Linear Algebra. Test 1: Solutions

1.1 The 4x4 matrix *A* reduces to the identity matrix I by the following four row operations:

- E_{21}: subtract row 1 from row 2;
- E_{31}: add 2*row 1 to row 3;
- E_{32}: subtract 3*row 2 from row 3;
- E_{41}: subtract 2*row 1 from row 4;
- E_{43}: add row 3 to row 4;

Write and compute A^{-1} (3 points) and A (2 points).

$$E_{43}E_{41}E_{32}E_{31}E_{21}A = I \;\Rightarrow\; A^{-1} = E_{43}E_{41}E_{32}E_{31}E_{21},$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 5 & -3 & 1 & 0 \\ 3 & -3 & 1 & 1 \end{pmatrix}$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{43}^{-1}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 2 & 0 & -1 & 1 \end{pmatrix}$$

1.2 The 4x4 matrix *A* reduces to the identity matrix I by the following four row operations:

- E_{21}: add 2*row 1 to row 2;
- E_{31}: subtract row 1 from row 3;
- E_{32}: subtract 3*row 2 from row 3;
- E_{41}: add row 1 to row 4;
- E_{43}: subtract 2*row 3 from row 4;

Write and compute A^{-1} (3 points) and A (2 points).

$$E_{43}E_{41}E_{32}E_{31}E_{21}A = I \;\Rightarrow\; A^{-1} = E_{43}E_{41}E_{32}E_{31}E_{21},$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -7 & -3 & 1 & 0 \\ 15 & 6 & -2 & 1 \end{pmatrix}$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{43}^{-1}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

2.1 Consider matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 6 & -3 \\ 0 & -3 & 2 \end{bmatrix}$$

Find the factorization $A = LDU$ (3 points)
Find A inverse (2 points).

$$A = LDU = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{3}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 2 & 3 \\ -3 & 3 & 5 \end{pmatrix}$$

2.2 Consider matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

Find the factorization $A = LDU$ (3 points)
Find A inverse (2 points).

$$A = LDU = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -7 & 4 & -2 \\ 4 & -2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

3.1 Consider matrix A

$$A = \begin{bmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Reduce the matrix A to its ordinary Echelon form U (1 point).

$$A = \begin{bmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a solution for each free variable and describe every solution to $Ax = 0$ (1 point).

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \quad \vec{x}_{N1} = c_1 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{x}_{N2} = c_2 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

For which nontrivial right-hand sides (find a condition on b_1, b_3, b_3, b_4) $Ax = b$ is solvable (1 point)?

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} \Rightarrow$$

Provide an example of vector $b \neq 0$ that makes this system solvable (1 point).

$$\Rightarrow b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Find the complete solution for $Ax = b$ using b from previous part (3.d) (1 point).

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 2 & 1 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_{N1} + \vec{x}_{N2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

3.2 Consider matrix A

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Reduce the matrix A to its ordinary Echelon form U (1 point).

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a solution for each free variable and describe every solution to $Ax = 0$ (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \quad \vec{x}_{N1} = c_1 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{x}_{N2} = c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

For which nontrivial right-hand sides (find a condition on b_1, b_3, b_3, b_4) $Ax = b$ is solvable (1 point)?

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 2b_2 - b_1 \\ b_2 - b_1 \end{bmatrix} \Rightarrow$$

Provide an example of vector $b \neq 0$ that makes this system solvable (1 point).

$$\Rightarrow b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Find the complete solution for $Ax = b$ using b from previous part (3.d) (1 point).

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_{N1} + \vec{x}_{N2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$