Case 2: Option Pricing and Delta Hedging

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1 Monte Carlo simulation

Geometric Brownian Motion is described by the following stochastic differential equation (SDE):

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sigma S(t)dW(t) \\
S(0) &= S_0,
\end{aligned}
$$

where W is a Brownian Motion under the risk-neutral probability measure.

Let the expiry time of an option be T, and let

$$
\begin{aligned}
N &= \frac{T}{\Delta t} \\
S_n &= S(n\Delta t).
\end{aligned}
$$

Then, given an initial price S_0, M realizations of the path of a risky asset are generated using the algorithm (Euler method):

$$S_{n+1} = S_n + S_n(r\Delta t + \sigma\sqrt{\Delta t}\varphi),$$

where φ is a normally distributed random variable with mean 0 and unit variance.

As in computer assignment 1, we want to price a European call option with maturity 3 months (0.25 years) and strike price equal to USD 3,300. The per annum 3-months interest rate equals 0.5% (with quarterly compounding). In this exercise, you can use the volatility that you have calculated in the second exercise of computer assignment 1 as the estimate of the volatility of monthly log-returns of S.

The price of an option can be calculated by computing the discounted value of the average pay-off, i.e.

$$V(S_0) = e^{-rT} \frac{\sum_{m=1}^{M} \text{payoff}^m(S_N)}{M},$$

where r is a continuously compounded interest rate.

(a) Choose Δt equal to 1 month (i.e. $N = 3$). Use the Euler method to calculate the option price with different choices for M: 100, 500, 1000, 5000, and 10000. Compare the results with your solutions of exercise 2e of computer assignment 1 (250 steps) and the result of the Black-Scholes formula.

Please note that you are not allowed to use programming packages for Monte Carlo simulations.

(b) Choose Δt equal to be 1 trading day. Assume that there are 63 trading days per quarter. Use the Euler method to calculate the option price with different choices for M: 100, 500, 1000, 5000, and 10000. Compare the results to the outcomes of a).

For special cases of constant coefficients, we can avoid time stepping errors for geometric Brownian Motion, since we can integrate the stochastic differential equation exactly to get:

$$S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}\varphi}.$$

(c) Generate 100 independent draws from the standard normal distribution. Use the provided formula to calculate S_T for each of the 100 draws. Then calculate the value of the call-option. Repeat the exercise but then take 500, 1000, 5000 and 10000 draws.

(d) Compare the results of c) to the Black & Scholes formula. In this exercise we have tried to approximate the Black-Scholes prices in a number of different ways. Why is the approach of part c) the most accurate approach?

2 Dynamic hedging

In the previous exercise we used the risk-neutral valuation method to approximate the value of a call-option in the Black-Scholes world. We have learned in week 1 of the course that we can also employ the replication method to derive the no-arbitrage price of a derivative contract. This method was explicitly illustrated in a binomial tree setting. In this exercise we are going to apply that method in a perfect Black-Scholes world.

We choose the practical situation where a trader sells a European call option on the S&P-500-index with maturity 3 months and strike price USD 3,300 to an institutional investor. We assume the absence of dividends and a perfect Black-Scholes world. The contract size of the call option contract is 100 and the trader sells 10 contracts. The value of the S&P-500 index at trade date is EUR 3,300. Use the volatility (per month) that you have also used in computer assignment 1. The 3-month per annum interest rate is equal to 0.5% (with quarterly compounding).

(a) Calculate the amount of money the trader receives from the institutional investor. As we live in a perfect Black-Scholes world, you can do this by using the Black-Scholes option pricing formula.

The trader has a considerable short position in the call option on his book, i.e. if the underlying value goes up, the trader will be unhappy. So, the trader decides to setup a hedge portfolio. The purpose of this hedge portfolio is to minimize the risks of the short position in the call-option. To be more precise: the trader wants to replicate a **long position in the call-option**. If the replication strategy works, the value of the replicating portfolio at the option's maturity date cancels out against the value of the option at its maturity date.

(b) Setup the dynamic hedging strategy where the hedge portfolio is adjusted on a weekly basis (use 13 weeks). Generate 5,000 paths for the S&P-500 index (for the avoidance of doubt: each path has 13 weeks), evaluate the value development of both the hedging strategy and the call-option, and provide a histogram of the P&L of the trader after three months (for the total position of shorting the call-option and hedging this with a portfolio of stocks and cash). We live in a perfect Black-Scholes world so you can use the Black-Scholes delta (which is $N(d_1)$) to setup and adjust the replicating portfolio.

Note 1: Replication takes place in the real-world, i.e. under the real-world probability measure $\mathbb{P}$. Hence, you need a model for the underlying value, the S&P-500-index under $\mathbb{P}$:

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\
S(0) &= S_0.
\end{aligned}
$$

You can assume that the per annum growth rate of the S&P-500 index, μ, is equal to 4%.

Note 2: we haven't discussed the 'Greeks' yet in the knowledge clips. The Greeks (delta, gamma, vega, rho, theta) measure the sensitivity of an option price for changes in a variable or a parameter. Delta, for instance, measures the sensitivity of an option price for a change in the underlying value of the option. Mathematically, delta is calculated as:

$$\delta = \frac{\partial C}{\partial S}.$$

If we have an analytical option pricing formula available, as in the Black-Scholes model, then we can derive δ by calculating this partial derivative. This turns out to be $N(d_1)$, where N stands for the cumulative distribution function of the standard normal distribution and,

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

(c) Repeat exercise (b) but now with a daily hedging frequency. Comment on the difference between the result of this exercise and (b).

(d) Repeat exercise (c) but now with $\mu = 8\%$ instead of $\mu = 4\%$. Comment on the difference between the result of this exercise and (c).

(e) Suppose now that the option is priced in the market at an annualized volatility that is 5%-points higher than the real-world volatility that you used to construct the hedge strategy. What is the impact of the higher option price, compared to the results of part (d)?

(f) The situation as described in (e), i.e. options being more expensive than expected given the volatility of the underlying, is already apparent in financial markets for more than three decades. That seems counter-intuitive. What is a possible explanation for this empirical phenomenon?

General information cases

The two cases of this course should be done in groups of two students. For each case, you have to write a short report where you must include the results, a short discussion/concluding section and the appendix with the codes. The reports should be submitted electronically, before the corresponding deadlines.

The preferred implementation tool is MatLab. During the allocated computer sessions, you will be able to get support and ask questions regarding your MatLab implementations. However, it is advised that you work on the assignments outside teaching hours and not only during the allocated sessions, as they will require significant time and effort.

In principle, you are not bound to MatLab and are free to choose the programming language/environment in which you would like to write your computer programs (so another alternative could be, e.g., R or Python). However, in this case we cannot guarantee implementation support.

Reporting requirements

- Report should be written in English. This means that you also should use English decimal notation in the text and in graphs.
- There is no maximum to the number of words or pages but please try to be concise but please make sure that it is clear to me what you did and what your interpretation of the results is.
- Figures and graphs should have captions such that they can be read independently from the text.
- Use your spelling checker.
- Put names and VU student numbers of all group members on the front page.
- Final reports should be uploaded on Canvas before 23:59pm CET on the deadline day.

Deadline

The deadline is for this case as follows:

- Case 2: Friday October 9, 2020 at 23:59 CET

Last but not least:

ENJOY AND GOOD LUCK!!