Individual Assignment Set 1

Question 1:

Suppose we want to price a 1-year forward FX contract where we want to sell US dollars and buy EUR. The current EURUSD exchange rate is denoted by FX_t, the EUR 1-year risk free rate is denoted by r_t^{EUR} and the USD 1-year is denoted by r_t^{USD}. Note that these interest rates are per annum rates with annual compounding. This means that EUR 1, invested against r_t^{EUR}, has grown to $(1 + r_t^{EUR})$ after 1 year.

(a) Use this information to derive the no-arbitrage forward FX exchange rate, such that there is no exchange of cash at time t, the inception date of the contract.
(b) Suppose that the market forward FX exchange rate is higher than the no-arbitrage price. Provide the arbitrage strategy and show that this strategy is a money machine.

Solution:

We employ the replication method. Suppose you would like to sell 1 USD for an amount of EUR in 1 year from now. We can replicate this by borrowing $\frac{1}{1+r_t^{USD}}$ right now. We want to know the guaranteed amount of EUR that we can generate with this borrowed amount of money. We can convert to borrowed USD into EUR as follows:

$$\frac{1}{1 + r_t^{USD}} \cdot \frac{1}{FX_t}.$$

This amount of money grows to:

$$\frac{1}{1 + r_t^{USD}} \cdot \frac{1}{FX_t}.(1 + r_t^{EUR}),$$

in one year from now.

So, by the end of year one we have:
- -1 USD
- $\frac{1}{1+r_t^{USD}} \cdot \frac{1}{FX_t}.(1 + r_t^{EUR})$ EUR

Hence, by no arbitrage the FX-forward contract should trade at an forward FX rate of:

$$FFX_{t:t+1} = FX_t \cdot \frac{1 + r_t^{USD}}{1 + r_t^{EUR}}.$$

Suppose that:

$$FFX_{t:t+1} > FX_t \cdot \frac{1 + r_t^{USD}}{1 + r_t^{EUR}}.$$

This means that entering into an forward FX contract delivers less EUR in one year than employing the replicating strategy. Hence, you go long the strategy and short the FX-forward contract. Hence,

- You enter into a forward FX-contract in which you buy 1 USD and sell $\frac{1}{FFX_{t:t+1}}$ EUR in 1 year from now.
- You borrow $\frac{1}{1+r_t^{USD}}$ right now
- Convert this into $\frac{1}{1+r_t^{USD}} \cdot \frac{1}{FX_t}$ EUR

Note that the starting value of this portfolio is equal to 0.

After one year (in EUR):

$$\frac{1}{1+r_t^{USD}} \cdot \frac{1}{FX_t} . (1+r_t^{EUR}) - \frac{1}{FFX_{t:t+1}} > 0.$$

Hence, we have an arbitrage opportunity.

Question 2:

If we want to calculate the time t no-arbitrage price of a payoff V at a future time $T > t$ using the pricing kernel method we need to evaluate:

$$V_t = E_t^P\left(V_T \frac{\pi_T}{\pi_t}\right).$$

See slides and clips for more details. Denote $r_{t:T}^F$ as the risk free rate between times t and T. Show that:

$$E_t^P\left(\frac{\pi_T}{\pi_t}\right) = \frac{1}{1+r_{t:T}^F}.$$

Solution:

The no-arbitrage formula should also hold for a risk free asset, for instance an asset that pays off 1 at time $t = T$. This gives:

$$V_t = E_t^P\left(1\frac{\pi_T}{\pi_t}\right) = E_t^P\left(\frac{\pi_T}{\pi_t}\right).$$

We also know that this risk free asset has rate of return equal to $r_{t:T}^F$. So, we know that the current price of the risk free pay off should be:

$$V_t = \frac{1}{1+r_{t:T}^F}.$$

Equating both expressions for V_t gives us the desired result.

Question 3:

Suppose we want to price a 1-year equity forward contract using the risk-neutral valuation method. The underlying value is denoted by S, is EUR denominated and pays no dividends. The EUR 1-year risk free rate is denoted by r_t^{EUR}. Note that this interest rates is a per annum rate with annual compounding. Derive the no-arbitrage reference price of the forward contract such that there is no exchange of cash at inception. As said, you are asked to apply the risk-neutral valuation method.

Solution:

From the slides and the clips we know that the no-arbitrage value, with the given interest rate, can be found by evaluating:

$$V_t = \frac{1}{1 + r_t^{EUR}} E_t^Q(V_{t+1}),$$

where t denotes the inception date of the forward contract and $t + 1$ its maturity date. Inserting the payoff of the equity forward:

$$V_t = \frac{1}{1 + r_t^{EUR}} E_t^Q(S_{t+1} - K_{t:t+1}),$$

We want to choose $K_{t:t+1}$ in such a way that there is no exchange of cash at the start, i.e. $V_t = 0$.

Hence, we need to solve:

$$0 = \frac{1}{1 + r_t^{EUR}} \left(E_t^Q(S_{t+1}) - K_{t:t+1}\right),$$

Reshuffling:

$$K_{t:t+1} = E_t^Q(S_{t+1}).$$

We know that under the risk-neutral probability measure all assets earn the risk free rate (see slides and clips):

$$K_{t:t+1} = S_t \cdot (1 + r_t^{EUR}).$$