"JUST THE MATHS"

UNIT NUMBER

19.3

PROBABILITY 3
(Random variables)

by

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UNIT 19.3 - PROBABILITY 3 - RANDOM VARIABLES

19.3.1 DEFINING RANDOM VARIABLES

(i) The kind of experiments discussed in the theory of probability are usually what are known as **"random experiments"**.

For example, in an experiment which involves throwing a die, suppose that the die is "unbiased". This means that it is just as likely to show one face as any other.

Similarly, drawing 6 numbers out of a possible 45 for a lottery is a random experiment, provided it is just as likely for one number to be drawn as any other.

In general, an experiment is a random experiment if there is more than one possible outcome (or event) and any one of those possible outcomes may occur. We assume that the outcomes are mutually exclusive (see Unit 19.1, section 19.1.4).

The probabilities of the possible outcomes of a random experiment form a collection called the **"probability distribution"** of the experiment. These probabilities need not be the same as one another.

The complete list of possible outcomes is called the **"sample space"** of the experiment.

(ii) In a random experiment, each of the possible outcomes may, for convenience, be associated with a certain numerical value called a **"random variable"**. This variable, which we shall call x in general, makes it possible to refer to an outcome without having to use a complete description of it.

In tossing a coin, for instance, we might associate a head with the number 1 and a tail with the number 0; then we could state the probabilities of either a head or a tail being obtained by means of the formulae

$$P(x = 1) = 0.5 \text{ and } P(x = 1) = 0.5$$

Note:
There is no restriction on the way we define the values of x; it would have been just as correct to associate a head with -1 and a tail with 1. But it is customary to assign the values of random variables as logically as possible. For example, in discussing the probability that two 6's would be obtained in 5 throws of a dice, we could sensibly use $x = 1, 2, 3, 4, 5$ and 6, respectively, for the results that a 1,2,3,4,5 and 6 would be thrown.

(iii) It is usually necessary to distinguish between random variables which are **"discrete"** and those which are **"continuous"**.

Discrete random variables may take only certain specified values, while continuous random variables may take any value with in a certain specified range.

Examples of discrete random variables include those associated with the tossing of coins, the throwing of dice and numbers of defective components in a batch from a production line.

Examples of continuous variables include those associated with persons' height or weight, lifetimes of manufactured components and rainfall figures.

Note:
For a random variable, x, the associated probabilities form a function, $P(x)$, called the **"probability function"**.

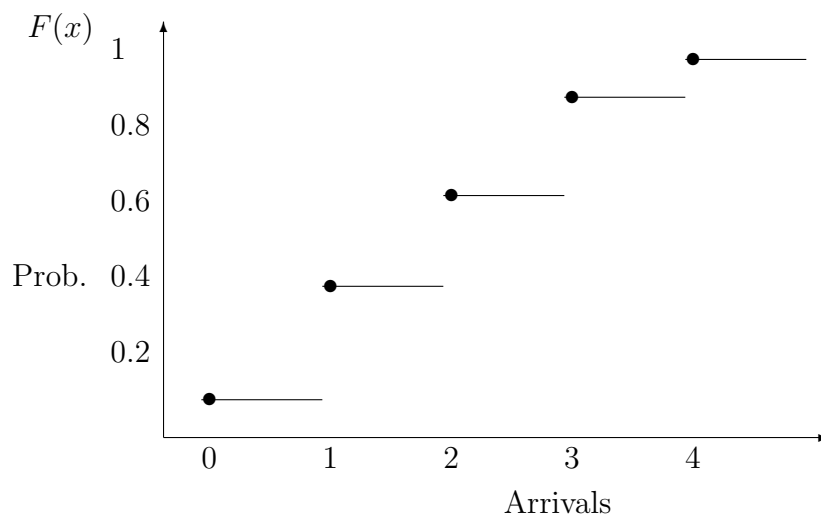## 19.3.2 PROBABILITY DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS

(a) Probability Distribution Functions

A **"probability distribution function"**, which will normally be denoted here by $F(x)$, is the relationship between a random variable, x, and the probability of obtaining any outcome for which the random variable can take values up to and including x. In other words, it is the probability, $P(\leq x)$, that the random variable for the outcome is less than or equal to x.

(i) Probability distribution functions for discrete variables

By way of illustration, suppose that the number of ships arriving at a container terminal during any one day can be 0,1,2,3 or 4, with respective probabilites 0.1, 0.3, 0.35, 0.2 and 0.05. The probabilites for outcomes other than those specified is taken to be zero.

The graph of the probability distribution function is as follows:

The value of the probability distribution function at a value, x, of the random variable is the sum of the probabilities to the left of, and including, x.

In view of the discontinuous nature of the graph, we have used "bullet" marks to indicate which end of each horizontal line belongs to the graph.
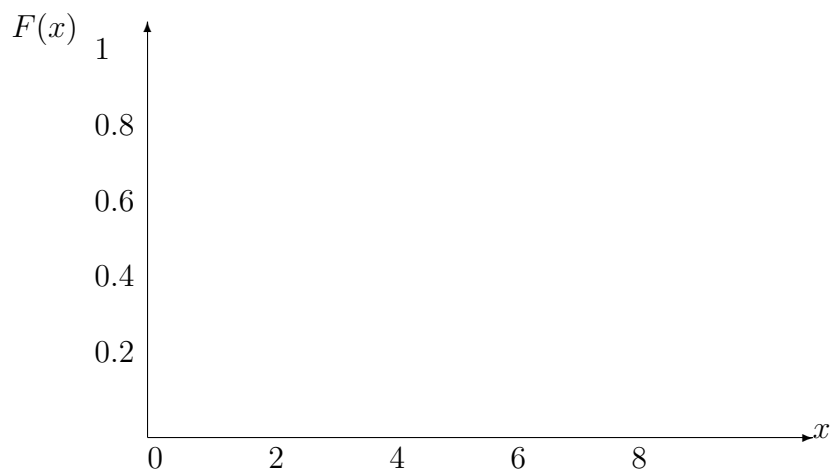
(ii) Probability distribution functions for continuous variables

For a continuous random variable, the probability distribution function is defined in a similar way as for a discrete variable. It measures (as before) the probability that the value of the random variable is less than or equal to x.

By way of illustration, we shall quote, here, the example of an **"exponential distribution"** in which it may be shown that the lifetime of a certain electronic component (in thousands of hours) is represented by a probability distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

The graph of the probability distribution function is as follows:

(b) Probability Density Functions

In the case of continuous random variables, a second function, $f(x)$ called the **"probability density function"** is defined to be the derivative, with respect to x, of the probability distribution function, $F(x)$.
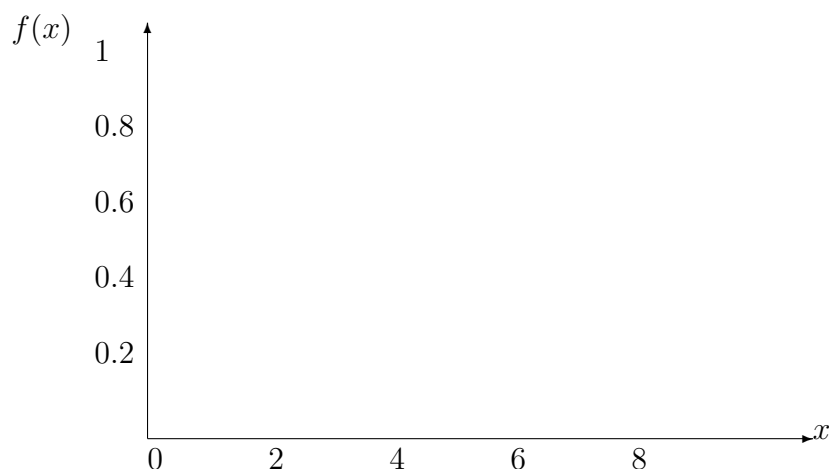
That is,

$$f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)].$$

The probability density function measures the **concentration** of possible values of x.

In the previous example, the probability density function is therefore given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0 \end{cases}$$

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The graph of the probability density function is as follows:

We may observe that most components have short lifetimes, while a small number can survive for much longer.

(c) Properties of probability distribution and probability density functions

The following properties are a consequence of the appropriate definitions:

(i)
$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

Proof:

It is impossible for a random variable to have a value less than $-\infty$ and it is certain to have a value less than ∞.

(ii) If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

Proof:

The outcomes of an experiment with random variable values up to and including x_2 includes those outcomes with random variable values up to and including x_1 so that $F(x_2)$ is as least as great as $F(x_1)$.

Note:
Results **(i)** and **(ii)** imply that, for any value of x, the probability distribution function is either constant or increasing between 0 and 1.

(iii) The probability that an outcome will have a random variable value, x, within the range $x_1 < x \leq x_2$ is given by the expression

$$F(x_2) - F(x_1).$$

Proof:

From, the outcomes of an experiment with random variable values up to and including x_2, suppose we exclude those outcomes with random variable values up to and including x_1. The residue will be those outcomes with random variable values which lie within the range $x_1 < x \leq x_2$.

Thus, the difference between the values of the probability distribution function at two particular points is the probability that the value of the random variable will either lie between those two points or will be equal to the higher of the two.

Note:
For a continuous random variable, this is also equal to the area under the graph of the probability density function between the two given points, by virtue of the definition that $f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)]$.

That is,

$$\int_{x_1}^{x_2} f(x) \, \mathrm{d}x.$$

(iv)
$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 1.$$

Proof:

The total area under the probability density function must be 1 since the random variable must have a value somewhere.

EXAMPLE

For the distribution of component lifetimes (in thousands of hours) given earlier by

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0, \end{cases}$$

determine the proportion of components which last longer than 3000 hours but less than or equal to 6000 hours.

Solution

The probability that components have lifetimes up to and including 3000 hours is given by

$$F(3) = 1 - e^{-\frac{3}{2}}.$$

The probability that components have lifetimes up to and including 600 hours is given by

$$F(6) = 1 - e^{-\frac{6}{2}} = 1 - e^{-3}.$$

The probability that components last longer than 3000 hours but less than or equal to 6000 hours is thus given by

$$F(6) - F(3) = e^{-\frac{3}{2}} - e^{-3} \simeq 0.173$$

The required proportion is thus approximately one in six.

19.3.3 EXERCISES

1. A coin is tossed three times and the random variable, x, represents the number of heads minus the number of tails.

   Construct a definition for the probability distribution function, $F(x)$,

   (a) if the coin is 'fair' (perfectly balanced);
   (b) if the coin is biased so that a head is twice as likely to occur as a tail.

2. Construct a definition for the probability distribution function, $F(x)$, for the sum, x, of numbers obtained when a pair of dice is tossed.

3. A certain assembly process is such that the probability of success at each attempt is 0.2. The probability, $P(x)$ that x independent attempts are needed to achieve success is given by

$$P(x) \equiv (0.2)(0.8)^{x-1} \quad x = 1,\ 2,\ 3,\dots$$

Plot a grap of the probability distribution function, $F(x)$, and determine the probability that success will be achieved by

(a) less than four independent attempts;

(b) more than three but less than or equal to five independent attempts.

4. The probability density function of a random variable, x, is given by

$$f(x) \equiv \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4; \\ 0 & \text{elswhere}, \end{cases}$$

where c is a constant.

Determine

(a) the value of c;

(b) the probability distribution function, $F(x)$;

(c) the probability that $x > 1$.

5. The shelf life (in hours) of a certain perishable packaged food is a random variable, x, with probability density function, $f(x)$ given by

$$f(x) \equiv \begin{cases} 20000(x + 100)^{-3} & \text{when } x > 0; \\ 0 & \text{otherwise}. \end{cases}$$

Determine the probabilities that one of these packages will have a shelf life of

(a) at least 200 hours;

(b) at most 200 hours;

(c) more than 80 hours but less than or equal to 120 hours.

19.3.4 ANSWERS TO EXERCISES

1. (a)

$$F(x) \equiv \begin{cases} \frac{1}{8} & \text{for } -3 \le x < -1; \\ \frac{1}{2} & \text{for } -1 \le x < 1; \\ \frac{7}{8} & \text{for } 1 \le x < 3; \\ 1 & \text{for } x \ge 3. \end{cases}$$

(b)
$$F(x) \equiv \begin{cases} \frac{1}{27} & \text{for } -3 \le x < -1; \\[1mm] \frac{7}{27} & \text{for } -1 \le x < 1; \\[1mm] \frac{19}{27} & \text{for } 1 \le x < 3; \\[1mm] 1 & \text{for } x \ge 3. \end{cases}$$

2.
$$F(x) \equiv \begin{cases} \frac{1}{36} & \text{for } 2 \le x < 3; \\ \frac{1}{12} & \text{for } 3 \le x < 4; \\[1mm] \frac{1}{6} & \text{for } 4 \le x < 5: \\[1mm] \frac{5}{18} & \text{for } 5 \le x < 6; \\[1mm] \frac{5}{12} & \text{for } 6 \le x < 7; \\[1mm] \frac{7}{12} & \text{for } 7 \le x < 8. \\[1mm] \frac{13}{18} & \text{for } 8 \le x < 9; \\[1mm] \frac{5}{6} & \text{for } 9 \le x < 10; \\[1mm] \frac{11}{12} & \text{for } 10 \le x < 11; \\[1mm] \frac{35}{36} & \text{for } 11 \le x < 12; \\[1mm] 1 & \text{for } x \ge 12. \end{cases}$$

3. (a) 0.488 (b) 0.3123

4. (a) $c = \frac{1}{4}$;
   (b)
$$F(x) = \begin{cases} 0 & \text{when } x < 0; \\[1mm] \frac{1}{2}\sqrt{x} & \text{when } 0 \le x \le 4; \\[1mm] 1 & \text{when } x > 4. \end{cases}$$

   (c) $\frac{1}{2}$

5. (a) $\frac{1}{9}$; (b) $\frac{3}{4}$; (c) 0.102