Analytical Geometry and Linear Algebra II, Lab 7

Eigenvalues and Eigenvectors

Diagonalization of a Matrix (Спектральное разложение)

Fast A^N calculation

Where it can be used

- Machine learning (transform data in more suitable form)
- Make some calculations easier (matrix^{100} — piece of cake)
- Predict the behavior of linear systems (physics, biology, etc)
- Design the controller for a system
- Estimate the complexity of calculations
- ...

Definition

In linear algebra, an **eigenvector or characteristic vector** of a linear transformation is a non-zero *vector that changes by only a scalar factor when that linear transformation is applied to it.* Wiki

$A\mathbf{x} = \lambda\mathbf{x}$, where

x – eigenvector (should be non-zero),

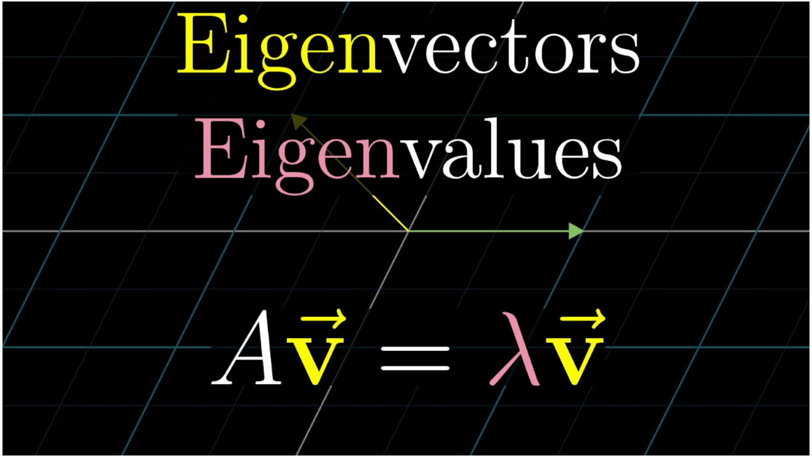
λ – eigenvalue,

A – *square* matrix.

For $n \times n$ matrix – max amount of λ is a number of n .

EigenValues concept

Video



Calculation (1)

Classical approach (max 4x4)

Algorithm

There are 2 steps:

1. Find λ (eigenvalue) — $det(A - \lambda I) = 0$
   - 2×2 matrix: $det(A - \lambda I) = \lambda^2 - trace(A)\lambda + det(A) = 0$, where $trace(A)$ – sum of diag values of A;
   - 3×3 matrix: $det(A - \lambda I) = \lambda^3 - trace(A)\lambda^2 - \frac{1}{2}(trace(A^2) - trace(A)^2)\lambda - det(A) = 0$
2. Find $\mathbf{x}$ for each λ — $(A - \lambda_i I)\mathbf{x} = 0$

Example

Case study, 2×2 matrix: $A = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix}$

1. $trace(A) = 4 + (-3) = 1$,
   $det(A) = 4(-3) - 3(-2) = -6$, hence
   $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$, $\rightarrow$
   $\rightarrow \lambda_1 = 3,\ \lambda_2 = -2$
2. 2.1 $A - 3I = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$; $x_{\lambda=3} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

   2.2 $A + 2I = \begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}$; $x_{\lambda=2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Task 1

Find the eigenvalues and eigenvectors:

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Task 1

Find the eigenvalues and eigenvectors:

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Answer

1. $\lambda_1 = -5,\ \lambda_2 = 9$

   $x_{\lambda=-5} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, x_{\lambda=9} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$
2. $\lambda_1 = 3 + 1i,\ \lambda_2 = 3 - 1i$

   $x_{\lambda=3+1i} = \begin{bmatrix} i \\ 1 \end{bmatrix}, x_{\lambda=3-1i} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Calculation (2)

Real life approach (Iterative algorithms)

Due to the reason that computers appeared recently, eigenpairs weren't used frequently.

Nowadays, it can be found easily by iteration method, which implemented in most programming languages.

Method	Applies to	Produces	Cost per step	Convergence
Lanczos algorithm	Hermitian	m largest/smallest eigenpairs		
Power iteration	general	eigenpair with largest value	$O(n^2)$	linear
Inverse iteration	general	eigenpair with value closest to μ		linear
Rayleigh quotient iteration	Hermitian	any eigenpair		cubic
Preconditioned inverse iteration[11] or LOBPCG algorithm	positive-definite real symmetric	eigenpair with value closest to μ		
Bisection method	real symmetric tridiagonal	any eigenvalue		linear
Laguerre iteration	real symmetric tridiagonal	any eigenvalue		cubic[12]
QR algorithm	Hessenberg	all eigenvalues	$O(n^2)$	cubic
		all eigenpairs	$6n^3 + O(n^2)$	
Jacobi eigenvalue algorithm	real symmetric	all eigenvalues	$O(n^3)$	quadratic
Divide-and-conquer	Hermitian tridiagonal	all eigenvalues	$O(n^2)$	
		all eigenpairs	$(4/3)n^3 + O(n^2)$	

Eigenvector and eigenvalue iterative algorithms Wiki

Eigenpair properties and features

- $\sum \lambda = trace(A)$
- $det(A) = \prod_{i=1}^{n} \lambda_i$
- $A_{new} = A_{old} + aI, \ \rightarrow$ eigenvectors won't change, $\lambda_{new} = \lambda_{old} + a$
- The matrix A is invertible if and only if every eigenvalue is nonzero.
- If matrix is triangular – the eigenvalues are on the main diagonal
- If matrix is symmetric – λ is *definitely* real
- If matrix is not symmetric – λ *can* contain imaginary part
- $Ax = \lambda x \rightarrow A^2x = Ax \ (left\ mult) \rightarrow A^2x = Ax(\lambda\ is\ const) = \lambda^2 x$

Diagonalization

Key idea / **Follow each eigenvector separately / $\boldsymbol{n}$ simple problems**

Eigenvector matrix $\boldsymbol{X}$
Assume independent $\boldsymbol{x}$'s
Then X is invertible

$$AX = A\begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix}$$

$$\boxed{\begin{aligned} \boldsymbol{AX} &= \boldsymbol{X\Lambda} \\ \boldsymbol{X^{-1}AX} &= \boldsymbol{\Lambda} \\ \boldsymbol{A} &= \boldsymbol{X\Lambda X^{-1}} \end{aligned}}$$

$$\begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Diagonalization properties

Some matrices are not diagonalizable

They don't have n independent vectors

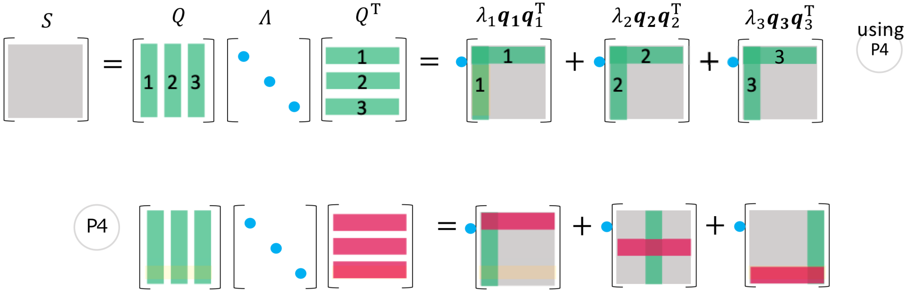
$$A = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} \text{ has } \lambda = \mathbf{3} \text{ and } \mathbf{3}$$

That A has double eigenvalue, single eigenvector

Only one $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Diagonalization of Symmetric matrices $S = Q\Lambda Q^{\mathrm{T}}$

All symmetric matrices S must have **real eigenvalues** and **orthogonal eigenvectors**. The eigenvalues are the diagonal elements of Λ and the eigenvectors are in Q.



A matrix is broken down to a sum of rank 1 matrices.

Task 2

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

- Find eigenpairs;
- Write down A in diagonal from;
- Draw several vectors: one, which are parallel to an eigenvector, other – not.
- Multiply chosen vectors on A, draw the new ones.

Task 2

Answer

(1) $Ax = \lambda x$; $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$D = 16 - 4 \cdot 3 = 4$

$\lambda_{1,2} = \frac{4 \pm \sqrt{4}}{2} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 3$

(1) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t;\ x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Assume, t =1

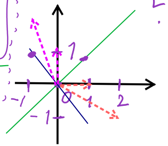
(2) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(2) $S^{-1} = S^T$ if orthonormal. In our case - no. Hence, we need to find inverse matrix

$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = S^{-1} A S$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$



It is scale + shear

Case studies

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Task 3

True or false: If the columns of X (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) X is invertible (d) X is diagonalizable.

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True or false: If the columns of X (eigenvectors of A) are linearly independent, then

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(c) X is invertible (d) X is diagonalizable.

Answer

(a) False: We are not given the λ's (b) True (c) True (d) False: For this we would need the eigenvectors of X

Task 4

If the eigenvectors of A are the columns of I, then A is a ______ matrix. If the eigenvector matrix X is triangular, then X^{-1} is triangular. Prove that A is also triangular.

Task 4

If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix X is triangular, then X^{-1} is triangular. Prove that A is also triangular.

Answer

With $X = I$, $A = X\Lambda X^{-1} = \Lambda$ is a diagonal matrix. If X is triangular, then X^{-1} is triangular, so $X\Lambda X^{-1}$ is also triangular.

A^k

A^k becomes easy $\quad A^k = (X\Lambda X^{-1})\,(X\Lambda X^{-1})\cdots(X\Lambda X^{-1})$

Same eigenvectors in X $\quad \boxed{\boldsymbol{A^k = X\Lambda^k X^{-1}}} \quad \boldsymbol{\Lambda^k} = (\text{eigenvalues})^{\boldsymbol{k}}$

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}^{\mathbf{4}} = X\Lambda^4 X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{1^4} & 0 \\ 0 & \mathbf{3^4} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 81 \\ 0 & 81 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{80} \\ \mathbf{0} & \mathbf{81} \end{bmatrix}$$

Question : **When does $\boldsymbol{A^k} \to$ zero matrix ?**

Answer : $\boxed{\textbf{All } \ |\boldsymbol{\lambda_i}| < \mathbf{1}}$

Applications (1)

Fast Calculations

Task: Find 50th Fibonacci value

Dummy approach: calculate it by iterative summarization.

Smart approach: use magic and diagonalization

$u_{k+1} = u_k + u_{k-1}$ Fibbonaci

Let's represent it as a Ax=b

$\begin{bmatrix} 1 & 1 \\ ? & \end{bmatrix}\begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} u_{k+1} \\ ? \end{bmatrix}$

We need matrix 2x2 -> let's make a useless second eqn.

$\begin{cases} u_k + u_{k-1} = u_{k+1} \\ u_k = u_k \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} \Rightarrow A\begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix}$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; Au_0 = u_1; u_2 = Au_1 = AAu_0 = A^2 u_0 \Rightarrow u_{k+1} = A^{k+1} u_0$

$F_{50} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \lambda_1^{50} & 0 \\ 0 & \lambda_2^{50} \end{bmatrix}\begin{bmatrix} x_1 & x_2 \end{bmatrix}^{-1}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= u_0}$

Let's diagonalize A. For such idea find λ, S

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ldots \lambda_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} \approx \begin{matrix} 1.618 \\ -0.618 \end{matrix} \quad x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

Lecture 22. Diagonalization and Powers of A

Task 5

$A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ______ . Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

Task 5

$A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ______. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

Answer: Markov matrix is a prob. matrix, where a summary in each column should be 1. It has a property, that $\lambda_{max} = 1$.

$A^k = X\Lambda^k X^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; A_1 is a Markov matrix so $\lambda_{\max} = 1$ and $A_1^k \to A_1^\infty$, A_2 has $\lambda = .6 \pm .3$ so $A_2^k \to 0$.

Applications (2)

Computer Vision

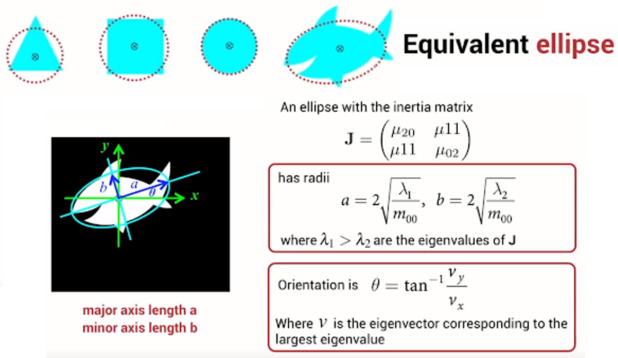
Task: we want to know the orientation of the object

Needed terms: Centroid, Image moments

Solution: use equivalent ellipse method. We consider an ellipse:

- centred at the object's centroid;
- has same moments of inertia about centroid.

Afterwards, we find an ellipse using eigenvalues and eigenvectors



Feature extraction masterclass video

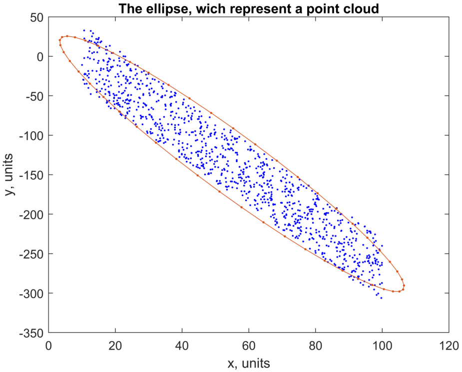
Applications (2.5)

How to visualize a point cloud as an ellipse

Task: We have a matrix with points. I want to make a model, which represent it in easier manner. Also, I want to visualize it.

Solution: We can find covariance matrix of our point cloud (It's topic from probabilistic and statistic course) and centroid of our point cloud. The matrix eigenpairs provide all info (minor and major axes length and orientation)

Application: Eigenvectors is a basis, so we can put all our points in this basis and work with it. More info in the next semesters.



More details in matlab code below

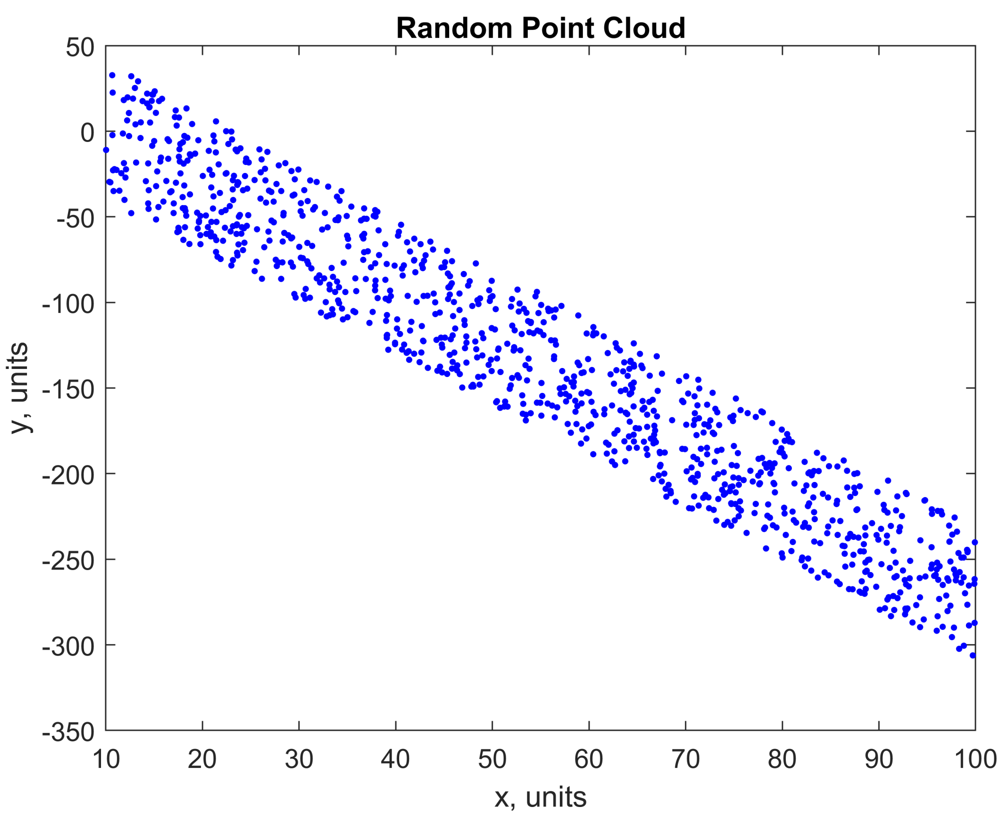
Lab 9

How to represent a point cloud as an ellipse

The core idea, that major and minor axis of ellipse are eigenvectors of our covariance matrix of point cloud

Generate some points around a line

```
intercept = -10; slope = -3;
npts = 1000; noise = 80;
xs = 10 + rand(npts, 1) * 90;
ys = slope * xs + intercept + rand(npts, 1) * noise;
% Plot the randomly generated points
figure;
plot(xs, ys, 'b.', 'MarkerSize', 8)
title("Random Point Cloud")
xlabel("x, units")
ylabel("y, units")
```



Find eigenpairs of the matrix

```
A = [xs ys];
covmat = cov(A)
```

```
covmat = 2×2
```

```
10^3 ×
    0.6656   -2.0010
   -2.0010    6.5510
```

```
[e,b] = eig(covmat)
```

```
e = 2×2
   -0.9558   -0.2942
   -0.2942    0.9558
b = 2×2
10^3 ×
    0.0497         0
         0    7.1669
```

```
% Just for curiosity - eigenvectors from A'A is almost the same as from cov(A),
% but not eigenvalues
covmat_A = A'*A
```

```
covmat_A = 2×2
10^7 ×
    0.3681   -0.9488
   -0.9488    2.5138
```

```
[e_A,b_A] = eig(covmat_A)
```

```
e_A = 2×2
   -0.9352   -0.3542
   -0.3542    0.9352
b_A = 2×2
10^7 ×
    0.0088         0
         0    2.8732
```

```
error = e-e_A
```

```
error = 2×2
   -0.0206    0.0600
    0.0600    0.0206
```

```
% We are interested in both correct eigenvalue and eigenvector, hence we
% will use data from covatiance matrix
```

Find centroid of a point cloud, major and minor axes and orientation of an ellipse

```
% formulas were given on the previous slide
b = 2*sqrt(diag(b))
```

```
b = 2×1
   14.1018
  169.3146
```

```
ang = rad2deg(atan2(e(1,2),e(2,2)))
```

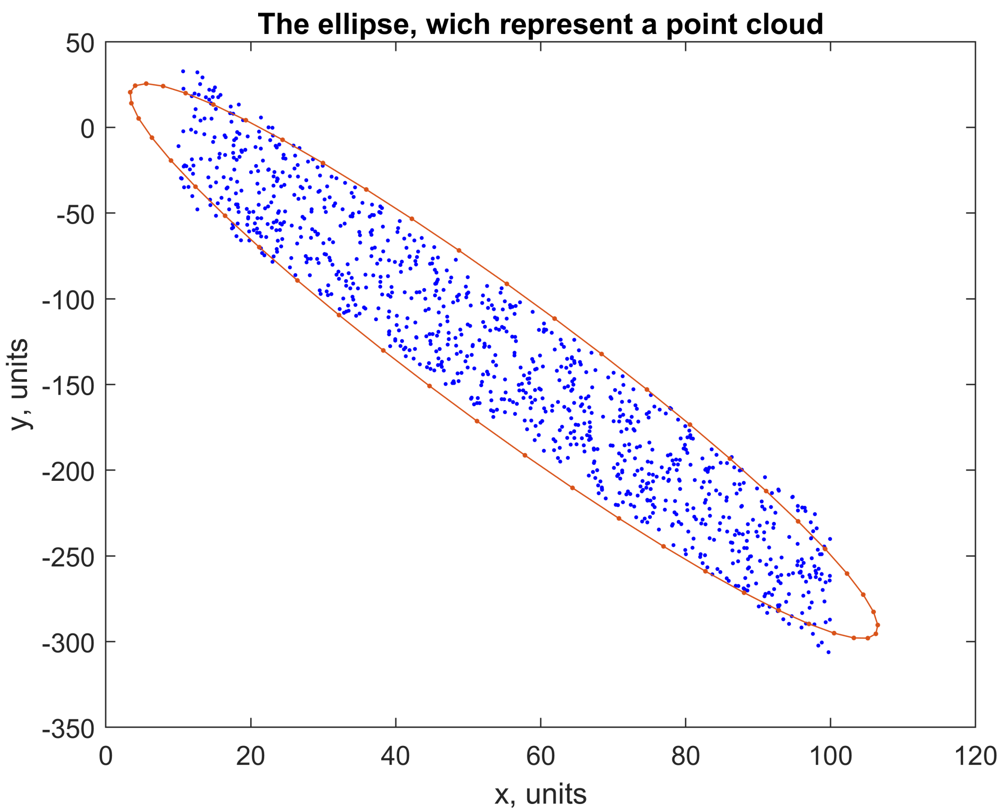
```
ang = -17.1073
```

```
centroid = mean([xs,ys])
```

```
centroid = 1×2
   54.9191 -136.3597
```

Plot

```
figure; plot(xs, ys, 'b.', 'MarkerSize', 5)
title("The ellipse, wich represent a point cloud")
xlabel("x, units")
ylabel("y, units")
hold on
p = calcEllipse(centroid(1), centroid(2), b(1),b(2) , deg2rad(ang), 50);
plot(p(:,1), p(:,2), '.-')
```



Applications (3)

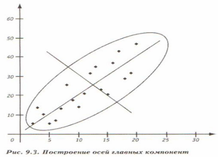
Machine learning + optimization

Task: we have data, which depicted on figure. We need to *find local minimum* of it.

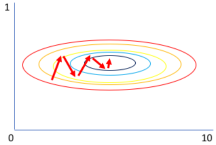
Dummy approach: let's use gradient descent w/o preprocessing.

Result of dummy approach: It can disconvergent, or solved very slow, because of big difference between step size in x and y direction.

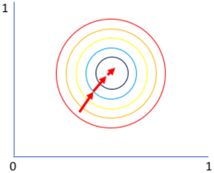
Smart approach: let's firstly represent it as a *circle* (**transform all data in eigenbasis**) and make gradient descent on it. In this case we have almost the same step size for x and y direction.



Represent data as an ellipse



Gradient descent Issue



Represent data as a circle

Applications (4)

Predict the behavior of linear systems

Task: I have a system and want to understand, how it will works. Afterwards, I want to control it (design a controller).

Solution:

- Extimate Stability using Eigenpairs. Looking on eigenvalues we can predict stability of our linear system;
- Coupled Oscillators. Example of Eigenvalues and Eigenvectors in the context of coupled oscillators (masses connected by springs)

Task 6

(Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Task 6

(Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Answer

Two problems: The nullspace and column space can overlap, so x could be in both.

There may not be r independent eigenvectors in the column space.

Reference material

- Lecture 21, Eigenvalues and Eigenvectors
- Lecture 22, Diagonalization and Powers of A
- *"Linear Algebra and Applications", pdf pages 270–306*

  Eigenvalues and Eigenvectors 5.1–5.3
- *"Introduction to Linear Algebra", pdf pages 299–329*

  Eigenvalues and Eigenvectors 6.1–6.2
- The eigenvalue problem | Lectures 32 – 38

  Video from Matrix Algebra for Engineers course

Deserve “A” grade!

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Room 105 (Underground robotics lab)