"JUST THE MATHS"

UNIT NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

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UNIT 14.1 - PARTIAL DIFFERENTIATION 1 - PARTIAL DERIVATIVES OF THE FIRST ORDER

14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1.

$$z = f(x, y),$$

which means that the variable, z, depends (for its values) on two variables, x and y.

2.

$$w = F(x, y, z),$$

which means that the variable, w, depends (for its values) on three variables, x, y and z.

Normally, the variables on the right-hand side of examples like those above may be chosen independently of one another and, as such, are called the **"independent variables"**. By contrast, the variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

An illustration of this type would be $x^2 + y^2 + z^2 = 16$.

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, giving a dependent variable in terms of them, may not actually be independent of one another. This would occur if those variables were already, themselves, dependent on a quantity not specifically mentioned in the formula.

For example, in the formula

$$z = xy^2 + \sin(x - y),$$

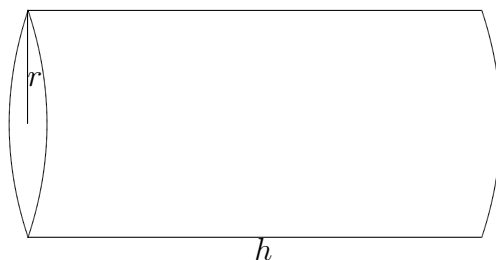suppose it is also known that $x = t - 1$ and $y = 3t + 2$.

Then the variables x and y are not independent of each other. In fact, by eliminating t, we obtain

$$y = 3(x + 1) + 2 = 3x + 5.$$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION

Consider the formulae for the volume, V, and the surface area, S, of a solid right-circular cylinder with radius, r, and height, h.

The relevant formulae are

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi rh,$$

so that both V and S are functions of the two variables, r and h.

But suppose it were possible for r to be held constant while h is allowed to vary. Then the corresponding rates of increase of V and S with respect to h are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

2

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r \text{ const.}} = 2\pi r.$$

These two expressions are called the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose it were possible for h to be held constant while r is allowed to vary. Then the corresponding rates of increase of V and S with respect to r are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h \text{ const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These two expressions are called the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

In the defining illustration above, the notation used for the partial derivatives of V and S was an adaptation of the notation for what will, in future, be referred to as **ordinary** derivatives.

It was, however, rather cumbersome; and the more standard notation which uses the symbol ∂ rather than d is indicated by restating the earlier results as

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

In this notation, it is understood that each independent variable (except the one with respect to which we are differentiating) is held constant.

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

   (a)

   $$z = (x^2 + 3y)^5;$$

   Solution

   $$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4 . 2x = 10x(x^2 + 3y)^4$$

   and

   $$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4 . 3 = 15(x^2 + 3y)^4.$$

   (b)

   $$w = ze^{3x-7y};$$

   Solution

   $$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

   $$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

   and

   $$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

   (c)

   $$z = x\sin(2x^2 + 5y).$$

   Solution

   $$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2\cos(2x^2 + 5y)$$

   and

   $$\frac{\partial z}{\partial y} = 5x\cos(2x^2 + 5y).$$

4

2. If
$$z = f(x^2 + y^2),$$

show that
$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. Given the formula

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

as an implicit relationship between two independent variables x and y and a dependent variable z, determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

Differentiating the formula partially with respect to x and y in turn, we obtain

$$-\sin(x + 2z).\left(1 + 2\frac{\partial z}{\partial x}\right) + 2y\left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x\left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.
Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$

14.1.3 EXERCISES

1. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following cases:

   (a)
   $$z = 2x^2 - 4xy + y^3;$$

   (b)
   $$z = \cos(5x - 3y);$$

   (c)
   $$z = e^{x^2 + 2y^2};$$

   (d)
   $$z = x\sin(y - x).$$

2. If
   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$

   show that
   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

3. Determine $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ in the following cases:

   (a)
   $$w = x^5 + 3xyz + z^2;$$

   (b)
   $$w = ze^{2x - 3y};$$

   (c)
   $$w = \sin(x^2 - yz).$$

6

14.1.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{\partial z}{\partial x} = 4x - 4y \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x + 3y^2;$$

(b)
$$\frac{\partial z}{\partial x} = -5\sin(5x - 3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\sin(5x - 3y);$$

(c)
$$\frac{\partial z}{\partial x} = 2xe^{x^2 + 2y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = 4ye^{x^2 + 2y^2};$$

(d)
$$\frac{\partial z}{\partial x} = \sin(y - x) - x\cos(y - x) \quad \text{and} \quad \frac{\partial z}{\partial y} = x\cos(y - x).$$

2.
$$\frac{\partial z}{\partial x} = \ln\left(\frac{x}{y}\right) + \frac{x + y}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \ln\left(\frac{x}{y}\right) - \frac{x + y}{y}.$$

3. (a)
$$\frac{\partial w}{\partial x} = 5x^4 + 3yz, \quad \frac{\partial w}{\partial y} = 3xz, \quad \frac{\partial w}{\partial z} = 3xy + 2z;$$

(b)
$$\frac{\partial w}{\partial x} = 2ze^{2x-3y}, \quad \frac{\partial w}{\partial y} = -3ze^{2x-3y}, \quad \frac{\partial w}{\partial z} = e^{2x-3y};$$

(c)
$$\frac{\partial w}{\partial x} = 2x\cos(x^2 - yz), \quad \frac{\partial w}{\partial y} = -z\cos(x^2 - yz), \quad \frac{\partial w}{\partial z} = -y\cos(x^2 - yz).$$