Chapter 10
Synthesis of Planar Mechanisms

Abstract Design in Mechanical Engineering must give answers to different requests that are based on two cornerstones, analysis and synthesis of mechanisms. Analysis allows determining whether a given system will comply with certain requirements or not. Alternatively, synthesis is the design of a mechanism so that it complies with previously specified requirements. For instance, mechanism synthesis allows finding the dimensions of a four-bar mechanism in which the output link generates a desired function with a series of precision points or a mechanism in which a point follows a given trajectory. Therefore, synthesis makes it possible to find the mechanism with a response previously defined. Currently, there is a set of methods and rules that makes it possible to find the solution to many mechanism design problems. However, since this is quite a newly-developed discipline, there are still many problems that need to be solved. The concept of synthesis was defined in Chap. 1 as follows: synthesis refers to the creative process through which a model or pattern can be generated, so that it satisfies a certain need while complying with certain kinematic and dynamic constraints that define the problem (Fig. 10.1). Other definitions can be added but all of them will somehow express the idea of creating mechanisms that can carry out a certain type of motion or, in a more general way, mechanisms that comply with a set of given requirements. There are several classifications for different types of synthesis but, basically, most authors agree on grouping the synthesis of mechanisms in two main branches: structural synthesis and dimensional synthesis.

10.1 Types of Synthesis

There are several classifications for different types of synthesis but, basically, most authors agree on grouping the synthesis of mechanisms in two main branches: structural synthesis and dimensional synthesis (Fig. 10.1).

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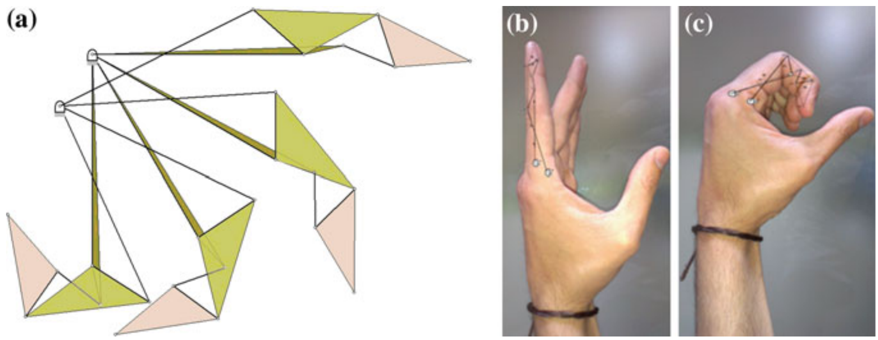


Fig. 10.1 **a** Six-bar linkage with one degree of freedom that can be used to build a finger prosthesis. Overlapping image of the mechanism on a real finger in two different positions (**b** and **c**)

10.1.1 Structural Synthesis

This synthesis deals with the topological and structural study of mechanisms. It only considers the interconnectivity pattern of the links so that the results are unaffected by the changes in the geometric properties of the mechanisms.

Structural synthesis includes the following:

- Synthesis of type or Reuleaux synthesis:
  Once the requirements have been defined the following questions are answered: What type of mechanism is more suitable? What type of elements will it be made of? Can it be formed by linkages, gears, flexible elements or cams? (Fig. 10.2) Different configurations are developed according to the pre-established requirements. The criteria to value the different characteristics of the mechanism are set.

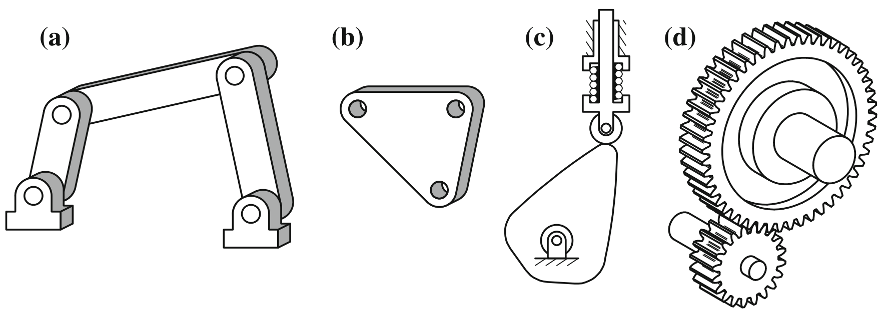


Fig. 10.2 Examples of elements that can be considered in synthesis of type. From *left* to *right*, **a** four bar linkage, **b** tertiary link, **c** cam and **d** pair of gears

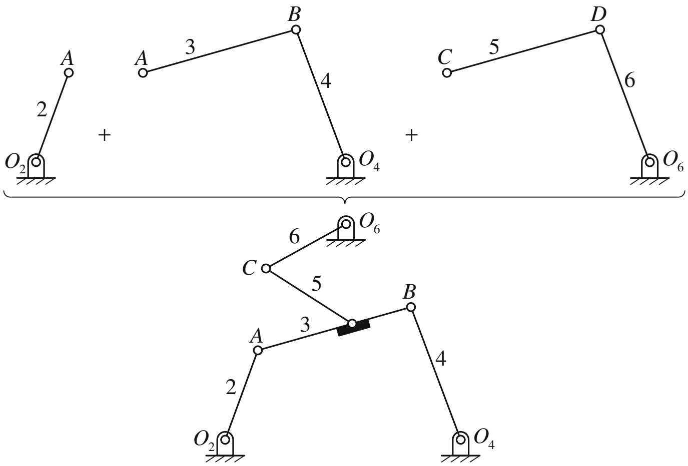


Fig. 10.3 Mechanism generation by addition of two RRR-type dyads

– Synthesis of number or Grübler synthesis:
  In the case of a linkage, it determines the number of links and their configuration. It works with concepts that were defined in the first chapters of this book such as link and link types, kinematic pair, kinematic chain, mechanism, inversion, degree of freedom and mobility criteria among others.

There are several methods to obtain new mechanisms. We will explain one of the most commonly used: the method of dyad addition. We call a group of elements formed by two links and three joints, a dyad. The two links and one of the joints form a kinematic pair and the other two joints connect the pair to two another links.

We can name the different types of dyads according to their joints. For example, RRR-type dyads shown in Fig. 10.3 have three rotational joints. In both cases one of the links is joined to the frame while the other one is joined to a point on a link that is moving. Figure 10.3 shows how new mechanism $\{O_2, A, B, O_4, C, D, O_6\}$ is generated by addition of these two dyads. The first RRR dyad, defined by $\{A, B, O_4\}$, is joined to point A on crank $\{O_2, A\}$ and to point O_4 on the frame. The second RRR dyad, defined by $\{C, D, O_6\}$, is joined to point C on dyad $\{A, B, O_4\}$ and to point O_6 on the frame.

Figure 10.4 shows an RRP-type and an RPR-type dyad. These dyads have two rotational joints and one prismatic joint but in the first case the dyad is formed by a rotational pair and in the second case by a prismatic one. In Fig. 10.4, it can be seen how new mechanism $\{O_2, A, B, C, D, O_6\}$ is generated by addition of dyads. Dyads RRP, defined by $\{A, B, \mathit{frame}\}$, and RPR, defined by $\{C, D, O_6\}$, are added to crank $\{O_2, A\}$.

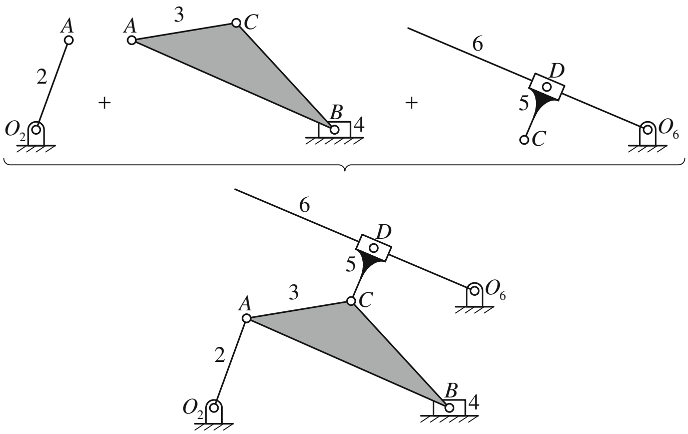


Fig. 10.4 Mechanism generation by addition of an RRP-type and an RPR-type dyads

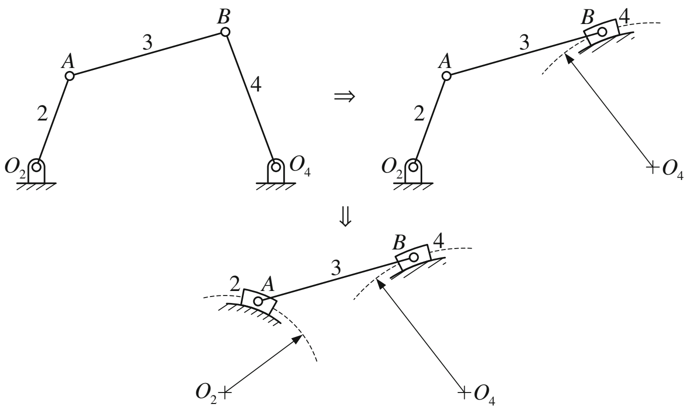


Fig. 10.5 Mechanism generation by kinematic equivalence

We can also obtain new mechanisms by equivalence kinematics. Figure 10.5 shows how new mechanisms are generated starting from a four-bar linkage and substituting rotation pairs for prismatic ones.

Another commonly used method is the one called degeneration kinematics in which, when we make the length of links 2 and 4 infinite in a four bar mechanism, original rotational pairs 12 and 14 turn into prismatic pairs. Figure 10.6a shows how four-bar mechanism $\{O_2, A, B, O_4\}$ is converted into a crank-shaft mechanism (Fig. 10.6b) or into a double-slide mechanism (Fig. 10.6c).

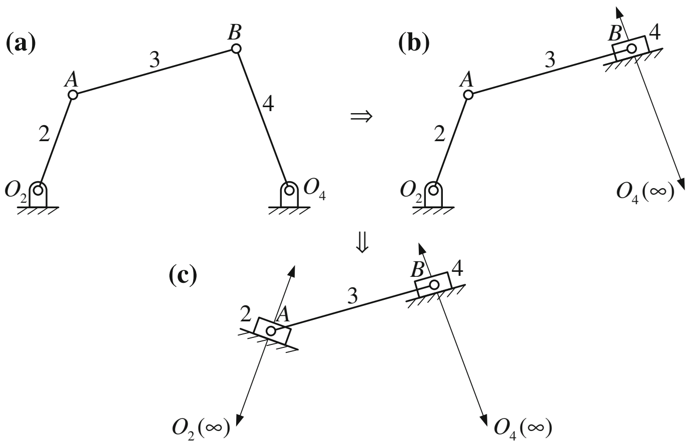


Fig. 10.6 Mechanism generation by kinematic degradation from four-bar (**a**) to crank-shaft (**b**) or into double-slide mechanism (**c**)

10.1.2 Dimensional Synthesis

It focuses on the problem of obtaining the dimensions of a predefined mechanism that has to comply with certain given requirements. It will be necessary to define the dimension of the links and the position of the supports, among others.

Dimensional synthesis can be divided into:

- Function generation.

Pre-established conditions refer to the relation between the input and output motions. These are defined by variables φ nd ψ that identify their positions (Fig. 10.7a). These parameters, normally used in synthesis of mechanisms, are equivalent to angles θ_2 and θ_4 used so far in this book. The relationship between φ and ψ can be defined by means of (Table 10.1) in which n pairs of these values are specified. These pairs can be set manually or according to a mathematical function. In this case, a series of precision points is used to generate the mechanism with exact correspondence between points or with a maximum error measured by means

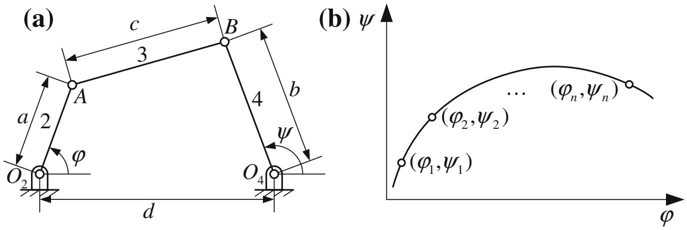


Fig. 10.7 **a** Four-bar mechanism. **b** Relationship between the input and the output

Table 10.1 Precision points for the synthesis of a mechanism with function generation

φ	φ_1	φ_2	…	φ_n
ψ	ψ_1	ψ_2	…	ψ_n

of an error function. If an exact synthesis is desired, the precision points are not used and the input and output positions have to fulfill, along the whole cycle, the relationship defined by a continuous mathematical function. Figure 10.7b shows the values of φ_i and ψ_i that establishes the relationship between the input and the output according to Table 10.1 for a length of links given $(r_1 = d, r_2 = a, r_3 = c, r_4 = b)$.

- Trajectory generation:

It studies and provides methods in order to obtain mechanisms in which one of the points describes a given trajectory. Figure 10.8 shows a mechanism generated with the WinMecC program. The desired trajectory has been defined by means of eight points. The trajectory followed by a point on the coupler of the obtained mechanism is drawn. The arrows show the error vectors between the points on the obtained trajectory and the desired one.

In general, these problems can be solved with three different types of methods:

– Graphical methods. These methods are very didactic and help us to understand the problem in an easy way. However, they offer a limited range of possibilities.
– Analytical methods. They solve the problem by means of mathematical equations based on the requirements.

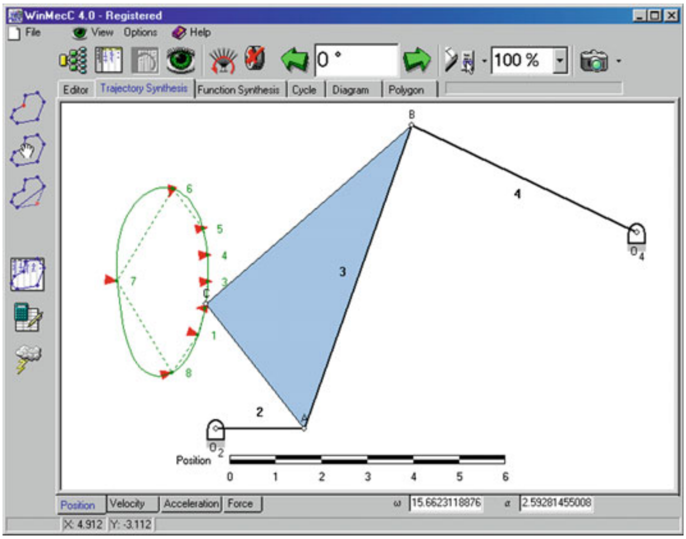


Fig. 10.8 Trajectory generation using WinMecC software

– Optimization-technique-based methods. They can find the optimal solution to the problem by means of the minimization of an objective function and the establishment of a series of restrictions. Different optimization techniques can be used.

10.2 Function Generation Synthesis

It can be defined as the part of synthesis that studies how the position of the input and output links in a mechanism relate to each other. In particular, when using dimensional synthesis, the dimensions of a mechanism are worked out from the imposition of such relationships.

The easiest way to set these relationships is by defining a univocal relationship between a series of n precision points, P_j, of the input link and the correspondent ones of the output link. Another option is to define a continuous function that relates the position of both links, $\psi = f_d(\varphi)$. In both cases, an error function $e(\varphi)$ can be defined between desired, $\psi = f_d(\varphi)$, and generated, $\psi = f_g(\varphi)$, functions to keep the error value within a certain limit (Fig. 10.9).

Function generation can be used to design mechanisms that carry out mathematical operations: addition, differentiation, integration or a combination of them. The first computers were mechanical devices based on this type of mechanisms.

In the mechanism in (Fig. 10.10), the relationship between the positions of links 4 and 6 is defined by mathematical function $\psi = \varphi^2$.

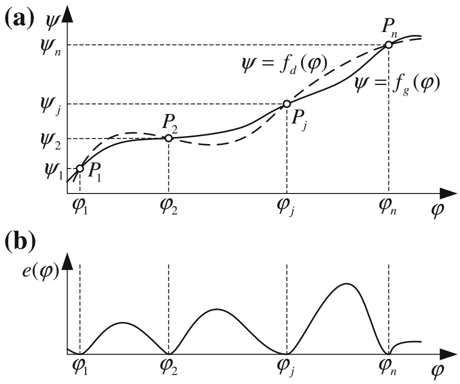


Fig. 10.9 **a** Precision points $P_1, P_2 \ldots P_n$ on the desired $\psi = f_d(\varphi)$ and generated $\psi = f_g(\varphi)$ functions. **b** Error function

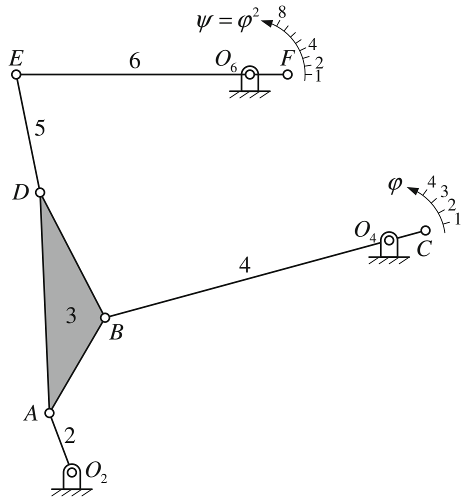


Fig. 10.10 Mechanical computing mechanism to calculate the square of a number

10.2.1 Graphical Methods

As in other areas of the Theory of Mechanisms, different graphical methods have been developed to solve the problem of function generation synthesis. The main benefits of these methods have already been mentioned: they are simple and highly educational. The designer can visually see the procedure and check the result.

In this section, we will explain a method based on the properties of the pole. With this method, in order to obtain the planar motion of a link we have to follow the next steps (Fig. 10.11):

1. Identify any two points, A and B, belonging to the link.
2. Take two different positions of the link which will give us two positions for A and B as well. These points will be named A_1, B_1 and A_2, B_2.

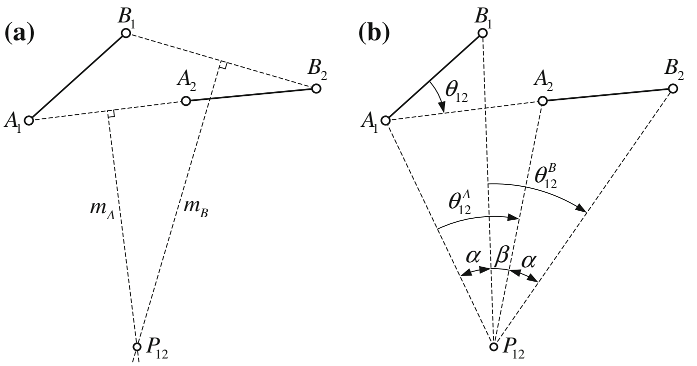


Fig. 10.11 **a** Pole P_{12} at perpendicular bisector intersections. **b** *Triangles* $A_1P_{12}B_1$ and $A_2P_{12}B_2$ rotate with angle $\theta_{12} = \theta_{12}^A = \theta_{12}^B$

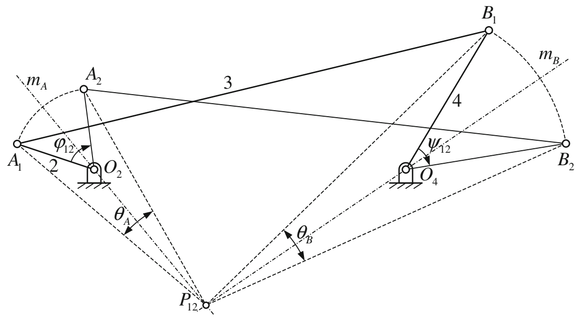


Fig. 10.12 Dimensional synthesis of a four-bar mechanism in which rotated angle of output ψ_{12} corresponds to input rotated angle φ_{12}

3. Take segments $\overline{A_1A_2}$ and $\overline{B_1B_2}$ and draw their perpendicular bisectors m_A and m_B.
4. Pole P_{12} is at their intersection point (Fig. 10.11a).
5. The link can be moved from position 1 to position 2 by carrying out a turn with angle $\theta_{12} = \theta_{12}^A = \theta_{12}^B$. Since triangles $A_1P_{12}B_1$ and $A_2P_{12}B_2$ in Fig. 10.11b are congruent, we can write (Eq. 10.1):

$$\left.\begin{aligned} \theta_{12}^A &= \alpha + \beta \\ \theta_{12}^B &= \beta + \alpha \end{aligned}\right\} \Rightarrow \theta_{12}^A = \theta_{12}^B = \theta_{12} \tag{10.1}$$

Although the pole has other properties, the latter is enough to develop a graphical method of function generation synthesis. As an example, we will generate a four-bar mechanism in which a rotated angle of the input link between positions 1 and 2, φ_{12}, corresponds to a rotated angle of output link ψ_{12}.

We will follow the next steps (Fig. 10.12):

1. We take an arbitrary point and name it pole P_{12}. We draw line m_A and any point O_2 in it.
2. Next we choose a value for the length of the crank and, taking point O_2 as the origin, we draw points A_1 and A_2 in symmetric positions with respect to line m_A. The angle formed by the points and the origin, has to be equal to specified angle φ_{12}.
3. Points A_1, A_2 and P_{12} are connected so that they form angle $\theta_A = A_1\widehat{P_{12}}A_2$.
4. Taking P_{12} as the origin, we draw a new arbitrary line, m_B, and take point O_4 in it.
5. Taking point P_{12} as the origin, we draw two lines that comply with two conditions, the angle they form is equal to $\theta_B = B_1\widehat{P_{12}}B_2 = \theta_A$ and m_B is their bisector.
6. Taking point O_4 as the origin we draw two lines with the condition that the angle they form is equal to ψ_{12} and that m_B is their bisector.
7. The intersection points of these four lines define points B_1 and B_2.

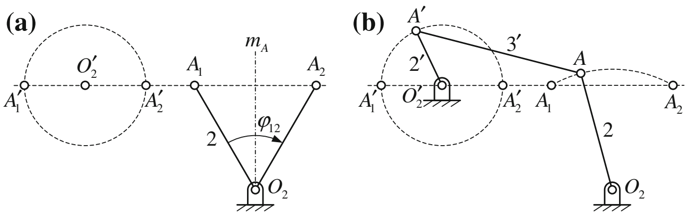


Fig. 10.13 **a** Calculation of the lengths of new links 2′ and 3′. **b** Four-bar mechanism with full rotation of input link 2′ obtained by adding one dyad to input link 2 of the original mechanism

8. The solution obtained is mechanism $\{O_{2,}, A_1, B_1, O_4\}$ as well as its second position, given by $\{O_{2,}, A_2, B_2, O_4\}$.

There are other interesting graphical methods like the Overlay one. This method, widely used, allows carrying out the dimensional synthesis of a four-bar mechanism by means of a trial and error procedure.

The results of these synthesis methods, as well as the results obtained with the trajectory generation synthesis methods that will be introduced further on in this chapter, are frequently four-bar mechanisms that do not comply with Grashoff's condition. That is, none of the links in the mechanism can carry out a full revolution. This condition is necessary if the mechanism has to be powered by a motor with continuous rotation.

To solve this problem, there is a method that adds an RRR dyad to the generated mechanism, so that a new input link, which can turn a full revolution, is defined. The steps to follow are:

1. Consider link 2 of the mechanism in Fig. 10.13a and its rotation interval φ_{12}. This is the input link for a mechanism that has already been synthesized. Draw segment $\overline{A_1A_2}$ as well as its perpendicular bisector, m_A.
2. Extend segment $\overline{A_1A_2}$ and draw point O'_2 on it.
3. Draw a circle with its center in O'_2 and its diameter equal to the length of segment $\overline{A_1A_2}$. This way we obtain points A'_1 and A'_2 (Fig. 10.13a).
4. Figure 10.13b shows the result, a four-bar mechanism in which link 2′ can rotate fully and link 2 is a rocker.

10.2.2 *Freudenstein's Method*

This analytical method starts with the vector loop equation already used in Chap. 3 to solve kinematic problems by means of Raven's method (Fig. 10.14b). Although this method is valid for any type of mechanism we will develop its explanation by applying it to the four-bar mechanism shown in Fig. 10.14a.

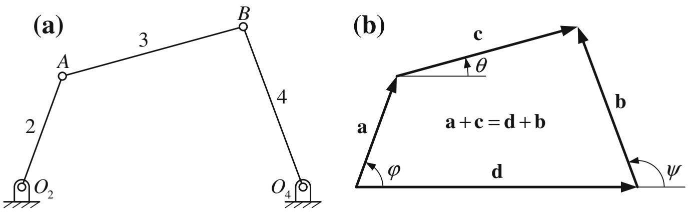


Fig. 10.14 **a** Four-bar mechanism. **b** Vector loop equation of the four-bar mechanism

Once we have defined the vectors in Fig. 10.14b, we can write its vector loop equation (Eq. 10.2) as

$$\mathbf{a} + \mathbf{b} = \mathbf{d} + \mathbf{b} \tag{10.2}$$

The projections on the ordinate and abscissa axis (Eq. 10.3) result are

$$\left.\begin{array}{l} a\cos\phi + c\cos\theta = d + b\cos\psi \\ a\sin\phi + c\sin\theta = b\sin\psi \end{array}\right\} \tag{10.3}$$

In order to find the relation $\psi = f(\varphi)$, we need to clear variable θ (Eq. 10.4) out of the system:

$$\left.\begin{array}{l} c\cos\theta = d + b\cos\psi - a\cos\phi \\ c\sin\theta = b\sin\psi - a\sin\phi \end{array}\right\} \tag{10.4}$$

Rising both equations to the second power, adding and remembering that (Eq. 10.5):

$$\cos(\psi - \phi) = \cos\psi\cos\phi + \sin\psi\sin\phi \tag{10.5}$$

We obtain (Eq. 10.6):

$$c^2 = d^2 + b^2 + a^2 + 2bd\cos\psi - 2ab\cos(\psi - \phi) - 2da\cos\phi \tag{10.6}$$

Next, we define coefficients R_i (Eq. 10.7) as:

$$\left.\begin{array}{l} R_1 = \dfrac{d}{a} \\ R_2 = \dfrac{d}{b} \\ R_3 = \dfrac{d^2 + b^2 + a^2 - c^2}{2ab} \end{array}\right\} \tag{10.7}$$

Finally, using these coefficients in Eq. (10.6), we obtain (Eq. 10.8):

$$R_1 \cos\psi - R_2 \cos\phi + R_3 = \cos(\psi - \phi) \tag{10.8}$$

Equation (10.8) is known as Freudenstein's equation and it is an effective tool to carry out function generation synthesis. We can obtain the length of links *a*, *b*, *c* and *d* in a four-bar mechanism, provided that we know three related positions of the input and output links. These positions are defined by pairs $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ and (φ_3, ψ_3) which are known as precision points.

When these values are substituted in Freudenstein's equation, we obtain a system with three equations (Eq. 10.9) and three unknowns: R_1, R_2 and R_3:

$$\left.\begin{aligned} R_1 \cos\psi_1 - R_2 \cos\phi_1 + R_3 &= \cos(\psi_1 - \phi_1) \\ R_1 \cos\psi_2 - R_2 \cos\phi_2 + R_3 &= \cos(\psi_2 - \phi_2) \\ R_1 \cos\psi_3 - R_2 \cos\phi_3 + R_3 &= \cos(\psi_3 - \phi_3) \end{aligned}\right\} \tag{10.9}$$

This system is linear and independent. It can easily be solved to obtain unknowns R_1, R_2 and R_3. Using the calculated values in the mathematical definitions of these parameters, we can find the length of links *a*, *b*, *c* and *d* by assigning an arbitrary value to one of them, for example, $d = 1$. In this case, the size of the mechanism obtained will depend on the value given to *d*, but it can be escalated to any size.

As said before, this method can be used with other type of mechanisms by following the same steps we have followed in the four-bar mechanism.

If we want to increase the number of precision points, we will also need to increase the number of unknowns in the equation. In the case of four precision points, we can use Freudenstein's equation before defining constants R_i. Hence the equation used has four unknowns: *a*, *b*, *c* and *d*. Substituting the values of the four precision points in Eq. (10.8), we obtain a system with four equations and four unknowns that can be easily solved.

10.3 Trajectory Generation Synthesis

In this part of the synthesis we will study the relationship between the trajectory described by a point in a link and the motion of another link, usually the input one.

Specifically in the case of dimensional synthesis we look for the dimensions of a given mechanism that complies with the condition that one of its points describes a certain trajectory. Based on these definitions, the pursued objective can be different.

The problem can be addressed from a general approach carrying out a dimensional synthesis of a given type of mechanism in which a certain trajectory is desired for a given point.

The problem can also be approached in a more specific way by carrying out the dimensional synthesis of a mechanism that complies with a certain condition such as a point following a trajectory with a straight segment, a circular arc or any other mathematical curve.

The most interesting problem and, hence, the most commonly addressed one is the synthesis of a mechanism in which the trajectory of a point is constrained to go over a number of precision points. In this case, the difference between the desired trajectory and the one obtained (Fig. 10.9a) is measured by defining an error function.

Several graphical, analytical and optimization-based methods have been proposed in order to solve this problem. We will develop some of them in the following sections.

It is very common to develop the different methods by applying them to a four-bar mechanism and paying special attention to the trajectory described by a point *P* of the coupler link. The trajectory curves that can be described by point *P* are called coupler curves and have been studied for a long time. They have been classified into different types depending on their geometry.

10.3.1 Graphical Methods

We will introduce two different methods and apply them to a four-bar mechanism. Due to their simplicity, these methods can be useful in those cases in which the conditions are restrained to two or three precision points.

10.3.1.1 Synthesis for Two Positions of the Coupler Link

The first method that can be used is based on the properties of the pole, which were already explained in this chapter. We can find the desired mechanism by following the next steps (Fig. 10.15):

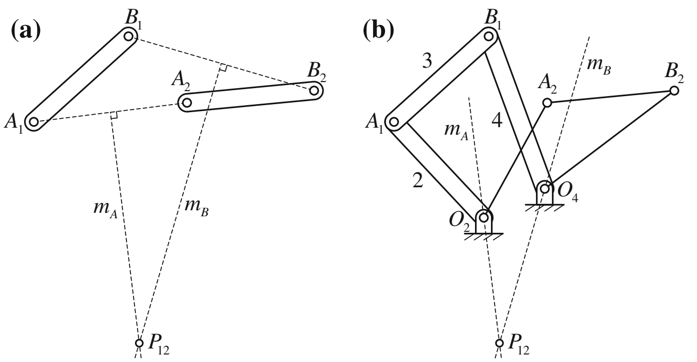


Fig. 10.15 **a** Defining pole position. **b** Synthesis for two positions of the coupler link

1. Two A_1B_1 and A_2B_2 positions of link AB are considered known.
2. We draw a segment between points A_1 and A_2. Then, we draw its perpendicular bisector, m_A.
3. We join points B_1 and B_2 with a new segment and we draw its perpendicular bisector, m_B.
4. The intersection point of these two bisectors defines the position of pole P_{12} (Fig. 10.15a).
5. We select an arbitrary point on m_A to define point O_2 and do the same to define point O_4 on m_B. The displacement of point A from position A_1 to A_2 can be considered a rotation about point O_2 and the displacement of point B from position B_1 to B_2 can be considered a rotation about point O_4
6. Segments O_2A and O_4B define the lengths of links 2 and 4 respectively (Fig. 10.15b). This way we obtain the mechanism we are looking for, $\{O_2, A, B, O_4\}$.

10.3.1.2 Synthesis for Three Positions of the Coupler Link

This method allows finding a four-bar mechanism in which the coupler link passes through the three specified positions (Fig. 10.16). The steps to follow are the next ones:

1. The three positions of link AB are considered known and they are identified as A_1B_1, A_2B_2 and A_3B_3.
2. We draw a segment between points A_1 and A_2 and then its perpendicular bisector $m_{A_{12}}$. The same way we draw a segment between points A_2 and A_3 and then its perpendicular bisector $m_{A_{23}}$.
3. The intersection point of both bisectors is point O_2 (Fig. 10.16a).

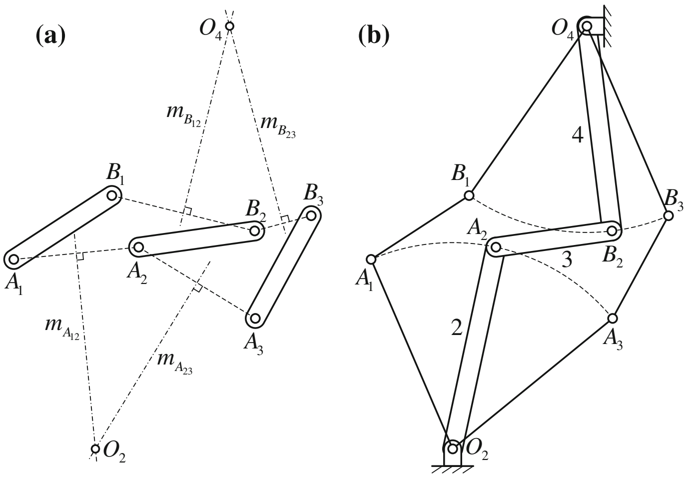


Fig. 10.16 **a** Defining fixed points O_2 and O_4. **b** Synthesis for three positions of the coupler link

4. We operate the same way drawing segments B_1B_2 and B_2B_3 with their perpendicular bisectors $m_{B_{12}}$ and $m_{B_{23}}$.
5. Their intersection point defines the position of point O_4.
6. Hence we obtain the mechanism we were looking for, $\{O_2, A, B, O_4\}$ (Fig. 10.16b).

10.3.2 Analytical Methods

The trajectory described by a point in a mechanism can be obtained as an analytical expression. In this section we will develop a generic example with a four-bar mechanism and then we will apply the three-precision point method to the same mechanism.

10.3.2.1 Trajectory Generation

In the four-bar mechanism shown in Fig. 10.17, a sextic equation of the following form (Eq. 10.10) can be formulated for the trajectory of point P:

$$F(x, y, a, b, d, g, f, \gamma) = 0 \tag{10.10}$$

On one side, we define the x and y coordinates of point P by means of segments a and g (Fig. 10.17). On the other side, we define the same coordinates using segments d, b and f. This way we obtain the system of equations (Eq. 10.11):

$$\left.\begin{aligned} x &= a\cos\phi + g\cos(\alpha+\theta) \\ y &= a\sin\phi + g\sin(\alpha+\theta) \\ x &= d + b\cos\psi + f\cos(\alpha+\theta+\gamma) \\ y &= b\sin\psi + f\sin(\alpha+\theta+\gamma) \end{aligned}\right\} \tag{10.11}$$

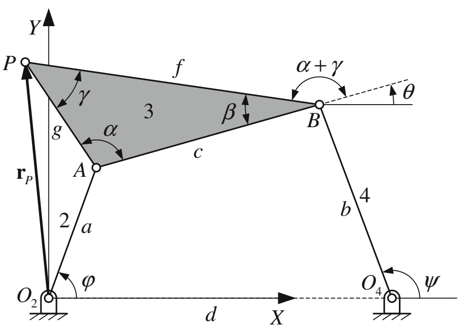


Fig. 10.17 Parameters used to solve a trajectory synthesis problem of a four-bar mechanism

In the system of equations (Eq. 10.11) we want to remove variables φ and ψ. To do so, the first two equations are cleared in terms of $\cos\varphi$ and $\sin\varphi$. After rising the two equations to the second power and adding them, φ disappears.

Similarly, the third and fourth equations can also be cleared in terms of $\cos\psi$ and $\sin\psi$. Rising the two equations to the second power and adding them, variable ψ disappears.

Thus, we obtain (Eq. 10.12):

$$\left.\begin{array}{l} a^2 = x^2 + y^2 - 2gx\cos(\alpha+\theta) - 2gy\sin(\alpha+\theta) \\ b^2 = (x-d)^2 + y^2 + f^2 - 2f(x-d)\cos(\alpha+\theta+\gamma) \\ \quad -2fy\sin(\alpha+\theta+\gamma) \end{array}\right\} \tag{10.12}$$

In order to obtain Eq. (10.10), we have to remove those terms depending on $\alpha+\theta$ from Eq. (10.12). In the final equation we will have six unknowns (a, b, d, g, f, γ) and two known values (x, y). Therefore, we can solve the problem with a maximum of six precision points.

Equation (10.10) can be used not only to represent the trajectory of a point on the coupler but also to compare it with the desired trajectory in order to calculate the error.

10.3.2.2 Trajectory Generation with Three Precision Points

A method based on complex numbers can be proposed to carry out the synthesis of a four-bar mechanism when we know the three different positions, P_1, P_2 and P_3, of the point of interest, P, on the coupler link (Fig. 10.18).

The position of the point of interest, P, can be defined by setting the following vector loops (Eqs. 10.13 and 10.14):

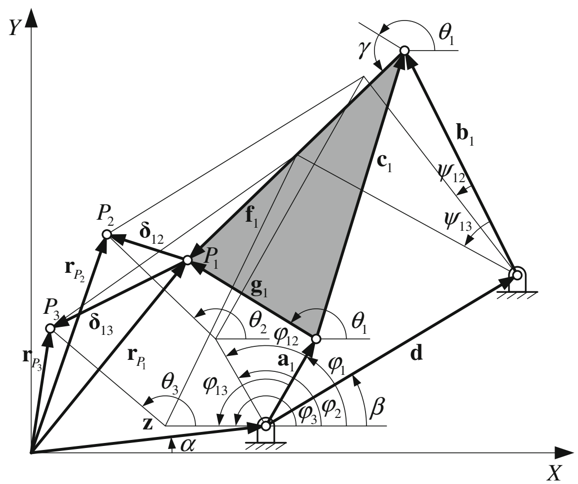


Fig. 10.18 Trajectory generation with three precision points

$$\mathbf{r}_{P_1} = \mathbf{z} + \mathbf{a}_1 + \mathbf{g}_1 \tag{10.13}$$

$$\mathbf{r}_{P_1} = \mathbf{z} + \mathbf{d} + \mathbf{b}_1 + \mathbf{f}_1 \tag{10.14}$$

Since they have no common unknowns, we can analyze each equation separately. We start with Eq. (10.13). Particularizing the expression for positions 1 and 2 yields (Eq. 10.15):

$$\left.\begin{aligned} \mathbf{r}_{P_1} &= z_1 \mathrm{e}^{i\alpha} + a\mathrm{e}^{i\varphi_1} + g\mathrm{e}^{i\theta_1} \\ \mathbf{r}_{P_2} &= z_1 \mathrm{e}^{i\alpha} + a\mathrm{e}^{i\varphi_2} + g\mathrm{e}^{i\theta_2} \end{aligned}\right\} \tag{10.15}$$

In order to obtain displacement $\boldsymbol{\delta}_{12}$ (Eq. 10.16), we subtract one equation from the other:

$$\boldsymbol{\delta}_{12} = \mathbf{r}_{P_2} - \mathbf{r}_{P_1} = a(\mathrm{e}^{i\varphi_2} - \mathrm{e}^{i\varphi_1}) + g(\mathrm{e}^{i\theta_2} - \mathrm{e}^{i\theta_1}) \tag{10.16}$$

Equation (10.16) can be written as:

$$\boldsymbol{\delta}_{12} = a\mathrm{e}^{i\varphi_1}\left(\frac{\mathrm{e}^{i\varphi_2}}{\mathrm{e}^{i\varphi_1}} - 1\right) + g\mathrm{e}^{i\theta_1}\left(\frac{\mathrm{e}^{i\theta_2}}{\mathrm{e}^{i\theta_1}} - 1\right) = \mathbf{a}_1(\mathrm{e}^{i\varphi_{12}} - 1) + \mathbf{g}_1(\mathrm{e}^{i\theta_{12}} - 1) \tag{10.17}$$

where angles φ_{12} and θ_{12} (Eq. 10.18) are:

$$\left.\begin{aligned} \varphi_{12} &= \varphi_2 - \varphi_1 \\ \theta_{12} &= \theta_2 - \theta_1 \end{aligned}\right\} \tag{10.18}$$

Displacement $\boldsymbol{\delta}_{13}$ (Eqs. 10.19 and 10.20) can be obtained by proceeding the same way:

$$\boldsymbol{\delta}_{13} = \mathbf{r}_{P_3} - \mathbf{r}_{P_1} = a(\mathrm{e}^{i\varphi_3} - \mathrm{e}^{i\varphi_1}) + g(\mathrm{e}^{i\theta_3} - \mathrm{e}^{i\theta_1}) \tag{10.19}$$

Thus:

$$\boldsymbol{\delta}_{13} = a\mathrm{e}^{i\varphi_1}\left(\frac{\mathrm{e}^{i\varphi_3}}{\mathrm{e}^{i\varphi_1}} - 1\right) + g\mathrm{e}^{i\theta_1}\left(\frac{\mathrm{e}^{i\theta_3}}{\mathrm{e}^{i\theta_1}} - 1\right) = \mathbf{a}_1(\mathrm{e}^{i\varphi_{13}} - 1) + \mathbf{g}_1(\mathrm{e}^{i\theta_{13}} - 1) \tag{10.20}$$

where angles φ_{13} and θ_{13} (Eq. 10.21) are:

$$\left.\begin{aligned} \varphi_{13} &= \varphi_3 - \varphi_1 \\ \theta_{13} &= \theta_3 - \theta_1 \end{aligned}\right\} \tag{10.21}$$

Hence, we obtain the system of equations (Eqs. 10.22 and 10.23):

$$\boldsymbol{\delta}_{12} = \mathbf{a}_1(\mathrm{e}^{i\varphi_{12}} - 1) + \mathbf{g}_1(\mathrm{e}^{i\theta_{12}} - 1) \tag{10.22}$$

$$\boldsymbol{\delta}_{13} = \mathbf{a}_1(\mathrm{e}^{i\varphi_{13}} - 1) + \mathbf{g}_1(\mathrm{e}^{i\theta_{13}} - 1) \tag{10.23}$$

From which we can clear $\mathbf{a}_1$ (Eq. 10.24) and $\mathbf{g}_1$ (Eq. 10.25):

$$\mathbf{a}_1 = \frac{\begin{vmatrix} \boldsymbol{\delta}_{12} & \mathrm{e}^{i\theta_{12}} - 1 \\ \boldsymbol{\delta}_{13} & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{i\varphi_{12}} - 1 & \mathrm{e}^{i\theta_{12}} - 1 \\ \mathrm{e}^{i\varphi_{13}} - 1 & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}} \tag{10.24}$$

$$\mathbf{g}_1 = \frac{\begin{vmatrix} \mathrm{e}^{i\varphi_{12}} - 1 & \boldsymbol{\delta}_{12} \\ \mathrm{e}^{i\varphi_{13}} - 1 & \boldsymbol{\delta}_{13} \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{i\varphi_{12}} - 1 & \mathrm{e}^{i\theta_{12}} - 1 \\ \mathrm{e}^{i\varphi_{13}} - 1 & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}} \tag{10.25}$$

In order to obtain $\mathbf{b}_1$ and $\mathbf{f}_1$, we write Eq. (10.14) for positions 1 and 2 (Eq. 10.26):

$$\left.\begin{aligned} \mathbf{r}_{P_1} &= z\mathrm{e}^{i\alpha} + d\mathrm{e}^{i\beta} + b\mathrm{e}^{i\psi_1} + f\mathrm{e}^{i(\theta_1+\gamma)} \\ \mathbf{r}_{P_2} &= z\mathrm{e}^{i\alpha} + d\mathrm{e}^{i\beta} + b\mathrm{e}^{i\psi_2} + f\mathrm{e}^{i(\theta_2+\gamma)} \end{aligned}\right\} \tag{10.26}$$

Displacement $\boldsymbol{\delta}_{12}$ (Eqs. 10.27 and 10.28) is found by subtracting one expression from the other:

$$\boldsymbol{\delta}_{12} = \mathbf{r}_{P_2} - \mathbf{r}_{P_1} = b(\mathrm{e}^{i\psi_2} - \mathrm{e}^{i\psi_1}) + f(\mathrm{e}^{i(\theta_2+\gamma)} - \mathrm{e}^{i(\theta_1+\gamma)}) \tag{10.27}$$

Thus:

$$\boldsymbol{\delta}_{12} = b\mathrm{e}^{i\psi_1}\left(\frac{\mathrm{e}^{i\psi_2}}{\mathrm{e}^{i\psi_1}} - 1\right) + f\mathrm{e}^{i(\theta_1+\gamma)}\left(\frac{\mathrm{e}^{i\theta_2}}{\mathrm{e}^{i\theta_1}} - 1\right) = \mathbf{b}_1(\mathrm{e}^{i\psi_{12}} - 1) + \mathbf{f}_1(\mathrm{e}^{i\gamma_{12}} - 1) \tag{10.28}$$

where angles ψ_{12} and θ_{12} (Eq. 10.29) are:

$$\left.\begin{aligned} \psi_{12} &= \psi_2 - \psi_1 \\ \theta_{12} &= \theta_2 - \theta_1 \end{aligned}\right\} \tag{10.29}$$

Angle θ of vectors $\mathbf{g}_1$ and $\mathbf{f}_1$ is the same since they belong to the same link.

Displacement $\boldsymbol{\delta}_{13}$ (Eqs. 10.30 and 10.31) is obtained by following the same process:

$$\boldsymbol{\delta}_{13} = \mathbf{r}_{P_3} - \mathbf{r}_{P_1} = b(\mathrm{e}^{i\psi_3} - \mathrm{e}^{i\psi_1}) + f(\mathrm{e}^{i(\theta_3+\gamma)} - \mathrm{e}^{i(\theta_1+\gamma)}) \tag{10.30}$$

$$\boldsymbol{\delta}_{13} = b\mathrm{e}^{i\psi_1}\left(\frac{\mathrm{e}^{i\psi_3}}{\mathrm{e}^{i\psi_1}} - 1\right) + f\mathrm{e}^{i(\theta_1+\gamma)}\left(\frac{\mathrm{e}^{i\theta_3}}{\mathrm{e}^{i\theta_1}} - 1\right) = \mathbf{b}_1(\mathrm{e}^{i\psi_{13}} - 1) + \mathbf{f}_1(\mathrm{e}^{i\theta_{13}} - 1) \tag{10.31}$$

where angles ψ_{13} and θ_{13} (Eq. 10.32) are:

$$\left.\begin{array}{l} \psi_{13} = \psi_3 - \psi_1 \\ \theta_{13} = \theta_3 - \theta_1 \end{array}\right\} \tag{10.32}$$

Hence, we obtain the system of equations (Eq. 10.33):

$$\left.\begin{array}{l} \boldsymbol{\delta}_{12} = \mathbf{b}_1(\mathrm{e}^{i\psi_{12}} - 1) + \mathbf{f}_1(\mathrm{e}^{i\theta_{12}} - 1) \\ \boldsymbol{\delta}_{13} = \mathbf{b}_1(\mathrm{e}^{i\psi_{13}} - 1) + \mathbf{f}_1(\mathrm{e}^{i\theta_{13}} - 1) \end{array}\right\} \tag{10.33}$$

From which $\mathbf{b}_1$ (Eq. 10.34) and $\mathbf{f}_1$(Eq. 10.35) can be cleared:

$$\mathbf{b}_1 = \frac{\begin{vmatrix} \boldsymbol{\delta}_{12} & \mathrm{e}^{i\theta_{12}} - 1 \\ \boldsymbol{\delta}_{13} & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{i\psi_{12}} - 1 & \mathrm{e}^{i\theta_{12}} - 1 \\ \mathrm{e}^{i\psi_{13}} - 1 & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}} \tag{10.34}$$

$$\mathbf{g}_1 = \frac{\begin{vmatrix} \mathrm{e}^{i\psi_{12}} - 1 & \boldsymbol{\delta}_{12} \\ \mathrm{e}^{i\psi_{13}} - 1 & \boldsymbol{\delta}_{13} \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{i\psi_{12}} - 1 & \mathrm{e}^{i\theta_{12}} - 1 \\ \mathrm{e}^{i\psi_{13}} - 1 & \mathrm{e}^{i\theta_{13}} - 1 \end{vmatrix}} \tag{10.35}$$

In Eq. (10.36), vectors $\mathbf{z}$ and $\mathbf{d}$ can be deduced from Eqs. (10.13) and (10.14) and $\mathbf{c}_1$ can be obtained from the mechanism vector loop equation:

$$\left.\begin{array}{l} \mathbf{z} = \mathbf{r}_{P_1} - \mathbf{a}_1 - \mathbf{g}_1 \\ \mathbf{d} = \mathbf{r}_{P_1} - \mathbf{z} - \mathbf{b}_1 - \mathbf{f}_1 \\ \mathbf{c}_1 = \mathbf{d} + \mathbf{b}_1 - \mathbf{a}_1 \end{array}\right\} \tag{10.36}$$

Therefore, we can obtain the dimensions of the links of a four-bar mechanism, when we know the three positions of a point, P, of the coupler link as well as the angles rotated by the links between the positions.

10.4 Optimal Synthesis of Mechanisms

Traditionally, engineers try to solve their design problems, particularly the problems of synthesis of mechanisms that concern us, by considering different alternatives with the intention of reaching the best solution. Thanks to the development of mathematical programming techniques and the rapid advances in computers and software, it is now possible to formulate the design problem as an optimization problem with the objective of minimizing an objective function provided that the design conditions are met. We can, thus, obtain the optimal solution.

In general, the solution of an optimization problem determines the value of the variables $(x_1, x_2, \ldots, x_n)$ that minimize objective function $f(x)$ subject to a set of constraints (Eq. 10.37). This can be written as:

$$\begin{aligned} &\min f(x_1, x_2, \ldots, x_n) \\ &\text{Subject to:} \\ &\quad h_j(x_1, x_2, \ldots, x_n) \leq 0 \quad j = 1, 2, \ldots, m \\ &\quad g_k(x_1, x_2, \ldots, x_n) = 0 \quad k = 1, 2, \ldots, p \end{aligned} \tag{10.37}$$

Function $f(x)$ is called objective function and functions $h_j(x)$ and $g_k(x)$ are called constraints of the problem. We can have both inequality and equality constraints. In the context of engineering design, the above mentioned concepts are defined as:

- Objective function: A function that expresses a fundamental aspect of the problem. An extreme value (minimum or maximum) is sought along the process of optimization. This function is often called merit function. Multifunctional functions, in which several features are optimized, can also be formulated. In this case, each one of them is weighted depending on their importance.
- Independent design variables: Such variables represent the geometry of the model. They are usually the dimensions of the mechanism such as the length or width of the links.
- Dependent variables: These are parameters that have to be included in the formulation of the objective function or the constraints but that depend on the design variables.
- Constraints: They are mathematical functions that define the relationships between the design variables that have to be met by every set of values that define a possible design. These relationships can be of three types.
  - Inequality restrictions: They are usually limitations to the behavior of the mechanism or security restrictions to prevent failure under certain conditions.
  - Variable limits: They are a specific case of the previous ones.
  - Equality restrictions: They are conditions that have to be met strictly in order for the design to be acceptable.

Depending on the aspect being considered, design problems can be classified in different ways. The two most commonly used classifications are:

- Classification based on the existence of restrictions.
  - Unconstrained optimization problems.
  - Constrained optimization problems.
- Classification based on the nature of the equations (objective function and constraints).
  - Linear optimization problem, when both the objective function and the constraints are linear.
  - Nonlinear optimization problem, when the objective function and/or the constraints are nonlinear.

Based on the types of constraints that shape the synthesis problem, we face a nonlinear constrained problem that suggests a mathematical programming problem with nonlinear objective functions and nonlinear constraints.

10.5 Analysis of the Objective Function

Previously in this chapter, we defined the function generation synthesis as the establishment of a relationship between the input and output links in a mechanism. Hence, objective function $f(x)$ has to be developed for the particular mechanism to be studied.

10.5.1 Function Generation Synthesis

In this case we will study the function synthesis of a four-bar mechanism (Fig. 10.19) in which we will define a relationship between parameter ψ that defines the position of the output link and parameter φ of the input link.

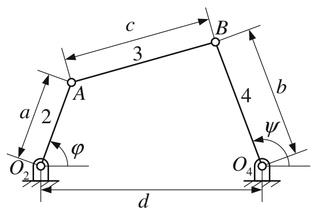


Fig. 10.19 Four-bar mechanism with the parameters used in function generation synthesis

We will use Freudenstein's equation (Eq. 10.8) and the method developed in Appendix B of this book to obtain the relationship between the output angle, ψ, and input one, φ (Eq. 10.38):

$$\psi = 2\arctan\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \psi(a, b, c, d, \varphi) \tag{10.38}$$

where coefficients A, B and C (Eq. 10.39) are:

$$\left.\begin{array}{l} A = \cos\varphi - R_1 - R_2\cos\varphi + R_3 \\ B = -2\sin\varphi \\ C = R_1 - (R_2 + 1)\cos\varphi + R_3 \end{array}\right\} \tag{10.39}$$

where R_1, R_2 and R_3 are known functions of a, b, c, and d.

The objective function will be the minimization of the quadratic difference between the obtained output angle, ψ, and the desired one, ψ^d. If we calculate the values of the output angle by means of the input angle using (Eq. 10.38), the optimization problem (Eq. 10.40) can be formulated as:

$$\begin{array}{l} \min \sum_{i=1}^{N} \left(\psi_i(X, \varphi_i^d) - \psi_i^d\right)^2 \\ \text{Subject to:} \\ \quad x_i \in [li_i, ls_i] \quad \forall \ x_i \in X = [a, b, c, d] \end{array} \tag{10.40}$$

When input angles φ are unknown, these values become design variables in the optimization problem. This problem (Eq. 10.41) is more complex and can be formulated as:

$$\begin{array}{l} \min \sum_{i=1}^{N} \left(\psi_i(X) - \psi_i^d\right)^2 \\ \text{Subject to:} \\ \quad x_i \in [li_i, ls_i] \quad \forall \ x_i \in X = [a, b, c, d, \varphi_1, \varphi_2, \ldots, \varphi_N] \end{array} \tag{10.41}$$

10.5.2 Trajectory Synthesis

There are many different methods to measure the error between two curves. In this section we present two of these methods.

10.5.2.1 Mean Square Error

To explain this method, we will address the synthesis of a four-bar mechanism in which a point of the coupler has to pass through a series of positions previously established. The difference between these desired positions and the exact position of the point of the synthesized mechanism will give us the error.

To do so, we have to study the kinematics of the mechanism shown in Fig. 10.20. We need to know the trajectory of the point by means of its coordinates. We use a coordinate system with its origin in O_2 and the x-axis defined by line O_2O_4:

$$r^r_{C_x} = a\cos\varphi + r^n_{CA}\cos\theta - r^t_{CA}\sin\theta \tag{10.42}$$

$$r^r_{C_y} = a\sin\varphi + r^n_{CA}\sin\theta + r^t_{CA}cos\,\theta \tag{10.43}$$

In Eqs. (10.42) and (10.43), we do not know the values for variable θ. Therefore, we have to express it in terms of known angle φ. This can be done by using the same approach that helped us to obtain Freudenstein's equation (Eq. 10.8) in Sect. 10.2.2 but removing variable ψ instead of θ:

The loop vector (Eq. 10.44) is:

$$\mathbf{a} + \mathbf{c} = \mathbf{d} + \mathbf{b} \tag{10.44}$$

Projecting the vectors on the X_r and Y_r axis yields (Eq. 10.45):

$$\left.\begin{array}{l} a\cos\varphi + c\cos\theta = d + b\cos\psi \\ a\sin\varphi + c\sin\theta = b\sin\psi \end{array}\right\} \tag{10.45}$$

In order to find the sought relationship, $\theta = \theta(\varphi)$, we need to remove variable ψ from Eq. (10.45).

Then, we can write the coordinates of point C in terms of the absolute coordinate system $\{O, X, Y\}$ (Eq. 10.46) by multiplying the relative coordinates of point C by the rotation matrix and adding the absolute coordinates of O_2:

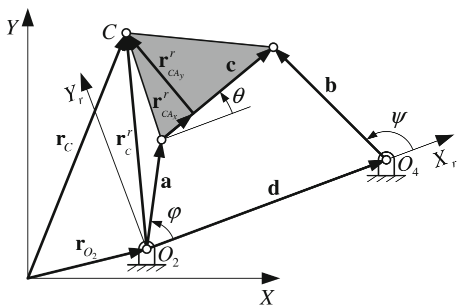


Fig. 10.20 Trajectory synthesis of a four-bar mechanism

$$\begin{pmatrix} r_{C_x} \\ r_{C_y} \end{pmatrix} = \begin{pmatrix} r_{O_{2x}} \\ r_{O_{2y}} \end{pmatrix} + \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} r^r_{C_x} \\ r^r_{C_y} \end{pmatrix} \tag{10.46}$$

Thus, we can express the coordinates of the coupler point in terms of the absolute coordinate system and the design variables are $a, b, c, d, r^r_{C_x}, r^r_{C_y}, \beta, r_{O_{2x}}$ and $r_{O_{2y}}$ for input angle φ.

The goal is to minimize the mean square error between the positions of the point of the coupler and the desired positions (Eq. 10.47). This is expressed as:

$$\min \frac{1}{N} \sum_{i=1}^{N} \left((r^d_{C_x,i} - r_{C_x,i})^2 + (r^d_{C_y,i} - r_{C_y,i})^2 \right) \tag{10.47}$$

where N is the number of desired points.

Once the objective function has been set, we have to add certain constraints that have to be met. In this example, they are the following:

- The values of the design variables have to be within a range.
- It is desirable that four-bar mechanisms have at least one link that can fully rotate, for which Grashoff's condition has to be met.
- Input angles, φ_i, have to be in sequential order. This means that the trajectory points have to be reached as a positive or negative sequence of the input angles.

We can include the last two restrictions in the objective function as $h_1(X)$ and $h_2(X)$ multiplied by constants M_1 and M_2 (Eq. 10.48). The value of these constants is high so that the objective function is penalized if they are not met.

Hence, the function remains:

$$\begin{gathered} \min \frac{1}{N} \sum_{i=1}^{N} \left((r^d_{C_x,i} - r_{C_x,i}(X))^2 + (r^d_{C_y,i} - r_{C_y,i}(X))^2 \right) + M_1 h_1(X) + M_2 h_2(X) \\ \text{Subject to:} \\ x_i \in [li_i, ls_i] \quad \forall \; x_i \in X = [a, b, c, d, r^r_{C_x}, r^r_{C_y}, \beta, r_{O_{2x}}, r_{O_{2y}}, \varphi_1, \varphi_2, \ldots, \varphi_N] \end{gathered} \tag{10.48}$$

where:

$$h_1(X) = \begin{cases} 0 & \text{if } X \text{ meets Grashoff's condition} \\ 1 & \text{otherwise} \end{cases}$$

$$h_2(X) = \begin{cases} 0 & \text{if } X \text{ meets } \varphi_i \text{ in sequential order condition} \\ 1 & \text{otherwise} \end{cases}$$

10.5.2.2 Turning Function

An intrinsic equation of a curve defines the curve by using properties that do not depend on its location and orientation. Therefore an intrinsic equation defines the curve without specifying its position with respect to an arbitrary coordinate system as we did in the previous section.

Commonly used intrinsic quantities are arc length s, tangential angle θ, curvature κ or radius of curvature ρ. Torsion τ is also used for 3-dimensional curves. Some examples of intrinsic equations are the natural equation that defines a curve by its curvature and torsion, the Whewell equation obtained as a relation between the arc length and the tangential angle and the Cesàro equation obtained as a relation between the arc length and the curvature.

The use of intrinsic properties greatly simplifies the problem. In this section we will use a slightly different definition of the Whewell equation.

First, we define turning function $\Theta_F(s)$ as the relation between the tangential angle and the normalized arc length. This function measures tangential angle θ of shape F at point P, as a function of the normalized arc, $s = l_P/L$, where l_P is the distance along the shape anti-clockwise from reference point O to point P (Fig. 10.21a) and L is the perimeter of the curve. If shape F is convex, its turning function $\Theta_F(s)$ will be monotone increasing. If F is a closed curve, the turning function verifies $\Theta_F(s+n) = \Theta_F(s) + 2\pi n$, where n represents the number of cycles or complete turns along the curve.

When the path is polygonal, the turning function is piecewise-constant, with steps corresponding to the vertices of shape F. Figure 10.21a shows polygonal shape F with four vertices and in Fig. 10.21b its turning function is shown.

Some general properties of turning functions, which are also applicable to non-polygonal shapes, are the following:

- If reference point O is changed to position O^*, the turning function shifts horizontally. The normalized arc distance between O and O^*, which can be calculated as $\Delta s_O = \Delta l_0/L$, gives us the shift value where Δl_0 is the distance along

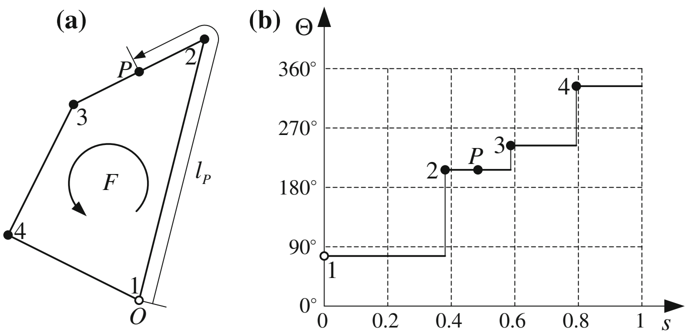


Fig. 10.21 **a** Polygonal shape F. **b** Turning function of shape F

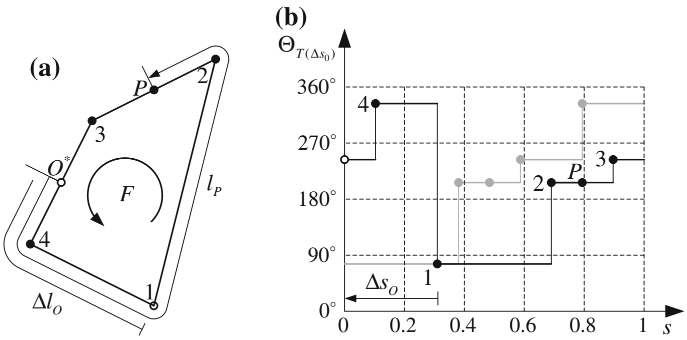


Fig. 10.22 **a** Polygonal shape F with another reference point O^*. **b** Shifted turning function

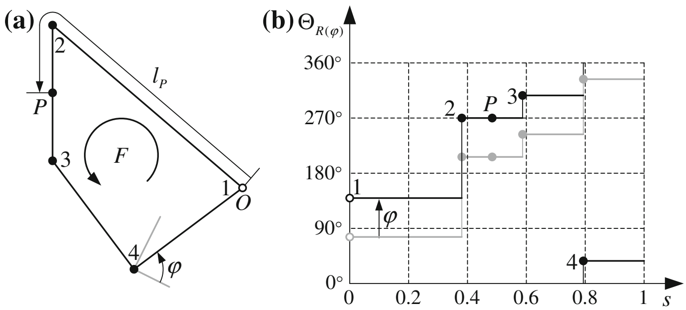


Fig. 10.23 **a** Polygonal shape F rotates angle φ. **b** Rotated turning function

the shape anti-clockwise from reference point O to new reference point O^*. The new turning function, $\Theta_{T(\Delta s_0)}(s) = \Theta(s + \Delta s_0)$, is shown in Fig. 10.22b.

- If curve F is rotated with affine transformation $R(\varphi)$ defined by means of angle φ, the turning function shifts vertically. The new turning function, $\Theta_{R(\varphi)}(s) = \Theta(s) + \varphi$, is shown in Fig. 10.23b.

Equation (10.49) defines the error between any two shapes A and B using their respective turning functions:

$$e_p(A,B) = \|\Theta_A(s) - \Theta_B(s)\|_p = \left(\int_0^1 |\Theta_A(s) - \Theta_B(s)| ds \right)^{\frac{1}{p}} \tag{10.49}$$

where $\|\|_p$ is L_p norm.

This error is related to the area between both turning functions and defines a metric that satisfies the following conditions for all $p > 0$, and specifically for $p = 2$:

- $e_p(A,B) \geq 0$, it is positive everywhere.
- $e_p(A,B) = 0$ if and only if $A = B$, it has the identity property.
- $e_p(A,B) = e_p(B,A)$, it is symmetric.
- $e_p(A,B) + e_p(B,C) \geq e_p(A,C)$, it also obeys the triangle inequality.

Therefore, the smaller the error is, the more similar shapes A and B are. However, relative rotation between both shapes and the reference point used in each one affect the error. To avoid this drawback, the minimum error of function (Eq. 10.49) for any angle φ and normalized arc distance Δs can be found using equation (Eq. 10.50).

$$
\begin{aligned}
e_p(A,B) &= \min_{\Delta s \in R \varphi \in [0,2\pi]} \left(\int_0^1 |\Theta_A(s+\Delta s) - \Theta_B(s) + \varphi| ds \right)^{\frac{1}{p}} \\
&= \min_{\Delta s \in R \varphi \in [0,2\pi]} (g(\varphi, \Delta s))^{\frac{1}{p}}
\end{aligned}
\tag{10.50}
$$

When $p = 2, e_2(A,B)$ is a quadratic function of φ for any fixed Δs value.

The value for function $g(\varphi, \Delta s)$ (Eq. 10.51) is obtained from equation (Eq. 10.50):

$$
g(\varphi, \Delta s) = \int_0^1 |\Theta_A(s+\Delta s) - \Theta_B(s) + \varphi|^2 ds \tag{10.51}
$$

The minimum value of $g(\varphi, \Delta s)$ with respect to parameter φ (Eq. 10.52) can be determined as:

$$
\begin{aligned}
\min_{\varphi \in [0,2\pi]} g(\varphi, \Delta s) &= \int_0^1 |\Theta_A(s+\Delta s) - \Theta_B(s)|^2 ds - \varphi_{opt}^2 \\
\varphi_{opt}(\Delta s) &= \int_0^1 \Theta_B(s) ds - \int_0^1 \Theta_A(s) ds - 2\pi \Delta s
\end{aligned}
\tag{10.52}
$$

where φ_{opt} is the optimum value that shifts turning function $\Theta_A(s+\Delta s)$ vertically. This way, the minimization problem of $e_2(A,B)$ (Eq. 10.53) becomes:

$$
\begin{aligned}
e_2(A,B) &= \min_{\Delta s \in R \varphi \in [0,2\pi]} \sqrt{\int_0^1 |\Theta_A(s+\Delta s) - \Theta_B(s) + \varphi|^2 ds} \\
&= \min_{\Delta s \in R} \sqrt{\min_{\varphi \in [0,2\pi]} g(\varphi, \Delta s)}
\end{aligned}
\tag{10.53}
$$

In this example, curves A and B are polygonal shapes defined by their vertex coordinates. Consequently, turning functions $\Theta_A(s)$ and $\Theta_B(s)$ are defined by means of their coordinates $\Theta(s) = \{(s^1, \varphi^1), (s^2, \varphi^2), \ldots, (s^n, \varphi^n)\}$, where n is the number of vertexes of the polygonal shape (Fig. 10.21b). Parameter $\Delta s = s_A^i - s_B^j$ can only take the values of the horizontal shifts that make discontinuities of turning functions $\Theta_A(s + \Delta s)$ and $\Theta_B(s)$ have the same horizontal coordinate. This set of values has an infinite size, depending on the turning function coordinates and their distribution along the interval [0, 1].

With this approach to the optimal trajectory synthesis problem, objective function $f(x)$, is $e_2(A, B)$.

10.6 Optimization Method Based on Evolutionary Algorithms

The method proposed in this section is based on evolutionary algorithms which have proved to be successful to solve optimization problems. These algorithms are based on the mechanisms of natural selection and the laws of natural genetics.

The main benefit of this method stems from the simple implementation of the algorithms, its low computational cost and the absence of needing to know if the search of space is continuous, differentiable or other mathematical constraints that are required in traditional search algorithms.

The evolutionary methods start by generating an initial population which evolves by means of natural mechanisms into new populations that improve the objective function. In this initial population, which can be created randomly, each individual ("chromosome") is a solution to the problem. This individual will consist of parameters ("genes") that represent the design variables that are used to define a problem.

In our case, we represent each individual (Eq. 10.54) as vector $\mathbf{X}$ with n real variables $x_1, x_2, \ldots, x_n$:

$$\mathbf{X} = [x_1, x_2, \ldots, x_n] \quad x_i \in \mathbb{R} \tag{10.54}$$

Objective function $f()$ of the problem to be solved, allows evaluating fitness $f(\mathbf{X})$ of each individual. The optimal solution to the problem will be the individual with minimal fitness value.

The population has to evolve towards populations in which the individuals are better, that is, their fitness values are lower. This is achieved through natural selection, reproduction, mutation and other genetic operators. These operators are implemented as follows:

Selection consists of choosing two individuals of the population which will form a couple for reproduction. This selection can be performed in various ways, the simplest one being purely random with uniform probability distribution. Selection can also be random but applying a weight to each individual depending on the value obtained in the objective function (fitness). In this case, the probability distribution is not uniform but the best individuals are more likely to be chosen.

In our case, we will form couples for reproduction by choosing the best individuals plus another two individuals picked up randomly with uniform distribution of probability in the entire population. These elements will form vector **V** (disturbed vector) (Eq. 10.55) as:

$$\begin{gathered} \mathbf{X}_i : i \in [1, NP] \\ \mathbf{V} = \mathbf{X}_{Best} + F(\mathbf{X}_{r1} - \mathbf{X}_{r2}) \end{gathered} \tag{10.55}$$

where $\mathbf{X}_{Best}$ is the best of a population of NP individuals and $\mathbf{X}_{r1}$ and $\mathbf{X}_{r2}$ are the two individuals randomly chosen. Factor F is a real value that regulates the disturbance of the best individual. This scheme is called differential evolution.

Vector **V** can be modified with a genetic operator called mutation that changes the value of its genes. This operator consists of choosing a value randomly among v_j and $v_j \pm \Delta_j$, so that if parameter v_j mutates, range Δ_j will be added or subtracted depending on the direction of the mutation.

Once vector **V** is defined, it is crossed over with individual $\mathbf{X}_i$ of the current population to obtain new individual $\mathbf{X}_i^N$ (offspring) who is a candidate to be part of the following population. This operation is called crossover. Different methods can be used to allocate the genes of the offspring. This is done randomly. In natural reproduction, the genes of the parents are exchanged to form the genes of the child or children. This is better illustrated in the scheme shown in Fig. 10.24.

In order to define the new population, we compare offspring fitness $f(\mathbf{X}_i^N)$ with its antecessor fitness, $f(\mathbf{X}_i)$. The individual with less fitness value is "better" and takes part in the new population. This way the population does not increase or decrease in number.

The scheme of the whole process can be seen in the diagram shown in Fig. 10.25.

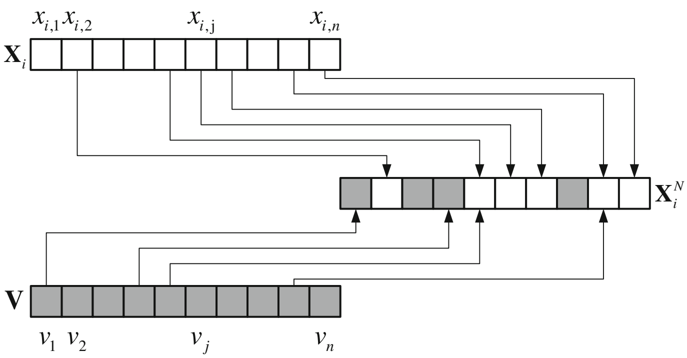


Fig. 10.24 Diagram of the crossover reproduction process

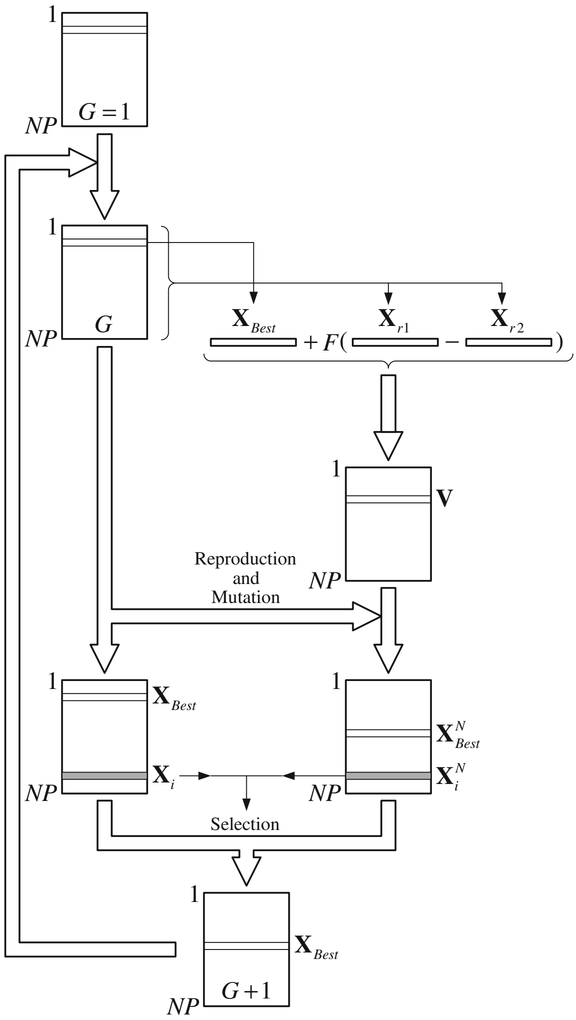


Fig. 10.25 Process scheme of the implementation of the described evolutionary algorithm

All the operators described above, take place depending on a probability that is defined as $CP \in [0, 1]$ for the crossover and $MP \in [0, 1]$ for the mutation.

The described method has been implemented in the synthesis module of the WinMecC program to optimize planar mechanisms. This module solves both function generation and trajectory generation synthesis problems in mechanisms of different types and with any number of links (Fig. 10.26).

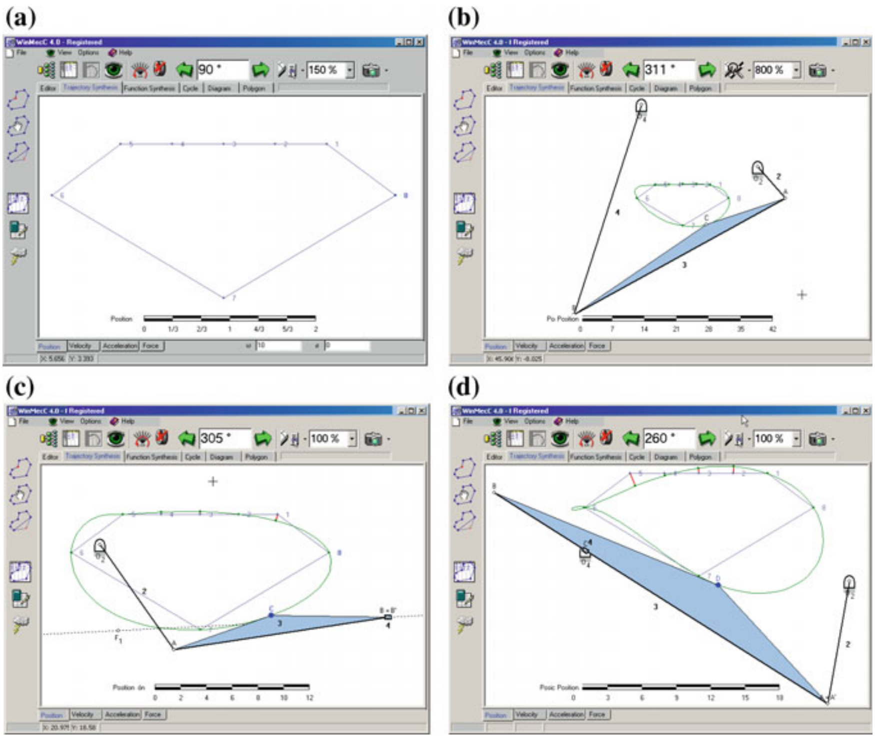


Fig. 10.26 **a** Definition of the desired path in the WinMecC synthesis module. **b** Synthesis of a four-bar mechanism. **c** Synthesis of a crank-shaft mechanism. **d** Synthesis of a slider-crank mechanism

10.7 Results

In this section we show the results of two problems solved with the proposed evolutionary algorithm. In both cases the turning function has been used to define the objective function of the optimization problems.

10.7.1 Closed Path Generation

In this example, we address the solution of a trajectory generation problem that was originally introduced by Kunjur and Krishnamurty (1997). The authors originally propose that couple point *C* of a four-bar mechanism passes through 18 points that describe a closed curve.

The problem is defined as follows:

- Design variables:

$$X = [a, b, c, d, r^r_{C_x}, r^r_{C_y}, \beta, r_{O_{2x}}, r_{O_{2y}}, \varphi_1]$$

- Points of the objective trajectory:

$$\left\{\mathbf{r}^d_{C,i}\right\} = \left\{\begin{array}{l}(0.5, 1.1), (0.4, 1.1), (0.3, 1.1), (0.2, 1.0), (0.1, 0.9), (0.05, 0.75),\\ (0.02, 0.6), (0, 0.5), (0, 0.4), (0.03, 0.3), (0.1, 0.25), (0.15, 0.2),\\ (0.2, 0.3), (0.3, 0.4), (0.4, 0.5), (0.5, 0.7), (0.6, 0.9), (0.6, 1.0)\end{array}\right\}$$

$$\{\varphi_i\} = \{\varphi_1, \varphi_1 + i \cdot 20^\circ \cdot \pi/180^\circ\} \quad i = 1, 2, \ldots, 17$$

- Limits of the variables:

$$\begin{aligned} a, b, c, d &\in [0, 50] \\ r^r_{C_x}, r^r_{C_y} &\in [-50, 50] \\ \beta, \varphi_1 &\in [0, 2\pi] \\ r_{O_{2x}}, r_{O_{2y}} &\in [-10, 10] \end{aligned}$$

The results shown in Table 10.2 together with Kunjur's results, belong to Cabrera et al. (2011) and Nadal et al. (2015). The three of them use genetic algorithms to solve the problem. The first two authors use the mean square error to synthesize the best mechanism while the third one uses turning functions.

Table 10.2 Solutions to the trajectory synthesis problem proposed by Kunjur with different methods

	Kunjur	Cabrera et al. (2011)	Nadal et al. (2015)
a	0.274853	0.297057	0.239834
b	2.138209	0.849372	3.164512
c	1.180253	3.913095	0.941660
d	1.879660	4.453772	2.462696
$r^r_{C_x}$	−0.833592	−2.067338	0.033708
$r^r_{C_y}$	−0.378770	1.6610626	0.483399
β (rad)	4.354224	2.7387359	4.788536
$r_{O_{2x}}$	1.132062	−1.309243	0.569026
$r_{O_{2y}}$	0.663433	2.806964	0.350557
φ_1 (rad)	2.558625	4.853543	2.472660
Mean square error	0.043	0.0196	0.113595
Turning function error	0.250734	0.261502	0.184365

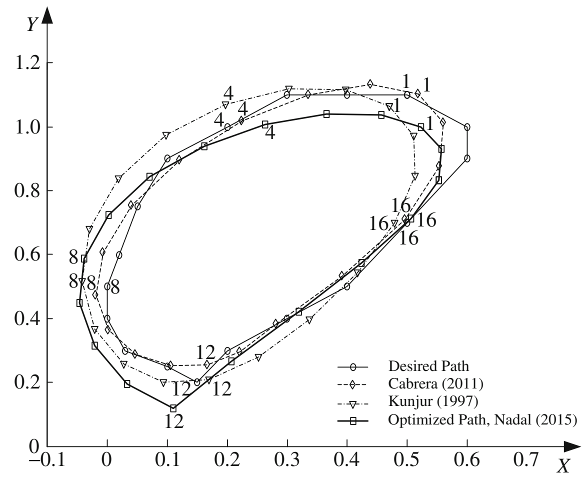


Fig. 10.27 Desired path and paths traced by the coupler of the synthesized mechanisms found with different methods

In Fig. 10.27 we can see the curves described by the three different mechanisms compared to the desired curve.

10.7.2 Open Path Generation

This example shows the results of an open path generation synthesis. The aim of the problem is that the coupler point of the synthesized mechanism describes a right angle path during the motion. The original problem was presented by Sedano et al. (2012). The original problem was defined as follows:

- Design variables:

$$\chi = \left[r_1, r_2, r_3, r_4, r_{cx}, r_{cy}, \theta_0, x_0, y_0, \theta_2^1\right]$$

- Target points:

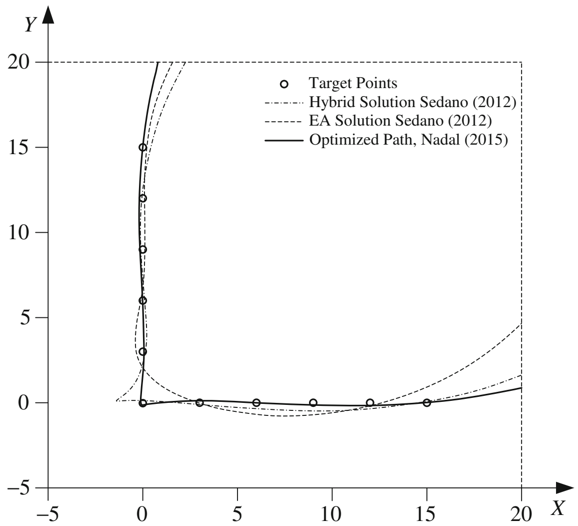
$$\left\{C_d^i\right\} = \left\{ \begin{matrix} (0,15),(0,12),(0,9),(0,6),(0,3),(0,0), \\ (3,0),(6,0),(9,0),(12,0),(15,0) \end{matrix} \right\}$$

As in the previous case, design variables in the original problem included linear and angular positions of the fixed link (Table 10.2 column 1).

Table 10.3 Comparative results for an open path generation example

	Sedano EA	Sedano hybrid	Nadal et al. (2015)
r_1	24.3	28.16	33.06872
r_2	14.81	16.94	21.37403
r_3	24.48	23.05	29.03806
r_4	31.7	32.68	26.49428
r_{cx}	3.126	−0.422	2.51363
r_{cy}	−2.528	−7.1175	−9.72539
θ_0 (rad)	3.4516	3.4216	3.81408
x_0	12.672	19.0533	25.69221
y_0	15.753	12.6922	18.10155
θ_2^1 (rad)	1.68	5.50	5.346
Mean square error	2.439	0.2025	0.1099
Turning function error	0.25202	0.263317	0.02746

Fig. 10.28 The best path traced by the coupler with three different bibliography examples



The best mechanism found by the evolutionary algorithm in the last iteration is listed in Table 10.3 along with the best results found by Sedano et al. (2015). In this case, we can observe that the results obtained by the evolutionary algorithm with turning functions, improve those achieved by Sedano et al. The transformed mechanism (Table 10.3 column 4) obtains the best right angle, as it is shown in Fig. 10.28.

References

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Kunjur A, Krishnamurty S (1997) Genetic algorithms in mechanical synthesis. J Appl Mech Robot 4(2):18–24

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