Individual Assignment Set 3

Fall 2020

Exercise 1.1. If W is a Brownian Motion, find

$$\mathbb{E}_0[W_s^2 W_t - W_t^3],$$

when $t \geq s$.

Solution:
First note that, $\mathbb{E}_0[W_t^3] = 0$. We know that $W_t \sim N(0, \sqrt{t})$ and that all odd central moments of a symmetric distribution equal zero, i.e.

$$\mathbb{E}_0\left[(W_t - \mathbb{E}(W_t))^3\right] = 0.$$

Second,

$$\begin{aligned}
\mathbb{E}_0\left[W_s^2 W_t\right] &= \mathbb{E}_0[W_s^3] + \mathbb{E}_0[W_s^2(W_t - W_s)] \\
&= 0 + \mathbb{E}_0[\mathbb{E}_s[W_s^2(W_t - W_s)]] = \mathbb{E}_0[W_s^2 \mathbb{E}_s[(W_t - W_s)]] \\
&= 0
\end{aligned}$$

Exercise 1.2. Define the stochastic process Z by $Z(t) = W^4(t),\ t \geq 0$, where $W(t),\ t \geq 0$ is a Brownian Motion. Derive the stochastic differential equation of Z. What is the initial value?

Solution:

$$dZ_t = 4W_t^3 dW_t + \frac{1}{2} 12 W_t^2 dt.$$

Initial value is equal to zero, because the intial of Brownian Motion is zero.

Exercise 1.3. One of the reasons we care about Itô formula is that it allows us to show that processes are martingales. Recall that if $dX_t = a_t dW_t$, where W_t is a Brownian Motion and a_t is a predictable function, i.e. there is no drift term, then X_t is a martingale with respect to the filtration of W_t. Check that $e^{t/2}\cos(W_t)$ is a martingale using Itô's lemma.

Solution:
Define function $f(t, S) = e^{t/2}\cos(S)$. Then the partial derivatives of f are:

$$\begin{aligned}
\partial_t f &= \frac{1}{2} e^{t/2}\cos(S) \\
\partial_S f &= -e^{t/2}\sin(S) \\
\partial_{SS} f &= -e^{t/2}\cos(S).
\end{aligned}$$

Substitute these partial derivatives in the Itô formula to get:

$$\begin{aligned} de^{t/2}\cos(W_t) &= \partial_t f(t, W_t)dt + \partial_S f(t, W_t)dW_t + \frac{1}{2}\partial_{SS} f(t, W_t)dt \\ &= \frac{1}{2}e^{t/2}\cos(W_t)dt - e^{t/2}\sin(W_t)dW_t - \frac{1}{2}e^{t/2}\cos(W_t)dt \\ &= -e^{t/2}\sin(W_t)dW_t. \end{aligned}$$

Exercise 1.4. Compute the stochastic differential for Z when $Z(t) = \frac{1}{X(t)}$ and X has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

Solution:

$$dZ = -\frac{1}{X(t)^2}dX + \frac{2}{2}\frac{1}{X(t)^3}dX(t)^2 = (-\alpha + \sigma^2)Z(t)dt - \sigma Z(t)dW(t).$$