Appendix A

Technical Lemmas

Lemma A.1. *Let $a > 0$. Then: $x \geq 2a\log(a) \;\Rightarrow\; x \geq a\log(x)$. It follows that a necessary condition for the inequality $x < a\log(x)$ to hold is that $x < 2a\log(a)$.*

Proof. First note that for $a \in (0, \sqrt{e}]$ the inequality $x \geq a\log(x)$ holds unconditionally and therefore the claim is trivial. From now on, assume that $a > \sqrt{e}$. Consider the function $f(x) = x - a\log(x)$. The derivative is $f'(x) = 1 - a/x$. Thus, for $x > a$ the derivative is positive and the function increases. In addition,

$$\begin{aligned} f(2a\log(a)) &= 2a\log(a) - a\log(2a\log(a)) \\ &= 2a\log(a) - a\log(a) - a\log(2\log(a)) \\ &= a\log(a) - a\log(2\log(a)). \end{aligned}$$

Since $a - 2\log(a) > 0$ for all $a > 0$, the proof follows. $\square$

Lemma A.2. *Let $a \geq 1$ and $b > 0$. Then: $x \geq 4a\log(2a) + 2b \;\Rightarrow\; x \geq a\log(x) + b$.*

Proof. It suffices to prove that $x \geq 4a\log(2a) + 2b$ implies that both $x \geq 2a\log(x)$ and $x \geq 2b$. Since we assume $a \geq 1$ we clearly have that $x \geq 2b$. In addition, since $b > 0$ we have that $x \geq 4a\log(2a)$ which using Lemma A.1 implies that $x \geq 2a\log(x)$. This concludes our proof. $\square$

Lemma A.3. *Let X be a random variable and $x' \in \mathbb{R}$ be a scalar and assume that there exists $a > 0$ such that for all $t \geq 0$ we have $\mathbb{P}[|X - x'| > t] \leq 2e^{-t^2/a^2}$. Then, $\mathbb{E}[|X - x'|] \leq 4a$.*

Proof. For all $i = 0, 1, 2, \ldots$ denote $t_i = a\,i$. Since t_i is monotonically increasing we have that $\mathbb{E}[|X - x'|]$ is at most $\sum_{i=1}^{\infty} t_i\, \mathbb{P}[|X - x'| > t_{i-1}]$. Combining this with the assumption in the lemma we get that $\mathbb{E}[|X - x'|] \leq 2a\sum_{i=1}^{\infty} i\,e^{-(i-1)^2}$. The proof now follows from the inequalities

$$\sum_{i=1}^{\infty} i\,e^{-(i-1)^2} \leq \sum_{i=1}^{5} i\,e^{-(i-1)^2} + \int_{5}^{\infty} x e^{-(x-1)^2}\,dx < 1.8 + 10^{-7} < 2\,.$$

$\square$

Lemma A.4. *Let X be a random variable and $x' \in \mathbb{R}$ be a scalar and assume that there exists $a > 0$ and $b \geq e$ such that for all $t \geq 0$ we have $\mathbb{P}[|X - x'| > t] \leq 2b\, e^{-t^2/a^2}$. Then, $\mathbb{E}[|X - x'|] \leq a(2 + \sqrt{\log(b)})$.*

Proof. For all $i = 0, 1, 2, \ldots$ denote $t_i = a\,(i + \sqrt{\log(b)})$. Since t_i is monotonically increasing we have that

$$\mathbb{E}[|X - x'|] \leq a\sqrt{\log(b)} + \sum_{i=1}^{\infty} t_i\, \mathbb{P}[|X - x'| > t_{i-1}].$$

Using the assumption in the lemma we have

$$\begin{aligned}
\sum_{i=1}^{\infty} t_i\, \mathbb{P}[|X - x'| > t_{i-1}] &\leq 2\,a\,b \sum_{i=1}^{\infty} (i + \sqrt{\log(b)}) e^{-(i-1+\sqrt{\log(b)})^2} \\
&\leq 2\,a\,b \int_{1+\sqrt{\log(b)}}^{\infty} x e^{-(x-1)^2} dx \\
&= 2\,a\,b \int_{\sqrt{\log(b)}}^{\infty} (y+1) e^{-y^2} dy \\
&\leq 4\,a\,b \int_{\sqrt{\log(b)}}^{\infty} y e^{-y^2} dy \\
&= 2\,a\,b \left[-e^{-y^2}\right]_{\sqrt{\log(b)}}^{\infty} \\
&= 2\,a\,b/b = 2\,a.
\end{aligned}$$

Combining the preceding inequalities we conclude our proof. □

Lemma A.5. *Let m, d be two positive integers such that $d \leq m - 2$. Then,*

$$\sum_{k=0}^{d} \binom{m}{k} \leq \left(\frac{e\,m}{d}\right)^d.$$

Proof. We prove the claim by induction. For $d = 1$ the left-hand side equals $1 + m$ while the right-hand side equals em; hence the claim is true. Assume that the claim holds for d and let us prove it for $d + 1$. By the induction assumption we have

$$\begin{aligned}
\sum_{k=0}^{d+1} \binom{m}{k} &\leq \left(\frac{e\,m}{d}\right)^d + \binom{m}{d+1} \\
&= \left(\frac{e\,m}{d}\right)^d \left(1 + \left(\frac{d}{e\,m}\right)^d \frac{m(m-1)(m-2)\cdots(m-d)}{(d+1)d!}\right) \\
&\leq \left(\frac{em}{d}\right)^d \left(1 + \left(\frac{d}{e}\right)^d \frac{(m-d)}{(d+1)d!}\right).
\end{aligned}$$

Using Stirling's approximation we further have that

$$
\begin{aligned}
&\le \left(\frac{em}{d}\right)^d \left(1+\left(\frac{d}{e}\right)^d \frac{(m-d)}{(d+1)\sqrt{2\pi d}(d/e)^d}\right) \\
&= \left(\frac{em}{d}\right)^d \left(1+\frac{m-d}{\sqrt{2\pi d}(d+1)}\right) \\
&= \left(\frac{em}{d}\right)^d \cdot \frac{d+1+(m-d)/\sqrt{2\pi d}}{d+1} \\
&\le \left(\frac{em}{d}\right)^d \cdot \frac{d+1+(m-d)/2}{d+1} \\
&= \left(\frac{em}{d}\right)^d \cdot \frac{d/2+1+m/2}{d+1} \\
&\le \left(\frac{em}{d}\right)^d \cdot \frac{m}{d+1},
\end{aligned}
$$

where in the last inequality we used the assumption that $d \le m-2$. On the other hand,

$$
\begin{aligned}
\left(\frac{em}{d+1}\right)^{d+1} &= \left(\frac{em}{d}\right)^d \cdot \frac{em}{d+1} \cdot \left(\frac{d}{d+1}\right)^d \\
&= \left(\frac{em}{d}\right)^d \cdot \frac{em}{d+1} \cdot \frac{1}{(1+1/d)^d} \\
&\ge \left(\frac{em}{d}\right)^d \cdot \frac{em}{d+1} \cdot \frac{1}{e} \\
&= \left(\frac{em}{d}\right)^d \cdot \frac{m}{d+1},
\end{aligned}
$$

which proves our inductive argument. □

Lemma A.6. *For all $a \in \mathbb{R}$ we have*

$$\frac{e^a + e^{-a}}{2} \le e^{a^2/2}.$$

Proof. Observe that

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}.$$

Therefore,

$$\frac{e^a + e^{-a}}{2} = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!},$$

and

$$e^{a^2/2} = \sum_{n=0}^{\infty} \frac{a^{2n}}{2^n\, n!}.$$

Observing that $(2n)! \ge 2^n\, n!$ for every $n \ge 0$ we conclude our proof. □