Linear Algebra Done Right (Axler) Exercises

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1 Chapter 2

Dimension

Exercise 1. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin, and $\mathbb{R}^2$

Solution. Suppose there exists another subspace V of $\mathbb{R}^2$ and let L_m the line with slope m and $L_\prime$ the line of the form $(0, y)$ notice that $L_m \cap V$ is a subspace of $\mathbb{R}^2$, then $dim L_m \cap V \leq dim\mathbb{R}^2 = 2$.

We have three cases:

i). $dim L_m \cap V = 2$

ii). $dim L_m \cap V = 1$

iii). $dim L_m \cap V = 0$

i) If $dim L_m \cap V = 2$ by theorem 2.39 $L_m \cap V = \mathbb{R}^2$, which is not possible since $L_m \subset \mathbb{R}^2$, for $L_\prime$ is analogous.

ii) If $dim L_m \cap V = 1$, we will prove that $L_m \cap V = L_m$, which implies directly that $L_m = V$

Let $l \in L_m \cap V$, then $l \in L_m$.

For the other side let $l \in L_m$, then there exist a $x \in \mathbb{R}$ such that $l = (x, mx)$, let also v the basis of $L_m \cap V$ of length 1, since $v \in L_m$ then there exists $y \in \mathbb{R}$ such that $v = (y, my)$ finally there must exist a $k \in \mathbb{R}$ such that $x = ky$, so

$$l = (x, mx) = (ky, mky) = k(y, my) = kv,$$

knowing that v is a basis of $L_m \cap V$, $l \in L_m \cap V$, so $L_m = V$.

iii) If $dim L_m \cap V = 0$, then $L_m \cap V = \emptyset$ i.e. V and every line with scope m are disjoint and also disjoint with $L_\prime$, Let $\mathbb{L}$ the set of all lines in $\mathbb{R}^2$, by lemma 1.1

$$\bigcup_{l \in \mathbb{L}} l = \mathbb{R}^2,$$

then $V \subset R^2 = \bigcup_{l \in \mathbb{L}} l$ which is a contradiction since every line in $\mathbb{R}^2$ and V are disjoint. $\square$