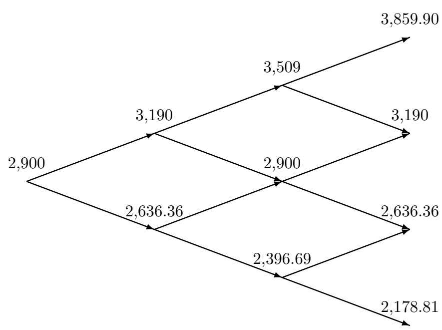
Stochastic Processes: The Fundamentals

Exam

December 13, 2019

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 70% of your final grade.

Question 1: Binomial trees (22 points)
We want to price a European and American put-option on the S&P-500 index. Strike price of both options is USD 2,900 and the remaining time to maturity is 3 months. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a 3-step binomial tree. The tree is given in the following picture:



In this tree I have used $u = 1.10$ and $d = 1/u$.

(a) Assume that the real-world probability of an upward movement is $\frac{4}{7}$. Calculate the variance of the 3-months return on the S&P-500 index (**5 pts**).

$$\mathbb{E}_t\left(R^S_{t:t+6w}\right) = \left(\frac{4}{7}\right)^3 \cdot 33.1\% + 3\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right) \cdot 10.0\% + 3\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)^2 \cdot -9.1\% + \left(\frac{3}{7}\right)^3 \cdot -24.9\% = 5.55\%.$$

And:

$$\mathbb{E}_t\left(\left(R^S_{t:t+6w}\right)^2\right) = \left(\frac{4}{7}\right)^3 \cdot (33.1\%)^2 + 3\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right) \cdot (10.0\%)^2 + 3\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)^2 \cdot (-9.1\%)^2 + \left(\frac{3}{7}\right)^3 \cdot (-24.9\%)^2$$

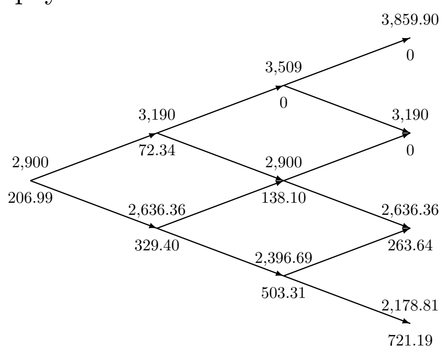
So:

$$\mathrm{Var}_t\left(R^S_{t:t+6w}\right) = 0.0321 - (0.0555)^2 = 0.0290.$$

(b) Is the stochastic process of the S&P-500 index a Markov process? Please motivate your answer (**4 pts**).

Yes, the probability distribution of S_{t+1} conditional on $\mathcal{F}_t$ only depends on S_t and not on the realisations of S before time t.

(c) Calculate the no-arbitrage value of European and American put-option using the risk-neutral valuation method. As was already mentioned, the strike price of the option is USD 2,900 and the remaining time to maturity is three months. Assume that the risk-free interest rate per 1 month is 0%, i.e. an investment of USD 1,000 in a money market account that pays this interest rate delivers USD 0 interest after one month (**7 pts**).



Suppose that the market price of the European put-option is USD 250.

(d) Provide for the path that leads to the lowest value for the S&P-500 index after 3 months, the arbitrage strategy in detail. I.e. give the exact strategy at times $t = 0, 1, 2$ and show numerically that the payoff of the strategy is positive in this particular path. (**6 pts**).

- Sell the option: generates USD 250 in cash
- $\Delta = 0.464$: sell 0.464 units of the index, this generates USD 1346.51 in cash
- So: short positions in stock and put-option, and cash amount 1346.51 + 250
- At time $t = 1$: sell 0.261 units of the index, this delivers 688.91 in cash
- At time $t = 2$: sell 0.274 units of the index, this delivers 657.60 in cash
- Total cash position now is 2943.01
- At maturity date of the option: option expires in the money, you buy the stock from the other party in the contract against 2900, you cancel out your short position in equity with the received stock and you are left ith 43.01 in cash.

Question 2: Algebras (10 points)
Let a, b, c be three distinct points.

(a) Write down all algebras on $\Omega = \{a, b\}$ (**3 pts**).

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$

- $\mathcal{F}_1 = \{\emptyset, \Omega, \{a\}, \{b\}\}$

(b) Write down all algebras on $\Omega = \{a, b, c\}$ (**4 pts**).

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- $\mathcal{F}_1 = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$
- $\mathcal{F}_2 = \{\emptyset, \Omega, \{b\}, \{a, c\}\}$
- $\mathcal{F}_3 = \{\emptyset, \Omega, \{c\}, \{a, b\}\}$
- $\mathcal{F}_4 = \{\emptyset, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

(c) Give an explicit counterexample which shows that the union of algebras is not necessarily an algebra (**3 pts**).

The union of $\mathcal{F}_1$ and $\mathcal{F}_2$ of question (b) is obviously not an algebra as $\{a, b\}$ is not part of the union.

Question 3: Brownian Motion (18 points)
Let $W(t)$ be a Brownian Motion and define

$$B(t) = \int_0^t \operatorname{sign}(W(s))dW(s),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

(a) Compute $d[B(t)W(t)]$ and conclude that $\mathbb{E}_0[B(t)W(t)]$ is equal to zero. *Hint: write $d[B(t)W(t)]$ in integral form and then take expectations.* (**6 pts**)

Solution:

Apply Ito's lemma:

$$\begin{aligned} dB(t)W(t) &= B(t)dW(t) + W(t)dB(t) + dB(t)dW(t) \\ &= B(t)dW(t) + W(t)\operatorname{sign}(W(t))dW(t) + \operatorname{sign}(W(t))dt \end{aligned}$$

Write this in integral form:

$$B(t)W(t) = B(0)W(0) + \int_0^t B(s)dW(s) + \int_0^t W(s)\operatorname{sign}(W(s))dW(s) + \int_0^t \operatorname{sign}(W(s))ds.$$

Taking expectations:

$$
\begin{aligned}
\mathbb{E}_0(B(t)W(t)) &= \mathbb{E}_0\left(\int_0^t \text{sign}(W(s))ds\right) \\
&= \int_0^t \mathbb{E}_0\left(\text{sign}(W(s))\right)ds \\
&= \int_0^t \mathbb{E}_0\left[1_{\{W_s\geq 0\}} - 1_{\{W_s<0\}}\right]ds \\
&= \frac{1}{2}t - \frac{1}{2}t = 0.
\end{aligned}
$$

(b) Verify that: (**5 pts**)

$$
dW^2(t) = 2W(t)dW(t) + dt.
$$

Solution:

Apply Ito's lemma:

$$
dW^2(t) = 2W(t)dW(t) + \frac{1}{2}\cdot 2\cdot (dW_t)^2 = 2W(t)dW(t) + dt.
$$

(c) Compute $d[B(t)W^2(t)]$ and use the result to compute $\mathbb{E}_0[B(t)W^2(t)]$ (**7 pts**).

Solution:

$$
\begin{aligned}
d[B(t)W^2(t)] &= B(t)dW^2(t) + W^2(t)dB(t) + dB(t)dW^2(t) \\
&= B(t)(2W(t)dW(t) + dt) + W^2(t)\text{sign}(W(t))dW(t) + \text{sign}(W(t))dW(t)(2W(t)dW(t) + dt) \\
&= 2B(t)W(t)dW(t) + B(t)dt + \text{sign}(W(t))W^2(t)dW(t) + 2\cdot\text{sign}(W(t))W(t)dt.
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbb{E}_0[B(t)W^2(t)] &= \mathbb{E}_0\left[\int_0^t B(s)ds\right] + 2\mathbb{E}_0\left[\int_0^t \text{sign}(W(s))W(s)ds\right] \\
&= \int_0^t \mathbb{E}_0(B(s))ds + 2\int_0^t \mathbb{E}_0\left[\text{sign}(W(s))W(s)ds\right]ds \\
&= 2\int_0^t \left(\mathbb{E}_0(W(s)1_{\{W(s)\geq 0\}}) - \mathbb{E}_0(W(s)1_{\{W(s)<0\}})\right)
\end{aligned}
$$

Question 4: Stochastic differential equations (19 points)
Suppose we have the standard Black-Scholes market that consists of a stock S and a money market account B. The stochastic differential equations of this market are given by:

$$
\begin{aligned}
&dS_t = \mu S_t dt + \sigma S_t dW_t^S, \quad S_0 = s \\
&dB_t = rB_t dt, \quad B_0 = 1,
\end{aligned}
$$

where W^S is a Brownian Motion under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

(a) Show that the solution of the SDE for the stock price S is given by (**6 pts**):

$$S_t = se^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Solution:

Note that S is a function of t and W. So, we can apply Ito's lemma to this function:

$$\begin{aligned}
dS_t &= se^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t}\left(\mu - \frac{1}{2}\sigma^2\right)dt + se^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t}\sigma dW_t + \frac{1}{2}se^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t}\sigma^2 dt \\
&= S_t\left(\mu - \frac{1}{2}\sigma^2\right)dt + S_t\sigma dW_t + \frac{1}{2}S_t\sigma^2 dt \\
&= \mu S_t dt + \sigma S_t dW_t.
\end{aligned}$$

Let us now introduce stochastic volatility into the model:

$$\begin{aligned}
&dS_t = \mu S_t dt + \sigma_t S_t dW_t^S, \quad S_0 = s \\
&d\sigma_t^2 = -\kappa(\sigma_t^2 - \sigma^2)dt + \gamma dW_t^V, \quad \sigma_0^2 = \sigma^2 \\
&d[W^S, W^V]_t = \rho dt \\
&dB_t = rB_t dt, \quad B_0 = 1,
\end{aligned}$$

where W^S and W^V are Brownian Motions under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

It is possible to derive that:

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2\int_0^t e^{\kappa u}du + \gamma\int_0^t e^{\kappa u}dW_u^V\right\}.$$

(b) Derive $\mathrm{Var}_0(\sigma_t^2)$ (**6 pts**).

Solution:

$$
\begin{aligned}
\mathrm{Var}_0(\sigma_t^2) &= \mathrm{Var}_0(e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2\int_0^t e^{\kappa u}du + \gamma\int_0^t e^{\kappa u}dW_u^V\right\}) \\
&= \mathrm{Var}_0(e^{-\kappa t}(\gamma\int_0^t e^{\kappa u}dW_u^V)) \\
&= e^{-2\kappa t}\gamma^2\mathrm{Var}_0(\int_0^t e^{\kappa u}dW_u^V) \\
&= e^{-2\kappa t}\gamma^2\mathbb{E}_0((\int_0^t e^{\kappa u}dW_u^V)^2) \\
&= e^{-2\kappa t}\gamma^2\int_0^t e^{2\kappa u}du \\
&= \frac{\gamma^2}{2\kappa}(1-e^{-2\kappa t}).
\end{aligned}
$$

Now we are interested in the process of the squared variance.

(c) Define $Y_t = \sigma_t^2$. Derive dY_t^2. Is this process also mean-reverting? (**7 pts**)

Solution:

$$
\begin{aligned}
dY_t^2 &= 2Y_t dY_t + \frac{1}{2}2d[Y,Y]_t \\
&= 2Y_t\left(-\kappa(\sigma_t^2-\sigma^2)dt + \gamma dW_t^V\right) + \gamma^2 dt \\
&= 2Y_t\left(-\kappa(Y_t-\sigma^2)dt + \gamma dW_t^V\right) + \gamma^2 dt \\
&= -2\kappa Y_t^2 + 2Y_t\kappa\sigma^2 dt + \gamma^2 dt + 2\gamma Y_t dW_t^V \\
&= -2\kappa\left(Y_t^2 - \frac{(Y_t\kappa\sigma^2 + \frac{1}{2}\gamma^2)}{\kappa}\right)dt + 2\gamma Y_t dW_t^V \\
&= -\tilde{\kappa}\left(Y_t^2 - \tilde{\mu}_t\right)dt + 2\gamma Y_t dW_t^V
\end{aligned}
$$

This process does not revert to a constant mean.

Question 5: Bachelier process (21 points)
Consider the following market:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma dW_t, \quad S_0 = s \\
dB_t &= rB_t dt, \quad B_0 = 1, \quad r > 0,
\end{aligned}
$$

where W is a Brownian Motion under the real-world probability measure $\mathbb{P}$, S denotes the stock price process and B the money market account.

The solution of the stock price SDE is given by:

$$
S_t = e^{\mu t}S_0 + \sigma e^{\mu t}\int_0^t e^{-\mu s}dW_s.
$$

(a) Prove that this is indeed the solution of the stock price SDE (**8 pts**). *Hint: write the solution as*

$$X_t = Y_t + Z_t \cdot R_t,$$

where,

$$\begin{aligned} Y_t &= e^{\mu t} \cdot S_0 \\ Z_t &= e^{\mu t} \cdot \sigma \\ R_t &= \int_0^t e^{-\mu s} dW_s. \end{aligned}$$

First compute the differentials dZ, dY and dR. Then use the multidimensional Itô formula to the function $f(y, z, r) = y + z \cdot r$.

Solution:

$$\begin{aligned} dY_t &= \mu \cdot e^{\mu t} \cdot S_0 dt \\ dZ_t &= \mu \cdot e^{\mu t} \cdot \sigma dt \\ dR_t &= e^{-\mu t} dW_t. \end{aligned}$$

Then:

$$\begin{aligned} df &= 1 \cdot dY + R \cdot dZ + Z \cdot dR \\ &= \mu \cdot e^{\mu t} \cdot S_0 dt + \int_0^t e^{-\mu s} dW_s \mu \cdot e^{\mu t} \cdot \sigma dt + e^{\mu t} \cdot \sigma e^{-\mu t} dW_t \\ &= \mu \left(e^{\mu t} \cdot S_0 + e^{\mu t} \sigma \int_0^t e^{-\mu s} dW_s \right) dt + \sigma dW_t \\ &= \mu S_t dt + \sigma dW_t. \end{aligned}$$

It is possible to derive that:

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r)\left(\frac{S_t}{B_t}\right) dt + \sigma \frac{1}{B_t} dW_t.$$

We can rewrite this SDE as:

$$d\left(\frac{S_t}{B_t}\right) = \frac{\sigma}{B_t}\left(dW_t + \frac{(\mu - r)S_t}{\sigma} dt\right).$$

Using Girsanov's theorem we can derive the process for the discounted stock price under the risk-neutral probability measure.

$$d\left(\frac{S_t}{B_t}\right) = \frac{\sigma}{B_t} d\tilde{W}_t.$$

(b) Derive the SDE for the stock price under the risk-neutral measure by calculating $d(Y_t \cdot B_t)$, where $Y_t := \left(\frac{S_t}{B_t}\right)$. Is the resulting process for S a martingale? (**5 pts**).

Solution:

$$
\begin{aligned}
d(Y_t \cdot B_t) &= Y_t dB_t + B_t dY_t + dB_t dY_t \\
&= \left(\frac{S_t}{B_t}\right) rB_t dt + B_t \frac{\sigma}{B_t} d\tilde{W}_t + 0 \\
&= rS_t dt + \sigma d\tilde{W}_t.
\end{aligned}
$$

Hence, the process for S is not a martingale under the risk-neutral probability measure.

The probability distribution of the stock price at time T conditional on time t information is given by:

$$
S_T | S_t \sim N\left(e^{r(T-t)} S_t, \frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)\right).
$$

(c) Use this to calculate the no-arbitrage price at time t of a European put option with strike price K and maturity date $T > t$ (**8 pts**).

Solution:

We apply the first fundamental theorem of asset pricing for the put-option price $P(t, S_t)$:

$$
\begin{aligned}
P(t, S_t) &= e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}(\max(K - S_T, 0)) \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} \max(K - S_T, 0) q(S_T) dS_T \\
&= e^{-r(T-t)} \int_{-\infty}^{K} (K - S_T) q(S_T) dS_T \\
&= e^{-r(T-t)} \left(\int_{-\infty}^{K} K q(S_T) dS_T - \int_{-\infty}^{K} S_T q(S_T) dS_T\right).
\end{aligned}
$$

We know that $S_T | S_t$ has a normal distribution with parameters given above. Hence,

$$
Y_T := \frac{S_T - e^{r(T-t)} S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} \sim N(0, 1).
$$

Let's use this to calculate the first integral:

$$
\begin{aligned}
\int_{-\infty}^{K} K q(S_T) dS_T &= K \int_{-\infty}^{\frac{K - e^{r(T-t)} S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}} \phi(Y_T) dY_T \\
&= K \Phi\left(\frac{K - e^{r(T-t)} S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right).
\end{aligned}
$$

For the second integral we have:

$$\begin{aligned}\int_{-\infty}^{K} S_T q(S_T) dS_T &= \int_{-\infty}^{\frac{K-e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}} \left(e^{r(T-t)}S_t + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)} Y_T \right) \phi(Y_T) dY_T \\ &= e^{r(T-t)}S_t \Phi\left(\frac{K-e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}} \right) + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)} \int_{-\infty}^{\frac{K-e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}} Y_T \phi(Y_T) dY_T.\end{aligned}$$

For the final part we use that:

$$\int_{-\infty}^{d} x\phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} xe^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} (-e^{-\frac{1}{2}x^2})|_{-\infty}^{d} = -\phi(d).$$

Hence, the no-arbitrage price of the put-option is given by:

$$P(t, S_t) = (K - e^{r(T-t)}S_t)\Phi(d) + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}\phi(d),$$

where

$$d = \frac{K-e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}.$$