Trapezoidal trajectories

Among many other combinations, a possible approach for planning a trajectory is to use linear segments joined with parabolic blends.

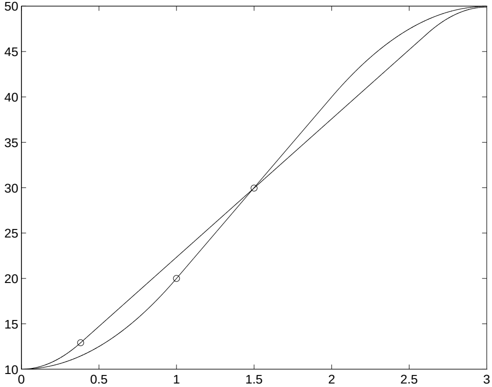
In the linear tract, the velocity is constant while, in the parabolic blends, it is a linear function of time: trapezoidal profiles, typical of this type of trajectory, are then obtained.

In trapezoidal trajectories, the duration is divided into three parts:

1. in the first part, a constant acceleration is applied, then the velocity is linear and the position a parabolic function of time
2. in the second, the acceleration is null, the velocity is constant and the position is linear in time
3. in the last part a (negative) acceleration is applied, then the velocity is a negative ramp and the position a parabolic function.

Trapezoidal trajectories

Usually, the acceleration and the deceleration phases have the same duration ($t_a = t_d$). Therefore, symmetric profiles, with respect to a central instant $(t_f - t_i)/2$, are obtained.



Trapezoidal trajectories

The trajectory is computed according to the following equations.

1) **Acceleration phase, $t \in [0 \div t_a]$.**

The position, velocity and acceleration are described by

$$
\begin{aligned}
q(t) &= a_0 + a_1 t + a_2 t^2 \\
\dot{q}(t) &= a_1 + 2a_2 t \\
\ddot{q}(t) &= 2a_2
\end{aligned}
$$

The parameters are defined by constraints on the initial position q_i and the velocity $\dot{q}_i$, and on the desired constant velocity $\dot{q}_v$ that must be obtained at the end of the acceleration period. Assuming a null initial velocity and considering $t_i = 0$ one obtains

$$
\begin{aligned}
a_0 &= q_i \\
a_1 &= 0 \\
a_2 &= \frac{\dot{q}_v}{2t_a}
\end{aligned}
$$

In this phase, the acceleration is constant and equal to $\dot{q}_v/t_a$.

Trapezoidal trajectories

2) **Constant velocity phase, $t \in [t_a \div t_f - t_a]$.**
Position, velocity and acceleration are now defined as

$$\begin{aligned} q(t) &= b_0 + b_1 t \\ \dot{q}(t) &= b_1 \\ \ddot{q}(t) &= 0 \end{aligned}$$

where, because of continuity,

$$b_1 = \dot{q}_v$$

Moreover, the following equation must hold

$$q(t_a) = q_i + \dot{q}_v \frac{t_a}{2} = b_0 + \dot{q}_v t_a$$

and then

$$b_0 = q_i - \dot{q}_v \frac{t_a}{2}$$

Trapezoidal trajectories

3) **Deceleration phase, $t \in [t_f - t_a \div t_f]$.**

The position, velocity and acceleration are given by

$$\begin{aligned} q(t) &= c_0 + c_1 t + c_2 t^2 \\ \dot{q}(t) &= c_1 + 2 c_2 t \\ \ddot{q}(t) &= 2 c_2 \end{aligned}$$

The parameters are now defined with constrains on the final position q_f and velocity $\dot{q}_f$, and on the velocity $\dot{q}_v$ at the beginning of the deceleration period.

If the final velocity is null, then:

$$\begin{aligned} c_0 &= q_f - \frac{\dot{q}_v}{2} \frac{t_f^2}{t_a} \\ c_1 &= \dot{q}_v \frac{t_f}{t_a} \\ c_2 &= -\frac{\dot{q}_v}{2 t_a} \end{aligned}$$

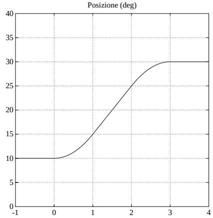
Trapezoidal trajectories

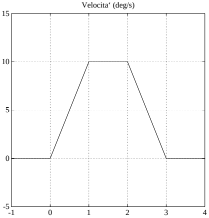
Summarizing, the trajectory is computed as

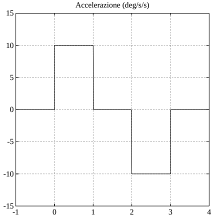
$$q(t) = \begin{cases} q_i + \dfrac{\dot{q}_v}{2t_a} t^2 & 0 \le t < t_a \\ q_i + \dot{q}_v (t - \frac{t_a}{2}) & t_a \le t < t_f - t_a \\ q_f - \dfrac{\dot{q}_v}{t_a} \dfrac{(t_f - t)^2}{2} & t_f - t_a \le t \le t_f \end{cases}$$

Trapezoidal trajectories

Typical position, velocity and acceleration profiles of a trapezoidal trajectory.







Trapezoidal trajectories

Some additional constraints must be specified in order to solve the previous equations.
A typical constraint concerns the duration of the acceleration/deceleration periods t_a that, for symmetry, must satisfy the condition

$$t_a \leq t_f/2$$

Moreover, the following condition must be verified (for the sake of simplicity, consider $t_i = 0$):

$$\begin{aligned} \ddot{q} t_a &= \frac{q_m - q_a}{t_m - t_a} \\ q_a &= q_i + \frac{1}{2}\ddot{q} t_a^2 \end{aligned} \qquad \left\{ \begin{array}{lcl} q_a &=& q(t_a) \\ q_m &=& (q_i + q_f)/2 \\ t_m &=& t_f/2 \end{array} \right.$$

from which

$$\ddot{q} t_a^2 - \ddot{q} t_f t_a + (q_f - q_i) = 0 \qquad (7)$$

Finally:

$$\dot{q}_v = \frac{q_f - q_i}{t_f - t_a}$$

Trapezoidal trajectories

Any pair of values $(\ddot{q}, t_a)$ verifying (7) can be considered.

Given the acceleration $\ddot{q}$ (for example $\ddot{q}_{max}$), then

$$t_a = \frac{t_f}{2} - \frac{\sqrt{\ddot{q}^2 t_f^2 - 4\ddot{q}(q_f - q_i)}}{2\ddot{q}}$$

from which we have also that the minimum value for the acceleration is

$$|\ddot{q}| \geq \frac{4|q_f - q_i|}{t_f^2}$$

if the value $|\ddot{q}| = \frac{4|q_f - q_i|}{t_f^2}$ is assigned, then $t_a = t_f/2$ and the constant velocity tract does not exist.

If the value $t_a = t_f/3$ is specified, the following velocity and acceleration values are obtained

$$\dot{q}_v = \frac{3(q_f - q_i)}{2t_f} \qquad\qquad \ddot{q} = \frac{9(q_f - q_i)}{2t_f^2}$$

Trapezoidal trajectories

Another way to compute this type of trajectory is to define a maximum value $\ddot{q}_a$ for the desired acceleration and then compute the relative duration t_a of the acceleration and deceleration periods.

If the maximum values ($\ddot{q}_{max}$ and $\dot{q}_{max}$, known) for the acceleration and velocity must be reached, it is possible to assign

$$\left\{ \begin{array}{rcll} t_a & = & \dfrac{\dot{q}_{max}}{\ddot{q}_{max}} & \text{acceleration time} \\ \dot{q}_{max}(T - t_a) & = & q_f - q_i = L & \text{displacement} \\ T & = & \dfrac{L\ddot{q}_{max} + \dot{q}_{max}^2}{\ddot{q}_{max}\dot{q}_{max}} & \text{time duration} \end{array} \right.$$

and then ($t_f = t_i + T$)

$$q(t) = \left\{ \begin{array}{ll} q_i + \frac{1}{2}\ddot{q}_{max}(t - t_i)^2 & t_i \leq t \leq t_i + t_a \\ q_i + \ddot{q}_{max} t_a (t - t_i - \frac{t_a}{2}) & t_i + t_a < t \leq t_f - t_a \\ q_f - \frac{1}{2}\ddot{q}_{max}(t_f - t - t_i)^2 & t_f - t_a < t \leq t_f \end{array} \right. \qquad (8)$$

Trapezoidal trajectories

In this case, the linear tract exists if and only if

$$L \geq \frac{\dot{q}_{max}^2}{\ddot{q}_{max}}$$

Otherwise

$$\begin{cases} t_a &= \sqrt{\dfrac{L}{\ddot{q}_{max}}} & \text{acceleration time} \\ T &= 2t_a & \text{total time duration} \end{cases}$$

and (still $t_f = t_i + T$)

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_{max}(t - t_i)^2 & t_i \leq t \leq t_i + t_a \\ q_f - \frac{1}{2}\ddot{q}_{max}(t_f - t)^2 & t_f - t_a < t \leq t_f \end{cases} \tag{9}$$

Trapezoidal trajectories

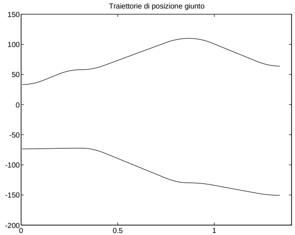
With this modality for computing the trajectory, the time duration of the motion from q_i to q_f is not specified. In fact, the period T is computed on the basis of the maximum acceleration and velocity values.

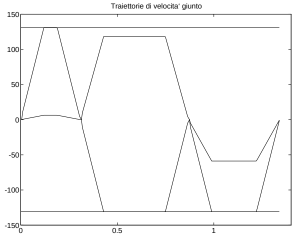
If more joints have to be co-ordinated with the same constraints on the maximum acceleration and velocity, the joint with the largest displacement must be individuated. For this joint, the maximum value $\ddot{q}_{max}$ for the acceleration is assigned and then the corresponding values t_a and T are computed.

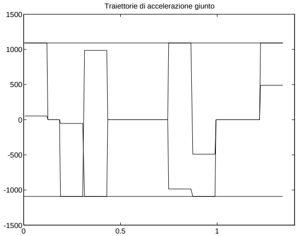
For the remaining joints, the acceleration and velocity values must be computed on the basis of these values of t_a and T, and on the basis of the given displacement L_i:

$$\ddot{q}_i = \frac{L_i}{t_a(T - t_a)}, \qquad \dot{q}_i = \frac{L_i}{T - t_a}$$

Trapezoidal trajectories

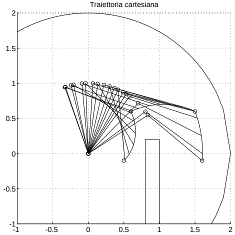


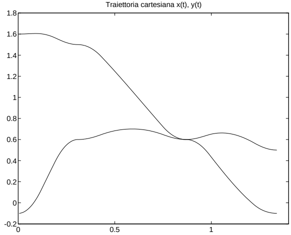


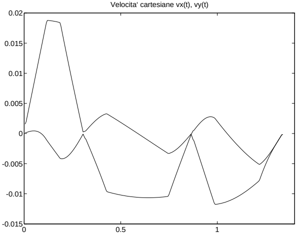


Trapezoidal trajectories

The trajectories in the workspace are:

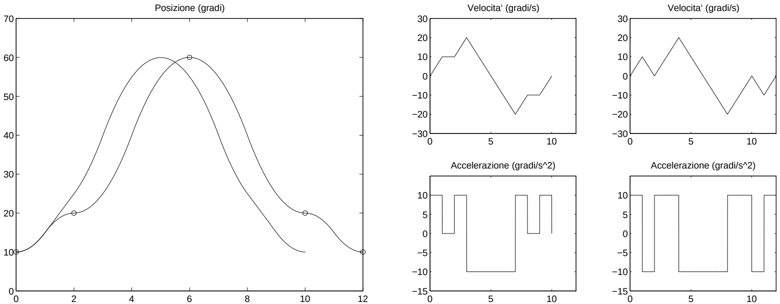






Trapezoidal trajectories

If a trajectory interpolating more consecutive points is computed with the above technique, a motion with null velocities in the via-points is obtained. Since this behavior may be undesirable, it is possible to “anticipate” the actuation of a tract of the trajectory between points q_k and q_{k+q} *before* the motion from q_{k-1} to q_k is terminated. This is possible by adding (starting at an instant $t_k - t'_a$) the velocity and acceleration contributions of the two segments $[q_{k-1} - q_k]$ and $[q_k - q_{k+1}]$.
Obviously, another possibility is to compute the parameters of the functions defining the trapezoidal trajectory in order to have desired boundary conditions (i.e. velocities) for each segment.

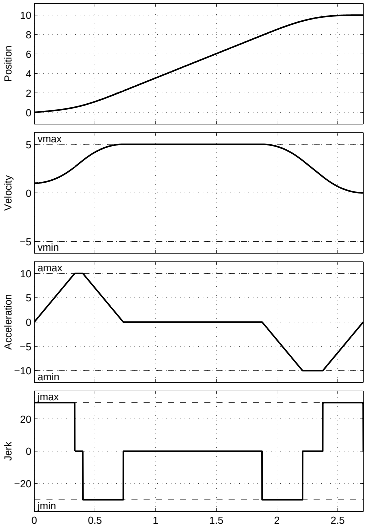


Trapezoidal trajectories

A trapezoidal velocity motion profile presents a discontinuous acceleration. For this reason, this trajectory may generate efforts and stresses on the mechanical system that may result detrimental or generate undesired vibrational effects.

Therefore, a smoother motion profile must be defined, for example by adopting a continuous, linear piece-wise, acceleration profile. In this manner, the resulting velocity is composed by linear segments connected by parabolic blends.

The shape of the velocity profile is the reason of the name *double S* for this trajectory, known also as *bell* trajectory or *seven segments* trajectory, because it is composed by seven different tracts with constant jerk.



Piecewise polynomial trajectories

Other functions can be obtained by properly composing segments defined with polynomial functions of different degree (*piecewise polynomial functions*).
In these cases, it is necessary to define an adequate number of conditions (boundary conditions, point crossing, continuity of velocity, acceleration, ...), as done e.g. for the computation of trapezoidal (linear segments with second or higher degree polynomials blends) and 'double S' trajectories.
For example, in *pick-and-place* operations by an industrial robot it may be of interest to have motions with very smooth initial and final phases. In such a case, one can use a motion profile obtained as the connection of three polynomials $q_l(t)$, $q_t(t)$, $q_s(t)$ (i.e. *lift-off, travel, set-down*) with (for example):

$$
\begin{array}{lcl}
q_l(t) & \Longrightarrow & \text{4-th degree polyn.} \\
q_t(t) & \Longrightarrow & \text{3-rd degree polyn.} \\
q_s(t) & \Longrightarrow & \text{4-th degree polyn.}
\end{array}
$$