"JUST THE MATHS"

UNIT NUMBER

6.1

COMPLEX NUMBERS 1
(Definitions and algebra)

by

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UNIT 6.1 - COMPLEX NUMBERS 1 - DEFINITIONS AND ALGEBRA

6.1.1 THE DEFINITION OF A COMPLEX NUMBER

Students who are already familiar with the Differential Calculus may appreciate that equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

which are called "Differential Equations", have wide-reaching applications in science and engineering. They are particularly applicable to problems involving either electrical circuits or mechanical vibrations.

It is possible to show that, in order to determine a formula (without derivatives) giving the variable y in terms of the variable x, one method is to solve, first, the quadratic equation whose coefficents are a, b and c and whose solutions are therefore

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
Students who are **not** already familiar with the Differential Calculus should consider only the quadratic equation whose coefficients are a, b and c, ignoring references to differential equations.

ILLUSTRATION

One method of solving the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13 = 2\sin x$$

would be to solve, first, the quadratic equation whose coefficients are 1, -6 and 13.

Its solutions are
$$\frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$
which clearly do not exist since we cannot find the square root of a negative number in elementary arithmetic.

However, if we assume that the differential equation represents a genuine scientific problem with a genuine scientific solution, we cannot simply dismiss the result obtained from the quadratic formula.

The difficulty seems to be, not so much with the -16 but with the minus sign in front of the 16. We shall therefore write the solutions in the form

$$\frac{6 \pm 4\sqrt{-1}}{2} = 3 \pm 2\sqrt{-1}.$$

Notes:
(i) The symbol $\sqrt{-1}$ will be regarded as an **"imaginary"** number.

(ii) In mathematical work, $\sqrt{-1}$ is normally denoted by i but, in scientific work it is denoted by j in order to avoid confusion with other quantities (eg. electric current) which could be denoted by the same symbol.

(iii) Whenever the imaginary quantity $j = \sqrt{-1}$ occurs in the solutions of a quadratic equation, those solutions will always be of the form $a + bj$ (or $a + jb$), where a and b are ordinary numbers of elementary arithmetic.

DEFINITIONS

1. The term **"complex number"** is used to denote any expression of the form $a + bj$ or $a + jb$ where a and b are ordinary numbers of elementary arithmetic (including zero) and j denotes the imaginary number $\sqrt{-1}$; i.e. $j^2 = -1$.

2. If the value a happens to be zero, then the complex number $a + bj$ or $a + jb$ is called **"purely imaginary"** and is written bj or jb.

3. If the value b happens to be zero, then the complex number $a + bj$ or $a + jb$ is defined to be the same as the number a and is called **"real"**. That is $a + j0 = a + 0j = a$.

4. For the complex number $a + bj$ or $a + jb$, the value a is called the **"real part"** and the value b is called the **"imaginary part"**. Notice that the imaginary part is b and **not** jb.

5. The complex numbers $a \pm bj$ are said to form a pair of **"complex conjugates"** and similarly $a \pm jb$ form a pair of complex conjugates. Alternatively, we may say, for instance, that $a - jb$ is the complex conjugate of $a + jb$ and $a + jb$ is the complex conjugate of $a - jb$.

Note:
In some work on complex numbers, especially where many complex numbers may be under

discussion at the same time, it is convenient to denote real and imaginary parts by the symbols x and y respectively, rather than a and b. It is also convenient, on some occasions, to denote the whole complex number $x + jy$ by the symbol z in which case the conjugate, $x - jy$, will be denoted by $\overline{z}$.

6.1.2 THE ALGEBRA OF COMPLEX NUMBERS

INTRODUCTION

An **"Algebra"** (coming from the Arabic word AL-JABR) refers to any mathematical system which uses the concepts of equality, addition, subtraction, multiplication and division. For example, the algebra of real numbers is what we normally call **"arithmetic"**; but algebraical concepts can be applied to other mathematical systems of which the system of complex numbers is one.

In meeting a new mathematical system for the first time, the concepts of equality, addition, subtraction, multiplication and division need to be properly defined, and that is the purpose of the present section. In some cases, the definitions are fairly obvious, but need to be made without contradicting ideas already established in the system of real numbers which complex numbers include.

(a) EQUALITY

Unlike a real number, a complex number does not have a "value"; and so the word "equality" must take on a meaning, here, which is different from that used in elementary arithmetic. In fact two complex numbers are defined to be equal if they have the same real part <u>and</u> the same imaginary part.

That is

$$a + jb = c + jd \ \ \text{if} \ \ \text{and} \ \ \text{only} \ \ \text{if} \ \ a = c \ \ \text{and} \ \ b = d$$

EXAMPLE

Determine x and y such that

$$(2x - 3y) + j(x + 5y) = 11 - j14.$$

Solution

From the definition of equality, we may

<div align="center">

EQUATE REAL AND IMAGINARY PARTS.

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Thus,

$$\begin{aligned} 2x - 3y &= 11, \\ x + 5y &= -14 \end{aligned}$$

These simultaneous linear equations are satisfied by $x = 1$ and $y = -3$.

(b) ADDITION AND SUBTRACTION

These two concepts are very easily defined. We simply add (or subtract) the real parts and the imaginary parts of the two complex numbers whose sum (or difference) is required.

That is,

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

EXAMPLE

$$(-7 + j2) + (10 - j5) = 3 - j3 = 3(1 - j)$$

and

$$(-7 + j2) - (10 - j5) = -17 + j7.$$

(c) MULTIPLICATION

The definition of multiplication essentially treats j in the same way as any other algebraic symbol, but uses the fact that $j^2 = -1$.

Thus,

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad);$$

but this is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1.

$$(5 + j9)(2 + j6) = (10 - 54) + j(18 + 30) = -44 + j48.$$

2.

$$(3 - j8)(1 + j4) = (3 + 32) + j(-8 + 12) = 35 + j4.$$

3.

$$(a + jb)(a - jb) = a^2 + b^2.$$

Note:
The third example above will be useful in the next section. It shows that **the product of a complex number and its complex conjugate is always a real number consisting of the sum of the squares of the real and imaginary parts**.

(d) DIVISION

The objective here is to make a definition which provides the real and imaginary parts of the complex expression

$$\frac{a + jb}{c + jd}.$$

Once again, we make this definition in accordance with what would be obtained algebraically by treating j in the same way as any other algebraic symbol, but using the fact that $j^2 = -1$.

The method is to multiply both the numerator and the denominator of the complex ratio by the conjugate of the denominator giving

$$\frac{a+jb}{c+jd} \cdot \frac{c-jd}{c-jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2}.$$

The required definition is thus

$$\frac{a+jb}{c+jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2},$$

which, again, is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1.
$$\frac{5+j3}{2+j7} = \frac{5+j3}{2+j7} \cdot \frac{2-j7}{2-j7}$$

$$= \frac{(10+21)+j(6-35)}{2^2+7^2} = \frac{31-j29}{53}.$$

Hence, the real part is $\frac{31}{53}$ and the imaginary part is $-\frac{29}{53}$.

2.
$$\frac{6+j}{j2-4} = \frac{6+j}{j2-4} \cdot \frac{-j2-4}{-j2-4}$$

$$= \frac{(-24+2)+j(-4-12)}{(-2)^2+(-4)^2} = \frac{-22-j16}{20}.$$

Hence, the real part is $-\frac{22}{20} = -\frac{11}{10}$ and the imaginary part is $-\frac{16}{20} = -\frac{4}{5}$.

6.1.3 EXERCISES

1. Simplify the following:
   (a) j^3; (b) j^4; (c) j^5; (d) j^{15}; (e) j^{22}.

2. If $z_1 = 2 - j5$, $z_2 = 1 + j7$ and $z_3 = -3 - j4$, determine the following in the form $a + jb$:
   (a)
   $$z_1 - z_2 + z_3;$$
   (b)
   $$2z_1 + z_2 - z_3;$$
   (c)
   $$z_1 - (4z_2 - z_3);$$
   (d)
   $$\frac{z_1}{z_2};$$
   (e)
   $$\frac{z_2}{z_3};$$
   (f)
   $$\frac{z_3}{z_1}.$$

3. Determine the values of x and y such that
   $$(3x - 5y) + j(x + 3y) = 20 + j2.$$

4. Determine the real and imaginary parts of the expression
   $$(1 - j3)^2 + j(2 + j5) - \frac{3(4 - j)}{1 - j}.$$

5. If $z \equiv x + jy$ and $\bar{z} \equiv x - jy$ are conjugate complex numbers, determine the values of x and y such that
   $$4z\bar{z} - 3(z - \bar{z}) = 2 + j.$$

7

6.1.4 ANSWERS TO EXERCISES

1. (a) $-j$; (b) 1; (c) j; (d) $-j$; (e) -1.
2. (a)

$$4 - j16;$$

(b)

$$8 + j;$$

(c)

$$-5 - j37;$$

(d)

$$-0.66 - j0.38;$$

(e)

$$-1.24 - j0.68;$$

(f)

$$0.48 - j0.79$$

3.

$$x = 5 \quad \text{and} \quad y = -1.$$

4. The real part $= -20.5$; the imaginary part $= -8.5$
5.

$$x = \pm\frac{1}{2} \quad \text{and} \quad y = -\frac{1}{2}.$$