



Analytical Geometry and Linear Algebra II, Lab 12

Similar matrices

Singular value decomposition - SVD

Left and right inverses. Pseudoinverse



How I spent last weekend



Jobless Reincarnation ranobe: 4342/6859 pages. Tome 19th



Catamaran sailing training with KFU



Similar Matrices

All the matrices $A = B^{-1}CB$ are "similar". They all share the eigenvalues of C .

Example

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \lambda = 3, 1; S^{-1}AS = \Lambda_A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, S - \text{eigenvectors}$$

Let's find other similar matrix: $M^{-1}AM = T$, $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, M is random matrix. $T = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$, which λ are also 3, 1.



Similar Matrices

Properties

Because matrices are similar if and only if they represent the *same linear operator with respect to (possibly) different bases*, similar matrices share all properties of their shared underlying operator:

- Rank
- Characteristic polynomial, and attributes that can be derived from it:
 - Eigenvalues
 - Determinant
 - Trace

[More info](#) can be found here. This topic was used for checking understanding LA by the students ([paper](#)).



The primary goal of SVD

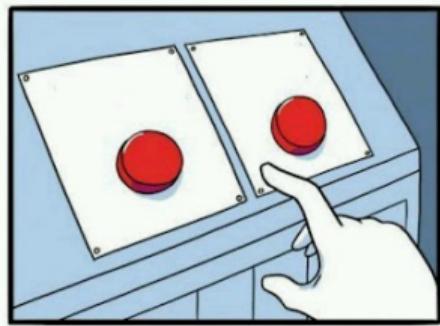
To “X-RAY” matrix
(To understand the structure of matrix)

Singular Value Decomposition (SVD)

3 ways of explanation

1. Linear transformation – [Kutz video](#)
2. Algebraic – [MIT video \(Strang\)](#), [Aaron Greiner video](#)
3. As a tool for DS – [Stanford video](#), [Brunton video](#)

According to Kholodov words, SVD was created for *finding an inverse for any matrices*. It is needed in linear transformation related operations. Other properties were found afterwards.



Singular Value Decomposition

$$E' = AE \quad \begin{matrix} \rightarrow \text{Change of basis} \\ (\text{AGLA1}) \end{matrix}$$

Old basis. We assume, it is orthonormal basis

↑
New basis ↑
Transition matrix

Let's rewrite it in common SVD notation

$$AV = E'$$

$\boxed{U \Sigma I}$

$$\begin{bmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_1 & 0 \\ 0 & \tilde{\sigma}_2 \end{bmatrix} = \begin{bmatrix} u_{1x}\tilde{\sigma}_1 & u_{2x}\tilde{\sigma}_2 \\ u_{1y}\tilde{\sigma}_1 & u_{2y}\tilde{\sigma}_2 \end{bmatrix}$$

Matrix order appears because we want stretch columns

Stretching component of E'

Rotation component of E'

$$A = U \Sigma V^H$$

$$A^T A = (U \Sigma V^H)^T (U \Sigma V^H) = V \Sigma^2 V^H; \quad A A^T = U \Sigma^2 U^H \dots$$

$$A^T A V = V \Sigma^2 V^H V$$

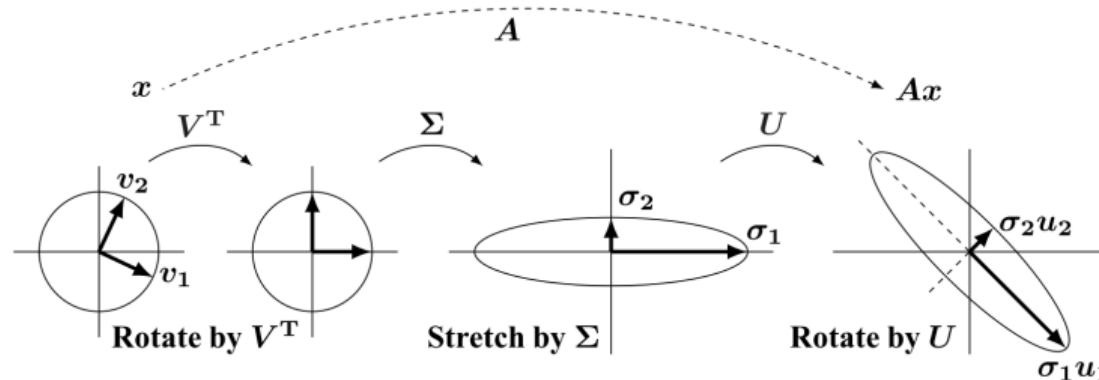
Common eigen problem. We can find both V and σ .

If singular values are distinct, then U and V are unique



Singular Value Decomposition

Geometrical explanation



$$A = (\text{Orthogonal}) (\text{Diagonal}) (\text{Orthogonal})$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$$

Four numbers a, b, c, d in A produce four numbers $\theta, \sigma_1, \sigma_2, \phi$ in the SVD



Singular Value Decomposition (SVD)

How to calculate it (2 common ways)

First approach

1. Find eigenpairs for $A^T A$. Result is Σ and V . ($A^T A = V \Sigma^2 V^H$)
2. Find U , using Σ and V ($AV\Sigma^{-1} = U$)

Second approach

1. Find eigenpairs for $A^T A$ and AA^T . ($A^T A = V \Sigma^2 V^H$) ($AA^T = U \Sigma^2 U^H$)



Singular Value Decomposition (SVD)

Obtain SVD for $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$, using second approach (task was taken)

1. Eigenpairs of AA^T .

$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}. \lambda_1 = 25, \lambda_2 = 9. x_{\lambda_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, x_{\lambda_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}. U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2. Eigenpairs of A^TA .

$$A^TA = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}. \lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0. V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \end{bmatrix}$$

$$3. \text{ Result. } A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$



Task 1

Find the matrices U, Σ, V for $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$. The rank is $r = 2$.



Task 1

Find the matrices U, Σ, V for $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$. The rank is $r = 2$.

Answer

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

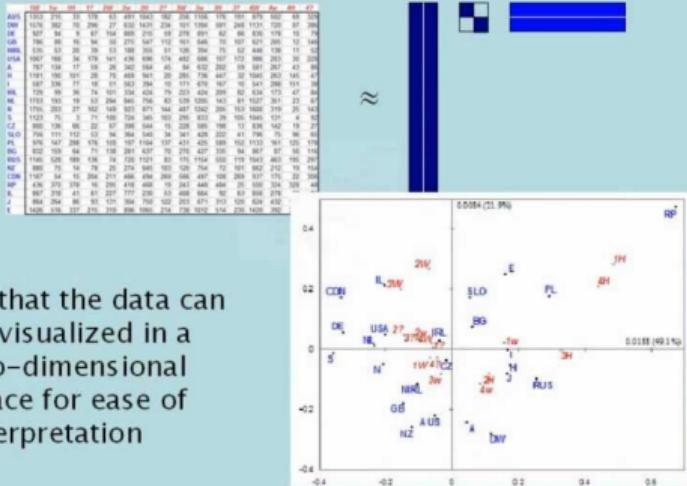


It had to be *U*

Video

Visualizing high-dimensional data

This allows data matrices of high-dimensionality to be approximated optimally by one of rank 2:



so that the data can
be visualized in a
two-dimensional
space for ease of
interpretation



Singular Value Decomposition (SVD)

Properties

- It is always possible to decompose a real matrix A into SVD
- U, Σ, V are unique
- U, V - column orthonormal
- By convention Σ contains singular values in sorted order $\sigma_1 \geq \sigma_2 \dots$



Task 2

Find the eigenvalues and the singular values of this 2 by 2 matrix A .

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{with} \quad A^T A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \quad \text{and} \quad AA^T = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}.$$

The eigenvectors $(1, 2)$ and $(1, -2)$ of A are not orthogonal. How do you know the eigenvectors v_1, v_2 of $A^T A$ are orthogonal? Notice that $A^T A$ and AA^T have the same eigenvalues (25 and 0).



Task 2

Answer

The matrix A has trace 4 and determinant 0. So its eigenvalues are 4 and 0—*not used in the SVD!* The matrix $A^T A$ has trace 25 and determinant 0, so $\lambda_1 = 25 = \sigma_1^2$ gives $\sigma_1 = 5$.

The eigenvectors v_1, v_2 of $A^T A$ (a symmetric matrix !) are orthogonal :

$$\begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{25} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{0} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly AA^T has orthogonal eigenvectors u_1, u_2 :

$$\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{25} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \mathbf{0} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Lab 12: SVD

Dimension reduction, toy example

```
A = [1 2 3;4 5 6; 7 8 9]
```

```
A = 3x3
 1   2   3
 4   5   6
 7   8   9
```

```
Rank = rank(A)
```

```
Rank = 2
```

```
[U,S,V] = svd(A)
```

```
U = 3x3
 -0.2148   0.8872   0.4082
 -0.5206   0.2496  -0.8165
 -0.8263  -0.3879   0.4082
```

```
S = 3x3
 16.8481      0      0
      0   1.0684      0
      0      0   0.0000
```

```
V = 3x3
 -0.4797   -0.7767  -0.4082
 -0.5724   -0.0757   0.8165
 -0.6651    0.6253  -0.4082
```

```
% Find full A again
```

```
A_full = U*S*V'
```

```
A_full = 3x3
 1.0000   2.0000   3.0000
 4.0000   5.0000   6.0000
 7.0000   8.0000   9.0000
```

```
% Reduce 3 el from S
```

```
A_2 = U(:,1:2)*S(1:2,1:2)*V(:,1:2)'
```

```
A_2 = 3x3
 1.0000   2.0000   3.0000
 4.0000   5.0000   6.0000
 7.0000   8.0000   9.0000
```

```
% reduce all columns except 1 one
```

```
A_1 = U(:,1)*S(1,1)*V(:,1)'
```

```
A_1 = 3x3
 1.7362   2.0717   2.4073
 4.2072   5.0202   5.8332
 6.6781   7.9686   9.2592
```

```
%result - dim the same, but info and rank changes
```

```
Rank_new = rank(A_1)
```

```
Rank_new = 1
```



Task 3

Construct the matrix with rank one that has $A\mathbf{v} = 12\mathbf{u}$ for $\mathbf{v} = \frac{1}{2}(1, 1, 1, 1)$ and $\mathbf{u} = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 = \underline{\hspace{2cm}}$.



Task 3

Construct the matrix with rank one that has $A\mathbf{v} = 12\mathbf{u}$ for $\mathbf{v} = \frac{1}{2}(1, 1, 1, 1)$ and $\mathbf{u} = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 = \underline{\hspace{2cm}}$.

Answer

A rank-1 matrix with $A\mathbf{v} = 12\mathbf{u}$ would have \mathbf{u} in its column space, so $A = \mathbf{u}\mathbf{w}^T$ for some vector \mathbf{w} . I intended (but didn't say) that \mathbf{w} is a multiple of the unit vector $\mathbf{v} = \frac{1}{2}(1, 1, 1, 1)$ in the problem. Then $A = 12\mathbf{u}\mathbf{v}^T$ to get $A\mathbf{v} = 12\mathbf{u}$ when $\mathbf{v}^T\mathbf{v} = 1$.



Singular Value Decomposition (SVD)

Where it can be used

- For working with big datasets
- Image compression (on page 19)
- Pseudo-inverse (next slide)
- Least square (on page 23)
- Principal Component Analysis (PCA) ([video](#))
- Eigenfaces algorithms ([video](#))



Pseudoinverse

- $J^\#$ always exists, and is the unique matrix satisfying

$$\begin{aligned} JJ^\# J &= J & J^\# JJ^\# &= J^\# \\ (JJ^\#)^T &= JJ^\# & (J^\# J)^T &= J^\# J \end{aligned}$$

- if J is full (row) rank, $J^\# = J^T(JJ^T)^{-1}$; else, it is computed numerically using the SVD (Singular Value Decomposition) of J (`pinv` of Matlab)



Computation of pseudoinverses

- show that the pseudoinverse of J is equal to

$$J = U\Sigma V^T \quad \Rightarrow \quad J^\# = V\Sigma^\# U^T \quad \Sigma^\# = \begin{pmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_\rho} & \\ \hline & & & 0_{(M-\rho) \times (M-\rho)} \\ & & & 0_{(N-M) \times M} \end{pmatrix}$$

for any rank ρ of J



Task 4

Put numbers into the singular value decomposition of A :

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$$

Put numbers into the pseudoinverse $V\Sigma^+U^T$ of A . Compute AA^+ and A^+A :



Task 4

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$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$$

Put numbers into the pseudoinverse $V\Sigma^+U^T$ of A . Compute AA^+ and A^+A :

Answer

$$A = [3 \ 4 \ 0] = [\mathbf{u}_1] [\sigma_1 \ 0 \ 0] [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]^T.$$

Pseudoinverse $A^+ = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ 0 \end{bmatrix} [\mathbf{u}_1]^T.$

$$A = [1] [5 \ 0 \ 0] V^T \text{ and } A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}; A^+ A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}; AA^+ = [1]$$

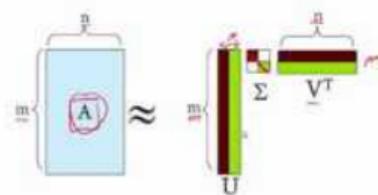


SVD Applications

Video: Users-to-Movies

Singular Value Decomposition

$$A \approx U\Sigma V^T = \sum_i \sigma_i u_i \circ v_i^T$$





Task 5

7.4 A If A has rank n (full column rank) then it has a **left inverse** $L = (A^T A)^{-1} A^T$. This matrix L gives $LA = I$. Explain why the pseudoinverse is $A^+ = L$ in this case.

If A has rank m (full row rank) then it has a **right inverse** $R = A^T (AA^T)^{-1}$. This matrix R gives $AR = I$. Explain why the pseudoinverse is $A^+ = R$ in this case.

Find L for A_1 and find R for A_2 . Find A^+ for all three matrices A_1, A_2, A_3 :

$$A_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$



Task 5

Answer

Solution If A has independent columns then $A^T A$ is invertible—this is a key point of Section 4.2. Certainly $L = (A^T A)^{-1} A^T$ multiplies A to give $LA = I$: a left inverse.

$AL = A(A^T A)^{-1} A^T$ is the projection matrix (Section 4.2) on the column space. So L meets the requirements on A^+ : LA and AL are projections on $C(A)$ and $C(A^T)$.

If A has rank m (full row rank) then AA^T is invertible. Certainly A multiplies $R = A^T(AA^T)^{-1}$ to give $AR = I$. In the opposite order, $RA = A^T(AA^T)^{-1} A$ is the projection matrix onto the row space (column space of A^T). So R equals the pseudoinverse A^+ .

The example A_1 has full column rank (for L) and A_2 has full row rank (for R):

$$A_1^+ = (A_1^T A_1)^{-1} A_1^T = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_2^+ = A_2^T (A_2 A_2^T)^{-1} = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Notice $A_1^+ A_1 = [1]$ and $A_2 A_2^+ = [1]$. But A_3 has no left or right inverse. Its rank is not full. Its pseudoinverse brings the column space of A_3 to the row space.

$$A_3^+ = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}^+ = \frac{\mathbf{v}_1 \mathbf{u}_1^T}{\sigma_1} = \frac{1}{10} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$



SVD Applications

Image compression

Task: We want to compress our image for reducing the size.

Solution: We can represent our picture as a matrix.

Next step is using SVD for reducing matrix rank.

Code on page 24

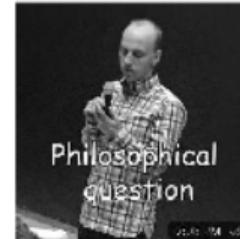
Full-Rank Logo



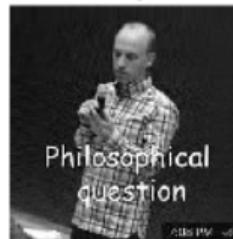
Rank 154 picture



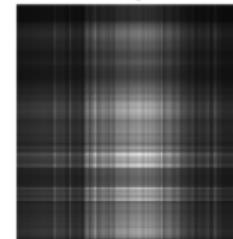
Rank 103 picture



Rank 52 picture



Rank 2 picture



Rank 1 picture





Task 6

All matrices in this problem have rank one. The vector b is (b_1, b_2) .

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad AA^T = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad A^TA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \quad A^+ = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix}$$

- (a) The equation $A^TA\hat{x} = A^Tb$ has many solutions because A^TA is ____.
- (b) Verify that $x^+ = A^+b = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^TAx^+ = A^Tb$.
- (c) Add $(1, -1)$ to that x^+ to get another solution to $A^TA\hat{x} = A^Tb$. Show that $\|\hat{x}\|^2 = \|x^+\|^2 + 2$, and x^+ is shorter.



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All matrices in this problem have rank one. The vector b is (b_1, b_2) .

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad AA^T = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad A^TA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \quad A^+ = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix}$$

- (a) The equation $A^TA\hat{x} = A^Tb$ has many solutions because A^TA is ____.
- (b) Verify that $x^+ = A^+b = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^TAx^+ = A^Tb$.
- (c) Add $(1, -1)$ to that x^+ to get another solution to $A^TA\hat{x} = A^Tb$. Show that $\|\hat{x}\|^2 = \|x^+\|^2 + 2$, and x^+ is shorter.

Answer

- (a) A^TA is singular
- (b) This x^+ in the row space does give $A^TAx^+ = A^Tb$
- (c) If $(1, -1)$ in the nullspace of A is added to x^+ , we get another solution to $A^TA\hat{x} = A^Tb$. But this \hat{x} is longer than x^+ because the added part is orthogonal to x^+ in the row space and $\|\hat{x}\|^2 = \|x^+\|^2 + \|\text{added part from nullspace}\|^2$.



Reference material

- Lecture 28: Similar Matrices and Jordan Form.
- Lecture 29: Singular Value Decomposition
- Lecture 33: Left and Right Inverses; Pseudoinverse
- 6. Singular Value Decomposition (SVD)
- "*Introduction to Linear Algebra*", pdf pages 375–411
 7 Singular Value Decomposition (SVD)
- "*Linear Algebra and Applications*", pdf pages 335–345
 5.6 Similarity Transformations
- "*Linear Algebra and Applications*", pdf pages 377–386
 6.3 Singular Value Decomposition

Deserve “A” grade!

– Oleg Bulichev

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↗ @Lupasic

🚪 Room 105 (Underground robotics lab)



Appendix: Line fitting pdf

Next Slide

Lab 12: SVD

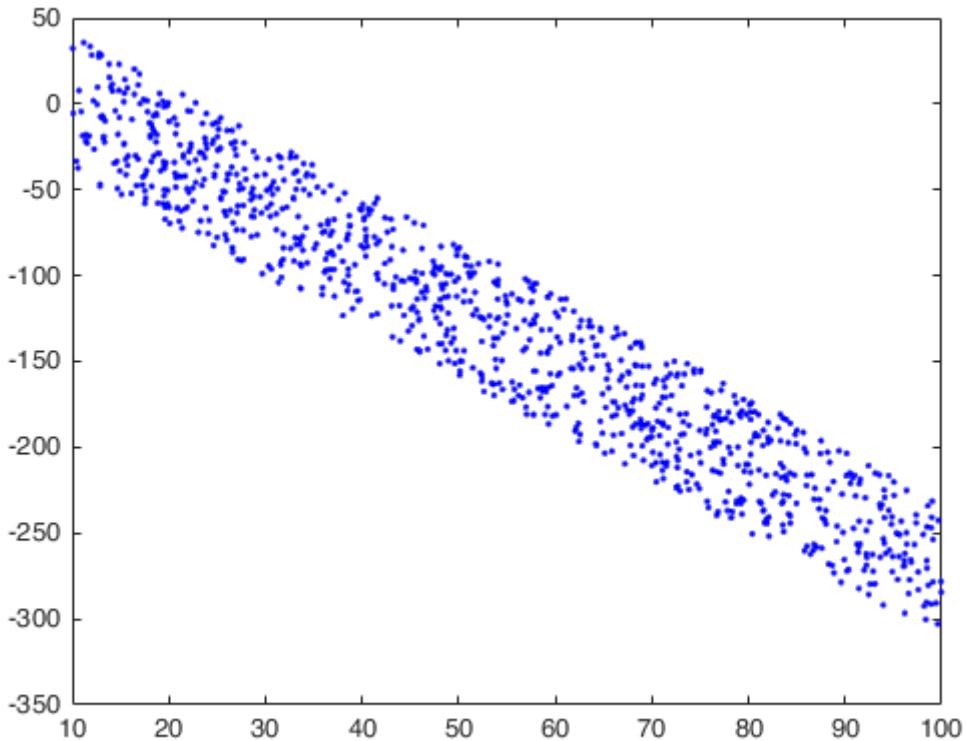
Two ways to fit point cloud to a line

1 - take SVD of our matrix $Ax=0$ (we need to put b in to A matrix), take the smallest V vector it will be a solution.

2 - Classical least square solution via pseudo inverse

2nd approach is more accurate, it can be seen by error comparison

```
% Generate some points around a line
intercept = -10; slope = -3;
npts = 1000; noise = 80;
xs = 10 + rand(npts, 1) * 90;
ys = slope * xs + intercept + rand(npts, 1) * noise;
% xs = [1;2;3]
% ys = [1;2;1]
% npts = 3;
% Plot the randomly generated points
figure; plot(xs, ys, 'b.', 'MarkerSize', 5)
```



```
% Fit these points to a line - 1st approach
A = [xs, ys, -1 * ones(npts, 1)];
[U, S, V] = svd(A);
fit = V(:, end-1)
```

```
fit =
-0.9326
```

```
-0.3590
0.0362
```

```
% Get the coefficients a, b, c in ax + by + c = 0
a = fit(1); b = fit(2); c = fit(3);

% Compute slope m and intercept i for y = mx + i
slope_est = -a/b;
intercept_est = c/b;

% Plot fitted line on top of old data
ys_est = slope_est * xs + intercept_est;
figure; plot(xs, ys, 'b.', 'MarkerSize', 5);
hold on; plot(xs, ys_est, 'r-')
% Error
sum_err_line = sum((ys_est-ys).^2)
```

```
sum_err_line = 7.0883e+05
```

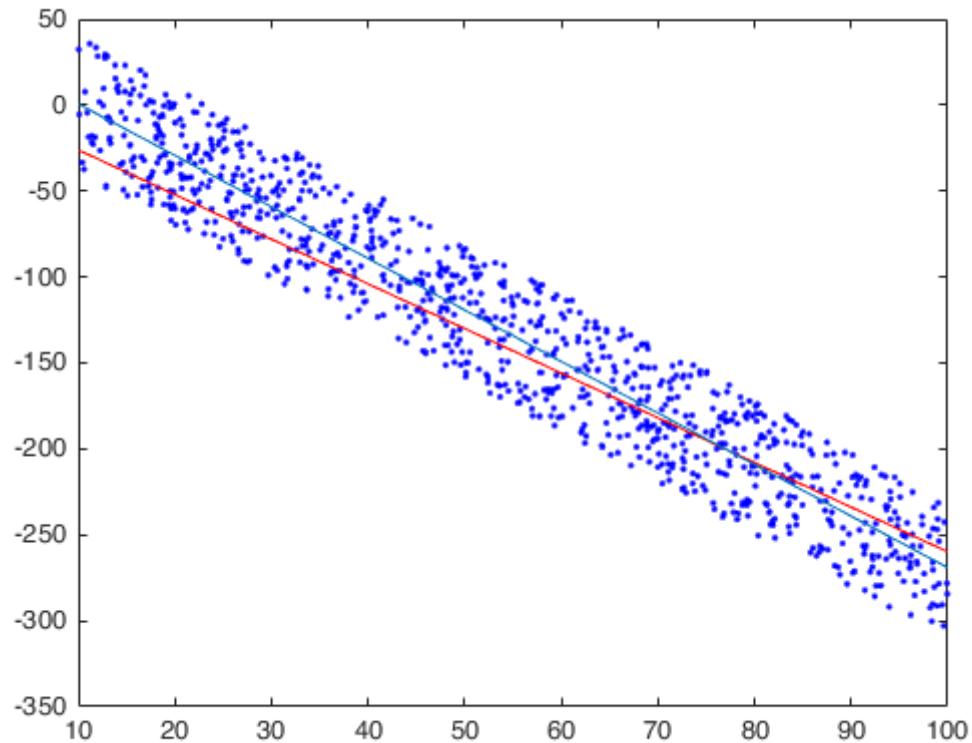
```
% Fit these points to a line - 2nd approach
fit1 = pinv([A(:,1) 1 * ones(npts, 1)])*A(:,2)
```

```
fit1 =
-2.9980
30.6925
```

```
k = fit1(1); b = fit1(2);
slope_est1 = k;
intercept_est1 = b;
ys_est1 = slope_est1 * xs + intercept_est1;
% Error
sum_err_line1 = sum((ys_est1-ys).^2)
```

```
sum_err_line1 = 5.2035e+05
```

```
% Blue one - 2nd approach, Red one - 1st
hold on; plot(xs, ys_est1)
```





Appendix: Image compressing pdf

Next Slide

Lab 12: SVD

Image compressing - Gorodetskii and Cell

Cell is needed for understanding what does information means

```
pic_name = ['tsar.jpg';'cell.jpg']

pic_name = 2x8 char array
    'tsar.jpg'
    'cell.jpg'

for pic_num=1:size(pic_name,1)
    logo_num = im2double(rgb2gray(imread(pic_name(pic_num,:))));
    [U, S, V] = svd(logo_num);
    % Compute SVD of this picture
    [U, S, V] = svd(logo_num);
    S_myau = S;
    % Plot the magnitude of the singular values (log scale)
    sigmas = diag(S);
    figure; plot(log10(sigmas)); title('Singular Values (Log10 Scale)');
    % It shows how much information will be after reducing matrix rank
    figure; plot(cumsum(sigmas) / sum(sigmas)); title('Cumulative Percent of Total Sigmas');

    % Show full-rank picture
    figure; subplot(2, 3, 1), imshow(logo_num), title('Full-Rank Logo');

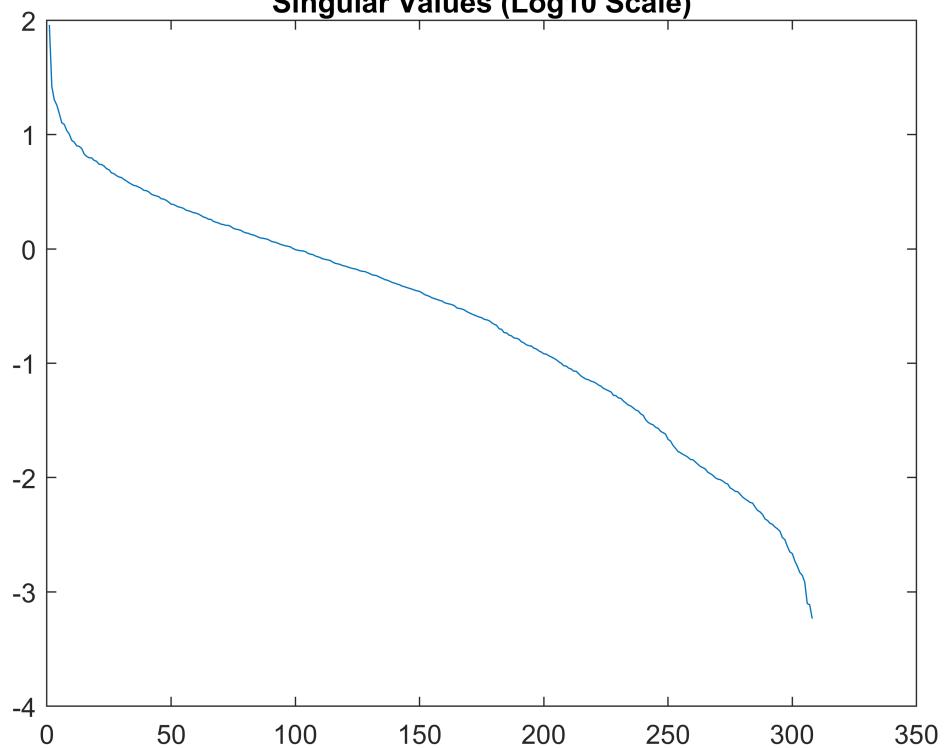
    % Compute low-rank approximations of the picture, and show them
    ranks = [ceil(rank(S)/2), ceil(rank(S)/3), ceil(rank(S)/6), 2, 1];
    for i = 1:length(ranks)
        % Keep largest singular values, and nullify others.
        approx_sigmas = sigmas; approx_sigmas(ranks(i):end) = 0;

        % Form the singular value matrix, padded as necessary
        ns = length(sigmas);
        approx_S = S; approx_S(1:ns, 1:ns) = diag(approx_sigmas);

        % Compute low-rank approximation by multiplying out component matrices.
        approx_logo = U * approx_S * V';

        % Plot approximation
        subplot(2, 3, i + 1), imshow(approx_logo), title(sprintf('Rank %d picture', ranks(i)));
    end
end
```

Singular Values (Log10 Scale)



Cumulative Percent of Total Sigmas

