Analytical Geometry and Linear Algebra II, Lab 9

System of linear differential equations

1st order system of linear differential equations
Algorithm

1. Write equation in $u' = Au$ form;

2. Find eigenpairs of A

3. Substitute λ and x_{λ_i} to $u(t) = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$

4. Find all c_i using $u(0)$ – which is vector. $u(0) = \underbrace{u(t = 0)}_{\text{from above}}$. Solve it.

5. Put c_i to $u(t)$.

Example 1   Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

This is a vector equation for $\boldsymbol{u}$. It contains two scalar equations for the components y and z. They are "coupled together" because the matrix A is not diagonal:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \quad \text{means that} \quad \frac{dy}{dt} = z \text{ and } \frac{dz}{dt} = y.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it. Add and subtract equations:

$$\frac{d}{dt}(y + z) = z + y \qquad \text{and} \qquad \frac{d}{dt}(y - z) = -(y - z).$$

The combination $y + z$ grows like e^t, because it has $\lambda = 1$. The combination $y - z$ decays like e^{-t}, because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$, looking for these special combinations. The eigenvectors and eigenvalues of A will do it for us.

This matrix A has eigenvalues 1 and -1. The eigenvectors $\boldsymbol{x}$ are $(1,1)$ and $(1,-1)$. The pure exponential solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ take the form $e^{\lambda t}\boldsymbol{x}$ with $\lambda_1 = 1$ and $\lambda_2 = -1$:

$$\boldsymbol{u}_1(t) = e^{\lambda_1 t}\boldsymbol{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{u}_2(t) = e^{\lambda_2 t}\boldsymbol{x}_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \qquad (4)$$

Notice: These $\boldsymbol{u}$'s satisfy $A\boldsymbol{u}_1 = \boldsymbol{u}_1$ and $A\boldsymbol{u}_2 = -\boldsymbol{u}_2$, just like $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The factors e^t and e^{-t} change with time. Those factors give $d\boldsymbol{u}_1/dt = \boldsymbol{u}_1 = A\boldsymbol{u}_1$ and $d\boldsymbol{u}_2/dt = -\boldsymbol{u}_2 = A\boldsymbol{u}_2$. **We have two solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$. To find all other solutions, multiply those special solutions by any numbers C and D and add:**

Complete solution $\boldsymbol{u}(t) = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \qquad (5)$

With these two constants C and D, we can match any starting vector $\boldsymbol{u}(0) = (u_1(0), u_2(0))$. Set $t = 0$ and $e^0 = 1$. Example 1 asked for the initial value to be $\boldsymbol{u}(0) = (4, 2)$:

$$\boldsymbol{u}(0) \text{ decides } C, D \qquad C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + D\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{yields} \quad C = 3 \quad \text{and} \quad D = 1.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is completely solved. The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

---

1. Write $\boldsymbol{u}(0)$ as a **combination** $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ **of the eigenvectors of** A.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by **its growth factor** $e^{\lambda_i t}$.

3. The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n. \qquad (6)$$

Not included: If two λ's are equal, with only one eigenvector, another solution is needed. (It will be $te^{\lambda t}\boldsymbol{x}$.) Step 1 needs to diagonalize $A = X\Lambda X^{-1}$: a basis of n eigenvectors.

Example 2 Solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

Typical example		
Equation for $\boldsymbol{u}$	$\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{u}$	starting from $\boldsymbol{u}(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}$.
Initial condition $\boldsymbol{u}(0)$		

The eigenvectors are $\boldsymbol{x}_1 = (1, 0, 0)$ and $\boldsymbol{x}_2 = (1, 1, 0)$ and $\boldsymbol{x}_3 = (1, 1, 1)$.

Step 1 The vector $\boldsymbol{u}(0) = (9, 7, 4)$ is $2\boldsymbol{x}_1 + 3\boldsymbol{x}_2 + 4\boldsymbol{x}_3$. Thus $(c_1, c_2, c_3) = (\boldsymbol{2, 3, 4})$.

Step 2 The factors $e^{\lambda t}$ give exponential solutions $e^t\boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.

Step 3 The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = 2e^t\boldsymbol{x}_1 + 3e^{2t}\boldsymbol{x}_2 + 4e^{3t}\boldsymbol{x}_3$.

The coefficients $2, 3, 4$ came from solving the linear equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = \boldsymbol{u}(0)$:

$$\begin{bmatrix} & & \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} \quad \text{which is} \quad X\boldsymbol{c} = \boldsymbol{u}(0). \quad (7)$$

Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

Task 1

Find two λ's and x's so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

Answer

Eigenvalues 4 and 1 with eigenvectors $(1, 0)$ and $(1, -1)$ give solutions $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\boldsymbol{u}_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\boldsymbol{u}(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} +$

$2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbf{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(0) = (3, 1)$?

Task 2

Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbf{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(0) = (3, 1)$?

Answer

A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1, 1)$. Eigenvectors with $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1, -1)$. For the solution to $\boldsymbol{u}' = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Idea

$u'' + Bu' + Cu = 0$ is equivalent to $\begin{bmatrix} u \\ u' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}$

Everything else is the same as in first order system.

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (that is *Newton's Law*). The force includes the damping $-by'$ and the elastic force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative of y brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0 \quad \text{becomes} \quad (m\lambda^2 + b\lambda + k) e^{\lambda t} = 0. \qquad (8)$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1 y_1 + c_2 y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a *vector equation for y and y'*: first derivative only. Suppose the mass is $m = 1$. Two equations for $\boldsymbol{u} = (y, \ y')$ give $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{matrix} dy/dt = y' \\ dy'/dt = -ky - by' \end{matrix} \quad \text{converts to} \quad \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \qquad (9)$$

The first equation $dy/dt = y'$ is trivial (but true). The second is equation (8) connecting y'' to y' and y. Together they connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve $\boldsymbol{u}' = A\boldsymbol{u}$ by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

The equation for the λ's is the same as (8)! It is still $\lambda^2 + b\lambda + k = 0$, since $m = 1$. The roots λ_1 and λ_2 are now *eigenvalues of A*. The eigenvectors and the solution are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of $\boldsymbol{u}(t)$ has $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$—the same solution as before. It can't be anything else. In the second component of $\boldsymbol{u}(t)$ you see the velocity dy/dt. The vector problem is completely consistent with the scalar problem. The 2 by 2 matrix A is called a *companion matrix*—a companion to the second order equation with y''.

Example 3 *Motion around a circle with $y'' + y = 0$ and $y = \cos t$*

This is our master equation with mass $m = 1$ and stiffness $k = 1$ and $d = 0$: no damping. Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\lambda^2 + 1 = 0$. *The roots are $\lambda = i$ and $\lambda = -i$.* Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos t$.

As a first-order system, the initial values $y(0) = 1$, $y'(0) = 0$ go into $\boldsymbol{u}(0) = (1, 0)$:

$$\text{Use } y'' = -y \qquad \frac{d\boldsymbol{u}}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \qquad (10)$$

The eigenvalues of A are again the same $\lambda = i$ and $\lambda = -i$ (no surprise). A is anti-symmetric with eigenvectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The combination that matches $\boldsymbol{u}(0) = (1, 0)$ is $\frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$. Step 2 multiplies the $\boldsymbol{x}$'s by e^{it} and e^{-it}. Step 3 combines the pure oscillations into $\boldsymbol{u}(t)$ to find $y = \cos t$ as expected:

$$\boldsymbol{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}. \qquad \text{This is } \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

All good. The vector $\boldsymbol{u} = (\cos t, -\sin t)$ goes around a circle (Figure 6.3). The radius is 1 because $\cos^2 t + \sin^2 t = 1$.

Solve $y'' + 4y' + 3y = 0$ by linear algebra.

Task 3

Solve $y'' + 4y' + 3y = 0$ by linear algebra.

Answer

To use linear algebra we set $\boldsymbol{u} = (y, y')$. Then the vector equation is $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\begin{array}{l} dy/dt = y' \\ dy'/dt = -3y - 4y' \end{array} \quad \text{converts to} \quad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{u}.$$

This A is a "companion matrix" and its eigenvalues are again -1 and -3:

Same quadratic $\quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$

The eigenvectors of A are $(1, \lambda_1)$ and $(1, \lambda_2)$. Either way, the decay in $y(t)$ comes from e^{-t} and e^{-3t}. With constant coefficients, calculus leads to linear algebra $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Task 4

Find solution in general form ($c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$):

$$u''' + 2u'' - u' - 2u = 0, \ u(0) = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

$$u''' + 2u'' - u' - 2u = 0, \ u(0) = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

Answer

1. Equation: $\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$;

2. Eigenpairs: $\lambda_1 = -2, \ \lambda_2 = -1, \ \lambda_3 = 1$; $x_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \ x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

3. $c_1 = 1, \ c_2 = -1, \ c_3 = 3$;

4. General form: $u(t) = 1e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + (-1)e^{-1t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3e^{1t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Reference material

- Lecture 23, Differential Equations and exp(At)

- *"Introduction to Linear Algebra", pdf pages 436*
  Circulant Matrix 8.3

- *"Introduction to Linear Algebra", pdf pages 330–348*
  Systems of Differential Equations 6.3

Deserve "A" grade!

– Oleg Bulichev

✉ o.bulichev@innopolis.ru

✈ @Lupasic

🚪 Room 105 (Underground robotics lab)