Analytical Geometry and Linear Algebra II, Lab 2

Gaussian elimination recap

A=LU, A=LDV, PA=LU factorization

Innopolis University

Gaussian elimination

Algorithm

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Gaussian elimination

Algorithm

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \to \color{red}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 5 \end{bmatrix}}$$

A=LU factorization

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A}$$

A=LU factorization

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U}$$

A=LU factorization

Application (1)

$$\begin{cases} 2x_1 + 3x_2 - x3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases}$$

A=LU factorization

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$$\begin{cases} 2x_1 + 3x_2 - x3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases} \quad \text{Ans:} \quad \begin{cases} x_1 = 4 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

A=LU factorization

Application: Theory (2)

$$Ax = b$$

A=LU factorization

Application: Theory (2)

$$Ax = b$$

$$LUx = b \rightarrow Lc = b \rightarrow Ux = c \rightarrow \text{Ans: } x$$

A=LU factorization

Application: Theory (3)

Description

Try to solve such systems using:

- Gauss-Jordan elimination
- LU factorization

Compare time consuption

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & 10 & 2 \end{bmatrix}, B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A=LU factorization

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$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \overbrace{\underbrace{\begin{bmatrix} \frac{-1}{3} & \frac{16}{29} & 1 \\ \frac{-2}{3} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 10 & 2 \\ 0 & \frac{29}{3} & \frac{13}{3} \\ 0 & 0 & \frac{-21}{2} \end{bmatrix}}_{U}}^{\textit{Matlab}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -16 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 21 \end{bmatrix}}_{U}, B_2 = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{-4}{7} \end{bmatrix}, B_3 = \begin{bmatrix} \frac{73}{49} \\ \frac{109}{147}\ -\frac{185}{147} \end{bmatrix}$$

Task 1

Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

Task 1

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Answer

$c = 2$ leads to zero in the second pivot position: exchange rows and not singular.
$c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

Task 2

This nonsymmetric matrix will have the same L as in Problem **13**:

Find L **and** U **for**

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

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Answer

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}. \text{ Need } \begin{array}{c} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{array}$$

Task 3

If A and B have nonzeros in the positions marked by x, which zeros (marked by 0) *stay zero* in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

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Answer

For the first matrix A, L keeps the 3 lower zeros at the start of rows. But U may not have the upper zero where $A_{24} = 0$. For the second matrix B, L keeps the bottom left zero at the start of row 4. U keeps the upper right zero at the start of column 4. One zero in A and two zeros in B are filled in.

Reference material

- Lecture 4 MIT course
- Gilbert Strang Book 2.6

Deserve "A" grade!

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Room 105 (Underground robotics lab)