"JUST THE MATHS"

UNIT NUMBER

9.3

MATRICES 3
(Matrix inversion & simultaneous equations)

by

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UNIT 9.3 - MATRICES 3

MATRIX INVERSION AND SIMULTANEOUS LINEAR EQUATIONS

9.3.1 INTRODUCTION

In Matrix Algebra, there is no such thing as **division** in the usual sense of this word since we would effectively be dividing by a table of numbers with the headings of the table removed.

However, we may discuss an equivalent operation called **inversion** which is roughly similar to the process in elementary arithmetic where division by a value, a, is the same as multiplication by $\frac{1}{a}$.

For example, consider the solution of a single linear equation in one variable, x, namely

$$mx = k,$$

for which the solution is obviously

$$x = \frac{k}{m}.$$

An apparently over-detailed solution may be set out as follows:

(a) Pre-multiply both sides of the given equation by m^{-1}, giving

$$m^{-1}.(mx) = m^{-1}k.$$

(b) Use the Associative Law of Multiplication in elementary arithmetic to rearrange this as

$$(m^{-1}.m)x = m^{-1}k.$$

(c) Use the property that $m^{-1}.m = 1$ to conclude that

$$1.x = m^{-1}k.$$

(d) Use the fact that the number 1 is the multiplicative identity in elementary arithmetic to conclude that

$$x = m^{-1}k.$$

We shall see later how an almost identical sequence of steps, with matrices instead of numbers, can be used to solve a set of simultaneous linear equations with what is called the **"multiplicative inverse"** of a matrix.

Matrix inversion is a concept which is developed from the rules for matrix multiplication.

9.3.2 MATRIX REPRESENTATION OF SIMULTANEOUS LINEAR EQUATIONS

In this section, we consider the matrix equivalent of three simultaneous linear equations in three unknowns; this case is neither too trivial, nor too difficult to follow. Let the equations have the form:

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1, \\
a_2x + b_2y + c_2z &= k_2, \\
a_3x + b_3y + c_3z &= k_3.
\end{aligned}
$$

Then, from the properties of matrix equality and matrix multiplication, these can be written as one matrix equation, namely

$$
\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix},
$$

which will be written for short in the form

$$
\text{MX} = \text{K}.
$$

Note:
If we could find a matrix, say N, such that NM = I, we could pre-multiply the above statement by it to give
$$
\text{N(MX)} = \text{NK}.
$$
That is,
$$
\text{(NM)X} = \text{NK}.
$$
In other words,
$$
\text{IX} = \text{NK}.
$$
Hence,
$$
\text{X} = \text{NK}.
$$

Clearly N exhibits a similar behaviour to the number m^{-1} encountered earlier and, for this reason, a better notation for N would be M^{-1}. This is what we shall use in the work which follows.

9.3.3 THE DEFINITION OF A MULTIPLICATIVE INVERSE

The **"multiplicative inverse"** of a square matrix, M, is another matrix, denoted by M^{-1}, which has the property

$$M^{-1}.M = I.$$

Notes:
(i) It is certainly **possible** for the product of two matrices to be an identity matrix, as illustrated, for example, in the exercises on matrix products in Unit 9.2.

(ii) It is usually acceptable to call M^{-1} the "inverse" of M rather than the "multiplicative inverse", unless we wish to distinguish this kind of inverse from the additive inverse defined in Unit 9.1.

(iii) It can be shown that, when $M^{-1}.M = I$, it is also true that

$$M.M^{-1} = I,$$

even though matrices do not normally commute in multiplication.

(iv) It is easily shown that a square matrix cannot have more than one inverse; for, suppose a matrix A had two inverses, B and C.

Then,
$$C = CI = C(AB) = (CA)B = IB = B,$$

which means that C and B are the same matrix.

9.3.4 THE FORMULA FOR A MULTIPLICATIVE INVERSE

So far, we have established the fact that a set of simultaneous linear equations is expressible in the form

$$MX = K$$

and their solution is expressible in the form

$$X = M^{-1}K.$$

We therefore need to establish a method for determining the inverse, M^{-1}, in order to solve the equations for the values of the variables involved.

Development of this method will be dependent on the result known as "Cramer's Rule" for solving simultaneous equations by determinants. (see Units 7.2 and 7.3)

(a) The inverse of a 2 x 2 matrix

Taking

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

and

$$M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

we require that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned}
a_1 P + b_1 R &= 1, \\
a_2 P + b_2 R &= 0, \\
a_1 Q + b_1 S &= 0, \\
a_2 Q + b_2 S &= 1.
\end{aligned}$$

It is easily verified that these equations are satisfied by the solution

$$P = \frac{b_2}{|M|} \quad Q = -\frac{b_1}{|M|} \quad R = -\frac{a_2}{|M|} \quad S = \frac{a_1}{|M|},$$

where

$$|M| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and is called **"the determinant of the matrix M"**.

Summary
The formula for the inverse of a 2 x 2 matrix is given by

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix},$$

in which we interchange the diagonal elements of M and reverse the sign of the other two elements.

EXAMPLES

1. Write down the inverse of the matrix
$$M = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}.$$

Solution

Since $|M| = 41$, we have
$$M^{-1} = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.$$

Check
$$M^{-1}.M = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix} . \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y &= 1, \\ x - 2y &= 5. \end{aligned}$$

Solution

The equations can be written in the form $MX = K$, where
$$M = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Initially, we **must** check that M is non-singluar by evaluating its determinant:
$$|M| = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -6 - 1 = -7.$$

The inverse matrix is now given by
$$M^{-1} = -\frac{1}{7} \begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Thus, the solution of the simultaneous equations is given by
$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

That is,
$$x = 1 \text{ and } y = -2.$$

Note:

Example 2 should be regarded as a model solution to this type of problem but the student should not expect every exercise to yield whole number answers.

(b) The inverse of a 3 x 3 Matrix

The most convenient version of Cramer's rule to use here may be stated as follows:

The simultaneous linear equations

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= k_1, \\
a_2 x + b_2 y + c_2 z &= k_2, \\
a_3 x + b_3 y + c_3 z &= k_3
\end{aligned}
$$

have the solution

$$
\frac{x}{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.
$$

METHOD

(i) We observe, first, that the last determinant in the Cramer's Rule formula contains the same arrangement of numbers as the matrix M; as in (a), we call it **"the determinant of the matrix** M" and we denote it by | M |.

In this determinant, we let the symbols A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 denote the **"cofactors"** (or **"signed minors"**) of the elements a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3 respectively.

(ii) We now observe that, for each of the elements k_1, k_2 and k_3 in the first three determinants of Cramer's Rule, the cofactor is the same as the cofactor of one of the elements in the final determinant | M |.

(iii) Expanding each of the first three determinants in Cramer's Rule along the column of k-values, the solutions for x, y and z can be written as follows:

$$
x = \frac{1}{|\,\mathrm{M}\,|} \left(k_1 A_1 + k_2 A_2 + k_3 A_3 \right);
$$

$$y = \frac{1}{\mid M \mid} \left(k_1 B_1 + k_2 B_2 + k_3 B_3 \right);$$

$$z = \frac{1}{\mid M \mid} \left(k_1 C_1 + k_2 C_2 + k_3 C_3 \right);$$

or, in matrix format,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\mid M \mid} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

Comparing this statement with the statement

$$X = M^{-1} K,$$

we conclude that

$$M^{-1} = \frac{1}{\mid M \mid} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

Summary
Since it is known that Cramer's Rule is applicable to any number of equations in the same number of unknowns, similar working would occur for larger or smaller systems of equations. In general, the inverse of a square matrix is **the transpose of the matrix of cofactors times the reciprocal of the determinant of the matrix**.

Notes:
(i) If it should happen that $\mid M \mid = 0$, then the matrix M does not have an inverse and is said to be **"singular"**. In all other cases, it is said to be **"non-singular"**.

(ii) The transpose of the matrix of cofactors is called the **"adjoint"** of M, denoted by AdjM. There is always an adjoint though not always an inverse; but, when the inverse does exist, we can say that

$$M^{-1} = \frac{1}{\mid M \mid} \text{AdjM}.$$

(iii) The inverse of a matrix of order 2×2 fits the above scheme also. The cofactor of each element will be a single number associated with a **"place-sign"** according to the following pattern:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}.$$

Hence, if

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

then,

$$M^{-1} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

As before, we see that the matrix part of the result can be obtained by interchanging the diagonal elements of M and reversing the signs of the other two elements.

EXAMPLE

Use matrices to solve the simultaneous linear equations

$$\begin{aligned}
3x + y - z &= 1, \\
x - 2y + z &= 0, \\
2x + 2y + z &= 13.
\end{aligned}$$

Solution

The equations can be written in the form MX = K, where

$$M = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix}.$$

Initially, we **must** check that M is non-singluar by evaluating its determinant:

$$\mid M \mid = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 3(-2-2) - 1(1-2) + (-1)(2+4) = -17.$$

If C denotes the matrix of cofactors, then

$$C = \begin{bmatrix} -4 & 1 & 6 \\ -3 & 5 & -4 \\ -1 & -4 & -7 \end{bmatrix}.$$

Notes:

(i) The framed elements indicate those for which the place sign is positive and hence no changes of sign are required to convert the minor of the corresponding elements of M into their cofactor. It is a good idea to work these out first.

(ii) The remaining four elements are those for which the place sign is negative and so the cofactor is minus the value of the minor in these cases. If these, too, are worked out together, there is less scope for making mistakes.

(iii) It is important to remember that, in finding the elements of C, we **do not multiply the cofactors of the elements in** M **by the elements themselves**.

The transpose of C now provides the adjoint matrix and hence the inverse as follows:

$$
M^{-1} = \frac{1}{\mid M \mid} AdjM = \frac{1}{-17} C^{T} = \frac{1}{-17} \begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}.
$$

Thus, the solution of the simultaneous equations is given by

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -17 \\ -51 \\ -85 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.
$$

That is,

$$
x = 1 \quad y = 3 \quad \text{and} \quad z = 5.
$$

Note:

Once again, the example should be regarded as a model solution to this type of problem but the student should not expect every exercise to yield whole number answers.

9.3.5 EXERCISES

1. Determine, where possible, the multiplicative inverses of the following matrices:

(a) $\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$; (b) $\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$; (c) $\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 4 & -2 \\ -3 & 2 & -7 \\ 1 & 0 & 5 \end{bmatrix}$; (e) $\begin{bmatrix} 6 & 3 & 1 \\ 0 & -5 & 2 \\ 1 & 1 & 4 \end{bmatrix}$; (f) $\begin{bmatrix} 11 & 3 & 5 \\ 6 & 2 & 2 \\ 10 & 3 & 4 \end{bmatrix}$.

2. Use a matrix inverse method to solve the following sets of simultaneous linear equations:

(a)

$$
\begin{aligned}
4x - 3y &= 9, \\
3x - 2y &= 7;
\end{aligned}
$$

(b)

$$
\begin{aligned}
x + 2y &= -2, \\
5x - 4y &= 3;
\end{aligned}
$$

(c)

$$
\begin{aligned}
3x + 2y + z &= 3, \\
5x + 4y + 3z &= 3, \\
6x + y + z &= 5;
\end{aligned}
$$

(d)

$$
\begin{aligned}
x + 2y + 2z &= 0, \\
2x + 5y + 4z &= 2, \\
x - y - 6z &= 4;
\end{aligned}
$$

(e)

$$
\begin{aligned}
3x + 2y - 2z &= 16, \\
4x + 3y + 3z &= 2' \\
2x - y + z &= -1;
\end{aligned}
$$

(f)

$$
\begin{aligned}
2(i_3 - i_2) + 5(i_3 - i_1) &= 4, \\
(i_2 - i_3) + 2i_2 + (i_2 - i_1) &= 0, \\
5(i_1 - i_3) + 2(i_1 - i_2) + i_2 &= 1.
\end{aligned}
$$

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3. If $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & -4 \\ 0 & 5 & -2 \end{bmatrix}$,

determine a matrix, B, such that A.B = C.

9.3.6 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{26}\begin{bmatrix} 9 & -5 \\ -7 & 1 \end{bmatrix}$; (b) No Inverse, (c) $-\frac{1}{4}\begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}$;

(d) $\frac{1}{46}\begin{bmatrix} 10 & -20 & -24 \\ 8 & 7 & 13 \\ -2 & 4 & 14 \end{bmatrix}$; (e) $-\frac{1}{121}\begin{bmatrix} -22 & -11 & 11 \\ 2 & 23 & -12 \\ 5 & -3 & -30 \end{bmatrix}$; (f) There is no inverse.

2. (a) $x = 3$ and $y = 1$; (b) $x = \frac{1}{7}$ and $y = -\frac{13}{14}$; (c) $x = 1$, $y = 1$ and $z = -2$;

(d) $x = -\frac{3}{2}$, $y = 2$ and $z = -\frac{5}{4}$; (e) $x = 2$, $y = \frac{3}{2}$ and $z = -\frac{7}{2}$;

(f) $i_1 = \frac{11}{6}$, $i_2 = 1$ and $i_3 = \frac{13}{6}$.

3.

$$B = \frac{1}{41}\begin{bmatrix} 11 & -2 & -1 \\ -13 & 21 & -10 \\ -3 & 8 & 4 \end{bmatrix} . \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & -4 \\ 0 & 5 & -2 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} -17 & 15 & 21 \\ 76 & -55 & -77 \\ 27 & 22 & -43 \end{bmatrix}.$$