Some distributions that are used in models for daily stock returns

Lennart Hoogerheide

In this short note we consider some of the distributions that are used for daily stock returns r_t. We assume that the (conditional) distribution (conditional upon the information available at time $t-1$) has mean μ_t (which may simply be 0) and variance h_t. In the notation we drop the dependence on $r_{t-1}, r_{t-2}, \ldots$ and the parameters, so that the probability density function is simply denoted by $p(r_t)$.

Note: as mentioned during the lecture, it is advised to make use of $f_t = \log(h_t)$ (the natural logarithm of the variance) in the framework of the Generalized Autoregressive Score (GAS) model, so that we need the derivative

$$\nabla_t = \frac{\partial \log p(r_t)}{\partial f_t} = \frac{\partial \log p(r_t)}{\partial \log(h_t)} = \frac{\partial \log p(r_t)}{\partial h_t} \frac{\partial h_t}{\partial \log(h_t)} = \frac{\partial \log p(r_t)}{\partial h_t} h_t.$$

In that case one can make use of scaling factor $S_t = 1$, so that $s_t = S_t \cdot \nabla_t = \nabla_t$.

1 Normal/Gaussian distribution

The normal/Gaussian distribution has probability density function

$$p(r_t) = (2\pi h_t)^{-1/2} \exp\left(-\frac{(r_t - \mu_t)^2}{2h_t}\right),$$

with (natural) logarithm

$$\log p(r_t) = -\frac{1}{2}\log(2\pi h_t) - \frac{(r_t - \mu_t)^2}{2h_t}.$$

This distribution has skewness 0 and kurtosis 3. The thin tails of the normal distribution (corresponding to the relatively small kurtosis) imply that this distribution is typically not a wise choice for daily returns, especially if one is interested in the far/deep tails (for example, for a 99% or 99.5% Value at Risk).

2 Student-t distribution

The Student-t distribution with degrees of freedom parameter $\nu > 2$ has probability density function

$$p(r_t) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}((\nu-2)\pi h_t)^{-1/2}\left(1+\frac{(r_t-\mu_t)^2}{(\nu-2)h_t}\right)^{-\frac{\nu+1}{2}}$$

with natural logarithm

$$\log p(r_t) = \log\Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log((\nu-2)\pi h_t) - \frac{\nu+1}{2}\log\left(1+\frac{(r_t-\mu_t)^2}{(\nu-2)h_t}\right).$$

This distribution has skewness 0 or $\nu > 3$, the skewness is undefined for $2 < \nu \leq 3$. The kurtosis is $3+\frac{6}{\nu-4}$ for $\nu > 4$, the kurtosis is ∞ for $2 < \nu \leq 4$. The fat tails of the Student-t distribution imply that this distribution can be a wise choice for daily returns, as long as the empirical distribution of the daily returns is (almost) symmetric. For $\nu \to \infty$ the distribution tends to the normal distribution.

3 Generalized Error Distribution (GED)

The Generalized Error Distribution (GED) with shape parameter $\nu > 0$ (note: in case of the GED this is **not** called a degrees of freedom parameter) has probability density function

$$p(r_t) = \left(2^{(1+(1/\nu))}\Gamma(1/\nu)\lambda\right)^{-1}\ h_t^{-1/2}\ \nu\exp\left(-\frac{1}{2}\left|\frac{r_t-\mu_t}{\lambda\, h_t^{1/2}}\right|^{\nu}\right)$$

with

$$\lambda = \left(\frac{\Gamma(1/\nu)}{2^{2/\nu}\Gamma(3/\nu)}\right)^{1/2}.$$

The natural logarithm is

$$\log p(r_t) = -\log\left(2^{(1+(1/\nu))}\Gamma(1/\nu)\lambda\right) - \frac{1}{2}\log(h_t) + \log(\nu) - \frac{1}{2}\left|\frac{r_t-\mu_t}{\lambda\, h_t^{1/2}}\right|^{\nu}.$$

Note: λ is not a separate parameter, it is simply a function of ν. This distribution has skewness 0. The kurtosis is $\frac{\Gamma(1/\nu)\Gamma(5/\nu)}{(\Gamma(3/\nu))^2}$. The GED has fatter tails (and larger kurtosis) than the normal distribution if $0 < \nu < 2$. The GED is the normal distribution if $\nu = 2$. The GED has thinner tails (and smaller kurtosis) than the normal distribution if $\nu > 2$.

The fat tails of the GED (if $0 < \nu < 2$) imply that this distribution can be a wise choice for daily returns, as long as the empirical distribution of the daily returns is (almost) symmetric.

4 Skewed Student-t distribution

The skewed Student-t distribution with $\nu > 2$ degrees of freedom and asymmetry parameter $\xi > 0$ has probability density function:

$$p(r_t) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}((\nu-2)\pi h_t)^{-1/2} \times s \times \left(\frac{2}{\xi + \frac{1}{\xi}}\right) \times \left(1 + \frac{\left(s\frac{r_t-\mu_t}{h_t^{1/2}} + m\right)^2}{\nu - 2}\xi^{-2I_t}\right)^{-\frac{\nu+1}{2}}$$

with

$$\begin{aligned}
I_t &= \begin{cases} 1 & \text{if } s\frac{r_t-\mu_t}{h_t^{1/2}} + m \geq 0 \\ -1 & \text{if } s\frac{r_t-\mu_t}{h_t^{1/2}} + m < 0 \end{cases} \\
m &= \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})} \times \sqrt{\frac{\nu-2}{\pi}} \times \left(\xi - \frac{1}{\xi}\right) \\
s &= \sqrt{\left(\xi^2 + \frac{1}{\xi^2} - 1\right) - m^2}.
\end{aligned}$$

The natural logarithm is

$$\begin{aligned}
\log p(r_t) &= \log\Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log((\nu-2)\pi h_t) \\
&\quad + \log(s) + \log\left(\frac{2}{\xi + \frac{1}{\xi}}\right) \\
&\quad - \frac{\nu+1}{2}\log\left(1 + \frac{\left(s\frac{r_t-\mu_t}{h_t^{1/2}} + m\right)^2}{\nu-2}\xi^{-2I_t}\right).
\end{aligned}$$

Note: m and s are not separate parameters, these are simply functions of ν and ξ.

For $\xi = 1$, this distribution reduces to the symmetric Student-t distribution with degrees of freedom parameter ν (with $m = 0$ and $s = 1$). For $\xi > 1$ the distribution has a positive skewness (as long as $\nu > 3$). For $0 < \xi < 1$ the distribution has a negative skewness (as long as $\nu > 3$).

The fat tails and possible asymmetry imply that this distribution can be a wise choice for daily returns.

5 Exponential Generalized Beta distribution of the second kind (EGB2)

The Exponential Generalized Beta distribution of the second kind (EGB2) with shape parameters $p > 0$ and $q > 0$ has probability density function:

$$p(r_t) = \frac{\sqrt{\Omega}\exp\left(p\left(\sqrt{\Omega}\frac{r_t-\mu_t}{\sqrt{h_t}} + \Delta\right)\right)}{\sqrt{h_t}B(p,q)\left(1+\exp\left(\sqrt{\Omega}\frac{r_t-\mu_t}{\sqrt{h_t}} + \Delta\right)\right)^{p+q}}$$

with

$$\begin{aligned} \Delta &= \psi(p) - \psi(q) \\ \Omega &= \psi'(p) + \psi'(q) \end{aligned}$$

where $\psi()$ is the digamma function (the first order derivative of the logarithm of the Gamma function $\log\Gamma(.)$) and $\psi'()$ is the trigamma function (the second order derivative of the logarithm of the Gamma function $\log\Gamma(.)$). $B(p,q)$ is the Beta function $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

The natural logarithm is

$$\begin{aligned} \log p(r_t) &= \frac{1}{2}\log(\Omega) + p\left(\sqrt{\Omega}\frac{r_t-\mu_t}{\sqrt{h_t}} + \Delta\right) \\ &\quad -\frac{1}{2}\log(h_t) - \log B(p,q) - (p+q)\log\left(1+\exp\left(\sqrt{\Omega}\frac{r_t-\mu_t}{\sqrt{h_t}} + \Delta\right)\right). \end{aligned}$$

The distribution is symmetric if $p = q$. If $p = q \to \infty$, then the distribution tends to a normal distribution. The skewness is positive if $p > q$. The skewness is negative if $p < q$. The skewness lies between -2 and 2. The kurtosis can take values up to 9.[1]

The fat tails (with kurtosis up to 9) and possible asymmetry imply that this distribution can be a wise choice for daily returns (as long as one does not need an extremely fat-tailed distribution).

[1]The skewness is equal to

$$\frac{\psi''(p) - \psi''(q)}{(\psi'(p) + \psi'(q))^{3/2}}.$$

The kurtosis is equal to

$$3 + \frac{\psi'''(p) + \psi'''(q)}{(\psi'(p) + \psi'(q))^2},$$

where $\psi''()$ and $\psi'''()$ are the third and fourth order derivatives of the logarithm of the Gamma function $\log\Gamma()$.