Classification

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Content

Binary Classification

Suppose we want to build an email spam classification software:

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Hello
I want to know about different topics that relate to qualitative reinforcement learning and make abstraction&aggregation... to solve problem compactly . I have read some survey to know exactly but sometimes I doubt about some topics are related to or not. for example Qualitative Spatial Representation and Reasoning. can anyone tell me different categorized topics? I need to know the general classification of them.

...

Binary Classification

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- The most simple classification problem is the binary classification, in which $y = \{0, 1\}$.
- In a binary classification problem we have positive examples $y = 1$ (spam) and negative examples $y = 0$ (no spam).
- The $\mathbf{x}$ may be some features of some piece of email.

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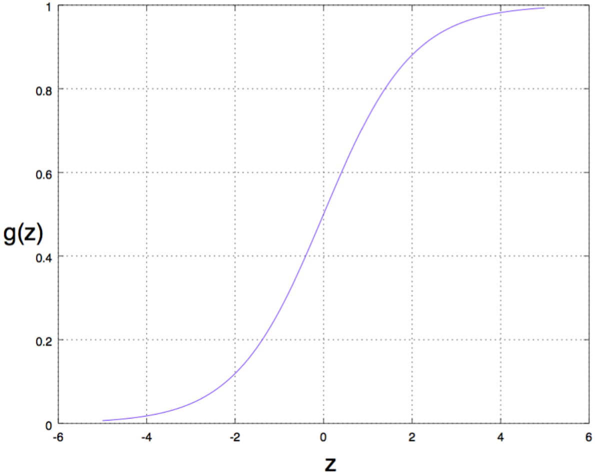
- where $g(z) = \frac{1}{1+e^{-z}}$

Binary Classification

$$g(z) = \frac{1}{1+e^{-z}} \qquad g : \mathbb{R} \to (0,1)$$



Derivative of the Sigmoid Function

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&= g(z)\big(1-g(z)\big).
\end{aligned}
$$

A Probabilistic Approach

Lets assume that:

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\begin{aligned}
P(y=1|x;\theta) &= h_\theta(x) \\
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Note that this can be written more compactly as:

$$p(y|x;\theta) = (h_\theta(x))^y (1 - h_\theta(x))^{1-y}$$

Likelihood of the Training Data's Labels

Assuming that the m training examples were generated independently, we can then write:

$$
\begin{aligned}
L(\theta) &= p(\mathbf{y}|\mathbf{X};\theta) \\
&= \prod_{i=1}^{m} p(y^{(i)}|\mathbf{x}^{(i)};\theta) \\
&= \prod_{i=1}^{m} (h_\theta(x^{(i)}))^{y^{(i)}} (1 - h_\theta(x^{(i)}))^{1-y^{(i)}}
\end{aligned}
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Using gradient ascent we get an update rule like this:

$$\theta := \theta + \alpha \nabla_\theta \ell(\theta)$$

Maximizing the Likelihood

Working with one example, the derivatives are as follows:

$$\frac{\partial}{\partial \theta_j}\ell(\theta) = \quad y \log h(x) + (1-y)\log(1-h(x))$$

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&= (y-h_\theta(\mathbf{x}))x_j
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The LMS Update Rule for Classification

Given that:

$$\frac{\partial}{\partial \theta_j} \ell(\theta) = (y - h_\theta(\mathbf{x}))x_j,$$

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$$\theta_j := \theta_j + \alpha(y^{(i)} - h_\theta(\mathbf{x}^{(i)})) x_j^{(i)},$$

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where $h_\theta(\mathbf{x}^{(i)}) = g(\theta^\top \mathbf{x}^{(i)}) = \frac{1}{1+e^{(-\theta^\top \mathbf{x}^{(i)})}}$ is now defined as a non-linear function of $\theta^\top \mathbf{x}^{(i)}$.

So, we end up with the same update rule for a different algorithm and learning problem.

LMS Algorithms for Classification

Batch Gradient Descent

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left[y^{(i)} - g(\theta^\top x^{(i)}) \right] x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

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Stochastic Gradient Descent

Loop {

for $i = 1$ to m {

$$\theta_j := \theta_j + \alpha \left[y^{(i)} - g(\theta^\top x^{(i)}) \right] x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

}

LMS Algorithms for Classification

Mini-Batch Gradient Descent

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{k} \left[y^{(i)} - h_\theta(x^{(i)}) \right] x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

Here we use mini-batches containing 10 to 1000 examples. This is $k \in [10, 1000]$.

Error of a Binary Classifier (1/2)

True Positive = A positive example correctly identified as positive.

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Error of a Binary Classifier (2/2)

Precision

It answers the question: How many of the examples identified as positives are indeed positives?

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Recall (Sensitivity)

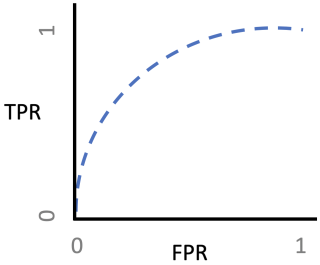
It answers the question: How many of the positive examples were correctly identified as positive?

$$\text{Recall} = \frac{TP}{P} = \frac{TP}{TP+FN}.$$

Receiver Operating Characteristic (ROC) curve

The Receiver Operating Characteristic curve is a graph showing the performance of a binary classifier with all the classification thresholds.

It shows two parameters: the true positive rate (TPR) and the false positive rate (FPR).



$$TPR = \frac{TP}{TP+FN}$$

$$FPR = \frac{FP}{FP+TN}$$

The Perceptron Learning Algorithm

Consider modifying the logistic regression to output either 1 or 0:

$$g(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} \geq 0 \\ 0 & \text{if } \mathbf{z} < 0 \end{cases}$$

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$$g(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} \geq 0 \\ 0 & \text{if } \mathbf{z} < 0 \end{cases}$$

By making $h_\theta(x) = g(\theta^\top \mathbf{x})$, then we have the update rule:

$$\theta_j := \theta_j + \alpha(y^{(i)} - h_\theta(x^{(i)}))x_j^{(i)}.$$

Newton's Method for Finding a Zero of a function

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ and we want to find a value of θ such that $f(\theta) = 0$, with $\theta \in \mathbb{R}$.

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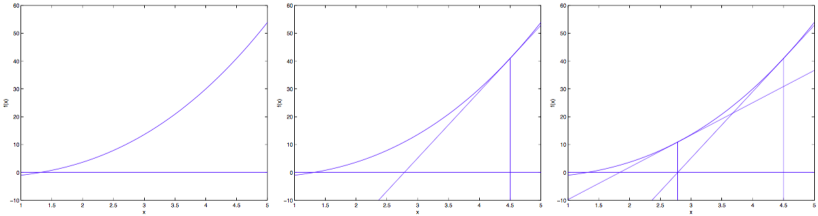

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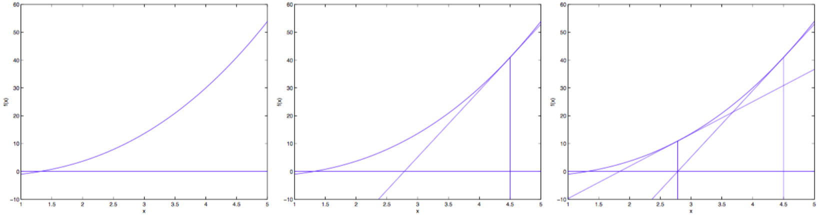

Newton's method performs the following update rule:

$$\theta := \theta - \frac{f(\theta)}{f'(\theta)}.$$

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Now, suppose we want to maximize a function ℓ. The maxima of ℓ correspond to points where its first derivative $\ell'(\theta)$ is zero.

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Now, suppose we want to maximize a function ℓ. The maxima of ℓ correspond to points where its first derivative $\ell'(\theta)$ is zero.

So, by letting $f(\theta) = \ell'(\theta)$, we can use the same algorithm to maximize ℓ:

$$\theta := \theta - \frac{\ell'(\theta)}{\ell''(\theta)}.$$

Newton-Raphson Method

In our regression setting θ is vector-valued. The generalization of Newton's method to this multidimensional setting is given by

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where H is the Hessian matrix

$$H_{ij} = \frac{\partial^2 \ell(\theta)}{\partial \theta_i \partial \theta_j}.$$

Thank you!

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