LINEAR ALGEBRA. LECTURE 8

Complex Vectors and Matrices

Complex Numbers

Start with the imaginary number i. Everybody knows that $x^2 = -1$ has no real solution. When you square a real number, the answer is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition and multiplication, with one difference. ***Replace*** i^2 ***by*** -1.

A complex number (say $3 + 2i$) ***is the sum of a real number*** (3) ***and a pure imaginary number*** ($2i$). Addition keeps the real and imaginary parts separate. Multiplication uses $i^2 = -1$:

$$\textbf{Add:}\quad (3+2i)+(3+2i) = 6+4i$$

$$\textbf{Multiply:}\quad (3+2i)(1-i) = 3+2i-3i-2i^2 = 5-i.$$

If I add $3 + i$ to $1 - i$, the answer is 4. The real numbers $3 + 1$ stay separate from the imaginary numbers $i - i$. We are adding the vectors $(3, 1)$ and $(1, -1)$.

The number $(1 + i)^2$ is $1 + i$ times $1 + i$. The rules give the surprising answer $2i$:

$$(1+i)(1+i) = 1 + i + i + i^2 = 2i.$$

In the complex plane, $1+i$ is at an angle of 45°. It is like the vector $(1, 1)$. When we square $1 + i$ to get $2i$, the angle doubles to 90°. If we square again, the answer is $(2i)^2 = -4$. The 90° angle doubled to 180°, the direction of a negative real number.

A real number is just a complex number $z = a + bi$, with zero imaginary part: $b = 0$. A pure imaginary number has $a = 0$:

The ***real part*** is $a = \text{Re}\,(a + bi)$. The ***imaginary part*** is $b = \text{Im}\,(a + bi)$.

The Complex Plane

Complex numbers correspond to points in a plane. Real numbers go along the x axis. Pure imaginary numbers are on the y axis. ***The complex number*** $\mathbf{3 + 2i}$ ***is at the point with coordinates*** $(3, 2)$. The number zero, which is $0 + 0i$, is at the origin.

Adding and subtracting complex numbers is like adding and subtracting vectors in the plane. The real component stays separate from the imaginary component. The vectors go head-to-tail as usual. The complex plane $\mathbf{C}^1$ is like the ordinary two-dimensional plane $\mathbf{R}^2$, except that we multiply complex numbers and we didn't multiply vectors.

Now comes an important idea. ***The complex conjugate of*** $\mathbf{3 + 2i}$ ***is*** $\mathbf{3 - 2i}$. The complex conjugate of $z = 1 - i$ is $\overline{z} = 1 + i$. In general the conjugate of $z = a + bi$ is $\overline{z} = a - bi$. **(Some writers use a "*bar*" on the number and others use a "*star*":** $\overline{z} = z^*$.) The imaginary parts of z and "z bar" have opposite signs. In the complex plane, $\overline{z}$ is the image of z on the other side of the real axis.

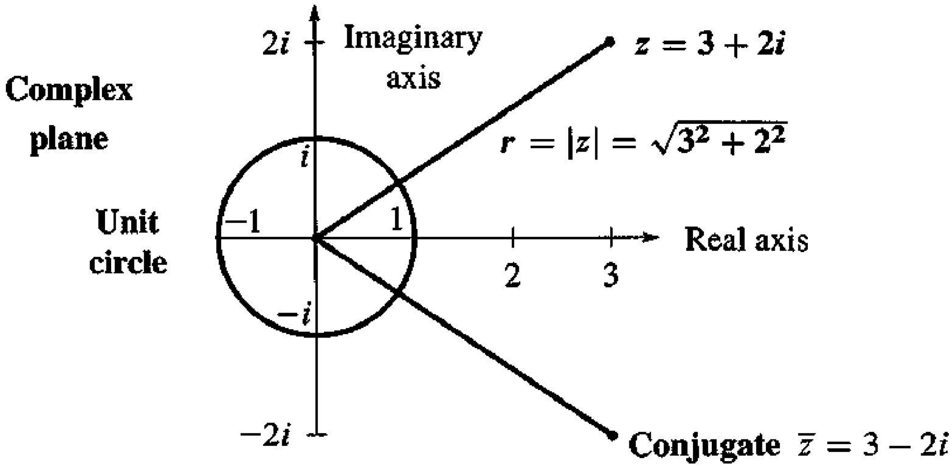

Figure 1: The number $z = a + bi$ corresponds to the point (a, b) and the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Something special happens when $z = 3 + 2i$ combines with its own complex conjugate $\overline{z} = 3 - 2i$. The result from adding $z + \overline{z}$ or multiplying $z\overline{z}$ is always real:

$$\begin{aligned} z + \overline{z} &= \textbf{real} \qquad (3 + 2i) + (3 - 2i) = 6 \quad \text{(real)} \\ z\overline{z} &= \textbf{real} \qquad (3 + 2i) \times (3 - 2i) = 9 + 6i - 6i - 4i^2 = 13 \quad \text{(real).} \end{aligned}$$

The sum of $z = a + bi$ and its conjugate $\overline{z} = a - bi$ is the real number $2a$. The product of z times $\overline{z}$ is the real number $a^2 + b^2$:

$$\textbf{Multiply } z \textbf{ times } \overline{z} \qquad (a + bi)(a - bi) = a^2 + b^2. \tag{2}$$

The next step with complex numbers is $1/z$. How to divide by $a + ib$? The best idea is to multiply by $\overline{z}/\overline{z}$. That produces $z\overline{z}$ in the denominator, which is $a^2 + b^2$:

$$\frac{1}{a + ib} = \frac{1}{a + ib}\frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2} \qquad \frac{1}{3 + 2i} = \frac{1}{3 + 2i}\frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{13}.$$

In case $a^2 + b^2 = 1$, this says that $(a + ib)^{-1}$ is $a - ib$. ***On the unit circle,* $1/z$ equals $\overline{z}$.** Later we will say: $1/e^{i\theta}$ is $e^{-i\theta}$ (the conjugate). A better way to multiply and divide is to use the polar form with distance r and angle θ.

The Polar Form $re^{i\theta}$

The square root of $a^2 + b^2$ is $|z|$. This is the ***absolute value*** (or ***modulus***) of the number $z = a + ib$. The square root $|z|$ is also written r, because it is the distance from 0 to z. ***The real number* r *in the polar form gives the size of the complex number* z:**

The absolute value of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$. This is called r.

The absolute value of $z = 3 + 2i$ is $|z| = \sqrt{3^2 + 2^2}$. This is $r = \sqrt{13}$.

The other part of the polar form is the angle θ. The angle for $z = 5$ is $\theta = 0$ (because this z is real and positive). The angle for $z = 3i$ is $\pi/2$ radians. The angle for a negative $z = -9$ is π radians. ***The angle doubles when the number is squared.*** The polar form is excellent for multiplying complex numbers (not good for addition).

When the distance is r and the angle is θ, trigonometry gives the other two sides of the triangle. The real part (along the bottom) is $a = r\cos\theta$. The imaginary part (up or down) is $b = r\sin\theta$. Put those together, and the rectangular form becomes the polar form:

$$\textbf{\textit{The number}} \quad \boldsymbol{z = a + ib} \qquad \textbf{\textit{is also}} \quad \boldsymbol{z = r\cos\theta + ir\sin\theta}. \quad \textbf{\textit{This is}}\ \boldsymbol{re^{i\theta}}$$

Note: $\boldsymbol{\cos\theta + i\sin\theta}$ ***has absolute value*** $\boldsymbol{r = 1}$ ***because*** $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$. Thus $\cos\theta + i\sin\theta$ lies on the circle of radius 1—*the unit circle.*

Example 1 Find r and θ for $z = 1 + i$ and also for the conjugate $\bar{z} = 1 - i$.

Solution The absolute value is the same for z and $\bar{z}$. For $z = 1+i$ it is $r = \sqrt{1+1} = \sqrt{2}$:

$$|z|^2 = 1^2 + 1^2 = 2 \qquad \text{and also} \qquad |\bar{z}|^2 = 1^2 + (-1)^2 = 2.$$

The distance from the center is $\sqrt{2}$. What about the angle? The number $1 + i$ is at the point $(1, 1)$ in the complex plane. The angle to that point is $\pi/4$ radians or $45°$. The cosine is $1/\sqrt{2}$ and the sine is $1/\sqrt{2}$. Combining r and θ brings back $z = 1 + i$:

$$r\cos\theta + ir\sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + i.$$

The angle to the conjugate $1 - i$ can be positive or negative. We can go to $7\pi/4$ radians which is $315°$. Or we can go *backwards through a negative angle,* to $-\pi/4$ radians or $-45°$. ***If*** $\boldsymbol{z}$ ***is at angle*** $\boldsymbol{\theta}$***, its conjugate*** $\boldsymbol{\bar{z}}$ ***is at*** $\boldsymbol{2\pi - \theta}$ ***and also at*** $\boldsymbol{-\theta}$.

We can freely add 2π or 4π or -2π to any angle! Those go full circles so the final point is the same. This explains why there are infinitely many choices of θ. Often we select the angle between zero and 2π radians. But $-\theta$ is very useful for the conjugate $\bar{z}$.

Powers and Products: Polar Form

Computing $(1 + i)^2$ and $(1 + i)^8$ is quickest in polar form. That form has $r = \sqrt{2}$ and $\theta = \pi/4$ (or $45°$). If we square the absolute value to get $r^2 = 2$, and double the angle to get $2\theta = \pi/2$ (or $90°$), we have $(1 + i)^2$. For the eighth power we need r^8 and 8θ:

$$\boldsymbol{(1+i)^8} \qquad r^8 = 2\cdot 2\cdot 2\cdot 2 = 16 \text{ and } 8\theta = 8\cdot\frac{\pi}{4} = 2\pi.$$

This means: $(1 + i)^8$ has absolute value 16 and angle 2π. *The eighth power of* $1 + i$ *is the real number* 16.

Powers are easy in polar form. So is multiplication of complex numbers.

The polar form of z^n has absolute value r^n. The angle is n times θ:

$$\textbf{\textit{The nth power of}} \quad z = r(\cos\theta + i\sin\theta) \quad \textbf{\textit{is}} \quad z^n = r^n(\cos n\theta + i\sin n\theta). \tag{3}$$

In that case z multiplies itself. In all cases, ***multiply r's and add the angles***:

$$r(\cos\theta + i\sin\theta) \text{ times } r'(\cos\theta' + i\sin\theta') = rr'\big(\cos(\theta+\theta') + i\sin(\theta+\theta')\big). \tag{4}$$

One way to understand this is by trigonometry. Concentrate on angles. Why do we get the double angle 2θ for z^2?

$$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = \cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta.$$

The real part $\cos^2\theta - \sin^2\theta$ is $\cos 2\theta$. The imaginary part $2\sin\theta\cos\theta$ is $\sin 2\theta$. Those are the "double angle" formulas. They show that θ in z becomes 2θ in z^2.

There is a second way to understand the rule for z^n. It uses the only amazing formula in this section. Remember that $\cos\theta + i\sin\theta$ has absolute value 1. The cosine is made up of even powers, starting with $1 - \frac{1}{2}\theta^2$. The sine is made up of odd powers, starting with $\theta - \frac{1}{6}\theta^3$. The beautiful fact is that $e^{i\theta}$ combines both of those series into $\cos\theta + i\sin\theta$:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \quad \text{becomes} \quad e^{i\theta} = 1 + i\theta + \frac{1}{2}i^2\theta^2 + \frac{1}{6}i^3\theta^3 + \cdots$$

Write -1 for i^2 to see $1 - \frac{1}{2}\theta^2$. ***The complex number*** $e^{i\theta}$ ***is*** $\cos\theta + i\sin\theta$***:***

$$\textbf{\textit{Euler's Formula}} \quad e^{i\theta} = \cos\theta + i\sin\theta \quad \text{gives} \quad z = r\cos\theta + ir\sin\theta = re^{i\theta} \tag{5}$$

Hermitian and Unitary Matrices

The main message of this section can be presented in one sentence: ***When you transpose a complex vector z or matrix A, take the complex conjugate too.*** Don't stop at z^{T} or A^{T}. Reverse the signs of all imaginary parts. From a column vector with $z_j = a_j + ib_j$, the good row vector is the *conjugate transpose* with components $a_j - ib_j$:

$$\textbf{Conjugate transpose} \qquad \bar{z}^{\mathrm{T}} = \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} = \begin{bmatrix} a_1 - ib_1 & \cdots & a_n - ib_n \end{bmatrix}. \qquad (1)$$

Here is one reason to go to $\bar{z}$. The length squared of a real vector is $x_1^2 + \cdots + x_n^2$. The length squared of a complex vector is *not* $z_1^2 + \cdots + z_n^2$. With that wrong definition, the length of $(1, i)$ would be $1^2 + i^2 = 0$. A nonzero vector would have zero length—not good. Other vectors would have complex lengths. Instead of $(a + bi)^2$ we want $a^2 + b^2$, the *absolute value squared*. This is $(a + bi)$ times $(a - bi)$.

For each component we want z_j times $\bar{z}_j$, which is $|z_j|^2 = a_j^2 + b_j^2$. That comes when the components of z multiply the components of $\bar{z}$:

$$\begin{array}{l}\textbf{Length}\\ \textbf{squared}\end{array} \quad \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + \cdots + |z_n|^2. \quad \textbf{This is} \quad \bar{z}^{\mathrm{T}} z = \|z\|^2. \qquad (2)$$

Now the squared length of $(1, i)$ is $1^2 + |i|^2 = 2$. The length is $\sqrt{2}$. The squared length of $(1 + i, 1 - i)$ is 4. The only vectors with zero length are zero vectors.

The length $\|z\|$ is the square root of $\bar{z}^{\mathrm{T}} z = z^{\mathrm{H}} z = |z_1|^2 + \cdots + |z_n|^2$

Before going further we replace two symbols by one symbol. Instead of a bar for the conjugate and T for the transpose, we just use a superscript H. Thus $\bar{z}^{\mathrm{T}} = z^{\mathrm{H}}$. This is "$z$ Hermitian," the *conjugate transpose* of z. The new word is pronounced "Hermeeshan." The new symbol applies also to matrices: The conjugate transpose of a matrix A is A^{H}.

Another popular notation is A^*. The MATLAB transpose command $'$ automatically takes complex conjugates (A' is A^{H}).

The vector z^{H} is $\bar{z}^{\mathrm{T}}$. The matrix A^{H} is $\bar{A}^{\mathrm{T}}$, the conjugate transpose of A:

$$\boldsymbol{A}^{\mathrm{H}} = \textbf{"A Hermitian"} \quad \text{If} \quad A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix} \quad \text{then} \quad A^{\mathrm{H}} = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

Complex Inner Products

For real vectors, the length squared is $x^{\mathrm{T}}x$—*the inner product of x with itself.* For complex vectors, the length squared is $z^{\mathrm{H}}z$. It will be very desirable if $z^{\mathrm{H}}z$ is the inner product of z with itself. To make that happen, the complex inner product should use the conjugate transpose (not just the transpose). The inner product sees no change when the vectors are real, but there is a definite effect from choosing $\bar{u}^{\mathrm{T}}$, when u is complex:

DEFINITION The inner product of real or complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$:

$$\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n. \tag{3}$$

With complex vectors, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ is different from $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$. *The order of the vectors is now important.* In fact $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = \overline{v}_1 u_1 + \cdots + \overline{v}_n u_n$ is the complex conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$. We have to put up with a few inconveniences for the greater good.

Example 1 The inner product of $\boldsymbol{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$.

Example 1 is surprising. Those vectors $(1, i)$ and $(i, 1)$ don't look perpendicular. But they are. ***A zero inner product still means that the*** (complex) ***vectors are orthogonal.*** Similarly the vector $(1, i)$ is orthogonal to the vector $(1, -i)$. Their inner product is $1 - 1 = 0$. We are correctly getting zero for the inner product—where we would be incorrectly getting zero for the length of $(1, i)$ if we forgot to take the conjugate.

Note We have chosen to conjugate the first vector $\boldsymbol{u}$. Some authors choose the second vector $\boldsymbol{v}$. Their complex inner product would be $\boldsymbol{u}^{\mathrm{T}}\overline{\boldsymbol{v}}$. It is a free choice, as long as we stick to it. We wanted to use the single symbol $^{\mathrm{H}}$ in the next formula too:

The inner product of $A\boldsymbol{u}$ with $\boldsymbol{v}$ equals the inner product of $\boldsymbol{u}$ with $A^{\mathrm{H}}\boldsymbol{v}$:

$$A^{\mathrm{H}} = \text{“adjoint” of } A \qquad (A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}). \tag{4}$$

The conjugate of $A\boldsymbol{u}$ is $\overline{A\boldsymbol{u}}$. Transposing it gives $\overline{\boldsymbol{u}}^{\mathrm{T}}\overline{A}^{\mathrm{T}}$ as usual. This is $\boldsymbol{u}^{\mathrm{H}}A^{\mathrm{H}}$. Everything that should work, does work. The rule for $^{\mathrm{H}}$ comes from the rule for $^{\mathrm{T}}$. That applies to products of matrices:

The conjugate transpose of AB is $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.

We constantly use the fact that $(a - ib)(c - id)$ is the conjugate of $(a + ib)(c + id)$.

Hermitian Matrices

Among complex matrices, the special class contains the ***Hermitian matrices***: $A = A^{\mathrm{H}}$. The condition on the entries is $a_{ij} = \overline{a_{ji}}$. In this case we say that "*A is* Hermitian." *Every real symmetric matrix is Hermitian*, because taking its conjugate has no effect. The next matrix is also Hermitian, $A = A^{\mathrm{H}}$:

Example 2 $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ The main diagonal is real since $a_{ii} = \overline{a_{ii}}$. Across it are conjugates $3+3i$ and $3-3i$.

This example will illustrate the three crucial properties of all Hermitian matrices.

If $A = A^{\mathrm{H}}$ and z is any vector, the number $z^{\mathrm{H}}Az$ is real.

Quick proof: $z^{\mathrm{H}}Az$ is certainly 1 by 1. Take its conjugate transpose:

$$(z^{\mathrm{H}}Az)^{\mathrm{H}} = z^{\mathrm{H}}A^{\mathrm{H}}(z^{\mathrm{H}})^{\mathrm{H}} \quad \text{which is } z^{\mathrm{H}}Az \text{ again.}$$

This used $A = A^{\mathrm{H}}$. So the number $z^{\mathrm{H}}Az$ equals its conjugate and must be real. Here is that "energy" $z^{\mathrm{H}}Az$ in our example:

$$\begin{bmatrix} \overline{z}_1 & \overline{z}_2 \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{2\overline{z}_1 z_1 + 5\overline{z}_2 z_2}_{\textit{diagonal}} + \underbrace{(3-3i)\overline{z}_1 z_2 + (3+3i) z_1 \overline{z}_2}_{\textit{off-diagonal}}.$$

The terms $2|z_1|^2$ and $5|z_2|^2$ from the diagonal are both real. The off-diagonal terms are conjugates of each other—so their sum is real. (The imaginary parts cancel when we add.)

Unitary Matrices

A ***unitary matrix*** U is a (complex) square matrix that has ***orthonormal columns.*** U is the complex equivalent of Q. The eigenvectors of A give a perfect example:

Unitary matrix $$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

This U is also a Hermitian matrix. I didn't expect that! The example is almost too perfect.

The matrix test for real orthonormal columns was $Q^{\mathrm{T}}Q = I$. When Q^{T} multiplies Q, the zero inner products appear off the diagonal. In the complex case, Q becomes U. The columns show themselves as orthonormal when U^{H} multiplies U. The inner products of the columns are again 1 and 0. They fill up $\boldsymbol{U^{\mathrm{H}}U = I}$:

Every matrix U with orthonormal columns has $U^{\mathrm{H}}U = I$.

If U is square, it is a unitary matrix. Then $U^{\mathrm{H}} = U^{-1}$.