Exercise Set 3

Fall 2020

Exercise 3.1. Consider the standard Black-Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$. Denote the corresponding arbitrage free price process by $V(t)$.

(a) Show that under the risk-neutral measure $\mathbb{Q}, V(t)$ has a local rate of return equal to the short rate of interest r. In other words, show that $V(t)$ is of the form:

$$dV(t) = rV(t)dt + g(t)dW(t).$$

(b) Show that under the risk-neutral measure $\mathbb{Q}$, the process $Z(t) = \frac{V(t)}{B(t)}$ is a martingale. More precisely, show that the stochastic differential equation for Z has zero drift term, i.e. it is of the form

$$dZ(t) = Z(t)\sigma_Z(t)dW(t).$$

Determine also the diffusion process $\sigma_Z(t)$ in terms of the pricing function and its derivatives.

Solution:

1. In the Black–Scholes model the price process is given by $V(t) = F(t, S(t))$, where F satisfies the Black-Scholes PDE. Applying the Itô lemma we get

$$dV(t) = \underbrace{\left[\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial S^2}\right]}_{rF} dt + \sigma S(t)\frac{\partial F}{\partial S}dW(t).$$

   Noting that the expression in brackets is rF due to the Black–Scholes PDE we get

$$dV(t) = rV(t) + g(t)dW(t),$$

   where $g(t) = \sigma S(t)\frac{\partial F}{\partial S}$.

2. We apply the Itô lemma to the process $Z(t) = \frac{V(t)}{B(t)}$

$$\begin{aligned}
dZ &= \frac{dV}{B} - \frac{V}{B^2}dB \\
&\text{note that the cross term and higher orders are zero} \\
&= \frac{rV(t) + g(t)dW(t)}{B} - \frac{V}{B^2}rBdt \\
&= \frac{g(t)}{B}dW(t) \\
&= Z(t)\frac{\sigma S(t)}{V(t)}\frac{\partial F}{\partial S}dW(t) = Z(t)\sigma_Z(t)dW(t),
\end{aligned}$$

where $\sigma_Z(t) = \frac{\sigma S(t)}{V(t)} \frac{\partial F}{\partial S}$. The absence of the drift term shows that $Z(t)$ is a martingale under the risk-neutral measure Q.

Exercise 3.2. Consider the standard Black-Scholes model. Using the Greeks, verify that a delta-hedged European call-option earns the risk-free rate.

Solution:
We consider the proces:

$$V(t, S(t)) = C(t, S(t)) - \Delta(t, S(t))S(t), \quad t \geq 0.$$

In the Black-Scholes world:

$$V(t, S(t)) = C(t, S(t)) - \frac{\partial C}{\partial S} S(t), \quad t \geq 0.$$

The SDE of V is given by:

$$\begin{aligned} dV &= d(C - \frac{\partial C}{\partial S} S) = dC - \frac{\partial C}{\partial S} dS \\ &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS)^2 - \frac{\partial C}{\partial S} dS \\ &= \left(\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) dt. \end{aligned}$$

Now, use the analytical formulas for Θ and Γ:

$$\begin{aligned} dV &= \left(-\frac{\sigma S n(d_1)}{2\sqrt{T-t}} - re^{-r(T-t)} K\mathcal{N}(d_2)) + \frac{1}{2} \sigma^2 S^2 \frac{n(d_1)}{S\sigma\sqrt{T-t}} \right) dt \\ &= \left(-re^{-r(T-t)} K\mathcal{N}(d_2)) \right) dt \\ &= r \left(S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) - S\mathcal{N}(d_1) \right) dt \\ &= r \left(C - \frac{\partial C}{\partial S} S \right) dt \\ &= rV dt. \end{aligned}$$

Exercise 3.3. Verify by substitution that the Black-Scholes formula for call prices:

$$\begin{aligned} C(t, S(t)) &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \\ d_1 &= \frac{\log \frac{S(t)}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

is a solution of the Black-Scholes PDE:

$$\begin{aligned} F_t(t, s) + rSF_S(t, S) + \frac{1}{2} S^2 \sigma^2 F_{SS}(t, S) - rF(t, S) = 0 \\ F(T, S) = (S - K)^+. \end{aligned}$$

Solution:
Let's compute the derivatives of V with respect to S and t. The first derivatives wrt S is called Delta, Δ and is by far the most important risk measure of the position.

$$\Delta = \frac{\partial V}{\partial S} = \mathcal{N}(d_1) + Sn(d_1)\frac{\partial d_1}{\partial S} - e^{-r(T-t)}Kn(d_2)\frac{\partial d_2}{\partial S},$$

where $n(d_{1,2})$ is a normal pdf at d_1 and d_2, respectively. Let's express $n(d_2)$ through $n(d_1)$

$$\begin{aligned}
n(d_2) &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{d_2^2}{2}) \\
&= \frac{1}{\sqrt{2\pi}}\exp(-\frac{(d_1 - \sigma\sqrt{T-t})^2}{2}) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right)\exp\left(\frac{2d_1\sigma\sqrt{T-t} - \sigma^2(T-t)}{2}\right) \\
&= n(d_1)\exp\left(\ln S/K + r(T-t) + \frac{\sigma^2(T-t)}{2} - \frac{\sigma^2(T-t)}{2}\right) \\
&= n(d_1)\frac{S}{K}e^{r(T-t)}
\end{aligned}$$

Partial derivatives $\partial d_{1,2}/\partial S$ are

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Substituting $n(d_2)$ and the partial derivatives $\partial d_{1,2}/\partial S$ into equation for Δ, we see that the last two terms cancel each other and the equation for Δ becomes very simple

$$\Delta = \frac{\partial V}{\partial S} = \mathcal{N}(d_1).$$

The second derivatives of V with respect to S is called Gamma and is equal

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \mathcal{N}(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{T-t}}$$

A derivative with respect to time is called Theta, Θ, and is a change of the option value when all other market parameters are constant

$$\begin{aligned}
\Theta &= \frac{\partial V}{\partial t} \\
&= Sn(d_1)\frac{\partial d_1}{\partial t} - re^{-r(T-t)}K\mathcal{N}(d_2) - e^{-r(T-t)}Kn(d_2)\frac{\partial d_2}{\partial t} \\
&\text{substituting } n(d_2) \text{ we get} \\
&= -re^{-r(T-t)}K\mathcal{N}(d_2) + Sn(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)
\end{aligned}$$

where

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial (d_1 - d_2)}{\partial t} = \frac{\sigma \partial \sqrt{T-t}}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}$$

and finally

$$\Theta = \frac{\partial V}{\partial t} = -\frac{\sigma S n(d_1)}{2\sqrt{T-t}} - r e^{-r(T-t)} K \mathcal{N}(d_2).$$

One can see that the Black–Scholes equation can be written in terms of Greeks

$$\Theta + rS\Delta + \frac{1}{2} S^2 \sigma^2(t, S)\Gamma - rV = 0$$

Substituting Θ, Δ, Γ, and V into the equation we get

$$\begin{aligned}
&-\frac{\sigma S n(d_1)}{2\sqrt{T-t}} - r e^{-r(T-t)} K \mathcal{N}(d_2) \\
&+ rS\mathcal{N}(d_1) \\
&+ \frac{1}{2} S^2 \sigma^2 \frac{n(d_1)}{S\sigma\sqrt{T-t}} \\
&- r\left(S\mathcal{N}(d_1) - e^{-r(T-t)} K \mathcal{N}(d_2)\right) = 0.
\end{aligned}$$

Comparing terms for $n(d_1)$, $\mathcal{N}(d_1)$, and $\mathcal{N}(d_2)$ one can easily see that the equality is fulfilled.

What remains is to check if the solution satisfies the boundary condition

$$V(T) = (S - K)^+.$$

To see this we note that the leading term of d_1 is $\ln(S/K)/\sigma\sqrt{T-t}$ when $t \to T$ as

$$d_1 = \frac{\ln S/K}{\sigma\sqrt{T-t}} + \frac{r + \sigma^2/2}{\sigma}\sqrt{T-t} \approx \frac{\ln S/K}{\sigma\sqrt{T-t}}$$

respectively as $t \to T$

$$d_1 \to \begin{cases} \infty & \text{if } S > K \\ -\infty & \text{if } S < K. \end{cases}$$

and similar for d_2. Further

$$\mathcal{N}(d_{1,2}) \to \begin{cases} 1 & \text{if } S > K \\ 0 & \text{if } S < K. \end{cases}$$

And finally

$$V(t \to T) = S\mathcal{N}(d_1) - K\mathcal{N}(d_2) \to \begin{cases} S - K & \text{if } S > K \\ 0 & \text{if } S < K. \end{cases}$$

Exercise 3.4. A so-called **binary option** is a claim which pays a certain amount if the stock price at a certain date falls within some prespecified interval. Otherwise nothing will be paid out. Consider a binary option which pays USD K to the holder at maturity date T if the stock price at time T is in the interval $[\alpha, \beta]$. Determine the no-arbitrage price of this option in the Black-Scholes world.

Solution:
We first consider a simpler version of the binary option, an option that pays 1 USD at time T, if the spot is above α at time T. Obviously, and option that pays K USD will be K times more expensive. The payoff function of such option is an indicator function $\mathbb{I}(x)$ of the form

$$\Phi_C(S_T) = \mathbb{I}(S_T > \alpha).$$

This option is called a binary (or digital) *call* option. A binary put option has payoff

$$\Phi_P(S_T) = \mathbb{I}(S_T < \alpha).$$

Obviously,

$$\Phi_C(S_T) + \Phi_P(S_T) = 1.$$

The price of the call option is

$$V_C(0;\alpha) = e^{-rT}\mathrm{E}\left[\mathbb{I}(S_T > \alpha)\right].$$

The payoff of the original binary option can be replicated as

$$\Phi_B(S_T;\alpha,\beta) = \mathbb{I}(S_T > \alpha) - \mathbb{I}(S_T > \beta).$$

The value at time $t = 0$ respectively is

$$V_B(0;\alpha,\beta) = e^{-rT}\mathrm{E}\left[\Phi_B(S_T;\alpha,\beta)\right] = V_C(0;\alpha) - V_C(0;\beta).$$

Thus, all we need is to compute the price of binary call option for an arbitrary strike α, $V_C(0;\alpha)$. The explicit solution for S_t in the Black–Scholes model is

$$S_t = S_0\exp(rt - \frac{1}{2}\sigma^2 t + \sigma W_t).$$

Expressing the expectation through an integral we get

$$\begin{aligned}
V_C(0;\alpha) &= e^{-rT}\mathrm{E}\left[\mathbb{I}(S_T > \alpha)\right] \\
&= e^{-rT}\int_{-\infty}^{\infty}\mathbb{I}\left(S_0\exp(rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x) - \alpha\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= e^{-rT}\int_{-\infty}^{\infty}\mathbb{I}\left(x > -\frac{\ln S/\alpha + rT - \sigma^2 T/2}{\sigma\sqrt{T}}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= e^{-rT}\int_{-\frac{\ln S/\alpha + rT - \sigma^2 T}{\sigma\sqrt{T}}}^{\infty}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= e^{-rT}\int_{-d_2(\alpha)}^{\infty}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= e^{-rT}N(d_2(\alpha)).
\end{aligned}$$

Respectively, the price of a binary option paying K USD if spot S_T is between $[\alpha, \beta]$ is

$$V_B(0; \alpha, \beta) = K \times e^{-rT} \left[N\left(d_2(\alpha)\right) - N\left(d_2(\beta)\right) \right],$$

where

$$d_2(y) = \frac{\ln S/y + rT - \sigma^2 T/2}{\sigma\sqrt{T}}.$$

Exercise 3.5. Suppose the current value of the S&P-500 index is equal to USD 3,000. An investor wants to go long in a 6-month forward contract on the S&P-500 index. Calculate the forward price with the risk-neutral valuation method such that there is no cash payment at the start of the contract. Assume that the per annum risk-free interest rate with semi-annual compounding is 2%.

Solution:
We need to prove that the forward price is given by

$$f(t;T) = e^{r(T-t)} S_t.$$

Let us first determine the current price of a contract that pays at time T the value of the stock at time, S_T. Obviously, if we buy the stock today, we can sell it at time T exactly at the price at that time, S_T. The value of the stock today is S_t, thus the value of such a contract should be also S_t. Otherwise, we can make an arbitrage. Thus,

$$V_S(t) = e^{-r(T-t)} \mathrm{E}[S_T] = S_t.$$

Forward contract, entered today, is a contract to buy (or sell) stock at time T at a predefined price $f(t;T)$. The payoff of the buy contract is

$$\Phi(S_T) = S_T - f(t;T).$$

Market standard is that the current value of the contract is zero. Thus,

$$\begin{aligned} V(t) = 0 = e^{-r(T-t)} \mathrm{E}_t\left[S_T - f(t;T)\right] &= e^{-r(T-t)} \mathrm{E}_t[S_T] - e^{-r(T-t)} \mathrm{E}_t[f(t;T)] \\ &= S_t - e^{-r(T-t)} f(t;T) \mathrm{E}_t[1] \\ &= S_t - f(t;T) e^{-r(T-t)}, \end{aligned}$$

from where the required formula for the forward price follows.

Exercise 3.6. Consider a roulette table where people can bet on red or black. There is also green on which people cannot bet. The chance of green is 1/37. The probability of red and black are 18/37 and 18/37. If you bet USD 1 on red (or black) and win, you get USD 2 back. Calculate the risk-neutral probabilities. Show that an equivalent martingale measure (other wording for risk-neutral measure) does not exist. What does this imply for the fundamental theorems regarding arbitrage?

Solution:

Under the objective measure $\mathbb{P}$ we have the probabilities

$$P^{\mathbb{P}}(\text{red}) = P^{\mathbb{P}}(\text{black}) = 18/37$$
$$P^{\mathbb{P}}(\text{green}) = 1/37$$

We have two "assets", a bet on red and a bet on black. The price of these "assets", V, is equal to the initial bet, \$1. The payoff is \$2, if we are right, and zero, if we are wrong. Thus, we can formally compute probabilities under risk-neutral measure $\mathbb{Q}$ using the following equations.

$$V(\text{red}) = \$2 \cdot P^{\mathbb{Q}}(\text{red}) + \$0 \cdot P^{\mathbb{Q}}(\text{black}) + \$0 \cdot P^{\mathbb{Q}}(\text{green}) = \$1$$
$$V(\text{black}) = \$0 \cdot P^{\mathbb{Q}}(\text{red}) + \$2 \cdot P^{\mathbb{Q}}(\text{black}) + \$0 \cdot P^{\mathbb{Q}}(\text{green}) = \$1$$

From here we see that the risk-neutral probabilities are

$$P^{\mathbb{Q}}(\text{red}) = P^{\mathbb{Q}}(\text{black}) = 1/2.$$

And the risk-neutral probability of green is

$$P^{\mathbb{Q}}(\text{green}) = 1 - P^{\mathbb{Q}}(\text{red}) - P^{\mathbb{Q}}(\text{black}) = 0.$$

You can see that $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent measures, because $P^{\mathbb{P}}(\text{green}) > 0$ and $P^{\mathbb{Q}}(\text{green}) = 0$.

This implies that we have an arbitrage. If we short green and black "assets", which actually means that we own the casino and put the proceedings, prices of the bets in a risk-free account, i.e. own pocket, we have zero investments, non-negative payoff, and a positive probability of profit.

Exercise 3.7. The fairly unknown company *G&S Inc.* has blessed the market with a new derivative, 'the Mean'. With 'effective period' given by $[T_1, T_2]$ the holder of a mean contract will, at the maturity T_2, obtain the amount

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)du.$$

Determine the no-arbitrage price, at time t, of the Mean contract. Assume that you live in a Black-Scholes world, and $t < T_1$.

Solution:
The arbitrage free price of a contingent claim $\mathcal{X}$, even if it is not simple, i.e. $\mathcal{X} \neq \Phi(S(T))$, but depends upon path of S_t, is given by

$$\begin{aligned} V(t) &= e^{-r(T_2-t)} \mathrm{E}_t^Q[\mathcal{X}] \\ &= e^{-r(T_2-t)} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathrm{E}_t^Q[S(u)]du. \end{aligned}$$

For GBM

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ S(t) &= s \end{aligned}$$

the expectation of $S(u)$ is

$$\mathrm{E}_t^Q[S(u)] = se^{r(u-t)}.$$

Thus we have

$$\begin{aligned} V(t) &= e^{-r(T_2-t)} \frac{s}{T_2 - T_1} \int_{T_1}^{T_2} e^{r(u-t)} du \\ &= e^{-r(T_2-t)} \frac{s}{T_2 - T_1} e^{-rt} \frac{1}{r} \left(e^{rT_2} - e^{rT_1}\right) \\ &= \frac{s}{r(T_2 - T_1)} \left(1 - e^{-r(T_2 - T_1)}\right) \end{aligned}$$

Exercise 3.8. Consider the following processes under the probability measure $\mathbb{P}$:

1. $X(t) = W(t) - 2t$
2. $X(t) = e^{W(t)}$
3. $X(t) = e^{W(t)+t}$.

Find the Radon-Nikodym derivative $R = \frac{d\mathbb{Q}}{d\mathbb{P}}$ that change the measure from $\mathbb{P}$ to $\mathbb{Q}$ such that the processes become martingales under measure $\mathbb{Q}$.

Solution:

1. SDE under $\mathbb{P}$ measure is

   $$dX_t = -2dt + dW_t^P.$$

   For X_t to be a martingale under $\mathbb{Q}$ measure, the corresponding SDE should be

   $$dX_t = dW_t^Q.$$

   Hence,

   $$dW_t^Q = -2dt + dW_t^P.$$

   Further,

   $$Z_t = e^{-\int_0^t \Theta_s dW_s^P - \frac{1}{2}\int_0^t \Theta_s^2 ds} = e^{2W_t^P - 2t}$$

   and

   $$R = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = Z_T = e^{2W_T^P(\omega) - 2T}.$$

   Under this measure $d\mathbb{Q}$, X_t will be a martingale on $0 \le t \le T$.

2. The differential dX_t is

$$dX_t = \frac{1}{2}X_t dt + X_t dW_t^P = X_t\left(\frac{1}{2}dt + dW_t^P\right).$$

We want it to be a martingale under $d\mathbb{Q}$ measure, i.e.

$$dX_t = X_t dW_t^Q.$$

This means,

$$dW_t^Q = \frac{1}{2}dt + dW_t^P.$$

And therefore,

$$R = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = Z_T = e^{-\frac{1}{2}W_T(\omega)-\frac{1}{8}T}$$

3. The differential dX_t is

$$dX_t = \frac{3}{2}X_t dt + X_t dW_t^P = X_t\left(\frac{3}{2}dt + dW_t^P\right).$$

Following the same reasoning as above we get

$$R = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = Z_T = e^{-\frac{3}{2}W_T(\omega)-\frac{9}{8}T}.$$

Exercise 3.9.
Consider the following market:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma dW_t, \quad S_0 = s \\ dB_t &= rB_t dt, \quad B_0 = 1, \quad r > 0, \end{aligned}$$

where W is a Brownian Motion under the real-world probability measure $\mathbb{P}$, S denotes the stock price process and B the money market account.

Suppose now that the solution of the stock price SDE is given by:

$$S_t = e^{\mu t}S_0 + \sigma e^{\mu t}\int_0^t e^{-\mu s}dW_s.$$

(a) Give the probability distribution of $S_t|S_0$.

We know that increments of Brownian Motion are normally distributed. The stochastic integral is a sum of increments in Brownian Motion and therefore normally distribution.

We also know that the expectation of the stochastic integral is equal to zero. Therefore the expectation of $S_t|S_0$ is given by:

$$\mathbb{E}_0(S_t) = e^{\mu t}S_0.$$

The variance of $S_t|S_0$ is given by:

$$\mathbf{Var}_0(S_t) = \sigma^2 e^{2\mu t}\mathbb{E}_0((\int_0^t e^{-\mu s} dW_s)^2).$$

Then using Ito-isometry:

$$\mathbf{Var}_0(S_t) = \sigma^2 e^{2\mu t} \int_0^t e^{-2\mu s} ds.$$

Solving the integral gives:

$$\mathbf{Var}_0(S_t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right).$$

(b) Give your personal opinion on this model for stock price behaviour.

You could have observed that:

- **Negative stock prices are possible in this model (not very realistic)**
- **Constant volatility (not very realistic)**

(c) Use Itô's lemma to derive the process for the discounted stock price $Y_t := S_t/B_t$.

Solution:

$$\begin{aligned} dY_t = d\left(\frac{S_t}{B_t}\right) &= \frac{1}{B_t}dS_t - \frac{S_t}{B_t^2}dB_t \\ &= \mu\left(\frac{S_t}{B_t}\right)dt + \sigma\frac{1}{B_t}dW_t - r\frac{S_t}{B_t}dt \\ &= (\mu - r)Y_t dt + \sigma\frac{1}{B_t}dW_t. \end{aligned}$$

(d) Apply Girsanov's theorem and conclude that the stock price process for S under the risk-neutral probability measure is given by:

$$dS_t = rS_t dt + \sigma d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian Motion under the risk-neutral probability measure $\mathbb{Q}$.

Solution:

$$\begin{aligned} dY_t &= (\mu - r)Y_t dt + \sigma \frac{1}{B_t} dW_t \\ &= \sigma \frac{1}{B_t} \left(dW(t) + \frac{(\mu - r)S_t}{\sigma} dt \right) \\ &= \sigma \frac{1}{B_t} d\tilde{W}_t. \end{aligned}$$

Then use the SDE for S:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma dW_t \\ &= \mu S_t dt + \sigma \left(d\tilde{W}_t - \frac{(\mu - r)S_t}{\sigma} dt \right) \\ &= r S_t dt + \sigma d\tilde{W}_t. \end{aligned}$$

In the same spirit as above, the solution of this SDE is given by:

$$S_t = e^{rt} S_0 + \sigma e^{rt} \int_0^t e^{-rs} d\tilde{W}_s.$$

(e) Calculate the no-arbitrage price at time t of a European digital call option with strike price K and maturity date $T > t$. This option pays 1 if the stock price at maturity is larger than strike price K and 0 in all other cases.

Using the result of question (a) we know that under $\mathbb{Q}$:

$$S_T | S_t \sim N \left(e^{r(T-t)} S_0, \frac{\sigma^2}{2r} \left(e^{2r(T-t)} - 1 \right) \right).$$

Using the FFTAP we know that the no-arbitrage price of the digital call option at time t is given by:

$$V_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left(\mathbb{I}_{\{S_T > K\}} \right).$$

Now work out the expectation:

$$\begin{aligned} V_t &= e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left(\mathbb{I}_{\{S_T > K\}} \right) \\ &= e^{-r(T-t)} \mathbb{Q}_t (S_T > K) \\ &= e^{-r(T-t)} \mathbb{Q}_t \left(\frac{S_T - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r} \left(e^{2r(T-t)} - 1 \right)}} > \frac{K - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r} \left(e^{2r(T-t)} - 1 \right)}} \right) \\ &= e^{-r(T-t)} \Phi \left(\frac{K - e^{r(T-t)} S_0}{\sqrt{\frac{\sigma^2}{2r} \left(e^{2r(T-t)} - 1 \right)}} \right). \end{aligned}$$