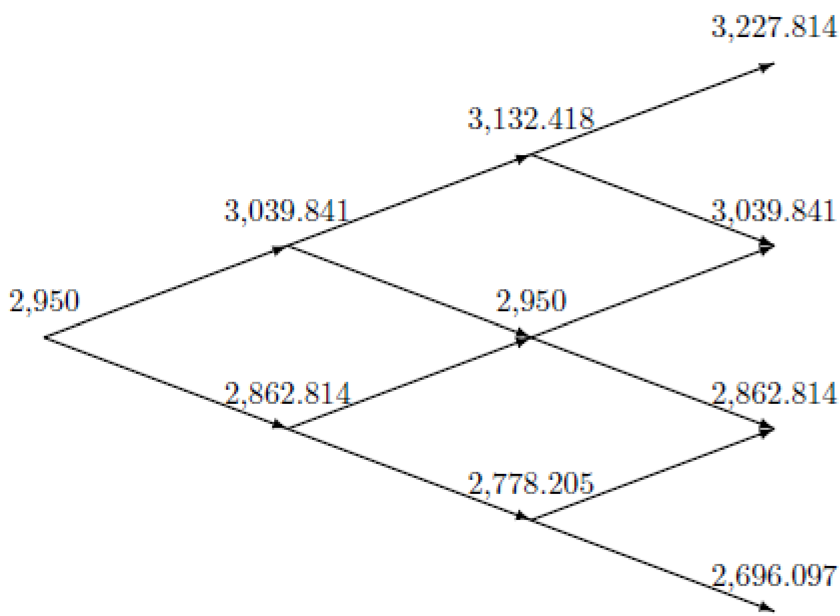
Individual Assignment Set 2

Question 1:

We want to price a European call-option on the S&P-500 index. Strike price of the option is USD 3,000 and the remaining time to maturity is **six weeks**. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a 3-step binomial tree. The tree is given in the following picture:



In this tree I have used $u = 1.030455$ and $d = 1/u$.

(a) Assume that the real-world probability of an upward movement is 11/20. Calculate the expectation of the 6-weeks return on the S&P-500 index.

Solution:

$$\boldsymbol{E(S_3) = \left(\frac{11}{20}\right)^3 \cdot 3{,}227.814 + 3 \cdot \left(\frac{11}{20}\right)^2 \left(\frac{9}{20}\right) \cdot 3{,}039.841 + 3 \cdot \left(\frac{11}{20}\right)\left(\frac{9}{20}\right)^2 \cdot 2{,}862.814 + \left(\frac{9}{20}\right)^3 \cdot 2{,}696.097 = 2{,}980.642.}$$

Then the expected return is given by:

$$\boldsymbol{\frac{2{,}980.642}{2{,}950} - 1 = 1.0387\%.}$$

(b) Is the stochastic process of the S&P-500 a martingale? Please motivate your answer.

<u>Solution:</u>

No, because the expected value of S_3 is higher than S_0.

Denote the index values at time *t* = 0, *t* = 2 weeks, *t* = 4 weeks and *t* = 6 weeks by S_0, S_1, S_2, and S_3, respectively.

(c) Calculate $E(S_3|S_1)$.

<u>Solution:</u>

This is a conditional expectation. Let us try to use the definition of slide 11, clip 5, week 2, in a discrete setting.

For $S_1 = 3{,}039.841$:

$$\sum_{\omega \in A} E(S_3|S_1)P(\omega) = \sum_{\omega \in A} S(3)(\omega)P(\omega).$$

The right hand side:

$$\left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{3}} \cdot \mathbf{3{,}227.814} + \ \mathbf{2} \cdot \left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{2}} \left(\frac{\mathbf{9}}{\mathbf{20}}\right) \cdot \mathbf{3{,}039.841} + \left(\frac{\mathbf{11}}{\mathbf{20}}\right)\left(\frac{\mathbf{9}}{\mathbf{20}}\right)^{\mathbf{2}} \cdot \mathbf{2{,}862.814} = \mathbf{1{,}683.47}.$$

The left hand side:

$$\left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{3}} \cdot E(S_3|S_1 = S_0 u) + \left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{2}} \left(\frac{\mathbf{9}}{\mathbf{20}}\right) \cdot E(S_3|S_1 = S_0 u) + \left(\frac{\mathbf{11}}{\mathbf{20}}\right)\left(\frac{\mathbf{9}}{\mathbf{20}}\right)^{\mathbf{2}} \cdot E(S_3|S_1 = S_0 u)$$

Now we can solve for $\boldsymbol{E(S_3|S_1 = S_0 u)}$:

$$E(S_3|S_1 = S_0 u) = 3{,}060.855.$$

In the same way we can derive $\boldsymbol{E(S_3|S_1 = S_0 d)}$:

$$E(S_3|S_1 = S_0 d) = 2{,}882.604.$$

Hence, the complete answer is:

$$E(S_3|S_1) = 3{,}060.855 \cdot 1_{\{S_1 = 3{,}039.841\}} + \ 2{,}882.604 \cdot 1_{\{S_1 = 2{,}862.814\}}.$$

The easier way of doing this:

Calculate conditional probabilities:

- $P(S_3 = S_1 * u * u|S_1) = \left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{2}}$
- $P(S_3 = S_1 * u * d|S_1) = 2 * \left(\frac{\mathbf{11}}{\mathbf{20}}\right) * \left(\frac{\mathbf{9}}{\mathbf{20}}\right)$
- $P(S_3 = S_1 * d * d|S_1) = \left(\frac{\mathbf{9}}{\mathbf{20}}\right)^{\mathbf{2}}$

Then:

$$E(S_3|S_1 = S_0u) = \left(\frac{\mathbf{11}}{\mathbf{20}}\right)^{\mathbf{2}} \cdot \mathbf{3,227.814} + \mathbf{2}\left(\frac{\mathbf{11}}{\mathbf{20}}\right)\left(\frac{\mathbf{9}}{\mathbf{20}}\right) \cdot \mathbf{3,039.841} + \left(\frac{\mathbf{9}}{\mathbf{20}}\right)^{\mathbf{2}} \cdot \mathbf{2,862.814}$$
$$= \mathbf{3060.855}.$$

…

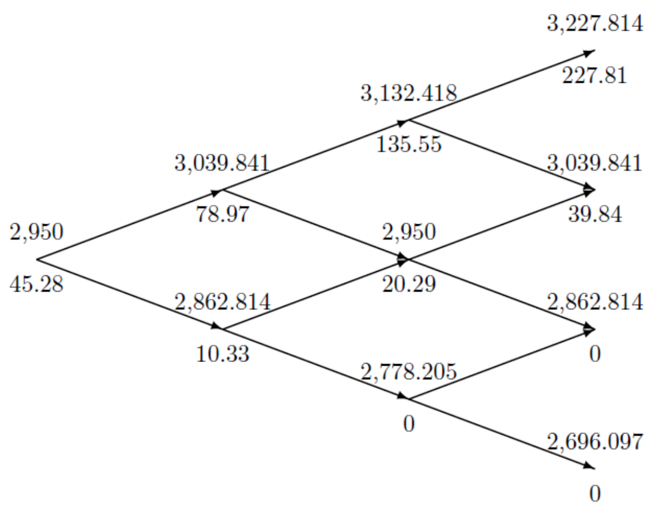
(d) Calculate the no-arbitrage value of the European call-option using the risk-neutral valuation method. As was already mentioned, the strike price of the option is USD 3,000 and the remaining time to maturity is six weeks. Assume that the risk-free interest rate per two weeks is 0.10%, i.e. an investment of USD 1,000 in a money market account that pays this interest rate delivers USD 1 interest after two weeks.

Solution:

We use the risk-neutral valuation. First, we need to calculate the risk-neutral probability of an upward movement (over a 2-week period).

$$\boldsymbol{q_u} = \frac{\mathbf{(1+0.1\%) - 0.970466}}{\mathbf{1.030455 - 0.970466}} = \mathbf{50.9165\%}.$$

Then we can work backwards in the tree:



(e) Suppose that the market price of the call-option is higher than the price you calculated in (d). This implies the existence of an arbitrage strategy. Give this strategy and show that the strategy generates a riskless profit.

Solution:

Suppose that the market price is USD 50 instead of USD 45.28.

At t = 0:

- **Short the call: receive USD 50 in cash**
- **Invest in 0.3877 (delta) S&P-500: value is USD 1143.921**
- **Borrow USD 1093.921 to finance the purchase of the stock index (against 0.10%)**

At t=1 (assume that the index goes down):

- **Value stock index holding: USD 1110.113**
- **Value cash: -1095.015**
- **Total portfolio value: 15.098**
- **New delta: 0.11808**
- **Sell 0.26969 units of the stock index against USD 772.0712**
- **New value stock holding USD 338.0421**
- **New value cash: USD -322.944**
- **Total portfolio value USD 15.098 (self-financing portfolio)**

At t = 2 (assume that the index goes down)

- **Value stock index holding: USD 328.051**
- **Value cash: USD -323.267**
- **Portfolio value: USD 4.784**
- **New delta: 0**
- **Sell remaining stock position**
- **New cash value: USD 4.784**

At t = 3 (up or down)

- **Portfolio value: USD 4.7892**

At the start you receive "too much" for selling the option. That's the reason why at the option's maturity you will always have a positive amount in cash. We could have shown this also for all other paths.