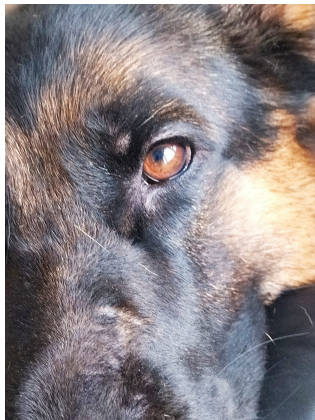




# How I spent last weekend



# Symmetric Matrices (1)



- 1 A symmetric matrix  $S$  has  $n$  **real eigenvalues**  $\lambda_i$  and  $n$  **orthonormal eigenvectors**  $q_1, \dots, q_n$ .
- 2 Every real symmetric  $S$  can be diagonalized:  $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$
- 3 The number of positive eigenvalues of  $S$  equals the number of positive pivots.

## Symmetric Matrices (2)



*Symmetric matrices  $S$  have orthogonal eigenvector matrices  $Q$ . Look at this again:*

**Symmetry**      $S = X\Lambda X^{-1}$  becomes  $S = Q\Lambda Q^T$  with  $Q^T Q = I$ .

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^T = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}. \quad (5)$$

*Columns  $q_1$  and  $q_2$  multiply rows  $\lambda_1 q_1^T$  and  $\lambda_2 q_2^T$  to produce  $S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$ .*



## Task 1

Write  $A$  as  $S + N$ , symmetric matrix  $S$  plus skew-symmetric matrix  $N$ :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \quad (S^T = S \text{ and } N^T = -N).$$

For any square matrix,  $S = \frac{1}{2}(A + A^T)$  and  $N = \underline{\hspace{2cm}}$  add up to  $A$ .



## Task 1

Write  $A$  as  $S + N$ , symmetric matrix  $S$  plus skew-symmetric matrix  $N$ :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = S + N \quad (S^T = S \text{ and } N^T = -N).$$

For any square matrix,  $S = \frac{1}{2}(A + A^T)$  and  $N = \text{_____}$  add up to  $A$ .

## Answer

$$\begin{aligned} A &= \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \\ &= \text{symmetric} + \text{skew-symmetric}. \end{aligned}$$

## Task 2



Find an orthogonal matrix  $Q$  that diagonalizes  $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$ . What is  $\Lambda$ ?



## Task 2

Find an orthogonal matrix  $Q$  that diagonalizes  $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$ . What is  $\Lambda$ ?

### Answer

$\lambda = 10$  and  $-5$  in  $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$ ,  $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$  have to be normalized to unit vectors in  $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ .



## Task 3



*True* (with reason) *or false* (with example).

- (a) A matrix with real eigenvalues and  $n$  real eigenvectors is symmetric.
- (b) A matrix with real eigenvalues and  $n$  orthonormal eigenvectors is symmetric.
- (c) The inverse of an invertible symmetric matrix is symmetric.
- (d) The eigenvector matrix  $Q$  of a symmetric matrix is symmetric.

## Task 3



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- (c) The inverse of an invertible symmetric matrix is symmetric.
- (d) The eigenvector matrix  $Q$  of a symmetric matrix is symmetric.

## Answer

- (a) False.  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  (b) True from  $A^T = Q\Lambda Q^T = A$  (d) False!
- (c) True from  $S^{-1} = Q\Lambda^{-1}Q^T$

# Positive Definite Matrices



This section concentrates on *symmetric matrices that have positive eigenvalues*. If symmetry makes a matrix important, this extra property (*all  $\lambda > 0$* ) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called *positive definite*.

# Positive Definite Matrices

*Applications from ML*



- Cholesky decomposition -  $A = LDL^H$  (A special case of  $A = LU$ )
- Least squares computation reduction
- Support Vector Machine (SVM), *kernel* - Positive-definite kernel
- Representer Theorem

# Positive Definite Matrices

*Five tests*



**Positive definite matrices** are the best. How to test  $S$  for  $\lambda_i > 0$ ?

Test 1    Compute the **eigenvalues** of  $S$ : All eigenvalues positive

Test 2    The **energy**  $x^T S x$  is positive for every vector  $x \neq 0$

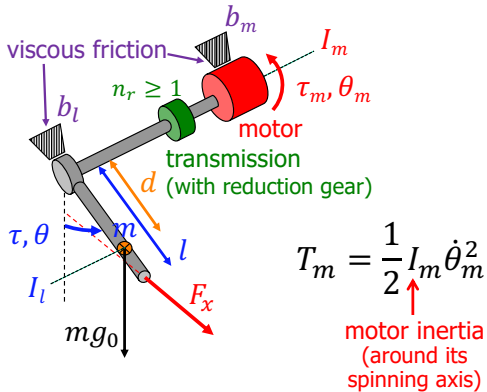
Test 3    The **pivots** in elimination on  $S$  are all positive

Test 4    The upper left **determinants** of  $S$  are all positive

Test 5     $S = A^T A$  for some matrix  $A$  with independent columns

# Dynamics of an actuated pendulum

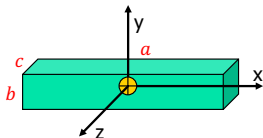
## a first example





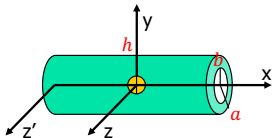
# Examples of body inertia matrices

homogeneous bodies of mass  $m$ , with axes of symmetry



parallelepiped with sides  
 $a$  (length/height),  $b$  and  $c$  (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2 + c^2) & & \\ & \frac{1}{12}m(a^2 + c^2) & \\ & & \frac{1}{12}m(a^2 + b^2) \end{pmatrix}$$



empty cylinder with length  $h$ ,  
and external/internal radius  $a$  and  $b$

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2 + b^2) & & \\ & \frac{1}{12}m(3(a^2 + b^2) + h^2) & \\ & & I_{zz} \end{pmatrix} \quad I_{zz} = I_{yy}$$



## Kinetic energy of a rigid body (cont)

$$= \frac{1}{2} m v_c^T v_c$$

↑  
translational  
kinetic energy  
(point mass  
at CoM)

+

rotational  
kinetic energy  
(of the whole body) →

$$= \frac{1}{2} \omega^T I_c \omega$$

↑  
body inertia matrix  
(around the CoM)



# Positive Definite Matrices

*Example*

Test  $S$  for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

# Positive Definite Matrices

*Solution: Test 1, 3, 4*

The pivots of  $S$  are 2 and  $\frac{3}{2}$  and  $\frac{4}{3}$ , all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of  $S$  are  $2 - \sqrt{2}$  and  $2 + \sqrt{2}$ , all positive. That completes tests **1**, **2**, and **3**.

# Positive Definite Matrices

*Solution: Test 5*

$A_2$  comes from  $S = LDL^T$  (the symmetric version of  $S = LU$ ). Elimination gives the pivots  $2, \frac{3}{2}, \frac{4}{3}$  in  $D$  and the multipliers  $-\frac{1}{2}, 0, -\frac{2}{3}$  in  $L$ . **Just put  $A_2 = L\sqrt{D}$ .**

$$LDL^T = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^T = A_2^T A_2.$$

*$A_2$  is the Cholesky factor of  $S$*

**Eigenvalues give the symmetric choice  $A_3 = Q\sqrt{\Lambda}Q^T$ .** This is also successful with  $A_3^T A_3 = Q\Lambda Q^T = S$ . All tests show that the  $-1, 2, -1$  matrix  $S$  is positive definite.

# Positive Definite Matrices

*Solution: Test 2*



$$\mathbf{x}^T S \mathbf{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

$$\|A_2 \mathbf{x}\|^2 = 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2$$

$$\|A_3 \mathbf{x}\|^2 = \lambda_1(\mathbf{q}_1^T \mathbf{x})^2 + \lambda_2(\mathbf{q}_2^T \mathbf{x})^2 + \lambda_3(\mathbf{q}_3^T \mathbf{x})^2$$

**Rewrite with squares**

**Using  $S = LDL^T$**

**Using  $S = Q\Lambda Q^T$**



## Task 4

What is the function  $f = ax^2 + 2bxy + cy^2$  for each of these matrices? Complete the square to write each  $f$  as a sum of one or two squares  $f = d_1(\quad)^2 + d_2(\quad)^2$ .

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



## Task 4

What is the function  $f = ax^2 + 2bxy + cy^2$  for each of these matrices? Complete the square to write each  $f$  as a sum of one or two squares  $f = d_1(\quad)^2 + d_2(\quad)^2$ .

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad f = [x \ y] \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

## Answer

$$f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2; \quad x^2 + 6xy + 9y^2 = (x + 3y)^2.$$

## Task 5



For what numbers  $c$  and  $d$  are  $S$  and  $T$  positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$



## Task 5

For what numbers  $c$  and  $d$  are  $S$  and  $T$  positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

### Answer

$S$  is positive definite for  $c > 1$ ; determinants  $c$ ,  $c^2 - 1$ , and  $(c - 1)^2(c + 2) > 0$ .

$T$  is *never* positive definite (determinants  $d - 4$  and  $-4d + 12$  are never both positive).





## Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have  $\mathbf{x}^T S \mathbf{x} > 0$ :

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = ( \quad , \quad , \quad ).$$



## Task 6

A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have  $\mathbf{x}^T S \mathbf{x} > 0$ :

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = ( \quad , \quad , \quad ).$$

## Answer

$\mathbf{x}^T S \mathbf{x}$  is zero when  $(x_1, x_2, x_3) = (0, 1, 0)$  because of the zero on the diagonal. Actually  $\mathbf{x}^T S \mathbf{x}$  goes *negative* for  $\mathbf{x} = (1, -10, 0)$  because the second pivot is *negative*.

# Positive Definite Matrices

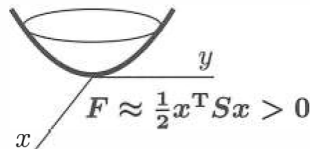
*Important Application: Test for a Minimum*

Does  $F(x, y)$  have a minimum if  $\partial F/\partial x = 0$  and  $\partial F/\partial y = 0$  at the point  $(x, y) = (0, 0)$ ?

For  $f(x)$ , the test for a minimum comes from calculus:  $df/dx$  is zero and  $d^2f/dx^2 > 0$ . Two variables in  $F(x, y)$  produce a symmetric matrix  $S$ . It contains *four second derivatives*. **Positive  $d^2f/dx^2$  changes to positive definite  $S$ :**

**Second  
derivatives**

$$S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x\partial y \\ \partial^2 F/\partial y\partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$



**$F(x, y)$  has a minimum if  $\partial F/\partial x = \partial F/\partial y = 0$  and  $S$  is positive definite.**

Reason:  $S$  reveals the all-important terms  $ax^2 + 2bxy + cy^2$  near  $(x, y) = (0, 0)$ . The second derivatives of  $F$  are  $2a, 2b, 2b, 2c$ . For  $F(x, y, z)$  the matrix  $S$  will be 3 by 3.

## Task 7



For  $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$  and  $F_2(x, y) = x^3 + xy - x$  find the second derivative matrices  $S_1$  and  $S_2$ :

**Test for minimum**  $S = \begin{bmatrix} \partial^2 F / \partial x^2 & \partial^2 F / \partial x \partial y \\ \partial^2 F / \partial y \partial x & \partial^2 F / \partial y^2 \end{bmatrix}$  is positive definite

$S_1$  is positive definite so  $F_1$  is concave up (= convex). Find the minimum point of  $F_1$ .  
*Find the saddle point of  $F_2$  (look only where first derivatives are zero).*

## Task 7



### Answer

$S_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$  is semidefinite;  $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$  on the curve  $\frac{1}{2}x^2 + y = 0$ ;

$S_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is indefinite at  $(0, 1)$  where first derivatives = 0. Then  $x = 0, y = 1$  is a saddle point of the function  $f_2(x, y)$ .



## Reference material

- Lecture 28, Positive Definite Matrices and Minima
- 5. Positive Definite and Semidefinite Matrices
- *"Introduction to Linear Algebra"*, pdf pages 349–374  
6.4 – Symmetric, 6.5 – Positive Definite matrices
- *"Linear Algebra and Applications"*, pdf pages 355–376  
Positive Definite Matrices 6.1, 6.2

# Deserve "A" grade!

– Oleg Bulichev

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🏢 Room 105 (Underground robotics lab)