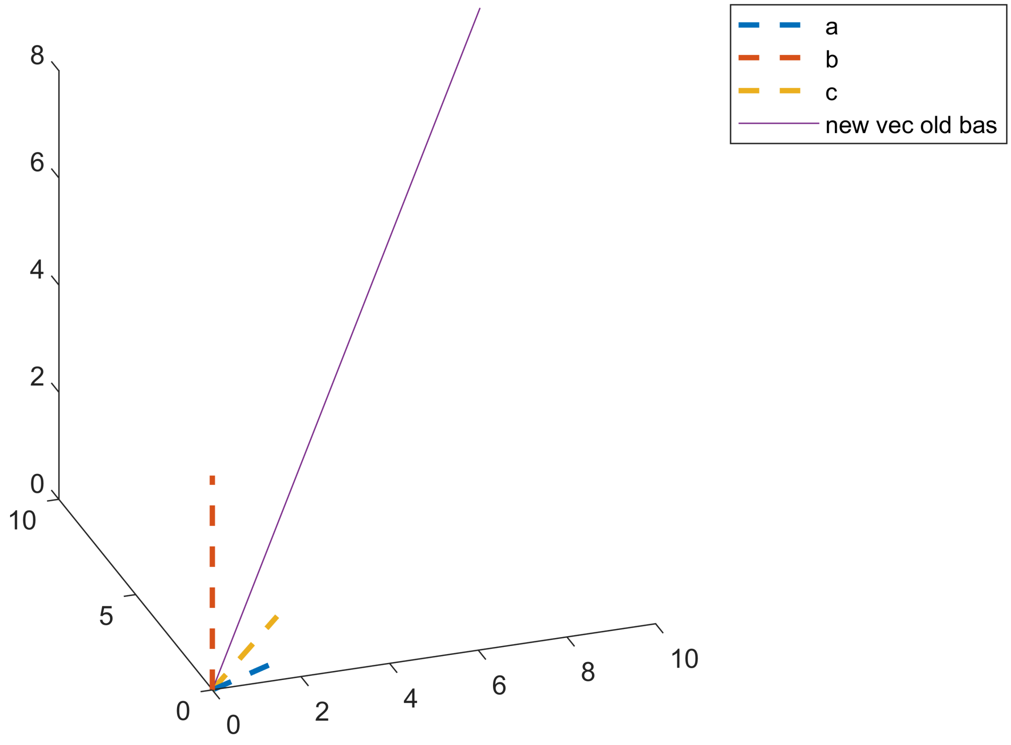
```
new_vec_old_bas_norm = new_vec_el(1)*a./norm(a) + new_vec_el(2)*b./norm(b) + new_vec_el(3)*c./r
```

```
new_vec_old_bas_norm = 3×1
    2.8150
    2.7519
    2.0000
```

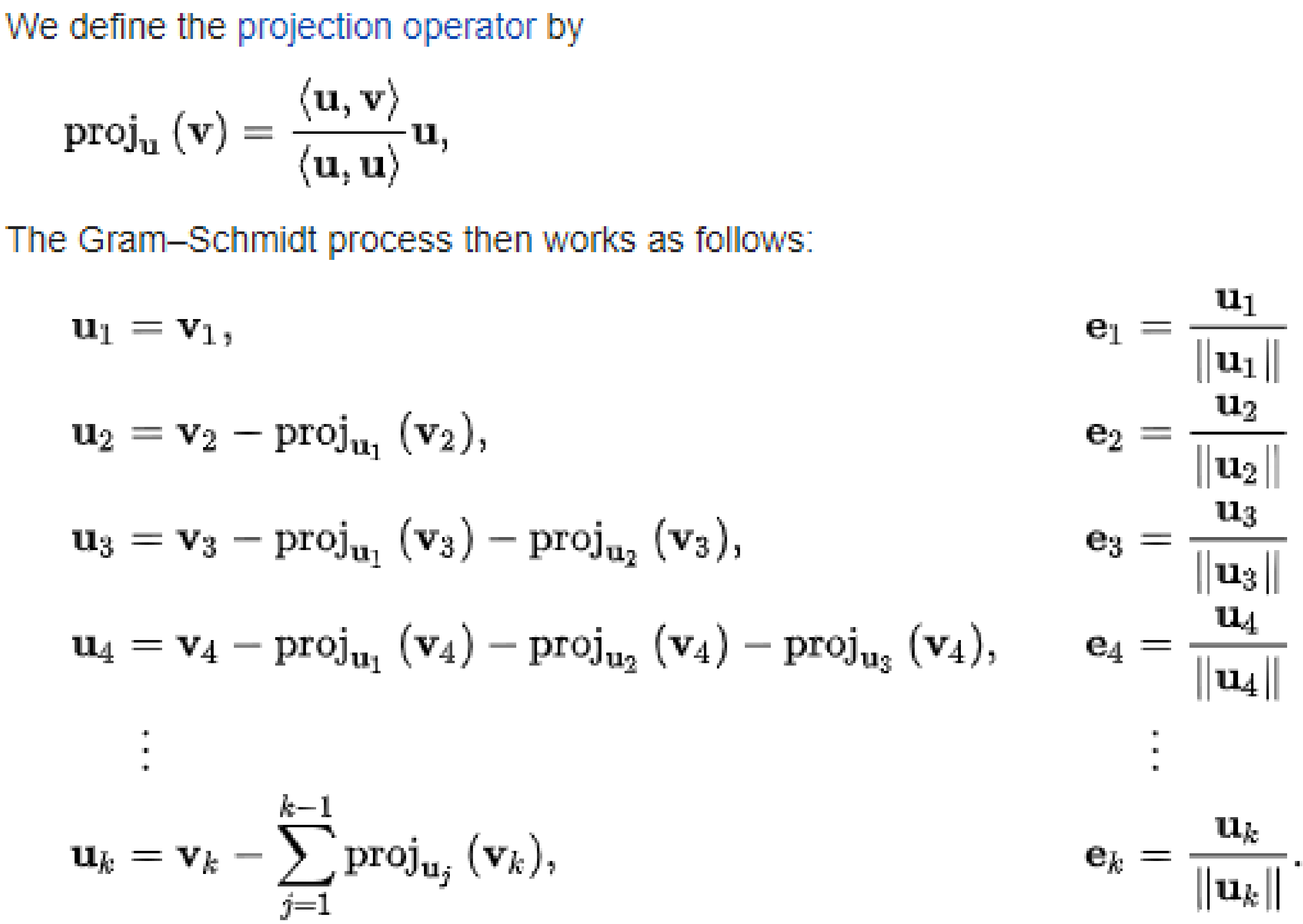
```

figure
plot3([zero(1) a(1)], [zero(2) a(2)],[zero(3) a(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) b(1)], [zero(2) b(2)],[zero(3) b(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) c(1)], [zero(2) c(2)],[zero(3) c(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) new_vec_old_bas(1)], [zero(2) new_vec_old_bas(2)],[zero(3) new_vec_old_bas(3)])
legend('a','b','c','new vec old bas')

view([-19.1 25.2])
```



There is a `Gram-Schmidt` algorithm, there is a link, which is understandable.



We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u},$$

The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

```
A = a;
B = b - (A'*b)./(A'*A)*A
```

```
B = 3×1
     0
     0
     4
```

```
C = c - (A'*c)./(A'*A)*A - (B'*c)./(B'*B)*B
```

```
C = 3×1
   -0.7000
    1.4000
         0
```

There is a new basis

```
New_BASIS = rats([A B C])
```

```
New_BASIS = 3×42 char array
    '       2              0            -7/10      '
    '       1              0             7/5       '
    '       0              4              0        '
```

```
% det_new_bas = det([A B C])
```

And it is `orthogonal`, because all of our dot products are equal to 0

```
dot(A,B)
```

```
ans = 0
```

```
round(dot(B,C))
```

```
ans = 0
```

```
round(dot(A,C))
```

```
ans = 0
```

Let's put our old vector scalar values in new basis

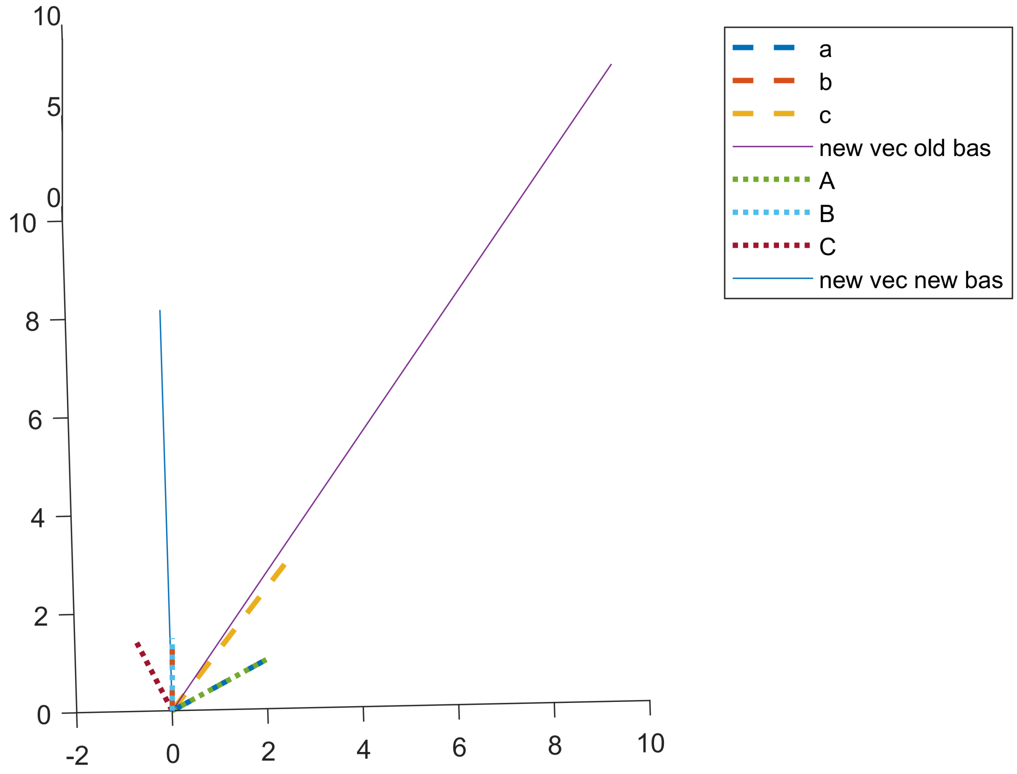
```
new_vec_new_bas = new_vec_el(1)*A + new_vec_el(2)*B + new_vec_el(3)*C
```

```
new_vec_new_bas = 3×1
   -0.1000
    5.2000
    8.0000
```

```

figure
plot3([zero(1) a(1)], [zero(2) a(2)],[zero(3) a(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) b(1)], [zero(2) b(2)],[zero(3) b(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) c(1)], [zero(2) c(2)],[zero(3) c(3)],'--',"LineWidth",2)
hold on
plot3([zero(1) new_vec_old_bas(1)], [zero(2) new_vec_old_bas(2)],[zero(3) new_vec_old_bas(3)])
hold on
plot3([zero(1) A(1)], [zero(2) A(2)],[zero(3) A(3)],':',"LineWidth",2)
hold on
plot3([zero(1) B(1)], [zero(2) B(2)],[zero(3) B(3)],':',"LineWidth",2)
hold on
plot3([zero(1) C(1)], [zero(2) C(2)],[zero(3) C(3)],':',"LineWidth",2)
hold on
plot3([zero(1) new_vec_new_bas(1)], [zero(2) new_vec_new_bas(2)],[zero(3) new_vec_new_bas(3)])
legend('a','b','c','new vec old bas','A','B','C','new vec new bas')

view([-1.45 69.72])
```



As it can be seen, there are not in the same position.

```
New_BASIS_NORM = ([A./norm(A) B./norm(B) C./norm(C)]);
```

Gram - Schmidt is complete, there is our **OrthoNormal basis**

```
New_BASIS_NORM_char = rats(New_BASIS_NORM)
```

```
New_BASIS_NORM_char = 3×42 char array
    '    305/341          0         -305/682   '
    '    305/682          0          305/341   '
    '       0             1             0      '
```

```
% det_new_bas_norm = det(New_BASIS_NORM)
```

Even with norm values, our vector is not equal from the old basis. That means, that using this tech we can only assume that we have the same space with some linear mapping between them.

```
new_vec_new_bas_norm = new_vec_el(1)*A./norm(A) + new_vec_el(2)*B./norm(B) + new_vec_el(3)*C./r
```

```
new_vec_new_bas_norm = 3×1
   -0.4472
    3.1305
    2.0000
```

Solving simple linear equation, we can find the same vector in new basis

```
new_vec_el_in_new_bas = linsolve(New_BASIS_NORM,new_vec_old_bas)
```

```
new_vec_el_in_new_bas = 3×1
   12.9692
    8.0000
    4.6957
```

```
new_vec_new_bas_norm_correct_el = new_vec_el_in_new_bas(1)*A./norm(A) + new_vec_el_in_new_bas(2
```

```
new_vec_new_bas_norm_correct_el = 3×1
    9.5000
   10.0000
    8.0000
```

```
new_vec_old_bas
```

```
new_vec_old_bas = 3×1
    9.5000
   10.0000
    8.0000
```

Gram-Schmidt

Task 1

Consider a subspace of all four-by-one column vectors with the following basis:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. Use the Gram-Schmidt process to construct an orthogonal basis for this subspace.

Gram-Schmidt

Task 1

Consider a subspace of all four-by-one column vectors with the following basis:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. Use the Gram-Schmidt process to construct an orthogonal basis for this subspace.

Answers

1. $W_{new} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 \\ 1 & \frac{1}{4} & -\frac{2}{3} \\ 1 & \frac{1}{4} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$ - Not normalized!!

QR Factorization

QR Factorization: How to obtain

$A = QR$ → Upper triangular matrix

↳ Orthogonal matrix

Algorithm:
1) Gram-Schmidt process (A->Q)
2) R = Q^{T}A

Case study:

$$A = \begin{array}{cc} a & b \\ \left[\begin{matrix}2\\2\\1\end{matrix}\right. & \left.\begin{matrix}3\\4\\1\end{matrix}\right] \end{array}$$

$$1)\ Q = \begin{array}{cc} \frac{a}{\|a\|} & \frac{\mathcal{B}}{\|\mathcal{B}\|} \\ \left[\begin{matrix}2/3\\2/3\\1/3\end{matrix}\right. & \left.\begin{matrix}-1/3\\2/3\\-2/3\end{matrix}\right] \end{array} \qquad \frac{a}{\|a\|} \bullet \frac{\mathcal{B}}{\|\mathcal{B}\|} = 0$$

$$2)\ R = Q^TA = \begin{bmatrix}2/3 & 2/3 & 1/3\\-1/3 & 2/3 & -2/3\end{bmatrix}\begin{bmatrix}2&3\\2&4\\1&1\end{bmatrix} = \begin{bmatrix}3&5\\0&1\end{bmatrix}$$

$$A = \begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\1/3 & -2/3\end{bmatrix}\begin{bmatrix}3&5\\\textcircled{0}&1\end{bmatrix} \qquad a \bullet \frac{\mathcal{B}}{\|\mathcal{B}\|} = 0$$

QR Factorization in Least Squares

$$A^TA\hat{x} = A^Tb$$

$$(QR)^T(QR)\hat{x} = (QR)^Tb$$

$$\cancel{R^T}\underbrace{Q^TQ}_{I}R\hat{x} = \cancel{R^T}Q^Tb$$

$$R\hat{x} = Q^Tb$$

Case study:

$$\begin{bmatrix}3&5\\0&1\end{bmatrix}\hat{x} = \begin{bmatrix}2/3 & 2/3 & 1/3\\-1/3 & 2/3 & -2/3\end{bmatrix}\begin{bmatrix}7\\3\\1\end{bmatrix}$$

$$\begin{bmatrix}3&5\\0&1\end{bmatrix}\hat{x} = \begin{bmatrix}7\\-1\end{bmatrix}$$

Quiz

Tasks:

1. *Least Squares*. There are several points

   $$A = \begin{bmatrix} -1 \\ 2 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ C = \begin{bmatrix} -1 \\ 0 \end{bmatrix};\ D = \begin{bmatrix} 2 \\ 1 \end{bmatrix};\ E = \begin{bmatrix} 0 \\ -1 \end{bmatrix};\ F = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   1.1 What curve type is the best for fitting these points? (no more than 2nd order polynomial eq.)

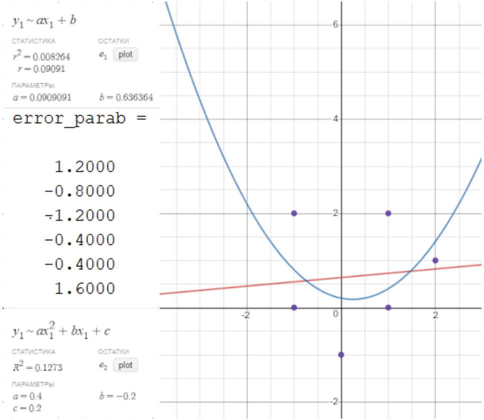
   1.2 Find the parameters of the curve, find summary error (SE).

2. *Gram-Schmidt*. There is a basis $\begin{bmatrix} 14 & 3 & -1 & -1 \\ 21 & 0 & 0 & 2 \\ 14 & 0 & 2 & -1 \\ -14 & 3 & 1 & 1 \end{bmatrix}$. Make this basis orthogonal.

Quiz

Answers

1. *Least Squares*
    1.1 Parabola (or any 2nd order polynomial curve)
    1.2 $a = 0.4,\ b = -0.2,\ c = 0.2;\ SE = 6.4$
2. *Gram-Schmidt*
    2.1 The same. It is already orthogonal ♡



$y_1 \sim ax_1 + b$

СТАТИСТИКА

$r^2 = 0.008264$

$r = 0.09091$

ОСТАТКИ

e_1 plot

ПАРАМЕТРЫ

$a = 0.0909091$ $b = 0.636364$

```
error_parab =

    1.2000
   -0.8000
   -1.2000
   -0.4000
   -0.4000
    1.6000
```

$y_1 \sim ax_1^2 + bx_1 + c$

СТАТИСТИКА

$R^2 = 0.1273$

ОСТАТКИ

e_2 plot

ПАРАМЕТРЫ

$a = 0.4$ $b = -0.2$

$c = 0.2$

Task 1

Task 1

Midterm Preparation

Task: find v, where v is intersection of $\begin{cases} S_1 = x + 7y - 3z = 13 \\ S_2 = x + y + 0z = 5 \end{cases}$

Task 1

Midterm Preparation

Task: find v, where v is intersection of $\begin{cases} S_1 = x + 7y - 3z = 13 \\ S_2 = x + y + 0z = 5 \end{cases}$

Answer

1. Using line representation as an intersection between two planes (AGLA1, lab 7).
2. Using knowledge about solving undetermined equations (AGLA2, lab 3).

Task 2

Midterm Preparation

Task: Find a vector v that is orthogonal to S, if subspace S of R^3 is formed by linear combination of vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

Task 2

Midterm Preparation

Task: Find a vector v that is orthogonal to S, if subspace S of R^3 is formed by linear combination of vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

Answer

1. Using knowledge about projection (AGLA2, lab 5).
2. Using knowledge that nullspace perpendicular to row space (AGLA2, lab 4).

Task 3

Midterm Preparation

Task:

1. For each real parameter λ construct a linear independent system that contains the maximum number of the following vectors.
2. Find the dimensions of the four fundamental subspaces associated with result of ”1”, depending on the parameter λ.

$$a = \begin{bmatrix} -\lambda \\ 1 \\ 2 \\ 3 \end{bmatrix};\ b = \begin{bmatrix} 1 \\ -\lambda \\ 3 \\ 2 \end{bmatrix};\ c = \begin{bmatrix} 2 \\ 3 \\ -\lambda \\ 1 \end{bmatrix};\ d = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -\lambda \end{bmatrix};\ e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Task 3

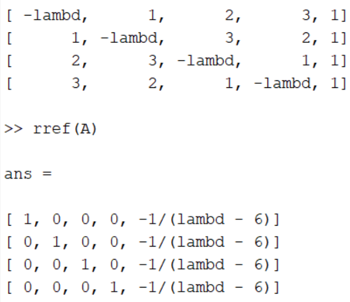
Midterm Preparation

Answer: $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, when $\lambda \neq 6$.

1. Using **rref** we can easily find max possible max rank, hence choose our basis.

   Hint: It can be solved, if you sum all vectors to the first column (all rows will be the same).

2. Using knowledge from previous lab (AGLA2, lab 4).



```
[ -lambd,      1,      2,      3, 1]
[      1, -lambd,      3,      2, 1]
[      2,      3, -lambd,      1, 1]
[      3,      2,      1, -lambd, 1]

>> rref(A)

ans =

[ 1, 0, 0, 0, -1/(lambd - 6)]
[ 0, 1, 0, 0, -1/(lambd - 6)]
[ 0, 0, 1, 0, -1/(lambd - 6)]
[ 0, 0, 0, 1, -1/(lambd - 6)]
```

Reference material

- Lecture 17
- *"Linear Algebra and Applications", pdf pages 205–221*
  Orthogonal Bases and Gram-Schmidt
- Gram-Schmidt Process | Lectures 19 and 20
  Video from Matrix Algebra for Engineers course
- QR Factorization

Deserve "A" grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru
@Lupasic
Room 105 (Underground robotics lab)