"JUST THE MATHS"

UNIT NUMBER

1.1

ALGEBRA 1
(Introduction to algebra)

by

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UNIT 1.1 - ALGEBRA 1 - INTRODUCTION TO ALGEBRA

DEFINITION

An **"Algebra"** is any Mathematical system which uses the concepts of Equality (=), Addition (+), Subtraction (-), Multiplication (× or .) and Division (÷).

Note:
The Algebra of Numbers is what we normally call **"Arithmetic"** and, as far as this unit is concerned, it is only the algebra of numbers which we shall be concerned with.

1.1.1 THE LANGUAGE OF ALGEBRA

Suppose we use the symbols a, b and c to denote numbers of arithmetic; then

(a) $a + b$ is called the **"sum of a and b"**.

Note:
$a + a$ is usually abbreviated to $2a$,
$a + a + a$ is usually abbreviated to $3a$ and so on.

(b) $a - b$ is called the **"difference between a and b"**.

(c) $a \times b$, $a.b$ or even just ab is called the **"product"** of a and b.

Notes:
(i)
$a.a$ is usually abbreviated to a^2,
$a.a.a$ is usually abbreviated to a^3 and so on.

(ii) $-1 \times a$ is usually abbreviated to $-a$ and is called the **"negation"** of a.

(d) $a \div b$ or $\frac{a}{b}$ is called the **"quotient"** or **"ratio"** of a and b.

(e) $\frac{1}{a}$, [also written a^{-1}], is called the **"reciprocal"** of a.

(f) $\mid a \mid$ is called the **"modulus"**, **"absolute value"** or **"numerical value"** of a. It can be defined by the two statements

$\mid a \mid = a$ when a is positive or zero;
$\mid a \mid = -a$ when a is negative or zero.

Note:
Further work on fractions (ratios) will appear later, but we state here for reference the rules for combining fractions together:

Rules for combining fractions together

1. $$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

2. $$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

3. $$\frac{a}{b}.\frac{c}{d}=\frac{a.c}{b.d}$$

4. $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{a.d}{b.c}$$

EXAMPLES

1. How much more than the difference of 127 and 59 is the sum of 127 and 59 ?
   Solution
   The difference of 127 and 59 is $127-59=68$ and the sum of 127 and 59 is $127+59=186$. The sum exceeds the difference by $186-68=118$.

2. What is the reciprocal of the number which is 5 multiplied by the difference of 8 and 2 ?
   Solution
   We require the reciprocal of $5.(8-2)$; that is, the reciprocal of 30. The answer is therefore $\frac{1}{30}$.

3. Calculate the value of $4\frac{2}{3}-5\frac{1}{9}$ expressing the answer as a fraction.
   Solution
   Converting both numbers to a single fraction, we require
   $$\frac{14}{3}-\frac{46}{9}=\frac{126-138}{27}=-\frac{12}{27}=-\frac{4}{9}.$$
   We could also have observed that the 'lowest common multiple' (see later) of the two denominators, 3 and 9, is 9; hence we could write the alternative solution
   $$\frac{42}{9}-\frac{46}{9}=-\frac{4}{9}.$$

4. Remove the modulus signs from the expression $\mid a-2 \mid$ in the cases when (i) a is greater than (or equal to) 2 and (ii) a is less than 2.

   Solution
   (i) If a is greater than or equal to 2,

$$\mid a-2 \mid= a-2;$$

   (ii) If a is less than 2,

$$\mid a-2 \mid= -(a-2) = 2-a.$$

1.1.2 THE LAWS OF ALGEBRA

If the symbols a, b and c denote numbers of arithmetic, then the following Laws are obeyed by them:

(a) The Commutative Law of Addition $\quad a+b=b+a$

(b) The Associative Law of Addition $\quad a+(b+c)=(a+b)+c$

(c) The Commutative Law of Multiplication $\quad a.b=b.a$

(d) The Associative Law of Multipication $\quad a.(b.c)=(a.b).c$

(e) The Distributive Laws $\quad a.(b+c)=a.b+a.c$ and $(a+b).c=a.c+b.c$

Notes:

(i) A consequence of the Distributive Laws is the rule for multiplying together a pair of bracketted expressions. It will be encountered more formally later, but we state it here for reference:

$$(a+b).(c+d) = a.c+b.c+a.d+b.d$$

(ii) The alphabetical letters so far used for numbers in arithmetic have been taken from the **beginning** of the alphabet. These tend to be reserved for fixed quantities called **constants**. Letters from the **end** of the alphabet, such as w, x, y, z are normally used for quantities which may take many values, and are called **variables**.

1.1.3 PRIORITIES IN CALCULATIONS

Suppose that we encountered the expression $5\times 6-4$. It would seem to be ambiguous, meaning either $30-4=26$ or $5\times 2=10$.

However, we may remove the ambiguity by using brackets where necessary, together with a rule for precedence between the use of the brackets and the symbols $+$, $-$, $\times$ and $\div$.

The rule is summarised in the abbreviation

B.O.D.M.A.S.

which means that the order of precedence is

B	brackets	( )	First Priority
O	of	$\times$	Joint Second Priority
D	division	$\div$	Joint Second Priority
M	multiplication	$\times$	Joint Second Priority
A	addition	$+$	Joint Third Priority
S	subtraction	$-$	Joint Third Priority

Thus, $5 \times (6 - 4) = 5 \times 2 = 10$
but $5 \times 6 - 4 = 30 - 4 = 26$.

Similarly, $12 \div 3 - 1 = 4 - 1 = 3$
whereas $12 \div (3 - 1) = 12 \div 2 = 6$.

1.1.4 FACTORS

If a number can be expressed as a product of other numbers, each of those other numbers is called a **"factor"** of the original number.

EXAMPLES

1. We may observe that
$$70 = 2 \times 7 \times 5$$
so that the number 70 has factors of 2, 7 and 5. These three cannot be broken down into factors themselves because they are what are known as **"prime"** numbers (numbers whose only factors are themselves and 1). Hence the only factors of 70, apart from 70 and 1, are 2, 7 and 5.

2. Show that the numbers 78 and 182 have two common factors which are prime numbers.

   The two factorisations are as follows:
$$78 = 2 \times 3 \times 13,$$

$$182 = 2 \times 7 \times 13.$$

The common factors are thus 2 and 13, both of which are prime numbers.

Notes:

(i) If two or more numbers have been expressed as a product of their prime factors, we may easily identify the prime factors which are common to all the numbers and hence obtain the **"highest common factor"**, h.c.f.

For example, $90 = 2 \times 3 \times 3 \times 5$ and $108 = 2 \times 2 \times 3 \times 3 \times 3$. Hence the h.c.f $= 2 \times 3 \times 3 = 18$

(ii) If two or more numbers have been expressed as a product of their prime factors, we may also identify the **"lowest common multiple"**, l.c.m.

For example, $15 = 3 \times 5$ and $20 = 2 \times 2 \times 5$. Hence the smallest number into which both 15 and 20 will divide requires two factors of 2 (for 20), one factor of 5 (for both 15 and 20) and one factor of 3 (for 15). The l.c.m. is thus $2 \times 2 \times 3 \times 5 = 60$.

(iii) If the numerator and denominator of a fraction have factors in common, then such factors may be cancelled to leave the fraction in its **"lowest terms"**.

For example $\frac{15}{105} = \frac{3\times5}{3\times5\times7} = \frac{1}{7}$.

1.1.5 EXERCISES

1. Find the sum and product of

   (a) 3 and 6; (b) 10 and 7; (c) 2, 3 and 6;

   (d) $\frac{3}{2}$ and $\frac{4}{11}$; (e) $1\frac{2}{5}$ and $7\frac{3}{4}$; (f) $2\frac{1}{7}$ and $5\frac{4}{21}$.

2. Find the difference between and quotient of

   (a) 18 and 9; (b) 20 and 5; (c) 100 and 20;

   (d) $\frac{3}{5}$ and $\frac{7}{10}$; (e) $3\frac{1}{4}$ and $2\frac{2}{9}$; (f) $1\frac{2}{3}$ and $5\frac{5}{6}$.

3. Evaluate the following expressions:

   (a) $6 - 2 \times 2$; (b) $(6 - 2) \times 2$;
   (c) $6 \div 2 - 2$; (d) $(6 \div 2) - 2$;
   (e) $6 - 2 + 3 \times 2$; (f) $6 - (2 + 3) \times 2$;
   (g) $(6 - 2) + 3 \times 2$; (h) $\frac{16}{-2}$; (i) $\frac{-24}{-3}$; (j) $(-6) \times (-2)$.

4. Place brackets in the following to make them correct:

   (a) $6 \times 12 - 3 + 1 = 55$; (b) $6 \times 12 - 3 + 1 = 68$;
   (c) $6 \times 12 - 3 + 1 = 60$; (d) $5 \times 4 - 3 + 2 = 7$;
   (e) $5 \times 4 - 3 + 2 = 15$; (f) $5 \times 4 - 3 + 2 = -5$.

5. Express the following as a product of prime factors:

   (a) 26; (b) 100; (c) 27; (d) 71;

   (e) 64; (f) 87; (g) 437; (h) 899.

6. Find the h.c.f of

   (a) 12, 15 and 21; (b) 16, 24 and 40; (c) 28, 70, 120 and 160;

   (d) 35, 38 and 42; (e) 96, 120 and 144.

7. Find the l.c.m of

   (a) 5, 6, and 8; (b) 20 and 30; (c) 7, 9 and 12;

   (d) 100, 150 and 235; (e) 96, 120 and 144.

1.1.6 ANSWERS TO EXERCISES

1. (a) 9, 18; (b) 17,70; (c) 11,36; (d) $\frac{41}{22}$, $\frac{6}{11}$; (e) $\frac{183}{20}$, $\frac{217}{20}$; (f) $\frac{154}{21}$, $\frac{545}{49}$.

2. (a) 9,2; (b) 15,4; (c) 80,5; (d) $-\frac{1}{10}$, $\frac{6}{7}$; (e) $\frac{37}{36}$, $\frac{117}{80}$; (f) $-\frac{25}{6}$, $\frac{2}{7}$.

3. (a) 2; (b) 8; (c) 1; (d) 1; (e) 10;
   (f) -4; (g) 10; (h) -8; (i) 8; (j) 12;

4. (a) $6 \times (12 - 3) + 1 = 55$; (b) $6 \times 12 - (3 + 1) = 68$;
   (c) $6 \times (12 - 3 + 1) = 60$; (d) $5 \times (4 - 3) + 2 = 7$;
   (e) $5 \times 4 - (3 + 2) = 15$; (f) $5 \times (4 - [3 + 2]) = -5$.

5. (a) 2×13; (b) $2 \times 2 \times 5 \times 5$; (c) $3 \times 3 \times 3$; (d) 71×1;

   (e) $2 \times 2 \times 2 \times 2 \times 2 \times 2$; (f) 3×29; (g) 19×23; (h) 29×31.

6. (a) 3; (b) 8; (c) 2; (d) 1; (e) 24.

7. (a) 120; (b) 60; (c) 252; (d) 14100; (e) 1440.