Exercise Set 2

Fall 2020

Exercise 2.1 Let $W_t, t \geq 0$ be a Brownian Motion, show that

(a) W_t^3 is not a martingale
(b) $W_t^3 - 3tW_t$ is a martingale.

Solution:

$$\begin{aligned}
\mathbb{E}[W_t^3|\mathcal{F}_s] &= \mathbb{E}[W_s^3 + 3W_s^2(W_t - W_s) + 3W_s(W_t - W_s)^2 + (W_t - W_s)^3|\mathcal{F}_s] \\
&= \mathbb{E}[W_s^3\mathcal{F}_s] + \mathbb{E}[3W_s^2(W_t - W_s)|\mathcal{F}_s] \\
&+ \mathbb{E}[3W_s(W_t - W_s)^2|\mathcal{F}_s] + \mathbb{E}[(W_t - W_s)^3|\mathcal{F}_s] \\
&= W_s^3 + 0 + 3W_s(t-s) + 0 \\
&= W_s^3 + 3(t-s)W_s.
\end{aligned}$$

Hence, W_t^3 is not a martingale.

$$\begin{aligned}
\mathbb{E}[W_t^3 - 3tW_t|\mathcal{F}_s] &= \mathbb{E}[W_t^3|\mathcal{F}_s] - \mathbb{E}[3tW_t|\mathcal{F}_s] \\
&= W_s^3 + 3(t-s)W_s - 3t\mathbb{E}[W_t|\mathcal{F}_s] \\
&= W_s^3 + 3(t-s)W_s - 3tW_s \\
&= W_s^3 - 3sW_s.
\end{aligned}$$

Hence, $W_t^3 - 3tW_t$ is a martingale.

Exercise 2.2. Compute the stochastic differential dZ when

(a) $Z(t) = e^{\alpha t}$
(b) $Z(t) = \int_0^t g(s)dW_s$, where g is an adapted stochastic process
(c) $Z(t) = e^{\alpha W_t}$

Solution:

$$dZ(t) = e^{\alpha t}\alpha dt = \alpha Z(t)dt.$$

$$dZ(t) = g(t)dW_t.$$

$$dZ(t) = e^{\alpha W_t}\alpha dW(t) + \frac{1}{2}e^{\alpha W_t}\alpha^2 dt = \frac{\alpha^2}{2}Z(t)dt + \alpha Z(t)dW(t).$$

Exercise 2.3. Let X and Y be given as solutions to the following system of stochastic differential equations:

$$
\begin{aligned}
dX &= \alpha X dt - Y dW, \quad X(0) = x_0, \\
dY &= \alpha Y dt + X dW, \quad Y(0) = y_0.
\end{aligned}
$$

Note that the initial values x_0, y_0 are deterministic constants.

(a) Prove that the process R defined by $R(t) = X^2(t) + Y^2(t)$ is deterministic.
(b) Compute $\mathbb{E}[X(t)]$.

Solution: We apply the Itô formula to $R = R(X, Y)$:

$$
\begin{aligned}
dR &= \partial_X R dX + \partial_Y R dY + \frac{1}{2}\partial_{XX} R dX^2 + \frac{1}{2}\partial_{YY} R dY^2 \\
&= 2X dX + 2Y dY + dX^2 + dY^2 \\
&= 2\alpha X^2 dt - 2XY dW + 2\alpha Y^2 dt + 2XY dW + Y^2 dt + X^2 dt \\
&= (2\alpha + 1)(X^2 + Y^2) dt = (2\alpha + 1) R dt.
\end{aligned}
$$

The absence of the diffusion term shows that the process $R(t)$ is deterministic.

Integrate the SDE for $X(t)$:

$$X(t) = X(0) + \alpha \int_0^t X(s) ds - \int_0^t Y(s) dW_s.$$

Take the expectation on both sides:

$$\mathbb{E}_0[X(t)] = X(0) + \alpha \int_0^t \mathbb{E}_0[X(s)] ds.$$

Now replace the expectations:

$$m(t) = m(0) + \alpha \int_0^t m(s) ds.$$

Or in differential form:

$$dm(t) = \alpha m(t) dt.$$

We know that the solution to this differential equation is:

$$m(t) = m(0) e^{\alpha t} = X(0) e^{\alpha t}.$$

And hence,

$$\mathbb{E}_0[X(t)] = m(t) = X(0) e^{\alpha t}.$$

Exercise 2.4. Is the Ornstein-Uhlenbeck process,

$$dX(t) = \theta(\mu - X(t)) dt + \sigma dW(t), \quad \theta, \mu, \sigma > 0,$$

a martingale?

Solution: Let us first define $Y(t) = X(t) - \mu$. Then

$$dY(t) = dX(t) - 0 = \theta(\mu - X(t))dt + \sigma dW(t) = -\theta Y(t)dt + \sigma dW(t).$$

Obviously, if $X(t)$ is a martingale, then $Y(t)$ is also a martingale.

Define $Z(t) = e^{\theta t} Y(t)$. Using Itô's Lemma we get:

$$dZ(t) = \theta e^{\theta t} Y(t)dt + e^{\theta t} dY(t) = \sigma e^{\theta t} dW(t).$$

In integral form:

$$Z(t) = Z(0) + \sigma \int_0^t e^{\theta u} dW(u).$$

For $t > s$ we can write:

$$Z(t) = Z(s) + \sigma \int_s^t e^{\theta u} dW(u).$$

Substituting $Y(t)$ into the equation gives:

$$Y(t) = e^{-\theta(t-s)} Y(s) + \sigma \int_s^t e^{-\theta(t-u)} dW(u).$$

Taking the conditional expectation we get:

$$\mathbb{E}[Y(t)|\mathcal{F}_s] = e^{-\theta(t-s)} Y(s) + 0 = e^{-\theta(t-s)} Y(s).$$

Or for $X(t)$

$$\mathbb{E}[X(t)|\mathcal{F}_s] = \mu + e^{-\theta(t-s)}(X(s) - \mu) + 0 = e^{-\theta(t-s)}(X(s) - \mu).$$

Unless $\theta = 0$, $\mathbb{E}[X(t)|\mathcal{F}_s] \neq X(s)$ and thus the Ornstein-Uhlenbeck process is not a martingale.

Exercise 2.5. Let $W_1(t)$ and $W_2(t)$ be two uncorrelated Wiener processes, i.e. $dW_1 dW_2 = 0$. And let stochastic process $Y(t)$ be defined as:

$$Y(t) = W_1(t)W_2(t).$$

Use the multi-dimensional Itô's lemma to find $dY(t)$. Additionally, based on the expression of $dY(t)$, argue if $Y(t)$ is a martingale or not.

Solution:

$$\begin{aligned} dY(t) &= W_2(t)dW_1(t) + W_1(t)dW_2(t) + dW_1(t)dW_2(t) \\ &= W_2(t)dW_1(t) + W_1(t)dW_2(t). \end{aligned}$$

Stochastic process $Y(t)$ is a martingale (no drift term).

Exercise 2.6. Suppose we have the following SDE for stochastic variance:

$$d\sigma_t^2 = -\kappa(\sigma_t^2 - \sigma^2)dt + \sigma^\sigma \sigma_t dW_t^V, \quad \sigma_0^2 = \sigma^2,$$

where W^V is Brownian Motions under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

(a) Derive $d\exp(\kappa t)\sigma_t^2$ and show that:

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2 \int_0^t e^{\kappa u} du + \int_0^t e^{\kappa u}\sigma^\sigma \sigma_u dW_u^V\right\}.$$

(b) Derive the expression for σ_{t+h}^2 conditional on σ_t^2.

(c) Compute $\mathbb{E}_t[\sigma_{t+h}^2]$.

Solution:

$$\begin{aligned}
d\exp(\kappa t)\sigma_t^2 &= \kappa\exp(\kappa t)\sigma_t^2 dt + \exp(\kappa t)d\sigma_t^2 \\
&= \kappa\exp(\kappa t)\exp(\kappa t)(-\kappa(\sigma_t^2 - \sigma^2)dt + \sigma^\sigma\sigma_t dW_t^V) \\
&= \kappa\sigma^2\exp(\kappa t)dt + \exp(\kappa t)\sigma^\sigma\sigma_t dW_t^V.
\end{aligned}$$

Write this in integral form:

$$\exp(\kappa t)\sigma_t^2 - \exp(\kappa 0)\sigma_0^2 = \kappa\sigma^2\int_0^t \exp(\kappa u)du + \int_0^t \exp(\kappa u)\sigma^\sigma\sigma_u dW_u^V.$$

Then it is trivial to see that:

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2 \int_0^t e^{\kappa u} du + \int_0^t e^{\kappa u}\sigma^\sigma \sigma_u dW_u^V\right\}.$$

$$\sigma_{t+h}^2 = e^{-\kappa(t+h)}\left\{\sigma_0^2 + \kappa\sigma^2 \int_0^{t+h} e^{\kappa u} du + \int_0^{t+h} e^{\kappa u}\sigma^\sigma \sigma_u dW_u^V\right\}.$$

Hence,

$$\sigma_{t+h}^2 = e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2\int_t^{t+h} e^{\kappa u}du + e^{-\kappa(t+h)}\int_t^{t+h} e^{\kappa u}\sigma^\sigma\sigma_u dW_u^V.$$

Finally, we can calculate the conditional expectation:

$$\begin{aligned}
\mathbb{E}_t(\sigma_{t+h}^2) &= e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2\int_t^{t+h} e^{\kappa u}du + 0 \\
&= e^{-\kappa h}\sigma_t^2 + e^{-\kappa(t+h)}\kappa\sigma^2\frac{1}{\kappa}(e^{\kappa(t+h)} - e^{\kappa t}) \\
&= \sigma^2 + e^{-\kappa h}(\sigma_t^2 - \sigma^2).
\end{aligned}$$