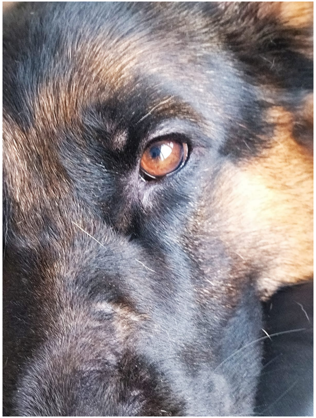
Analytical Geometry and Linear Algebra II, Lab 10

Symmetric matrices

Positive definite matrices and minima

Innopolis University

How I spent last weekend



Positive Definite Matrices

Positive Definite Matrices

Five tests

Positive definite matrices are the best. How to test S for $\lambda_i > 0$?

Test 1	Compute the **eigenvalues** of S: All eigenvalues positive
Test 2	The **energy** $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{S} \boldsymbol{x}$ is positive for every vector $\boldsymbol{x} \neq \mathbf{0}$
Test 3	The **pivots** in elimination on S are all positive
Test 4	The upper left **determinants** of S are all positive
Test 5	$\boldsymbol{S} = \boldsymbol{A}^{\mathrm{T}} \boldsymbol{A}$ for some matrix A with independent columns

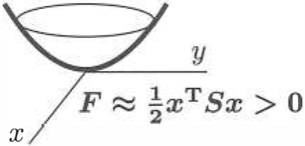
Positive Definite Matrices

Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S. It contains *four second derivatives*. **Positive d^2f/dx^2 changes to positive definite S:**

Second derivatives $$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$



$F(x, y)$ has a minimum if $\partial F/\partial x = \partial F/\partial y = 0$ and S is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

Reference material

- Lecture 28, Positive Definite Matrices and Minima
- *"Introduction to Linear Algebra", pdf pages 349–374*

  6.4 – Symmetric, 6.5 – Positive Definite matrices
- *"Linear Algebra and Applications", pdf pages 355–376*

  Positive Definite Matrices 6.1, 6.2

Deserve “A” grade!

– Oleg Bulichev

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Room 105 (Underground robotics lab)