INTELLIGENT SYSTEMS LABORATORY

Lesson 4
Probability Distributions

Mathematics and Statistics for Data Science

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Content

- Probability distributions & their applications
- Joint probability distributions

Random Variables

- A random variable assigns a numerical value to each outcome in a sample space.
- There are two types: *discrete* and *continuous*.
  - A discrete random variable is one whose possible values can be ordered, and there are gaps between adjacent values.
  - The possible values of a continuous random variable always contain an interval, that is, all the points between some two numbers.

Probability Distribution (Discrete)

- The list of possible values of a discrete random variable X, along with the probabilities for each, provides a complete description of the population from which X is drawn.
- This is known as probability distribution.
- The probability distribution of a discrete random variable X is the function $p(x) = P(X = x)$.
- A cumulative distribution function specifies the probability that X is *less than or equal to* a given value, i.e. $F(x) = P(X \leq x)$.

$$\mu_X = \sum_x x\,P(X = x)$$

Mean

(Discrete Random Variable)

$$\sigma_X^2 = \sum_x (x - \mu_X)^2 P(X = x)$$

Variance

(Discrete Random Variable)

$$\sigma_X = \sqrt{\sigma_X^2}$$

Standard Deviation

(Discrete Random Variable)

Example 1

- A certain industrial process is brought down for recalibration whenever the quality of the items produced falls below specifications. Let X represent the number of times the process is recalibrated during a week, and assume that X has the following probability distribution.

x	0	1	2	3	4
$p(x)$	0.35	0.25	0.20	0.15	0.05

- Find mean, variance, and standard deviation of X.

Example 1 - Solution

- Find mean, variance, and standard deviation of X.

x	0	1	2	3	4
$p(x)$	0.35	0.25	0.20	0.15	0.05

$$\mu_X = 0(0.35) + 1(0.25) + 2(0.20) + 3(0.15) + 4(0.05) = 1.30$$

$$\begin{aligned}\sigma_X^2 &= (0 - 1.30)^2 P(X = 0) + (1 - 1.30)^2 P(X = 1) + (2 - 1.30)^2 P(X = 2) \\ &\quad + (3 - 1.30)^2 P(X = 3) + (4 - 1.30)^2 P(X = 4) \\ &= (1.69)(0.35) + (0.09)(0.25) + (0.49)(0.20) + (2.89)(0.15) + (7.29)(0.05) \\ &= 1.51\end{aligned}$$

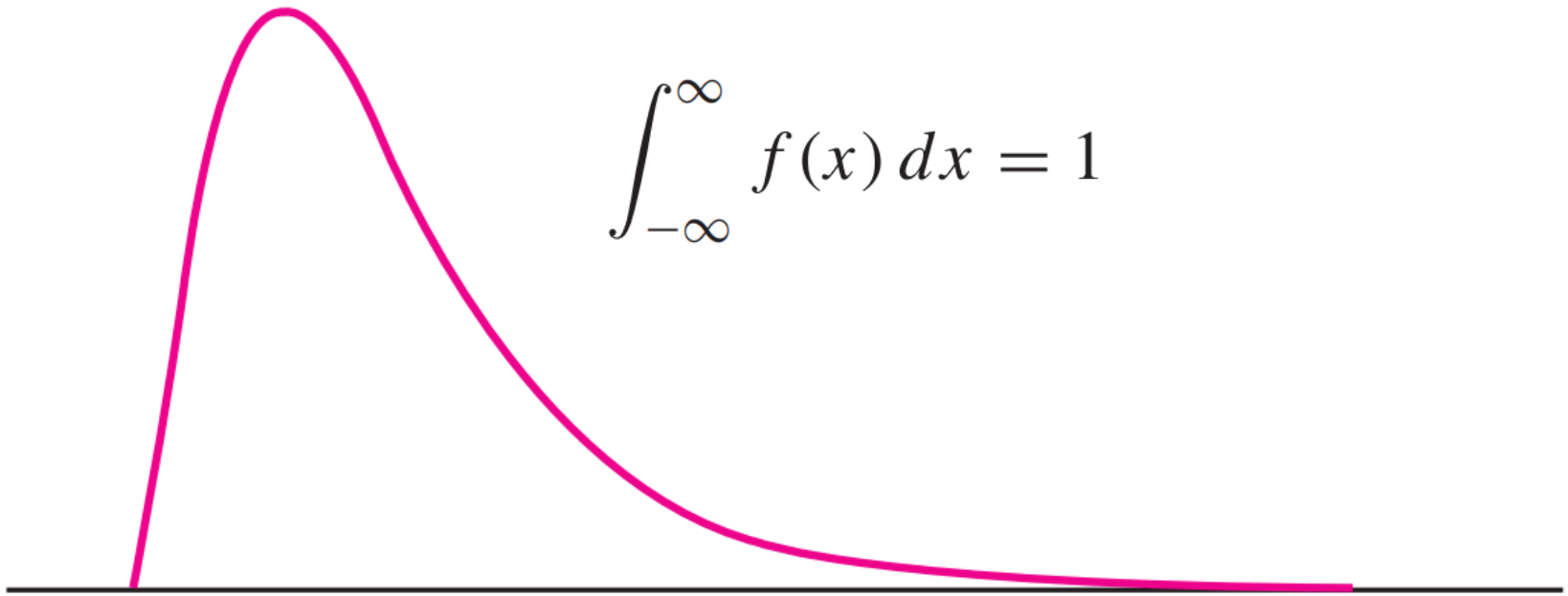
$$\sigma_X = \sqrt{1.51} = 1.23$$

Probability Distribution (Continuous)

- A random variable is continuous if its probabilities are given by areas under a curve.
- The curve is called a probability density function (or probability distribution) for the random variable.
- Let X be a continuous random variable with probability density function *f(x)*. Then:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Continuous Distribution - Example



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Cumulative Distribution Function

- Let X be a continuous random variable with probability density function f(x), the cumulative distribution function of X is:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

$$\mu_X = \int_{-\infty}^{\infty} x f(x)\, dx$$

Mean

(Continuous Random Variable)

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\, dx$$

Variance

(Continuous Random Variable)

$$\sigma_X = \sqrt{\sigma_X^2}$$

Standard Deviation

(Continuous Random Variable)

Bernoulli Trial

- An experiment that can result in one of two outcomes; for examples, "success" and "failure".
  - The probability of "success" is denoted by p.
  - The probability of "failure" is therefore 1 - p.
- Such a trial is called a Bernoulli trial with success probability p.

Bernoulli Distribution

- For any Bernoulli trial, we define a random variable X where if the experiment results in success, then X = 1; Otherwise, X = O.
- It follows that X is a discrete random variable, with probability distribution p(x) defined by:

$$p(0) = P(X = 0) = 1 - p$$

$$p(1) = P(X = 1) = p$$

- The random variable X is said to have the Bernoulli distribution.

$$\mu_X = p$$

$$\sigma_X^2 = p(1-p)$$

Mean & Variance

(Bernoulli Distribution)

N Bernoulli Trials

- In practice, we might take several samples from a very large lot and count the number of "successes" among them.
- This amounts to conducting several independent Bernoulli trials and counting the number of successes.
- The number of successes is then a random variable, which is said to have a binomial distribution.

Binomial Distribution

- If a total of n Bernoulli trials are conducted, and:
  - The trials are independent
  - Each trial has the same success probability p
  - X is the number of successes in the n trials
- X has the binomial distribution with parameters n and p.

$$p(x) = P(X = x) = \begin{cases} \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = np$$

$$\sigma_X^2 = np(1 - p)$$

Mean & Variance

(Binomial Distribution)

Example 2

- A company allows a discount on any invoice that is paid within 30 days. Of all invoices, 10% receive the discount. In a company audit, 12 invoices are sampled at random.

  - Find the probability that fewer than 4 of 12 invoices receive a discount.
  - Find the probability that more than 1 of 12 invoices receives a discount.
  - Find the mean and variance.

Normal Distribution

- The normal distribution (also called Gaussian distribution) is by far the most commonly used distribution in statistics.
- This distribution provides a good model for many, although not all, continuous populations.
- The probability density function of a normal random variable with mean μ and variance σ^2 is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x-\mu)^2/(2\sigma^2)}$$

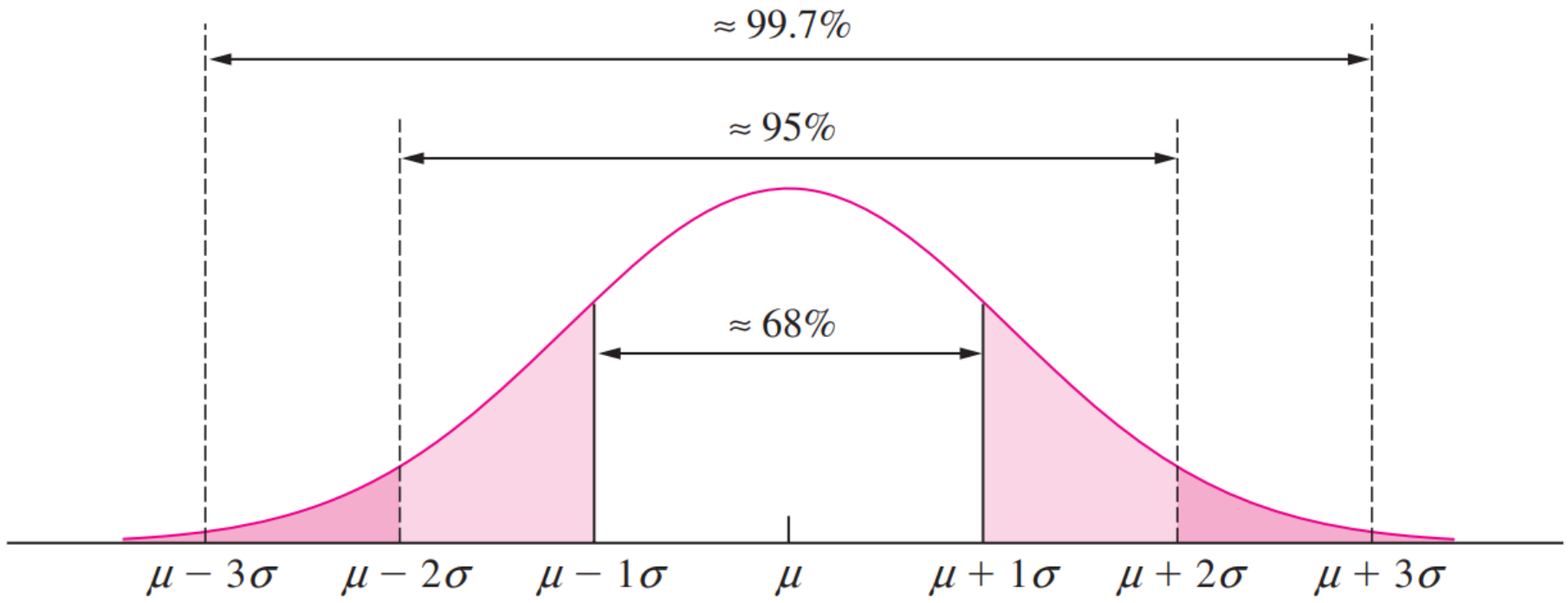
$$\mu_X = \mu$$

$$\sigma_X^2 = \sigma^2$$

Mean & Variance

(Normal Distribution)

Probability Density Function



The z-score

- When dealing with normal populations, we often convert from units in which the population items were originally measured to standard units.
- If x is an item sampled from a normal population with mean μ and variance σ2, the standard unit equivalent of x is the number z, where:

$$z = \frac{x - \mu}{\sigma}$$

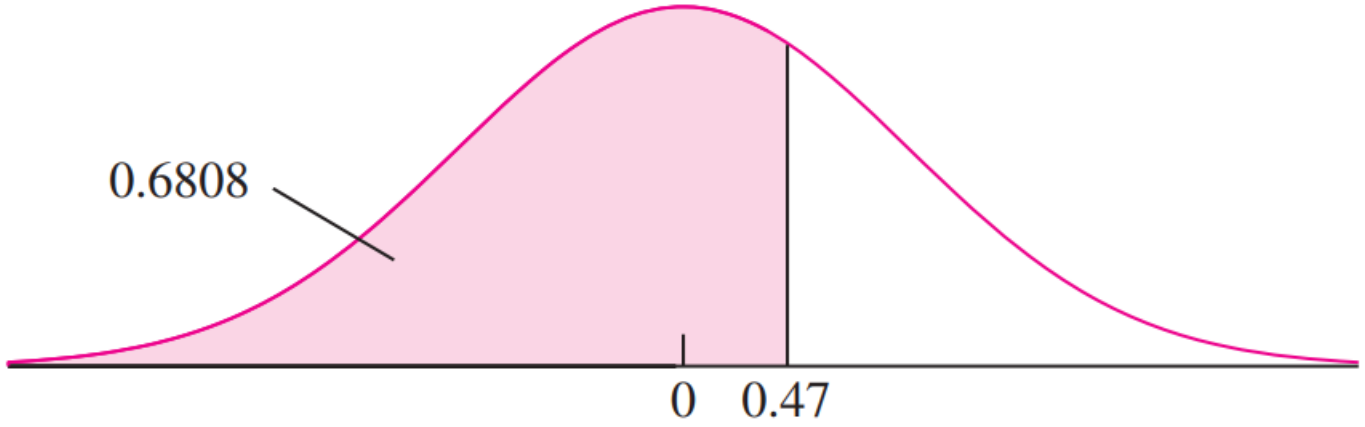
- The number z is called the z-score.

Example 3

- Aluminum sheets used to make beverage cans have thicknesses that are normally distributed with mean 10 and standard deviation 1.3. A particular sheet is 10.8 thousandths of an inch thick.
- Find the z-score.

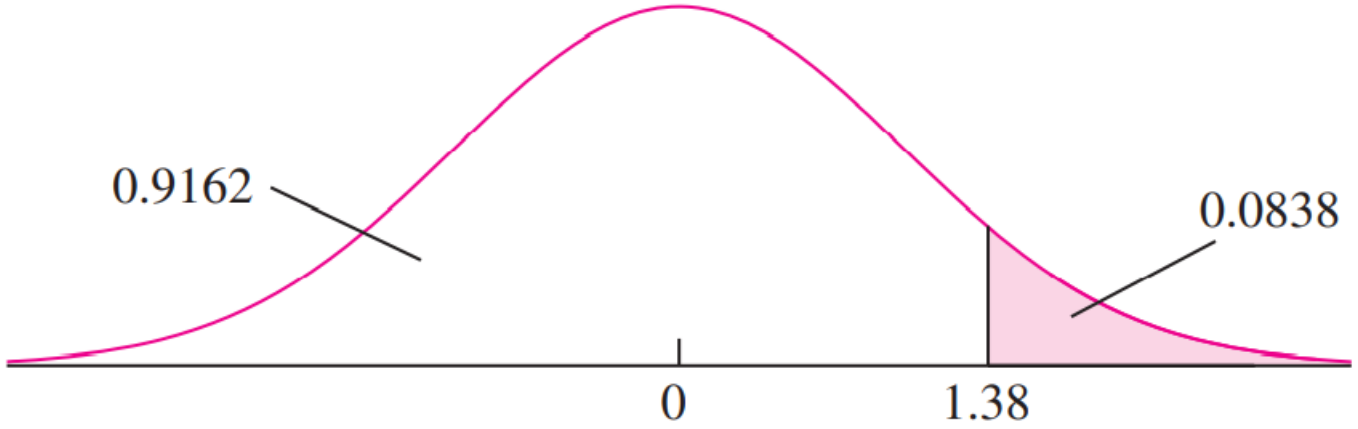
Example 4

- Find the area under the normal curve to the left of z = 0.47.



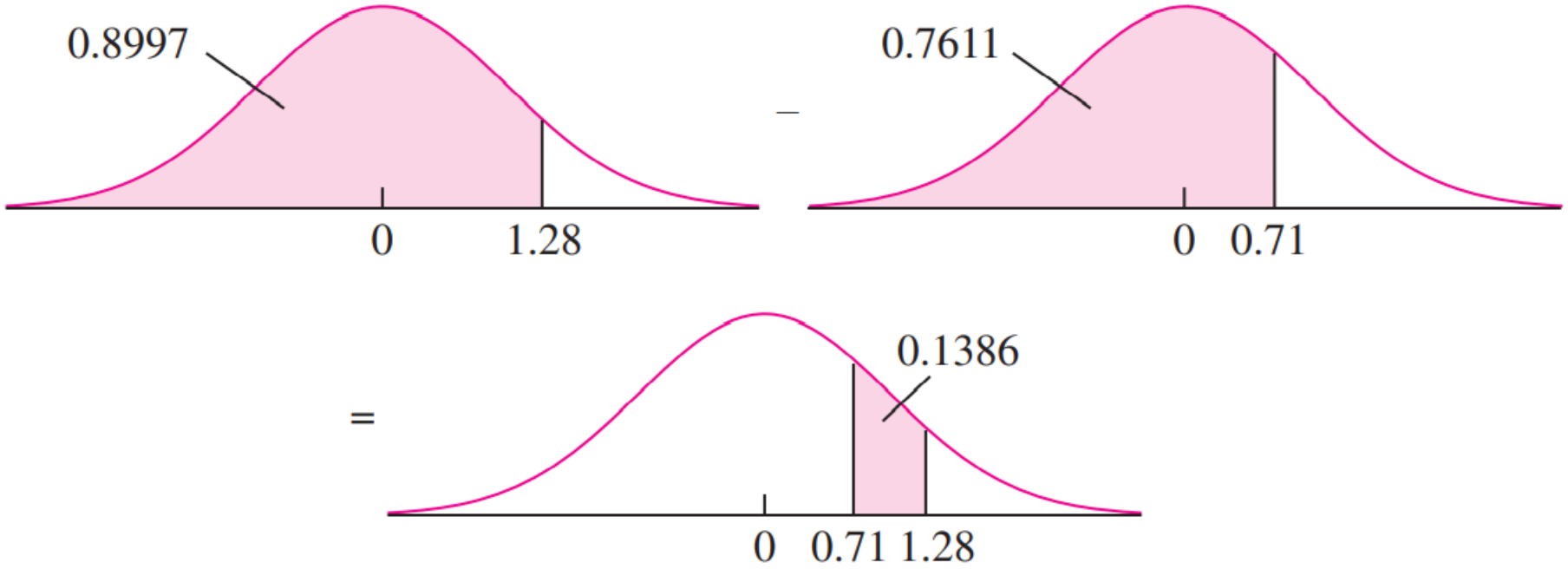
Example 5

- Find the area under the normal curve to the right of z = 1.38.



Example 6

- Find the area under the normal curve between z = 0.71 and z = 1.28.



Central Limit Theorem

- If we draw a large enough sample from a population, then the distribution of the sample mean is *approximately normal*, no matter what population the sample was drawn from.
- This allows us to compute probabilities for sample means using the z table, even though the population from which the sample was drawn is not normal.

Joint Probabilities – An Example

x	y	$P(X = x \text{ and } Y = y)$
129	15	0.12
129	16	0.08
130	15	0.42
130	16	0.28
131	15	0.06
131	16	0.04

Jointly Discrete

- If X and Y are jointly discrete random variables, the joint probability distribution of X and Y is:

$$p(x,y) = P(X = x \text{ and } Y = y)$$

- Marginal probability distribution of X and of Y:

$$p_X(x) = P(X = x) = \sum_y p(x,y)$$

$$p_Y(y) = P(Y = y) = \sum_x p(x,y)$$

Jointly Continuous

- If X and Y are jointly continuous random variables, the joint probability distribution of X and Y is:

$$P(a \leq X \leq b \text{ and } c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

- Marginal probability distribution of X and of Y:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Example 7

- Given the discrete joint probability distribution below:

x	y	$P(X = x \text{ and } Y = y)$
129	15	0.12
129	16	0.08
130	15	0.42
130	16	0.28
131	15	0.06
131	16	0.04

  - Find the probability that X = 129.
  - Find the probability that Y = 16.

Example 7 - Solution

- Given the discrete joint probability distribution below:

x	y	$P(X = x \text{ and } Y = y)$
129	15	0.12
129	16	0.08
130	15	0.42
130	16	0.28
131	15	0.06
131	16	0.04

  - P(X = 129) = 0.20
  - P(Y = 16) = 0.40