Stochastic Processes: The Fundamentals

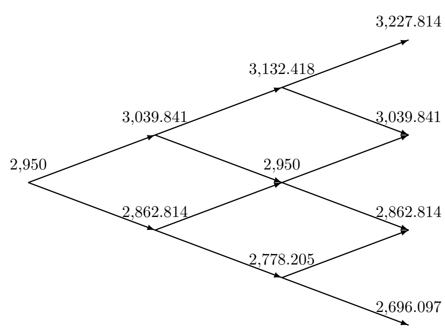
Exam

October 23, 2019

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 70% of your final grade.

Question 1: Binomial trees (22 points)

We want to price a European call-option on the S&P-500 index. Strike price of the option is USD 3,000 and the remaining time to maturity is six weeks. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a 3-step binomial tree. The tree is given in the following picture:



In this tree I have used $u = 1.030455$ and $d = 1/u$.

(a) Assume that the real-world probability of an upward movement is $\frac{11}{20}$. Calculate the expectation of the 6-weeks return on the S&P-500 index (**4 pts**).

The expected value of the S&P-500 index after six weeks is given by:

$$\left(\frac{11}{20}\right)^3 \cdot 3,227.814 + 3 \cdot \left(\frac{11}{20}\right)^2 \left(\frac{9}{20}\right) \cdot 3,039.841 + 3 \cdot \left(\frac{11}{20}\right) \left(\frac{9}{20}\right)^2 \cdot 2,8262.814 + \left(\frac{9}{20}\right)^3 \cdot 2,696.097 = 2,980.642.$$

Then the expected 6-weeks returns is given by:

$$\frac{2,980.642}{2,950} - 1 = 1.0387\%.$$

(b) Is the stochastic process of the S&P-500 index a martingale? Please motivate your answer (**4 pts**).

No, because the expected return is not equal to zero.

Denote the index values at time $t = 0$, $t = 2$ weeks, $t = 4$ weeks and $t = 6$ weeks by S_0, S_1, S_2, and S_3, respectively.

(c) Calculate $\mathbb{E}(S_3|S_1)$ (**3 pts**).

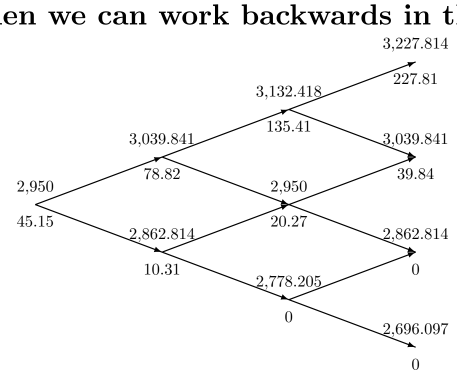
See solution of individual assignment number 2.

(d) Calculate the no-arbitrage value of the European call-option using the risk-neutral valuation method. As was already mentioned, the strike price of the option is USD 3,000 and the remaining time to maturity is six weeks. Assume that the risk-free interest rate per two weeks is 0.10%, i.e. an investment of USD 1,000 in a money market account that pays this interest rate delivers USD 1 interest after two weeks (**6 pts**).

First we need to calculate the risk-neutral probability of an upward movement (over a 2-week period):

$$q_u = \frac{(1 + 0.1\%) - 0.970466}{1.030455 - 0.970466} = 50.9165\%.$$

Then we can work backwards in the tree, using the risk-neutral probabilities.



(e) Suppose the market price of the call-option is higher than the price you calculated in (d). This implies the existence of an arbitrage strategy. Give this strategy and show that the strategy generates a riskless profit (**5 pts**).

Short the call option and then setup a portfolio that goes long in the underlying and takes a position in the risk-free asset. Positions in the stock and risk-free asset are updated after two weeks and four weeks.

Question 2: Algebras (10 points)
Let $X := \{a, b, c\}$, $A_1 := \{\emptyset, X, \{a\}, \{b, c\}\}$ and $A_2 := \{\emptyset, X, \{b\}, \{a, c\}\}$.

(a) Is A_1 an algebra? Please motivate your answer (**3 pts**).

Yes. In the exam you should check all properties of the definition of an algebra.

(b) Is A_2 an algebra? Please motivate your answer (**3 pts**).

Yes. In the exam you should check all properties of the definition of an algebra.

(c) Is $A_1 \cup A_2$ an algebra? Please motivate your answer (**4 pts**).

No. $\{a,b\}$ is not in the set.

Question 3: Brownian Motion (16 points)
Define the stochastic process Z by $Z(t) = W^4(t)$, $t \geq 0$, where $W(t)$, $t \geq 0$ is a Brownian Motion.

(a) Derive the stochastic differential equation of Z. What is the initial value? (**6 pts**)

Solution:

$$dZ_t = 4W_t^3 dW_t + \frac{1}{2} 12 W_t^2 dt.$$

Initial value is equal to zero.

(b) Compute $\mathbb{E}_0[Z(t)]$ and conclude that Z is not a martingale (**5 pts**).

Solution:

$$Z_t = Z_0 + 4\int_0^t W_u^3 dW_u + 6\int_0^t W_u^2 du.$$

Now take expectations:

$$\begin{aligned}
\mathbb{E}_0(Z_t) &= Z_0 + 0 + \mathbb{E}_0\left(6\int_0^t W_u^2 du\right) \\
&= 0 + 0 + 6\int_0^t \mathbb{E}_0(W_u^2) du \\
&= 6\int_0^t u du \\
&= 3t^2.
\end{aligned}$$

(c) Provide a stochastic process Y that consists of a random part which is given by Z and a deterministic part. Choose the deterministic part such that Y is a martingale (**5 pts**).

Solution:

$$Y_t = Z_t - 3t^2, \quad t \geq 0.$$

Question 4: Stochastic differential equations (20 points)
Suppose we have a market that consists of a stock S and a money market account B. The stochastic differential equations of this market are given by:

$$\begin{aligned}
&dS_t = \mu S_t dt + \sigma_t S_t dW_t^S, \quad S_0 = s \\
&d\sigma_t^2 = -\kappa(\sigma_t^2 - \sigma^2)dt + \sigma^\sigma \sigma_t dW_t^V, \quad \sigma_0^2 = \sigma^2 \\
&d[W^S, W^V]_t = \rho dt \\
&dB_t = rB_t dt, \quad B_0 = 1,
\end{aligned}$$

where W^S and W^V are Brownian Motions under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

(a) Is this market complete? Please motivate your answer. How could this market be completed? (**4 pts**)

No, there are two sources of risk and just two assets. We need to introduce another asset that relies on W^S, W^V or both. An option that has S as underlying is an example of an asset that could complete this market.

Let us now focus on the SDE for the variance.

(b) Derive $d\exp(\kappa t)\sigma_t^2$ and show that (**6 pts**):

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2\int_0^t e^{\kappa u}du + \int_0^t e^{\kappa u}\sigma^\sigma\sigma_u dW_u^V\right\}.$$

Solution:

$$\begin{aligned}
d\exp(\kappa t)\sigma_t^2 &= \kappa\exp(\kappa t)\sigma_t^2 dt + \exp(\kappa t)d\sigma_t^2 \\
&= \kappa\exp(\kappa t)\exp(\kappa t)(-\kappa(\sigma_t^2 - \sigma^2)dt + \sigma^\sigma\sigma_t dW_t^V) \\
&= \kappa\sigma^2\exp(\kappa t)dt + \exp(\kappa t)\sigma^\sigma\sigma_t dW_t^V.
\end{aligned}$$

Write this in integral form:

$$\exp(\kappa t)\sigma_t^2 - \exp(\kappa 0)\sigma_0^2 = \kappa\sigma^2\int_0^t \exp(\kappa u)du + \int_0^t \exp(\kappa u)\sigma^\sigma\sigma_u dW_u^V.$$

Then it is trivial to see that:

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2\int_0^t e^{\kappa u}du + \int_0^t e^{\kappa u}\sigma^\sigma\sigma_u dW_u^V\right\}.$$

(c) Derive the expression for σ_{t+h}^2 conditional on σ_t^2 (**5 pts**).

Solution:

$$\sigma_{t+h}^2 = e^{-\kappa(t+h)}\left\{\sigma_0^2 + \kappa\sigma^2\int_0^{t+h} e^{\kappa u}du + \int_0^{t+h} e^{\kappa u}\sigma^\sigma\sigma_u dW_u^V\right\}.$$

Hence,

$$\sigma_{t+h}^2 = e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2\int_t^{t+h} e^{\kappa u}du + e^{-\kappa(t+h)}\int_t^{t+h} e^{\kappa u}\sigma^\sigma\sigma_u dW_u^V.$$

(d) Compute $\mathbb{E}_t[\sigma_{t+h}^2]$ (**5 pts**).

Solution:

$$
\begin{aligned}
\mathbb{E}_t(\sigma^2_{t+h}) &= e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2 \int_t^{t+h} e^{\kappa u} du + 0 \\
&= e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2 \frac{1}{\kappa}(e^{\kappa(t+h)} - e^{\kappa t}) \\
&= \sigma^2 + e^{-\kappa h}(\sigma_t^2 - \sigma^2).
\end{aligned}
$$

Question 5: Bachelier process (22 points)
Consider the following market:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma dW_t, \quad S_0 = s \\
dB_t &= rB_t dt, \quad B_0 = 1, \quad r > 0,
\end{aligned}
$$

where W is a Brownian Motion under the real-world probability measure $\mathbb{P}$, S denotes the stock price process and B the money market account.

Suppose now that the solution of the stock price SDE is given by:

$$
S_t = e^{\mu t} S_0 + \sigma e^{\mu t} \int_0^t e^{-\mu s} dW_s.
$$

(a) Give the probability distribution of $S_t|S_0$ (**5 pts**).

We know that increments of Brownian Motion are normally distributed. The stochastic integral is a sum of increments in Brownian Motion and therefore normally distribution.

We also know that the expectation of the stochastic integral is equal to zero. Therefore the expectation of $S_t|S_0$ is given by:

$$
\mathbb{E}_0(S_t) = e^{\mu t} S_0.
$$

The variance of $S_t|S_0$ is given by:

$$
\mathbf{Var}_0(S_t) = \sigma^2 e^{2\mu t} \mathbb{E}_0((\int_0^t e^{-\mu s} dW_s)^2).
$$

Then using Ito-isometry:

$$
\mathbf{Var}_0(S_t) = \sigma^2 e^{2\mu t} \int_0^t e^{-2\mu s} ds.
$$

Solving the integral gives:

$$
\mathbf{Var}_0(S_t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right).
$$

(b) Give your personal opinion on this model for stock price behaviour (**3 pts**).

You could have observed that:

- **Negative stock prices are possible in this model (not very realistic)**
- **Constant volatility (not very realistic)**

(c) Use Itô's lemma to derive the process for the discounted stock price $Y_t := S_t/B_t$ (**3 pts**).

Solution:

$$\begin{aligned}
dY_t = d\left(\frac{S_t}{B_t}\right) &= \frac{1}{B_t}dS_t - \frac{S_t}{B_t^2}dB_t \\
&= \mu\left(\frac{S_t}{B_t}\right)dt + \sigma\frac{1}{B_t}dW_t - r\frac{S_t}{B_t}dt \\
&= (\mu - r)Y_t dt + \sigma\frac{1}{B_t}dW_t.
\end{aligned}$$

(d) Apply Girsanov's theorem and conclude that the stock price process for S under the risk-neutral probability measure is given by:

$$dS_t = rS_t dt + \sigma d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian Motion under the risk-neutral probability measure $\mathbb{Q}$ (**4 pts**).

Solution:

$$\begin{aligned}
dY_t &= (\mu - r)Y_t dt + \sigma\frac{1}{B_t}dW_t \\
&= \sigma\frac{1}{B_t}\left(dW(t) + \frac{(\mu - r)S_t}{\sigma}dt\right) \\
&= \sigma\frac{1}{B_t}d\tilde{W}_t.
\end{aligned}$$

Then use the SDE for S:

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma dW_t \\
&= \mu S_t dt + \sigma\left(d\tilde{W}_t - \frac{(\mu - r)S_t}{\sigma}dt\right) \\
&= rS_t dt + \sigma d\tilde{W}_t.
\end{aligned}$$

In the same spirit as above, the solution of this SDE is given by:

$$S_t = e^{rt}S_0 + \sigma e^{rt}\int_0^t e^{-rs}d\tilde{W}_s.$$

(e) Calculate the no-arbitrage price at time t of a European digital call option with strike price K and maturity date $T > t$. This option pays 1 if the stock price at maturity is larger than strike price K and 0 in all other cases (**7 pts**).

Using the result of question (a) we know that under $\mathbb{Q}$:

$$S_T | S_t \sim N\left(e^{r(T-t)} S_0, \frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)\right).$$

Using the FFTAP we know that the no-arbitrage price of the digital call option at time t is given by:

$$V_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}\left(\mathbb{I}_{\{S_T > K\}}\right).$$

Now work out the expectation:

$$\begin{aligned}
V_t &= e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}\left(\mathbb{I}_{\{S_T > K\}}\right) \\
&= e^{-r(T-t)} \mathbb{Q}_t (S_T > K) \\
&= e^{-r(T-t)} \mathbb{Q}_t \left(\frac{S_T - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} > \frac{K - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right) \\
&= e^{-r(T-t)} \Phi\left(\frac{K - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right).
\end{aligned}$$