



# How I spent last weekend



(a) Cross-country skiing

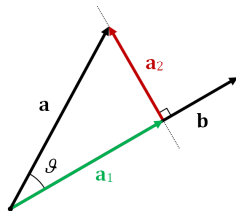


(b) Sous vide

# Projection

## Definition

The *vector projection* of a vector **a** on (or onto) a nonzero vector **b**, sometimes denoted  $\text{proj}_{\mathbf{b}} \mathbf{a}$  is the orthogonal projection of **a** onto a straight line parallel to **b**.



Projection of **a** on **b** (**a<sub>1</sub>**), and rejection of **a** from **b** (**a<sub>2</sub>**)

## Where it can be used:

- Maps
- Blueprints
- Fitting algorithms (Least squares)
- Reduce matrix dimension
- RL fitness functions

# Projection (1)



2D case

Classical way

Project "b" on "a<sub>1</sub>"

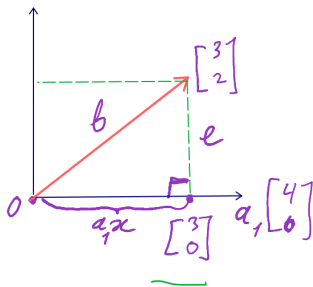
$$e = b - a_1 x$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T b = a_1^T a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x \quad \text{— classic formula from school}$$



Particular example

$$\frac{a_1^T b}{a_1^T a_1} = x$$

$$\frac{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}} = \frac{12}{16} = \frac{3}{4}$$

$$\text{projection } p = a_1 x = \frac{3}{4} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

# Projection (2)



2D case

Projection matrix

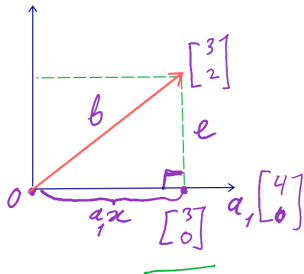
Project "b" on "a<sub>1</sub>"

Like affine transformation matrix

$$p \cdot b = x a_1 = a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x \rightarrow p = \frac{a_1 a_1^T}{a_1^T a_1}$$

Projection matrix



Particular example

$$P = \frac{\begin{bmatrix} 4 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \end{bmatrix}}{16} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p = P b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

# Projection (3)

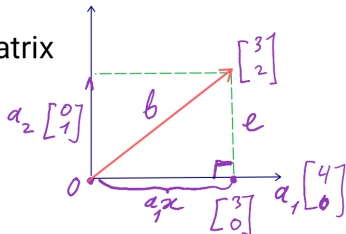


3D case

Projection matrix

Project "b"

on ["a<sub>1</sub>", "a<sub>2</sub>"]



$$A^T(b - Ax) = 0$$

$$A^T Ax = A^T b$$

$$\underbrace{A(A^T A)^{-1} A^T}_{\text{projection matrix } P} b = Ax$$

projection matrix

It will be Identity for full rank square matrix

Particular example

$$P = [a_1 \ a_2] \left( \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \begin{bmatrix} a_1 \ a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It's correct, because we project b on a plane, where it lies

$$p = Pb = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

## Projection (4)

2D case

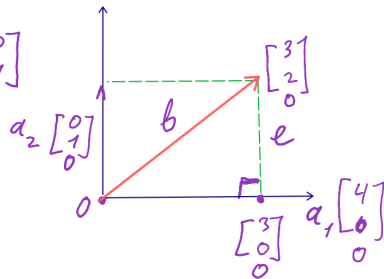
Project "b"

on "d" perpendicular to "a<sub>1</sub>"

$$P_{d_1} = I - P = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_{d_1} b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$



3D case

Project "b"

on "d" perpendicular to  
["a<sub>1</sub>", "a<sub>2</sub>"]

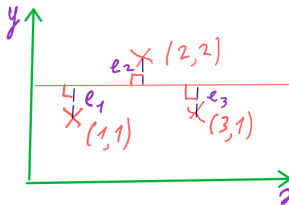
$$P_{d_2} = I - P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; P_{d_2} b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Least squares (1)



Task description: a guy draw the line on a floor. We want to know the equation of this line. We send a robot, which follows the line and obtain a dataset from GPS.



$$\|e\|^2 = \|Ax - y\|^2 \rightarrow \min$$

Equation of the line is  $y=kx+t$ . Hence, there are 2 unknown variables. Be we obtained 3 equations. Our system is overdetermined (more equations (points), than variables).

We have to reformulate our task.  
Let's fit all points on the line, where the distance between points and our resulted line will be minimized.

There are two ways of thinking:

- 1) Calculus way (derivatives)
- 2) LA way (projections)

$$\frac{dE^2}{dx} = 0$$

$$A^T A x = A^T y$$



## Least squares (2)



Case study:  $y = kx + t \Rightarrow \begin{cases} t + k \cdot 1 = 1 \\ t + k \cdot 2 = 2 \\ t + k \cdot 3 = 1 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\begin{matrix} A & x & y \end{matrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ k \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \end{bmatrix}$$

Let's find errors (subs  $t$  and  $k$  to  $Ax = y$ )  $\begin{cases} 4/3 + 0 \cdot 1 = 1 \\ 4/3 + 0 \cdot 2 = 2 \\ 4/3 + 0 \cdot 3 = 1 \end{cases} \Rightarrow e = \begin{cases} -1/3 \\ 2/3 \\ -1/3 \end{cases}$

## Task 1



**4.2 A** Project the vector  $\mathbf{b} = (3, 4, 4)$  onto the line through  $\mathbf{a} = (2, 2, 1)$  and then onto the plane that also contains  $\mathbf{a}^* = (1, 0, 0)$ . Check that the first error vector  $\mathbf{b} - \mathbf{p}$  is perpendicular to  $\mathbf{a}$ , and the second error vector  $\mathbf{e}^* = \mathbf{b} - \mathbf{p}^*$  is also perpendicular to  $\mathbf{a}^*$ .

Find the 3 by 3 projection matrix  $P$  onto that plane of  $\mathbf{a}$  and  $\mathbf{a}^*$ . Find a vector whose projection onto the plane is the zero vector.



## Task 1

**4.2 A** Project the vector  $\mathbf{b} = (3, 4, 4)$  onto the line through  $\mathbf{a} = (2, 2, 1)$  and then onto the plane that also contains  $\mathbf{a}^* = (1, 0, 0)$ . Check that the first error vector  $\mathbf{b} - \mathbf{p}$  is perpendicular to  $\mathbf{a}$ , and the second error vector  $\mathbf{e}^* = \mathbf{b} - \mathbf{p}^*$  is also perpendicular to  $\mathbf{a}^*$ .

Find the 3 by 3 projection matrix  $P$  onto that plane of  $\mathbf{a}$  and  $\mathbf{a}^*$ . Find a vector whose projection onto the plane is the zero vector.

### Answer

**Solution** The projection of  $\mathbf{b} = (3, 4, 4)$  onto the line through  $\mathbf{a} = (2, 2, 1)$  is  $\mathbf{p} = 2\mathbf{a}$ :

Onto a line 
$$\mathbf{p} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = \frac{18}{9} (2, 2, 1) = (4, 4, 2).$$

The error vector  $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 0, 2)$  is perpendicular to  $\mathbf{a}$ . So  $\mathbf{p}$  is correct.

The plane of  $\mathbf{a} = (2, 2, 1)$  and  $\mathbf{a}^* = (1, 0, 0)$  is the column space of  $A = [\mathbf{a} \ \mathbf{a}^*]$ :

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \quad A^T A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \quad (A^T A)^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

Then  $\mathbf{p}^* = P\mathbf{b} = (3, 4.8, 2.4)$ . The error  $\mathbf{e}^* = \mathbf{b} - \mathbf{p}^* = (0, -.8, 1.6)$  is perpendicular to  $\mathbf{a}$  and  $\mathbf{a}^*$ . This  $\mathbf{e}^*$  is in the nullspace of  $P$  and its projection is zero! Note  $P^2 = P$ .

## Task 2



- 11 Project  $\mathbf{b}$  onto the column space of  $A$  by solving  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  and  $\mathbf{p} = A \hat{\mathbf{x}}$ :

$$(a) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad (b) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find  $\mathbf{e} = \mathbf{b} - \mathbf{p}$ . It should be perpendicular to the columns of  $A$ .

## Task 2



11 Project  $\mathbf{b}$  onto the column space of  $A$  by solving  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  and  $\mathbf{p} = A \hat{\mathbf{x}}$ :

$$(a) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad (b) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find  $\mathbf{e} = \mathbf{b} - \mathbf{p}$ . It should be perpendicular to the columns of  $A$ .

### Answer

$$(a) \quad \mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} = (2, 3, 0), \quad \mathbf{e} = (0, 0, 4), \quad A^T \mathbf{e} = \mathbf{0} \quad (b) \quad \mathbf{p} = (4, 4, 6), \quad \mathbf{e} = \mathbf{0}.$$

## Task 3



(Quick and Recommended) Suppose  $A$  is the 4 by 4 identity matrix with its last column removed.  $A$  is 4 by 3. Project  $\mathbf{b} = (1, 2, 3, 4)$  onto the column space of  $A$ . What shape is the projection matrix  $P$  and what is  $P$ ?

## Task 3

(Quick and Recommended) Suppose  $A$  is the 4 by 4 identity matrix with its last column removed.  $A$  is 4 by 3. Project  $\mathbf{b} = (1, 2, 3, 4)$  onto the column space of  $A$ . What shape is the projection matrix  $P$  and what is  $P$ ?

### Answer

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{p} = P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

## Task 4



- (a) If  $P$  is the 2 by 2 projection matrix onto the line through  $(1, 1)$ , then  $I - P$  is the projection matrix onto \_\_\_\_\_.
- (b) If  $P$  is the 3 by 3 projection matrix onto the line through  $(1, 1, 1)$ , then  $I - P$  is the projection matrix onto \_\_\_\_\_.



## Task 4



- (a) If  $P$  is the 2 by 2 projection matrix onto the line through  $(1, 1)$ , then  $I - P$  is the projection matrix onto \_\_\_\_\_.
- (b) If  $P$  is the 3 by 3 projection matrix onto the line through  $(1, 1, 1)$ , then  $I - P$  is the projection matrix onto \_\_\_\_\_.

## Answer

- (a)  $I - P$  is the projection matrix onto  $(1, -1)$  in the perpendicular direction to  $(1, 1)$
- (b)  $I - P$  projects onto the plane  $x + y + z = 0$  perpendicular to  $(1, 1, 1)$ .

## Task 5



- (a) Find the projection matrix  $P_C$  onto the column space of  $A$  (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

- (b) Find the 3 by 3 projection matrix  $P_R$  onto the row space of  $A$ . Multiply  $B = P_C A P_R$ . Your answer  $B$  should be a little surprising—can you explain it?

## Task 5



- (a) Find the projection matrix  $P_C$  onto the column space of  $A$  (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

- (b) Find the 3 by 3 projection matrix  $P_R$  onto the row space of  $A$ . Multiply  $B = P_C A P_R$ . Your answer  $B$  should be a little surprising—can you explain it?

### Answer

- (a) The column space is the line through  $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  so  $P_C = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$ .
- (b) The row space is the line through  $\mathbf{v} = (1, 2, 2)$  and  $P_R = \mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v}$ . Always  $P_C A = A$  (columns of  $A$  project to themselves) and  $A P_R = A$ . Then  $P_C A P_R = A$ !

## Reference material



- Lecture 15 and 16
- "*Linear Algebra and Applications*", pdf pages 181–204  
Projections onto lines and Least squares
- The Least-Squares Problem  
Video from Matrix Algebra for Engineers course

# Deserve "A" grade!

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🏢 Room 105 (Underground robotics lab)