Homework 1

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Exercise 6.22 For each of the following regression models, indicate whether it is a general linear regression model. If not, state whether it can be expressed in the form (6.7) by a suitable transformation:

(6.7) $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \varepsilon_i$

a. $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 \log_{10} X_{i2} + \beta_3 X_{i1}^2 + \varepsilon_i$

Yes, this is expressed in the form of (6.7).

b. $Y_i = \varepsilon_i \, e^{(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}^2)}$

From the start, this is not expressed in the form of (6.7).

Suppose
$$\log_e Y_i = \log\left(\varepsilon_i \cdot e^{(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}^2)}\right)$$
$$\log_e Y_i = \log_e\left(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}^2\right) + \log_e(\varepsilon_i)$$

Let $Y_i' = \log_e Y_i$ & $\varepsilon_i' = \log_e(\varepsilon_i)$

$$Y_i' = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}^2 + \varepsilon_i'.$$

c. $Y_i = \log_{10}(\beta_1 X_{i1}) + \beta_2 X_{i2} + \varepsilon_i$

This is already expressed in terms of (6.7). β_0 does not need to be present.

d. $Y_i = \beta_0 e^{(\beta_1 X_{i1})} + \varepsilon_i \quad = \quad \begin{matrix} \log(\beta_0) + \log_e(\beta_1 X_{i1}) + \log_e(\varepsilon_i) \\ \log_e(\beta_0) + \beta_1 X_{i1} + \varepsilon_i' \end{matrix}$

This is expressed in terms of (6.7) and can not be modified by $\log_e$.

e. $Y_i = \left[1 + e^{(\beta_0 + \beta_1 X_{i1} + \varepsilon_i)}\right]^{-1} = \dfrac{1}{\left[1 + e^{(\beta_0 + \beta_1 X_{i1} + \varepsilon_i)}\right]}$

This not expressed in terms of (6.7).

$\log_e(Y_i) = Y_i'$

$$\begin{aligned} Y_i' &= \log_e(1) - \log_e\left(1 + e^{(\cdots)}\right) \\ &= 0 - \left(\log_e(1) + \log_e e^{(\beta_0 + \beta_1 X_{i1} + \varepsilon_i)}\right) \\ &= 0 - 0 - (\beta_0 + \beta_1 X_{i1} + \varepsilon_i) \\ &= -(\beta_0 + \beta_1 X_{i1} + \varepsilon_i) \end{aligned}$$

6.24 Consider the multiple regression model:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2} + \varepsilon_i \qquad i = 1, \ldots, n.$$

where ε_i are indep. $N(0, \sigma^2)$

a. State the least squares criterion and derive the least squares normal equations.

normal equations (1.9): $\sum Y_i = n b_0 + b_1 \sum X_i$, $\sum X_i Y_i = b_0 \sum X_i + b_1 \sum X_i^2$

sum of squared residuals: $\hat{Y} = \beta_0 + \beta_1 X$, $\sum (Y_i - \hat{Y}_i)^2 = \sum (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))^2$

Let $Q = \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})^2$

$$\frac{\partial Q}{\partial \beta_0} = \sum \frac{\partial}{\partial \beta_0} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})^2$$
$$= -2 \sum (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))$$

$\frac{\partial}{\partial \beta_0} (\cdots - \beta_0 \cdots)^2 = (2)(-1) = -2$

$$\frac{\partial Q}{\partial \beta_1} = \sum \frac{\partial}{\partial \beta_0} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))^2$$
$$= -2 \sum X_{i1} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))$$

$\frac{\partial}{\partial \beta_1} \cdot (\cdots + \beta_1 X_{i1} \cdots)^2 = -2 X_{i1} \cdot \beta_1$

$$\frac{\partial Q}{\partial \beta_2} = \sum \frac{\partial}{\partial \beta_2} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))^2$$
$$= -2 \sum X_{i1}^2 (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))$$

$\frac{\partial}{\partial \beta_2} - (\cdots + \beta_2 X_{i2}^2 \cdots)^2 = X_1^2 \cdot (-2) \beta_2$

$$\frac{\partial Q}{\partial \beta_3} = \sum \frac{\partial}{\partial \beta_3} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))^2$$
$$= -2 \sum X_{i2} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2}))$$

Now we set the partial derivates equal to 0, to get our least squares normal equations.

β_0

$$\left[-2 \sum (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2})) = 0\right] \cdot -\tfrac{1}{2}$$
$$\sum Y_i - \sum \beta_0 - \sum \beta_1 X_{i1} - \sum \beta_2 X_{i1}^2 - \sum \beta_3 X_{i2} = 0$$
$$\sum Y_i - n b_0 - b_1 \sum X_{i1} - b_2 \sum X_{i1}^2 - b_3 \sum X_{i2} = 0 \quad \text{(i)}$$

$\sum_{i=1}^{n} \beta_0 = n \cdot \beta_0$

β_1

$$\left[-2 \sum X_{i1} (Y_i - (\beta_0 + \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})) = 0\right] \cdot -\tfrac{1}{2}$$
$$\sum X_{i1} Y_i - b_0 \sum X_{i1} - b_1 \sum X_{i1}^2 - b_2 \sum X_{i1}^3 - b_3 \sum X_{i1} X_{i2} = 0 \quad \text{(ii)}$$

β_2

$$\left[-2 \sum X_{i1}^2 (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2})) = 0\right] \cdot -\tfrac{1}{2}$$
$$\sum X_{i1}^2 Y_i - b_0 \sum X_{i1}^2 - b_1 \sum X_{i1}^3 - b_2 \sum X_{i1}^4 - b_3 \sum X_{i2} X_{i1}^2 = 0 \quad \text{(iii)}$$

β_3

$$\left[-2 \sum X_{i2} (Y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2})) = 0\right] \cdot -\tfrac{1}{2}$$
$$\sum X_{i2} Y_i - b_0 \sum X_{i2} - b_1 \sum X_{i1} X_{i2} - b_2 \sum X_{i2} X_{i1}^2 - b_3 \sum X_{i2}^2 = 0 \quad \text{(iv)}$$

b. State the likelihood function and explain why the maximum likelihood estimators will be the same as the least squares estimators

The method of maximum likelihood uses the density of the probability distribution at Y_i (i.e., the height of the curve at Y_i).

$$L(\beta_0, \beta_1, \beta_2, \beta_3, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})^2}$$

$$\log_e L = -\frac{n}{2} \log_e 2\pi - \frac{n}{2} \log_e \sigma^2 - \frac{1}{2\sigma^2} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})^2$$

$$\frac{\partial (\log_e L)}{\partial \beta_0} = 0 - 0 - \frac{1}{\sigma^2} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) \quad \text{(i)}$$

$$\frac{\partial (\log_e L)}{\partial \beta_1} = -\frac{1}{\sigma^2} \sum X_{i1} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) \quad \text{(ii)}$$

$$\frac{\partial (\log_e L)}{\partial \beta_2} = -\frac{1}{\sigma^2} \sum X_{i1}^2 (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) \quad \text{(iii)}$$

$$\frac{\partial (\log_e L)}{\partial \beta_3} = -\frac{1}{\sigma^2} \sum X_{i2} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) \quad \text{(iv)}$$

$$\frac{\partial (\log_e L)}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) \quad \text{(v)}$$

$$\left[-\frac{1}{\sigma^2} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0\right] \cdot -\sigma^2$$

$$\sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0 \quad \text{(i)}$$
$$\sum X_{i1} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0 \quad \text{(ii)}$$
$$\sum X_{i1}^2 (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0 \quad \text{(iii)}$$
$$\sum X_{i2} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0 \quad \text{(iv)}$$
$$\frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = 0$$

$$\left[\frac{1}{2\sigma^4} \sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2}) = \frac{n}{2\sigma^2}\right] \frac{2\sigma^4}{n} \quad \text{(v)}$$

$$\frac{\sum (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i1}^2 - \beta_3 X_{i2})}{n} = \hat{\sigma}^2$$