Mathematics Diary

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Abstract

This Mathematics Diary is made up of the things (on Mathematics) that the author thinks it worth writing down, including the *notes*, *important* and *interesting results*, *examples*, *problems* and *ideas* etc. produced during the learning and in the daily life (but one can also simply recognize this as a normal notebook). These things are recorded basically in the *time* order, which makes it like a *diary* (though I don't write down the actual date for the items), so it would sometimes seem like a mess.

There are also some special notations in this diary. The reason for why we don't use common words here is, it's inconvenient, not so beautiful and sometimes difficult to determine the type of everything. After all, this diary is mixed and doesn't have a strong logical system. Currently everything in the diary is separated into four types:

Thorium Classification for *theorems*, *propositions*, *rules*, *axioms*, *formulas*, *lemmas*, *corollaries*, $\cdots$ Any kind of statments. Sometimes a thorium is followed by a *proof*.

Erbium Classification for *examples*, *problems*, etc. Erbiums are often followed by a *proof* or a *solution*.

Fluorine Classification for *figures*, *diagrams*, *tables*, *lists*, etc.

And the fourth type is a paragraph without a caption like above. This is for *free notes* or *remarks*. The section counter (§) organizes things into many units, which function like the stories in a real diary.

The diary will update from time to time. The date appeared under the author's name is the date when it was last updated.

§1

Thorium 1.

$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$

Thorium 2.

$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$

Thorium 3.

$$(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4$$

§2

Thorium 4 (Binomial Theorem).

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + y^n \\ (x+y)^r &= x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \cdots\end{aligned}$$

§3

Thorium 5.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Thorium 6.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+2)}{6}$$

Thorium 7.

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

§4

Thorium 8.

$$\sum_{k=0}^{n-1} k^n = \frac{r^n - 1}{r - 1}$$

§5

Thorium 9 (Fundamental Theorem of Algebra). Every single-variable polynomial with complex coefficients has at least one complex root.

Equivalently:

Thorium 10. Every this kind of polynomial with degree n has, counted with multiplicity, exactly n complex roots.

§6

Thorium 11. If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

§7

Thorium 12 (Vieta's Formulas). Suppose the n roots of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ are $r_1, \cdots, r_n$, then

$$\begin{cases} r_1 + \cdots + r_n = -a_{n-1}/a_n \\ (r_1 r_2 + \cdots + r_1 r_n) + (r_2 r_3 + \cdots + r_2 r_n) + \cdots + r_{n-1} r_n = a_{n-2}/a_n \\ \vdots \\ r_1 r_2 \cdots r_n = (-1)^n a_0/a_n \end{cases}$$

Equivalently:

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} r_{i_j} \right) = (-1)^k \frac{a_{n-k}}{a_n}$$

§8

$$\frac{1}{n}\sum_{i=1}^{n} a_i = \text{arithmetic mean} = \text{AM}$$

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} = \text{geometric mean} = \text{GM}$$

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}} = \text{harmonic mean} = \text{HM}$$

Thorium 13.

$$\text{AM} \quad \geq \quad \text{GM} \quad \geq \quad \text{HM}$$

§9

Thorium 14.

$$\sin^2\alpha + \cos^2\alpha = 1$$

Thorium 15.

$$\sec^2\alpha - \tan^2\alpha = 1$$

Thorium 16.

$$\csc^2\alpha - \cot^2\alpha = 1$$

§10

Thorium 17.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Thorium 18.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Thorium 19.

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

§11

Thorium 20.

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

Thorium 21.

$$\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1\end{aligned}$$

Thorium 22.

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

§12

Thorium 23.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Thorium 24.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Thorium 25.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \\ &= \csc\alpha - \cot\alpha\end{aligned}$$

§13

Thorium 26.

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\pm\beta}{2}$$

Thorium 27.

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Thorium 28.

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

§14

Thorium 29. If $\alpha+\beta+\gamma=\pi$, then

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\cdot\tan\beta\cdot\tan\gamma$$

Thorium 30. If $\alpha + \beta + \gamma = \frac{\pi}{2}$, then

$$\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha \cdot \cot\beta \cdot \cot\gamma$$

§15

Thorium 31. Let $t = \tan\frac{\alpha}{2}$, then

$$\sin\alpha = \frac{2t}{1+t^2}$$
$$\cos\alpha = \frac{1-t^2}{1+t^2}$$
$$\tan\alpha = \frac{2t}{1-t^2}$$

§16

Thorium 32 (Law of Sines).

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\triangle}{abc} = \frac{1}{2R}$$

where $\triangle$ is the *area* of the triangle and R the *circumradius*.

§17

Thorium 33 (Law of Cosines).

$$a^2 + b^2 = c^2 + 2ab\cos C$$

§18

Thorium 34 (Law of Tangents).

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$$

§19

Thorium 35.

$$\begin{aligned}(s-a)\tan\frac{A}{2} &= (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}\\ = \zeta &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\end{aligned}$$

where s is the *semi-perimeter* of the triangle and ζ the *inscriberadius*.

§20

Thorium 36.

$$\sum_{k=1}^{n}\sin(k\alpha) = \frac{\cos\frac{\alpha}{2} - \cos(n+\frac{1}{2})\alpha}{2\sin\frac{\alpha}{2}}$$

§21

Thorium 37.

$$\sin^4\alpha - \cos^4\alpha = \sin^2\alpha - \cos^2\alpha$$

Thorium 38.

$$\sec^2\alpha + \csc^2\alpha = \sec^2\alpha\csc^2\alpha$$

Thorium 39.

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$$

§22

Thorium 40.

$$\lim_{x\to a} f(x) = L \quad\Longleftrightarrow\quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

§23

Thorium 41. If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exists as $x \to a$, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

§24

Thorium 42 (The Squeeze Theorem / The Sandwich Theorem). If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Erbium 1. Show that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

Proof. Since

$$-1 \leq \sin \frac{1}{x} \leq 1$$

and $x^2 > 0$ $(x \neq 0)$, we have

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

We also know that

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0$$

By the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

□

§25

A function f is called *continuous* at a if

$$\lim_{x \to a} f(x) = f(a)$$

§26

Thorium 43. If f is continuous at $b = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

In other words,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

§27

Thorium 44. Suppose a function f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

§28

Thorium 45. If f is differentiable at a, then it's continuous at a.

§29

Thorium 46.

$$\frac{\mathrm{d}}{\mathrm{d}x} c = 0$$

Thorium 47.

$$\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$$

Thorium 48.

$$(cf)' = c(f')$$

Thorium 49.

$$(f \pm g)' = f' \pm g'$$

Thorium 50.

$$(fg)' = f'g + fg'$$

Thorium 51.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

§30

Thorium 52.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Thorium 53.

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

§31

Erbium 2. Find

$$\lim_{x\to 0} \frac{\sin(7x)}{4x}$$

Solution.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin(7x)}{4x} \\
&= \frac{7}{4} \lim_{x\to 0} \frac{\sin(7x)}{7x} \\
&= \frac{7}{4} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \\
&= \frac{7}{4}
\end{aligned}$$

□

§32

Thorium 54.

$$\sin' x = \cos x$$

Thorium 55.

$$\cos' x = -\sin x$$

Thorium 56.

$$\tan' x = sec^2 x$$

Thorium 57.

$$\csc' x = -\csc x \cot x$$

Thorium 58.

$$\sec' x = \sec x \tan x$$

Thorium 59.

$$\cot' x = -\csc^2 x$$

§33

Thorium 60 (The Chain Rule).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

or

$$(f \circ g)' = (f' \circ g)g'$$

§34

Erbium 3. Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$$

Solution.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} \\
&= \lim_{x\to 0} \frac{\tan x - \sin x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}} \cdot \lim_{x\to 0} \frac{\tan x - \sin x}{x^3} \\
&= \frac{1}{2} \lim_{x\to 0} \frac{\sin x(\sec x - 1)}{x^3} \\
&= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sec x - 1}{x^2} \\
&= \frac{1}{2} \lim_{x\to 0} \frac{\sin^2 x}{x^2 \cos x(1+\cos x)} \\
&= \frac{1}{2} \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x\to 0} \frac{1}{\cos x(1+\cos x)} = \frac{1}{4}
\end{aligned}$$

□

§35

Thorium 61 (The Extreme Value Theorem). If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

§36

Thorium 62. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

§37

A *critical number* of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

§38

Thorium 63 (Rolle's Theorem). Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

§39

Thorium 64 (The Mean Value Theorem). Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Erbium 4. Prove the inequality

$$|\sin a - \sin b| \leq |a - b|$$

Proof. If $a = b$, then $|\sin a - \sin b| = |a - b| = 0$. Otherwise, by the Mean Value Theorem, there exists a number c for all a, b such that

$$\frac{|\sin a - \sin b|}{|a - b|} = |\sin' c| = |\cos c| \leq 1$$

□

§40

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a *slant asymptote*.

§41

A function F is called an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all x in I.

§42

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. Let $x_0(=a), x_1, x_2, \cdots, x_n(=b)$ be the *end points* of these subintervals and $x_1^*, x_2^*, \cdots, x_n^*$ be any *sample points* in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the *definite integral* of f from a to b is

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

§43

Thorium 65. If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.

§44

Thorium 66.

$$\int_a^b c\ \mathrm{d}x = c(b-a)$$

Thorium 67.

$$\int_a^b f(x) + g(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x$$

Thorium 68.

$$\int_a^b cf(x)\ \mathrm{d}x = c\int_a^b f(x)\ \mathrm{d}x$$

Thorium 69.

$$\int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x$$

§45

Thorium 70. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\ \mathrm{d}x \geq \int_a^b g(x)\ \mathrm{d}x$$

Thorium 71. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\ \mathrm{d}x \leq M(b-a)$$

Erbium 5. If f is continuous on $[a, b]$, show that

$$\left| \int_a^b f(x)\ \mathrm{d}x \right| \leq \int_a^b |f(x)|\ \mathrm{d}x$$

Proof. We have

$$-|f(x)| \leq f(x) \leq |f(x)|$$

Since f is continuous, it is integrable (§43 Thorium 65). By Thorium 70,

$$-\int_a^b |f(x)|\ \mathrm{d}x \leq \int_a^b f(x)\ \mathrm{d}x \leq \int_a^b |f(x)|\ \mathrm{d}x$$

Thus

$$\left| \int_a^b f(x)\ \mathrm{d}x \right| \leq \int_a^b |f(x)|\ \mathrm{d}x$$

□

§46 The Fundamental Theorem of Calculus

Thorium 72 (FTC-1). If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ \mathrm{d}t$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x)$$

Or using Leibniz's notation for derivatives,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\ \mathrm{d}t = f(x)$$

Thorium 73 (FTC-2). If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where F is any antiderivative of f.

Erbium 6. Find

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^{x^4} \sec t\,\mathrm{d}t$$

Solution. Let $u = x^4$. By the Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^{x^4} \sec t\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x} \int_1^{u} \sec t\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}u} \int_1^{u} \sec t\,\mathrm{d}t \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

By FTC-1,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^{x^4} \sec t\,\mathrm{d}t = \sec u \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 \sec x^4$$

□

§47

$$F(x)\Big]_a^b \quad = \quad F(x)\Big]_{x=a}^b \quad = \quad F(b) - F(a)$$

§48

$$\int f(x)\,\mathrm{d}x$$

is used for an antiderivative of f and is called an *indefinite integral*.

Thorium 74.

$$\int_a^b f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x\Big]_a^b$$

§49

Thorium 75.

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$

Thorium 76.

$$\int f(x) + g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x + \int g(x)\ \mathrm{d}x$$

Thorium 77.

$$\int k\ \mathrm{d}x = kx + C$$

Thorium 78.

$$\int x^r\ \mathrm{d}x = \frac{x^{r+1}}{r+1} + C \qquad (r \neq -1)$$

Thorium 79.

$$\int \sin x\ \mathrm{d}x = -\cos x + C$$

Thorium 80.

$$\int \cos x\ \mathrm{d}x = \sin x + C$$

Thorium 81.

$$\int \sec^2 x\ \mathrm{d}x = \tan x + C$$

Thorium 82.

$$\int \csc^2 x\ \mathrm{d}x = -\cot x + C$$

Thorium 83.

$$\int \sec x \tan x\ \mathrm{d}x = \sec x + C$$

Thorium 84.

$$\int \csc x \cot x\ \mathrm{d}x = -\csc x + C$$

§50

Thorium 85 (The Net Change Theorem). The integral of a rate of change is the net change:

$$\int_a^b F'(x)\ \mathrm{d}x = F(b) - F(a)$$

§51

Thorium 86 (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\ \mathrm{d}x = \int f(u)\ \mathrm{d}x$$

Erbium 7. Find

$$\int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

Solution. Let $u = 1 - 4x^2$. Then $\mathrm{d}u = -8x\mathrm{d}x$, so $x\mathrm{d}x = -\frac{1}{8}\mathrm{d}u$ and

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x &= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u = -\frac{1}{8}\int u^{-\frac{1}{2}}\,\mathrm{d}u \\ &= -\frac{1}{8}\cdot 2\sqrt{u} + C \\ &= -\frac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$

□

§52

Thorium 87 (The Substitution Rule for Definite Integrals). If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

Erbium 8. Evaluate

$$\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2}$$

Solution. Let $u = 3-5x$, then $\mathrm{d}u = -5\mathrm{d}x$, so $\mathrm{d}x = -\frac{1}{5}\mathrm{d}u$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\begin{aligned}\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2} &= -\frac{1}{5}\int_{-2}^{-7}\frac{\mathrm{d}u}{u^2} \\ &= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Big]_{-2}^{-7} \\ &= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}\end{aligned}$$

□

§53

Thorium 88. If f' is continuous on $[a,b]$, then

$$\int_a^b f(x)f'(x)\,\mathrm{d}x = \frac{f(b)^2 - f(a)^2}{2}$$

Proof. Let $u = f(x)$, then $\mathrm{d}u = f'(x)\mathrm{d}x$. So by the Substitution Rule,

$$\begin{aligned} \int_a^b f(x)f'(x)\,\mathrm{d}x &= \int_{f(a)}^{f(b)} u\,\mathrm{d}u \\ &= \frac{u^2}{2}\Bigg]_{f(a)}^{f(b)} \\ &= \frac{f(b)^2 - f(a)^2}{2} \end{aligned}$$

$\square$

§54

Thorium 89. The area between the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x$$

Thorium 90. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)\,\mathrm{d}x$$

Thorium 91 (Method of Cylindrical Shells). The volume of the solid obtained by rotating about the y-axis the region under the curver $y = f(x)$ from a to b is

$$V = \int_a^b 2\pi x f(x)\,\mathrm{d}x$$

$$\boxed{\int_a^b \underbrace{2\pi x}_{\text{circumference}} \quad \underbrace{f(x)}_{\text{height}} \quad \underbrace{\mathrm{d}x}_{\text{thinkness}}}$$

§55

The *average value* of f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Erbium 9. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x+h]$?

Solution. The average value of f on $[x, x+h]$ is

$$\bar{f} = \frac{1}{h} \int_x^{x+h} f(t)\ \mathrm{d}t$$

By FTC-2, if F is an antiderivative of f, then

$$\bar{f} = \frac{F(x+h) - F(x)}{h}$$

So

$$\lim_{h\to 0} \bar{f} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$

□

§56

Thorium 92 (The Mean Value Theorem for Integrals). If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x$$

That is,

$$\int_a^b f(x)\ \mathrm{d}x = f(c)(b-a)$$

§57

Thorium 93. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

§58

Thorium 94. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Or using Leibniz's notation,

$$\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

§59

Thorium 95.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

Thorium 96.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

Thorium 97.

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2}$$

Thorium 98.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad (-1 < x < 0 \text{ or } 0 < x < 1)$$

Thorium 99.

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}} \qquad (-1 < x < 0 \text{ or } 0 < x < 1)$$

Thorium 100.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{1+x^2}$$

§60

Thorium 101 (L'Hôpital's Rule). Suppose f and g are differentiable on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$).

§61

Thorium 102 (Cauchy's Mean Value Theorem). Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

§62

Thorium 103 (Integration by Parts).

$$\int f(x)g'(x)\ \mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\ \mathrm{d}x$$

or

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$$

Erbium 10. Find

$$\int \cos^3 x\ \mathrm{d}x$$

Solution. Let $u = \sin x$ so $\mathrm{d}u = \cos x \mathrm{d}x$. Then

$$\begin{aligned}\int \cos^3 x\ \mathrm{d}x = \int \cos^2 x \cdot \cos x\ \mathrm{d}x &= \int (1 - \sin^2 x) \cos x\ \mathrm{d}x \\ &= \int (1 - u^2)\ \mathrm{d}u = u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C\end{aligned}$$

□

§63

Thorium 104.

$$\int \sin^n x\ \mathrm{d}x = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\ \mathrm{d}x$$

Thorium 105.

$$\int_0^{\frac{\pi}{2}} \sin^n x\ \mathrm{d}x = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\ \mathrm{d}x$$

Thorium 106.

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, \mathrm{d}x = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

Thorium 107.

$$\int \tan^n x \, \mathrm{d}x = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, \mathrm{d}x$$

§64

Thorium 108.

$$\int \ln^n x \, \mathrm{d}x = x \ln^n x - n \int \ln^{n-1} x \, \mathrm{d}x$$

§65

Thorium 109.

$$\int \sec x \, \mathrm{d}x = \ln |\sec x + \tan x| + C$$

Proof. First we multipy numerator and denominator by $\sec x + \tan x$:

$$\begin{aligned} \int \sec x \, \mathrm{d}x &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, \mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, \mathrm{d}x \end{aligned}$$

If we substitude $u = \sec x + \tan x$, then $\mathrm{d}u = (\sec x \tan x + \sec^2 x) \, \mathrm{d}x$, so the integral becomes

$$\int \frac{1}{u} \, \mathrm{d}u = \ln |u| + C$$

Thus we have

$$\sec x \, \mathrm{d}x = \ln |\sec x + \tan x| + C$$

□

§66

Erbium 11. Calculate

$$\int \sec^6 x \, \mathrm{d}x$$

First integrate by parts by letting $u = \sec^4 x$ and $\mathrm{d}v = \sec^2 x\mathrm{d}x$ so that $\mathrm{d}u = 4\sec^5 x \sin x\mathrm{d}x = 4\sec^4 x \tan x\mathrm{d}x$ and $v = \tan x$:

$$\begin{aligned}\int \sec^6 x \, \mathrm{d}x &= \sec^4 x \tan x - \int 4\sec^4 x \tan x \cdot \tan x \, \mathrm{d}x \\ &= \sec^4 x \tan x - 4\int \sec^4 x \tan^2 x \, \mathrm{d}x\end{aligned}$$

Put $t = \tan x$ so $\mathrm{d}t = \sec^2 x\mathrm{d}x$ and $\sec^2 x = t^2 + 1$, then

$$\begin{aligned}\int \sec^6 x \, \mathrm{d}x &= \sec^4 \tan x - 4\int (t^2+1)t^2 \, \mathrm{d}t \\ &= \sec^4 x \tan x - \frac{4}{5}t^5 - \frac{4}{3}t^3 + C \\ &= (\tan^2 x + 1)^2 \tan x - \frac{4}{5}\tan^5 x - \frac{4}{3}\tan^3 x + C \\ &= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C\end{aligned}$$

§67

Thorium 110.

$$\int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x| + C$$

Proof.

$$\begin{aligned}\int \csc x \, \mathrm{d}x &= \int \frac{\sin x}{\sin^2 x} \, \mathrm{d}x = \int \frac{-1}{1-\cos^2 x} \, \mathrm{d}(\cos x) \\ &= \frac{1}{2}\int \frac{1}{\cos x + 1} - \frac{1}{\cos x - 1} \, \mathrm{d}(\cos x) \\ &= \frac{1}{2}(\ln|\cos x + 1| - \ln|\cos x - 1|) + C \\ &= \ln\sqrt{\left|\frac{1+\cos x}{1-\cos x}\right|} + C = \ln|\csc x - \cot x| + C\end{aligned}$$

□

§68

Thorium 111 (The Inverse Substitution Rule).

$$\int f(x) \, \mathrm{d}x = \int f(g(t))g'(t) \, \mathrm{d}t$$

§69

Erbium 12. The function $y = e^{x^2}$ and $y = x^2e^{x^2}$ don't have elementary antiderivatives, but

$$\int (2x^2+1)e^{x^2}\,\mathrm{d}x$$

does. Calculate this integral.

Solution.

$$\begin{aligned}\int (2x^2+1)e^{x^2}\,\mathrm{d}x &= \int e^{x^2}\,\mathrm{d}x + \int 2x^2e^{x^2}\,\mathrm{d}x\\ &= xe^{x^2} - \int 2x^2e^{x^2}\,\mathrm{d}x + \int 2x^2e^{x^2}\,\mathrm{d}x\\ &= xe^{x^2} + C\end{aligned}$$

$\square$

§70

A integral is called an *improper integral* if the interval is infinite (type I) or the integrand has an infinite discontinuity (type II). Definition of an improper integral of type I:

(a) If $\int_a^t f(x)\,\mathrm{d}x$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$$

provided this limit exists.

(b) If $\int_t^b f(x)\,\mathrm{d}x$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x$$

provided this limit exists.

(c) If both $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}t = \int_a^\infty f(x)\,\mathrm{d}x + \int_{-\infty}^a f(x)\,\mathrm{d}x$$

Any real number a can be used.

Definition of an improper integral of type II:

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\,\mathrm{d}x$$

if this limit exists.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\,\mathrm{d}x$$

if this limit exists.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ are convergent, then we define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

§71

Erbium 13. Calculate the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R.

Solution. The initial kinetic energy of the rocket is $\frac{1}{2}mv_0^2$. According to Newton's Law of Gravitation, the gravitational force given by the planet to the rocket which is a distance h away from the center of the planet is

$$F = G\frac{mM}{h^2}$$

where G is the gravitational constant. So

$$\begin{aligned}\frac{1}{2}mv_0^2 &= \int_R^\infty G\frac{mM}{h^2}\,\mathrm{d}h \\ &= GmM\left[-\frac{1}{h}\right]_R^\infty = \frac{GmM}{R} \\ v_0 &= \sqrt{\frac{2GM}{R}}\end{aligned}$$

$\square$

§72

Thorium 112.

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x \quad \text{is convergent if } p>1 \text{ and divergent if } p \le 1$$

So the region $\mathscr{R} = \{(x,y) : x \ge 1, 0 \le y \le 1/x\}$ has infinite area. But by rotating $\mathscr{R}$ about the x-axis we obtain a solid with finite volume

$$\begin{aligned} V &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 \mathrm{d}x \\ &= \lim_{t\to\infty} \left[-\frac{\pi}{x}\right]_1^t \\ &= \pi \end{aligned}$$

§73

Erbium 14. Evaluate

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,\mathrm{d}x$$

Solution. Let $e^x = \sec^2\theta$, then

$$\begin{aligned} e^x\mathrm{d}x &= 2\sec^2\theta\tan\theta\mathrm{d}\theta \\ \mathrm{d}x &= 2\tan\theta\mathrm{d}\theta \end{aligned}$$

So

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,\mathrm{d}x = \int_0^{\sec^{-1}\sqrt{10}} \frac{2\sec^2\theta\tan^2\theta}{\sec^2\theta+8}\,\mathrm{d}\theta$$

Substitude $y = \tan\theta$, so $\mathrm{d}y = \sec^2\theta\mathrm{d}\theta$ and $\sec^2\theta = y^2+1$, so

$$\begin{aligned} \int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,\mathrm{d}x &= \int_0^3 \frac{2y^2}{y^2+9}\,\mathrm{d}y = \int_0^3 2 - \frac{18}{y^2+9}\,\mathrm{d}y \\ &= 2y - 6\tan^{-1}\frac{y}{3}\Big]_0^3 = 6 - \frac{3\pi}{2} \end{aligned}$$

□

§74

Erbium 15. Evaulate

$$\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x$$

Solution. Integrate by parts by taking

$$\begin{aligned} u = xe^{2x} \quad &\implies \quad \mathrm{d}u = (1+2x)e^{2x}\mathrm{d}x \\ \mathrm{d}v = \frac{\mathrm{d}x}{(1+2x)^2} \quad &\implies \quad v = -\frac{1}{1+2x} \end{aligned}$$

We get

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{xe^{2x}}{(1+2x)^2}\,\mathrm{d}x &= -\frac{1}{2}\frac{xe^{2x}}{1+2x} + \int \frac{1}{2}e^{2x}\,\mathrm{d}x\Bigg]_0^{\frac{1}{2}} \\ &= -\frac{1}{2}\frac{xe^{2x}}{1+2x} + \frac{1}{4}e^{2x}\Bigg]_0^{\frac{1}{2}} = \frac{e}{8} - \frac{1}{4} \end{aligned}$$

□

§75

Erbium 16. Show that

$$\int_0^\infty \frac{\ln x}{1+x^2}\,\mathrm{d}x = 0$$

Proof. Write

$$\int_0^\infty \frac{\ln x}{1+x^2}\,\mathrm{d}x = I$$

Substitude $u = 1/x$,

$$I = \int_\infty^0 \frac{\ln u^{-1}}{1+u^{-2}} \cdot (-u^{-2})\,\mathrm{d}u = -\int_0^\infty \frac{\ln u}{1+u^2}\,\mathrm{d}u = -I$$

Thus

$$I = -I \quad \implies \quad I = 0$$

□