Stochastic Processes: The Fundamentals Cheatsheet

Arbitrage

An arbitrage portfolio is a portfolio h with the properties

$$\begin{cases} V_0^h = 0 \\ V_1^h > 0 \,\text{with}\, \mathrm{P} = 1 \end{cases}$$

Thus it is a portfolio with guaranteed profit.

Law of one price

The law of one price says that if portfolios or instruments have the same payoff these portfolios or instruments should have the same price. This principle is applied a lot when pricing derivatives: the no-arbitrage of derivatives can in quite a few cases be derived from the price of base instruments like stocks, bond and cash.
The pricing method is also known as the **replication method**"construct a portfolio of base instruments that replicate the payoff of the derivative perfectly.

Capital Asset Pricing model

Where do prices in market come from? One example is the **Capital Asset Pricing Model**

$$\mathbb{E}(R^i) = R^f + \beta^i \mathbb{E}(R^M - R^f)$$

Which states that the expected return R on a stock i is fully determined by its β. There is thus only one factor that drives the cross-section of expected stock returns.
From modelling perspective: all investors have the same view on the probability distribution of all risky payoffs and care only about mean and variance. The composition of the risky part of each portfolio is exactly the same: this is the market portfolio.

Equity forwards

The long position has the obligation to buy the underlying stock at a future moment for a price agreed upon today. The reference price is usually set in a way such that there is no exchange of cash at start. Payoff equals

$$V(T) = S(T) - K$$

And is the same as call option without the optionality! FX-forward contract is equity forward contract in currency. Two parties agree to exchange money in different currencies at a future point in time. Obligation for both parties. Currencies and amounts are set at start of contract such that there is no exchange of cash at start. Payoff of long position in FX-forward contract is given by

$$V(T) = N_T^{USD}\left(\frac{1}{K^{FFX}} - \frac{1}{FX_T}\right)$$

Given that we normally want the cash involvement at start to be zero, we can use

$$V(t) = 0 = e^{-r(T-t)}\mathbb{E}_t\big[S_T - K\big] = S_t - Ke^{-r(T-t)}$$

Call and put option

Call option is same as equity forward but with optionality and thus pay off structure

$$V(T) = \max(S(T) - K, 0)$$

Put option identical but for decaying prices. It is obvious that from a no-arbitrage perspective, the call and put options should trade at a positive value. Otherwise the profit and loss of the investor would always be positive at maturity.

Risk valuation

Consider two investment opportunities: 1 with guaranteed no loss and no gain and one with 90% probability of gain and 10% probability of 100% loss. The expected values (payoffs) would be the same, but the price of the two investment opportunities would depend on how risk is valued by investor. Two essential concepts:

- Compensation for putting away money: **nominal interest rates**
- Compensation for pain of losing money: **risk premia**

For a risk neutral investor: The fair price is given by discounting the expected payoff by the nominal risk free rate. For a risk averse investor, we would require compensation and thus a lower price of the product. the more risk averse an investor is, the less he is willing to pay for the risky investment opportunity.

Linear pricing rule

A positive linear pricing rule is described by

$$\nu = \eta G$$

Here, G is a payoff matrix and η maps the payoffs to the no arbitrage prices in ν. A positive linear pricing rule implies the existence of a positive pricing kernel. We can define **risk-neutral probabilities** via

$$q_i = \frac{\eta_i}{\sum_{i=1}^m \eta_i}$$

Pricing kernel method

The following statements are equivalent (**Fundamental Theorem of Finance**)

- No arbitrage
- The existence of a positive linear pricing rule
- The existence of a finite positive optimal demand positive for some agent who prefers to more to less

In addition, we have the the **replication theorem** that says

- The existence of a positive linear pricing rule
- The existence of a positive pricing kernel
- The existence of positive risk-neutral probabilities and an associated risk free rate

Using the pricing kernel, we have that the price of a future payoff is given by

$$V_t = \mathbb{E}_t^{\mathbb{P}}\left(V_T \frac{\pi_T}{\pi_t}\right)$$

Where π represents the pricing kernel. A high value of the pricing kernel means that an invetor appreciates a positive payoff. To find no arbitrage price of equity forward, one solves

$$0 = \mathbb{E}_t^{\mathbb{P}}\left((S_T - K_{t,T})\frac{\pi_T}{\pi_t}\right)$$

And find

$$K_{t,T} = S_t e^{r(T-t)}$$

Risk-neutral valuation

We can use the risk-neutral valuation method to derive the no-arbitrage price of a derivative:

$$q_u = \frac{(1+r)-d}{u-d}$$

Binomial Tree model

Binomial tree model assumes a simple stochastic process for the underlying asset of the derivative. Two assets only: stock and bond and we are discrete in time. We have that

$$d \leq 1+r \leq u$$

The alternative would lead to arbitrage. In one timestep, we get that $S \to uS$. To get no-arbitrage price, we use replication method and find

$$X_0 = \frac{1}{1+r}\left[\tilde{\mathrm{p}}V_1(H) + (1-\tilde{\mathrm{p}})V_1(T)\right]$$

Here, the probability for an upmove is given by

$$\tilde{\mathrm{p}} = q = \frac{1+r-d}{u-d}$$

Which are also called **risk-neutral probabilities**. In the case of a multi-period binomial tree model, little changes. We get more discounting and start calculating at the final step and work backwards to find the initial price.

Probability theory

- Conditional expectation of X given $\mathcal{G}$ is denoted by $\mathbb{E}[X|\mathcal{G}]$
- $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable
- Ω is set of all possible outcomes: **sample space**
- $\mathcal{F}$ is a **collection of events**
- $\mathcal{F}$ is an **algebra** is:
  - $\emptyset \in \mathcal{F}$
  - $A \in \mathcal{F} \to A^c \in \mathcal{F}$
  - $A_i \in \mathcal{F}$ for $i = \{1, ...N\} \to \cup_i^N A_i \in \mathcal{F}$

An increasing set of algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_1 ...$ is called a **filtration**. A finite **probability space** consists of three elements:

- Ω: nonempty set called the sample space
- $\mathcal{F}$: an algebra of subsets of Ω
- $\mathbb{P}$: a probability measure on $(\Omega, \mathcal{F})$

Probability theory

In the case of the binomial three, S_i is a **discrete time stochastic process** with **discrete values**. The probability to be at node $k = 0, ..., n$ after n steps is

$$\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

If X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, its expectation is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$$

An **adapted stochastic process** is a collection of random variables. This means that the information at time t is sufficient to evaluate the random variable $X(t)$. A stochastic process X_n is a **martingale** if

- X_n depends only on the first n iterations, i.e. is adapted to $\mathcal{F}_n$
- $\mathbb{E}_m[X_n] = X_m$, i.e. the best estimate for X_n is X_m where $m \leq n$

A stochastic process X_n adapted to filtration $\mathcal{F}_n$ is a **Markov process** if the distribution of X_{n+1} conditioned on $\mathcal{F}_n$ is the same as the distribution of X_{n+1} conditioned on X_n. This means that the future behavior of X_n does not depend on how the process arrived at the point X. The future depends only on the current value. In the multi-period binomial tree, S is a Markov process.

Continuous random variables

One way to move to a continuous framework is to make more and more steps in the binomial tree. However there is a crucial difference between this extended space Ω_N of random steps: the interval of real numbers $[0, 1]$ is countable and thus is Ω_N countable, whereas an interval of real numbers consists of uncountable number of points. In the infinite sample space we are not able to determine the probability of a subset A by summing the probabilities of all elements in A. We should define the **probability of subsets**. We can define the probability of some simple subsets and apply the properties to determine the probability of more complicated subsets.

Borel algebra

The Borel σ-algebra of subsets of $[0,1]$ is the collection of all sets whose probability is determined by the Lebesgue probability measure. Borel algebra includes single points:

$$X_n = [a - 1/n, a + 1/n]$$

which converges to a single point a. Open intervals also belong to the Borel algebra. A random variable X on a measurable space $(\Omega, \mathcal{F})$ is a mapping

$$X : \Omega \to \mathbb{R}$$

such that X if $\mathcal{F}$-measurable and $\mathbb{R}$ is the state space of X.
If B is the σ-algebra generated by closed intervals in $\mathbb{R}$, then the distribution measure μ_X for a random variable X is the probability measure that assigns to each Borel subset B of $\mathbb{R}$ the mass.

$$\mu_X(B) = \mathbb{P}\{\omega \in \Omega; X(\omega) \in B\}$$

Riemann integral

If $f(x)$ is a continuous function defined for all x in the closed interval $[a,b]$, the **Riemann integral** $\int_a^b f(x)dx$ is defined as follows

$$\sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

Where M_k can be either $max(f(x))$ of $min(f(x))$ **inside** the subinterval $[x_{k-1}, x_k]$ depending on whether we evaluate the upper or lower Riemann sum.

Lebesgue sum

If we also partition the y-axis, we reach the Lebesgue sum.

$$LS_\Pi = \sum_{k=1}^{\infty} y_k \mathbb{P}(A_k)$$

Going from finite to infinite

$$\mathbb{E}(X) = \int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$$

Conditional expectations

Conditional probability $\mathbb{P}(A|B)$ for event A given B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Likelihood for unaffected events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Conditional expectation

$$\mathbb{E}[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X(\omega) d\mathbb{P}(\omega)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a random variable. Let $\mathcal{G}$ be a σ-algebra such that $\mathcal{G} \subseteq \mathcal{F}$. If Z is a random variable such that

- Z is $\mathcal{G}$-measurable
- For all $B \in \mathcal{G}$ it holds that

$$\int_B Z(\omega) d\mathbb{P}(\omega) = \int_B X(\omega) d\mathbb{P}(\omega)$$

then we say that Z is the conditional expectation of X given the σ-algebra $\mathcal{G}$ and we denote is as

$$\mathbb{E}[X|\mathcal{G}]$$

Random walk

The **Black-Scholes** model is built around the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

It is a continuous time model. Brownian motion is a continuous time random walk with independent increments in nonoverlapping time intervals. Each increment has expectation zero

$$\mathbb{E}(M_{k_{i+1}} - M_k) = \mathbb{E}\left(\sum_{j=k_i+1}^{k_{i+1}} X_j\right) = \sum_{j=k_i+1}^{k_{i+1}} \mathbb{E}(X_j) = 0$$

The variance is equal to the time difference:

$$Var(M_{k_{i+1}} - M_k) = k_{i+1} - k_i$$

And expectation of M_l is M_k (Martingale)

Brownian Motion

To approximate a Brownian motion process, we scale down the step size and speed up the time.

$$W^n(t) = \frac{1}{\sqrt{n}} M_{nt}$$

This leads to the scales symmetric random walk. It has independent increments and expectations of increments of zero. The variance of increments equals the time span and it is a martingale. The quadratic variation also equals the time span. The quadratic variation for W^{100} up to time 4 is given by

$$[W^{100}, W^{100}](4) = \sum_{j=1}^{400} \left[W^{100} \frac{j}{100} - W^{100} \frac{j-1}{100}\right]^2$$

$$= \sum_{j=1}^{400} \left[\frac{1}{10} X_j\right]^2 = \sum_{j=1}^{400} \left[\frac{1}{100}\right] = 4$$

In the limit we have a continuous time stochastic process. At each time point t the random variable W_t^∞ has a normal distribution with expectation zero and variance t. And this is Brownian motion: the limit of scaled random walks $W^n(t)$ as $n \to \infty$. The variance of a Brownian motion is not a martingale. The process $W^2(t) - t$ is however a Martingale.

Stochastic Integral

A general version of a stochastic integral is

$$\int_0^T \sigma(T) dW(t)$$

If $\sigma(t)$ is a continuous function on $[0,T]$, we approximate it using a piecewise function $\sigma_n(t)$. The stochastic integral becomes the limit

$$Z(T) = lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma_n(t_i, t_{i+1})(W(t_{i+1} - W(t_i))$$

Thus we find that $Z_n(T) \to Z(T)$. And as a consequence, we have

$$Z_n(s,t) = W(t_n) - W(t_0)$$

Stochastic Integral

The stochastic integral is a random variable because it depends on random variables. The expected value of a stochastic integral is 0 and the variance if, by Itô's isometry equal to

$$\mathbb{E}\Big[\Big(\int_0^T g(s)dW(s)\Big)^2)\Big] = \int_0^T \mathbb{E}\big[g(s)^2\big]ds$$

Using these Brownian Motion we can get more interesting processes. e.g.

$$dX(t) = \mu(\cdot)dt + \sigma(\cdot)dW(t)$$

Which is Brownian motion with a drift (μ) and diffusion (σ) term.

More on Brownian motion

The variance of Brownian motion

$$\mathbb{E}\Big[W^2(t)|\mathcal{F}_s\Big] \geq W^2(s)$$

and is thus not a Martingale. However, $\mathbb{E}\Big[W^2(t)|\mathcal{F}_s\Big] = \mathbb{E}\Big[(W(t) - W(S))^2 + 2(W(t) - W(s))(W(s) + W^2(s)|\mathcal{F}_s\Big]$ shows that

$$W^2(t) - t$$

is a martingale. Brownian motion is also a Markov process. The covariance of two random variables $W(s)$ and $W(u)$ for any two times is

$$\mathbb{E}\Big[W(s)W(u)\Big] = min(s, u)$$

And the correlation

$$\rho = \frac{\mathbb{E}\Big[W(s)W(u)\Big]}{\sqrt{Var\Big[W(s)\Big]Var\Big[W(u)\Big]}} = \sqrt{\frac{s}{u}}$$

The quadratic variation is found using the limit $n \to \infty$ of

$$\Big[W, W\Big]_T = \sum_{j=0}^{n-1}\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]^2$$

More on Brownian motion

Then it is easy to see

$$FV_T(f) = \int_0^T |f'(t)|dt$$

There is no value for which $\frac{d}{dt}W(t)$ is defined because the Brownian motion is erratic / pointy / spiky. MOst functions have continuous derivatives and hence their quadratic variation is zero. For Brownian Motion this does not hold and we in general have

- First order variation: ∞ for all $T > 0$
- Quadratic variation: $\Big[W, W\Big]_T = T$

We can define conditional and cumulative distribution functions using the notion of normal increments.

$$p(t, x, T, y)dy = \frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-x)^2}{2(T-t)}}dy$$

$$\mathbb{P}\big[W_T < z|W_t = x\big] = N\left(\frac{z-x}{\sqrt{T-t}}\right)$$

Because of the symmetry of the distribution of the increments, we have

$$\mathbb{E}\Big[W_t^{2n+1}\Big] = 0$$

If W_t is a Brownian motion, so are $-W_t$ and $\frac{1}{\sqrt{c}}W_{ct}$.

Stochastic vs ordinary calculus

Recall the general Taylor expansion of $f(x)$:

$$dF(t, x) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \mathcal{O}(dt, dx)$$

Here, $\mathcal{O}(dt, dx)$ contains all cross terms that are smaller than dt and dx: $dtdx, dx^2, dt^2$. These are ignored in ordinary calculus. If $X(t)$ is a stochastic process in differential form

$$dX(t) = \mu dt + \sigma dW(t)$$

we need to add the stochastic second order term

$$dF(t, x) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}dx^2 + ...$$

Itô's lemma

We can substitute the differential equation for $X(t)$ into the Taylor series. We set $dt^2 = 0$, $dtdW = 0$ and $dW^2 = dt$ to get **Itô's formula**

$$dF = \frac{\partial F}{\partial t}dt + \mu\frac{\partial F}{\partial X}dt + \sigma\frac{\partial F}{\partial X}dW + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 dt$$

An intuition for setting $dW^2 = dt$ is that the variance of the sum of Brownian motion increments converges to t. Example of using Itô's formula. Consider

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

and $Y(t) = f(t) = ln(X(t))$. What is $dY(t)$? Using Itô's formula we have that

$$\frac{\partial f}{\partial x} = \frac{1}{x},\ \frac{\partial^2 f}{\partial x^2} - \frac{-1}{x^2},\ \frac{\partial f}{\partial x} = 0$$

Such that

$$dY(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)$$

$$Y(T) = Y(0) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\left(W(T) - W(0)\right)$$

Suppose we have M stochastic processes and we want to model them with N Wiener processes. We can write the processes in vector form

$$d\vec{X}(t) = \vec{\mu}dt + \hat{\sigma}d\vec{W}(t)$$

$$dF = \frac{\partial F}{\partial t}dt + \sum_i \frac{\partial F}{\partial X_i}dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 F}{\partial X_i \partial X_j}dX_i dX_j$$

Where W_i and W_j are in the most general form correlated

$$\mathbb{E}_t\Big[dW_i(t) \cdot dW_j(t)\Big] = \rho_{ij}dt$$

dS in BS world

Under the neutral probability, we have

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

And in the BS world

$$V(t) = F(t, S(t))$$

Stochastics

A stochastic process X_t is called an $\mathcal{F}_t$-martingale if it satisfies

$$\mathbb{E}_s\big[X(t)\big] = \mathbb{E}_s\big[X(t)|\mathcal{F}_s\big] = X(s)$$

A stochastic process Z defined as

$$Z(t) - \int_0^t g(u)dW(u)$$

also is an $\mathcal{F}_t$-martingale. This is relevant as we can try to model stock returns as continuous time process

$$dS(t) = \sigma dW(t)$$

$$S(t) - S(0) = \sigma \int_0^t dW(u)$$

We need to add a drift term (risk premia) and continuous time (to avoid negative returns of over 100%)

$$logS(t+h) = logS(t) + \int_t^{t+h} \mu du + \int_t^{t+h} \sigma dW(u)$$

Radon-Nikodyn theorem

The **Radon-Nikodyn Theorem** relates two measures. Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let Z be an almost surely nonnegative random variable with $\mathbb{E}(Z) = 1$. For $A \in \mathcal{F}$, we define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$$

Then, $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}(X) = \mathbb{E}(XZ)$$

If Z is strictly positive, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent.

- Two measures are equivalent: both measures agree on which sets in $\mathcal{F}$ have zero probability
- Measures may disagree on how probable the possibilities are
- We talk of real world and risk neutral world
- The risk neutral measure $\tilde{\mathbb{P}}$ is for the purpose of pricing derivatives

Greeks

The sensitivies of the option portfolio:

$$\Delta = \frac{\partial V}{\partial S} = \mathcal{N}(d_1)$$

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}$$

$$\rho = \frac{\partial V}{\partial r} = K(T-t)e^{-r(T-t)}\mathcal{N}(d_2)$$

$$\Theta = \frac{\partial V}{\partial t} = -\frac{S\phi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\mathcal{N}(d_2)$$

$$\nu = \frac{\partial V}{\partial \sigma} = S\phi(d_1)\sqrt{T-t}$$

Casino

If we cannot develop an equivalent risk free measure, there is an arbitrage opportunity. Consider Roulette with three colors: red, black and green with probabilities $\mathbb{P}_{red} = \mathbb{P}_{black} = 18/37$ and $\mathbb{P}_{green} = 1/37$ and suppose we can only bet on red and black. The payoff structure is as follows: we bet 1 and receive 2 if the outcome is our color, otherwise we receive 0.
We can compute probabilities under the risk-neutral measure $\mathbb{Q}$

$$V(red) = 2 \times P^{\mathbb{Q}}_{red} + 0 \times P^{\mathbb{Q}}_{black} + 0 \times P^{\mathbb{Q}}_{green} = 1$$

$$V(black) = 0 \times P^{\mathbb{Q}}_{red} + 2 \times P^{\mathbb{Q}}_{black} + 0 \times P^{\mathbb{Q}}_{green} = 1$$

From this we find that the risk-neutral probabilities are $P^{\mathbb{Q}}_{red} = P^{\mathbb{Q}}_{black} = 1/2$. However, then we have that $P^{\mathbb{Q}}_{green} = 0$. Hence, $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent since under $\mathbb{Q}$ we cannot have green, which can happen under $\mathbb{P}$. So there is an arbitrage opportunity by going short in the assets: this is exactly what a casino does!

Replication

We can hedge a call option using $\Delta(t) = \frac{\partial C}{\partial S}$. It is important for the success of the replication method that there is just one source of risk driving the randomness in our market and that we have two primary assets. We say that the market is complete.

Black-Scholes

For a stock with price S following geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

and Itô's formula we can get the differential of a calll option

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 =$$

$$\left(\frac{\partial C}{\partial t} + \mu S(T)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S(t)\frac{\partial C}{\partial S}dW(t)$$

The solution to this is the famous Black-Scholes model

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

With

$$d_1 = \frac{lnS - lnK + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

If an investor sold a call option, the portfolio can be made riskless by investing $\frac{\partial C}{\partial S}$ units in stock. We usually refer to the money market account or bond interchangeably:

$$dB(t) = rB(t)dt$$

Under the probability measure $\mathbb{Q}$, the no-arbitrage price of a European call option is given by

$$C(t, S(t)) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_t(max(S(T) - K, 0))$$

To evaluate the risk-neutral expectation we need to find the process S under the risk-neutral probability measure.

Radon-Nikodyn theorem

If we have that

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

or

$$d\tilde{\mathbb{P}}(\omega) = Z(\omega) d\mathbb{P}(\omega)$$

We say that Z is the **Radon-Nikodym derivative** of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ and we write

$$Z(\omega) = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

Which implies

$$\tilde{\mathbb{P}}(A) = \sum_{\omega \in A} Z(\omega) \mathbb{W}(\omega)$$

This shows that is we multiply the original probability with an outcome-dependent factor Z, we will get the adjusted probability.

Girsanov's theorem

In general we can derive processes under the new probability measure $\tilde{\mathbb{P}}$ by applying **Girsanov's theorem**:
Let $W(t)$ be a Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and het $\mathcal{F}(t)$ be a filtration for this Brownian Motion. Let $\Theta(t)$ be an adapted process and define

$$Z(t) = exp\left(-\int_0^t \Theta(u) dW(u) - \frac{1}{2}\int_0^t \Theta(t)^2 du\right)$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u) du$$

Let $Z = Z(T)$. Then $\mathbb{E}(Z) = 1$ and under the new probability measure $\tilde{\mathbb{P}}$ the process $\tilde{W}(t)$ is a Brownian Motion. It is obvious that $\tilde{W}(t)$ was not a Brownian motion under $\mathbb{P}$. Girsanov's theorem is a tool that can be used to change the mean of a process.

Girsanov's theorem

So in short, if W is a $\mathbb{P}$-Brownian Motion and Θ a process adapted to the filtration $\mathcal{F}$, then

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u) du$$

is a $\tilde{\mathbb{P}}$ Brownian Motion and the adjusted probability measure $\tilde{\mathbb{P}}$ can be found through the Radon-Nikodym derivative process:

$$Z(t) = exp\left(-\int_0^t \Theta(u) dW(u) - \frac{1}{2}\int_0^t \Theta(t)^2 du\right)$$

FFTAP

To see the formal connection between Girsanov's theorem and arbitrage, we look at the **first fundamental theorem of asset pricing**: A financial market is arbitrage-free under the probability measure $\mathbb{P}$ if and only if there exists another probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$, under which all discount asset prices are martingales. This means that the existence of $\mathbb{Q}$ implies that the time t no arbitrage price of a financial instrument V can be calculated using

$$\frac{V(t)}{B(t)} = \mathbb{E}_t^{\mathbb{Q}}\left(\frac{V(T)}{B(T)}\right)$$

If we found a probability measure $\tilde{\mathbb{P}}$ under which the discounted stock price is a martingale, the FFTAP says that

- The market of stock and bond is free of arbitrage
- We can use this probability measure to find no-arbitrage prices of other assets in this market

Put-call parity

The no-arbitrage prices of put and call options are related to each other via the put-call parity

$$P(t, S(t)) = C(t, S(t)) - S(t) + Ke^{-r(T-t)}$$

Martingale Representation Theorem

Let $W(t)$ be a Brownian Motion on a probability space on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}$ be the filtration generated by this Brownian Motion. Let $M(t)$ be a martingale with respect to this filtration. Then there is an adapted process $\Gamma(u)$ such that

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u)$$

Or,

$$dM(t) = \Gamma(t) dW(t)$$

The Martingale Representation theorem justifies the risk-neutral pricing formulas but it does not provide a practical method of finding the replicating portfolio $\Delta(t)$. Hence, the Martingale Representation Theorem guarantees that a process $\tilde{\Gamma}$ exists and therefore a replicating strategy exists, but it does not provide a method for finding $\tilde{\Gamma}(t)$.

SFTAP

The **Second fundamental Theorem of Asset Pricing**: Consider a market model that has a risk-neutral probability measure (does not allow for arbitrage opportunities). The model is complete if and only if the risk-neutral probability measure is unique. Thus, a market model is complete if every derivative security can be hedged. Options are redundant assets if they can be replicated perfectly. A market that is driven by two Brownian Motions (two sources of risk) is not complete.
If a market model is incomplete, we are unable to apply the law of one price. Then there are multiple prices for the option (i.e. the no arbitrage price is not uniquely defined). We can make such a market complete by adding a second stock that depends on at least one of the two Brownian Motions. Thus: the number of assets in the replicating strategy should be the number of sources of randomness plus 1.

Feynman-Kac

The **Feynman-Kac** theorem says that the solution of the Partial Differential Equation

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0$$

with boundary condition

$$F(T,x) = \Phi(x)$$

Is given by

$$F(t,X_t) = \mathbb{E}_t\big[\Phi(X_T)\big]$$

with X satisfying

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \;\; X(t) = x$$

Feynman-Kac thus tells us how partial differential equations can be solved with stochastic differential equations. For almost every problem we thus have two methods to solve:

- Monte Carlo simulations
-

Claim in BS world

In the Black Scholes world, the price process is given by

$$V(t) = F(t,S(t))$$

Where F satisfies the Black-Scholes PDE. We can apply Itô's lemma to the PDE

$$dV(t) = \left(\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial S^2}\right)dt + \sigma S(t)\frac{\partial F}{\partial S}dW(t)$$

Here the first term is somply rF due to the Black Scholes PDE.