"JUST THE MATHS"

UNIT NUMBER

10.8

DIFFERENTIATION 8
(Higher derivatives)

by

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UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most of the examples (seen in earlier Units) on differentiating a function of x with respect to x, the result obtained has been **another** function of x. In general, this **will** be the case and the possibilty arises of differentiating again with respect to x.

This would occur, for example, in the case when the formula

$$y = f(x)$$

represents the distance, y, travelled by a moving object at time, x.

The **speed** of the moving object is the rate of increase of distance with respect to time; that is, $\frac{dy}{dx}$. But a second quantity called **acceleration** is defined as the rate of increase of speed with respect to time. It is therefore represented by the symbol

$$\frac{d}{dx}\left[\frac{dy}{dx}\right];$$

but this is usually written as

$$\frac{d^2y}{dx^2}$$

and is pronounced "d two y by dx squared".

We could, if necessary, differentiate over and over again to obtain the derivatives of order three, four, etc., namely

$$\frac{d^3y}{dx^3} \quad \text{and} \quad \frac{d^4y}{dx^4}, \quad \text{etc.}$$

EXAMPLES

1. If $y = \sin 2x$, show that

$$\frac{d^2y}{dx^2} + 4y = 0.$$

Solution

Firstly,

$$\frac{dy}{dx} = 2\cos 2x,$$

so that, on differentiating a second time, we obtain

$$\frac{d^2y}{dx^2} = -4\sin 2x = -4y.$$

Hence, the result follows.

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

   Solution

   $$\frac{dy}{dx} = 4x^3;$$

   $$\frac{d^2y}{dx^2} = 12x^2;$$

   $$\frac{d^3y}{dx^3} = 24x;$$

   $$\frac{d^4y}{dx^4} = 24.$$

   We now have a constant function, so that all future derivatives will be zero.

   Note:

   In general, every derivative of $y = x^n$ after the n-th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{d^2y}{dx^2}$ in terms of t.

   Solution

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

   giving

   $$\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}.$$

   In order to differentiate again with respect to x, we observe that, in the formula for the first derivative with respect to x, we need to replace y with $\frac{dy}{dx}$.

   That is,

   $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d\left[\frac{dy}{dx}\right]}{dx}.$$

Hence,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

In the present example, therefore, we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\right]}{6t} = \frac{\frac{-1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x), \ f'''(x), \ f^{(iv)}(x), \ \text{etc.}$$

10.8.2 EXERCISES

1. Obtain expressions for $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in the following cases:

   (a)
   $$y = 4x^3 - 7x^2 + 5x - 17;$$

   (b)
   $$y = (3x - 2)^{10};$$

   (c)
   $$y = x^2 e^{4x};$$

   (d)
   $$y = \frac{x - 1}{x + 1}.$$

2. If $y = \sin 3x$, evaluate $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ when $x = \frac{\pi}{4}$.

3. If $x^2 + y^2 - 2x + 2y = 23$ determine the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the point where $x = -2$ and $y = 3$.

4. If $x = 3(1 - \cos\theta)$ and $y = 3(\theta - \sin\theta)$, show that

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(a)

$$\frac{dy}{dx} = \tan\frac{\theta}{2};$$

(b)

$$\frac{d^2y}{dx^2} = \frac{1}{12\sin\frac{\theta}{2}\cos^3\frac{\theta}{2}}.$$

5. If $y = 3e^{2x}\cos(2x - 3)$, verify that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0.$$

10.8.3 ANSWERS TO EXERCISES

1. (a)

$$\frac{d^2y}{dx^2} = 12x^2 - 14;$$

(b)

$$\frac{d^2y}{dx^2} = 810(3x - 2)^8;$$

(c)

$$\frac{d^2y}{dx^2} = e^{4x}\left[16x^2 + 16x + 2\right];$$

(d)

$$\frac{d^2y}{dx^2} = -\frac{4}{(x+1)^3}.$$

2.

$$-\frac{9}{\sqrt{2}}.$$

3.

$$\frac{3}{4} \text{ and } -\frac{25}{64}.$$

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