Analytical Geometry and Linear Algebra II, Lab 12

Similar matrices

Singular value decomposition - SVD

Left and right inverses. Pseudoinverse

Similar Matrices

All the matrices $A = BCB^{-1}$ are "similar". They all share the eigenvalues of C.

Example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \to \lambda = 3,\ 1;\ S^{-1}AS = \Lambda_A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, S – eigenvectors

Let's find other similar matrix: $M^{-1}AM = T$, $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, M is random matrix. $T = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$, which λ are also $3,\ 1$.

Similar Matrices

Properties

Because matrices are similar if and only if they represent the *same linear operator with respect to (possibly) different bases*, similar matrices share all properties of their shared underlying operator:

- Rank
- Characteristic polynomial, and attributes that can be derived from it:
  - Eigenvalues
  - Determinant
  - Trace

More info can be found here. This topic was used for checking understanding LA by the students (paper).

The primary goal of SVD

To “X-RAY” matrix

(To understand the structure of matrix)

Singular Value Decomposition (SVD)

3 ways of explanation

1. Linear transformation – Kutz video
2. Algebraic – MIT video (Strang), Aaron Greiner video
3. As a tool for DS – Stanford video, Brunton video

According to Kholodov words, SVD was created for *finding an inverse for any matrices*. It is needed in linear transformation related operations. Other properties were found afterwards.

Singular Value Decomposition

$E' = AE$ → Change of basis (AGLA1)

Old basis. We assume, it is orthonormal basis

New basis

Transition matrix

Let's rewrite it in common SVD notation

$$AV = E'$$

$$E' = U \Sigma I$$

$$\begin{bmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} u_{1x}\sigma_1 & u_{2x}\sigma_2 \\ u_{1y}\sigma_1 & u_{2y}\sigma_2 \end{bmatrix}$$

Matrix order appears because we want stretch columns

Rotation component of E'

Stretching component of E'

$$A = U \Sigma V^H$$

$$A^TA = (U\Sigma V^H)^T (U\Sigma V^H) = V\Sigma^2 V^H; \quad AA^T = U\Sigma^2 U^H \ldots$$

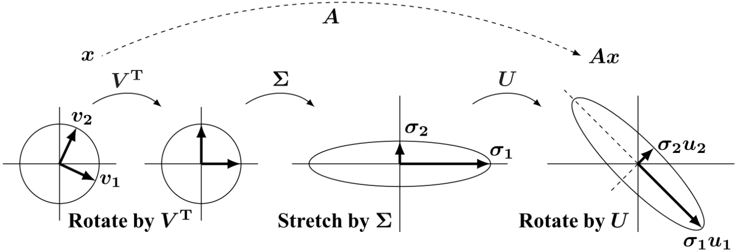
$$A^TAV = V\Sigma^2 \overbrace{V^H V}^{I}$$

Common eigen problem. We can find both V and sigma.

If singular values are distinct, then U and V are unique

Singular Value Decomposition

Geometrical explanation



$$\boldsymbol{A} = (\textbf{Orthogonal})\,(\textbf{Diagonal})\,(\textbf{Orthogonal})$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Four numbers $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in A produce four numbers $\boldsymbol{\theta}, \boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\phi}$ in the SVD

Singular Value Decomposition (SVD)

How to calculate it (2 common ways)

First approach

1. Find eigenpairs for A^TA. Result is Σ and V. ($A^TA = V\Sigma^2V^H$)
2. Find U, using Σ and V ($AV\Sigma^{-1} = U$)

Second approach

1. Find eigenpairs for A^TA and AA^T. ($A^TA = V\Sigma^2V^H$) ($AA^T = U\Sigma^2U^H$)

Singular Value Decomposition (SVD)

Obtain SVD for $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$*, using second approach (task was taken)*

1. Eigenpairs of AA^T.

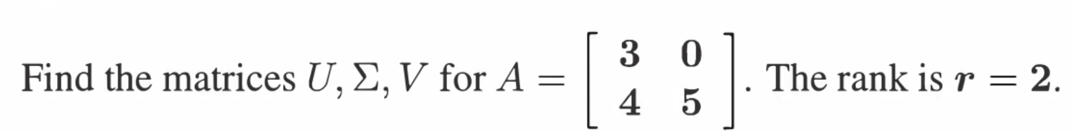
$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.\ \lambda_1 = 25, \lambda_2 = 9.\ x_{\lambda_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, x_{\lambda_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.\ U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2. Eigenpairs of A^TA.

$$A^TA = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}.\ \lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0.\ V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \end{bmatrix}$$

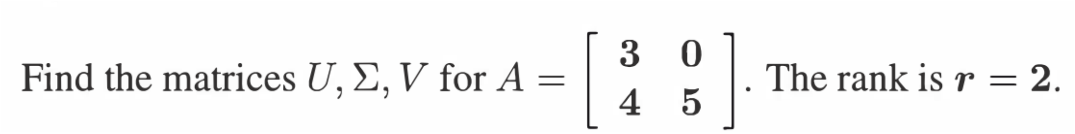
3. Result. $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

Task 1



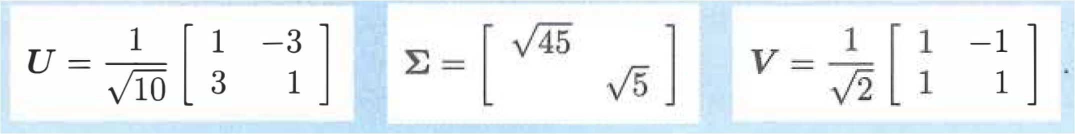
Find the matrices U, Σ, V for $A = \begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} \end{bmatrix}$. The rank is $\boldsymbol{r} = \mathbf{2}$.

Task 1



Find the matrices U, Σ, V for $A = \begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} \end{bmatrix}$. The rank is $\boldsymbol{r} = \mathbf{2}$.

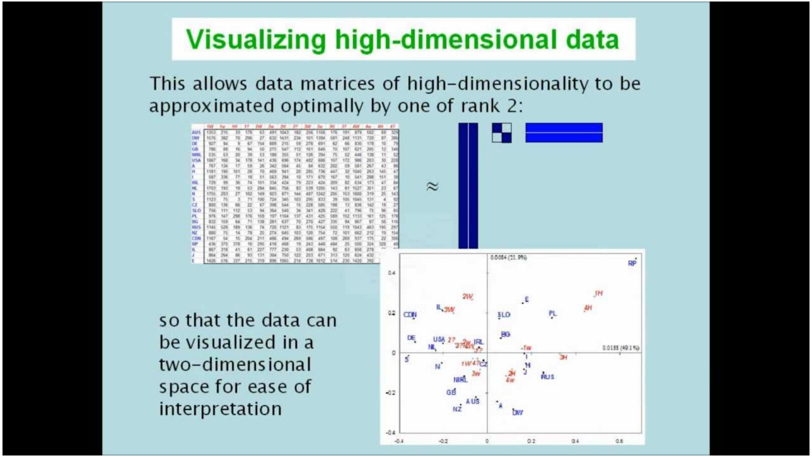
Answer



$$\boldsymbol{U} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix} \qquad \boldsymbol{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It had to be *U*

Video



Visualizing high-dimensional data

This allows data matrices of high-dimensionality to be approximated optimally by one of rank 2:

≈

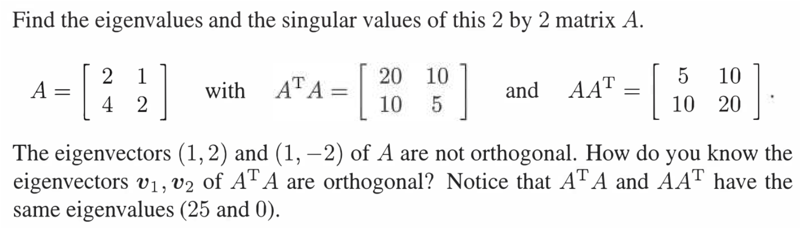
so that the data can be visualized in a two-dimensional space for ease of interpretation

Singular Value Decomposition (SVD)

Properties

- It is always possible to decompose a real matrix A into SVD
- $U,\ \Sigma,\ V$ are unique
- $U,\ V$ – column orthonormal
- By convention Σ contains singular values in sorted order $\sigma_1 \geq \sigma_2 \ldots$

Task 2



Find the eigenvalues and the singular values of this 2 by 2 matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{with} \quad A^{\mathrm{T}}A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \quad \text{and} \quad AA^{\mathrm{T}} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}.$$

The eigenvectors $(1, 2)$ and $(1, -2)$ of A are not orthogonal. How do you know the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$ are orthogonal? Notice that $A^{\mathrm{T}}A$ and AA^{T} have the same eigenvalues (25 and 0).

Task 2

Answer

The matrix A has trace 4 and determinant 0. So its eigenvalues are 4 and 0—*not used in the SVD*! The matrix $A^{\mathrm{T}}A$ has trace 25 and determinant 0, so $\lambda_1 = 25 = \sigma_1^2$ gives $\sigma_1 = 5$.

The eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$ (a symmetric matrix !) are orthogonal :

$$\begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{25} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{0} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly AA^{T} has orthogonal eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2$:

$$\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{25} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \mathbf{0} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Lab 12: SVD

Dimension reduction, toy example

```
A = [1 2 3;4 5 6; 7 8 9]
```

```
A = 3×3
     1     2     3
     4     5     6
     7     8     9
```

```
Rank = rank(A)
```

```
Rank = 2
```

```
[U,S,V] = svd(A)
```

```
U = 3×3
   -0.2148    0.8872    0.4082
   -0.5206    0.2496   -0.8165
   -0.8263   -0.3879    0.4082
S = 3×3
   16.8481         0         0
         0    1.0684         0
         0         0    0.0000
V = 3×3
   -0.4797   -0.7767   -0.4082
   -0.5724   -0.0757    0.8165
   -0.6651    0.6253   -0.4082
```

```
% Find full A again
A_full = U*S*V'
```

```
A_full = 3×3
    1.0000    2.0000    3.0000
    4.0000    5.0000    6.0000
    7.0000    8.0000    9.0000
```

```
% Reduce 3 el from S
A_2 =U(:,1:2)*S(1:2,1:2)*V(:,1:2)'
```

```
A_2 = 3×3
    1.0000    2.0000    3.0000
    4.0000    5.0000    6.0000
    7.0000    8.0000    9.0000
```

```
% reduce all columns exept 1 one
A_1=U(:,1)*S(1,1)*V(:,1)'
```

```
A_1 = 3×3
    1.7362    2.0717    2.4073
    4.2072    5.0202    5.8332
    6.6781    7.9686    9.2592
```

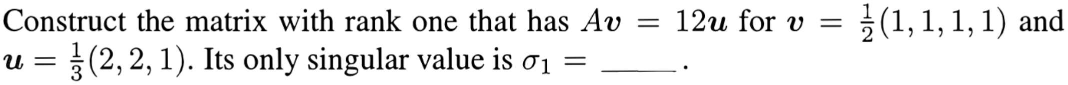
```
%result - dim the same, but info and rank changes
Rank_new = rank(A_1)
```

```
Rank_new = 1
```

Task 3

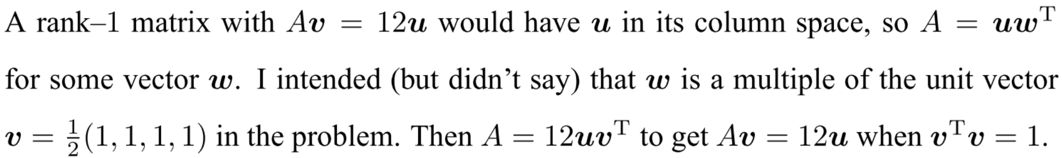
Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1, 1, 1, 1)$ and $\boldsymbol{u} = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 =$ ______ .

Task 3



Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ and $\boldsymbol{u} = \frac{1}{3}(2,2,1)$. Its only singular value is $\sigma_1 =$ _____ .

Answer



A rank–1 matrix with $A\boldsymbol{v} = 12\boldsymbol{u}$ would have $\boldsymbol{u}$ in its column space, so $A = \boldsymbol{u}\boldsymbol{w}^{\mathrm{T}}$ for some vector $\boldsymbol{w}$. I intended (but didn't say) that $\boldsymbol{w}$ is a multiple of the unit vector $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ in the problem. Then $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to get $A\boldsymbol{v} = 12\boldsymbol{u}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 1$.

Singular Value Decomposition (SVD)

Where it can be used

- For working with big datasets
- Image compression (on page 18)
- Pseudo-inverse (next slide)
- Least square (on page 22)
- Principal Component Analysis (PCA) (video)
- Eigenfaces algorithms (video)

Pseudoinverse

- $J^{\#}$ always exists, and is the unique matrix satisfying

$$J\,J^{\#}J = J \qquad J^{\#}J\,J^{\#} = J^{\#}$$
$$\left(J\,J^{\#}\right)^T = J\,J^{\#} \qquad \left(J^{\#}J\right)^T = J^{\#}J$$

- if J is full (row) rank, $J^{\#} = J^T(J\,J^T)^{-1}$; else, it is computed numerically using the SVD (Singular Value Decomposition) of J (`pinv` of Matlab)

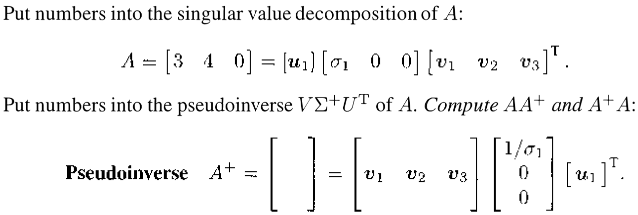
Computation of pseudoinverses

- show that the pseudoinverse of J is equal to

$$J = U\Sigma V^T \quad \Rightarrow \quad J^\# = V\Sigma^\# U^T \qquad \Sigma^\# = \left(\begin{array}{cccc} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_\rho} & \\ & & & 0_{(M-\rho)\times(M-\rho)} \\ \hline \multicolumn{4}{c}{0_{(N-M)\times M}} \end{array}\right)$$

for any rank ρ of J

Task 4



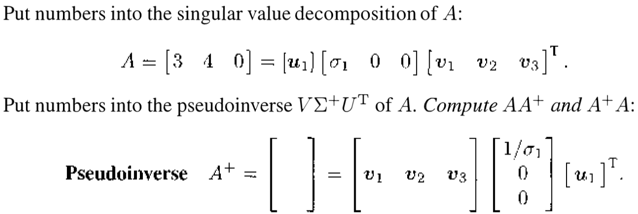
Put numbers into the singular value decomposition of A:

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}^{\mathrm{T}}.$$

Put numbers into the pseudoinverse $V\Sigma^{+}U^{\mathrm{T}}$ of A. *Compute* AA^{+} *and* $A^{+}A$:

$$\textbf{Pseudoinverse} \quad A^{+} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \end{bmatrix}^{\mathrm{T}}.$$

Task 4



Put numbers into the singular value decomposition of A:

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} u_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathrm{T}}.$$

Put numbers into the pseudoinverse $V\Sigma^+U^{\mathrm{T}}$ of A. *Compute AA^+ and A^+A*:

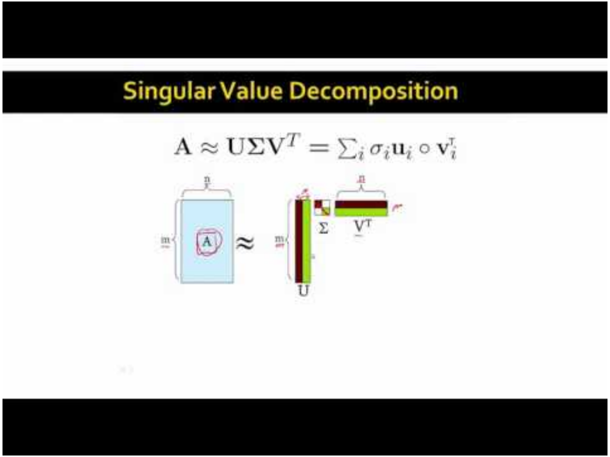
$$\textbf{Pseudoinverse} \quad A^+ = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix}^{\mathrm{T}}.$$

Answer

$$A = [\,1\,]\,[\,5 \quad 0 \quad 0\,]V^{\mathrm{T}} \text{ and } A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix};\ A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ AA^+ = [\,1\,]$$

SVD Applications

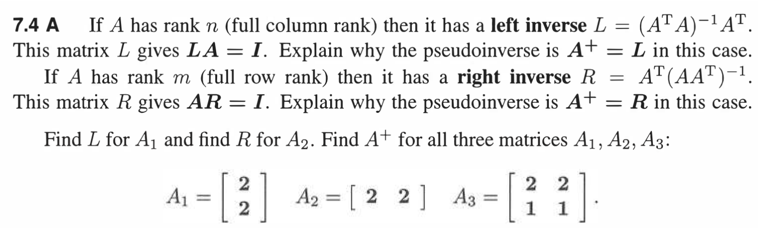
Video: Users-to-Movies



Singular Value Decomposition

$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$

Task 5



7.4 A If A has rank n (full column rank) then it has a **left inverse** $L = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. This matrix L gives $\boldsymbol{LA = I}$. Explain why the pseudoinverse is $\boldsymbol{A^+ = L}$ in this case.

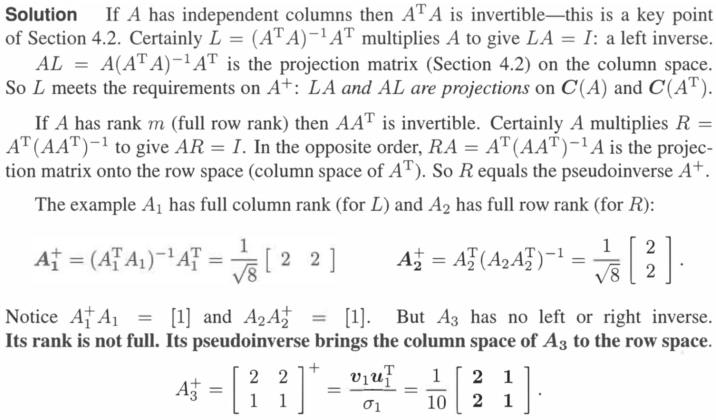
If A has rank m (full row rank) then it has a **right inverse** $R = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$. This matrix R gives $\boldsymbol{AR = I}$. Explain why the pseudoinverse is $\boldsymbol{A^+ = R}$ in this case.

Find L for A_1 and find R for A_2. Find A^+ for all three matrices A_1, A_2, A_3:

$$A_1 = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix} \quad A_2 = \begin{bmatrix} \mathbf{2} & \mathbf{2} \end{bmatrix} \quad A_3 = \begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

Task 5

Answer



Solution If A has independent columns then $A^{\mathrm{T}}A$ is invertible—this is a key point of Section 4.2. Certainly $L=(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ multiplies A to give $LA=I$: a left inverse. $AL=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is the projection matrix (Section 4.2) on the column space. So L meets the requirements on A^+: *LA and AL are projections* on $\boldsymbol{C}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

If A has rank m (full row rank) then AA^{T} is invertible. Certainly A multiplies $R=A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ to give $AR=I$. In the opposite order, $RA=A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A$ is the projection matrix onto the row space (column space of A^{T}). So R equals the pseudoinverse A^+.

The example A_1 has full column rank (for L) and A_2 has full row rank (for R):

$$\boldsymbol{A_1^+}=(A_1^{\mathrm{T}}A_1)^{-1}A_1^{\mathrm{T}}=\frac{1}{\sqrt{8}}\begin{bmatrix}2 & 2\end{bmatrix} \qquad \boldsymbol{A_2^+}=A_2^{\mathrm{T}}(A_2A_2^{\mathrm{T}})^{-1}=\frac{1}{\sqrt{8}}\begin{bmatrix}2\\2\end{bmatrix}.$$

Notice $A_1^+A_1=[1]$ and $A_2A_2^+=[1]$. But A_3 has no left or right inverse. **Its rank is not full. Its pseudoinverse brings the column space of A_3 to the row space.**

$$A_3^+=\begin{bmatrix}2 & 2\\1 & 1\end{bmatrix}^+=\frac{\boldsymbol{v}_1\boldsymbol{u}_1^{\mathrm{T}}}{\sigma_1}=\frac{1}{10}\begin{bmatrix}\mathbf{2} & \mathbf{1}\\\mathbf{2} & \mathbf{1}\end{bmatrix}.$$

SVD Applications

Image compression

Task: We want to compress our image for reducing the size.

Solution: We can represent our picture as a matrix.

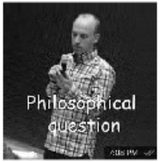
Next step is using SVD for reducing matrix rank.

Code on page 23

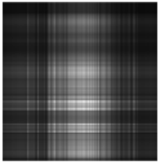
Full-Rank Logo

Rank 154 picture

Rank 103 picture



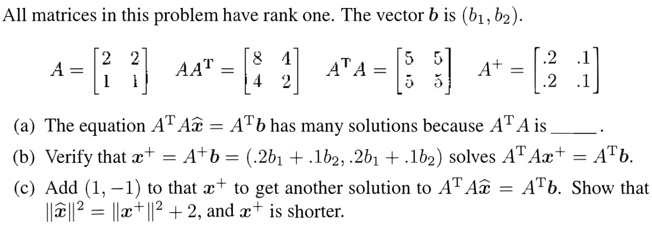
Rank 52 picture



Rank 2 picture

Rank 1 picture

Task 6



All matrices in this problem have rank one. The vector $\boldsymbol{b}$ is (b_1, b_2).

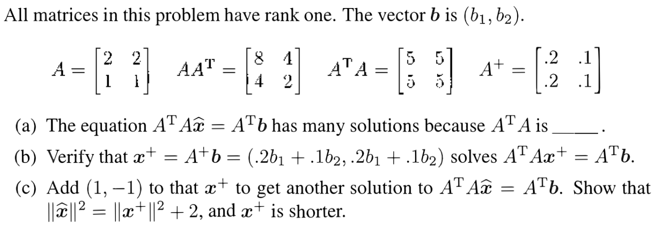
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad AA^{\mathrm{T}} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \quad A^{+} = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix}$$

(a) The equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ has many solutions because $A^{\mathrm{T}}A$ is _____ .

(b) Verify that $\boldsymbol{x}^{+} = A^{+}\boldsymbol{b} = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^{\mathrm{T}}A\boldsymbol{x}^{+} = A^{\mathrm{T}}\boldsymbol{b}$.

(c) Add $(1, -1)$ to that $\boldsymbol{x}^{+}$ to get another solution to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Show that $\|\widehat{\boldsymbol{x}}\|^2 = \|\boldsymbol{x}^{+}\|^2 + 2$, and $\boldsymbol{x}^{+}$ is shorter.

Task 6



All matrices in this problem have rank one. The vector $\boldsymbol{b}$ is (b_1, b_2).

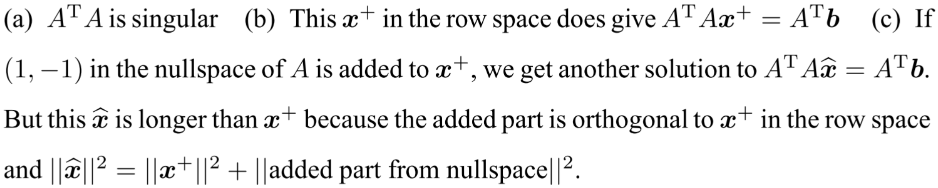
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad AA^{\mathrm{T}} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \quad A^{+} = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix}$$

(a) The equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ has many solutions because $A^{\mathrm{T}}A$ is _____ .

(b) Verify that $\boldsymbol{x}^{+} = A^{+}\boldsymbol{b} = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^{\mathrm{T}}A\boldsymbol{x}^{+} = A^{\mathrm{T}}\boldsymbol{b}$.

(c) Add $(1, -1)$ to that $\boldsymbol{x}^{+}$ to get another solution to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Show that $||\widehat{\boldsymbol{x}}||^2 = ||\boldsymbol{x}^{+}||^2 + 2$, and $\boldsymbol{x}^{+}$ is shorter.

Answer



(a) $A^{\mathrm{T}}A$ is singular (b) This $\boldsymbol{x}^{+}$ in the row space does give $A^{\mathrm{T}}A\boldsymbol{x}^{+} = A^{\mathrm{T}}\boldsymbol{b}$ (c) If $(1, -1)$ in the nullspace of A is added to $\boldsymbol{x}^{+}$, we get another solution to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. But this $\widehat{\boldsymbol{x}}$ is longer than $\boldsymbol{x}^{+}$ because the added part is orthogonal to $\boldsymbol{x}^{+}$ in the row space and $||\widehat{\boldsymbol{x}}||^2 = ||\boldsymbol{x}^{+}||^2 + ||\text{added part from nullspace}||^2$.

Reference material

- Lecture 28: Similar Matrices and Jordan Form.
- Lecture 29: Singular Value Decomposition
- Lecture 33: Left and Right Inverses; Pseudoinverse
- 6. Singular Value Decomposition (SVD)
- ”*Introduction to Linear Algebra*”, *pdf pages 375–411*
  7 Singular Value Decomposition (SVD)
- ”*Linear Algebra and Applications*”, *pdf pages 335–345*
  5.6 Similarity Transformations
- ”*Linear Algebra and Applications*”, *pdf pages 377–386*
  6.3 Singular Value Decomposition

Deserve “A” grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru

@Lupasic

Room 105 (Underground robotics lab)

Appendix: Line fitting pdf

Next Slide

Lab 12: SVD

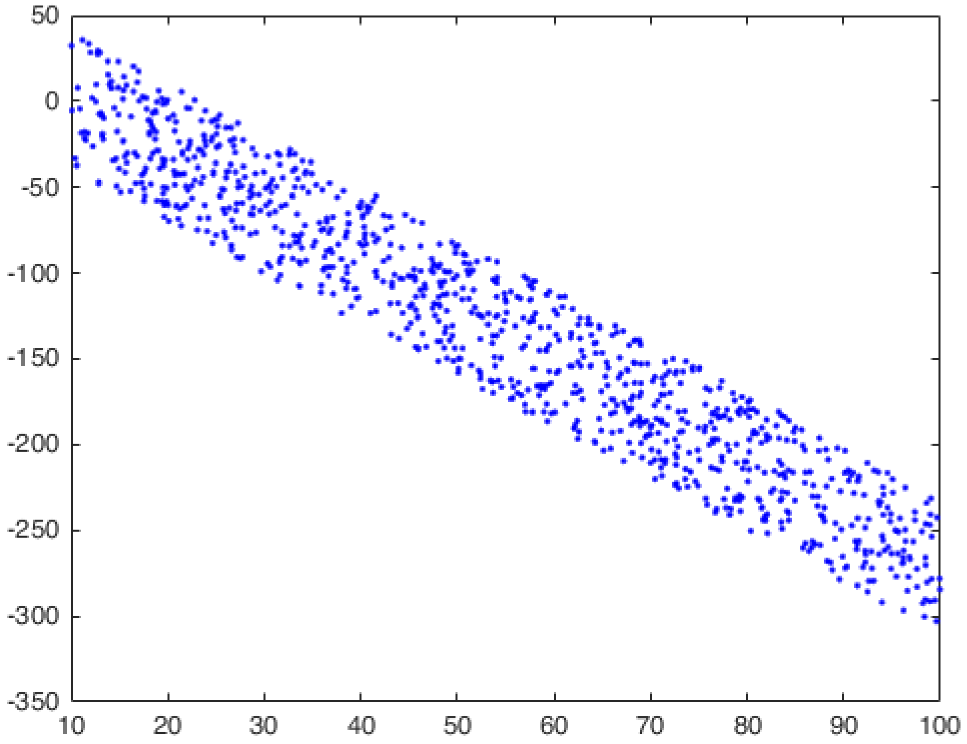
Two ways to fit point cloud to a line

1 - take SVD of our matrix Ax=0 (we need to put b in to A matrix), take the smallest V vector it will be a solution.

2 - Classical least square solution via pseudo inverse

2nd approach is more accurate, it can be seen by error comparison

```
% Generate some points around a line
intercept = -10; slope = -3;
npts = 1000; noise = 80;
xs = 10 + rand(npts, 1) * 90;
ys = slope * xs + intercept + rand(npts, 1) * noise;
% xs = [1;2;3]
% ys = [1;2;1]
% npts = 3;
% Plot the randomly generated points
figure; plot(xs, ys, 'b.', 'MarkerSize', 5)
```



```
% Fit these points to a line - 1st approach
A = [xs, ys, -1 * ones(npts, 1)];
[U, S, V] = svd(A);
fit = V(:, end-1)
```

```
fit =
   -0.9326
```

```
   -0.3590
    0.0362
```

```
% Get the coefficients a, b, c in ax + by + c = 0
a = fit(1); b = fit(2); c = fit(3);

% Compute slope m and intercept i for y = mx + i
slope_est = -a/b;
intercept_est = c/b;

% Plot fitted line on top of old data
ys_est = slope_est * xs + intercept_est;
figure; plot(xs, ys, 'b.', 'MarkerSize', 5);
hold on; plot(xs, ys_est, 'r-')
% Error
sum_err_line = sum((ys_est-ys).^2)
```

```
sum_err_line = 7.0883e+05
```

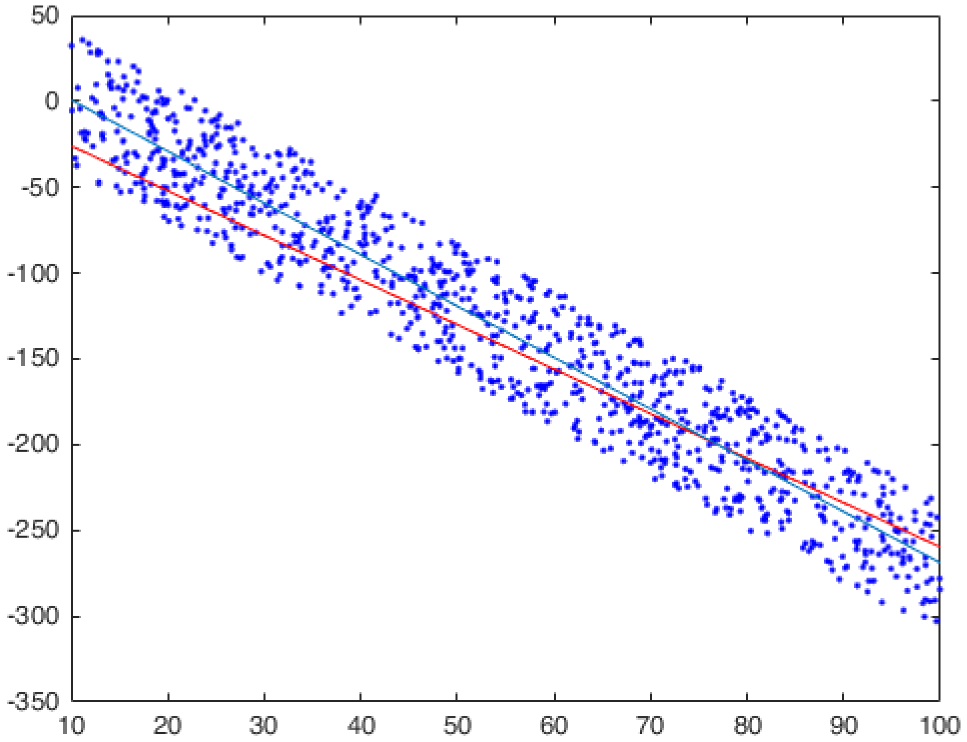
```
% Fit these points to a line - 2nd approach
fit1 = pinv([A(:,1) 1 * ones(npts, 1)])*A(:,2)
```

```
fit1 =
   -2.9980
   30.6925
```

```
k = fit1(1); b = fit1(2);
slope_est1 = k;
intercept_est1 = b;
ys_est1 = slope_est1 * xs + intercept_est1;
% Error
sum_err_line1 = sum((ys_est1-ys).^2)
```

```
sum_err_line1 = 5.2035e+05
```

```
% Blue one - 2nd approach, Red one - 1st
hold on; plot(xs, ys_est1)
```


50
0
-50
-100
-150
-200
-250
-300
-350
10
20
30
40
50
60
70
80
90
100

Appendix: Image compressing pdf

Next Slide

Lab 12: SVD

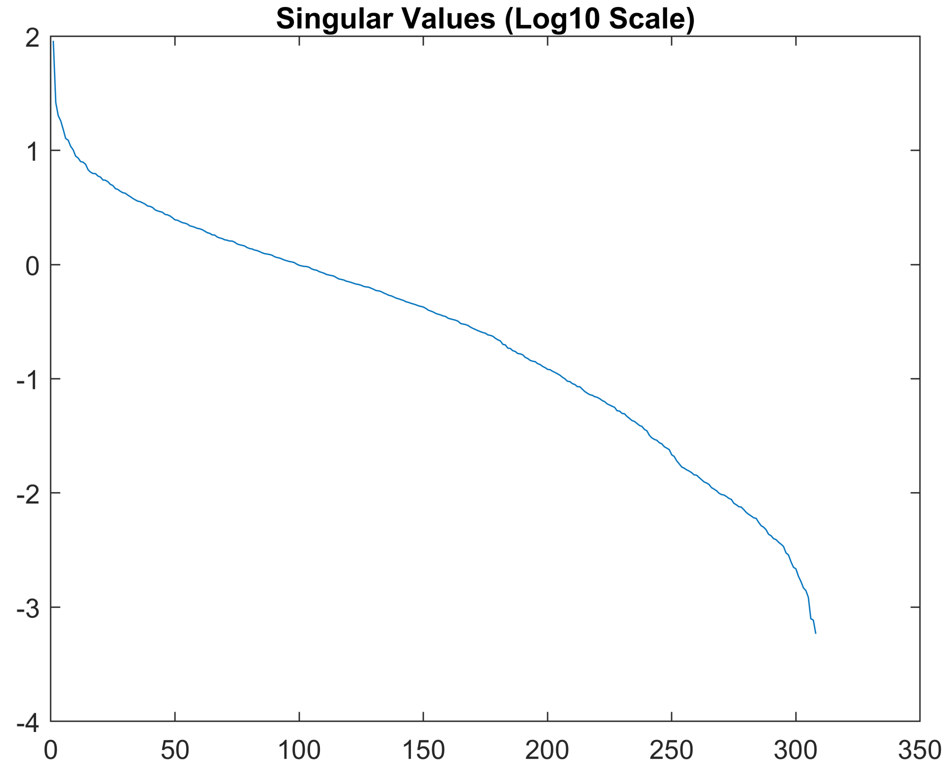
Image compressing - Gorodetskii and Cell

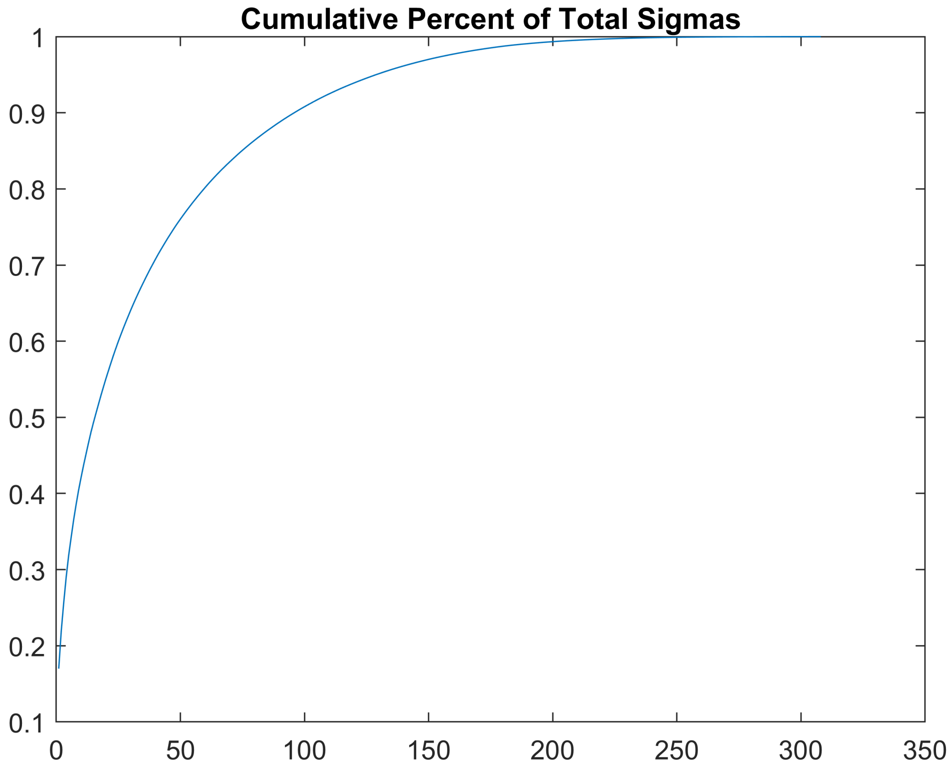
Cell is needed for understanding what does information means

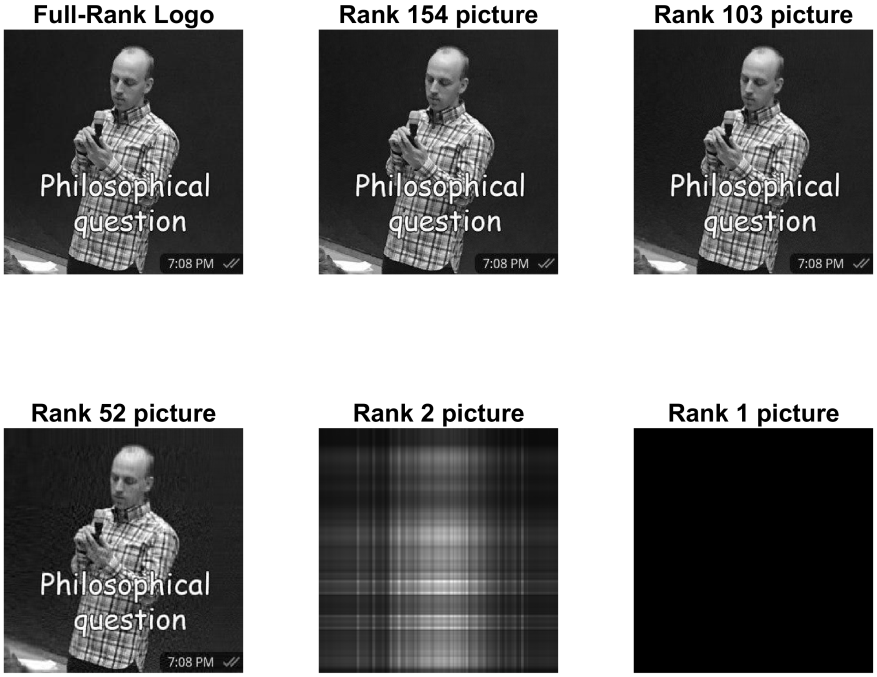
```
pic_name = ['tsar.jpg';'cell.jpg']
```

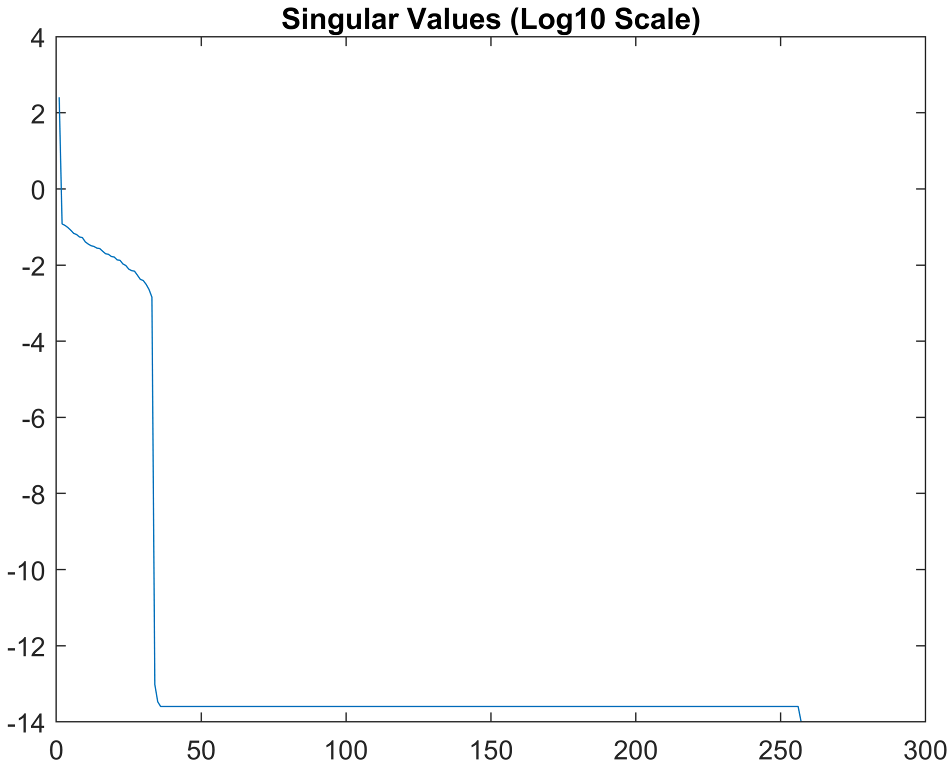
```
pic_name = 2×8 char array
    'tsar.jpg'
    'cell.jpg'
```

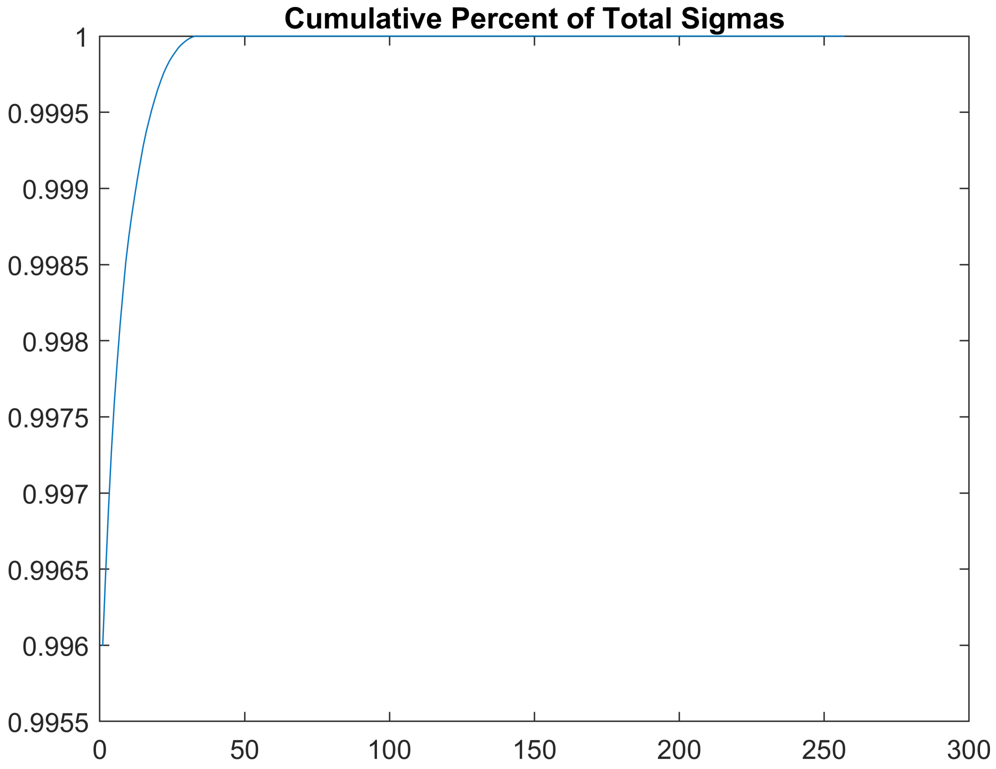
```
for pic_num=1:size(pic_name,1)
    logo_num = im2double(rgb2gray(imread(pic_name(pic_num,:))));
    [U, S, V] = svd(logo_num);
    % Compute SVD of this picture
    [U, S, V] = svd(logo_num);
    S_myau = S;
    % Plot the magnitude of the singular values (log scale)
    sigmas = diag(S);
    figure; plot(log10(sigmas)); title('Singular Values (Log10 Scale)');
    % It shows how much information will be after redusing matrix rank
    figure; plot(cumsum(sigmas) / sum(sigmas)); title('Cumulative Percent of Total Sigmas');

    % Show full-rank picture
    figure; subplot(2, 3, 1), imshow(logo_num), title('Full-Rank Logo');

    % Compute low-rank approximations of the picture, and show them
    ranks = [ceil(rank(S)/2), ceil(rank(S)/3), ceil(rank(S)/6), 2, 1];
    for i = 1:length(ranks)
        % Keep largest singular values, and nullify others.
        approx_sigmas = sigmas; approx_sigmas(ranks(i):end) = 0;

        % Form the singular value matrix, padded as necessary
        ns = length(sigmas);
        approx_S = S; approx_S(1:ns, 1:ns) = diag(approx_sigmas);

        % Compute low-rank approximation by multiplying out component matrices.
        approx_logo = U * approx_S * V';

        % Plot approximation
        subplot(2, 3, i + 1), imshow(approx_logo), title(sprintf('Rank %d picture', ranks(i)))
    end
end
```


Singular Values (Log10 Scale)
2
1
0
-1
-2
-3
-4
0
50
100
150
200
250
300
350

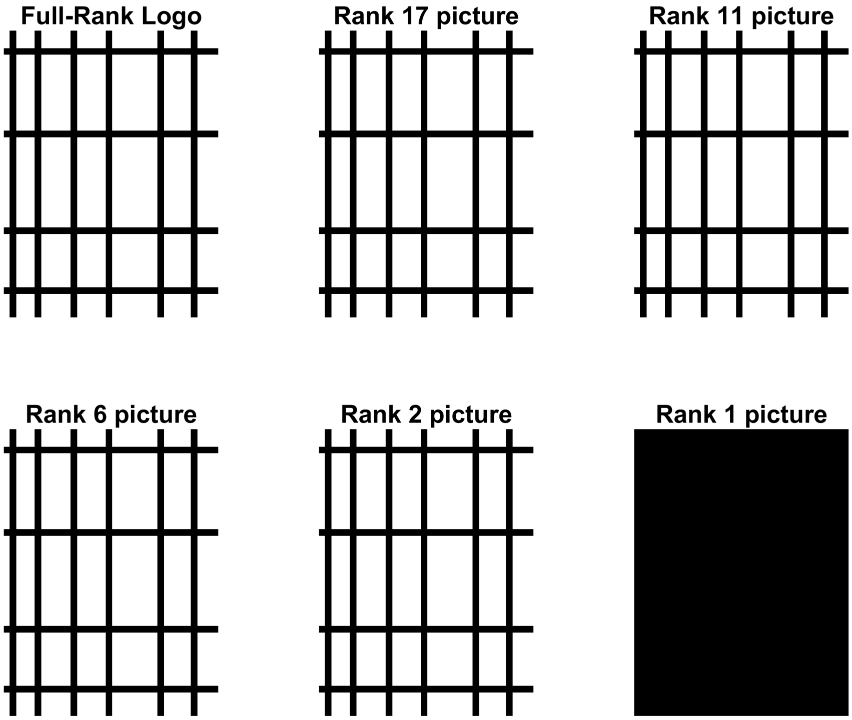

Cumulative Percent of Total Sigmas
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
250
300
350


Full-Rank Logo
Rank 154 picture
Rank 103 picture
Rank 52 picture
Rank 2 picture
Rank 1 picture
Philosophical question
7:08 PM


Singular Values (Log10 Scale)
4
2
0
-2
-4
-6
-8
-10
-12
-14
0
50
100
150
200
250
300



Cumulative Percent of Total Sigmas



Full-Rank Logo

Rank 17 picture

Rank 11 picture

Rank 6 picture

Rank 2 picture

Rank 1 picture