Exercise 2.5.1: solution

Exercise. Let us denote

$$\mathrm{prox}_g(y) = \arg\min_{x\in\mathcal{X}} g(x) + \frac{1}{2}\|x-y\|^2$$

the proximal operator of g at y.
Fix $\gamma > 0$. Show that if f and g are convex, then the fixed points of the nonlinear equation

$$x = \mathrm{prox}_{\gamma g}(x - \gamma\nabla f(x))$$

are the minimizers of the function $F = f + g$.
► *Let us denote $p = \mathrm{prox}_g(y)$. By definition, p minimizes $h(x) = g(x) + \frac{1}{2}\|x-y\|^2$. By Fermat's rule, we get the inclusion $0 \in \partial g(p) + (p - y)$. Note that $x \mapsto \frac{1}{2}\|x-y\|^2$ is differentiable so that the subdifferential of the sum is the sum of the subdifferentials (Prop 2.4.1).*
We now apply this result for $p = \mathrm{prox}_{\gamma g}(x - \gamma\nabla f(x))$:

$$0 \in \gamma\partial g(p) + (p - x + \gamma\nabla f(x))$$

Suppose now that x is a fixed point of the nonlinear equation. This means that $p = x$ above and thus, $0 \in \gamma\partial g(x) + \gamma\nabla f(x)$. Equivalently, we can use Fermat's rule again and since $\gamma > 0$, we have $x \in \arg\min_y f(y) + g(y)$.