Eidgenössische Technische Hochschule Zürich

Analysis II: Several Variables

Lecture Notes — Spring Semester 2024

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Preface

This notes are the continuation of 'Analysis I: One variable' and follow the same format and general spirit.

The text is partially based in some notes originally crafted in German for the academic year 2016/2017 by Manfred Einsiedler and Andreas Wieser, they were designed for the Analysis I and II courses in the Interdisciplinary Natural Sciences, Physics, and Mathematics Bachelor programs. In the academic year 2019/2020, a substantial revision was undertaken by Peter Jossen.

For the academic year 2023/2024, Joaquim Serra wrote this English version. Although some chapter build on German original, with changes, the new version is completely different in several aspects: some chapters (such as the measure and integral, or most of global integral theorem) have been completely rewritten. Also the parts that have been have been reorganized: alternative proofs of some materials, rewriting and expansion in certain areas, and a more concise presentation in others. This version strictly aligns with the material presented in class, offering a streamlined educational experience.

The courses Analysis I/II and Linear Algebra I/II are fundamental to the mathematics curriculum at ETH and other universities worldwide. They lay the groundwork upon which most future studies in mathematics and physics are built.

Throughout Analysis I/II, we will delve into various aspects of differential and integral calculus. Although some topics might be familiar from high school, our approach requires minimal prior knowledge beyond an intuitive understanding of variables and basic algebraic skills. Contrary to high-school methods, our lectures emphasize the development of mathematical theory over algorithmic practice. Understanding and exploring topics such as differential equations and multidimensional integral theorems is our primary goal. However, students are encouraged to engage with numerous exercises from these notes and other resources to deepen their understanding and proficiency in these new mathematical concepts.

Contents

Chapter 9

Metric spaces

In Analysis I, we focused primarily on functions that operate between real numbers, $\mathbb{R}$ to $\mathbb{R}$. Consequently, we deeply explored the real number line, $\mathbb{R}$, and its properties.

Now, in Analysis II, we expand our scope to functions that map from $\mathbb{R}^n$ to $\mathbb{R}^m$, where n and m are positive integers. In this context, we start by delving into the properties of $\mathbb{R}^n$ (or $\mathbb{R}^m$, as we can use them interchangeably since n and m are arbitrary).

We'll discover that $\mathbb{R}^n$ follows the axioms of a metric space when equipped with the standard Euclidean distance, which we'll define later. This means that $\mathbb{R}^n$ is a metric space. Understanding this concept is vital because it lays the foundation for many fundamental definitions and results we want to establish for $\mathbb{R}^n$ and can be extended to a broader range of metric spaces.

Additionally, this will allow us to revisit (and review) the crucial concept of convergence, which we have seen in Analysis I for $\mathbb{R}$, within the broader context of metric spaces. We'll emphasize the essential properties of metric spaces and their topology, particularly focusing on compactness.

Furthermore, we'll introduce and explore standard results related to normed vector spaces. As we will see, these spaces are more general than Euclidean space $\mathbb{R}^n$, but they possess a higher degree of structure (i.e., more axioms) than metric spaces.

9.1 Basics of Metric Spaces

In this class, our primary focus will be on $\mathbb{R}^n$, but we'll find that certain definitions and proofs become clearer when viewed within the broader context of metric spaces, which have fewer axioms.

However, when we're dealing with a general metric space X, it often helps to initially visualize it as our familiar 2-dimensional or 3-dimensional spaces, $\mathbb{R}^2$ or $\mathbb{R}^3$. This approach can simplify the intuitive understanding of various arguments. If an argument relies on only a limited set of fundamental properties of $\mathbb{R}^3$ (excluding aspects like angles and vector space structure), it may be applicable to general metric spaces as well.

9.1.1 The Euclidean space $\mathbb{R}^n$

For some integer $n \geq 1$, we denote by $\mathbb{R}^n$ the set of all ordered n-tuples of real numbers. A general element $x \in \mathbb{R}^n$ is thus of the form $x = (x_1, \ldots, x_n)$, where the x_i's are real numbers. (Even more rigorously $\mathbb{R}^n := \big\{x \colon \{1, \ldots, n\} \to \mathbb{R}\big\}$.)

$\mathbb{R}^n$ is a vector space over the field of real numbers with the coordinate-wise addition and
1 multiplication by a scalar. Rigorously, given $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

$$x + y := (x_1 + y_1, \ldots, x_n + y_n), \qquad \lambda x := (\lambda x_1, \ldots, \lambda x_n).$$

Definition 9.1: Euclidean Structure of $\mathbb{R}^n$

Given $x, y \in \mathbb{R}^n$ we define the standard **scalar product** of x and y as

$$x \cdot y = \langle x, y \rangle := \sum_{1 \leq i \leq n} x_i y_i,$$

the **Euclidean norm** of x as

$$\|x\| := \sqrt{\sum_{1 \leq i \leq n} x_i^2},$$

and the **Euclidean distance** of x and y as

$$d(x, y) := \|x - y\| = \sqrt{\sum_{1 \leq i \leq n} (x_i - y_i)^2}.$$

We key property of the Euclidean distance that we want to abstract is

Proposition 9.2: Triangle Inequality in $\mathbb{R}^n$

For all $x, y, z \in \mathbb{R}^n$

$$\|x - z\| \leq \|x - y\| + \|y - z\|.$$

Equivalently, for all $x, y \in \mathbb{R}^n$, $\|x + y\| \leq \|x\| + \|y\|$.

Proof. To prove the equivalence of the two statements consider $a = x - y$ and $b = y - z$ so that $a + b = x - z$.

We will see later a more general proof that applies to all inner product spaces (see Corollary 9.101), based on the Cauchy-Schwarz inequality (see Proposition 9.100). Lets us give here a hands-on argument.

Pick two points in $\mathbb{R}^n$, $x = (x_1, \dots x_n)$ and $y = (y_1, \dots, y_n)$ we would like to show

$$\Big(\sum_{i=1}^{n}(x_i + y_i)^2\Big)^{1/2} \leq \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} + \Big(\sum_{i=1}^{n} y_i^2\Big)^{1/2},$$

1 Taking squares, this is equivalent to

$$\sum_{i=1}^{n} x_i^2 + 2x_i y_i + y_i^2 \leq \sum_{i=1}^{n} x_i^2 + 2\Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} + \sum_{i=1}^{n} y_i^2,$$

and, canceling terms, to

$$\sum_{i=1}^{n} x_i y_i \leq \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n} y_i^2\Big)^{1/2}. \tag{9.1}$$

Therefore, the Proposition will follow if we can establish the validity of (9.1), or (squaring it) of:

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)\Big(\sum_{j=1}^{n} x_j y_j\Big) = \sum_{i,j=1}^{n} x_i x_j y_i y_j \leq \sum_{i,j=1}^{n} x_i^2 y_j^2.$$

But this last inequality is easily established summing over all pairs $i, j \in \{1, \dots, n\}$ the inequalities

$$2x_i x_j y_i y_j \leq x_i^2 y_j^2 + x_i^2 y_j^2 \quad \Leftrightarrow \quad (x_i y_j - x_j y_i)^2 \geq 0;$$

and observing that $\sum_{i,j=1}^{n} x_i^2 y_j^2 = \sum_{i,j=1}^{n} x_j^2 y_i^2$. □

9.1.2 Definition of metric Space

DEFINITION 9.3: METRIC SPACE

A **metric space** (X, d) is a nonempty set X together with a nonnegative function $d : X \times X \to [0, \infty)$, called the **distance** (or **metric**) on X, which satisfies:

(1) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$ (Definiteness).

(2) For all $x, y \in X$, $d(x, y) = d(y, x)$ (Symmetry).

(3) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

9.4. — A metric d on a set X assigns to each pair of points their **distance**. In this interpretation, the definiteness condition states that the only point at zero distance from a given point $x \in X$ is x itself. The symmetry condition states that the distance from $x \in X$ to $y \in X$ is the same as from y to x. Interpreting the distance between two points as the length of a shortest path from one point to the other, the triangle inequality states that the length of a shortest path from x to y is at most the length of a path one takes by first going to y and then from there to z.

1 EXERCISE 9.5. — Prove using induction over the number of points that $N \geq 3$ that if d satisfies the triangle inequality then:

$$d(x_1, x_N) \leq \sum_{i=1}^{N-1} d(x_i, x_{i+1}).$$

9.6. — When there is no possible confusion, we will often say "Let X be a metric space...", leaving the distance function unspecified. This is a shorter version of the more precise sentence "Let (X, d) be a metric space...".

Furthermore, we may refer to the set X as a **space** and the elements of X as **points**. This is because we have in mind that X is some sort of geometric space, like a subset of the plane or the surface of a sphere. In this setting, "spaces" and "points" will be synonymous with "sets" and "elements".

9.7. — Notice that the Euclidean space $(\mathbb{R}^n, d)$, with $d(x, y) := \|x - y\|$, is a metric space. In particular $\mathbb{R}$, equipped with the absolute value distance $|x - y|$ is a metric space.

EXERCISE 9.8. — Let (X, d) be a metric space and let $\phi\colon [0, \infty) \to [0, \infty)$ a function which is concave, increasing, $\phi(0) = 0$, and not identically zero. Show that $(X, \phi \circ d)$ is again a metric space. For example one can take $\phi(t) = \sqrt{t}, \phi(t) = \arctan t$ or $\phi(t) = \frac{t}{1+t}$. Notice that the last two choices always give bounded distances.

Example 9.9. — When $X \subset \mathbb{R}$, we can take the **standard metric** d defined by

$$d(x,y) = |x-y| \text{ for all } x, y \in X.$$

Example 9.10. — Let X be a set, and $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for $x, y \in X$. Then, (X, d) is a metric space. Indeed, d is definite and symmetric by definition. Furthermore, d satisfies the triangle inequality: Let x, y, z be points in X. If $d(x,z) = 0$, then $d(x,z) \leq d(x,y) + d(y,z)$ is trivially satisfied. If $d(x,z) = 1$, then $x \neq z$, and y is at least different from one point in $\{x, z\}$, so the triangle inequality also holds. This metric d is called the **discrete metric** on the set X.

Example 9.11. — Let $X = \mathbb{R}^2$, and define the **Manhattan metric** on X by

$$d_{\mathrm{NY}}(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

where we put $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It can be verified (exercise) that d_{NY} satisfies all 1 axioms of a metric. The reason why d_{NY} is called the Manhattan metric is that in grid-like places such as Manhattan, one can reach from (x_1, x_2) to (y_1, y_2) in the following way: first move 'horizontaly' (i.e., with constant second coordinate from $x = (x_1, x_2)$ to (y_1, x_2) and then 'vertically' (with constant first coordinate) from (y_1, x_2) to $y = (y_1, y_2)$, or vice versa: $x = (x_1, x_2)$ to (x_1, y_2), and then to $y = (y_1, y_2)$. Since all streets in Manhattan run either from west to east or from north to south, d_{NY} measures the relevant distance between two points.

Exercise 9.12. — Let X be the set of all continuous real-valued functions defined on $[0,1] \subset \mathbb{R}$. For $f, g \in X$ set

$$d_1(f,g) := \max\{|f(x) - g(x)| \mid x \in [0,1]\} \quad \text{and} \quad d_2(f,g) := \int_0^1 |f(x) - g(x)| dx.$$

Show that (X, d_1) and (X, d_2) are metric spaces.

Example 9.13. — If (X, d) is a metric space and $X_0 \subset X$ is some subset then X_0 inherits an structure of metric space from X. Indeed, one can easily verify that (X_0, d_0), where d_0 the restriction of d to $X_0 \times X_0 \subset X \times X$ is a metric space.

For a more concrete instance of this take $X = \mathbb{R}^3$ with the Euclidean distance d and let X_0 be the sphere $\left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = R^2\right\}$, for some $R > 0$. Then for any pair of points

x, y in the sphere we have

$$\begin{aligned} d_0(x,y) &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} = \sqrt{(x-y)\cdot(x-y)} \\ &= \sqrt{x\cdot x + y\cdot y - 2x\cdot y} = \sqrt{2R^2 - 2x\cdot y} \end{aligned}$$

An arguably more natural metric d_1 on the sphere X_0 can be define measuring the length of the geodesic arc joining x and y. One can see that this metric is given by:

$$d_1(x,y) = R\arccos\left(\frac{x\cdot y}{R^2}\right) \in [0,\pi R].$$

9.1.3 Sequences, limits, and completeness

The definition of sequence in a set was given in Analysis I. We recall it next:

Definition 9.14: Sequences in a set

Let X be a set, a **sequence** in X is a function $x : \mathbb{N} \to X$. The image $x(n)$ of $n \in \mathbb{N}$ is also denoted as x_n and referred to as the n-th **term** of x.
Instead of $x : \mathbb{N} \to X$, one often writes $(x_n)_{n\in\mathbb{N}}$, $(x_n)_{n\geq 0}$, or $(x_n)_{n=0}^{\infty}$.

We introduce the following vocabulary, which is useful if used precisely.

1 Definition 9.15: "Eventually"

Let $(x_n)_{n\geq 0} \subset X$ be a sequence and let $\mathcal{P} : X \to \{\text{true}\ ,\ \text{false}\}$ be a property that an element in X might have or not. Then one says that "x_n satisfies $\mathcal{P}$ eventually" there exist $N \in \mathbb{N}$ such that $\mathcal{P}(x_n)$ is true for all $n \geq N$. In other words, if $\mathcal{P}(x_n)$ holds true along the sequence with only finitely many exceptions.

Definition 9.16: Convergent sequence

Let (X,d) be a metric space, $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ **converges** to x, or that x is the **limit** of the sequence $(x_n)_{n\in\mathbb{N}}$, if

$$\lim_n d(x_n, x) = 0.$$

In other words, for any $\varepsilon > 0$ eventually $d(x_n, x) < \varepsilon$.

9.17. — When the metric space (X,d) is clear from the context, we may write $\lim_n x_n = x$ or even $x_n \to x$ to express that $(x_n)_{n\in\mathbb{N}}$ converges to x.

Exercise 9.18. — In the setting of Exercise 9.8, show that a sequence converges in (X,d) if and only if it converges in $(X, \phi\circ d)$.

Lemma 9.19: Uniqueness of the limit

In a metric space, a convergent sequence has only one limit.

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Proof. Let (X, d) be a metric space and let $A, B \in X$ be limits of some sequence $(x_n)_{n=0}^{\infty}$, we mean to show that $A = B$. Take $\varepsilon > 0$, then, we can find $N_A, N_B \in \mathbb{N}$ such that $d(x_n, A) < \frac{\varepsilon}{2}$ for all $n \geq N_A$, and $d(x_n, B) < \frac{\varepsilon}{2}$ for all $n \geq N_B$. Then, for $N := \max\{N_A, N_B\}$, we have that

$$d(A, B) \leq d(A, x_N) + d(x_N, B) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where we used the triangular inequality. Since $\varepsilon > 0$ was arbitrary, it follows that $d(A, B) = 0$, and thus $A = B$ because of the definitness of d. □

We recall the notions of subsequences and accumulation points

Definition 9.20: Subsequence

Let $(x_n)_{n=0}^{\infty}$ be a sequence in a set X.
A **subsequence** of $(x_n)_{n=0}^{\infty}$ is a sequence of the form $(x_{f(k)})_{k=0}^{\infty}$, where $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. It is standard to denote subsequences by

$$(x_{n_k})_{k\in\mathbb{N}}, \quad (x_{n_k})_{k\geq 0}, \quad (x_{n_k})_{k=0}^{\infty} \quad (\text{i.e., } n_k := f(k))$$

Definition 9.21: Accumulation points

Let (X, d) be a metric space.

- Given $Y \subset X$, we say that that $x \in X$ is an **accumulation point** of Y if there exists a sequence $(y_n)_{n\geq 0} \subset Y$ converging to x.

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- Given a sequence $(x_n)_{n\geq 0}$ in X, we say that x is an **accumulation point** of $(x_n)_{n\geq 0}$ if some subsequence converges to x.

Lemma 9.22: Accumulations points of a converging sequence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X, and let $x \in X$. $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

Proof. We first prove the “only if” part. Let $(x_{f(n)})_{n\in\mathbb{N}}$ be a subsequence, i.e., $f : \mathbb{N} \to \mathbb{N}$ is some strictly increasing map. Given $\varepsilon > 0$ there is N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. Hence, $d(x_{f(n)}, x) < \varepsilon$ for all $n \geq N$, indeed $f(n) \geq n$ since f is strictly increasing.

We now prove the “if” part we can simply use that $(x_n)_{n\geq 0}$ itself is a subsequence (choose $f(n) = n$) and hence it converges to x. □

9.23. — A stronger version of the previous Lemma that is useful in some contexts asserts the following: a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space converges to x if and only if every partial sequence $(x_{f(n)})_{n\in\mathbb{N}}$ (f increasing) has a sub-subsequence $(x_{g(f(n))})_{n\in\mathbb{N}}$ (g increasing) converging to x.

While the proof of the "only if" part is similar ($g(f(n) \geq n$) the "if" part is less trivial than in the previous lemma. One can argue by contraposition: If x_n does **not** converge to x then we want to find a subsequence such that we **cannot** extract a sub-subsequence converging to x.

To do so we start by the negation of "x_n converges to x. Recall:

$$x_n \to x \qquad \Leftrightarrow \qquad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad d(x_n, x) < \varepsilon$$

The negation of this is:

$$\exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \qquad d(x_n, x) \geq \varepsilon$$

In other words, the set of $\{n \in N \mid d(x_n, x) \geq \varepsilon\}$ and hence there is an increasing sequence $(n_k)_{k\geq 0}$ such that $d(x_{n_k}, x) \geq \varepsilon$. Notice that any sub-subsequence will still remain at distance $\geq \varepsilon$ from x and hence will not converge to x.

2 This stronger version can be used, for example, to prove that a continuous function $f : [0,1] \to \mathbb{R}$ has a unique minimum point if and only if all sequences $(x_n)_{n\geq 0} \subset [0,1]$ such that $f(x_n) \to \min_{[0,1]}$ converge to the same limit point.

Lemma 9.24: Convergence: in $\mathbb{R}^n$ versus coordinate-wise

A sequence in $\mathbb{R}^n$ converges (in the Euclidean distance) if and only if it converges coordinate-wise.

Proof. Let $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ be a sequence. For $j = 1, \ldots, n$, we denote with $x_{k,j}$ the j-th component of the vector x_k.

Assume that $x_k \to x \in \mathbb{R}^n$. By definition, given $\varepsilon > 0$ and any j, it holds

$$|x_{k,j} - z_j| \leq \left(\sum_{i=1}^{n} (x_{k,i} - z_i)^2\right)^{1/2} = \|x_k - x\| \leq \varepsilon \quad \text{eventually in } k.$$

This proves that, for each $j = 1, \ldots, n$, $x_{k,j} \to x_j$ when $k \to \infty$ (as sequences of real numbers, with the standard absolute value distance).

Assume now that for each $j = 1, \ldots, d$ it holds

$$x_{k,j} \to x_j \qquad \text{as } k \to \infty,$$

for some numbers $z_j \in \mathbb{R}$. We prove that $x_k \to x$ in $\mathbb{R}^d$ where $z := (z_1, \dots, z_d)$. Given $\varepsilon > 0$, for each $j \in \{1, \dots, d\}$, there exists an $N_j \in \mathbb{N}$ such that

$$|x_{k,j} - x_j| < \frac{\varepsilon}{n} \text{ for all } k \geq N_j.$$

This means that

$$\Big(\sum_{i=1}^{n} (x_{k,i} - x_i)^2\Big)^{1/2} \leq \sqrt{n}\frac{\varepsilon}{n} < \varepsilon \text{ for all } k \geq \max\{N_1, \dots, N_j\},$$

which proves that $x_k \to x$ with respect to the Euclidean distance. □

We introduce the concept of **completeness** for metric spaces. This concept does not conflict with the notion of completeness that we gave for $\mathbb{R}$. We will soon show that $\mathbb{R}$, as well as $\mathbb{C}$, is complete as a metric space. In contrast, the metric space $\mathbb{Q}$ is not complete.

Definition 9.25: Cauchy sequence

A sequence $(x_n)_{n=0}^{\infty}$ in a metric space (X, d) is a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all pair of integers m, n with $n \geq N$
2 and $m \geq N$.

Exercise 9.26. — Prove the following elementary facts about Cauchy sequences in a metric space (X, d):

- A Cauchy sequence is bounded (meaning that $\{d(x_n, x_0)\} \subset \mathbb{R}$ is a bounded set)
- Every convergent sequence is a Cauchy sequence.
- A Cauchy sequence converges if and only if it has a converging subsequence.

Definition 9.27: Complete metric space

A metric space (X, d) is called **complete** if every Cauchy sequence in (X, d) converges.

Example 9.28. — The interval $(0, 1) \subset \mathbb{R}$, endowed with the standard distance $d(x, y) = |x - y|$, is *not* complete. However, $\mathbb{N} \subset \mathbb{R}$ is complete, as well as $[0, \infty)$.

Exercise 9.29. — Show that $\mathbb{Q}$, with the distance inherited from the standard distance on $\mathbb{R}$, is **not** a complete metric space.

Exercise 9.30. — Show that the space X of all bounded sequences $(x_n)_{n\geq 0}$ of real numbers, equipped with the distance

$$d\big((x_n)_{n\geq 0}), (y_n)_{n\geq 0}\big) = \sup_{n\geq 0} |x_n - y_n|$$

is complete. Show also that the subspace X_0 of sequences with limit 0 is complete.

Theorem 9.31: Completeness of $\mathbb{R}^n$

For all $n \geq 1$, $\mathbb{R}^n$ (with the Euclidean distance) is complete. In particular, $\mathbb{R}$ and $\mathbb{C}$ are complete.

Proof. Similar to the proof Lemma 9.24, a sequence (x_k) in $\mathbb{R}^n$ is a Cauchy if and only if $x_{k,j}$, $j = 1, \dots, n$ are Cauchy (in $\mathbb{R}$). It then follows from Theorem 2.124 □

9.32. — Completion of metric space (extra material) Let (X, d) be a metric space. We write $\mathcal{C}_X$ for the set of all Cauchy sequences in X and define an equivalence relation on $\mathcal{C}_X$ by

$$(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty} \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

2 The quotient set $\overline{X} = \mathcal{C}_X/_{\sim}$, equipped with the metric $\overline{d}$ given by

$$\overline{d}([(x_n)_{n=0}^{\infty}], [(y_n)_{n=0}^{\infty}]) = \lim_{n\to\infty} d(x_n, y_n),$$

is called the **completion** of (X, d). The injection $\iota : X \to \overline{X}$, mapping $x \in X$ to the class of the constant sequence with value x, is called the **canonical embedding**. For all $x, y \in X$, we have

$$d(x, y) = \overline{d}(\iota(x), \iota(y)),$$

which implies that ι is injective.

Exercise 9.33. — Show that the objects introduced in 9.32 are well-defined. In particular, verify that $\overline{d}$ is indeed a metric on $\overline{X}$.

Exercise 9.34. — As the name suggests, the completion $(\overline{X}, \overline{d})$ of a metric space is complete, meaning that every Cauchy sequence in $\overline{X}$ converges. A sequence in $\overline{X}$ is essentially a sequence of sequences, i.e.,

$$[(x_{m,n})_{n=0}^{\infty}]_{m=0}^{\infty}.$$

Show that $[(x_{n,n})_{n=0}^{\infty}]$ is a limit of this sequence.

Exercise 9.35. — Let (X, d) be a metric space with completion $(\overline{X}, \overline{d})$. Let (Y, d_Y) be a complete metric space, and let $f : X \to Y$ be a function such that

$$d(x, y) = d_Y(f(x), f(y))$$

for all $x, y \in X$. Show that there exists a unique function $\overline{f} : \overline{X} \to Y$ such that $f = \overline{f} \circ \iota_X$ and

$$\overline{d}(\overline{x}, \overline{y}) = d_Y(\overline{f}(\overline{x}), \overline{f}(\overline{y}))$$

for all $\overline{x}, \overline{y} \in \overline{X}$. This can be interpreted as: "$\overline{X}$ is the *smallest* complete metric space containing X."

9.1.4 *The Reals as the Completion of Rationals (extra material; cf. Grundstrukturen)

In the first semester, we defined $\mathbb{R}$ as any complete ordered field, postulating its existence (Definitions 2.18 and 2.19).The idea of completion of metric spaces allows one to easily construct a model of $\mathbb{R}$. This constructions shows, in particular, the existence of a complete ordered field. One can also prove with a bit of patience (although it is not hard to do so) that actually there is only one model of $\mathbb{R}$, in the sense that any two complete ordered fields must be isomorphic. 2

The completion of $\mathbb{Q}$ serves as a model for a field of real numbers. First, note that the construction of the completion of $\mathbb{Q}$ does not necessarily require a field of real numbers (as the target space for the standard metric on $\mathbb{Q}$). The set of all Cauchy sequences $\mathcal{C}$ in $\mathbb{Q}$ is the set of all sequences of rational numbers $(q_n)_{n=0}^{\infty}$ such that

$$\forall k \in \mathbb{N}\, \exists N \in \mathbb{N} : m, n \geq N \implies |q_n - q_m| < 2^{-k}.$$

The set $\mathcal{C}_{\mathbb{Q}}$ is a vector space over $\mathbb{Q}$ with component-wise operations, and

$$\mathcal{N} = \left\{ (q_n)_{n=0}^{\infty} \in \mathcal{C} \mid \lim_{n \to \infty} q_n = 0 \right\}$$

is a linear subspace. The equivalence relation $(p_n - q_n)_{n=0}^{\infty} \in \mathcal{N}$ in 9.32 translates to $(p_n - q_n)_{n=0}^{\infty} \in \mathcal{N}$. We define the set $\mathbb{R}$ as the quotient

$$\mathbb{R} = \mathcal{C}/_{\sim} = \mathcal{C}/\mathcal{N}$$

in the sense of linear algebra. Thus, $\mathbb{R}$ is a vector space over $\mathbb{Q}$. We denote the injective linear map $\iota : \mathbb{Q} \to \mathbb{R}$ by the canonical embedding, which assigns to $q \in \mathbb{Q}$ the class of the constant sequence with value q. From now on, elements of $\mathbb{R}$ are called *real numbers*, and we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$ via the canonical embedding ι.

We define a product on $\mathbb{R}$ by component-wise multiplication. That is, for elements $x = [(p_n)_{n=0}^{\infty}]$ and $y = [(q_n)_{n=0}^{\infty}]$, we define

$$x \cdot y = [(p_n q_n)_{n=0}^{\infty}].$$

It can be verified that this gives a well-defined commutative operation on $\mathbb{R}$, satisfying the distributive law with respect to addition, and compatible with the multiplication of rational numbers via the canonical embedding. In particular, $1_{\mathbb{R}} = \iota(1) = [(1)_{n=0}^{\infty}]$ is the multiplicative identity in $\mathbb{R}$. If $x = [(q_n)_{n=0}^{\infty}]$ is non-zero, then $(q_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge to zero. Therefore, $q_n \neq 0$ for all but finitely many $n \in \mathbb{N}$. The class of the sequence $(p_n)_{n=0}^{\infty}$ given by

$$p_n = \begin{cases} 1 & \text{if } q_n = 0 \\ q_m^{-1} & \text{otherwise} \end{cases}$$

serves as a multiplicative inverse for x. This shows that $\mathbb{R}$ is a field with the given operations.

We use the usual order relation on $\mathbb{Q}$ to construct an order relation on $\mathbb{R}$. For elements $x = [(p_n)_{n=0}^{\infty}]$ and $y = [(q_n)_{n=0}^{\infty}]$ in $\mathbb{R}$, we declare

$$x \leq y$$

if there exists a sequence $(r_n)_{n=0}^{\infty} \in \mathcal{N}$ such that $p_n - r_n \leq q_n$ for all $n \in \mathbb{N}$. It is left to
2 the diligent reader to verify that this indeed defines a well-defined order relation on $\mathbb{R}$ that is compatible with the field structure on $\mathbb{R}$. Thus, $\mathbb{R}$ is equipped with the structure of an ordered field.

It remains to show that the ordered field $\mathbb{R}$ is complete in the sense of Definition 2.198.It is easy to see that $\mathbb{R}$ satisfies the Archimedean Principle: Let $x = [(q_n)_{n=0}^{\infty}] \in \mathbb{R}$ be positive. Then, $(q_n)_{n=0}^{\infty}$ is not a null sequence. Thus, there exists a $k \in \mathbb{N}$ such that $|q_n| > 2^{-k}$ for infinitely many $n \in \mathbb{N}$. However, $(q_n)_{n=0}^{\infty}$ is also a Cauchy sequence, so there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies |q_n - q_m| < 2^{-k-1}$. This shows that $|q_n| > 2^{-k-1}$ and even $q_n > 2^{-k-1}$ for all but finitely many $n \in \mathbb{N}$, since $x > 0$. This demonstrates $\iota(2^{-k-1}) \leq x$, or simply $2^{-k-1} \leq x$ as we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$. Thus, the Archimedean Principle holds, as stated in Corollary 2.66. Now, let $X, Y \subset \mathbb{R}$ be non-empty subsets such that $x \leq y$ for all $x \in X$ and $y \in Y$. We want to find a real number $z = [(r_n)_{n=0}^{\infty}] \in \mathbb{R}$ between X and Y. To do this, we first choose arbitrary $a_0, b_0 \in \mathbb{Q}$ such that $[a_0, b_0] \cap X \neq \emptyset$ and $[a_0, b_0] \cap Y \neq \emptyset$, and set $r_0 = \frac{1}{2}(a_0 + b_0)$. If $x \leq r_0 \leq Y$ for all $x \in X$ and $y \in Y$, we set $z = r_0$ and we are done. Otherwise, we define a_1 and b_1 as

$$\begin{cases} a_1 = r_0 \text{ and } b_1 = b_0 & \text{if } [a_0, r_0] \cap X \neq \emptyset \text{ and } [a_0, r_0] \cap Y \neq \emptyset \\ a_1 = a_0 \text{ and } b_1 = r_0 & \text{if } [r_0, b_0] \cap X \neq \emptyset \text{ and } [r_0, b_0] \cap Y \neq \emptyset \end{cases}$$

and set $r_1 = \frac{1}{2}(a_1 + b_1)$. By continuing this process, we either find an r_n such that $x \leq r_n \leq Y$ for all $x \in X$ and $y \in Y$, and we set $z = r_n$, or we obtain sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and

$(r_n)_{n=0}^{\infty}$ with $|b_n - a_n| \leq 2^{-n}|a_0 - b_0|$ and

$$[a_n, b_n] \cap X \neq \emptyset \quad \text{and} \quad [a_n, b_n] \cap Y \neq \emptyset.$$

As the diligent reader can verify, this implies that $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and $(r_n)_{n=0}^{\infty}$ are all Cauchy sequences, and the real number

$$z = [(a_n)_{n=0}^{\infty}] = [(r_n)_{n=0}^{\infty}] = [(b_n)_{n=0}^{\infty}]$$

satisfies the inequalities $x \leq z \leq y$ for all $x \in X$ and $y \in Y$.

2

9.2 Topology of Metric Spaces

9.2.1 Open and closed sets

Definition 9.36: Open balls

Let (X, d) be a metric space, $x \in X$, and $r > 0$ a real number. In this context, we write

$$B(x, r) := \{y \in X \mid d(x, y) < r\}$$

and refer to the set $B(x, r)$ as the **open ball** with center x and radius r.

Definition 9.37: Open and Closed sets

Let (X, d) be a metric space:

- A subset $E \subset X$ is called **open** if, for every $x \in E$, there exists $r > 0$ such that $B(x, r) \subset E$.
- The collection of all open sets, $\mathcal{T}_d = \{U \subset X \mid U \text{ open}\}$, is called the **topology** generated by d.
- A subset $E \subset X$ is called **closed** if $X \setminus E$ is open.

3

9.38. — In particular, $\emptyset$ and X are always both open and closed. In general, a subset U needs not to be neither open nor closed.

It is not true in general that the only "clopen" sets in a space are the empty set $\emptyset$ and the space itself X. A set is termed "clopen" if it is both open and closed. For illustration, take the space $X = (0, 1) \cup (2, 3)$, equipped with the standard metric from $\mathbb{R}$. Here, the intervals $(0, 1)$ and $(2, 3)$ are clopen: they are open and closed in X. This example underscores the presence of other clopen sets beyond just $\emptyset$ and X. The significance of clopen sets will become more apparent in our discussions on connectedness. As we will see, connected spaces are precisely characterized by the absence of nontrivial (neither empty nor the whole space) clopen sets.

9.39. — Consider the set $X = [0, 2]$, equipped with the standard metric inherited from $\mathbb{R}$. In this context, the subset $[0, 1)$ is open within X (an exercise worth verifying). However, when considered as subset of the whole $\mathbb{R}$, $[0, 1)$ is neither open nor closed. This example illustrates that statements regarding the openness of a set like $[0, 1)$ require clarity about the ambient space (X, d) being considered. In practice, though, such nuances are often glossed over when the context is clear, and delving into these subtleties is usually unnecessary for typical discussions.

Exercise 9.40. — Let (X, d) be a metric space. Show that

- The open ball $B(x, r)$ is an open set.
- Every finite subset of X is closed.

Proposition 9.41: Arbitrary unions and Finite intersections

Let $(U_i)_{i\in I}$ be any family of open subsets of X, then $\bigcup_{i\in I} U_i$ is also open. If I is a finite set, also $\bigcap_{i\in I} U_i$ is open.

Proof. Set

$$U = \bigcup_{i\in I} U_i$$

3 and let $x \in U$. Then there exists $i \in I$ with $x \in U_i$, and since U_i is open, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_i$, implying $B(x, \varepsilon) \subset U$. Thus, U is open. Finally, let $(U_i)_{i\in I}$ be a finite family of open subsets of X. Set

$$U = \bigcap_{i\in I} U_i$$

and let $x \in U$. Then $x \in U_i$ for all $i \in I$, and for each $i \in I$, there exists $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subset U_i$. For $\varepsilon := \min\{\varepsilon_i \mid i \in I\}$, we have $\varepsilon > 0$ and $B(x, \varepsilon) \subset U_i$ for all $i \in I$. Thus, $B(x, \varepsilon) \subset U$, completing the proof. □

Proposition 9.42: Arbitrary intersections and Finite unions

Let $(A_i)_{i\in I}$ be any family of closed subsets of X, then $\bigcap_{i\in I} A_i$ is also closed. If I is a finite set, also $\bigcup_{i\in I} A_i$ is closed.

Proof. Apply Proposition 9.41 to the (open) complements of the closed sets. □

Example 9.43. — The intersection of infinitely many open sets may not be open. Take for example $\mathbb{R}$ with the standard metric. The intersection of the family of open sets

$$\{(-1/n, 1/n) \mid n \in \mathbb{N}\}$$

gives $\{0\}$, which is not open. Taking complements you obtain an example where an infinite union of closed sets is not closed.

Definition 9.44: Interior, closure, and boundary

Let X be a metric space and $E \subset X$. We define:

- The **interior** $E^\circ := \bigcup\{U \subset E \mid U \text{ is open}\}$, which is the largest open set contained in E. We will also use the notation $\operatorname{int}(E)$ to refer to the interior of E.
- The **closure** $\overline{E} := \bigcap\{A \supset E \mid A \text{ is closed}\}$, which is the smallest closed set containing E.
- The **(topological) boundary** $\partial E := \overline{E} \setminus E^\circ$.

Exercise 9.45. — Using Proposition 9.41, prove that E° is always open while $\overline{E}$ and ∂E are always closed.

Exercise 9.46. — For balls $B(x,r)$ in $\mathbb{R}^n$ (with the Euclidean distance) prove that $\overline{B(x,r)} = \{y \in \mathbb{R}^n \mid d(x,y) \leq r\}$ and deduce $\partial B(x,r) = \{y \in \mathbb{R}^n \mid d(x,y) = r\}$.

Lemma 9.47: Open and Closed trough sequences

3 *Let (X,d) be a metric space.*

(1) A subset $U \subset X$ is open if and only if, for every convergent sequence in X with a limit in U, the sequence eventually lies in U.

(2) A subset $A \subset X$ is closed if and only if, for every convergent sequence $(x_n)_{n=0}^\infty$ in X with $x_n \in A$ for all $n \in \mathbb{N}$, the limit also lies in A. In other words, if and only if A coincides with the set of all its accumulation points.

Proof. Let $U \subset X$ be an open subset of X, and let $(x_n)_{n=0}^\infty$ be a sequence in X with a limit x in U. Then, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$, and since $(x_n)_{n=0}^\infty$ converges to x, there exists an $N \in \mathbb{N}$ such that $x_n \in B(x,\varepsilon)$ for all $n \geq N$. Conversely, let $V \subset X$ be a non-open subset. Then there exists a point $x \in V$ such that $B(x,\varepsilon) \setminus V \neq \emptyset$ for every $\varepsilon > 0$. For each $n \in \mathbb{N}$, we can find $x_n \in B(x, 2^{-n}) \setminus V$. The sequence $(x_n)_{n=0}^\infty$ in $X \setminus V$ converges to $x \in V$, and satisfies $x_n \notin V$ for every $n \in \mathbb{N}$. This completes the proof of the first statement.

Let $A \subset X$ be closed, and let $(x_n)_{n=0}^\infty$ be a convergent sequence in X with $x_n \in A$ for all $n \in \mathbb{N}$. Let x be the limit of the sequence $(x_n)_{n=0}^\infty$. Then, $U = X \setminus A$ is open and cannot contain the limit x of $(x_n)_{n=0}^\infty$, as otherwise almost all elements of the sequence $(x_n)_{n=0}^\infty$ would have to lie in U. Therefore, the limit x belongs to A. Finally, suppose $A \subset X$ is not closed. Then $U = X \setminus A$ is not open, and according to the previous argument, there exists a sequence $(x_n)_{n=0}^\infty$ in $A = X \setminus U$ with a limit $x \in U$. □

Exercise 9.48. — Let (X, d) be a complete metric space and $E \subset X$ a closed subset. Show that E is complete as well.

Proposition 9.49: Topological notion of convergence

Let (X, d) be a metric space a sequence $(x_n)_{n \geq 0}$ converges to x if and only if for all open sets U containing x, x_n eventually lies in U.

Proof. Notice that we can rewrite the definition of convergence as follows: $x_n \to x$ if and only if for all $\varepsilon > 0$, x_n eventually lies in $B(x, \varepsilon)$. Now, if U is any open set containing x then by definition of open set there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ and hence $x_n \to x$ implies that x_n eventually lies in U, establishing the "only if" direction. For the "if" part, we observe that for any given $\epsilon > 0$ we can take $U = B(x, \varepsilon)$ (open balls are open) and hence x_n eventually lies in $B(x, \varepsilon)$. □

Corollary 9.50: Distances with same convergent sequences

Let X be a set endowed with two different distances d_1 and d_2. Then (X, d_1) and (X, d_2) have the same convergent sequences if and only if the topologies generated by d_1 and d_2 coincide.

3 *Proof.* By Corollary 9.49 the notion of convergent sequence only depends on the collection of open sets. That is, it only depends on the distance through the topology it generate. Hence distances generating the same open sets have the same convergent sequences. This proves the "if" part of the statement.

We prove the "only if" part. Suppose that a set $U \subset X$ is open with respect to d_1, but not with respect to d_2. This means that there is $x \in U$ such that we can find

$$x_k \in B_{d_2}(x, 2^{-k}) \cap (X \setminus U) \neq \emptyset \text{ for all } k \geq 0.$$

By construction, the sequence $\{x_k\}$ is the convergent with respect to d_2 and then also with respect to d_1 (they have the same convergent sequences by assumption). By Proposition 9.49 (U is open in d_1!) we discover that $x_k \in U$ eventually, which contradicts how we constructed $\{x_k\}$.

□

Exercise 9.51. — Let (X, d) be a metric space, and $A \subset X$ a subset.

(i) Assume X is complete and A is closed. Show that the subspace A is also complete.

(ii) Assume A is complete. Show that $A \subset X$ is closed.

9.2.2 Continuity

We now aim to generalize the concept of continuity to functions defined between metric spaces.

Definition 9.52: Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. We say that f is continuous if one of the following **equivalent** conditions hold:

3 (1) We say f is **$\varepsilon - \delta$ continuous** if, for all $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d_X(x, x') < \delta, \quad x' \in X \implies d_Y(f(x), f(x')) < \varepsilon$. In other words, $f(B(x, \delta)) \subset B(f(y), \varepsilon)$.

(2) We say f is **sequentially continuous** if, for every convergent sequence $(x_n)_n$ in X with limit $x = \lim_{n\to\infty} x_n$, the sequence $(f(x_n))_n$ converges in Y, with $f(x) = \lim_{n\to\infty} f(x_n)$.

(3) We say f is **topologically continuous** if, for every open subset $U \subset Y$, the preimage $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open in X.

Proposition 9.53: The three faces of Continuity

Let X and Y be metric spaces, and let $f : X \to Y$ be a function. The following conditions are equivalent:

(1) The function f is $\varepsilon - \delta$ continuous.

(2) The function f is sequentially continuous.

(3) The function f is topologically continuous.

Proof. (1) $\implies$ (2): Let $(x_n)_{n=0}^{\infty}$ be a convergent sequence in X with limit $x \in X$, and let $\varepsilon > 0$. There exists a $\delta > 0$ such that $f(x') \in B(f(x), \varepsilon)$ for all $x' \in B(x, \delta)$. Since $(x_n)_{n=0}^{\infty}$
4 converges to x, there exists an $N \in \mathbb{N}$ such that $x_n \in B(x, \delta)$ for all $n \geq N$. In particular, for $n \geq N$, $f(x_n) \in B(f(x), \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{n\to\infty} f(x_n) = f(x)$, and thus f is sequentially continuous as claimed.

$\neg$(3) $\implies$ $\neg$(2): Assume f is not topologically continuous. Then exists $U \subset Y$ open such that $f^{-1}(U)$ is not open. Therefore, there is $x \in f^{-1}(U)$ and a sequence $(x_n)_{n\geq 0} \subset X \setminus f^{-1}(U)$ with $x_n \to x$. Then $f(x) \in U$ and $f(x_n) \in Y \setminus U$ for all n, but U is open this gives that $f(x_n)$ cannot converge to $f(x)$. In other words, we have found a sequence such that $x_n \to x$, but $f(x_n)$ does not converge to $f(x)$.

(3) $\implies$ (1): Let $x \in X$ and $\varepsilon > 0$. The preimage $f^{-1}(B(f(x), \varepsilon))$ contains the point x and is open by assumption, as $B(f(x), \varepsilon) \subset Y$ is open. Thus, there exists a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$. Therefore, f is ε-δ-continuous as claimed. □

Definition 9.54: Uniform and Lipchitz continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. We say that $f\colon X \to Y$ is

- **Uniformly continuous** if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $d_X(x, x') < \delta$.
- **Lipschitz continuous** if there is a constant $L > 0$ such that
$$d_Y(f(x), f(x')) \leq L\, d_X(x, x') \quad \text{for all } x, y \in X.$$
The constant L is called **Lipschitz constant** of f

Example 9.55. — In any metric space (X, d) for any given $x_0 \in X$ the function $f(x) = d(x, x_0)$ is Lipschitz (with constant 1). Indeed, the triangle inequality (using also symmetry) yields:

$$-d(x, x') \leq d(x, x_0) - d(x', x_0) \leq d(x, x').$$

Exercise 9.56. — Let (X, d) be a metric space, and let $E \subset X$ be a non-empty subset. 4 For $x \in X$, define

$$f_E(x) = \inf\{d(x, z) \mid z \in E\}.$$

Show that the function $f_E : X \to \mathbb{R}$ is 1-Lipschitz continuous, and that $E \subset X$ is closed if and only if $E = \{x \in X \mid f_E(x) = 0\}$.

Exercise 9.57. — Let (X, d_X) and (Y, d_Y) be metric spaces. Assume that Y is complete. Show that if $E \subset X$ and $f\colon E \to Y$ is a uniformly continuous function defined on a subset then there is a unique continuous extension $\bar{f}\colon \bar{E} \to Y$, which is also uniformly continuous.

9.2.3 Banach's Fixed-Point Theorem

Fixed-point theorems investigate when a mapping $f : X \to X$ possess a fixed point, i.e., a point $x \in X$ for which $f(x) = x$. They can be rather powerful tools to prove existence theorems.

We prove in this section Banach's Fixed-Point Theorem. The idea behind this result is flexible and useful: for example we will use it later on to prove the Implicit Function Theorem and the existence and uniqueness of solutions to ODEs via Picard's iteration.

Theorem 9.58: Banach's Fixed-Point Theorem

Let (X, d) be a ***complete*** *metric space and let $T : X \to X$ be a Lipschitz map with Lipschitz constant $\lambda < 1$. In other words, assume that for some $\lambda \in (0, 1)$ it holds*

$$d(T(x), T(x')) \le \lambda\, d(x, x') \qquad \textit{for all } x, x' \in X.$$

Then, there ***exists a unique*** *element $a \in X$ such that $T(a) = a$.*

The function T is a **Lipschitz contraction**. A point $x \in X$ with $T(x) = x$ is called a **fixed point** of the mapping T, and the theorem states that a Lipschitz contraction has a unique fixed point, provided the ambient space is complete (as in Definition 9.27).

Proof. First, we show uniqueness of a putative fixed point. Let $a \in X$ and $a' \in X$ be fixed points of T. Then,

$$d(a, a') = d(T(a), T(a')) \le \lambda d(a, a'),$$

which, since $\lambda < 1$, implies $d(a, a') = 0$ and thus $a = a'$.

We turn to prove the existence of a fixed point. Choose any $x_0 \in X$ and define a sequence $(x_n)_{n=0}^{\infty}$ recursively by $x_{n+1} = T(x_n)$, for $n \ge 0$. We claim that the sequence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Iterating the contractivity assumption we find that, for all integers $p \ge 0$,

4

$$\begin{aligned} d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) &\le \lambda\, d(x_n, x_{n-1}) \\ &\le \lambda^2\, d(x_{n-1}, x_{n-2})(x_0)) \le \ldots \le \lambda^n\, d(x_1, x_0). \end{aligned}$$

Pick now any integers $m \ge n \ge N$, then using this observation and the triangular inequality we find

$$d(x_m, x_n) \le \sum_{p=n}^{m-1} \lambda^p d(x_1, x_0) \le d(x_1, x_0) \sum_{p \ge N}^{\infty} \lambda^p = \frac{\lambda^N}{1 - \lambda} d(x_0, x_1).$$

We crucially used that $\lambda < 1$ to sum the geometric series. Now given any $\varepsilon > 0$ we can find some N so large that $\frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon$, thus proving that $(x_n)_{n=0}^{\infty}$ is Cauchy (this estimate is uniform in n, m as long as they are larger than N!).

Now we use the completeness assumption to infer that $x_n \to a$ for some $a \in X$. Since T is continuous, we have

$$T(a) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = a$$

which shows that a is a fixed point of T. □

9.59. — We remark that the proof is constructive and in concrete situation can be implemented in a an algorithm to find approximate fixed points.

Exercise 9.60. — Find examples for:

(1) A Lipschitz contraction $T : X \to X$ on a non-complete metric space X that has no fixed point.

(2) A complete metric space (X, d) and an isometry (i.e., a mapping $T : X \to X$ with $d(T(x_1), T(x_2)) = d(x_1, x_2))$ that has no fixed point

9.2.4 Compactness

A closed and bounded interval of the real line is is called a **compact** interval, as we saw in Analysis I. We proved some fundamental properties of continuous functions on compact intervals: boundedness, existence of maxima and minima, and uniform continuity. We intend to investigate these and other properties in the broader context of metric spaces. We start giving a general definition of compactness that works in metric spaces.

Let us immediately clarify a possible source of confusion.

4 Achtung! 9.61: Closed & Bounded VS Compact

- In a general metric space, it is **not** necessarily true that a closed and bounded set is compact.
- Nevertheless when considering $\mathbb{R}^n$ with its Euclidean structure, which is the main focus of this course, it will turn out that a closed and bounded set **is** indeed compact, and viceversa.

Interlude: "Cover"

Let $E \subset X$ and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of X, where I is some set of indices. We say that $\mathcal{U}$ **covers** E if

$$E \subset \bigcup \mathcal{U} = \bigcup_{i \in I} U_i.$$

Any family $\mathcal{V} \subset \mathcal{U}$ that still covers X is called a **subcover** of $\mathcal{U}$.

Definition 9.62: Compactness

Let (X, d) be a metric space. A subset $K \subset X$ is called **compact** if one of the following equivalent conditions hold:

(1) K is **sequentially compact**: every sequence $(x_n)_{n\in\mathbb{N}}$ in K has a subsequence that is convergent in K. 4

(2) K is **topologically compact**: every family of open sets $\{U_i\}_{i\in I}$ that cover K, has a finite subcover.

(3) K is **complete** (i.e., every Cauchy sequence contained in K has a limit in K) and **totally bounded**: for every $r > 0$, there exist finitely many $x_1, \ldots, x_n \in K$ such that the balls $B(x_1, r), \ldots, B(x_n, r)$ cover K.

9.63. — The definition of topological compactness does not explicitly use the distance function d, but it is formulated only in terms of the collection of open sets (i.e., the topology). For this reason it is called “topological”.

9.64. — The Bolzano-Weierstrass Theorem ensures that a closed and bounded interval of $\mathbb{R}$ is sequentially compact.

Example 9.65. — $\mathbb{Q} \cap [0, 2]$, endowed with the standard distance, is not topologically compact. Consider the covering

$$\mathbb{Q} \cap [0, 2] = (\mathbb{Q} \cap [0, \sqrt{2})) \cup \bigcup_{p\in\mathbb{Q},\ p>\sqrt{2}} (\mathbb{Q} \cap (p, 2]).$$

5 Any finite subcover will miss some rationals $> \sqrt{2}$.

Exercise 9.66. — Let X be a metric space. Show that if X is totally bounded, then it is bounded, i.e., $\sup_{x,x'\in X} d(x, x') < +\infty$.

Example 9.67. — The half-open interval $X = (0, 1] \subset \mathbb{R}$ is not compact. Indeed, the open cover $\mathcal{U} = \{(2^{-n}, 1] \mid n \in \mathbb{N}\}$ has no finite subcover.

The main result of this section is Theorem 9.69, which shows that the definition of compact metric space is indeed well-posed.

Exercise 9.68. — Show that a totally bounded metric space is bounded, and find an example of a bounded metric space that is not totally bounded.

Theorem 9.69: The three faces of Compactness

Let (X,d) be a metric space and $K \subset X$ a subset, the following statements are equivalent:

(1) K is sequentially compact.

(2) K is topologically compact.

(3) K is totally bounded and complete (in the sense of Cauchy sequences).

We will prove that $(1) \implies (2) \implies (3) \implies (1)$.

Proof that $(1) \implies (2)$. We start with a preliminary observation. Consider the function

$$r(x) := \min\Big\{1, \sup\{r > 0 : B(x,r) \subset U \text{ for some } U \in \mathcal{U}\}\Big\},$$

defined for all $x \in K$.

Notice that $r(x) > 0$ since $\mathcal{U}$ is open cover. For each $x \in K$ we choose — once and for all — some $U(x) \in \mathcal{U}$ which is almost optimal in the sense that $B(x, r(x)/2) \subset U(x)$.

Let us now proceed with the construction of our finite subcover. Pick any $U_0 \in \mathcal{U}$. If $K \subset U_0$ then we are done, otherwise there is some $x_1 \in K \backslash U_0$. In this case we set $U_1 := U(x_1)$.

Now we check if $K \subset U_0 \cup U_1$, in which case we have found our finite subcover. If not,
5 there is some $x_2 \in K \setminus (U_0 \cup U_1)$ and we set $U_2 := U(x_2)$.

Now we check again if $K \subset U_0 \cup U_1 \cup U_2$, in which case we have found our finite subcover. If not, there is some $x_3 \in K \setminus (U_0 \cup U_1 \cup U_2)$ and we set $U_3 := U(x_3)$.

If this procedure stops at a certain point, it means that we have found our finite open subcover. So let's assume that it is goes on indefinitely, and find a contradiction. We obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the property that

$$x_m \notin U_0 \cup \ldots \cup U_n \text{ for all } m > n \geq 0.$$

By assumption (1), we have $x_{n_k} \to z$ (as $k \to \infty$) for a suitable subsequence. On the other hand, for each $k \geq 0$ we have $z \notin U_{n_k}$ (the complement of a open set is closed, that is sequentially closed), and in particular $z \notin B(x_{n_k}), r(x_{n_k})/2)$. Combining this information with $x_{n_k} \to z$, we find $r(x_{n_k}) \to 0$ and in particular

$$r(x_{n_k}) = \sup\{r > 0 : B(x_{n_k}, r) \subset U \text{ for some } U \in \mathcal{U}\} \in (0,1).$$

This is a contradiction as $r(z) > 0$, so for k large enough we would have

$$B(x_{n_k}, 100r(x_{n_k})) \subset B(z, r(z)/2) \subset U(z) \in \mathcal{U},$$

which, by definition of $r(x_{n_k})$, implies $100r(x_{f(n)}) \leq r(x_{f(n)})$, impossible.

□

In order to prove (2) $\implies$ (3) we single out a rephrasing of (2) which is useful to keep in mind.

Lemma 9.70: Nesting Principle

Let X be a metric space, then $K \subset X$ is topologically compact if and only if has the following property, called the "Nesting property". For every collection $\mathcal{A} = \{A_i\}_{i \in I}$ of closed subsets of X, it holds:

> *"If every intersection of finitely many sets in $\mathcal{A}$ has a non-empty intersection with K, then $K \cap \bigcap_{i \in I} A_i$ is non-empty."*

Proof of Lemma 9.70. Assume K is compact, and let $\mathcal{A}$ be a collection of closed subsets of X with empty intersection. The collection of complements $\mathcal{U} = \{X \setminus A : A \in \mathcal{A}\}$ is then an open cover of K, so there exists a finite subcover $K \subset U_1 \cup \cdots \cup U_n$ of it. Set $A_i := K \setminus U_i$. Then $K \cap A_1 \cap \cdots \cap A_n = \emptyset$. Thus, K satisfies the Nesting property.

Now, assume K satisfies the Nesting property, and let $\mathcal{U}$ be an open cover of K. Then $\mathcal{A} = \{X \setminus U : U \in \mathcal{U}\}$ is a collection of closed subsets with an empty intersection with K. According to the Nesting property, there must exist $A_1, \ldots, A_n \in \mathcal{A}$ with an empty intersection. So if we consider $U_i = X \setminus A_i$, then $X = U_1 \cup \cdots \cup U_n$ is a finite subcover of K. Since the cover $\mathcal{U}$ was arbitrary, this shows that K is compact. □

5 *Proof that* (2) $\implies$ (3). We first prove that X is totally bounded. Pick any $r > 0$, consider the open covering $\mathcal{U} := \{B(x, r) : x \in X\}$ and extract a finite subcover $\{B(x_1, r), \ldots, B(x_N, r)\}$.

Now we prove that X must be complete, hence we pick a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ and show that it has a limit point. For each $k \geq 0$ there is $n(k)$ so large that

$$n, m \geq n(k) \implies d(x_n, x_m) < 2^{-k}.$$

For each k, consider the closed balls $A_k := \overline{B}(x_{n(k)}, 2^{-k})$; any finite intersection of them is nonempty, indeed for every $k_1, \ldots, k_N$ one has that $x_m \in A_{k_1} \cap \ldots \cap A_{k_N}$, provided $m \geq \max\{k_1, \ldots, k_N\}$. Hence we can apply the Nesting Principle (see Lemma 9.70) and find some $z \in \bigcap_{k \geq 0} A_k$. We claim that $x_n \to z$. Indeed if $m \geq n(k)$ it holds

$$d(x_m, z) \leq d(x_m, x_{n(k)}) + d(x_{n(k)}, z) \leq 2^{1-k},$$

and 2^{1-k} is arbitrarily small. □

Before continuing the proof we need an auxiliary Lemma.

Lemma 9.71: Diagonal Argument

Let $\mathbb{N} \supset N_0 \supset N_1 \supset N_2 \supset \ldots$ be a an infinite family of nested sets. Assume further that each N_k has infinite many elements. Then there exists $f \colon \mathbb{N} \to \mathbb{N}$ strictly increasing such that $f(k) \in N_k$ for all $k \geq 0$.

Proof of Lemma 9.71. Set $f(0)$ equal to an arbitrary element of N_0. Then set inductively $f(k) := \min\{m \in N_k : m > f(k-1)\}$, this set is non-empty because each N_k is infinite. □

Proof that (3) $\implies$ (1). We pick any sequence $(x_n)_{n\in\mathbb{N}}$ and show that admits a Cauchy subsequence, by completeness, this will prove (1).

By assumption, we can cover K by a finite number of balls of radius 1; it follows that $(x_n)_{n\in\mathbb{N}}$ will fall infinitely many times (at least) one of these balls. Let this ball be $B(z_1, 1)$ for some $z_1 \in X$. Accordingly, we define the set of indices $N_0 := \{j \in \mathbb{N} : x_j \in B(z_1, 1)\}$, which is infinite.

Now we proceed to do same thing to the restricted sequence $(x_n)_{n\in N_0}$, but we shorten the radius of from 1 to 1/2. Accordingly, we find a ball $B(z_2, 1/2)$ such that the set $N_1 := \{j \in N_0 : x_j \in B(z_2, 1/2)\}$ is infinite.

We proceed inductively, halving the radius each time, and construct a descending family
5 of infinite sets $\mathbb{N} \supset N_0 \supset N_1 \supset N_2 \supset \ldots$ with the property that

$$\forall k \geq 0, \exists z_k \in X, j \in N_k \implies d(z_k, x_j) < 2^{-k}.$$

We apply to these sets Lemma 9.71 and find $f : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $f(k) \in N_k$ for all $k \geq 0$. Then, the subsequence $(x_{f(k)})_{k\in\mathbb{N}}$ is Cauchy: for $n, m \geq k$ it holds

$$\begin{aligned} f(n) \in N_n, f(m) \in N_m &\implies f(n), f(m) \in N_k \\ &\implies x_{f(n)}, x_{f(m)} \in B(z_k, 2^{-k}) \implies d(x_{f(n)}, x_{f(m)}) < 2^{1-k}. \end{aligned}$$

□

We conclude the chain of implications thus proving Theorem 9.69.

Let us next give some simple consequences of the three faces of compactness result.

Corollary 9.72: Closed in Compact is Compact

Let X be a metric space, $A \subset X$ a closed subset and $K \subset X$ a compact subset. Then $A \cap K$ is compact.

Proof. We check that $A \cap K$ is sequentially compact. Take any sequence $(x_n)_{n\in\mathbb{N}} \subset A \cap K$, by compactness of K, it has a converging subsequence $x_{f(n)} \to x$, for some $x \in K$. On the
6 other hand, since A is closed it contains its accumulation points, so we also have $x \in A$. □

Corollary 9.73: Compact is always Closed

In a metric space, every compact subset is closed.

Proof. Take any sequence in E which is convergent to some $x \in X$. By compactness a suitable subsequence is converging to some $x' \in E$, by uniqueness of the limit $x = x'$, so E is closed (see (2) in Lemma 9.47). □

We also easily get the following version of the Heine Borel theorem in the Euclidean space $\mathbb{R}^n$. In its statement, a set $E \subset \mathbb{R}^n$ is called bounded if there there exist $N \in \mathbb{N}$ such that $E \subset [-2^N, 2^N]^n$ (prove that this is equivalent to $sup\{d(x,x') \mid (x,x') \in E \times E\} < \infty$).

Theorem 9.74: Compact subsets of $\mathbb{R}^n$ (Heine-Borel)

A subset $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. If K is compact, then it is closed by Corollary 9.73; and bounded, because it is totally bounded.

To show the converse, we show that K is complete and totally bounded. Since $\mathbb{R}^n$ is complete and K is closed, then K is complete as well (see Exercise 9.51).

On the other hand, by assumption K is bounded so there exist $N \in \mathbb{N}$ such that $K \subset [-2^N, 2^N]^n$. Given $r > 0$, take some large integer $N' \in \mathbb{N}$ large so that such that $2^{-N'} < r/\sqrt{n}$. Then the union of the balls $B_r(x)$ with x running in the finite grid

$$\left\{2^{-N'}y \mid y = (y_1, \dots y_n) \in \mathbb{Z}^n \text{ with } -2^{N+N'} \leq y_i \leq 2^{N+N'}\right\}$$

covers K, proving that it is totally bounded.

□

6 Example 9.75. — We stress that the Heine-Borel Theorem **fails** for general metric spaces: take $\mathbb{R}$ with the bounded distance $d(x,y) := \arctan|x-y|$ (see Exercise 9.8). Then in this metric space the set $\mathbb{N}$ would be closed and bounded, but it is trivial to construct sequences that do not converge.

Exercise 9.76. — Show that an open set $U \subset \mathbb{R}^n$ is complete if and only if $U = \mathbb{R}^n$ or $U = \emptyset$.

9.2.5 Compactness and continuity

Theorem 9.77: Continuous image of compact is compact

Let X and Y be metric spaces, let $f : X \to Y$ be a continuous function, and let $K \subset X$ be a compact subset. Then, $f(K)$ is a compact subset of Y.

Proof. We will give two proofs. The first one employs sequential compactness. Suppose that $(y_n)_{n\geq 0}$ is a sequence of points in $f(K)$. Then $y_n = f(x_n)$ for some $(x_n)_{n\geq 0}$. Since K is compact there is a convergent subsequence $x_{n_k} \to x \in K$. But then since f is continuous $y_{n_k} \to y = f(x) \in f(K)$.

The second one employs instead topological compactness. Let $\mathcal{U}$ be an open cover of $f(A)$. For each $U \in \mathcal{U}$, the set $f^{-1}(U) \subset X$ is open due to the continuity of f. The collection $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of A. Since A is compact, there exist $U_1, \dots, U_n \in \mathcal{U}$ such that $\{f^{-1}(U_i) \mid 1 \leq i \leq n\}$ is a cover of K. This implies that $\{U_i \mid 1 \leq i \leq n\}$ is a cover of $f(K)$, and since $\mathcal{U}$ was arbitrary, it shows that $f(K)$ is compact. □

Proposition 9.78: Continuous in a Compact is Uniformly Continuous

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a continuous function. If X is compact, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$. Due to the continuity of f, for each $x \in X$, there exists $\delta_x > 0$ such that $f(B(x, \delta_x)) \subset B(f(x), \frac{\varepsilon}{2})$. The collection $\{B(x, \frac{1}{2}\delta_x) \mid x \in X\}$ forms an open cover of X. Since X is compact by assumption, there exists a finite subcover of this collection. This implies the existence of $x_1, \ldots, x_n \in X$ such that

$$X = B(x_1, \tfrac{1}{2}\delta_{x_1}) \cup \cdots \cup B(x_n, \tfrac{1}{2}\delta_{x_n}).$$

Let $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$. For $x, x' \in X$ with $d_X(x, x') < \delta$, there exists k such that $x \in B(x_k, \frac{1}{2}\delta_{x_k})$. This implies $x' \in B(x_k, \delta_{x_k})$, leading to

6 $$d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_k)) + d_Y(f(x_k), f(x')) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which completes the proof. □

Corollary 9.79: Weierstrass

Let X a metric space, $f \colon X \to \mathbb{R}$ be a continuous function, and $K \subset X$ a compact subset. Then, f admits a maximum point, i.e., there exist $\bar{x} \in X$ such that $f(\bar{x}) = \sup_K f = \sup\{f(x) \mid x \in K\}$. An analogous statement holds for the minimum. In particular, f must be bounded.

Proof. $f(K) \subset \mathbb{R}$ is compact by Theorem 9.77 and nonempty, so $\sup f(K) \in f(K)$, any element $\bar{x} \in f^{-1}(\sup f(X))$ works.

The fact that $\sup f(K) \in f(K)$ is readily proved: by definition of supremum there is a sequence $(s_n) \subset f(K)$ such that $s_n \to \sup f(K)$, but then $\sup f(K) \in f(K)$, since $f(K)$ is closed and so it contains its accumulation points. □

9.2.6 Connectedness

Definition 9.80: Connectenedess

Let X be a metric space. A nonempty subset $E \subsetneq X$ is called **disconnected** if there exist two disjoint open sets U_1 and U_2 such that $E \cap U_i \neq \varnothing$ for $i = 1, 2$ and $E \subset U_1 \cup U_2$. Conversely, E is called **connected** if, for any pair open sets U_1 and U_2 such that $E \cap U_i \neq \varnothing$ for $i = 1, 2$ and $E \subset U_1 \cup U_2$ it is required that $U_1 \cap U_2$ is not empty.
A subset $F \subset E$ is called a **connected component of** E if F is non-empty, connected, and every $G \subset E$ such that $F \subsetneq G$, G is disconnected.

Exercise 9.81. — Let X be a metric space, and let E_1 and E_2 be connected subsets. If the intersection $E_1 \cap E_2$ is non-empty, then the union $E_1 \cup E_2$ is connected. Can you generalize this property to arbitrary unions? Use this property to define the connected component of a point as the union of all connected sets that contain the point.

Proposition 9.82: Connected subsets of $\mathbb{R}$

A non-empty subset $E \subset \mathbb{R}$ is connected in $\mathbb{R}$ if and only if E is an interval.

Proof. Assume $E \subset \mathbb{R}$ is not an interval. Then there exist real numbers $x_1 < y < x_2$ with $x_1, x_2 \in X$ and $y \notin X$. Then the two disjoint open subset 6

$$U_1 = (-\infty, y) \quad \text{and} \quad U_2 = (y, \infty)$$

cover E, so E is not connected.

To prove the opposite implication we also argue by contraposition. Suppose that E is disconnected and let U_1, U_2 disjoint open sets, both intersecting E, such that $E \subset U_1 \cup U_2$. Choose $x_1 \in E \cap U_1$ and $x_2 \in E \cap U_2$. Without loss of generality, assume $x_1 < x_2$.

Now let us define

$$t_* := \sup\{t \geq x_1 : [x_1, t] \subset U_1\}, \tag{9.2}$$

since $x_2 \in U_2$ we have $t_* \in [x_1, x_2)$. On the one hand t_* is the supremum (and hence the limit) of points in $\mathbb{R} \setminus U_2$, a closed set, and hence $t_* \in \mathbb{R} \setminus U_2$

On the other hand let us show that $t_* \notin U_1$. If $t_* \in U_1$ then $t_* < x_2$ and, since U_1 is open, we could enlarge a bit t_* while still satisfying (9.2). This violates the very definition of t_* in (9.2).

Therefore we have found two points x_1,x_2 in E and third point $t_* \in (x_1, x_2)$ with $t_* \in \mathbb{R} \setminus (U_1 \cup U_2) \subset \mathbb{R} \setminus E$. Hence, E cannot be an interval. □

Proposition 9.83: continuous image of connected

Let X and Y be metric spaces, and let $f : X \to Y$ be continuous. If $E \subset X$ is connected, then the image $f(E)$ is a connected subspace of Y.

Proof. Suppose $f(E)$ is disconnected. There there exists two nonempty, disjoint, open sets U_1, U_2 of Y such that $f(E)$ is covered by $U_1 \cup U_2$. But then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are two nonempty disjoint open sets covering E, contradicting its connectedness. □

Corollary 9.84: Intermediate Value Theorem

Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a continuous function, and $a, b \in I$. For every $y \in \mathbb{R}$ between $f(a)$ and $f(b)$, there exists an $x \in I$ between a and b such that $f(x) = y$.

Proof. Without loss of generality, we can assume $a < b$. Appling Propositions 9.82 and 9.83 $f([a, b])$ is connected. But then, using again Proposition 9.82, $f([a, b]) \subset \mathbb{R}$ must be an interval. As $f(a), f(b) \in f([a, b])$, all values between $f(a)$ and $f(b)$ lie in $f([a, b])$. □

Exercise 9.85. — Show the following generalization of the Intermediate Value Theorem: Let X be a connected topological space, and $f : X \to \mathbb{R}$ be a continuous function. Let $a, b \in X$. Then, for every $y \in \mathbb{R}$ between $f(a)$ and $f(b)$, there exists $x \in X$ such that $f(x) = y$.

Interlude: Paths and Curves

Let X be a metric space. A **path** or **curve** in X is a continuous function $\gamma\colon [0, 1] \to X$.
6 We call $\gamma(0)$ the **starting point** and $\gamma(1)$ the **ending point**. We also say that γ is a path from $\gamma(0)$ to $\gamma(1)$. A path γ with $\gamma(0) = \gamma(1)$ is called **closed path** or a **loop**. If $s\colon [a, b] \to [0, 1]$ is a bijective continuous function with continuous inverse we say that $\gamma \circ s$ is a **re-parametrization** of γ. Furthermore, exactly one of the following happens

- either, $s(a) = 0, s(b) = 1$, then we say that s is **orientation-preserving**,
- or, $s(a) = 1, s(b) = 0$, then we say that s is **orientation-reversing**.

Definition 9.86:

Let X be a metric space. We call $E \subset X$ **path-connected** if, for every two points $x, y \in E$, there exists a path $\gamma : [0, 1] \to E$ from $x = \gamma(0)$ to $y = \gamma(1)$.

Lemma 9.87:

Let X be a metric space. If $E \subset X$ is path-connected then it is connected.

Proof. Suppose that E is disconnected in the topological sense, then there exist non-empty, disjoint, open sets U_1 and U_2 such that $E \subset U_1 \cup U_2$. Let $x_1 \in U_1$ and $x_2 \in U_2$. If E were path-connected, there would exist a path $\gamma : [0, 1] \to E$ from x_1 to x_2. However, this implies that $V_1 = \gamma^{-1}(U_1)$ and $V_2 = \gamma^{-1}(U_2)$ are non-empty, disjoint open subsets of $[0, 1]$ with $V_1 \cup V_2 \supset [0, 1]$; a contradiction since $[0, 1]$ is connected. □

Exercise 9.88. — Sketch the subset $E \subset \mathbb{R}^2$ given by

$$E = \{0\} \times [-1,1] \cup \{(t, \sin(\tfrac{1}{t})) \mid t > 0\}$$

and show that E is connected but not path-connected.

Proposition 9.89:

Let $U \subset \mathbb{R}^n$ (with the Euclidean metric) be an open subset. Then U is path-connected if and only if U is connected.

Proof. If U is path-connected, then U is also connected according to Lemma 9.87. Now, assume U is connected, non-empty, and $x_0 \in U$ is a fixed point. We define the set

$$G = \{x \in U \mid \text{there exists a path in } U \text{ from } x_0 \text{ to } x\}$$

and want to show that $G = U$. Since U is connected and G is non-empty, it suffices to show that both G and $U \setminus G$ are open. 7

Let $x \in G$ and $\gamma : [0,1] \to U$ be a path from x_0 to x. Since U is open, there exists $r > 0$ such that $B(x,r) \subset U$. For any $y \in B(x,r)$, the straight path $t \mapsto (1-t)x + ty$, connecting x and y, lies in U. Concatenating these paths yields the path

$$t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ (2-2t)x + (2t-1)y & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

from x_0 to y. Thus, $y \in G$, and since y was arbitrary, we have $B(x,r) \subset G$. This shows that G is open. Using a similar argument, we can show that $U \setminus G$ is open. If $x \in U \setminus G$ and $r > 0$ with $B(x,r) \subset U$, then all points in $B(x,r)$ are not in G. If $y \in G \cap B(x,r)$, a concatenation of paths as above would connect x to x_0. Therefore, $B(x,r) \subset U \setminus G$, and $U \setminus G$ is open. Thus, G is closed.

□

Corollary 9.90: $\mathbb{R}^n$ and its balls are connected

For all $n \ge 1$ and $r > 0$, the metric space $\mathbb{R}^n$ (with the standard distance) and the subsets $B(x,r)$ and $\overline{B(x,r)}$ of $\mathbb{R}^n$ are connected.

9.3 Normed vector spaces

9.3.1 Definition of Normed Vector spaces

A **norm** on a real vector space V is a function from V to $\mathbb{R}$ that assigns to each vector a non-negative number, informally its **length**. In general there are many different norms on a vector space, and we can use any of them to construct a distance and turn V into a metric space. A particularly interesting class of norms is obtained from **scalar products**. Many notions of this sections would carry out to vector spaces over $\mathbb{C}$ with little modifications, but we will stick to real vector spaces.

Definition 9.91: Normed vector space

Let V be a vector space over $\mathbb{R}$. A **norm** on V is a mapping $\|\cdot\| : V \to [0, \infty)$ that satisfies the following three properties.

(1) (Definiteness) For all $v \in V$, $\|v\| = 0 \iff v = 0$.

(2) (Homogeneity) For all $v \in V$ and all $\alpha \in \mathbb{R}$, $\|\alpha v\| = |\alpha| \|v\|$.

(3) (Triangle Inequality) For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

7 The pair $(V, \|\cdot\|)$ is called a **normed vector space**.

Example 9.92. — Let $n \in \mathbb{N}$. The **maximum norm** or **infinity norm** $\|\cdot\|_\infty$, and the **1-norm** $\|\cdot\|_1$ on $\mathbb{R}^n$ are defined by

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\} \quad \text{and} \quad \|v\|_1 = \sum_{j=1}^{n} |v_j|$$

for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. The properties of definiteness and homogeneity, as well as the triangle inequality, can be verified by exercise.

Example 9.93. — If V is the vector space of continuous $\mathbb{R}$-valued functions on $[0, 1]$, we define analogously the 1-norm and the infinity norm as

$$\|f\|_1 = \int_0^1 |f| dx \quad \text{and} \quad \|f\|_\infty = \sup\{|f(x)| \, x \in [0, 1]\}.$$

9.94. — From now on, in order to keep the notation, simple, we will denote the Euclidean norm of a vector x in $\mathbb{R}^n$ by $|x|$, instead of $\|x\|$, that we will reserve for (less standard) norms. Notice that this notation does not create any ambiguity or collision with previously introduced notations. Indeed:

- If $n = 1$ the Euclidean norm coincides with the absolute value.
- If $n = 2$ and we are identifying the $\mathbb{R}^2$ and $\mathbb{C}$ via the usual map $(x_1, x_2) \mapsto x_1 + ix_2$, then Euclidean norm coincides with the complex absolute value.

9.95. — The Euclidean norm on $\mathbb{R}^n$ holds a special position among all norms on $\mathbb{R}^n$. On $\mathbb{R}^2$ or $\mathbb{R}^3$, it measures the "physical" length of vectors. However, many other norms $\|\cdot\|$ confer $\mathbb{R}^n$ the structure of normed vector space. A standard family of norms is given by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

where $p \in [1, +\infty)$ is a given number. Notice that the Euclidean norm $|x|$ corresponds to the $p = 2$ case.

One can check (exercise) that for any given $x \in \mathbb{R}^n$

$$\lim_{p \to +\infty} \|x\|_p = \max_{1 \le i \le n} |x_i|$$

is also a norm. That is why the maximum norm is commonly called infinity norm and denoted $\|\cdot\|_\infty$.

7 We immediately observe that Normed Vector Spaces are "automatically" Metric Spaces.

Lemma 9.96: Normed Vector Spaces are Metric Spaces

Let V be a vector space over $\mathbb{R}$ and $\|\cdot\|$ be a norm on V. Define the function

$$d \colon V \times V \to [0, \infty), \qquad d(v, w) := \|v - w\|,$$

then (V, d) is a metric space.

Proof. We check definiteness, symmetry, and the triangle inequality in the definition of a metric 9.3. For $v, w \in V$, we have $d(v, w) = \|v - w\| \ge 0$, and

$$d(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w$$

by the definiteness of the norm. Using homogeneity of the norm for $\alpha = -1$, we have for $v, w \in V$,

$$d(v, w) = \|v - w\| = \|(-1)(v - w)\| = \|w - v\| = d(w, v)$$

thus establishing the symmetry of d. Finally, using the triangle inequality of the norm, we obtain

$$d(u, w) = \|u - w\| = \|(u - v) + (v - w)\| \le \|u - v\| + \|v - w\| = d(u, v) + d(v, w)$$

for all $u, v, w \in V$. This shows the triangle inequality for d, so d is indeed a metric on V. □

We have seen that a normed space is naturally a metric space, now we check that the norm is indeed continuous. We show sequential continuity.

Lemma 9.97: The Norm is continuous with respect to its own distance

Let V be a $\mathbb{R}$-vector space, and let $\|\cdot\|$ be a norm on V. Let $(v_n)_{n=0}^{\infty}$ be a sequence in V converging with respect to the norm $\|\cdot\|$ to a limit $v \in V$. Then,

$$\lim_{n\to\infty} \|v_n\| = \|v\|.$$

Proof. By definition, $(v_n)_{n=0}^{\infty}$ converges to v, if and only if $d(v_n, v) = \|v_n - v\|)$ converges to 0. But using the triangle inequality:

$$-\|v_n - v\| \leq \|v_n\| - \|v\| \leq \|v_n - v\|.$$

Thus, if $\|v_n - w\| \to 0$, then necessarily $\|v_n\| \to \|v\|$. □

9.3.2 Inner Product Spaces are Normed Vector Spaces

Definition 9.98: Inner product space

7 Let V be a vector space over $\mathbb{R}$. An **inner product** on V is a map

$$\langle -, - \rangle : V \times V \to \mathbb{R}$$

that satisfies the following properties for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

(1) (Bilinearity) $\langle \alpha u + \beta v, w \rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$.

(2) (Symmetry) $\langle v, w\rangle = \langle w, v\rangle$.

(3) (Definiteness) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \iff v = 0$.

9.99. — An important example of an inner product is the **Euclidean inner product** or **standard inner product** on $\mathbb{R}^n$. It is given by

$$\langle -, - \rangle : V \times V \to \mathbb{R} \qquad \langle v, w\rangle = \sum_{k=1}^{n} v_k w_k$$

for $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. The proof of bilinearity and symmetry is left as an exercise. We verify definiteness. Let $v = (v_1, \ldots, v_d) \in \mathbb{R}^n$. Then,

$$\langle v, v\rangle = \sum_{k=1}^{d} v_k v_k = \sum_{k=1}^{n} v_k^2 \geq 0$$

is a non-negative real number. If $v = 0$, then $\langle v, v\rangle = 0$. If $\langle v, v\rangle = 0$, then each term $|v_k|^2$ must be zero, and thus $v_k = 0$ for all k, implying $v = 0$.

Proposition 9.100: Cauchy-Schwarz Inequality

Let V be a vector space over $\mathbb{R}$, let $\langle\cdot,\cdot\rangle$ be an inner product on V, and let $\|\cdot\| : V \to \mathbb{R}$ be given by $\|v\| = \sqrt{\langle v, v\rangle}$. Then the inequality holds

$$|\langle v, w\rangle| \leq \|v\|\|w\| \tag{9.3}$$

for all $v, w \in V$. Furthermore, equality in (9.3) holds if and only if v and w are linearly dependent.

Proof. Notice first that for all α, β positive real numbers (9.3) holds iff and only if

$$|\langle \alpha v, \beta w\rangle| \leq \|\alpha v\|\|\beta w\| \tag{9.4}$$

Now $v = 0$ or $w = 0$, then both sides of (9.3) are zero. So, putting $\alpha = 1/\|v\|$ and $\beta = 1/\|w\|$ we may assume without loss of generality that $\|v\| = \|w\| = 1$. Then:

$$\begin{aligned} 0 &\leq \|v - w\|^2 = \langle v - w, v - w\rangle = \langle v, v - w\rangle - \langle w, v - w\rangle \\ &= \langle v, v\rangle - \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle = \|v\|^2 - 2\langle v, w\rangle + \|w\|^2 \\ &= 2 - 2|\langle v, w\rangle| = 2(\|v\|\|w\| - \langle v, w\rangle). \end{aligned}$$

7

□

We prove that the norm induced by an inner product is indeed a norm on V.

Corollary 9.101:

Let V be a vector space over $\mathbb{R}$, let $\langle -, -\rangle$ be an inner product on V. The map defined by (9.5)

$$\|\cdot\| : V \to \mathbb{R}, \qquad \|v\| = \sqrt{\langle v, v\rangle}$$

satisfies the triangular inequality and is a norm.

Proof. Definiteness and homogeneity follow directly from the definiteness and bilinearity of the inner product. We only need to prove the triangle inequality. Let $v, w, \in V$. Using the Cauchy-Schwarz inequality, we have the estimate

$$\begin{aligned} \|v + w\|^2 = \langle v + w, v + w\rangle &= \|v\|^2 + \langle v, w\rangle + \langle w, v\rangle + \|w\|^2 \\ &= \|v\|^2 + 2(\langle v, w\rangle) + \|w\|^2 \leq \|v\|^2 + 2|\langle v, w\rangle| + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2, \end{aligned}$$

which implies the desired result after taking the square root. □

9.102. — Let V be a vector space over $\mathbb{R}$. If $\langle \cdot, \cdot \rangle$ is an inner product on V, we call the norm treated in Corollary 9.101

$$\| \cdot \| : V \to \mathbb{R}, \qquad \|v\| = \sqrt{\langle v, v \rangle} \tag{9.5}$$

the **induced norm** by $\langle \cdot, \cdot \rangle$. In particular, from the Euclidean inner product on $V = \mathbb{R}^n$, we can define a norm on $V = \mathbb{R}^n$. The **Euclidean norm** on $V = \mathbb{R}^n$ is given by

$$|x| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{k=1}^{n} |v_k|^2}$$

for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n$.

9.3.3 Equivalence of norms in finite dimensional normed spaces

Interlude: Finite and infinite dimensional vector spaces

A **basis** for a vector space V is a subset $\mathcal{B} = \{e_i\}_{i \in I} \subset V$ such that

- every $v \in V$ can be written as $v = \sum_{i \in I} v_i e_i$ for some coefficients $\{v_i\}_{i \in I} \subset \mathbb{R}$, only finitely many of which are nonzero (so that the previous sum always makes
7 sense, it is not a series!)
- the coefficients $\{v_i\}_{i \in I}$ are uniquely determined, in other words the following implication holds:

$$\sum_{i \in I} v_i e_i = 0, \text{ with finitely many non-zero } v_i \in \mathbb{R} \implies v_i = 0, \quad \forall i \in I.$$

For every vector space we can obtain a sequence of linearly independent vectors $e_1, e_2, e_3, \dots$. If this sequence necessarily stops, we obtain a finite basis and we say that the vector space is finite dimensional. If the sequence can be continued indefinitely we say that the vector space is infinite dimensional.
All bases of a finite dimensional vector space have the same number of vectors. This number is called the **dimension** of V.

Exercise 9.103. — Show that the space of polynomials with real coefficients $\mathbb{R}[x]$ is an infinite dimensional vector space and find a basis.

9.104. — If V has finite dimension $n \in \mathbb{N}$ and we fix a basis $\mathcal{B} = \{e_i\}_{1 \leq i \leq n}$ then map

$$\imath_{\mathcal{B}}(x_1, \dots, x_n) \mapsto x_1 e_1 + \ldots + x_n e_n$$

is a (vector space) isomorphism and allows to treat V as $\mathbb{R}^n$ for most practical tasks. In particular, if $(V, \|\cdot\|)$ is a normed space then $\imath_{\mathcal{B}}$ induces a norm in $\mathbb{R}^n$ defined as

$$\|x\| = \|\imath_{\mathcal{B}}(x)\|.$$

This is a motivation to prove results for $\mathbb{R}^n$ equipped with norms different from the Euclidean one. Indeed, all the concepts and results that can be stated for general norms will automatically hold in "abstract" finite dimensional normed vector space. We see next an important instance of this.

Definition 9.105: Equivalent (i.e., comparable) norms

Let V be a vector space over $\mathbb{R}$ and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. We call $\|\cdot\|_1$ and $\|\cdot\|_2$ **equivalent** if there are constants $A > 0$ and $B > 0$ such that

$$\|v\|_1 \leq A\|v\|_2 \quad \text{and} \quad \|v\|_2 \leq B\|v\|_1 \text{ for all } v \in V.$$

Example 9.106. — Let $n \in \mathbb{N}$. The 1-norm $\|\cdot\|_1$ and the maximum norm $\|\cdot\|_\infty$ given in Example 9.92 are equivalent, as the inequalities

$$\|v\|_\infty \leq \|v\|_1 \quad \text{and} \quad \|v\|_1 \leq n\|v\|_\infty$$

7 hold for all $v \in \mathbb{R}^n$. As we will show in Theorem 9.107, all norms on a finite-dimensional vector space over $\mathbb{R}$ are equivalent to each other. This is not the case for infinite-dimensional vector spaces. For example, the norms given in 9.92 on the space of continuous functions on $[0, 1]$ are not equivalent.

Theorem 9.107: All norms are equivalent

All norms on $\mathbb{R}^n$ are equivalent (i.e., comparable).

Proof. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$, and let $\|\cdot\|'$ denote another norm on $\mathbb{R}^n$. We show that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, which proves the Theorem.

Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$, and let $A = \max\{\|e_1\|', \|e_2\|', \ldots, \|e_n\|'\}$. For any vector $v = x_1e_1 + \cdots + x_ne_n \in V$, we have

$$\|v\|' \leq \sum_{k=1}^{n} |x_k| \cdot \|e_k\|' \leq A \cdot \sum_{k=1}^{n} |x_k| \leq A\sqrt{n}\sqrt{\sum_{k=1}^{n} |x_k|^2} = A\sqrt{n}\|v\|$$

which already shows one of the two required inequalities. For the second one, consider unit sphere (with respect to the Euclidean norm)

$$S := \{v \in \mathbb{R}^n : \|v\| = 1\}$$

and the real number $B = \inf\{\|v\|' \mid v \in S\}$. Let us show that $B > 0$. Indeed, by definition of "inf" there exists a sequence $(v_n)_{n=0}^{\infty}$ in S such that the sequence $\|v_n\| \to B$. Since $(v_n)_{n=0}^{\infty}$ belongs to a closed a bounded set, it contains a convergent subsequence by the Heine-Borel theorem, so by replacing $(v_n)_{n=0}^{\infty}$ with such a subsequence, we can ensure that $(v_n)_{n=0}^{\infty}$ converges to some $v \in S$ with respect to the Euclidean norm. But since

$$\|w - v_n\|' \leq A\sqrt{n}\|w - v_n\|$$

we deduce that the sequence $(v_n)_{n=0}^{\infty}$ also converges to v with respect to the norm $\|\cdot\|'$. Thus, by Lemma 9.97,

$$\|v\| = 1 \quad \text{and} \quad \|v\|' = B$$

and, in particular, $v \neq 0$ and $B > 0$, otherwise $\|\cdot\|'$ would not be a norm, giving zero length to a nonzero vector.

Finally any vector $x \neq 0$ in $\mathbb{R}^n$, $\frac{x}{\|x\|}$ is an element of S, and it satisfies

$$\frac{\|x\|'}{\|x\|} = \left\| \frac{1}{\|x\|} x \right\|' \geq B$$

7 Thus, $\|x\| \leq B^{-1}\|x\|'$ for all $x \neq 0$ in $\mathbb{R}^n$. Since this inequality is also trivially true for $x = 0$, it is shown that the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. □

9.108. — Notice that as a simple consequence of Theorem 9.107, in every finite dimensional vector space V all normed are equivalent. Indeed, we can fix a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of V use the inclusion map $\imath_{\mathcal{B}}$ defined in 9.104 to export the result for $\mathbb{R}^n$ to V.

Corollary 9.109: All norms lead to same topological properties

Topological properties, such as compactness or connectedness, of a subset $E \subseteq \mathbb{R}^n$, remain unchanged regardless of the norm chosen to define the metric in $\mathbb{R}^n$.

Proof. Since any two given norms in $\mathbb{R}^n$ are equivalent the associated distances d_1,d_2 satisfy, for some constant C, $C^{-1}d_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y)$ for all $x, y \in \mathbb{R}^n$. Hence, it is readily shown that U is open with respect to d_1 if and only if it is open with respect to d_2.

As a consequence, $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_2)$ have the same open sets and thus identical topological properties (all properties that we could define only in terms of the topologies, i.e., the collections of open sets) □

Chapter 10

Multidimensional Differentiation

In this chapter, we extend the concept of derivatives to functions defined on open subsets $U \subset \mathbb{R}^n$ and taking values in $\mathbb{R}^m$, for general positive integers n, m.

From now on, we consider always consider $\mathbb{R}^n$ to be equipped with the Euclidean norm (as in Definition 9.1). To keep the notation lighter we will write $|x|$ instead of $\|x\|$, to refer to the Euclidean norm of $x \in \mathbb{R}^n$. Notice that this does not create any ambiguity with the complex absolute value in $\mathbb{R}$ or the complex absolute value in $\mathbb{C}$ because they both coincide with the Euclidean norm. Therefore the notion of convergence for sequences $(x_k)_{k\geq 0}$ in $\mathbb{R}^n$ is the one given by the Euclidean distance

$$x_k \to y \qquad \Longleftrightarrow \qquad \lim_{k\to\infty} |x_k - y| = 0$$

In this section, vectors will be regarded as column vectors to adhere to the standard linear algebra conventions on matrix multiplication. That is vectors x in $\mathbb{R}^n$ and y in $\mathbb{R}^m$ will be
8 regarded as:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m.$$

Similarly, if $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^m$ a function, its components will be denoted always in column form

$$x \mapsto f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}.$$

Notice that for $i = 1, \ldots m$ we have $f_i : U \to \mathbb{R}$.

Definition 10.1: Limits of functions

If $f : U \to \mathbb{R}^m$ is some function and $x_\circ \in U$, we say that the limit of $f(x)$, as x converges to $x_\circ$, is $y_\circ \in \mathbb{R}^m$ and write

$$\lim_{x \to x_\circ} f(x) = y_\circ$$

if *for every* sequence $(x_k)_{k \geq 0} \subset U$ converging to $x_\circ$ we have

$$f(x_k) \to y_\circ, \quad \text{or, in other words,} \quad |f(x_k) - y_\circ| \to 0$$

as $k \to \infty$.

We will also make use the “little o and big O notations”, updated to the context of $\mathbb{R}^n$. If $U \subset \mathbb{R}^n$ is an open set, $x_\circ \in U$, and $f : U \to \mathbb{R}^m$ and $g : U \to \mathbb{R}$ are two functions, we write $f(x) = o(|g(x)|)$ as $x \to x_\circ$ if

$$\lim_{x \to x_\circ} \frac{|f(x)|}{|g(x)|} = 0.$$

Also,we write $f(x) = O(|g(x)|)$

$$\limsup_{x \to x_\circ} \frac{|f(x)|}{|g(x)|} < \infty.$$

8 We remark that this useful notation has to be used with care, since **it is not symmetric**:

$$f(x) = O(|x_1|) \Rightarrow f(x) = O(|x|), \text{ but } f(x) = O(|x|) \not\Rightarrow f(x) = O(|x_1|)!$$

In other words

$$f(x) = O(|x_1|) \text{ and } f(x) = O(|x|), \text{ but } O(|x_1|) \neq O(|x|).$$

10.1 The Differential

10.1.1 The derivative for real valued functions

The derivative of a real-variable function $f : \mathbb{R} \to \mathbb{R}$ at some point $x_\circ \in \mathbb{R}$ has various equivalent interpretations. Of course each of these interpretations provides the same number $f'(x_\circ)$, but the “meaning” we attach to this number is slightly different in each case:

- **Slope of tangent line to the graph.** We look at the graph of f, i.e. the curve $\{y = f(x)\} \subset \mathbb{R}^2$ and write the tangent line to the graph at $(x_\circ, f(x_\circ))$ in the form $y = ax + b$. Then we have $a = f'(x_\circ)$.
- **Zoom-in limit.** When we zoom-in more and more to inspect the graph of f around a point $(x_\circ, y_\circ = f(x_\circ)) \in \mathbb{R}^2$ we see in the limit the line $y = f'(x_\circ)x$. More precisely, in

each compact interval $[-C, C]$, the functions

$$f_r : x \mapsto \frac{f(x_\circ + rx) - f(x_\circ)}{r},$$

converge uniformly, as $r \downarrow 0$, to the linear function $x \mapsto f'(x_\circ)x$.

- **Coefficient in infinitesimal linear approximation.** The linear function that best approximates f in tiny intervals $x_\circ$. More precisely, assume we want to approximate $f(x) \sim a + bx$ around $x \sim x_\circ$, for some real numbers a, b. Then, choosing $b = f'(x_\circ)$, we get approximation error of size $o(|x - x_\circ|)$ as $x \to x_\circ$;
- **Stretching Factor.** Look at a short interval I around $x_\circ$ and the corresponding interval $f(I)$ around $f(x_\circ)$. These two intervals are related trough a “stretching factor” which tends to be $f'(x_\circ)$ as I is taken shorter. Look here for an animation.

To generalize derivatives to functions $f \colon \mathbb{R}^n \to \mathbb{R}^m$ we will start from the third point of view. As it will be clear later on, all these viewpoints (conveniently reinterpreted) are valid also in the several variables case.

10.1.2 Definition of differential

Definition 10.2: Differential of a Function

8 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be a function. Then f is called **differentiable** at $x_\circ \in U$ if there exists a *linear map* $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to 0} \frac{|f(x_\circ + x) - f(x_\circ) - L(x)|}{|x|} = 0$$

holds. If such L exists it is unique (exercise) and it is called the **differential** of f at the point $x_\circ$, and we denote it as

$$L = Df_{x_\circ}.$$

The function f is called **differentiable in U** if it is differentiable at every point in U.

Exercise 10.3 (Uniqueness of the differential). — Suppose that L and L' are two differentials of f at $x_\circ$. Show that $\lim_{x \to 0} L(x)/|x| = 0$ for all x and deduce that $(L - L')v = 0$ for all v, and hence $(L - L') = 0$.

10.4. — If $f : U \to \mathbb{R}^m$ is differentiable at the point $x_\circ \in U$, with differential $L = Df_{x_\circ}$, we can express f as

$$f(x_\circ + x) = f(x_\circ) + L(x) + R(x)$$

Here we recognize the affine-linear approximation $x \mapsto f(x_\circ) + L(x)$ to f, and a remainder term $R(x)$, for which, according to the definition of the total derivative, $R(x) = o(|x|)$ holds for $x \to 0$. Another notation often found in the literature for differential of f at the point $x_\circ$ is $df_{x_\circ}$

10.5. — For functions $f : \mathbb{R} \to \mathbb{R}$ the derivative $f'(x_\circ)$ is a real number. Notice that in this case $L(y) = Df_{x_\circ}(y) = f'(x_\circ)y$.

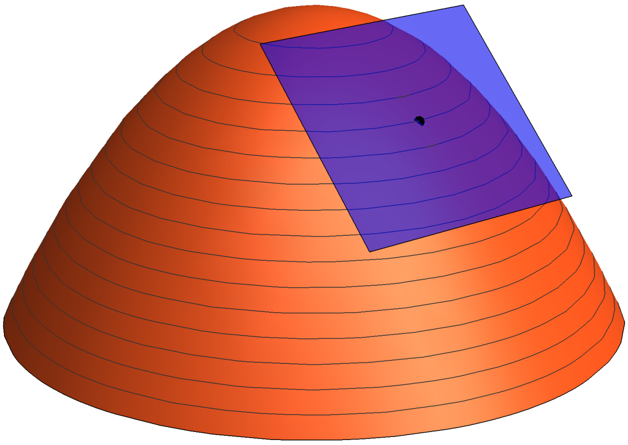

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Figure 10.1: For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the best infinitesimal linear approximation corresponds to the tangent plane of the graph in $\mathbb{R}^3$.

Applet 10.6 (Tangent Plane)**.** *As shown in the above image, we depict the tangent planes for the graphs of two functions $f : \mathbb{R}^2 \to \mathbb{R}$. Additionally, we visualize the partial derivatives and directional derivatives in Definition 10.8. Is there a directional derivative that vanishes at every point?*

Lemma 10.7: Differential component-wise

Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^m$ be a function and let $f_j : U \to \mathbb{R}$ denote j-th component of f. Then, f is differentiable at $x_\circ \in U$ if and only if, for all $1 \leq j \leq n$, the component f_j is differentiable at x_0. In this case, we have, for all $v \in \mathbb{R}^n$,

$$\big(Df_{x_\circ}(v)\big)_j = (Df_j)_{x_\circ}(v).$$

Proof. Assume that f_j is differentiable at $x_\circ$ for every j. There exists a linear function $L_j : \mathbb{R}^n \to \mathbb{R}$ and a remainder term $R_j : \mathbb{R}^n \to \mathbb{R}$ such that

$$f_j(x_\circ + x) = f_j(x_\circ) + L_j(x) + R_j(x)$$

with $R_j(x) = o(|x|)$ for $x \to 0$. We can summarize

$$f(x_\circ + x) = \begin{pmatrix} f_1(x_\circ + x) \\ \vdots \\ f_m(x_\circ + x) \end{pmatrix} = \begin{pmatrix} f_1(x_\circ) \\ \vdots \\ f_m(x_\circ) \end{pmatrix} + \begin{pmatrix} L_1(x) \\ \vdots \\ L_m(x) \end{pmatrix} + \begin{pmatrix} R_1(x) \\ \vdots \\ R_m(x) \end{pmatrix}$$

In summary, it can be written as $f(x_\circ + x) = f(x_\circ) + L(x) + R(x)$. In this expression, L is linear, and it holds that $R_j(x) = o(|x|)$ for $x \to 0$ and we conclude using Lemma 9.24. Thus, f is differentiable, and the claimed formula for $Df(x_\circ)$ holds. Similarly, if f is differentiable at the point $x_\circ$, then each component is differentiable and the claimed formula follows. □

Definition 10.8: Directional derivative

Let $U \subset \mathbb{R}^n$ be an open subset, $x_\circ \in U$, $v \in \mathbb{R}^n$ and and $f : U \to \mathbb{R}^m$. The **directional derivative of f in the direction v at $x_\circ$** is

$$\partial_v f(x_\circ) := \frac{d}{ds}\bigg|_{s=0} f(x_\circ + sv) = \lim_{s \to 0} \frac{f(x_\circ + sv) - f(x_\circ)}{s} \in \mathbb{R}^m,$$

provided that the limit exists. If $v = e_j$, for some $j \in \{1, \dots, n\}$, we denote:

8

$$\partial_{e_j} f(x_\circ) = D_j f(x_\circ) = \partial_j f(x_\circ) = \frac{\partial f(x_\circ)}{\partial x_j} = \begin{pmatrix} \partial_j f_1(x_\circ) \\ \partial_j f_2(x_\circ) \\ \vdots \\ \partial_j f_n(x_\circ) \end{pmatrix}$$

Of course, if the partial derivative in the j-th coordinate exists at every point in U, we obtain a function $\partial_j f : U \to \mathbb{R}^m$, which we call the **$j^{th}$ directional derivative of f**.

10.9. — The partial derivative and the directional derivative along any vector are the derivatives with respect to one of the independent variables, considering all other variables as constants (they are 'frozen'). For example, for the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x(y^2 + \sin(z))$, the partial derivatives with respect to all coordinate directions are given by

$$\begin{aligned} \partial_x f(x, y, z) &= y^2 + \sin(z) \\ \partial_y f(x, y, z) &= 2xy \\ \partial_z f(x, y, z) &= x \cos(z) \end{aligned}$$

for all $(x, y, z) \in \mathbb{R}^3$, as we can apply all known rules from Analysis I. If the total derivative exists, we can connect it with partial derivatives and derivatives along arbitrary vectors using the following proposition.

Proposition 10.10: Differentiable implies linear directional derivatives

Let $U \subset \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}^m$ be differentiable at $x_\circ \in U$. Then, for each $v \in \mathbb{R}^n$, the derivative of f in the direction v exists, and we have

$$\partial_v f(x_\circ) = Df_{x_\circ}(v) \in \mathbb{R}^m.$$

In particular, $\partial_{\alpha v+\beta w} f(x_\circ) = \alpha\partial_v f(x_\circ) + \beta\partial_w f(x_\circ)$, for all $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.

Proof. Assuming the total derivative $Df(x_\circ)$ exists, according to the definition of the derivative, $f(x_\circ + h) = f(x_\circ) + Df(x_\circ)(h) + o(|h|)$ holds for $h \to 0$. Choosing $h = sv$ for $s \to 0$ and $v \in \mathbb{R}^n$, we get

$$\partial_v f(x_\circ) = \lim_{s\to 0} \frac{f(x_\circ + sv) - f(x_\circ)}{s} = \lim_{s\to 0}(Df(x_\circ)(v) + o(1)) = Df(x_\circ)(v)$$

which concludes the proof. □

Exercise 10.11. — Let $U \subset \mathbb{R}^n$ be open, and let $f_1, f_2 : U \to \mathbb{R}$ be functions. Assume that f_1 and f_2 are differentiable at $x_\circ \in U$. Show that the functions $f_1 + f_2$ and $f_1 f_2$ are differentiable at $x_\circ$ and that

8

$$\begin{aligned} D(f_1 + f_2)(x_\circ) &= Df_1(x_\circ) + Df_2(x_\circ) \\ D(f_1 f_2)(x_\circ) &= f_1(x_\circ)Df_2(x_\circ) + f_2(x_\circ)Df_1(x_\circ) \end{aligned}$$

Formulate and prove analogous statements for directional derivatives in the direction of a fixed vector $v \in \mathbb{R}^n$.

Theorem 10.12: Sufficient condition for Differentiability

Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be a function. If for every $j \in \{1, \ldots, n\}$ the partial derivative $\partial_j f$ exists on the entire U and defines a continuous function, then f is differentiable on the entire U.

Proof. Due to Lemma 10.7, we can assume $m = 1$. Let's fix $x_\circ \in U$, and we need to show that f is differentiable at $x_\circ$. By replacing f with $x \mapsto f(x + x_\circ) - f(x_\circ)$, we can also assume that $x_\circ = 0$ and $f(0) = 0$. For $x = (x_1, \ldots, x_n) \in U$, we then have

$$\begin{array}{rcll} f(x) & = & f(x_1, x_2, x_3, \ldots, x_n) & -f(0, x_2, x_3 \ldots, x_n) \\ & & +f(0, x_2, x_3, \ldots, x_n) & -f(0, 0, x_3, \ldots, x_n) \\ & & +f(0, 0, x_3, \ldots, x_n) & - \qquad \cdots \\ & & + \qquad \cdots & -f(0, 0, \ldots, 0, x_n) \\ & & +f(0, 0, \ldots, 0, x_n) & -f(0, 0, \ldots, 0, 0). \end{array}$$

The function $[0, x_j] \to \mathbb{R}$ defined by $t \mapsto f(0, 0, \ldots, 0, t, x_{j+1}, \ldots, x_n)$ is continuously differentiable by hypothesis. Its derivative is given by the j-th partial derivative of f. Therefore, by the Mean Value Theorem, there exists an intermediate point $\xi_j \in [0, x_j]$ such that

$$\partial_j f(0, \ldots, 0, \xi_j, x_{j+1}, \ldots, x_n) x_j = f(0, \ldots, 0, x_j, x_{j+1}, \ldots, x_n) - f(0, \ldots, 0, 0, x_{j+1}, \ldots, x_n)$$

holds. For any choice of such intermediate points $\xi_j \in [0, x_j]$, we obtain

$$\begin{array}{rcl} f(x) & = & \partial_1 f(\xi_1, x_2, x_3, \ldots, x_n) x_1 \\ & & + \partial_2 f(0, \xi_2, x_3, \ldots, x_n) x_2 \\ & & + \qquad \cdots \\ & & + \partial_n f(0, 0, \ldots, 0, \xi_n) x_n. \end{array}$$

8 To show that the linear function $L : (v_1, \ldots, v_n) \mapsto \partial_1 f(0) v_1 + \cdots \partial_n f(0) v_n$ is the derivative $Df(0)$, we need to estimate the difference $R(x) := f(x) - L(x) = f(0 + x) - f(0) - L(x)$.

$$\begin{array}{rcl} R(x) & = & \big(\partial_1 f(\xi_1, x_2, x_3, \ldots, x_n) - \partial_1 f(0)\big) x_1 \\ & & + \big(\partial_2 f(0, \xi_2, x_3, \ldots, x_n) - \partial_2 f(0)\big) x_2 \\ & & + \qquad \cdots \\ & & + \big(\partial_n f(0, 0, \ldots, 0, \xi_n) - \partial_n f(0)\big) x_n \end{array}$$

According to the assumptions of the theorem and because $\frac{|x_j|}{|x|} \leq 1$ for all $x \in \mathbb{R}^n$, the asymptotics

$$\lim_{x \to 0} \frac{R(x)}{|x|} = 0$$

holds, demonstrating that f is differentiable at $x_\circ = 0$ with the derivative $Df(0) = L$. □

The following exercise shows that the mere existence of partial derivatives of a function f, without the continuity assumption, does not necessarily imply that the function is differentiable.

Exercise 10.13. — Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

9

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

for $(x, y) \in \mathbb{R}^2$. Show that the partial derivatives $\partial_x f$ and $\partial_y f$ exist everywhere in $\mathbb{R}^2$, but f is not differentiable at $(0, 0)$.

Definition 10.14: C^1 functions

We call a function $f : U \to \mathbb{R}^m$ on an open subset $U \subset \mathbb{R}^n$ **continuously differentiable** if all the partial derivatives $\partial_i f$, $i = 1, \ldots, n$ exist and continous in U. We will write in this case $f \in C^1(U, \mathbb{R}^m)$ and often $f \in C^1(U)$ when $m = 1$.

10.15. — Notice that $f \in C^1(U, \mathbb{R}^m)$ if and only if the map $Df \colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ mapping $x \mapsto Df_x$ is continuous.

10.16. — From Proposition 10.10, it follows in particular that the total derivative (when it exists) $Df(x_\circ)$ is uniquely determined by the partial derivatives. Specifically, for $v = a_1e_1 + \cdots + a_ne_n \in \mathbb{R}^n$,

$$Df_{x_\circ}(v) = \sum_{i=0}^{n} a_i Df_{x_\circ}(e_i) = \sum_{i=0}^{n} a_i \partial_i f(x_\circ).$$

The $m \times n$ matrix of the linear map $Df_{x_\circ} : \mathbb{R}^n \to \mathbb{R}^m$ is thus given with respect to the canonical bases by the matrix $n \times m$ matrix

9

$$(\partial_1 f(x_\circ), \partial_2 f(x_\circ) \ldots, \partial_n f(x_\circ)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_\circ) & \frac{\partial f_1}{\partial x_2}(x_\circ) & \cdots & \frac{\partial f_1}{\partial x_n}(x_\circ) \\ \frac{\partial f_2}{\partial x_1}(x_\circ) & \frac{\partial f_2}{\partial x_2}(x_\circ) & \cdots & \frac{\partial f_2}{\partial x_n}(x_\circ) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_\circ) & \frac{\partial f_m}{\partial x_2}(x_\circ) & \cdots & \frac{\partial f_m}{\partial x_n}(x_\circ) \end{pmatrix}$$

This matrix is referred to as the **Jacobian matrix** of f evaluated at the point $x_\circ$, commonly denoted by $Jf(x_\circ)$. It also is standard to denote the Jacobian matrix as $Df(x_\circ)$; however, to prevent confusion with $Df_{x_\circ}$ —linear map versus its matrix representation—, by now we will use the notation $Jf(x_\circ)$. While the difference between linear map and a matrix in canonical basis might appear to be a minor detail, it will become more significant in future applications such as in Differential Geometry or Physics.

Notice that, given $U \subset \mathbb{R}^n$ open, $f \in C^1(U, \mathbb{R}^n)$ if and only if the maps

$$Jf : U \to \mathrm{Mat}_{m,n}(\mathbb{R}) \cong \mathbb{R}^{m \times n}$$

defined by $x \mapsto Jf(x)$ is a continuous .

Example 10.17. — The Jacobian matrix $Jf(x, y, z)$ of the function $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by $f(x, y, z) = (x^2 + y, y^2 + z, x + z^2, xyz)$ is computed as follows:

$$Jf(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 1 & 0 \\ 0 & 2y & 1 \\ 1 & 0 & 2z \\ yz & xz & xy \end{pmatrix}$$

Where each entry of the matrix is the partial derivative of the function's output component with respect to an input variable.

Example 10.18. — Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (x^2 - \cos(xy), y^4 - \exp(x))$. The Jacobian matrix of f at (x,y) is then

$$\begin{pmatrix} 2x + \sin(xy)y & \sin(xy)x \\ -\exp(x) & 4y^3 \end{pmatrix},$$

which is continuous as a function of $(x,y) \in \mathbb{R}^2$.

9

10.1.3 The Chain Rule

Interlude: The Hilbert-Schmidt norm of a linear map (or matrix)

Let $M : \mathbb{R}^n \to \mathbb{R}^m$ is be linear map. We define the Hilbert-Schmidt norm of M (or of the matrix (M^i_j) as

$$\|M\|_2 = \sqrt{\operatorname{trace}(M^{\mathrm{T}}M)} = \sqrt{\textstyle\sum_{i=1}^n |Me_i|^2}.$$

Notice that

$$|My| \leq \|M\|_2 |y| \quad \text{for all } y \in \mathbb{R}^n$$

Writing $y = \sum_{i=1}^n y_i e_i$ we obtain:

$$\begin{aligned} |My|^2 &= |M \textstyle\sum_{i=1}^n y_i e_i|^2 = |\textstyle\sum_{i=1}^n y_i M e_i|^2 \\ &\leq \Big(\sum_{i=1}^n |y_i||Me_i| \Big)^2 \leq |y|^2 \sum_{i=1}^n |Me_i|^2 = \|M\|_2^2 |y|^2. \end{aligned}$$

Similarly, the Hilbert-Schmidt norm of the matrix in $\mathrm{Mat}_{m,n}(\mathbb{R})$) is defined as the norm of the associated linar map $\mathbb{R}^n \to \mathbb{R}^m$.

THEOREM 10.19: CHAIN RULE (POINTWISE VERSION)

Let $k, m, n \geq 1$, and let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. If $f : U \to V$ is differentiable at $x_\circ$ and $g : V \to \mathbb{R}^k$ is differentiable at $f(x_\circ)$, then $g \circ f$ is differentiable at $x_\circ$, and the differential of $(g \circ f)$ at $x_\circ$ is given by

$$D(g \circ f)_{x_\circ} = Dg_{f(x_\circ)} \circ Df_{x_\circ} \tag{10.1}$$

In other words, at the level of matrices

$$J(g \circ f)(x_\circ) = Jg(f(x_\circ)) \cdot Jf(x_\circ)$$

or equivalently (components):

$$\frac{\partial (g \circ f)_j}{\partial x_i}(x_\circ) = \sum_{\ell=1}^{m} \frac{\partial g_j}{\partial y_\ell}(f(x_\circ)) \frac{\partial f_\ell}{\partial x_i}(x_\circ)$$

9 *Intuitive idea of proof.* Since f is differentiable at $x_\circ$ we have

$$f(x) - f(x_\circ) \approx L(x - x_\circ),$$

for some L linear. Also, if g is differentiable at $y_\circ = f(x_\circ)$ we have

$$g(y) - g(y_\circ) \approx M(y - y_\circ),$$

for some M linear. Therefore, putting $y = f(x)$ we obtain

$$g(f(x)) - g(f(x_\circ)) = g(y) - g(y_\circ) \approx M(y - y_\circ) = M(f(x) - f(x_\circ)) \approx M(L(x - x_\circ)).$$

Since the composition of linear maps is linear this suggests that $g \circ f$ is differentiable at $x_\circ$ and that the differential of the composition is the composition of differentials In the next proof we will make such argument work rigorously. □

Proof. By the definition of differentiability of f at $x_\circ$ and g at $y_\circ = f(x_\circ)$, we have

$$f(x_\circ + x) = f(x_\circ) + L(x) + R(x) \quad \text{and} \quad g(y_\circ + y) = g(y_\circ) + M(y) + S(y)$$

with $L = Df_{x_\circ}$, $R(x) = o(|x|)$ as $x \to 0$, $M = Dg_{y_\circ}$, and $S(y) = o(|y|)$ as $y \to 0$.

Now, for $x \in \mathbb{R}^n$ with x sufficiently close to 0 (i.e., with $|x|$ small enough), put $y = f(x_\circ + x) - f(x_\circ) = Df_{x_\circ}(x) + R(x)$. We obtain the equation

$$\begin{aligned} g(f(x_\circ + x)) &= g(y_\circ + y) = g(y_\circ) + M(y) + S(y) \\ &= g(f(x_\circ)) + M(L(x)) + \underbrace{M(R(x)) + S(L(x) + R(x))}_{T(x)} \end{aligned}$$

and we want to show that $T(x) = o(|x|)$ as $x \to 0$. Since $R(x) = o(|x|)$ as $x \to 0$, and M is linear $|M(R(x))| \leq \|M\|_2 |R(x)| = o(|x|)$ as $x \to 0$.

It remains to show that $S(L(x) + R(x)) = o(|x|)$ as $x \to 0$. For this, notice that $|L(x) + R(x)| \leq \|L\|_2 |x| + |R(x)| \leq C|x|$ for $C = \|L\|_2 + 1$ for all x sufficiently close to zero (since $|R(x)| = o(|x|) \leq |x|$ as $x \to 0$). Hence, using $|S(y)| = o(|y|)$ as $y \to 0$,

$$\lim_{x \to 0} \frac{S(L(x) + R(x))}{|x|} = \lim_{x \to 0} \frac{S(L(x) + R(x))}{|L(x) + R(x)|} \frac{|L(x) + R(x)|}{|x|} \leq C \lim_{x \to 0} \frac{S(x)}{|x|} = 0.$$

The previous computation implicitly assumes that, along the sequence $x_k \to 0$ chosen to test the limit as $x \to 0$, $S(L(x_k) + R(x_k))$ does not vanish. However, the argument is easily adapted to the case that it could possibly vanish. The details are left as an exercise to the reader.

Thus, we conclude the differentiability of $g \circ f$ at $x_\circ$ and the equation (10.1). □

9 **Corollary 10.20: Chain Rule**

Let $k, m, n \geq 1$, and let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. If $f \in C^1(U, \mathbb{R}^m)$, $g \in C^1(V)$ and $f(U) \subset V$, then $g \circ f \in C^1(U)$ and

$$\partial_i (g \circ f) = \sum_{j=1}^{m} (\partial_j g) \circ f \, \partial_i f_j. \tag{10.2}$$

Example 10.21. — (How to use in practice the chain rule). Consider the functions

$$g(x, y) := e^{x+2y}, \quad f(x, y) = (\sin(x), \ln(1 + y))^T.$$

We aim to compute $\frac{\partial (g \circ f)}{\partial x}$ and $\frac{\partial (g \circ f)}{\partial y}$. In practice, we can operate in this way:

1. We call the two real variables of the map g with different names, so $g(u, v) = e^{u+2v}$ and we set
$$u = f_1(x, y) = \sin(x), \quad v = f_2(x, y) = \ln(y + 1).$$

2. Compute the necessary partial derivatives:
$$\frac{\partial g}{\partial u} = e^{u+2v}, \quad \frac{\partial g}{\partial v} = 2e^{u+2v}, \quad \frac{\partial u}{\partial x} = \cos(x), \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = \frac{1}{y+1}.$$

3. Apply the chain rule, and replace u, v with their expression in terms of x, y:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} = e^{u+2v}\cos(x)+2e^{u+v}\cdot 0 = e^{\sin(x)}\cdot\cos(x)(1+y)^2,\\ \frac{\partial g}{\partial y} &= \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} = e^{u+2v}\cdot 0+2\frac{e^{u+2v}}{1+y} = 2(1+y)e^{\sin(x)}.\end{aligned}$$

Notice that in this particular example the chain rule is not clearly more "economic" than differentiate directly

$$(g\circ f)(x,y) = (1+y)^2 e^{\sin(x)},$$

double check that the result is the same.

Exercise 10.22 (Euler's identity for homogeneous functions). — Assume $f \in C^1(\mathbb{R}^n\setminus\{0\})$ is positively homogeneous of degree $\lambda \in \mathbb{R}$, that is to say

$$f(rx) = r^\lambda f(x) \text{ for all } r > 0, x \neq 0.$$

Show that $\sum_{=1}^n x_j \partial_j f(x) = \lambda f(x)$.

Example 10.23. — Let's consider the special case $n = 1$ for the chain rule. Suppose $I \subset \mathbb{R}$ is an open interval, and $\gamma : I \to V \subset \mathbb{R}^m$ is a differentiable function with values in an open
9 subset $V \subset \mathbb{R}^m$. Further, let $f : V \to \mathbb{R}^k$ be differentiable. Then, the chain rule implies that
$f \circ \gamma$ is differentiable, and the formula

$$(f\circ\gamma)'(t) = Df_{\gamma(t)}(\gamma'(t))$$

holds for all $t \in I$.

10.1.4 The Mean Value Theorem

We formulate a generalization of the Mean Value Theorem for real-valued differentiable functions on an open set $U \subset \mathbb{R}^n$. To do this, we consider a given function f along a straight segment in the open set.

Theorem 10.24: Mean Value Theorem

Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ be differentiable. Let $x_\circ \in U$ and $h \in \mathbb{R}^n$ such that $x_\circ + th \in U$ for all $t \in [0,1]$. Then, there exists $t \in (0,1)$ such that for $\xi = x_\circ + th$, the equation

$$f(x_\circ + h) - f(x_\circ) = Df_\xi(h) = \partial_h f(\xi)$$

is satisfied.

Proof. The derivative of the straight path $\gamma : t \mapsto x_\circ + th$ for fixed $x_\circ, h \in \mathbb{R}^n$ is given by $\gamma'(t) = h$. Therefore, the function $g = f \circ \gamma : [0,1] \to \mathbb{R}$ satisfies all the conditions of the one-dimensional Mean Value Theorem due to the chain rule in Theorem 10.19. Hence, there exists $t \in (0,1)$ with $g(1) - g(0) = g'(t) = Df(x_\circ + th)(h)$ according to the chain rule, and thus

$$f(x_\circ + h) - f(x) = g(1) - g(0) = g'(t) = Df(\xi)(h)$$

for $\xi = x_\circ + th$. □

Definition 10.25: Local Lipschitz continuity

A function $f : X \to Y$ between metric spaces X, Y is called **locally Lipschitz continuous** if, for every $x_\circ \in X$, there exists $\epsilon > 0$ such that $f|_{B(x_\circ,\epsilon)}$ is Lipschitz continuous.

Corollary 10.26: Differentiability VS Lipschitz continuity

Let $U \subset \mathbb{R}^n$ be open and let $f \in C^1(U, \mathbb{R}^m)$. Then, f is locally Lipschitz continuous. If U is additionally **convex** *(i.e., for every two points $x, y \in U$ the segment $\{tx + (1-t)y \mid t \in [0,1]\}$ joining them is contained in U), and the Jacobi matrix Jf is bounded in the whole U, then f is Lipschitz continuous.*

Proof. It suffices to consider the case $m = 1$. First, assume first that U is convex and the
9 Jacobi matrix is bounded in U. That is there exists $M \geq 0$ such that $\|Df_\xi\|_2 = \|Jf(\xi)\|_2 \leq M$
for all $\xi \in U$. From the mean value Theorem 10.24, it follows for $x, y \in U$

$$|f(x) - f(y)| = |Df(\xi)(x-y) \leq M|x-y|$$

for some $\xi \in U$, since U is convex and thus contains the straight segment between x and y. This proves the second statement in the corollary.

The first statement follows from the second applied to the ball $U_0 = B(x_\circ, \epsilon)$ where $\epsilon > 0$ is chosen such that $\overline{B(x_\circ, \epsilon)} \subset U$. Indeed, U_0 is convex, and the mapping $\xi \mapsto Df(\xi)$ is a continuous function on the compact set $\overline{U_0} \subset U$, implying the boundedness of the differential on U_0. □

Corollary 10.27: Solutions of $Df = 0$

Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be differentiable with $Df(x) = 0$ for all $x \in U$. Then, f is constant on each connected component of U.
In particular, if U is connected then f is constant.

Proof. It suffices to consider the case $m = 1$. Assuming that U is non-empty, we choose $x_\circ \in U$ and consider the subset

$$U' = \{x \in U \mid f(x) = f(x_\circ)\}$$

of U.

On the one hand, since f is continuous,

$$U \setminus U' = f^{-1}\big((-\infty, f(x_\circ)) \cup (f(x_\circ), \infty)\big)$$

is open (it is the pre-image of an open set by a continuous function).

9 On other hand by the mean value theorem, it follows that U' is open: Indeed, for $x \in U'$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$, and since every point $y \in B(x, \epsilon)$ can be connected by a straight path to x, it follows from Theorem 10.24 that $f(y) = f(x) = f(x_\circ)$. Thus, $y \in U'$, and since $y \in B(x, \epsilon)$ was arbitrary, $B(x, \epsilon) \subset U'$.

Finally, since U is connected, and U' and $U \setminus U'$ are two disnoint open sets covering U we have $U' = U$ (as $x_\circ \in U'$ so it can not be empty). Hence, the corollary follows. □

10.2 Higher Derivatives

10.2.1 Definition and basic properties

Recall that for functions $f : \mathbb{R} \to \mathbb{R}$ we defined second and higher order derivatives recursively as follows $f^{(k+1)} = (f^{(k)})'$, $k \geq 0$, where $f^{(0)} = f$. Next we will introduce higher order derivatives for function $f : \mathbb{R}^n \to \mathbb{R}^n$.

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Definition 10.28: C^k functions

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be a function, and $k > 1$. We say that f belongs to $C^k(U, \mathbb{R}^m)$ if for all $i = 1, \dots, n$ the partial derivative $\partial_i f$ exists and belongs to $C^{k-1}(U, \mathbb{R}^m)$.

Notice that this recursive definition makes sense because $C^1(U, \mathbb{R}^m)$ have been previously defined.

When $f \in C^k(U, \mathbb{R}^m)$ we say that f is 'k-times continuously differentiable' or that 'f is of class C^k'.

Given $i_1, \dots, i_k$ in $\{1, \dots, n\}$, the k-th partial derivative $\partial_{i_1}\partial_{i_2} \cdots \partial_{i_k} f$ is recursively defined as $\partial_{i_1}(\partial_{i_2} \cdots \partial_{i_k} f)$ and it is a continuous function on U.

Consistently with the previous definition, it is convenient to define $C^0(U, \mathbb{R}^m)$ as the class of continuous functions $U \to \mathbb{R}^m$.

We define $C^\infty(U, \mathbb{R}^m)$ as the class of functions that belongs to $C^k(U, \mathbb{R}^m)$ for all $k \geq 1$.

Similarly to the $k = 1$ case, $C^k(U, \mathbb{R})$ is denoted $C^k(U)$ (also when k is replaced by ∞)

Proposition 10.29: Higher regularity of sums, products and compositions

Let $U \subset \mathbb{R}^n$ open, $f, g \in C^k(U, \mathbb{R})$, $h \in C^k(V, \mathbb{R}^n)$ with $V \subset \mathbb{R}^m$ open with $h(V) \subset U$, then

(1) $f + g, f \cdot g$ are of class C^k

(2) $f \circ h$ is of class C^k

Proof. Property (1) easily follows by the recursive (i.e., inductive) definition of C^k functions. Indeed, if f, g are of class C^k, for some $k \geq 1$ then, for all i

$$\partial_i(f+g) = \underbrace{\partial_i f}_{C^{k-1}} + \underbrace{\partial_i g}_{C^{k-1}} \quad \text{and} \quad \partial_i(fg) = \underbrace{\partial_i f}_{C^{k-1}} \cdot \underbrace{g}_{C^k} + \underbrace{f}_{C^k} \cdot \underbrace{\partial_i g}_{C^{k-1}} \tag{10.3}$$

This establishes first the case $k = 1$ and then $k > 1$ by induction over k.

To establish (2) we proceed by induction over k. On the one hand, since the composition of continuous is continuous the case $k = 0$ follows. On the other hand, if $k \geq 1$ by the chain rule (Corollary 10.20) we have that $f \circ h$ is differentiable and

$$\partial_j(f \circ h) = \sum_{\ell=1}^{m} \underbrace{\partial_\ell f \circ h}_{C^{k-1}} \underbrace{\partial_j h_\ell}_{C^{k-1}}.$$

Thus, using (1) we obtain that $\partial_i(h \circ f)$ is of class C^{k-1}, for $1 \leq j \leq n$, so $f \circ h \in C^k$. □

10.2.2 Schwartz's Theorem and Multi-indexes notation

Theorem 10.30: Schwarz's Theorem

Let $U \subset \mathbb{R}^n$ be open and let $f \in C^2(U, \mathbb{R}^m)$, then

$$\partial_j \partial_i f = \partial_i \partial_j f, \qquad \text{for all } \; i, j \in \{1, \ldots, n\}.$$

Intuitive idea of proof. For a smooth enough function $f : \mathbb{R}^2 \to \mathbb{R}$ we should have, for $h \in R$ with very small modulus,

$$\partial_1 f(x_1, x_2) \approx \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}$$

and then

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$$\begin{aligned} \partial_2(\partial_1 f(x_1, x_2)) &\approx \frac{\partial_1 f(x_1, x_2 + h) - \partial_1 f(x_1, x_2)}{h} \\ &\approx \frac{1}{h}\left(\frac{f(x_1 + h, x_2 + h) - f(x_1, x_2 + h)}{h} - \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}\right) \\ &= \frac{f(x_1 + h, x_2 + h) - f(x_1, x_2 + h) - f(x_1 + h, x_2) + f(x_1, x_2)}{h^2} \end{aligned}$$

Similarly,

$$\partial_2 f(x_1, x_2) \approx \frac{f(x_1, x_2 + h) - f(x_1, x_2)}{h}$$

and then

$$\begin{aligned} \partial_1(\partial_2 f(x_1, x_2)) &\approx \frac{\partial_2 f(x_1, x_2 + h) - \partial_2 f(x_1, x_2)}{h} \\ &\approx \frac{1}{h}\left(\frac{f(x_1 + h, x_2 + h) - f(x_1 + h, x_2)}{h} - \frac{f(x_1, x_2 + h) - f(x_1, x_2)}{h}\right) \\ &= \frac{f(x_1 + h, x_2 + h) - f(x_1, x_2 + h) - f(x_1 + h, x_2) + f(x_1, x_2)}{h^2} \end{aligned}$$

Therefore,

$$\partial_2 \partial_1 f(x_1, x_2) \approx \partial_1 \partial_2 f(x_1, x_2)$$

and we need to show that the approximation errors in the previous infinitessimally small as h goes to zero. Notice that this is nontrivial as we are dividing by h^2, which is 'doubly small'. The proof below will make thisheuristic rigurous.

□

Proof. If $i = j$ there is nothing to prove, so by symmetry of the statement with respect to i, j we may assume $i < j$. We want to show that for all $y = (y_1, y_2, \ldots, y_n) \in U$, $\partial_j \partial_i f(y) = \partial_j \partial_i f(y)$.

It suffices to consider the case $n = 2$, $m = 1$ and $i = 1$, $j = 2$. Indeed, for general $n \geq 2$ and m and fixed $i_1, i_2 \in \{1, \ldots, n\}$ with $i_1 \neq i_2$ we may apply the considered special case to

$$\widetilde{f}(x_1, x_2) = f_k(y_1, \ldots, \overbrace{x_1}^{i}, \ldots, \overbrace{x_2}^{j}, \ldots y_n) \quad \text{for fixed } k \in \{1, \ldots, m\}.$$

Thus, we assume without loss of generality $n = 2$ and $m = 1$. For $x = (x_1, x_2) \in U$ and a sufficiently small $h > 0$, we define a function F by

$$F(h) = f(x_1 + h, x_2 + h) - f(x_1 + h, x_2) - f(x_1, x_2 + h) + f(x_1, x_2).$$

Furthermore, for a sufficiently small but fixed $h > 0$, we consider the differentiable function $\varphi : [0, 1] \to \mathbb{R}$ given by $\varphi(t) = f(x_1 + th, x_2 + h) - f(x_1 + th, x_2)$ and obtain

$$F(h) = \varphi(1) - \varphi(0) = \varphi'(\xi_1) = \big(\partial_1 f(x_1 + \xi_1 h, x_2 + h) - \partial_1 f(x_1 + \xi_1 h, x_2)\big)h$$

for some $\xi_1 \in (0, 1)$ by the one-dimensional Mean Value Theorem **??**.

10

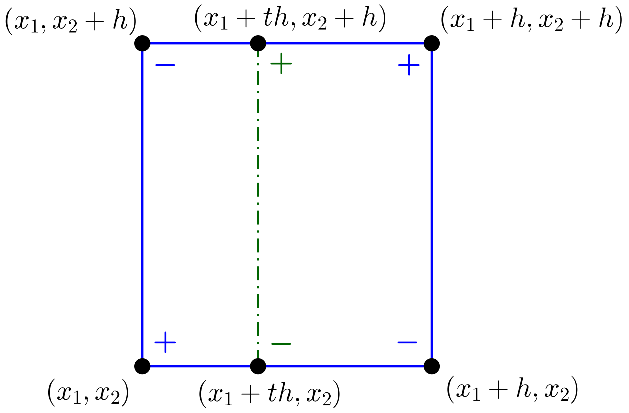


Figure 10.2: The function $h \mapsto F(h)$ is a signed sum of function values of f at the corners of a square (here marked by a solid line). The function $t \mapsto \varphi(t)$ corresponds to the difference of function values on a vertical segment through the square.

Applying the one-dimensional Mean Value Theorem again to $\psi : [0, 1] \to \mathbb{R}$ given by $\psi(t) = \partial_1 f(x_1 + \xi_1 h, x_2 + th)$ along with the chain rule, we obtain

$$F(h) = \big(\partial_1 f(x_1 + \xi_1 h, x_2 + h) - \partial_1 f(x_1 + \xi_1 h, x_2)\big)h = \partial_2 \partial_1 f(x_1 + \xi_1 h, x_2 + \xi_2 h)h^2$$

for some intermediate point $\xi_2 \in (0, 1)$.

Similarly, now defining $\widetilde{\varphi}(t) = f(x_1 + h, x_2 + th) - f(x_1, x_2 + th)$ and $\widetilde{\psi}(t) = \partial_1 f(x_1 + th, x_2 + \widetilde{\xi}_2 h)$ we obtain

$$F(h) = \partial_1 \partial_2 f(x_1 + \widetilde{\xi}_1 h, x_2 + \widetilde{\xi}_2 h)h^2.$$

for suitable $\widetilde{\xi}_1, \widetilde{\xi}_2 \in (0, 1)$. Dividing by $h^2 > 0$, we obtain

$$\partial_2 \partial_1 f(x_1 + \xi_1 h, x_2 + \xi_2 h) = \partial_1 \partial_2 f(x_1 + \widetilde{\xi}_1 h, x_2 + \widetilde{\xi}_2 h).$$

Since $\xi_1, \xi_2, \widetilde{\xi}_1, \widetilde{\xi}_2 \in (0, 1)$, the points $(\xi_1 h, \xi_2 h)$ and $(\widetilde{\xi}_1 h, \widetilde{\xi}_2 h)$ tend to $(0, 0)$ as h tends to 0. Therefore, due to the continuity of both partial derivatives, we conclude $\partial_2 \partial_1 f(x) = \partial_1 \partial_2 f(x)$. □

As a consequence of Schwartz's Theorem higher order derivatives are also independent of the order in which we take the partial derivatives

10

Corollary 10.31: Schwarz in C^k

Let $U \subset \mathbb{R}^n$ open and $f \in C^k(U)$. Then for every $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ and for every permutation $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ the derivative we have

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_{i_{\sigma(1)}} \partial_{i_{\sigma(2)}} \cdots \partial_{i_{\sigma(k)}} f,$$

Proof. It is a straightfoward consequence form Schwartz's Theorem, e.g. using that any permutation of $\{1, 2, \ldots, k\}$ can be obtained as a suitable composition the $k-1$ transpositions $\{(1, 2), (2, 3), \ldots (k-1, k)\}$. It is left as an exercise to the reader to give a proof by induction over k. □

Thanks to Schwartz's Theorem, multi-indexes are useful to express higher order derivatives.

Interlude: Multi-Indeces and Polynomials of several variables

Any $\alpha \in \mathbb{N}^n$ is called a **multi-index,** the length of a multi-index α is defined as

$$|\alpha| := \alpha_1 + \ldots + \alpha_n.$$

We say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \ldots, n$, and define the factorial

$$\alpha! := \alpha_1! \ldots \alpha_n! \quad \text{(recall that} \quad 0! = 1).$$

A polynomial of n variables $X_1, \ldots, X_n$ of degree k with real coefficients can be uniquely (and compactly) expressed as

$$\sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq k} c_\alpha X^\alpha := \sum_{|\alpha| \leq k} c_{(\alpha_1, \ldots, \alpha_n)} X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

where $c_{(\alpha_1, \ldots, \alpha_n)} \in \mathbb{R}$.

10

Many combinatorial formulas are simple when expressed in multi-index notation, such as the multinomial formula:

$$(X_1 + X_2 + \cdots + X_n)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} X^\alpha.$$

Thanks to Corollary 10.31 multiindex notation is also very useful to express partial derivatives. Indeed, if $f \in C^k$ and $|\alpha| \leq k$ we define

$$\partial^\alpha f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} f.$$

Exercise 10.32. — Prove the identity

$$\frac{n^k}{k!} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \frac{1}{\alpha!}.$$

10.2.3 Multidimensional Taylor Approximation

Theorem 10.33: Taylor's Theorem

Let $U \subset \mathbb{R}^n$ be open and let $f \in C^{k+1}(U), k \geq 0$. Let $x_0 \in U$ and $h \in \mathbb{R}^n$ such that $x_0 + th \in U$ for all $t \in [0,1]$. Then, we have

10 $$f(x_0 + h) = \sum_{\alpha \in \mathbb{N}^n,\ |\alpha| \leq k} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} + R_{k+1} f(x_0, h)$$

where the reminder is given by

$$R_{k+1} f(x_0, h) := \int_0^1 (k+1)(1-t)^k \sum_{\alpha \in \mathbb{N}^n,\ |\alpha| = k+1} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!}\, dt = O(|h|^{k+1}).$$

Proof. Since U is open, there exists $\varepsilon > 0$ such that $x + th \in U$ for all $t \in (-\varepsilon, 1+\varepsilon)$. We apply the one-dimensional Taylor approximation to $\varphi : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ given by $\varphi(t) = f(x + th)$. According to Taylor's Theorem, we obtain the Taylor approximation around 0 at 1

$$\varphi(1) = \sum_{m=0}^{k} \frac{\varphi^{(m)}(0)}{m!} + \int_0^1 \varphi^{(k+1)}(t) \frac{(1-t)^k}{k!} dt. \tag{10.4}$$

Applying the chain rule in Theorem 10.19 to φ, we get for $t \in (-\epsilon, 1+\epsilon)$ the derivatives

$$\varphi'(t) = \sum_{i=1}^{n} \partial_i f(x_0 + th) h_i = \sum_{|\alpha|=1} \partial^\alpha f(x_0 + th) h^\alpha.$$

Let us show that, for all $m \leq k+1$:

11 $$\varphi^{(m)}(t) = m! \sum_{|\alpha|=m} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!},$$

Indeed, using the chain rule and induction over $m \geq 1$, we have

$$\begin{aligned}
\varphi^{(m+1)}(t) = \frac{d}{dt} \varphi^{(m+1)}(t) &= \frac{d}{dt} \left(m! \sum_{|\alpha|=m} \partial^\alpha f(x_0 + th) \frac{h^\alpha}{\alpha!} \right) \\
&= m! \left(\sum_{|\alpha|=m} \sum_{i=1}^{n} \partial_i \partial^\alpha f(x_0 + th) \frac{h^\alpha h_i}{\alpha!} \right) \\
&= m! \left(\sum_{|\beta|=m+1} \partial^\beta f(x_0 + th) h^\beta \sum_{1 \leq i \leq n,\ \beta_i \geq 1} \frac{\beta_i}{\beta!} \right).
\end{aligned}$$

For the last equality we have used the following counting argument: If α is a multiindex with $|\alpha| = m$ then $\partial_i \partial^\alpha f = \partial^\beta f$ and $h_i h^\alpha = h^\beta$, where β is a multiindex with $|\beta| = m+1$.

Namely:

$$(\beta_1, \dots, \beta_i, \dots, \beta_n) = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_n). \tag{10.5}$$

In particular we have

$$\frac{1}{\alpha!} = \frac{\alpha_i + 1}{\alpha_1! \dots (\alpha_i + 1)! \dots \alpha_n!} = \frac{\beta_i}{\beta!} \tag{10.6}$$

Notice that different α's will give the same β for different i's. More precisely, for all β with $|\beta| = m + 1$ and $i \in \{1, 2, \dots n\}$ such that $\beta_i \geq 1$ we have $\partial_i \partial^\alpha f = \partial^\beta f$ and $h_i h^\alpha = h^\beta$ for exactly one i: with α and β satisfying (10.5) and (10.6). Hence,

$$\sum_{|\alpha|=m} \sum_{i=1}^{n} \frac{\partial_i \partial^\alpha f h_i h^\alpha}{\alpha!} = \sum_{|\beta|=m+1} \sum_{1 \leq i \leq n,\ \beta_i \geq 1} \frac{\beta_i \partial^\beta f h^\beta}{\beta!} \Big).$$

Substituting this into (10.4), we obtain the theorem. □

Corollary 10.34: Polynomial expansions determine higher derivatives

Let $x_0 \in U$, $f \in C^{k+1}(U)$ and $P(x)$ a polynomial of degree $k \geq 0$. Assume that

$$|f(x_0 + h) - P(h)| = o(|h|^k) \qquad \textit{as } h \to 0.$$

Then for all $|\alpha| \leq k$, $\partial^\alpha f(x_0) = \partial^\alpha P(0)$.

11 *Proof.* By Taylor's theorem we immediately get

$$\left| P(h) - \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq k} \partial^\alpha f(x_0) \frac{h^\alpha}{\alpha!} \right| = o(|h|^k),$$

but two polynomials of degree k whose difference is $o(|h|^k)$ must have exactly the same coefficients (exercise). □

This Corollary can be useful to compute Taylor polynomials for explicit functions, without having to care about factorials etc.

Example 10.35. — Let us compute the Taylor polynomial of degree 2 of $\sqrt{1 + x - y^2}$ around the origin. From Analysis I, you known that

$$\sqrt{1+t} = 1 + \tfrac{t}{2} - \tfrac{t^2}{8} + O(t^3), \quad t \to 0.$$

Plugging in $t = x - y^2$ we find

$$\begin{aligned} \sqrt{1 + x - y^2} &= 1 + \frac{x - y^2}{2} - \frac{(x - y^2)^2}{8} + O((x - y^2)^3) \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{2}y^2 + \frac{1}{4}xy^2 - \frac{1}{8}y^4 + O((x - y^2)^3). \end{aligned}$$

Now we want $(x,y) \to (0,0)$, and a reminder which is $o(r^2)$ where $r := \sqrt{x^2+y^2}$. Observing that

$$\frac{1}{4}xy^2 = O(r^3), \quad \frac{1}{8}y^4 = O(r^4), \quad O((x-y^2)^3) = O(r^3), \qquad \text{as } r \downarrow 0,$$

we find out expansion

$$\sqrt{1+x-y^2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{2}y^2 + O(r^3), \qquad \text{as } r \downarrow 0.$$

Exercise 10.36. — Compute the Taylor polynomials up to the quadratic order at $(0,0)$ of the following functions in two variables

$$\sin(xy), \quad \sqrt{1+x+y^2}, \quad \exp\arctan(x-y), \quad \frac{1}{1-x^2-y^2}, \dots$$

Don't use the general formula!

Exercise 10.37. — Prove the Taylor expansion of the determinant close to the identity is

$$\det(I+tX) = 1 + t\,\mathrm{tr}(X) + \frac{t^2}{2}\big(\mathrm{tr}(X)^2 - \mathrm{tr}(X^2)\big) + O(t^3),$$

11 and that the one of the inverse matrix function is

$$(I+tX)^{-1} = 1 - tX + t^2X^2 + O(t^3).$$

Applet 10.38 (Taylor Approximation)**.** *We observe how the first, second, or third-order Taylor approximations approximate the function $f(x,y) = \sin(x)\cos(y) + 2$.*

10.2.4 Real analytic functions of several variables

Definition 10.39: Real analytic functions

Let $U \subset \mathbb{R}^n$ be an open set. We say that $f \in C^\infty(U)$ is analytic if for every $x_0 \in U$ there exist $\varrho > 0$ and C such that

$$\sup_{B_\varrho(x_0)} |\partial^\alpha f| \le C|\alpha|!\,(n\varrho)^{-|\alpha|} \tag{10.7}$$

holds for every multiindex α (of arbitrarily large order).

Thanks to Taylor's theorem, a function is analytic if and only if it can be written as power series around any point in its domain of definition.

Theorem 10.40: Analytic expansion

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to R$ be analytic.
Given $x_0 \in U$, let ϱ and C be as in Definition 10.39. Consider the power series:

$$f_{x_0,k}(h) := \sum_{\alpha\in\mathbb{N}^n, |\alpha\le k|} \partial^\alpha f(x_0)\, \frac{h^\alpha}{\alpha!}.$$

with $k \to \infty$.
Then, for all $r \in (0, \varrho)$ the series is absolutely convergent the following sense: for all $k < \ell$

$$\sup_{h\in B_r(0)} \sum_{k\le|\alpha|\le\ell} \left|\partial^\alpha f(x_0)\, \frac{h^\alpha}{\alpha!}\right| \le \frac{C(r/\varrho)^k}{1-(r/\varrho)}. \tag{10.8}$$

In particular the series converges for $h \in B_r(0)$: $f_{x_0}(h) := \lim_{k\to\infty} f_{x_0,k}(h)$ is well-defined for all $h \in B_r(0)$ and

$$\sup_{h\in B_r(0)} \left|f_{x_0,k}(h) - f_{x_0}(h)\right| \le \frac{C(r/\varrho)^k}{1-(r/\varrho)}.$$

11 *Moreover the series represents f around x_0:*

$$f(x_0+h) = f_{x_0}(h) \quad \text{for all } h \in B_r(0).$$

Proof. Noticing $|h^\alpha| = h_1^{\alpha_1}\cdots h_n^{\alpha_n} \le |h|^{|\alpha|}$ and using (10.7) we obtain, for all $k \ge 0$:

$$\sum_{|\alpha|=k} \left|\partial^\alpha f(x_0)\, \frac{h^\alpha}{\alpha!}\right| \le \sum_{|\alpha|=k} C|h|^k (n\varrho)^{-k} \frac{k!}{|\alpha|}.$$

Using the multinomial type identity

$$n^k = (1+1+\cdots+1)^k = \sum_{|\alpha|=k} \frac{k!}{|\alpha|}$$

we obtain then that for $|h| < r$

$$\sum_{|\alpha|=k} \left|\partial^\alpha f(x_0)\, \frac{h^\alpha}{\alpha!}\right| \le C(r/\varrho)^k.$$

From here (10.8) follows summing the geometric series.

To prove the representation formula we estimate similarly the reminder in Taylor's theorem. For $|h| < r$ we have:

$$\begin{aligned}\big|R_{k+1}f(x_0,h)\big| &= \left|\int_0^1 (k+1)(1-t)^k \sum_{|\alpha|=k+1} \partial^\alpha f(x_0+th)\frac{h^\alpha}{\alpha!}\,dt\right| \\ &\le \int_0^1 (k+1)(1-t)^k \sum_{|\alpha|=k+1} \left|\partial^\alpha f(x_0+th)\frac{h^\alpha}{\alpha!}\right| dt \\ &\le \int_0^1 (k+1)(1-t)^k C(r/\varrho)^k\,dt \\ &= C(r/\varrho)^k\,.\end{aligned}$$

11

Therefore, for $h \in B_r(0)$

$$|f(x_0+h) - f_{x_0}(h)| \le |f(x_0+h) - f_{x_0,k}(h)| + |f_{x_0,k}(h) - f_{x_0}(h)| \le C(r/\varrho)^k + \frac{C(r/\varrho)^k}{1-(r/\varrho)}.$$

Sending $k \to \infty$ we conclude $f(x_0+h) = f_{x_0}(h)$ in $B_r(0)$. □

Corollary 10.41: Unique continuation principle

Assume that $U \subset \mathbb{R}^n$ is a connected open set and f and g are two real analytic functions in U. If at some point $x_0 \in U$ we have $\partial^\alpha f(x_0) = \partial^\alpha g(x_0)$ for every multi-index α then $f = g$ in the whole U.

In particular, if f and g coincide in a nonempty open subset $V \subset U$ then they must coincide in the whole U.

Proof. Let

$$U' := \{x \in U \mid \partial^\alpha f(x_0) = \partial^\alpha g(x_0) \quad \text{for all } \alpha \in \mathbb{N}^n\}$$

The complement of U' in U can be written as

12

$$U \setminus U' = \bigcup_{\alpha\in\mathbb{N}^n} (\partial^\alpha(f-g))^{-1}\big((-\infty,0)\cup(0,+\infty)\big).$$

Since each $\partial^\alpha(f-g)$ is a continuous function the previous set is a union of open sets, hence an open set.

Let us now show that U' is open. Indeed if $x \in U$ then since both f and g are analytic there exist $\rho > 0$ (the minimum of the two ρ's for f and g at x) and $C > 1$ (the maximum of the two ρ's for f and g at x) such that $f(x_0+h) = f_x(h)$ for $|h| < \varrho/2$ and $g(x_0+h) = g_x(h)$ for $|h| < \varrho/2$. But since by assumption $x \in U'$ we have $f_x(h) = g_x(h)$. It follows that $f - g = 0$ coincide in $B_{\varrho/2}(x_0)$ and hence $B_{\varrho/2}(x_0) \subset U'$.

Since U is connected and U' nonepty (it contains x_0) we obtain that $U \setminus U'$ must be empty. Thus, $f - g = 0$ in all of U. □

Chapter 11

Optimization and applications, Convexity

Differential calculus is crucial for solving optimization problems encountered in fields such as engineering, economics, and physics. These problems typically involve identifying the maximum or minimum values of a function subject to certain constraints. The Weierstrass theorem assures the existence of these optimal values for continuous functions defined over closed and bounded subsets of $\mathbb{R}^n$, but it does not provide a method for locating them. Thus, finding these optimal points necessitates additional techniques.

For a function like $f(t) = t^3 - 2t + 1$ defined on the compact interval $[0, 1]$, the search for minima (or maxima) involves checking critical points where $f'(t_0) = 0$. For this function, solving $f'(t) = 3t^2 - 2 = 0$ yields a single solution $t_0 = \sqrt{2/3}$ within $(0, 1)$. At this point, $f(\sqrt{2/3}) \approx -0.089$. However, the extremum could also be at the boundary points $t = 0$ and $t = 1$, with $f(0) = 1$ and $f(1) = 0$. The minimum of these three values is approximately -0.089, indicating that the minimum occurs at the interior point $\sqrt{2/3}$.

12 What about multi-variable optimization? Consider, for example, minimizing $f(x, y, z) = xy - y^2z + 6z^3$ within the closed unit ball $\overline{B_1} \subset \mathbb{R}^3$. To find potential interior points for minimal values, we extend the condition $f'(t) = 0$ to higher dimensions. We must also consider potential minimum points on the domain's boundary, in this case, the sphere $\partial B_1(0)$, defined by the equation:

$$\partial B_1(0) = \left\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = x_1^2 + x_2^2 + x_3^2 - 1 = 0\right\}.$$

This naturally leads to a 'constrained minimization problem': finding (x, y, z) that minimize $f(x, y, z)$ among all points satisfying $g(x, y, z) = 0$.

This section will delve into techniques for tackling both unconstrained and constrained optimization problems in multiple variables.

11.1 First order optimality condition for interior points

Definition 11.1: Gradient

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a C^1 function. We define the **gradient** of f at $x \in U$, denoted $\nabla f(x)$ as the (column) vector $(\partial_1 f(x), \partial_2 f(x), \dots, \partial_n f(x))^T$ In other words, $\nabla f(x) = Jf(x)^T$. Hence $\nabla f : U \to \mathbb{R}^n$ is continuous map associating a vector to each point in U (i.e., a so-called ‘vector field’).

11.2. — For $f \in C^1(U)$, $x_0 \in U$ is called a **critical point** of f if the gradient of f vanishes at x_0 (equivalently Df_{x_0} is the zero map).

Let $U \subseteq \mathbb{R}^n$ be open and non-empty. We discuss the relationship between derivatives and extrema of real-valued functions $f : U \to \mathbb{R}$. As with functions in one variable, the vanishing of the derivative is a necessary, but not sufficient, condition for the presence of an extremum.

11.3. — Recall that an element $x_0 \in U$ is called a **local maximum** of f if there exists $r > 0$ such that $f(x) \leq f(x_0)$ for all $x \in B(x_0, r)$. We say x_0 is an **isolated local maximum** 12 or **strict local maximum** if there exists $r > 0$ such that $f(x) < f(x_0)$ for all $x \in B(x_0, r)$ with $x \neq x_0$. The definition of a **local minimum** is analogous, and collectively, we refer to them as **local extrema**.

Proposition 11.4: The gradient vanishes at a local extremum

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be a C^1 function, and let $x_0 \in U$ be a point where f is differentiable and assumes a local extremum. Then $\partial_i f(x_0) = 0$ for all $i = 1, \dots, n$. In other words, $\nabla f(x_0) = 0$ and x_0 is a critical point of f.

Proof. Without loss of generality, assume that f attains a local minimum at x_0. For all $i \in \{1, \dots, n\}$ and sufficiently small $h > 0$, we have, by assumption,

$$f(x_0 + he_i) - f(x_0) \geq 0 \quad \text{and} \quad f(x_0 - he_i) - f(x_0) \geq 0$$

and thus$\partial_i f(x_0) = 0$, due to

$$\partial_i f(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + he_i) - f(x_0)}{h} \geq 0, \quad \partial_i f(x_0) = \lim_{h \to 0^+} \frac{f(x_0 - he_i) - f(x_0)}{-h} \leq 0.$$

□

11.2 First order optimality condition with constraints: Lagrange multipliers

In this section we discuss the method of Lagrange multipliers, a useful method to tackle constrained minimization problems of the type

$$\min\{f(x) \mid g_1(x) = \ldots = g_k(x) = 0\}. \tag{11.1}$$

Proposition 11.5: Lagrange Multipliers

Let $U \subset \mathbb{R}^n$ be open, $B_r(x_0) \subset U$ and $f, g_1, \ldots g_k$ functions in $C^1(U, \mathbb{R})$. Let

$$M := \{x \in U : g_1(x) = \ldots = g_k(x) = 0\} \neq \emptyset,$$

and suppose that $f|_M$ has a local minimum at $x_0 \in M$, that is to say:

$$f(x) \geq f(x_0) \text{ for all } x \in M \cap B_r(x_0).$$

Then there are real numbers $\lambda_0, \ldots, \lambda_k$ such that

$$\lambda_0 \nabla f(x_0) + \lambda_1 \nabla g_1(x_0) + \ldots + \lambda_k \nabla g_k(x_0) = 0, \quad \lambda_0^2 + \ldots + \lambda_k^2 = 1. \tag{11.2}$$

12 *In other words, the vectors $\nabla f(x_0), \nabla g_1(x_0), \ldots, \nabla g_k(x_0)$ are linearly dependent.*

Proof "à la De Giorgi". Possibly replacing $f(x)$ with $\tilde{f}(x) := f(x) + |x - x_0|^2$ (notice that $\nabla \tilde{f}(x_0) = \nabla f(x_0)$) and taking a smaller r (e.g., $\tilde{r} := r/2$), we may assume without loss of generality that x_0 is the only local local minimum in $M \cap \overline{B_r(x_0)}$ (in other words, it is a strict local minimum).

Step 1. For given $\varepsilon > 0$, consider the penalized function

$$f_\varepsilon(x) := f(x) + \tfrac{1}{2\varepsilon}(g_1^2(x) + \ldots + g_k(x)^2),$$

defined for $x \in \overline{B}_r(x_0)$. Take some sequence $\varepsilon_\ell \to 0$ as $\ell \to \infty$ and let x_ℓ be a point of minimum of f_{ε_ℓ} in the compact set $\overline{B}_r(x_0)$.

Step 2. We claim that $f_{\varepsilon_k}(x_\ell) \to f(x_0)$ and $x_\ell \to x_0$, as $\varepsilon \to 0$. Indeed, by minimality

$$f_{\varepsilon_\ell}(x_\ell) \leq f_{\varepsilon_\ell}(x_0) = f(x_0). \tag{11.3}$$

Hence

$$g_1^2(x_\ell) + \ldots + g_k(x_\ell)^2 \leq 2\varepsilon_\ell f_{\varepsilon_\ell}(x_\ell) \leq 2\varepsilon_\ell f(x_0) \to 0.$$

So, whenever a subsequence $(x_{\ell_m})_{m \geq 1}$ converges to $\overline{x} \in B_r(x_0)$ then $\bar{x} \in M \cap \overline{B_r(x_0)}$. Hence, we have

$$f(x_0) \leq f(\bar{x}) = \lim_m f(x_{\ell_m}) \leq \limsup_\ell f_{\varepsilon_\ell}(x_\ell) \leq f(x_0),$$

where we have used again (11.3). This implies implies $\bar{x} = x_0$ (recall that x_0 is a strict local minimum). This proves that $f_{\varepsilon_\ell}(x_\ell) \to f(x_0)$ and that $(x_k)_{k \geq 0}$ can only accumulate at x_0. On the other hand $\{x_k\} \subset \overline{B}_r(x_0)$ is bounded so it must have accumulation points. Hence we must have $x_k \to x_0$ as $\varepsilon_\ell \downarrow 0$ (there is not even need to pass a sub-sequence).

Step 3. In particular $x_\ell \in B_r(x_0)$ eventually, will be an interior critical point of f_{ε_ℓ}, so by Proposition 11.4,

$$0 = \varepsilon_\ell \nabla f_\varepsilon(x_\ell) = \varepsilon_\ell \nabla f(x_\ell) + g_1(x_\ell)\nabla g_1(x_\ell) + \ldots + g_k(x_\ell)\nabla g_k(x_\ell),$$

12 where we used that $\partial_i (g_j)^2 = g_j \partial_i g_j$ for $i = 1, \ldots, n$ and so $\nabla (g_j)^2 = g_j \nabla g_j$.

This means that the $k+1$ vectors $\{\nabla f(x_\ell), \nabla g_1(x_\ell), \ldots, \nabla g_k(x_\ell)\}$ are linearly dependent, hence there is a unit vector $\lambda^\ell \in \mathbb{R}^{k+1}$ such that

$$0 = \lambda_0^\ell \nabla f(x_\ell) + \lambda_1^\ell \nabla g_1(x_\ell) + \ldots + \lambda_k^\ell \nabla g_k(x_\ell). \tag{11.4}$$

Step 4. Since the unit k-sphere $\left\{y \in \mathbb{R}^{k+1} \mid |y|^2 = 1\right\}$ is compact (it is a closed bounded subset of $\mathbb{R}^{k+1}$) the sequence λ^ℓ has a converging subsequence to a limit point λ in the unit k-sphere. But then passing (11.4) to the limit along this convergent subsequence we obtain (11.2). □

11.6. — Often Proposition 11.5 is used in practice in the following way, under the extra assumption that

$$\text{For all } x \in U, \text{ the vectors } \nabla g_j(x),\ 1 \leq j \leq k,\ l\text{are linearly independent.}$$

In this situation we consider the so-called **Lagrangian function**

$$L : U \times \mathbb{R}^k \to \mathbb{R} \qquad\qquad L(x, \lambda) = f(x) - \sum_{j=1}^{k} \lambda_j g_j(x).$$

13 The components of $\lambda \in \mathbb{R}^k$ are called **Lagrange multipliers**, then Proposition 11.5 says that if x_0 is a local minimum for the constrained problem there exists $\lambda \in \mathbb{R}^k$ such that the equations

$$\partial_{x_i} L(x_0, \lambda) = 0 \quad \text{and} \quad \partial_{\lambda_j} L(x_0, \lambda) = 0$$

are satisfied for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$.

Example 11.7. — Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = y^3 - x^2$, and the compact set

$$K = \{(x, y) \in [-1, 1]^2 \mid F(x, y) = 0\}$$

and aim to find the minimum of the function $f(x, y) = 4y - 3x$ on K, which exist by Weierstrass theorem.

First, we find quickly that the only critical point of F inside K is $(0, 0)$. This will have to be studied by hand, $f(0, 0) = 0$.

Second, we find that $K \cap \partial[-1, 1]^2 = \{(1, 1), (-1, 1)\}$. Also here we study these points by hand finding

$$f(1, 1) = 1, \quad f(-1, 1) = 7.$$

Now we can use the method of Lagrange multipliers as in 11.5 to study the remaining portion of K, namely

$$M := K \setminus \{(0, 0), (1, 1), (-1, 1)\}.$$

The Lagrange function associated with f and F is given by

$$L(x, y, \lambda) = 4y - 3x - \lambda(y^3 - x^2).$$

The partial derivatives are calculated as

$$\begin{aligned}
\partial_x L(x, y, \lambda) &= -3 + 2\lambda x \\
\partial_y L(x, y, \lambda) &= 4 - 3\lambda y^2 \\
\partial_\lambda L(x, y, \lambda) &= -(y^3 - x^2).
\end{aligned}$$

13 From $-3 + 2\lambda x = 0$, we deduce $\lambda \neq 0$ and $x \neq 0$ with $\lambda = \frac{3}{2x}$. Similarly, from $4 - 3\lambda y^2 = 0$, we conclude that $y \neq 0$ and $\lambda = \frac{4}{3y^2}$. Thus, $\frac{3}{2x} = \frac{4}{3y^2}$ or equivalently $x = \frac{9}{8}y^2$. Furthermore, $\partial_\lambda L(x, y, \lambda) = -(y^3 - x^2) = 0$. Substituting $x = \frac{9}{8}y^2$, we obtain

$$0 = y^3 - \left(\tfrac{9}{8}y^2\right)^2 = y^3 - \tfrac{9^2}{8^2}y^4 = y^3\left(1 - \tfrac{9^2}{8^2}y\right).$$

Since $y \neq 0$, this yields $y = \frac{8^2}{9^2}$ and $x = \frac{9}{8}y^2 = \frac{8^3}{9^3}$. Therefore, using the Lagrange multipliers method, we find a single additional candidate for extremal values:

$$f\left(\tfrac{8^3}{9^3}, \tfrac{8^2}{9^2}\right) = 4\tfrac{8^2}{9^2} - 3\tfrac{8^3}{9^3} = 1.053\ldots.$$

The set of all points in K where f attains a local extremum on K is thus contained in

$$\{(0, 0), (1, 1), (1, -1), (\tfrac{8^3}{9^3}, \tfrac{8^2}{9^2})\}.$$

The global maximum of f is at the point $(1, -1)$ with a value of 7, and the global minimum is at the point $(0, 0)$ with a value of 0.

Applet 11.8 (Lagrange Multipliers and Normal Vectors)**.** *In this applet, we illustrate Proposition 11.5 with one constrain ($k = 1$). We see that gradient vectors ∇f and ∇g must be parallel at minima and maxima of the constrained problem.*

11.3 Application: Spectral theorem for symmetric matrices

With the method of Lagrange multipliers, we can relatively easily prove the following important theorem from linear algebra.

Theorem 11.9: Spectral theorem

Every symmetric matrix $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ is diagonalizable, and there exists an orthonormal basis of $\mathbb{R}^n$ consisting of real eigenvectors of A.

11.10. — Theorem 11.9 can be reformulated in the context of orthogonal matrices. To recall, an $n \times n$ matrix O is termed orthogonal if it satisfies $O^T O = \mathrm{Id}$, implying that $O^{-1} = O^T$. Orthogonal matrices are significant as they represent rotations in $\mathbb{R}^n$ and preserve the length of vectors. This preservation can be demonstrated by the equation

$$|Ov|^2 = (Ov)^T Ov = v^T O^T Ov = v^T v = |v|^2,$$

valid for any vector $v \in \mathbb{R}^n$. It highlights that applying an orthogonal matrix to a vector does not change the vector's magnitude.

A set of vectors $\{v_j\}$ forms an orthonormal basis if, and only if, the matrix composed of
13 their components $v_{j,i}$ is orthogonal. Consequently, Theorem 11.9 can be equivalently stated
as: For any symmetric matrix A, there exists an orthogonal matrix O such that $O^T AO$ yields
a diagonal matrix.

Theorem 11.9 will be an immediate consequence of the following:

Lemma 11.11: Finding eigenvectors through minimization

Let $n \geq 1$ and $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ be a symmetric matrix. Let $k \in \{0, 1, 2, \ldots, n-1\}$ and assume that $v_1, \ldots v_k$ are linearly independent real eigenvectors of A. Then, A has a real eigenvector v satisfying $\langle v, v_j \rangle = 0$ for all $j = 1, \ldots, k$.

Proof. Define $f : \mathbb{R}^n \to \mathbb{R}$ as:

$$f(x) = \langle x, Ax \rangle = \sum_{k,\ell=1}^{n} a_{k\ell} x_k x_\ell.$$

Consider the linear subspace

$$H = \{x \in \mathbb{R}^n \mid \langle x, v_j \rangle = 0 \text{ for all } 0 \leq j \leq k\},$$

and the unit sphere

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and define the compact set

$$K = \mathbb{S}^{n-1} \cap H.$$

Notice that

$$K = \{x \in \mathbb{R}^n \mid g_1(x) = g_2(x) = \cdots = g_k(x) = g_*(x) = 0\}$$

where

$$g_j(x) = \langle x, v_j \rangle = \sum_{k=1}^{n} x_k v_{j,k} \ , \qquad g_*(x) = |x|^2 - 1 = -1 + \sum_{k=1}^{n} (x_k)^2.$$

Since K is compact $f|_K$ attains its extremal values. Let $v \in K$ be an extremum point.

According Proposition 11.5, there exist $\lambda_0, \lambda_1, \ldots \lambda_k, \lambda_* \in \mathbb{R}$ with $\lambda_0^2 + \lambda_1^2 + \cdots \lambda_k^2 + \lambda_*^2 = 1$ such that

$$\lambda_0 \nabla f(v) + \sum_{1}^{k} \lambda_j \nabla g_j(v) + \lambda_* \nabla g_*(v) = 0. \tag{11.5}$$

Now we compute

$$\partial_i f(x) = \partial_i \left(\sum_{k,\ell=1}^{n} a_{k\ell} x_k x_\ell \right) = \sum_{\ell=1}^{n} a_{i\ell} x_\ell + \sum_{k=1}^{n} a_{ki} x_k,$$

13

$$\begin{aligned} \partial_i g_j(x) &= \partial_i \left(\sum_{k=1}^{n} x_k v_{j,k} \right) = v_{j,i} \\ \partial_i g_*(x) &= \partial_i \left(\sum_{k=1}^{n} x_k^2 \right) = 2x_i \end{aligned}$$

where we used $\partial_i x_k$ is zero unless $k = i$. Moreover, that A is symmetric, we obtain

$$\partial_i f(x) = 2 \sum_{\ell=1}^{n} a_{i\ell} x_\ell = 2(Ax)_i.$$

Hence we have shown

$$\nabla f(x) = 2Ax, \quad \nabla g_j(x) = v_j, \quad \nabla g(x) = 2x.$$

and thus (11.5) becomes

$$\lambda_0 Av + \sum_{j=1}^{k} \lambda_j v_j + \lambda_* v = 0.$$

Since v_j are linearly independent and $v \in K$ is nonzero and perpendicular to all of them we obtain that $\lambda_0 \neq 0$. Also, using that v_j are eigenvectors of A we have

$$\langle Av, v_j \rangle = (Av)^T v_j = v^T A^T v_j = v^T A v_j = \langle v, Av_j \rangle = \mu_j \langle v, v_j \rangle = 0$$

for all j, where μ_j denotes the eigenvalue associated to v_j. Hence, $w_1 := \lambda_0 Av + \lambda_* v$ and

$w_2 := \sum_{j=1}^{k} \lambda_j v_j$ are perpendicular. Since their sum is zero it must be $w_1 = w_2 = 0$ from which it follows that v is an eigenvector of A with real eigenvalue $-\lambda_*/\lambda_0$. □

Proof of Theorem 11.9. By applying Lemma 11.11 with $k = 0$, we obtain a first real eigenvector v_1, which can be assumed to be normalized, i.e., $|v_1| = 1$. Subsequently, applying Lemma 11.11 again with $k = 1$, we identify a second real eigenvector v_2 that is orthogonal to v_1 and also normalized. By repeating this procedure for k times, starting from $k = 0$ and progressing to $k = n - 1$, we ultimately find an orthonormal basis consisting of real eigenvectors. □

11.4 Second order optimality conditions

Definition 11.12: Gradient, Hessian, and Laplacian

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a C^2 function. We define the **Hessian matrix** of $f \in C^2(U)$ at $x \in U$ is the $n \times n$ matrix

$$H_{ij} f(x) = \partial_i \partial_j f(x)$$

for $i, j \in \{1, \dots, n\}$. Schwarz's theorem 10.30 entails that $Hf(x)$ is a symmetric matrix. An alternative standard notation for the Hessian matrix is $D^2 f(x)$.

13 The **Laplacian** of f is the trace of the Hessian

$$\Delta f(x) := \operatorname{tr} Hf(x) = \sum_{i=1}^{n} \partial_{ii} f(x)$$

Example 11.13. — Let $u \colon \mathbb{R}^n \to \mathbb{R}$ of class C^2, check that

$$\partial_i(\arctan(u)) = \frac{\partial_i u}{1+u^2}, \quad \partial_i(1/u) = -\frac{\partial_i u}{u^2}, \quad \partial_i(|Du|^2) = 2\sum_j \partial_j u \partial_{ij} u,$$

$$\Delta(|Du|^2) = 2\sum_{i,j} (\partial_{ij} u)^2 + 2\sum_i \partial_i u \partial_i (\Delta u).$$

If $O = (O_{ij}) \in O(n)$ and $v(x) := u(Ax)$ then

$$\partial_i v(x) = \sum_j O_{ij} \partial_j u(Ox) \text{ and } \Delta v(x) = \sum_i \partial_{ii} v(x) = \sum_j (\partial_{jj} u)(Ox) = \Delta u(Ox).$$

Exercise 11.14 (Polarisation formula). — Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Prove that the map

$$\mathbb{R}^n \ni e \mapsto \left.\frac{d^2}{dt^2}\right|_{t=0} f(te) = e^T Hf(0) e,$$

determines all the second derivatives $\partial_{ij} f$.

Interlude: The sign of a Square Symmetric Matrix

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called:

(1) **Positive definite** if all its eigenvalues are positive.

(2) **Negative definite** if all eigenvalues are negative.

(3) **Indefinite** if at least one eigenvalue is positive and at least one is negative.

(4) **Degenerate** if zero is an eigenvalue.

Proposition 11.15: Hessian test

Let $U \subset \mathbb{R}^n$ be open and $f \in C^3(U)$, and let $x_0 \in U$ with $\nabla f(x_0) = 0$. Let $Hf(x_0)$ be the Hessian matrix of f at point x_0.

(1) If $Hf(x_0)$ is positive definite, then f has a strict local minimum at x_0.

(2) If $Hf(x_0)$ is negative definite, then f has a strict local maximum at x_0.

(3) If $Hf(x_0)$ is indefinite and non-degenerate, then f has no local extremum at x_0. In this case x_0 is called a ***saddle point****.*

Proof. We will prove the first of the three statements. The proof of the other two are similar and are left as an exercise to the reader. 13

We notice that, since by assumption the first derivatives of f vanish at x_0, the quadratic Taylor expansion (provided by Theorem 10.33) can be written as

$$f(x_0+h) - f(x_0) = \frac{1}{2}\sum_{i,j=1}(Hf)_{ij}(x_0)h_ih_j + O(|h|^3). \tag{11.6}$$

Now, if the Hessian of f at x_0, $H := Hf(x_0)$ is positive definite, and we put λ_1 be its smallest (positive) eigenvalue we have

$$\sum_{i,j=1}^{n} H_{ij}v_iv_j \geq \lambda_1|v|^2$$

Indeed, using that O^THO is diagonal for some O orthogonal, we obtain, for any vector v (putting $w = O^Tv$)

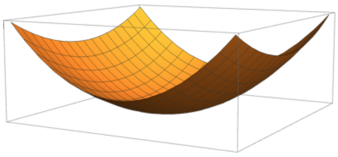
$$\sum_{i,j=1}^{n}(Hf)_{ij}v_iv_j = v^THv = w^T(O^THO)w = \sum_{\ell=1}^{n}\lambda_\ell|w_\ell|^2 \geq \lambda_1|w|^2 = \lambda_1|v|^2$$

Therefore (11.6) yields the existence of C large such that

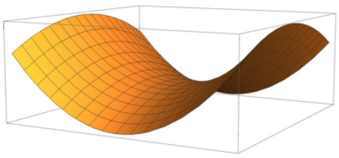
$$f(x_0+h) - f(x_0) \geq \frac{\lambda_1}{2}|h|^2 - C|h|^3 = |h|^2(1 - C|h|)$$

which is strictly positive for all $0 < |h| < \frac{1}{C}$. This means that f has a strict local minimum at x_0. □

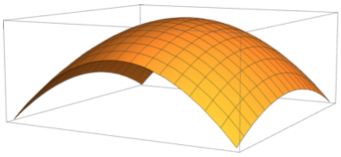
Example 11.16. — The behavior of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $0 \in \mathbb{R}^2$ illustrates the three cases in the proposition.



$f(x,y) = x^2 + y^2$



$f(x,y) = x^2 - y^2$



$f(x,y) = -x^2 - y^2$

The corresponding Hessian matrices are $\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}-2&0\\0&-2\end{smallmatrix}\right)$. When the Hessian matrix is degenerate, i.e., if 0 is an eigenvalue of $H(x_0)$, then the Hessian test is inconclusive and nothing can be said: for example the function $f(x,y) = ax^4 + by^4$ has a local maximum, a local minimum, or neither at 0 depending on the choice of a and b. But the Hessian matrix at 0 is the zero matrix regardless of the choices of a and b.

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Example 11.17. — Let $a, b \in \mathbb{R}$ be fixed parameters. We define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x\sin(y) + ax^2 + by^2$ for $(x,y) \in \mathbb{R}^2$ and consider the point $0 \in \mathbb{R}^2$. We have $Df(0) = 0$, and the Hessian matrix of f at 0 is given by

$$H = \begin{pmatrix} 2a & 1 \\ 1 & 2b \end{pmatrix}$$

with $\det H = 4ab - 1$. We obtain the following cases.

- If $a > 0$ and $4ab - 1 > 0$, then H is positive definite, and f has a local minimum at 0.
- If $a < 0$ and $4ab - 1 > 0$, then H is negative definite, and f has a local maximum at 0.
- If $\det H = 4ab - 1 = 0$, then the Hessian matrix is degenerate.
- If $4ab - 1 < 0$, then H is indefinite, and 0 is a saddle point.

Exercise 11.18. — Let $\alpha \in \mathbb{R}$. Find all points $(x,y) \in \mathbb{R}^2$ where the derivative of the function given by $f(x,y) = x^3 - y^3 + 3\alpha xy$ vanishes. Determine whether each point is an extremum and, if so, whether it is a local minimum or maximum.

11.5 The Fundamental Theorem of Algebra via minimization

In this section, we present a straightforward proof demonstrating the existence of complex roots for polynomials through a minimization approach. It's noteworthy to emphasize that this minimization method requires examination beyond merely the first or second-order expansions at potential extremum points: it necessitates consideration of higher-order expansions.

Theorem 11.19: Fundamental Theorem of Algebra

Every non-constant polynomial $f \in \mathbb{C}[z]$ has a root in $\mathbb{C}$. Hence, applying the division algorithm, f has n complex roots (counted with multiplicity).

Proof. Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n > 0$. Dividing f by a_n we may assume without loss of generality $a_n = 1$.

Consider the non-negative number

$$\mu := \inf\{|f(z)| : z \in \mathbb{C}\}.$$

Notice that, by the triangle inequality, in the region $|z| \geq 1$ we have

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$$|f(z)| = \Big|\sum_{k=0}^{n} a_k z^k| \geq |z^n| - \sum_{k=0}^{n-1} |a_k||z^k| \geq |z|^{n-1}(|z| - C)$$

for $C := \sum_{k=0}^{n} |a_k|$. So, if we fix any $R \geq C + 1 + 10\mu$ we have

$$|f(z)| \geq 1 + 10\mu \text{ as soon as } |z| \geq R.$$

This shows that the "battle for the infimum" is fought inside the compact ball $K := \{z \in \mathbb{C} : |z| \leq R\}$, that is to say

$$\min\{|f(z)| : z \in K\} = \inf\{|f(z)| : z \in K\} = \mu,$$

since in K^c we have $|f(z)| \geq 1+10\mu$. So let $z_0 \in K$ with $|f(z_0)| = \mu$. Since $|f(z)| \geq 1+10\mu > \mu$ on ∂K, we obtain that z_0 is an interior minimum point in K.

We claim that z_0 is a root of f, we set for brevity

$$g(z) := f(z_0 + z) = \sum_{k=0}^{n} a_k (z_0 + z)^k = \sum_{k=0}^{n} b_k z^k,$$

for some coefficients $b_k \in \mathbb{C}$. Assume by contradiction that $f(z_0) \neq 0$, that is to say $b_0 = g(0) \neq 0$. Also, let $\ell \geq 1$ be the smallest index ≥ 1 with $b_\ell \neq 0$. Now writing z in exponential form we have

$$g(re^{i\varphi}) = b_0 + b_\ell r^\ell e^{i\ell\varphi} + O(r^{\ell+1}) = b_0\left(1 + \tfrac{b_\ell}{b_0} r^\ell e^{i\ell\varphi}\right) + O(r^{\ell+1}) \qquad \text{as } r \downarrow 0,$$

Write $\frac{b_\ell}{b_0} = se^{i\psi}$ for some $s > 0$ and choose $\varphi = \frac{-\psi+\pi}{\ell}$, so that $e^{i(\ell\varphi+\psi)} = -1$, and

$$|b_0| = |g(0)| \leq |g(re^{i\varphi})| = |b_0\left(1 - sr^\ell\right) + O(r^{\ell+1})| \leq |b_0|\left(1 - sr^\ell\right) + Mr^{\ell+1}$$

for all sufficiently small $r > 0$ and some fixed constant $M > 0$. Reshuffling this expression we find

$$0 < s|b_0| \leq Mr$$

which is impossible if we let $r \downarrow 0$. Thus, it was a contradiction to assume $b_0 \neq 0$, and this means that $f(z_0) = 0$. □

Exercise 11.20. — Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be a complex-valued function that can be locally represented by power series (an analytic function). More precisely, for every $x_0 \in U$, there exists an $r > 0$ such that $B(x_0, r) \subset U$, and f on $B(x_0, r)$ is equal to a power series around x_0 with a convergence radius greater than or equal to r. Mimicking the proof of Theorem 11.19, show that the function $z \mapsto |f(z)|$ does not assume a minimum value on U. 14

Exercise 11.21. — For the sake of completeness, we present an elementary argument for the proof of Lemma 11.11 using the Fundamental Theorem of Algebra. Let $n \geq 1$ and $A \in \mathrm{Mat}_{n,n}(\mathbb{R})$ be a symmetric matrix.

(i) Show that all complex eigenvalues of A are real.

(ii) Prove Lemma 11.11 by showing that A has a complex eigenvector if and only if A has a real eigenvector.

The geometric understanding of the eigenvalues of A gained in our proof can also be utilized differently. As an example, one can prove a special case of the Courant-Fischer theorem.

(iii) Show that the values

$$\min_{x \in \mathbb{S}^{n-1}} x^t A x, \qquad \max_{x \in \mathbb{S}^{n-1}} x^t A x$$

represent the smallest and largest eigenvalue of A, respectively.

Exercise 11.22. — For $n \geq 2$, prove that two points $x, y \in \mathbb{S}^{n-1}$ have maximum distance if and only if $x = -y$. Consider the function $(x, y) \mapsto \|x - y\|^2$ on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \subset \mathbb{R}^{2n}$.

11.6 Convexity

Definition 11.23: Convex sets and functions

A nonempty subset $A \subset \mathbb{R}^n$ is called **convex** if for any two points $x, y \in A$ and any $t \in [0,1]$, the point $(1-t)x + ty$ also lies in A. In other words, the line segment connecting any two points in A lies entirely within A.
A function $f : A \to \mathbb{R}$ is called **convex** if for every $x, y \in A$ and any $t \in [0,1]$, the following inequality holds:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

This means that the line segment connecting any two points on the graph of f lies above or on the graph. In other words, the graph 'dips in the middle' (i.e, it is 'U-shaped').

Proposition 11.24: Nonnegative Hessian and convexity

Suppose that $U \subset \mathbb{R}^n$ is open and convex and let $f :\in C^2(U)$. Then f is convex if and only if one of the following conditions holds

(a) For all $x \in U$ the Hessian $Hf(x)$ is nonnegative definite (i.e., all of its eigenvalues are nonnegative) 14

(b) For all x and y in U we have

$$f(y) - f(x) \geq Df_x(y-x).$$

Proof. Assume f is convex. For any given $x \in U$ and $v \in \mathbb{R}^n$ with $|v|=1$, let $g(s) := f(x+sv)$, $s \in (-\varepsilon, \varepsilon)$. Using the convexity of g we obtain $g(a(1-t)+bt) \leq (1-t)g(a) + tg(b)$, for all $a, b \in (-\varepsilon, \varepsilon)$ and $t \in [0,1]$. But then by Corollary **??** we have $g''(s) \geq 0$ for all s close to 0.

Now the chain rule gives

$$g'(s) = \sum_i \partial_i f(x+se)v_i \quad g''(s) = \sum_{i,j} \partial_{ij} f(x+se)v_i v_j$$

So we have shown

$$g''(0) = \sum_{i,j} H_{ij} f(x) v_i v_j \langle v, Hf(x)v \rangle \geq 0.$$

Choosing v to be any normalized eigenvector, this proves that all the eigenvalues of $Hf(x)$ have are nonnegative, proving (a).

Now assume (a) and let us prove (b). Reciprocally assume and fix x, y in U with $x \neq y$. Put $v := y - x$ and define, as above, $g(s) := f(x+tv)$. Now g is defined for $s \in (-\varepsilon, 1+\varepsilon)$.

Now, by the same computation as above we obtain $g''(s) \geq 0$ and hence, by Taylor's theorem

$$f(y) - f(x) = g(1) - g(0) = g'(0) + \int_0^1 (1-t)g''(t)dt \geq g'(0) = Df_x(y-x).$$

Finally if (b) holds then for any $x, y \in U$ and $t \in [0,1]$, putting $z = (1-t)x + ty$ we have

$$f(x) - f(z) \geq Df_z(x-z) \qquad \text{and} \qquad f(y) - f(z) \geq Df_z(y-z)$$

Multiplying the first inequality by $(1-t)$ and the second buy t and summing we obtain

$$(1-t)f(x) + tf(y) - f(z) \geq Df_z((1-t)x + ty - z) = Df_z(0) = 0.$$

Hence, the convexity inequality follows. □

14 Exercise 11.25. — Show that condition (b) in Proposition 11.24 is equivalent to the convexity of when f is C^1. Deduce that for a C^1 function on an open convex set, every critical point is a point of minimum.

Proposition 11.26: Jensen inequality

Let $U \subset \mathbb{R}^n$ be open and convex and $f : U \to \mathbb{R}$ convex. Then for every collection of points $x_i \in U$ and positive 'weights' $w_i \in (0,1)$, $i = 1 \ldots N$, with $\sum_i w_i = 1$ we have

$$f\Big(\sum_i w_i x_i\Big) \leq \sum_i w_i f(x_i).$$

Proof. We will assume for simplicity that f is C^2 we will give a proof using Proposition 11.24 (b). It is left as an exercise to give proof for any convex function by induction over N.

Let $z = \sum_i w_i x_i$. Then

$$f(x_i) - f(z) \geq Df_z(x_i - z).$$

Multiplying by w_i and summing over i we obtain the desired conclusion. □

Chapter 12

Inverse and implicit function theorems and submanifolds

12.1 The inverse function Theorem

12.1.1 Small Lipschitz perturbations of the identity

The following lemma is the essential prelimiary step towards the inverse funtion theorem.

Lemma 12.1: small Lipschitz perturbations of the identity

Let $U \subset \mathbb{R}^n$ be an open set and assume that $F : U \to \mathbb{R}^n$ is a function of the form $F(x) = x + \phi(x)$, where ϕ is Lipschitz with constant $\lambda < 1$. Then

(1) Whenever $B_r(x) \subset U$ we have

$$B_{(1-\lambda)r}(F(x)) \subset F(B_r(x)) \subset B_{(1+\lambda)r}(F(x)).$$

In particular, $F(U)$ is open.

(2) F is injective and $F^{-1}\colon F(U) \to U$ is Lipschitz with constant $\frac{1}{1-\lambda}$.

15 *Proof.* We start with the first inclusion in (1), which is the core of this proof. Given a point $y \in B_{(1-\lambda)r}(F(x_0))$, for some $x_0 \in U$ such that $B_r(x_0) \subset U$ we want to find $x \in B_r(x_0)$ that such that

$$F(x) = y.$$

It is convenient to rewrite this equation as

$$x = y - \phi(x).$$

Then, solving the equation amounts to finding a fixed point of the map $x \mapsto y - \phi(x)$, and we can use an argument almost identical the one in the Banach fixed point theorem.

Indeed, consider the sequence of points defined by recurrence, starting at x_0,

$$x_{k+1} = y - \phi(x_k), \quad k \geq 0$$

Our goal is to show that x_k always belongs to U so that x_{k+1} is always well-defined and that the sequence $(x_k)_{k\geq 0}$ converges to a limit $x \in B_r(x_0)$, then $x = y - \phi(x)$ and we have solved $y = F(x)$.

Now, using the triangular inequality and the contraction property of ϕ we find

$$|x_{k+1} - x_k| \leq |\phi(x_k) - \phi(x_{k-1})| \leq \lambda |x_{k-1} - x_k| \leq \ldots \leq \lambda^k |x_1 - x_0| = \lambda^k |y - F(x_0)|,$$

which proves that $(x_k)_{k\geq 0}$ is Cauchy (if it is well defined).

But this also shows in turn that $(x_k)_{k\geq 0}$ is well defined because the sequence never leaves $B_r(x_0) \subset U$. Indeed:

$$|x_{k+1} - x_0| \leq \sum_{i=0}^{k} |x_{i+1} - x_i| \leq |y - F(x_0)| \sum_{i=0}^{k} \lambda^i < \frac{|y - F(x_0)|}{1 - \lambda} < r.$$

Hence x_k converges to x satisfying $|x - x_0| \leq |y - F(x_0)|/(1 - \lambda) < r$, which means that $x \in B_r(x_0)$, as we wanted to show.

In particular, since the point x_0 is arbitrary we have shown that $F(U)$ is open.

15 Finally, the second inclusion in (1) is readily checked

$$|F(y) - F(x_0)| = |y + \phi(y) - x_0 - \phi(x_0)| \leq |y - x| + \lambda |y - x| < r + \lambda r.$$

We turn to (2). First of all, if $F(x) = F(x')$, then $x - x' = \phi(x) - \phi(x')$, which is in contradiction with $\lambda < 1$. So F is injective and F^{-1} is a well-defined function.

Finally, for $x, x' \in U$ putting $y = F(x)$ and $y' = F(y)$ we have

$$y - y' = x - x' + \phi(x) - \phi(x')$$

and by the triangle inequality

$$|y - y'| \geq |x - x'| - |\phi(x) - \phi(x')| \geq (1 - \lambda)|x - x'| = (1 - \lambda)|F^{-1}(y) - F^{-1}(y')|,$$

which proves that F^{-1} is Lipschitz with constant $\frac{1}{1-\lambda}$. □

Exercise 12.2. — Proof that the map $y \mapsto F^{-1}(y) - y$ has Lipchitz constant $\frac{\lambda}{1-\lambda}$, so it is also a small Lipchitz perturbation of the identity when $\lambda < 1/2$.

Exercise 12.3. — In the proof of Lemma 12.1 we did not use the Euclidean structure of $\mathbb{R}^n$. In fact, this Lemma is true in any complete, normed vector space $(V, \|\cdot\|)$, with the same statement and the same proof. Check this claim.

Lemma 12.4: Automatic differentiability of the inverse

Let $U, V \subset \mathbb{R}^n$ be open sets and let $f\colon U \to V$ and $g\colon V \to U$ be bijective functions. Assume that f is differentiable at $x_0 \in U$ and Df_{x_0} is invertible, and that g is Lipschitz in V, and that

$$f(g(y)) = y, \qquad \text{for all } y \in V.$$

Then g must be differentiable at $y_0 = f(x_0)$ and $Dg_{y_0} = (Df_{x_0})^{-1}$.

Proof. Let us set $L := Df_{x_0}$. By the differentiability assumption, as $y \to y_0$,

$$y - y_0 = f(g(y)) - y_0 = f\big(x_0 + (g(y) - g(y_0))\big) - f(x_0) = L(g(y) - g(y_0)) + o(|g(y) - g(y_0)|).$$

Using that L is invertible, we re-write this as

$$g(y) - g(y_0) = L^{-1}(y - y_0) + o(\|L^{-1}\|_2 |g(y) - g(y_0)|). \tag{12.1}$$

We conclude noticing that, since g is Lipschitz, say with constant Λ,

$$o(\|L^{-1}\|_2 |g(y) - g(\bar{y})|) = o\Big(\frac{|g(y) - g(\bar{y})|}{|y - \bar{y}|}|y - \bar{y}|\Big) = o(\Lambda|y - \bar{y}|) = o(|y - \bar{y}|).$$

Thus (12.1) is saying that g is differentiable at $\bar{y}$, with differential L^{-1}. □

15

12.1.2 Inverse of differentiable maps

Lemma 12.5: Smoothness of the inverse

Let $U \subset \mathbb{R}^{n \times n}$ be the set of invertible matrices, and let $\theta\colon U \to U$ be defined as

$$\theta\colon X \mapsto X^{-1}.$$

Then $\theta \in C^\infty$.

Proof. Recall that the formula for the inverse matrix expresses the (p, q) entry of X^{-1} as a polynomial of $\{X_{i,j}\}$ divided by the polynomial $\det X$ (which is nonzero in U).

Indeed, each entry (p, q) of the inverse matrix X^{-1} can be computed as:

$$(X^{-1})_{p,q} = \frac{C_{q,p}}{\det(X)},$$

where $C_{q,p}$ is the cofactor of the element at position (q, p) in the matrix X. This cofactor is calculated as $(-1)^{q+p} \cdot \det(X_{q,p})$, where $\det(X_{q,p})$ is the determinant of the matrix obtained by removing the q-th row and p-th column from X.

Thus, since the map $X \mapsto X^{-1}$ is expressed as a quotients of polynomials in terms of the entries of X, it is infinitely differentiable on the domain excluding the points where the determinant, serving as the denominator, vanishes.

15 □

12.1.3 Inverse and Implicit function theorems

Theorem 12.6: Inverse function

Let $U \subset \mathbb{R}^n$ be an open set and $f \in C^1(U, \mathbb{R}^n)$. Let $x_0 \in U$ be such that $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, there is U_0 open containing x_0 such that

(1) f is injective in U_0.

(2) The set $V := f(U_0)$ is open.

(3) The inverse function $g := (f|_{U_0})^{-1}$, that is $g\colon V \to U_0$ satisfying

$$g(f(x)) = x, \quad \forall x \in U_0 \qquad \text{(or equivalently } f(g(y)) = y,\ \forall y \in V\text{)}$$

is of class C^1 and

$$Dg_{f(x)} = (Df_x)^{-1}, \quad \forall x \in U_0; \qquad Dg_y = (Df_{g(y)})^{-1}, \quad \forall y \in V.$$

Furthermore, if $f \in C^k(U_0, \mathbb{R}^n)$ for some $k \geq 1$, then also $g \in C^k(f(V, \mathbb{R}^n)$.

16 *Proof.* It is enough to prove the result in the case $x_0 = 0$ and $f(x_0) = 0$. Indeed the general case follows immediately from this "special" case applied to $\tilde{f}(x) := f(x - x_0) - f(x_0)$.

Let $L := Df_0$, we claim that $F := L^{-1} \circ f|_{B_r(0)}$ is a small Lipschitz perturbation of the identity, provided $r > 0$ is chosen small enough.

Notice that the function F is of class C^1 (being the composition of a C^1 function and a linear map). Write $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ the identity function (i.e., $\mathrm{Id}(x) = x$ for all $x \in \mathbb{R}^n$), and put $\phi := F - \mathrm{Id}$, i.e., $\phi(x) := F(x) - x$. We have

$$D\phi_0 = D(F - \mathrm{Id})_0 = DF_0 - D\mathrm{Id}_0 = L^{-1} \circ Df_0 - \mathrm{Id} = L^{-1} \circ L - \mathrm{Id} = 0$$

so choosing $r > 0$ small enough we will have, by continuity of the derivatives the Jacobi matrix $J\phi$ satisfies

$$\|J\phi(x)\|_2 \leq \frac{1}{2} \qquad \text{for all } x \in B_r(0).$$

Then the Mean Value Theorem (see the proof Corollary 10.26) implies that ϕ is Lipschitz with constant $1/2$ in $B_r := B_r(0)$

By Lemma 12.1, we find that $F(B_r(0))$ is open, that $F|_{B_r(0)}$ is injective and that the inverse function $F^{-1}\colon F(B_r) \to B_r$ is Lipschitz, with constant 2.

Now (1) holds because $f = L \circ F$, which is a composition of injective functions.

Also (2) holds because $f(B_r) = L(F(B_r))$ is open, where $F(B_r)$ is open and L linear invertible (an invertible linear map sends open sets to open sets).

For (3) we notice that $g = F^{-1} \circ L^{-1}$ will be Lipschitz in $f(B_r) = L(F(B_r))$, since it is the composition of Lipschitz maps. Indeed, since F^{-1} is 2-Lipschitz:

$$|g(y) - g(y')| = |F^{-1}(L^{-1}(y)) - F^{-1}(L^{-1}(y'))| \leq 2|L^{-1}(y) - L^{-1}(y')| \leq 2\|L^{-1}\|_2|y - y'|.$$

Since f is differentiable at every point of B_r, and g is Lipschitz, then Lemma 12.4 shows that g is differentiable for all $y \in f(B_r)$, and

$$Dg_y = (Df_{g(y)})^{-1} \qquad \Leftrightarrow \qquad Jg(y) = (Jf(g(y)))^{-1}, \tag{12.2}$$

for all $y \in f(B_r)$.

Finally, assume that $f \in C^k$ and that we known $f^{-1} \in C^\ell$ for some $\ell \in \{0, 1, 2 \ldots, k-1\}$. Equation (12.2) says that the map $y \mapsto Jg(y)$ is the composition of the following functions:

$$\begin{aligned} g\colon y \mapsto g(y) &\quad \text{which is of class } C^\ell, \\ Jf\colon x \mapsto Jf(x) &\quad \text{which is of class } C^{k-1}, \\ \theta\colon X \mapsto X^{-1} &\quad \text{which is of class } C^\infty \text{ on the set of invertible matrices by Lemma 12.5.} \end{aligned}$$

16

Thus $Jg \in C^\ell$, which means that $g \in C^{\ell+1}$. This proves the last part of the statement by induction from the $\ell = 0$ case. □

Definition 12.7: Diffeomorphism

Let $U, V \subset \mathbb{R}^n$ be open. A bijective, C^1 function $f : U \to V$ with a C^1 inverse $f^{-1} : V \to U$ is called a **diffeomorphism**. If f and f^{-1} are both k-times continuously differentiable for $k \geq 1$, we call f a C^k**-diffeomorphism**.

12.8. — Notice that if $f : U \to V$ is a diffeomorphism then it follows from $f^{-1} \circ f = \mathrm{Id}$ and the chain rule that $D(f^{-1})_{f(x)} Df_x = Id$ for all $x \in U$. In particular DF_x is always invertible for all $x \in U$.

An important consequence of the Inverse Function Theorem is the following

Corollary 12.9: Implicit Function Theorem

Let $0 < d < n$, $k \geq 1$ *be integers,* $U \subset \mathbb{R}^n$ *be open, let* $f \in C^1(U, \mathbb{R}^{n-d})$*. We write a point in* $\mathbb{R}^d \times \mathbb{R}^{n-d}$ *as* (x, y) *with* $x \in \mathbb{R}^d$ *and* $y \in \mathbb{R}^{n-d}$*.*

Assume that we have $(x_0, y_0) \in U$ *with* $f(x_0, y_0) = 0$ *such that the* $(n-d) \times (n-d)$ *matrix*

$$J_y f(x_0, y_0) := \big(\partial_{y_i} f_j(x_0, y_0)\big)_{1 \leq j,i \leq n-d}$$

is invertible. Then, for sufficiently small $r, s > 0$*, there is a function* g *from* $B_r(x_0) \subset \mathbb{R}^d$*, to* $B_s(y_0) \subset \mathbb{R}^{n-d}$*, such that, for all* (x, y) *in the cylinder* $U_0 := B_r(x_0) \times B_s(y_0) \subset U$*, it holds*

$$f(x, y) = 0 \qquad \Longleftrightarrow \qquad y = g(x).$$

Moreover, for all $x \in B_r(x_0)$*,*

$$Jg(x) = -\big((J_y f)(x, g(x))\big)^{-1} (J_x f)(x, g(x)),$$

where

$$J_x f(x_0, y_0) := \big(\partial_{x_i} f_j(x_0, y_0)\big)_{1 \leq j \leq n-d, 1 \leq i \leq d}$$

Furthermore, if $f \in C^k(U, \mathbb{R}^{n-d})$ *for some* $k \geq 1$*, then also* $g \in C^k(B_r(x_0), \mathbb{R}^d)$*.*

16 *Proof.* Consider the function $\Phi \in C^1(U, \mathbb{R}^n)$ given by

$$\Phi(x, y) := (x, f(x, y)), \qquad \text{for all } (x, y) \in U.$$

By assumption the Jacobi matrix $J\Phi(x_0, y_0)$ — which has size $n \times n$ — has a block decomposition

$$J\Phi(x_0, y_0) = \begin{pmatrix} \mathbf{1}_d & 0 \\ J_x f(x_0, y_0) & J_y f(x_0, y_0) \end{pmatrix},$$

so in particular it is invertible and we are under the assumptions of the Inverse function Theorem. Hence Φ has a C^1 inverse when restricted to a small cylinder 'centered at (x_0, y_0)':

$$U_0 := B_r(x_0) \times B_s(y_0) \subset U,$$

$r, s > 0$, which is mapped to the open set $V := \Phi(U_0) \subset \mathbb{R}^n$.

Let $\Psi : V \to U_0$ denote the inverse of the restriction of Φ to U_0. For given points (x, y) in U_0 put

$$(\xi, \eta) := \Phi(x, y) = (x, f(x, y)), \qquad \Longleftrightarrow \qquad (x, y) = \Psi(\xi, \eta)$$

Then $\xi = x$, so $\Psi(\xi, \eta)$ is of the form

$$\Psi(\xi, \eta) = (\xi, G(\xi, \eta)), \qquad \text{for all } (\zeta, \xi) \in V,$$

for some $G : V \to B_s(y_0)$ of class C^1 (or of class C^k if f is of class C^k) .

Thus, since

$$(x, y) = \Psi(x, \eta) = (x, G(x, \eta))$$

we obtain

$$f(x, y) = 0 \quad \Longleftrightarrow \quad \eta = 0 \quad \Longleftrightarrow \quad y = G(x, 0),$$

where the last implication $\Leftarrow$ follows using that Ψ is bijective.

In other words, defining $g(x) := G(x, 0)$ we obtain what we need.

Finally, the formula for $Jf(x_0)$ follows from differentiating the identity

$$f(x, g(x)) = 0$$

using chain rule. Indeed, we obtain, for all $1 \leq i \leq d$

$$0 = \partial_i(f(x, g(x))) = \partial_{x_i} f(x, f(x)) + \sum_{\ell=1}^{n-d} \partial_{y_\ell} f(x, g(x)) \partial_\ell g(x),$$

or in matricial form: 16

$$0 = J_x f(x, g(x)) + J_y f(x, g(x)) J_x g(x).$$

□

12.2 Submanifolds of $\mathbb{R}^n$

12.2.1 Definition of submanifold and different representations

Definition 12.10: Submanifold of $\mathbb{R}^n$

Given $0 < d < n$ and $k \geq 1$ integers, We say that $M \subset \mathbb{R}^n$ nonempty **is a d-dimensional submanifold of $\mathbb{R}^n$** of class C^k if, for every point $p_\circ \in M$ there exists a $U \subset \mathbb{R}^n$ open containing $p_\circ$, $V \subset \mathbb{R}^n$ open containing 0, and a C^k-diffeomorphism $\Psi : U \to V$, such that:

$$\Psi(M \cap U) = \{y \in V \mid y_{d+1} = y_{d+2} = \cdots = y_n = 0\}.$$

The map Ψ is called a **submanifold chart**.

For the next result, as well as for the submanifolds section it is useful to introduce the following:

Definition 12.11: Permutation of coordinates

We call a map $P : \mathbb{R}^n \to \mathbb{R}^n$ a **permutation of the coordinates** if for some permu-
17 tation $\sigma \colon \{0, 1, \ldots, n\} \to \{1, 2, \ldots n\}$ we have

$$P(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \quad \forall x \in \mathbb{R}^n$$

17

Proposition 12.12: Different local representations of submanifold

Let $0 < d < n$ and $k \geq 1$ be integers and $M \subset \mathbb{R}^n$ a nonempty subset. The following four are equivalent:

(1) $M \subset \mathbb{R}^n$ is d-dimensional submanifold.

*(2) **(Implicit representation).** For all $p_\circ \in M$ there are $U \subset \mathbb{R}^n$ open containing $p_\circ$ and $f \in C^k(U, \mathbb{R}^{n-d})$ such that $Df_{p_\circ}$ has maximal rank (i.e., has rank $n-d$)*

$$M \cap U = \{x \in U \mid f(x) = 0\}.$$

*(3) **(Parametric representation).** For all $p_\circ \in M$ there is $V \subset \mathbb{R}^d$ open, $y_\circ \in V$, and $G \in C^k(V, \mathbb{R}^n)$ such that $G(y_\circ) = p_\circ$, $DG_{y_\circ}$ has maximal rank (i.e., has rank d) and, in addition, for all $V' \subset V$ open there is $U' \subset \mathbb{R}^n$ open such that*

$$M \cap U' = G(V').$$

*(4) **(Graphical representation).** There are $U \subset \mathbb{R}^n$ open containing $p_\circ$, $V \subset \mathbb{R}^d$ open, and $g \in C^k(V, \mathbb{R}^{n-d})$ such that*

$$M \cap U = P(\mathrm{graph}(g)),$$

where $P \colon \mathbb{R}^n \to \mathbb{R}^n$ is some permutation of the coordinates and

$$\mathrm{graph}(g) := \Big\{x \in V \times \mathbb{R}^{n-d} \mid (x_{d+1}, \dots, x_n) = g(x_1, \dots, x_d)\Big\} \subset \mathbb{R}^n$$

Proof. We will show (1) $\implies$ (2) $\implies$ (4) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2): Since M is a submanifold, there is $U \subset \mathbb{R}^n$ open containing $p_\circ$ and a diffemorphism $\Psi : U \to V \subset \mathbb{R}^n$ such that

$$x \in M \cap U \qquad \iff \qquad \Psi_{d+1}(x) = \cdots = \Psi_n(x) = 0.$$

Hence, (2) follows defining $f : U \to \mathbb{R}^{n-d}$ as $f(x) := (\Psi_{d+1}(x), \cdots, \Psi_n(x))^T$. Notice that $Df_{p_\circ}$ has maximal rank since it is made of columns of the invertible matrix $D\Psi_{p_\circ}$.

(2) $\implies$ (4): If $Df_{p_\circ}$ has maximal rank $n-d$, then for a suitable permutation of the variables P, when we consider the map $\tilde{f} = f \circ P^{-1}$, the $(n-d) \times (n-d)$ matrix

$$(\partial_i \tilde{f}_j(q_\circ))_{1 \leq j \leq n-d, d+1 \leq i \leq n}$$

is invertible at $q_\circ = P(p_\circ)$.

Hence $\tilde{f}$ satisfies the assumptions of Implicit Function Theorem and hence there exist an open cylinder $U_0 \subset U$ centered at $q_\circ$, $V \subset \mathbb{R}^d$ open, $g : V \to \mathbb{R}^{n-d}$ of class C^k, such that for all $x \in U_0$,

$$\tilde{f}(x) = 0 \quad \iff \quad (x_{d+1}, \dots, x_d) = g(x_1, \dots, x_d) \iff P^{-1}(x) \in M,$$

choosing $x = Pz$ for z in a neighbourhood of $p_\circ$ we conclude.

(4) $\implies$ (3): By (3) we have $M \cap U = P(\text{graph}(g))$, for some $g : V \to \mathbb{R}^{n-d}$, where $V \subset \mathbb{R}^n$ is open. So defining $\tilde{G}(y) := (y, g(y))^T$ we have $M \cap U = (P \circ \tilde{G})(V)$. Let $y_\circ$ be the point such that $(P \circ \tilde{G})(y_\circ) = p_\circ$ and can take $G(y) := (P \circ \tilde{G})$. It is immediate to verify that DG_y has maximal rank for all $y \in V$ (in particular at the point $y_\circ$, which is mapped to $p_\circ$).

(3) $\implies$ (1): Given $G : V \to \mathbb{R}^n$, $V \subset \mathbb{R}^d$ as in (3) we can pick $n - d$ vectors $v_{d+1}, \dots v_n$ in $\mathbb{R}^n$ such that the matrix

$$\big(DG_{y_\circ} \mid v_{d+1} \mid \dots \mid v_n\big)$$

is invertible (i.e., has full rank).

Consider the map $\Phi \in C^1(V \times \mathbb{R}^{n-d}, \mathbb{R}^n)$ given by

$$\Phi(y_1, \dots, y_n) := G(y_1, \dots, y_d) + \sum_{j=d+1}^{n} y_j v_j.$$

Notice that $D\Phi_{(y_\circ, 0)}$ is invertible by construction.

Then, by the Inverse function Theorem there is $W \subset V \times \mathbb{R}^{n-k}$ open containing $(y_\circ, 0)$,
17 $U \subset \mathbb{R}^n$ open containing $p_\circ = \Phi((y_\circ, 0))$ and $\Psi : U \to W$ of class C^k such that

$$x = \Phi(\Psi(x)) = G(\Psi_1(x), \dots, \Psi_d(x)) + \sum_{j=d+1}^{n} \Psi_j(x) v_j.$$

Finally, since Φ, Ψ are both injective it follows that

$$x \in M \cap U = G(V) \cap U \quad \iff \quad \Psi_{d+1}(x) = \dots = \Psi_n(x) = 0.$$

□

12.2.2 Some examples of submanifolds of $\mathbb{R}^3$

Sphere (S^2) in $\mathbb{R}^3$ The sphere S^2 in $\mathbb{R}^3$ serves as a quintessential example of a submanifold showcasing simple representation through implicit, parametric, and graphical forms.

- **Implicit Form:** In its implicit form, the sphere S^2 can be described as the set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation:

  $$x^2 + y^2 + z^2 = 1.$$

  This equation represents the set of all points in $\mathbb{R}^3$ that are at a unit distance from the origin, defining the sphere.

- **Parametric Form:** A standard parametrization of the sphere is given

  $$\begin{aligned} x &= \sin(\theta)\cos(\phi), \\ y &= \sin(\theta)\sin(\phi), \\ z &= \cos(\theta), \end{aligned}$$

  where $\theta \in [0, \pi]$ is the polar angle and $\phi \in [0, 2\pi)$ is the azimuthal angle. This parametrization corresponds to the latitude and longidude coordinates used on the Earth.

  This form, however, *does not cover the poles* $\theta = 0$ and $\theta = \pi$ with a single parameterization due to the singularity at the poles. To fully cover S^2, additional local parametrizations these points need to be used.

- **Graphical Form:** For the upper hemisphere, the sphere can be graphically represented as:

  $$z = \sqrt{1 - x^2 - y^2},$$

  for $x^2 + y^2 \leq 1$. This equation describes how the z-coordinate depends on x and y coordinates. However, this form *only represents the upper hemisphere.* The lower hemisphere would similarly be represented by $z = -\sqrt{1 - x^2 - y^2}$, indicating the need for multiple functions to describe the entire surface. To cover all the surface with smooth 17 graphs, we must also introduce functions for the y and x coordinates as functions of the other two variables. Specifically, we can use:

  $$y = \pm\sqrt{1 - x^2 - z^2}$$

  for portions of the sphere where y is the dependent variable, and

  $$x = \pm\sqrt{1 - y^2 - z^2}$$

  for parts of the sphere with x as the dependent variable. These representations allow us to graphically depict the entire sphere by selecting the appropriate function based on the region of interest, ensuring smoothness across all points on S^2.

For the implicit representation $x^2+y^2+z^2 = 1$, the gradient vector, serving as the Jacobian matrix for scalar functions, is derived from $f(x, y, z) = x^2 + y^2 + z^2 - 1$. The gradient is:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

This gradient vector is non-zero everywhere on S^2, ensuring that the Jacobian has maximal rank (1 in this case) across the surface defined by $f(x, y, z) = 0$.

For the parametric representation:

$$\begin{aligned}
x &= \sin(\theta)\cos(\phi),\\
y &= \sin(\theta)\sin(\phi),\\
z &= \cos(\theta),
\end{aligned}$$

the Jacobian matrix J is given by:

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\phi) & -\sin(\theta)\sin(\phi) \\ \cos(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) \\ -\sin(\theta) & 0 \end{pmatrix}.$$

To verify this matrix has maximal rank (2, since it's a 3×2 matrix), consider the determinant of a 2×2 submatrix, for example:

$$\det\begin{pmatrix} \cos(\theta)\cos(\phi) & -\sin(\theta)\sin(\phi) \\ \cos(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) \end{pmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1,$$

which is non-zero for all $\theta \in (0,\pi)$, thus demonstrating the Jacobian has maximal rank for almost all values of θ and ϕ, excluding the poles where other charts are necessary.

Torus (T^2) in $\mathbb{R}^3$. A torus can be parametrically represented by:

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$$\begin{aligned}
x(u,v) &= (R + r\cos v)\cos u,\\
y(u,v) &= (R + r\cos v)\sin u,\\
z(u,v) &= r\sin v,
\end{aligned}$$

where $u, v \in [0, 2\pi)$, R is the distance from the center of the tube to the center of the torus, and r is the radius of the tube.

The Jacobi matrix is:

$$\begin{pmatrix} -\sin(u)(R + r\cos(v)) & -r\sin(v)\cos(u) \\ \cos(u)(R + r\cos(v)) & -r\sin(v)\sin(u) \\ 0 & r\cos(v) \end{pmatrix}.$$

Its rank is maximal (2) for all u, v.

Möbius Strip in $\mathbb{R}^3$. The Möbius strip is a so-called non-orientable surface that can be parametrically represented as:

$$\begin{aligned}
x(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u,\\
y(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u,\\
z(u,v) &= \frac{v}{2}\sin\frac{u}{2},
\end{aligned}$$

where $u \in [0, 2\pi)$ and $v \in [-1, 1]$.

The Jacobi matrix is:

$$\begin{pmatrix} -\sin(u)\left(1+\frac{v}{2}\cos\frac{u}{2}\right)-\frac{v}{2}\sin\frac{u}{2}\cos(u) & \frac{1}{2}\cos\frac{u}{2}\cos(u) \\ \cos(u)\left(1+\frac{v}{2}\cos\frac{u}{2}\right)-\frac{v}{2}\sin\frac{u}{2}\sin(u) & \frac{1}{2}\cos\frac{u}{2}\sin(u) \\ \frac{1}{2}\cos\frac{u}{2} & 0 \end{pmatrix}.$$

Its rank is maximal (2).

Catenoid in $\mathbb{R}^3$. The catenoid is a so-called ‘minimal surface’. It can be represented in parametric form by:

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$$\begin{aligned} x(u,v) &= \cosh(v)\cos(u), \\ y(u,v) &= \cosh(v)\sin(u), \\ z(u,v) &= v, \end{aligned}$$

where $u \in [0, 2\pi)$ and $v \in \mathbb{R}$, with cosh denoting the hyperbolic cosine function.

The Jacobi matrix is.

$$\begin{pmatrix} -\sin(u)\cosh(v) & \cos(u)\sinh(v) \\ \cos(u)\cosh(v) & \sin(u)\sinh(v) \\ 0 & 1 \end{pmatrix}.$$

It rank of is maximal (2) across all u, v.

Chapter 13

Multidimensional Integration

13.1 The Jordan measure in $\mathbb{R}^n$

In this section we introduce the Jordan measure and its first properties.

13.1. — The following notation will be useful throughout the section. If $X \subset \mathbb{R}^n$ is a some set $\varrho > 0$ and $a \in \mathbb{R}^n$ we will use the notations

$$\varrho X + a \qquad \text{or} \qquad a + \varrho X$$

18 to refer to the set $\{\varrho x + a \mid x \in X\}$. This is a compact way to refer to dilations and translations of X:

Notice that with this notation we can operate with sets easily according to the usual rules of sum and multiplication, and see what is the compounded effect of some dilations and translation: for example, given $\varrho, \varrho' > 0$ and $a, a' \in \mathbb{R}^n$ we have

$$\varrho'(a + \varrho X) + a' = (\varrho' a + a') + \varrho' \varrho X$$

since this simply boils down to

$$\big\{\varrho'(a + \varrho x) + c \mid x \in X\big\} = \big\{(\varrho' a + a') + \varrho' \varrho x \mid x \in X\big\}\,.$$

Definition 13.2: Dyadic cubes, dyadic subsets, and their measure

We call a subset of $\mathbb{R}^n$ of the form

$$Q = 2^{-p}(a + [0,1)^n) := \prod_{i=1}^{n} \big[2^{-p}a_i, 2^{-p}(a_i + 1)\big),$$

for some $a = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ and $p \in \mathbb{N}$, **dyadic cube** of side length 2^{-p}.
Any finite union (possibly empty) of dyadic cubes of the same side length is called **dyadic set** of **pixel size** 2^{-p}. Notice that such set is determined by giving $p, N \in \mathbb{N}$ and an injective map $a : \{1, 2, \dots, N\} \to \mathbb{Z}^k$:

$$E = \bigcup_{\ell=1}^{N} 2^{-k}(a(\ell) + [0,1)^n). \tag{13.1}$$

If Q is a dyadic cube of side length 2^{-k} we define its n-volume, denoted μ_n or μ as $\mu(Q) := 2^{-kn}$ (we will check below that it is well-defined). Also if E is a dyadic set of the form (13.1) we define $\mu(E) := 2^{-pn}N$.
By definition, the empty set, denoted $\varnothing$, is a dyadic set. It corresponds to $N = 0$ and
18 hence it has zero volume: $\mu(\varnothing) = 0$.

Lemma 13.3: dyadic refinements

Assume

$$E = \bigcup_{\ell=1}^{N} 2^{-p}(a(\ell) + [0,1)^n),$$

for some $a : \{1, 2, \dots, N\} \to \mathbb{Z}^n$ *injective.*
Then for every $q > p$ *we can decompose*

$$E = \bigcup_{\ell=1}^{2^{(q-p)n}N} 2^{-q}(b(\ell) + [0,1)^n),$$

for a suitable $b : \{1, 2, \dots, 2^{(q-p)n}N\} \to \mathbb{Z}^n$ *injective.*

Proof. It easily follows from the decomposition

$$[0,1)^n = \bigcup \left\{ 2^{-(q-p)}(z + [0,1)^n) \mid z \in \left(\mathbb{Z} \cap [0, 2^{q-p})\right)^n \right\}. \tag{13.2}$$

□

Proposition 13.4: Properties of μ

Unions, intersections, and differences of dyadic sets (with possibly different pixel sizes) are dyadic sets. (More precisely, if E and F are dyadic sets then also $E \cup F$, $E \setminus F$, $F \setminus E$, and $E \cap F$ are).
The measure $\mu = \mu_n$, is well-defined over dyadic sets satisfies the following properties:

1. *Additivity: $\mu(E \cup F) = \mu(E) + \mu(F)$ for all E, F dyadic and disjoint.*

2. *Normalization: $\mu([0,1)^n) = 1$*

3. *Translation and scaling invariance: $\mu(\varrho E + \tau) = \varrho^n \mu(E)$, for all E dyadic, ϱ belonging to $2^{-p}\mathbb{N}$ for some $p \in \mathbb{N}$, and τ belonging to $2^{-p}\mathbb{Z}^n$ for some $p \in \mathbb{N}$;*

Proof. First, we observe that the measure $\mu(E)$ is well-defined: it does not depend on the size of the pixel chosen to represent E. Indeed, if E is initially represented using pixels of size 2^{-p}, and these pixels are further subdivided into smaller pixels of size 2^{-q} where $q > p$, then each original cube Q_0 with side length 2^{-p} is divided into $2^{(q-p)n}$ smaller cubes Q_i of side length 2^{-q}, as shown in (13.2). Consequently, the measure of Q_0, which is by definition is 2^{-pn}, equals the aggregate measure of the smaller cubes, given by

18

$$2^{-pn} = 2^{(q-p)n} \cdot 2^{-qn} = \sum_{i=1}^{2^{(q-p)n}} \mu(Q_i).$$

Hence, the measure $\mu(E)$ for the dyadic set E remains consistent, regardless of the chosen pixel size for its representation.

By applying Lemma 13.3, we can adjust refine one of the two dyadic sets, if necessary, to ensure both sets have the same pixel size, specifically by choosing the smaller pixel size from the two. Consequently, we ascertain that the union and intersection of any two given dyadic sets are themselves dyadic. Furthermore, the measure μ exhibits additivity, which naturally stems from counting the number of pixels in each set. The normalization of μ is established by its definitional properties.

To show the translation and rotation invariance we first consider the case $E_\circ = 2^{-q}(a + [0,1)^n)$. Then, for all $m \in \mathbb{N}$, $p \in \mathbb{N}$ and $b \in \mathbb{Z}^n$ we have

$$\begin{aligned} \varrho E_\circ + \tau &= m2^{-p}E + 2^{-p}b = m2^{-p}2^{-q}(a + [0,1)^n) + 2^{-p}b = 2^{-(p+q)}(ma + 2^q b + m[0,1)^n) \\ &= \bigcup_{z \in \mathbb{Z}^n \cap [0,m)^n} 2^{-(p+q)}(ma + 2^q b + z + [0,1)^n) \end{aligned}$$

and hence

$$\mu(\varrho E_\circ + \tau) = 2^{-(p+q)n} \#\{\mathbb{Z}^n \cap [0,m)^n\} = 2^{-(p+q)n} m^n = \varrho^n 2^{-nq} = \varrho^n \mu(E).$$

From this property for cubes we deduce the property for general dyadic sets using additivity: if $E = \bigcup_{\ell=1}^{N} 2^{-k}(a(\ell) + [0,1)^n)$ then

$$\varrho E + \tau = \bigcup_{\ell=1}^{N} 2^{-k}(\varrho a(\ell) + \tau \varrho[0,1)^n)$$

and the union is disjoint. Therefore,

$$\mu(\varrho E + \tau) = \sum_{\ell=1}^{N} \mu\Big(2^{-k}(\varrho a(\ell) + \tau \varrho[0,1)^n)\Big) = \sum_{\ell=1}^{N} \varrho^n 2^{-kn} = \varrho^n \mu(E).$$

□

13.5. — One can show (exercise) that μ is the only measure (i.e., a map assigning to each subset in a given collection a nonnegative real number) defined on the dyadic sets and satisfying properties (1)-(3).

Exercise 13.6. — Show that, for two dyadic sets E_1 and E_2, the measure of their union
18 is given by:

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2)$$

This formula ensures that the overlapping part of E_1 and E_2 is not counted twice.

Show that, for N dyadic sets, the measure of their union is determined by the principle of inclusion-exclusion:

$$\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{k=1}^{N} (-1)^{k+1} \left(\sum_{1 \le i_1 < i_2 < \ldots < i_k \le N} \mu(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) \right).$$

Definition 13.7: Jordan measurable sets and their measure

Given a set $E \subset \mathbb{R}^n$ let us define its inner and outer volumes as:

$$\mu_{\text{in}}(E) = \sup\{\mu(F) \mid F \subset E,\ F \text{ dyadic set}\}$$

and

$$\mu_{\text{out}}(E) = \inf\{\mu(F) \mid E \subset F,\ F \text{ dyadic set}\}$$

If $\mu_{in}(E) = \mu_{out}(E)$ then we say that E is **Jordan measurable**. We then denote $\mu_{in}(E) = \mu_{out}(E)$ by $\mu(E)$, $\mu_n(E)$, or $\mathrm{vol}_n(E)$.

Definition 13.8: Jordan and Lebesgue null sets

We say that $E \subset \mathbb{R}^n$ is a **Jordan null** if $\mu_{\text{out}}(E) = 0$. Also, we say that E is **Lebesgue null** if for all $\varepsilon > 0$ exist countably many dyadic cubes Q_i (possibly with different side sizes) such that

$$E \subset \cup_i Q_i \qquad \text{and} \qquad \sum_i \mu(Q_i) < \varepsilon.$$

13.9. — Notice that by definition every Jordan null is jordan measurable (with zero volume) and is also Lebesgue null.

Clearly, Jordan null sets must be bounded, but this is not the most crucial difference with respect to Lebesgue null sets: Several bounded sets are Lebesgue null but not Jordan null. For example $E := \mathbb{Q}^n \cap [0,1]^n$ is a bounded subset of $\mathbb{R}^n$ is Lebesgue null as $\mathbb{Q}^n$ is countable so there exist a surjective map from $f : \mathbb{N} \to \mathbb{Q}^n$. For all $i \in N$ let Q_i be the dyadic cube of side length 2^{-N-i-1} that contains $f(i)$. Then $E \subset \cup_i Q_i$ but

$$\sum_i \mu(Q_i) = \sum_{i=0}^{\infty} (2^{-N-i})^n \leq 2^{-N}$$

for all $n \geq 1$. Since taking N large we can make $2^{-N} < \varepsilon$, so E is indeed Lebesgue null.

18

Lemma 13.10: Compact Lebesgue null is Jordan null

If $K \subset \mathbb{R}^n$ is compact and Lebesgue null then it is also Jordan null.

Proof. Suppose K is a Lebesgue null set. For any given $\varepsilon > 0$, it is possible to cover K with finitely many dyadic cubes Q_i such that $K \subset \bigcup_{i=1}^{\infty} Q_i$ and

$$\sum_{i=1}^{\infty} \mu(Q_i) < \frac{\varepsilon}{3^n}.$$

For each dyadic cube Q_i, define Q_i^* as the open cube with the same center as Q_i, but with each side length tripled. Explicitly, if $Q_i = 2^{-k_i}(a_i + [0,1)^n)$, then $Q_i^* = \bigcup_{h \in \{-1,0,1\}^n} 2^{-k_i}(a_i + h + [0,1)^n)$. In particular we have $\mu(Q_i^*) = 3^n \mu(Q_i)$.

Notice now that K is contained within the union of the interiors of these expanded cubes, $K \subset \bigcup_{i=1}^{\infty} \text{int}(Q_i^*)$. Given the compactness of K, a finite subcover can be selected, $K \subset \bigcup_{\ell=1}^{N} Q_{i_\ell}^*$. Thus,

$$\sum_{\ell=1}^{N} \mu(Q_{i_\ell}^*) < \varepsilon.$$

Therefore, by demonstrating that K can be covered by a finite collection of dyadic cubes with a total measure less than any arbitrary ε, we conclude that K is Jordan null.

□

18 Exercise 13.11. — Prove that finite (respectively countable) unions of Jordan (respectively Lebesgue) null sets are also null sets.

Proposition 13.12: Bounded Jordan measurable sets

A bounded subset E of $\mathbb{R}^n$ is Jordan measurable if and only if E is bounded and ∂E is a Lebesgue null set.

Before giving this proof we need three short lemmas.

Lemma 13.13: measure of interior and closure of dyadic set

Let E be a dyadic set. Then its both its interior $\mathrm{int}(E)$ and closure $\overline{E}$ are Jordan measurable and $\mu(\mathrm{int}(E)) = \mu(\overline{E}) = \mu(E)$.

Proof. Assume $E = \bigcup_{\ell=1}^{N} 2^{-p}(a(\ell) + [0,1)^n)$, where $a : \{1, \dots N\} \to \mathbb{Z}^n$ is injective and $p \in \mathbb{N}$.

Given $k \geq 2$, Consider the dyadic sets

$$F_k = \bigcup_{\ell=1}^{N} \bigcup_{h \in \{1,2,\dots,2^k-1\}^n} 2^{-p}(a(\ell) + 2^{-k}h + 2^{-k}[0,1)^n),$$

19 and

$$G_k = \bigcup_{\ell=1}^{N} \bigcup_{h \in \{0,1,2,\dots,2^k\}^n} 2^{-p}(a(\ell) + 2^{-k}h + 2^{-k}[0,1)^n),$$

Then

$$F \subset \mathrm{int}(E) \subset E \subset \overline{E} \subset G.$$

But

$$\mu(G_k) = N(2^k+1)^n 2^{-kpn} \quad \text{and} \quad \mu(F) = N(2^k-1)^n 2^{-kpn}$$

Sending $k \to \infty$ we have $\lim_k \mu(G_k) = \lim_k \mu(F_k) = 2^{-pn}$. This shows that both $\mathrm{int}(Q)$ and $\overline{Q}$ are Jordan measurable and have the same measure as Q. □

Lemma 13.14: A pixel is either inside or outside a dyadic set

Suppose that Q is a dyadic cube of side length 2^{-p} and F is some dyadic set of pixel size 2^{-p} then either $Q \subset F$ or $Q \cap F = \varnothing$.

Proof. By definition $Q = 2^{-p}(a_\circ + [0,1)^n)$ for some $a_\circ \in \mathbb{Z}^n$ and $F = \bigcup_{\ell=1}^{N} 2^{-p}(a(\ell) + [0,1)^n)$ with $a : \{1, \dots, N\} \to \mathbb{Z}^n$ is injective. So either $a_\circ$ belongs to the image of a, in which case $Q \subset F$, or not, in which case $Q \cap F = \varnothing$. □

Lemma 13.15: Sandwich

If $E \subset \mathbb{R}^n$ is a subset and there are sequences of sets F_k, G_k such that $F_k \subset E \subset G_k$ such that $\lim_k \mu_{\text{out}}(G_k) - \mu_{\text{in}}(F_k) = 0$ then E is Jordan measurable and

$$\mu(E) = \inf_k \mu_{\text{out}}(G_k) = \sup_k \mu_{\text{in}}(F_k). \tag{13.3}$$

.

Proof. For all $\epsilon > 0$ the exist $k \in \mathbb{N}$ and, $\widetilde{F}_k \subset F_k$ and $\widetilde{G}_k \supset G_k$ dyadic sets such that

$$\mu_{\text{out}}(G_k) - \mu_{\text{in}}(F_k) < \epsilon/3$$

$$\mu(\widetilde{G}_k) - \mu_{\text{out}}(G_k) < \epsilon/3$$

$$\mu_{\text{in}}(F_k) - \mu(\widetilde{F}_k) < \epsilon/3.$$

Hence, $\widetilde{F}_k \subset E \subset \widetilde{G}_k$ and $\mu(\widetilde{G}_k) - \mu(\widetilde{F}_k) < \epsilon$, and since ϵ can be make arbitrarily small this proves that the set is measurable. Also, (13.3) follows from $\mu(\widetilde{G}_k) \leq \mu(E) \leq \mu(\widetilde{F}_k)$. □

Proof of Proposition 13.12. Obersve first that the boundary ∂E of a bounded set is compact. So the boundary is Lebesgue null if and only if it is Jordan null.

For any given $p \in \mathbb{N}$ (large), put

19

$$Q_p(a) := 2^{-p}(a + [0,1)^n),$$

where $a \in \mathbb{Z}^n$, and define

$$F_p := \bigcup \{Q_p(a) \mid Q_p(a) \subset E, \ a \in \mathbb{Z}^n\}$$

$$G_p := \bigcup \{Q_p(a) \mid Q_p(a) \cap E \neq \varnothing, \ a \in \mathbb{Z}^n\}$$

Notice that

$$F_p \subset E \subset G_p$$

Moreover, by Lemma 13.14, F_p the largest dyadic set of pixel size 2^{-p} contained in E; while G_p is the smallest dyadic set of pixel size 2^{-p} containing E.

Suppose first that ∂E is a Jordan null set and let us show that E is Jordan measurable. Indeed, for each $\epsilon > 0$ there is $p \in \mathbb{N}$ and H_p dyadic with pixel size 2^{-p} such that

$$\partial E \subset H_p \qquad \text{and} \qquad \mu(H_p) < \epsilon$$

But notice that $G_p \setminus F_p \subset H_p$. Indeed, if $Q_p(a) \subset G_p \setminus F_p$ then $Q_p(a)$ contains simultaneously a point $x \in E$ and a point $y \in \mathbb{R}^n \setminus E$. Then $Q_p(a)$ also contains the point

$$z_* := (1 - t_*)x + t_* y \quad \text{with} \quad t_* := \sup\{t \in (0,1) \mid (1-t)x + ty \in E\}.$$

But z_* belongs to ∂E being an accumulation point of both E and $\mathbb{R}^n \backslash E$. Hence $z_* \in Q_p(a) \cap \partial E$ and thus $Q_p(a) \subset H_p$ (by Lemma 13.14).

Then, since μ is additive on dyadic sets, we have

$$\mu_{\text{out}}(E) - \mu_{\text{in}}(E) \le \mu(G_p) - \mu(F_p) \le \mu(H_p) < \epsilon$$

Since $\epsilon > 0$ can be made arbitrarily small it follows that E is Jordan measurable.

To establish the converse implication, we notice that if E is Jordan measurable, then there exists p such that

$$\mu(G_p) - \mu(F_p) \le \epsilon.$$

The only issue now to conclude that that ∂E is a nulls set, is that ∂E may not be contained in $G_p \setminus F_p$. (For example, when $E = [0,1)^n$, then $F_p == G_p = E$ for all p)

However, by Lemma 13.13, the $\operatorname{int}(F_p)$ and $\overline{G}_p$ are Jordan measurable and have the same measure as F_p and G_p, respectively. But $\partial E \subset \overline{G}_p \setminus \operatorname{int}(F_p)$ and $\mu(\overline{G}_p \setminus \operatorname{int}(F_p)) = \mu(\overline{G}_p) - \mu(\operatorname{int}(F_p)) < \epsilon$

Since $\epsilon > 0$ can be made arbitrarily small it follows that ∂E is a Jordan null set. □

In order to proof the proposition we will need the following

Lemma 13.16: Lipschitz maps preserve null sets

19 *Suppose that $m \le n$, and let $U \subset \mathbb{R}^m$ open, $E \subset U \subset \mathbb{R}^m$ a Jordan null set (in R^m, i.e. with respect to the measure μ_m), and $f : U \to \mathbb{R}^n$ a is a Lipschitz map. Then $f(E) \subset \mathbb{R}^n$ is also Jordan null*

Proof. Since $\mu(E) = 0$, given $\epsilon > 0$ exist disjoint dyadic cubes $Q_\ell := 2^{-p}(a(\ell) + [0,1)^m) \subset \mathbb{R}^m$ covering E such that $\sum_{\ell=1}^N \mu_m(Q_i) < \epsilon$.

If Λ is the Lipschitz constant of f then for all $x \in Q_\ell \cap U$ we have

$$|f(x) - f(2^{-p}a(\ell))| \le \Lambda |x - 2^{-p}a(\ell)| \le \sqrt{n}\Lambda 2^{-p},$$

where Λ is the Lipchitz constant of f (in U).

Hence, fixing $p_0 \ge 1$ (depending only on n and Λ) so that $\sqrt{n}(\Lambda+1) < 2^{p_0}$ and picking $y_\ell \in 2^{-p}\mathbb{Z}^n$ such that $y_\ell \in f(2^{-p}a(\ell)) + 2^{-p}[0,1)^n$, we obtain $f(Q_\ell \cap U) \subset \widetilde{Q}_\ell := y_\ell + 2^{p_0 - p}[0,1)^n$ (Indeed, for all $x \in Q_\ell \cap U$ we have $|f(x) - y_\ell| \le |f(x) - f(2^{-p}a(\ell))| + \sqrt{n}2^{-p} \le \sqrt{n}(\Lambda+1)2^{-p}$.)

Hence,

$$f(E) \subset \bigcup_{\ell=1}^N \widetilde{Q}_\ell$$

But notice that each $\widetilde{Q}_\ell$ is a union of dyadic cubes of $\mathbb{R}^n$ and

$$\sum_{\ell=1}^N \mu_n(\widetilde{Q}_\ell) \le N 2^{(p_0 - p)n} = \le N 2^{p_0 n} 2^{-pm} = 2^{p_0 n} \sum_{\ell=1}^N \mu_m(Q_\ell) < 2^{p_0 n}\epsilon.$$

Hence, since $\epsilon > 0$ can be made arbitrarily small we conclude that $f(E)$ is a null set (in $\mathbb{R}^n$). □

Lemma 13.17: Graphs of uniformly continuous maps are null

Suppose that $E \subset \mathbb{R}^n$ is a bounded set and $f : E \to \mathbb{R}$ is a uniformly continuous map (i.e., for all $\epsilon > 0$ exists $\delta > 0$ such that $|x - x'| < \delta$, $x, x' \in A \implies |f(x) - f(x')| < \epsilon$). Then, $\{(x, f(x)) \mid x \in E\} \subset \mathbb{R}^{n+1}$ is Jordan null.

Proof. Since E is bounded we have $E \subset [-N_\circ, N_\circ)^n$, for some $N_\circ \in \mathbb{N}$.

Fix $\epsilon = 2^{-q} > 0$, where $q \in \mathbb{N}$. Since f is uniformly continuous in E, there exists $\delta > 0$ such that if $x, x' \in A$ are such that $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$ Take $p \in \mathbb{N}$ such that $\sqrt{n}2^{-p} < \delta$. Assume without loss of generality that $p \geq q$.

We can then cover E by (a number $N \leq (2^{p+1}N_\circ)^n$ of) disjoint union of cubes $Q_\ell = 2^{-p}(a(\ell) + [0,1)^n)$, where $a(\ell) \in \mathbb{Z}^n$. We can assume that each cube Q_ℓ has nonempty intersection with E. For each ℓ, let x_ℓ be some point belonging to $E \cap Q_\ell$ and let $b_\ell \in \mathbb{Z}$ be such that $f(x_\ell) \in 2^{-q}(b_\ell + [0,1))$.

Then, for all $x \in E \cap Q_\ell$ we have 19

$$|f(x) - 2^{-q}b_\ell| < |f(x) - f(x_\ell)| + |f(x_\ell) - 2^{-p}b_\ell| \leq 22^{-q},$$

where we used that $|x - x_\ell| \leq \sqrt{n}2^{-p}$ for all $x \in Q_\ell$ (and that $\sqrt{n}2^{-p} < \delta$).

Therefore,

$$\{(x, f(x)) \mid x \in E\} \subset F := \bigcup_{\ell=1}^{N} Q_\ell \times 2^{-q}[b_\ell - 2, b_\ell + 2)$$

Now, for each ℓ, the 'vertical column above Q_ℓ', $Q_\ell \times 2^{-q}[b_\ell - 2, b_\ell, +2)$, can be written as:

$$\bigcup_{-22^{p-q} \leq k < 22^{p-q}} 2^{-p}((a_\ell, k) + [0,1)^{n+1}).$$

That is, this 'vertical columm above Q_ℓ' is the union of 42^{p-q} cubes of side length 2^{-p} in $\mathbb{R}^{n+1}$, the set F is dyadic and

$$\mu_{n+1}(F) = 4N2^{p-q}2^{-pn+1} = 4(2N_\circ)^n 2^{-q} < 4(2N_\circ)^n \epsilon.$$

Sending $\epsilon \to 0$ we obtain that the graph of f is Jordan null. □

Theorem 13.18: first properties of the Jordan measure

The Jordan measure $\mu = \operatorname{vol}_n$, defined over all Jordan measurable sets $E \subset \mathbb{R}^n$ satisfies the following properties.

1. *Additivity: If E, F are Jordan measurable $E \cup F$, $E \cap F$, $E \setminus F$ and $F \setminus E$ are Jordan measurable. Moreover, if E and F are disjoint: $\mu(E \cup F) = \mu(E) + \mu(F)$.*
2. *Normalization: $\operatorname{vol}_n([0,1)^n) = 1$*
3. *Volume of boxes: if*
$$E = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$
with $b_i > a_i$ for $1 \le i \le n$ then $\mu(E) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Proof. The boundary of any of the four sets $E \cup F$, $E \cap F$, $E \setminus F$ and $F \setminus E$ is a subset of $\partial E \cup \partial F$ (exercise). If E, F are Jordan measurable then ∂E and ∂F are Jordan null, by Lemma 13.12, and hence their union and the boundaries of the previous four sets are Jordan null. Hence, using Lemma 13.12 again each of the previous four sets is Jordan measurable.

19 Now assume that E_1, E_2 are Jordan measurable and disjoint. For any given $\epsilon > 0$ there are dyadic sets F_i, G_i such that

$$F_i \le E_i \le G_i \ , \qquad \mu(F_i) \le \mu(E_i) \le \mu(G_i) \le \mu(F_i) + \epsilon.$$

But then
$$F_1 \cup F_2 \subset E_1 \cup E_2 \subset G_1 \cup G_2$$
and F_1, F_2 are disjoint. Thus,

$$\mu(E_1 \cup E_2) \ge \mu(F_1) + \mu(F_2) \ge \mu(E_1) - \epsilon + \mu(E_2) - \epsilon;$$

and
$$\mu(E_1 \cup E_2) \le \mu(G_1) + \mu(G_2) \le \mu(E_1) + \epsilon + \mu(E_2) + \epsilon.$$

Therefore,
$$-2\epsilon \le \mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2) \le 2\epsilon.$$

Sending $\epsilon \to 0$ we conclude that additivity property $\mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2) = 0$.

Finally suppose that

$$E = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

with $b_i > a_i$ for $1 \leq i \leq n$. For $t \in \mathbb{R}$, recall that the floor and ceiling of t are respectively defined as:

$$\lfloor t \rfloor = \max \{k \in \mathbb{Z} \mid k \leq t\} \qquad \text{and} \qquad \lceil t \rceil = \min \{k \in \mathbb{Z} \mid k \geq t\}.$$

Given $p \in \mathbb{N}$ let us define

$$\lfloor t \rfloor_p := 2^{-p} \lfloor 2^p t \rfloor_p \qquad \text{and} \qquad \lceil t \rceil_p := 2^{-p} \lceil 2^p t \rceil_p .$$

For given $p \in \mathbb{N}$ consider the two dyadic sets

19
$$F_p = \prod_{i=1}^{n} \left[\lceil a_i \rceil_p, \lfloor b_i \rfloor_p \right)$$

and

$$G_p = \prod_{i=1}^{n} \left[\lfloor a_i \rfloor_p, \lceil b_i \rceil_p, \right).$$

Then, $F_p \subset E \subset G_p$ and (for p large enough so that $2^{1-p} < \min_i(b_i - a_i)$)

$$\prod_{i=1}^{n} (b_i - a_i - 2^{1-p}) \leq \mu(F_p) \leq \mu(E) \leq \mu(G_p) \leq \prod_{i=1}^{n} (b_i - a_i + 2^{1-p}).$$

Sending $p \to \infty$ we conclude $\mu(E) = \prod_{i=1}^{n} (b_i - a_i)$. □

Exercise 13.19. — Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map with continuous inverse then for every set $E \subset \mathbb{R}^n$ we have $f(\overline{E}) = \overline{f(E)}$, $f(\operatorname{int}(E)) = \operatorname{int}(f(E))$ and $f(\partial E) = \partial f(E)$.

Lemma 13.20: Translation and dilation invariance

For all E Jordan measurable, $\varrho > 0$ and $\tau \in \mathbb{R}^n$, $\varrho E + \tau$ is also Jordan measurable and $\mu(\varrho E + \tau) = \varrho^n \mu(E)$

Proof. Since the map $x \mapsto \varrho x + \tau$ is Lipschitz and $\partial(\varrho E + \tau) = \varrho \partial E + \tau$ we obtain, using
20 Lemma 13.12 that $\varrho E + \tau$ is Jordan measurable.

Now, putting $Q_p := 2^{-p}[0,1)^n$ the formula for the measure of boxes implies that $\mu(\varrho Q_p + \tau) = \mu([0, \varrho 2^{-p}]^n) = (\varrho 2^{-p})^n = \varrho^n \mu(Q_p)$ for every translation $\tau \in \mathbb{R}^n$ and dilation factor $\varrho > 0$. Then, since μ is additive, $\mu(\varrho F + \tau) = \varrho^n \mu(F)$ for all F dyadic (since F is, by definition, a disjoint union of dyadic cubes).

Finally, for all E Jordan measurable, given $\epsilon > 0$ there exist dyadic sets F_ϵ and G_ϵ such that $F_\epsilon \subset E \subset G_\epsilon$ and $\mu(F_\epsilon) \leq \mu(E) \leq \mu(G_\epsilon) \leq \mu(F_\epsilon) + \epsilon$. But then $(\varrho F_\epsilon + \tau) \subset (\varrho E + \tau) \subset (\varrho G_\epsilon + \tau)$ and thus $\varrho^n \mu(F_\epsilon) = \mu(\varrho F_\epsilon + \tau) \leq \mu(\varrho E + \tau) \leq \mu(\varrho G_\epsilon + \tau) = \varrho^n \mu(G_\epsilon) \leq \varrho^n(\mu(F_\epsilon) + \epsilon)$. Sending $\epsilon \to 0$ (since $\mu(F_\epsilon) \to \mu(E)$) we conclude that $\mu(\varrho E + \tau) = \varrho^n \mu(E)$. □

Proposition 13.21: Affine transformations

Given some invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$. If a set $E \subset \mathbb{R}^n$ is Jordan measurable then so is its image under L, which we denote LE. Moreover there exist a positive factor $\lambda(L)$ such that

$$\mu(LE) = \lambda(L)\mu(E) \quad \textit{for all } E \textit{ Jordan measurable.}$$

Proof of Proposition 13.21. By the exact same argument as in Lemma 13.20. If $E \subset \mathbb{R}^n$ is Jordan measurable the $L(E)$ is also Jordan measurable (since L is Lipschitz).

Let $Q_p := [0, 2^{-p})^n$. We know that $L(Q_0)$ is measurable. Since L^{-1} is Lipchitz and maps $L(Q_0)$ to Q_0 (and $\mu(Q_0) = 1 > 0$) the set $L(Q_0)$ cannot be a null set. Let us put $\lambda(L) := \mu(L(Q_0)) > 0$

Now since L is linear it follows that $L(Q_p + a) = 2^{-p}L(Q_0) + L(a)$. But then by the scaling and translation invariance of μ we obtain $\mu(L(Q_p + a)) = 2^{-pn}\lambda(L)$.

But then using the additivity of μ we deduce that for all F dyadic

$$\mu(L(F)) = \lambda(L)\mu(F).$$

Finally, the validity previous formula is extended to every Jordan measurable set exactly as in the proof of Lemma 13.20. □

20 **Proposition 13.22: The factor is the determinant**

For every invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ we have $\lambda(L) = |\det L|$

The proof of Proposition 13.22 uses the following three lemmas.

Lemma 13.23: The factor for special stretchings

Given $\lambda_1, \dots, \lambda_n$ given positive real numbers, consider the 'special stretching' (affine transformation) $Sx = (\lambda_1 x_1, \dots \lambda_n x_n)$. Then $\lambda(S) = \prod_{i=1}^n \lambda_i$)

Proof. It is an immediate consequence of the formula of the volume of boxes. □

Lemma 13.24: Balls are Jordan measurable

The unit ball $B_1(0) \subset \mathbb{R}^n$ is Jordan measurable.

Proof. The boundary of the ball is covered by the union of the two graphs

$$x_n = f(x_1, \dots, x_{n-1}) \qquad x_n = -f(x_1, \dots, x_{n-1})$$

where

$$f(x_1, \dots, x_{n-1}) = \sqrt{1 - (x_1^2 + \dots + x_{n-1}^2)},$$

is uniformly continuous in the closed unit ball of $\mathbb{R}^{n-1}$.

Hence, using Lemma 13.17 we obtain, $\partial B_1(0)$ is a null set and thus, by Lemma 13.12 we obtain that $B_1(0)$ is Jordan measurable. □

The next result is a standard consequence of the Spectral Theorem (11.9)

Lemma 13.25: Polar decomposition

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ an invertible linear map, i.e., an invertible $n \times n$ matrix with real entries acting on vectors). Then L admits the factorization

$$L = R_2 S R_1,$$

where R_i are $n \times n$ orthogonal matrices (i.e., $R_i^T R_i = \mathrm{I}_n$) and S is a diagonal matrix with positive entries.

Proof. Notice that the matrix $A = L^T L$ is symmetric and nonnegative definite (since $v^T A v = |Lv|^2 \geq 0$ for all $v \in \mathbb{R}^n$). Thus, applying the Spectral Theorem to A we obtain $O^T A O = D$, where $D = \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2)$ is a diagonal matrix with nonnegative entries and O is orthogonal. Since L is invertible the entries of D must be all positive.

Define then $S := \mathrm{diag}(|\lambda_1|, \ldots, |\lambda_n|)$ and notice that $S^2 = D$. We thus find $O^T A O = S^2$
and thus $S^{-1} O^T L^T L O S^{-1} = \mathrm{I}_n$. But since $S^{-1} = (S^{-1})^T$ (because S and S^{-1} are diagonal)
20 we have shown $(LOS^{-1})^T LOS^{-1} = \mathrm{I}_n$. In other words LOS^{-1} is an orthogonal matrix.
Hence, $LOS^{-1} =: R_2$, is orthogonal. Thus, $L = R_2 S O^T$, and the lemma follows putting
$R_1 = O^T$. □

Proof of Proposition 13.22. By Lemma 13.25 we have that $L = R_2 S R_1$, where S is an special stretching and R_i are orthogonal matrices (i.e., rotations and symmetries). Since rotations leave the unit ball invariant we have $\mu(B_1) = \mu(R_i B_1) = \lambda(R_i)\mu(B_1)$ and hence $\lambda(R_i) = 1$. On the other hand, it is an immediate consequence of the defining property of λ that it must be clearly multiplicative:

$$\lambda(L_2 \circ L_1) = \lambda(L_2)\lambda(L_1).$$

(Indeed for all E Jordan measurable, $\lambda(L_2 \circ L_1)\mu(E) = \mu((L_2 \circ L_1)(E)) = \mu((L_2(L_1(E))) = \lambda(L_2)\mu(L_1(E)) = \lambda(L_2)\lambda(L_1)\mu(E)$.)

Hence, $\lambda(L) = \lambda(R_2 S R_1) = \lambda(R_2)\lambda(S)\lambda(R_1) = \lambda S = |\mathrm{det} S|$.

Since orthogonal matrices have absolute value of determinant equal to one, and the determinant is also multiplicative, $|\mathrm{det} S| = |\mathrm{det} R_2 S R_1| = |\mathrm{det} L|$ and the lemma follows. □

Corollary 13.26: Isotropy of the volume

For all E Jordan measurable and R othogonal $\mu(RE) = \mu(E)$. In other words, μ is invariant by rotations and reflections.

13.2 Riemann integral, change of variables formula

We next define the Riemann integral for real functions of several variables.

First we define the characteristic functions of a set in $\mathbb{R}^n$.

Definition 13.27: Characteristic function of a set

Given $A \subset \mathbb{R}^n$ the **characteristic function** of A, that we will denote $\mathbf{1}_A; \mathbb{R}^n \to \mathbb{R}$ is the function defined as:

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A. \end{cases}$$

20

Definition 13.28: dyadic step functions and their integral

We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is a dyadic step function if for some $p, N \in \mathbb{N}$ and $a : \{1, \dots, N\} \to \mathbb{Z}^n$ injective and $b : \{1, \dots, N\} \to \mathbb{Z}$ we have

$$g(x) = \sum_{\ell=1}^{N} 2^{-p} b(\ell) \mathbf{1}_{Q_p(a(l))}(x),$$

where we use the notation $Q_p(a) := 2^{-p}(a + [0,1)^n)$ for dyadic cubes.

If $g : \mathbb{R}^n \to R$ is a dyadic step function we define its integral, denoted $\int g$ as:

$$\int g = \sum_{\ell=1}^{N} 2^{-p} b(\ell) \mu_n(Q_p(a(\ell))) = 2^{-np} \sum_{\ell=1}^{N} 2^{-p} b(\ell))$$

21 **Cancelled.**

Definition 13.29: Riemann integrable functions and their integral

Let $A \subset \mathbb{R}^n$ be subset and $f : A \to \mathbb{R}$ be a function.
We say that f is **Riemann integrable** over A if lower and upper sums, defined as

$$I_{\text{low}}(f, A) := \sup \left\{ \int g \mid g : \mathbb{R}^n \to \mathbb{R} \text{ dyadic step function such that } g \le f \mathbf{1}_A \right\}$$

and

$$I_{\text{up}}(f, A) := \inf \left\{ \int h \mid h : \mathbb{R}^n \to \mathbb{R} \text{ dyadic step function such that } h \ge f \mathbf{1}_A \right\},$$

coincide.
The common value is termed the **integral of f over A**, denoted by $\int_A f$.
To explicitly indicate the variables of integration, we use notations like $\int_A f(x)\, dx$ or $\int_A f(y)\, dy$, where $dx = dx_1 \dots dx_n$ and $dy = dy_1 \dots dy_n$, respectively. To explicitly indicate that the reference measure to define the integral is the Jordan measure, denoted μ, μ_n, or vol_n we will sometimes write $\int_A f d\mu$, $\int_A f d\mu_n$, or $\int_A f d\,\text{vol}_n$

22 **Lemma 13.30: First properties of the Riemann integral**

If A is Jordan measurable and $c : A \to \mathbb{R}$ is a constant function then it is Riemann integrable and $\int_A c = c\mu_n(A)$.
Also if, $f_1, f_2 : A \to \mathbb{R}$ are Riemann integrable and $c_1, c_2 \in \mathbb{R}$ then $c_1 f_1 + c_2 f_2$ if Riemann integrable and:

$$\int_A (c_1 f_1 + c_2 f_2) = c_1 \int_A f_1 + c_2 \int_A f_2.$$

Proof. For the first part of the lemma notice that if E is Jordan measurable then given $\epsilon > 0$ the exists dyadic sets $G_p \subset E$ and $H_p \supset E$ such that $\mu(G_p) \le \mu(E) \le \mu(H_p) \le \mu(G_p) + \epsilon$.

Then the step function $g := \lfloor c \rfloor_p \mathbf{1}_{G_p}$ and $h := \lceil c \rceil_p \mathbf{1}_{H_p}$ satisfy $g \le c\mathbf{1}_E$, $h \ge c\mathbf{1}_E$, and $\int g \le c\mu(E) \le \int h \le \int g + c\epsilon + 2^{-p})\mu(E)$. Sending $\epsilon \to 0$ and $p \to \infty$ we obtain that $(I) = (II) = c\mu(E)$.

For the linearity property, given that f_1 and f_2 are Riemann integrable over A, and c_1, c_2 are constants, the function $(c_1 f_1 + c_2 f_2)\mathbf{1}_A$ can be approximated (by below and by above) by linear combinations of dyadic step functions approximating f_1 and f_2. Since by definition the integral is linear over step functions we conclude. The details are left to the reader. □

Definition 13.31: Positive and negative parts of a function

If X is a set and $f : X \to \mathbb{R}$ is a function, we define the positive part $f^+ : X \to [0, +\infty)$ and negative part $f^- : X \to [0, +\infty)$ of f as:

$$f^+(x) := \max(f(x), 0), \qquad f^-(x) := \max(-f(x), 0) \qquad x \in X.$$

Notice that we can write $f = f^+ - f^-$.

Lemma 13.32: All boils down to integrating nonnegative functions

Given $A \subset \mathbb{R}^n$, a function $f : A \to \mathbb{R}$ is Riemann integrable (over A) if and only if f^+ a f^- are. Moreover:

$$\int_A f = \int_A f^+ - \int_A f^-$$

Proof. Left as an exercise. □

Lemma 13.33: Riemann integral as Jordan measure

Let $A \subset \mathbb{R}^n$ be a set. A function $f : A \to [0, \infty)$ is Riemann integrable if and only if the hypograph

$$\Gamma_A(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in A, \quad 0 \le y < f(x)\}$$

22

is Jordan measurable in $\mathbb{R}^{n+1}$.
In such case:

$$\int_A f = \mu_{n+1}(\Gamma_A(f)).$$

Proof. The lemma follows easily from the following observation: Given any dyadic step functions

$$g(x) = \sum_{\ell=1}^{N} 2^{-p} b(\ell) \mathbf{1}_{Q_p(a(l))}(x),$$

such that $g \le f$; then the dyadic set

$$G = \bigcup \{2^{-p}((a(l), k) + [0, 1)^{n+1}) \mid 1 \le \ell \le N, \ 0 \le k < b(\ell)\}$$

is contained in $\Gamma(f)$ and satisfies $\mu_{n+1}(G) = \int g$.

Similarly, given any dyadic step function h such that $f \le h$ we have an associated dyadic set H containing $\Gamma_A(f)$ and such that $\mu_{n+1}(H) = \int h$. □

Corollary 13.34: Integrability of uniformly continuous functions

Suppose that $E \subset \mathbb{R}^n$ is a Jordan measurable set and $f : \overline{E} \to [0, \infty)$ is a continuous function. Then, f is Riemann integrable over E.

Proof. Since E Jordan measurable it is bounded. Hence its closure $\overline{E}$ is compact. Therefore f is a continuous function on a compact set, hence it is bounded and uniformly continuous. Let $C := \max_{x\in\overline{E}} f(x)$.

Lemma 13.17 implies that

$$A_1 := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \overline{E},\ y = f(x)\big\}$$

is Jordan null in $\mathbb{R}^{n+1}$. And the set

$$A_2 := \big\{(x,0) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \overline{E}\big\}$$

is also Jordan null in $\mathbb{R}^{n+1}$.

Finally, using that ∂E is Jordan null in $\mathbb{R}^n$ is is not difficult to show that the set

$$A_3 := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \partial E,\ 0 \le y \le C\}$$

is also Jordan null in $\mathbb{R}^{n+1}$ (left as an exercise).

Now since,

$$\partial\Gamma_E(f) \subset A_1 \cup A_2 \cup A_3$$

we conclude that $\partial\Gamma(f)$ is Jordan null and hence $\Gamma(f)$ is Jordan measurable or, equivalently, f is Riemann integrable. □

22 We now give the change of variables formula

Theorem 13.35: Change of variables formula

Suppose that $U, V \subset \mathbb{R}^n$ are open sets and $\Phi : U \to V$ be a C^1 diffeomorphism. Then, if $A \subset U$ is a Jordan measurable set with $\overline{A} \subset U$ and $f : \overline{A} \to [0, +\infty)$ is a continuous function then

$$\int_A f(x)dx = \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}dy. \tag{13.4}$$

where $|\det J\Phi(x)|$ is the absolute value of the determinant of the Jacobi matrix of Φ at $x \in U$.

Proof. Since $\Phi \in C^1(U, \mathbb{R}^n)$ we have that Φ is locally Lipschitz continuous.

Let us show that $\Phi(\overline{A}) = \overline{\Phi(A)}$ is Jordan measurable.

Indeed, since A is Jordan measurable so is the compact set $\overline{A}$ (they have the same boundary). Now, since Φ is locally Lipchitz continuous for every $x \in \overline{A}$ there is $r_x > 0$ such that $f|_{B_{r_x}(x)}$ is Lipchitz continuous. But since the collection of open balls $\big\{B_{r_x}(x) \mid x \in \overline{A}\big\}$ is an open cover of A we can extract a finite subcover $B_{r_1}(x_1)$, ... $B_{r_N}(x_N)$.

Now since $\mu(\partial A) = 0$ we have $\mu(\partial A \cap B_{r_i}(x_i)) = 0$ for all $i = 1, \dots, N$ and hence $\Phi(\partial A \cap B_{r_i}(x_i)) = 0$ being the Lipchitz image of a Jordan null set.

But then since, $\partial\Phi(A) = \Phi(\partial A) = \bigcup_{i=1}^{N} \Phi(\partial A \cap B_{r_i}(x_i))$ we obtain that $\partial\Phi(A)$ is Jordan null and hence $\Phi(A)$ is Jordan measurable.

Therefore by Corollary 13.34 the integrals at the two sides of (13.4) are well-defined Riemann integrals.

For $p \in \mathbb{N}$ and $\delta \in (0, 1/2)$ define

$$Q_{p,\delta} := 2^{-p}[\delta, 1-\delta]^n.$$

Now, a key part of the proof is to show that for any given $\delta > 0$ (small) there exists $p_\delta \in \mathbb{N}$ such that for every $p \geq p_\delta$ and $x_\circ \in \overline{A}$ we have

$$(1-2\delta)^n 2^{-pn} |\det J\Phi(x_\circ)| = \mu(D\Phi_{x_\circ}(Q_{p,\delta})) \leq \mu\big(\Phi(x_\circ + [0, 2^{-p})^n)\big) \tag{13.5}$$

and

$$\sup\big\{|\det J\Phi(x)| \mid x \in x_\circ + [0, 2^{-p})^n\big\} \leq (1+\delta)|\det J\Phi(x_\circ)|. \tag{13.6}$$

We notice that the equality in (13.5) follows from (13.22). To establish the inequality we will show that

$$(\Phi(x_\circ) + D\Phi_{x_\circ}(Q_{p,\delta})) \subset \Phi(x_\circ + [0, 2^{-p})^n)$$

Similarly to the proof of the Inverse Function Theorem (IFT), for every $\lambda > 0$ (small)
22 there exists $r_\lambda > 0$ such that for all $x_\circ \in \overline{A}$ that the map

$$\Psi_{x_\circ}(h) := \Phi^{-1}\big(\Phi(x_\circ) + D\Phi_{x_\circ}(h)\big) - x_\circ$$

satisfies that $\Psi_{x_\circ} - \mathrm{Id}$ is λ-Lipchitz for $h \in B_{r_\lambda}(0)$. Indeed, by the chain rule we have

$$D(\Psi_{x_\circ} - \mathrm{Id})_0 = 0$$

and hence, reasoning exactly as in the proof of IFT, given $\lambda > 0$ there is $r_\lambda > 0$ such that

$$\Psi_{x_\circ} - \mathrm{Id} \quad \text{is } \lambda\text{-Lipschitz for } h \in B_{r_\lambda}(0).$$

The fact that $r_\lambda > 0$ can be chosen independent of $x_\circ$ is a consequence of the compactness of $\overline{A}$ —one can use similar argument to the proof of Proposition 9.78 ('Continuous function in compact is uniformly continuous').

Since $\Psi_{x_\circ}(0) = 0$ for $h \in Q_{p,\delta} \subset B_{r_\circ}$ we have

$$\big\|\Psi_{x_\circ}(h) - h\big\| \leq \lambda|h| \leq \lambda\sqrt{n}2^{-p}$$

Therefore, given $\delta > 0$ we can choose λ such that $\lambda\sqrt{n} < \delta$ and p_δ such that $2^{-p_\delta} \leq r_\lambda$ so that

$$\Psi_{x_\circ}(Q_{p,\delta}) \subset [0, 2^{-p})^n \quad \text{for all } p \geq p_\delta.$$

That is:

$$\Phi^{-1}\big(\Phi(x_\circ) + D\Phi_{x_\circ}(Q_{p,\delta})\big) \subset [0, 2^{-p})^n + x_\circ \text{for all } p \geq p_\delta.$$

Applying Φ both sides we conclude (13.5).

On the other hand (13.6) follows from the fact that $|\det J\Phi(x)|$ is positive (since Φ is a diffeomorphism) and uniformly continuous on the compact set $\overline{A}$. Indeed, we have

$$c := \inf_{x \in \overline{A}} |\det J\Phi(x)| > 0$$

and then by uniform continuity for every $\delta > 0$ exist $r_\delta > 0$ such that for all $x_\circ \in \overline{A}$ we have

$$\big||\det J\Phi(x)| - |\det J\Phi(x_\circ)|\big| < \delta c \qquad \text{for } x \in B_{r_\delta}(x_\circ) \cap \overline{A}$$

and hence

$$|\det J\Phi(x)| \leq |\det J\Phi(x_\circ)| + \delta c \leq (1+\delta)|\det J\Phi(x_\circ)| \qquad \text{for } x \in B_{r_\delta}(x_\circ) \cap \overline{A}.$$

Suppose now that $g : \mathbb{R}^n \to \mathbb{R}$ is a dyadic step function approximating $f\mathbf{1}_A$ by below. More precisely, given $\epsilon > 0$ let $g = \sum_{\ell=1}^N g_\ell \mathbf{1}_{Q_\ell}$ where $Q_\ell = 2^{-p}(a(\ell) + [0,1)^n)$, $a : \{1, \ldots, N\} \to \mathbb{Z}^n$ injective, satisfies $g \leq f\mathbf{1}_A$ in all of $\mathbb{R}^n$ and $\int_A f - \epsilon \leq \int g$. Put $x_\ell := 2^{-p}a(\ell)$.

22 Finally, combining (13.5)-blabla2 we obtain:

$$\sup\big\{|\det J\Phi(x)| \mid x \in x_\circ + [0, 2^{-p})^n\big\}\, 2^{-pn} \leq \frac{1+\delta}{(1-2\delta)^n}\mu\big(\Phi(x_\circ + [0,2^{-p})^n)\big)$$

for all $x_\circ \in \overline{A}$, provided $p \geq p_\delta$.

Therefore, observing

$$g_\ell \leq \inf_{x \in Q_\ell} f(x) = \inf_{y \in \Phi(Q_\ell)} f(\Phi^{-1}(y))$$

we obtain:

$$\begin{aligned}
\int_A f - \epsilon \leq \int g &= \sum_{\ell=1}^N g_\ell 2^{-np} \\
&\leq \frac{1+\delta}{(1-2\delta)^n} \sum_{\ell=1}^N \inf_{y \in \Phi(Q_\ell)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}\mu(\Phi(Q_\ell)) \\
&\leq \frac{1+\delta}{(1-2\delta)^n} \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|} dy
\end{aligned}$$

Since ϵ and δ can be made arbitrarily small (by taking p large enough), we obtain:

$$\int_A f \leq \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}\, dy.$$

This inequality also holds with A, f, and Φ replaced by $\widetilde{A} = \Phi(A)$, $\widetilde{f}(y) = \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}$, and $\widetilde{\Phi} = \Phi^{-1}$, yielding:

$$\int_{\widetilde{A}} \widetilde{f} \leq \int_{\widetilde{\Phi}(\widetilde{A})} \frac{\widetilde{f}(\widetilde{\Phi}^{-1}(x))}{|\det J\widetilde{\Phi}(\widetilde{\Phi}^{-1}(x))|}\, dx. \tag{13.7}$$

However, considering the definitions of $\widetilde{f}$ and $\widetilde{\Phi}$, we perform the computation:

$$\frac{\widetilde{f}(\widetilde{\Phi}^{-1}(x))}{|\det J\widetilde{\Phi}(\widetilde{\Phi}^{-1}(x))|} = \frac{\frac{f(\Phi^{-1}(\widetilde{\Phi}^{-1}(x)))}{|\det J\Phi(\Phi^{-1}(\widetilde{\Phi}^{-1}(x)))|}}{|\det J\widetilde{\Phi}(\widetilde{\Phi}^{-1}(x))|} = \frac{\frac{f(x)}{|\det J\Phi(x)|}}{|\det J(\Phi^{-1})(\Phi(x))|} = f(x)$$

because $\widetilde{\Phi}^{-1}(x) = \Phi(x)$ and $\det J\Phi(x)\ \det J(\Phi^{-1})(\Phi(x)) = 1$ (since $(J\Phi)^{-1}(x) = J(\Phi^{-1})(\Phi(x))$).

22 Therefore, the (13.7) can be rewritten as:

$$\int_{\widetilde{A}} \widetilde{f} = \int_{\Phi(A)} \frac{f(\Phi^{-1}(y))}{|\det J\Phi(\Phi^{-1}(y))|}\, dy \leq \int_{\widetilde{\Phi}(\widetilde{A})} \frac{\widetilde{f}(\widetilde{\Phi}^{-1}(x))}{|\det J\widetilde{\Phi}(\widetilde{\Phi}^{-1}(x))|}\, dx = \int_A f(x)dx.$$

This establishes the equality (13.4) and hence the theorem. □

Exercise 13.36. — Prove the following equivalent version of the change of variables formula: If $\Psi : V \to U$ is a diffeomorphism, $f : U \to \mathbb{R}$ a continuous function and $\overline{A} \subset U$ a compact Jordan measurable subset, then:

$$\int_A f(x)dx = \int_{\Psi^{-1}(A)} f(\Psi(y))|\det J\Psi(y)|\, dy.$$

23 13.3 Fubini Theorem, differentiation under the integral sign

We next give the Fubini Theorem

Theorem 13.37: Slicing formula (or Cavalieri's principle)

Suppose that $E \subset \mathbb{R}^{n-1} \times (-C, C) \subset \mathbb{R}^n$ is a Jordan measurable set. For $y \in \mathbb{R}$ let the 'slice' $S_y(E) \subset \mathbb{R}^{n-1}$ be defined as:

$$S_y(E) := \left\{ x \in \mathbb{R}^{n-1} \mid (x, y) \in E \right\}.$$

Then functions

$$\varphi(y) := \mu_{n,\text{in}}(S_y(E)) \qquad \textit{and} \qquad \psi(y) := \mu_{n,\text{out}}(S_y(E)),$$

which satisfy $0 \leq \varphi \leq \psi$ are both Riemann integrable in $\mathbb{R}$ and

$$\int_{-C}^{C} \varphi(y)\, dy = \int_{-C}^{C} \psi(y)\, dy = \mu_{n+1}(E).$$

Proof. Since E is Jordan measurable, for any $\epsilon > 0$ there $p \in \mathbb{N}$ and G_p, H_p dyadic with pixel size 2^{-p} such that $G_p \subset E \subset H_p$ and $\mu_n(G_p) \leq \mu_n(E) \leq \mu_n(H_p) < \mu_n(G_p) + \epsilon$.

Notice that for all $y \in \mathbb{R}$ the sets $S_y(G)$ and $S_y(H)$ are dyadic (also with pixel size 2^{-p}) and satisfy $S_y(G) \subset S_y(E) \subset S_y(H)$.

23 Notice also that the sets $S_y(G)$ and $S_y(H)$ remain constant for y within intervals of the fo $y \in 2^{-p}[k, k+1)$, for all $k \in \mathbb{Z}$.

Hence the functions

$$g_p(y) := \mu_{n-1}(S_y(G)) \qquad \text{and} \qquad h_p(y) := \mu_{n-1}(S_y(H))$$

are dyadic step functions in $\mathbb{R}$. Notice also that

$$\int_{\mathbb{R}} g_p = \mu_n(G_p) \qquad \text{and} \qquad \int_{\mathbb{R}} h_p = \mu_n(H_p).$$

Indeed, as stated above, within each interval $y \in 2^{-p}[k, k+1)$, the function $g_p(y)$ remains constant. It is calculated as $2^{-p(n-1)}$ times the number of n-dimensional cubes in G_p that intersect the hyperplane $\{x \in \mathbb{R}^n \mid x_n = t\}$. As t varies, this hyperplane effectively sweeps through all the cubes within G_p. Within this setup, the identity $\mu_n(G_p) = \int_{\mathbb{R}} g_p$ amounts to an elementary discrete counting argument: that the total number of cubes in G_p is equal to the sum, across all vertical levels, of the number of cubes at each level. A similar argument applies to h_p.

Since $S_y(G) \subset S_y(E) \subset S_y(H)$ we have

$$g_p(y) \leq \varphi(y) = \mu_{n,\text{in}}(S_y(E)) \leq \mu_{n,\text{out}}(S_y(E)) = \psi(y) \leq h_p(y)$$

for all $y \in \mathbb{R}$ and therefore:

$$\int g_p \le I_{\mathrm{low}}(\varphi, (-C, C)) \le I_{\mathrm{up}}(\varphi, (-C, C)) \le \int h_p = \int g_p + \epsilon$$

and

$$\int g_p \le I_{\mathrm{low}}(\psi, (-C, C)) \le I_{\mathrm{up}}(\psi, (-C, C)) \le \int h_p \le \int g_p + \epsilon.$$

Since $\epsilon > 0$ can be made arbitrarily small by taking $p \to \infty$ (and using that $\int g_p = \mu_{n+1}(G_p)$ converges to $\mu_{n+1}(E)$) it follows that both φ and ψ are Riemann integrable with

$$\int_{-C}^{C} \varphi = \int_{-C}^{C} \psi = \mu_{n+1}(E)$$

as claimed. □

With a similar proof we have

Theorem 13.38: Slicing formula for functions (Fubini)

23 *Suppose $A \subset \mathbb{R}^n$ is a bounded subset, i.e. $A \subset (-C, C)$ for some C, and that $f : A \to \mathbb{R}$ is Riemann integrable function. Let $\widetilde{f} : (-C, C)^n \to \mathbb{R}$ be defined as $\widetilde{f} = f\mathbf{1}_A$. For a given $i \in \{1, 2, \dots, n\}$ and $y \in \mathbb{R}$, define the 'slice' $f_{i,y} : \mathbb{R}^{n-1} \to \mathbb{R}$ as follows:*

$$f_{i,y}(x_1, \dots, x_{n-1}) := \widetilde{f}(x_1, \dots, x_{i-1}, y, x_i, \dots, x_{n-1}),$$

where y is inserted in the i-th position of the function argument, replacing the i-th variable. Then functions

$$\varphi(y) := I_{\mathrm{low}}(f_{i,y}, (-C, C)^{n-1}) \qquad \text{and} \qquad \psi(y) := I_{\mathrm{up}}(f_{i,y}, (-C, C)^{n-1}),$$

which satisfy $\varphi \le \psi$, are both Riemann integrable (in $\mathbb{R}$) and

$$\int_{-C}^{C} \varphi(y)\, dy = \int_{-C}^{C} \psi(y)\, dy = \int_A f.$$

Proof. Is a variation of the proof of Theorem 13.37. Left to the interested reader. □

THEOREM 13.39: FUBINI IN BOXES

Suppose that $f : K \to [0, \infty)$ *is a continuous function in the compact box*

$$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Then,

$$\begin{aligned}\int_K f(x)dx &= \int_{a_n}^{b_n} \left(\int_{a_{n-1}}^{b_{n-1}} \left(\cdots \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_n) dx_1 \right) dx_2 \cdots dx_{n-1} \right) dx_n \right) \\ &= \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_{n-1} dx_n.\end{aligned}$$

Moreover, we can integrate with respect to the different variables in any order: For any permutation σ *of* $\{1, 2, \ldots, n\}$*, we have:*

$$\int_{a_{\sigma(n)}}^{b_{\sigma(n)}} \cdots \int_{a_{\sigma(2)}}^{b_{\sigma(2)}} \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} f(x_1, x_2, \ldots, x_n)\, dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(n)}.$$ 23

Proof. It follows applying n times the slicing formula for functions, each time in one dimension less. □

13.40. — Observe that in the Slicing formula we have used the variable x_{n+1} to slice just for convenience, but we could have sliced in any other direction x_i. Indeed, if P is a permutation of the coordinates then we can apply our slicing results to $\widetilde{f} = f \circ P$ after noticing that $\int_{P^{-1}(A)} f \circ P = \int_A P$.

As a consequence in Fubini's Theorem the order in which we perfom the integrations is irrelevant. For example if $f = f(x_1, x_2, x_3)$ is a continuous function in $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ then

$$\begin{aligned}\int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3) dx_1 dx_2 dx_3 &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(x_1, x_2, x_3) dx_3 dx_1 dx_2 \\ &= \cdots = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3) dx_3 dx_2 dx_1\end{aligned}$$

24 Another very useful result is the following

Theorem 13.41: Differentiation under the integral sign

Suppose that $U \subset \mathbb{R}^n \times \mathbb{R}$ is open and that $f \in C^1(U)$. Let $K \subset \mathbb{R}^n$ be compact and Jordan measurable and suppose that $K \times [-a, b] \subset U$ for some $a < b$. Then for $y \in (a, b)$ the function

$$y \mapsto \int_K f(x,y)dx$$

is continuously differentiable in (a, b) and

$$\frac{d}{dy}\int_K f(x,y)dx = \int_K \frac{\partial}{\partial y} f(x,y)dx.$$

Proof. Fix $y \in (a, b)$ and let $h > 0$ be small enough so that $y + h \in (a, b)$. The function $\frac{\partial}{\partial y} f(x, y)$ is continuous and hence uniformly continuous in the compact $K \times [a, b]$.

On the other hand, by the Intermediate Value Theorem we have $f(x, y + h) - f(x, y) = \frac{\partial}{\partial y} f(x, \xi_{x,y,h})$ for some $\xi_{x,y,h}$ in the interval joining y and $y + h$.

Thus, for any given $\epsilon > 0$ exists $\delta > 0$ such that if $0 < |h| < \delta$ and $x \in K$ we have,

$$\left|\frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial}{\partial y} f(x,y)\right| = \left|\frac{\partial}{\partial y} f(x, \xi_{x,y,h}) - \frac{\partial}{\partial y} f(x,y)\right| < \epsilon.$$

24 Then, using the linearity of the integral

$$\frac{1}{h}\left(\int_K f(x,y+h)dx - \int_K f(x,y)dx\right) = \int_K \frac{f(x,y+h)-f(x,y)}{h}dx \tag{13.8}$$

but

$$\begin{aligned}\left|\int_K \frac{f(x,y+h)-f(x,y)}{h}dx - \int_K \frac{\partial}{\partial y} f(x,y)dx\right| \leq \\ \leq \int_K \left|\frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial}{\partial y} f(x,y)\right| dx \leq \epsilon \mu_n(K)\end{aligned} \tag{13.9}$$

Sending h (and also hence δ, ϵ) to zero we find

$$\lim_{h\to 0}\int_K \frac{f(x,y+h)-f(x,y)}{h}dx = \int_K \frac{\partial}{\partial y} f(x,y)dx$$

But then in view of (13.8) we obtain that $\frac{d}{dy}\int_K f(x,y)$ exists and equals $\int_K \frac{\partial}{\partial y} f(x,y)dx$. That the function $y \mapsto \int_K \frac{\partial}{\partial y} f(x,y)dx$ is continuous follows from (13.9). □

Exercise 13.42. — Show that Theorem 13.41 still holds (with the same proof) if we replace the assumption $f \in C^1(U)$ by the assumption that $f \in C^0(U)$ and $\partial_y f$ exists and is continuous in U.

13.4 Examples: Change of variables and Fubini in practice

Integrals polar coordinates in $\mathbb{R}^2$

Cartesian coordinates (x, y) and polar coordinates (r, θ) in $\mathbb{R}^2$ are related by the transformation:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \theta \in [0, 2\pi), \quad r \geq 0,$$

in other words

$$(x, y) = \Psi(r, \theta) := (r\cos\theta, r\sin\theta).$$

The Jacobian matrix of this transformation is

$$J_{\text{polar}} = J\Psi(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

and the determinant of this Jacobian matrix is:

$$\det(J_{\text{polar}}) = r$$

Example 13.43. — To compute the integral of the function $(x, y) \mapsto e^{-x^2-y^2}$ in the ball of radius one$B_1 \subset \mathbb{R}^2$ we notice that that the set Ψ is a diffeomorphism when restricted to
24 $V := (0, 2\pi) \times (0, 1))$ and that $U := \Psi((0, 2\pi) \times (0, 1))$ is such that $B_1 \setminus U$ is a null set.

Hence the change of variables formula gives

$$\begin{aligned} \int_{B_1} e^{-x^2-y^2} dxdy = \int_U e^{-x^2-y^2} dxdy = \int_V e^{-(r\cos\theta)^2-(r\sin\theta)^2} rdrd\theta \\ = \int_0^1 \int_0^{2\pi} e^{-r^2} rdr = \pi \int_0^1 e^{-r^2} 2rdr = \pi(1 - e^{-1}). \end{aligned}$$

Integrals in Spherical Coordinates in $\mathbb{R}^3$

Cartesian coordinates (x, y, z) and spherical coordinates (r, ϕ, θ) in $\mathbb{R}^3$ are related by the transformation:

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi.$$

The Jacobian matrix of this transformation is:

$$J_{\text{spherical}} = \begin{bmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix},$$

and the determinant of this Jacobian matrix is:

$$\det(J_{\text{spherical}}) = \rho^2 \sin\phi.$$

Example 13.44. — To compute the volume of a sphere of radius R, we integrate in spherical coordinates:

$$\text{Volume} = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{4}{3}\pi R^3.$$

Cylindrical Coordinates in $\mathbb{R}^3$

Cylindrical coordinates are a generalization of polar coordinates to three dimensions where a point in space is represented by (r, θ, z). Here, r and θ have the same interpretation as in polar coordinates, representing the radial distance from the origin and the angle from a reference direction in the plane, respectively, while z represents the height above the plane.

The transformation from Cartesian coordinates (x, y, z) to cylindrical coordinates is given by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The Jacobian matrix for this transformation is:

$$J_{\text{cylindrical}} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

24 and the determinant of this Jacobian matrix is:

$$\det(J_{\text{cylindrical}}) = r.$$

In cylindrical coordinates, the volume element is $r \, dr \, d\theta \, dz$, which is used for integrating functions over a volume in $\mathbb{R}^3$.

Example 13.45. — To compute the volume of a right circular cone in cylindrical coordinates, consider a cone with height h and base radius R. The cone is parametrized as follows:

$$0 \le r \le R, \quad 0 \le \theta < 2\pi, \quad 0 \le z \le \frac{h}{R}r.$$

The volume element in cylindrical coordinates is $r \, dr \, d\theta \, dz$, so the volume V of the cone can be computed by the integral:

$$\text{Volume} = \int_0^{2\pi} \int_0^{R} \int_0^{\frac{h}{R}r} r \, dz \, dr \, d\theta.$$

First, integrate with respect to z:

$$\text{Volume} = \int_0^{2\pi} \int_0^{R} r \left[\frac{h}{R}r\right] dr \, d\theta = \frac{h}{R} \int_0^{2\pi} \int_0^{R} r^2 \, dr \, d\theta.$$

Next, integrate with respect to r:

$$\text{Volume} = \frac{h}{R}\int_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^R d\theta = \frac{h}{R}\int_0^{2\pi}\frac{1}{3}R^3\,d\theta = \frac{h}{R}\cdot\frac{1}{3}R^3\cdot 2\pi.$$

which is the classical formula for the volume of a cone.

Volume Computation of a Torus

Consider a torus with a major radius R and a minor radius r. To compute the volume, we parametrize the torus and calculate the Jacobian determinant for the transformation from toroidal to Cartesian coordinates.

The torus can be parametrized by:

$$\begin{aligned} x &= (R+\rho\cos\theta)\cos\varphi, \\ y &= (R+\rho\cos\theta)\sin\varphi, \\ z &= \rho\sin\theta, \end{aligned}$$

where $0 \le \rho \le r$, $0 \le \theta < 2\pi$, and $0 \le \varphi < 2\pi$.

To transform from toroidal coordinates (ρ, θ, φ) to Cartesian coordinates (x, y, z), we compute the Jacobian determinant. The partial derivatives of x, y, z with respect to ρ, θ, φ form the Jacobian matrix: 24

$$J = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi & -(R+\rho\cos\theta)\sin\varphi \\ \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & (R+\rho\cos\theta)\cos\varphi \\ \sin\theta & \rho\cos\theta & 0 \end{bmatrix}.$$

The Jacobian determinant $|\det J|$ is then computed as:

$$\begin{aligned} |\det J| &= \rho\left((R+\rho\cos\theta)(\cos^2\theta+\sin^2\theta)\right) \\ &= \rho(R+\rho\cos\theta). \end{aligned}$$

Using this Jacobian, the volume of the torus is given by integrating over the parametric domain:

$$\text{Volume} = \int_0^{2\pi}\int_0^{2\pi}\int_0^r \rho(R+\rho\cos\theta)\,d\rho\,d\theta\,d\varphi.$$

First, we solve the integral with respect to ρ:

$$\int_0^r \rho(R+\rho\cos\theta)\,d\rho = \int_0^r (R\rho+\rho^2\cos\theta)\,d\rho.$$

This evaluates to

$$\left[\frac{1}{2}R\rho^2+\frac{1}{3}\rho^3\cos\theta\right]_0^r = \frac{1}{2}Rr^2+\frac{1}{3}r^3\cos\theta.$$

Now, integrate with respect to θ:

$$\int_0^{2\pi} \left(\frac{1}{2}Rr^2 + \frac{1}{3}r^3\cos\theta\right) d\theta = \pi R r^2,$$

because the integral of $\cos\theta$ over one period is 0.

Finally, integrate with respect to φ:

$$\int_0^{2\pi} \pi R r^2 \, d\varphi = 2\pi^2 R r^2.$$

Thus, the volume of the torus is $2\pi^2 Rr^2$.

13.5 Improper integrals: definition and examples

Definition 13.46:

Suppose that $U \subset \mathbb{R}^n$ is open and $f \in C^0(U)$ is nonnegative. We define the **improper integral** of f over U as $\int_U f$ as

$$\sup\left\{\int_K f \mid K \text{ compact, Jordan measurable, and contained in } U\right\}.$$

24 Notice that $\int_U f$ is always well-defined, although it can be $+\infty$.

Lemma 13.47: Improper integrals as limits

Suppose that $U \subset \mathbb{R}^n$ is open and $f \in C(U)$ is nonnegative.
Assume that K_ℓ, $\ell \geq 0$ is a increasing sequence of nested compact Jordan measurable sets with

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

such that

$$\bigcup_{\ell=0}^{\infty} \operatorname{int}(K_\ell) = U.$$

Then

$$\int_U f = \lim_{\ell\to\infty} \int_{K_\ell} f.$$

(In particular the limit exists.)

Proof. Notice that since

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

and $f \geq 0$ we have $\int_{K_\ell} f \leq \int_{K_{\ell+1}} f$ for all $\ell \geq 0$. Hence the sequence of integrals is monotonone and thus has a limit.

Nowe, for any $K \subset U$ compact and Riemann integrable, since $K \subset \bigcup_{\ell=0}^{\infty} \operatorname{int}(K_\ell)$ is an open cover of K (and the sets are nested) there exists $K_{\ell_\circ}$ such that $K \subset K_{\ell_\circ}$.

But then

$$\int_K f \le \int_{K_{\ell_0}} f \le \lim_{k\to\infty} \int_{K_\ell} f$$

Taking the supremum over $K \subset U$ compact and Jordan measurable we obtain

$$\int_U f \le \lim_{k\to\infty} \int_{K_\ell} f.$$

The opposite inequality is trivial as each K_ℓ is an admissible set in the supremum defining the improper integral. □

24

Example 13.48. — Let us compute the integral of the $I := \int_{\infty}^{\infty} e^{-x^2}dx = \lim_{A\to+\infty} \int_{-A}^{A} e^{-x^2}dx$.

To compute it, we can use the following classical trick of doubling the variables:

$$\left(\int_{-A}^{A} e^{-x^2}dx\right)^2 = \int_{-A}^{A} e^{-x^2}dx \int_{-A}^{A} e^{-y^2}dy = \int_{[-A,A]^2} e^{-x^2-y^2}dxdy,$$

and hence, sending $A \to +\infty$,

$$\begin{aligned} I^2 &= \int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \lim_{R\to\infty} \int_{B_R} e^{-x^2-y^2}dxdy = \lim_{R\to\infty} \int_0^R \int_0^{2\pi} e^{-r^2} r d\theta dr \\ &= \pi \lim_{R\to\infty} \int_0^R e^{-r^2} 2r dr = \pi. \end{aligned}$$

13.6 Length, Area, and integrals over submanifolds

The following definition is standard, convenient, and will be used from now on.

Definition 13.49: C^k functions on non-open domains

25 If $A \subset \mathbb{R}^n$ is not open we say that $f \in C^k(A, \mathbb{R}^m)$ if there exist $U \subset A$ open and $\tilde{f} \in C^k(U, \mathbb{R}^m)$ such that $\tilde{f} = f$ when restricted to A.

(In practice, assuming $f \in C^k(A, \mathbb{R}^n)$ is is the same as assuming that $f \in C^k(U, \mathbb{R}^n)$ for some U open containing A, but it is shorter to write.)

Here only length with total variation formula. Prove finite for Lipschitz curve.

Definition 13.50: Length of a C^1 curve

Let $\gamma \in C^1([a,b],\mathbb{R}^n)$ be a curve. We define the length of γ as

$$L(\gamma) := \int_a^b |\gamma'(t)|dt,$$

where the vector $\gamma'(t) = (\gamma_1'(t),\dots,\gamma_n'(t))$ is the **velocity** and the number $|\gamma'(t)| = \sqrt{\sum_{i=1}^n (\gamma_i'(t))^2}$ is the **speed** of the path at time t.

Exercise 13.51. — Show that if $s\colon [c,d] \to [a,b]$ is a C^1 bijective map with C^1 inverse, then $L(\gamma \circ s) = L(\gamma)$.

The following result brings some intuition into our definition of length

Theorem 13.52: Length as total variation

Let $\gamma \in C^1([a,b],\mathbb{R}^n)$ be a curve and define

$$V(\gamma) := \sup\Big\{\sum_{j=0}^N |\gamma(t_{j+1}) - \gamma(t_j)| : N \in \mathbb{N}, a = t_0 \le t_1 \le \ldots \le t_{N-1} \le t_N = b\Big\}.$$

25 *Then $L(\gamma) = V(\gamma)$.*

Proof. (Extra material.) We start showing $L(\gamma) \ge V(\gamma)$. Denote $\gamma(t) = (\gamma_1(t),\dots,\gamma_n(t))$, fix $N \in \mathbb{N}$, some partition $0 = t_0 \le t_1 \le \ldots \le t_{N-1} \le t_N = 1$ and any set of N unit vectors in $\nu_1,\dots,\nu_N \in \mathbb{R}^n$:

$$\begin{aligned}\int_0^1 |\gamma'(s)|\,ds = \sum_{j=0}^N \int_{t_j}^{t_{j+1}} |\gamma'(s)|\,ds &\ge \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \nu_j \cdot \gamma'(s)\,ds\\ &= \sum_{j=0}^N \big(\nu_j \cdot \gamma(t_{j+1}) - \nu_j \cdot \gamma(t_j)\big),\end{aligned}$$

where we used that $\frac{d}{ds}(\nu \cdot \gamma(s)) = \nu \cdot \gamma'(s))$ and the Fundamental Theorem of Calculus.

With the choice

$$\nu_j := \frac{\gamma(t_{j+1}) - \gamma(t_j)}{|\gamma(t_{j+1}) - \gamma(t_j)|}$$

the last term becomes

$$\sum_{j=0}^N \nu_j \cdot \big(\gamma(t_{j+1}) - \gamma(t_j)\big) = \sum_{j=0}^N |\gamma(t_{j+1}) - \gamma(t_j)|,$$

concluding the proof.

To prove the opposite inequality, let us show that of $L(\gamma) \le V(\gamma) + \varepsilon$ for all $\varepsilon > 0$ small.

Observe that the function $\gamma' : [a,b] \to \mathbb{R}^n$ is uniformly continuous (it is continuous on a compact) so there is $\delta > 0$ such that if $s,t \in [a,b]$ and $|s-t| < \delta$ we have

$$|\gamma'(s) - \gamma'(t)| < \frac{\varepsilon}{2\sqrt{n}(b-a)}$$

Choose a partition $a = t_0 < t_1 < t_2 < \ldots < t_{N-1} \leq t_N = b$ of $[a,b]$ such that $t_j - t_{j-1} < \delta$ and $\xi_j \in [t_{j-1}, t_i]$ for $1 \leq i \leq N$ such that

$$L(\gamma) < \sum_{j=1}^{N} (t_j - t_{j-1})|\gamma'(\xi_j)| + \frac{\varepsilon}{2}. \tag{13.10}$$

By the Mean Value Theorem for all $1 \leq i \leq n$ and $1 \leq j \leq N$ we have

$$\gamma_i(t_j) - \gamma_i(t_{j-1}) = \gamma_i'(\eta_{i,j})(t_j - t_{j-1}),$$

for some $\eta_{i,j} \in [t_{j-1}, t_j]$. But then since $|\eta_{i,j} - \xi_j| < \delta$

$$|\gamma_i'(\eta_{i,j}) - \gamma_i'(\xi_j)| \leq |\gamma'(\eta_{i,j}) - \gamma'(\xi_j)| \leq \frac{\varepsilon}{2\sqrt{n}(b-a)},$$

Therefore,

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$$|\gamma_i'(\xi_j)|(t_j - t_{j-1}) \leq |\gamma_i(t_j) - \gamma_i(t_{j-1})| + \frac{\varepsilon}{2\sqrt{n}(b-a)}$$

and thus (summing the squares for $i = 1, \ldots n$)

$$|\gamma'(\xi_j)|(t_j - t_{j-1}) \leq |\gamma(t_j) - \gamma(t_{j-1})| + \frac{\varepsilon(t_j - t_{j-1})}{2(b-a)}.$$

Inserting this in (13.10) we obtain:

$$L(\gamma) < \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})| + \varepsilon \leq V(\gamma) + \varepsilon,$$

which concludes the proof. □

For the discussion that follows it is convenient to recall the symmetries of the Euclidean space:

Definition 13.53: Euclidean isometries (or rigid motions)

A map $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form $F(p) = Rp + a$ where R is a orthogonal $n \times n$ matrix and $a \in \mathbb{R}^n$, is termed **Euclidean isometry** or **rigid motion**.

Notice that Euclidean are the maps that preserve the Euclidean distance: $d(F(p), F(q)) = d(p,q)$ for every pair of points $p, q \in \mathbb{R}^n$

13.54. — Besides Theorem (13.52), the appropriateness of $L(\gamma)$ as a definition of length is supported by the following properties:

1. **Invariance under Reparametrization:** $L(\gamma)$ remains unchanged under any smooth and bijective reparametrization of γ, indicating that it fundamentally depends on the path's image, not on the parametrization. This property confirms that $L(\gamma)$ reflects the geometric nature of the curve.

2. **Invariance under Euclidean Isometries:** For any rigid motion $F : \mathbb{R}^n \to \mathbb{R}^n$, characterized by $|F(x) - F(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$, it holds that $L(F \circ \gamma) = L(\gamma)$.

   Indeed, since $F(p) = Rp + a$, where R is an orthogonal matrix, the chain rule gives $|(F \circ \gamma)'(t)| = |R\gamma'(t)| = |\gamma'(t)|$ for all t.

3. **Additivity:** The length function L is additive. Specifically, if $\gamma : [a, b] \to \mathbb{R}^n$ and $c \in (a, b)$, then $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$.

4. **Normalization** The lenght of the segment $[0, 1] \times \{0\} \times \cdots \times \{0\}$ is 1.

These properties serve as the foundational guidelines for defining the area of surfaces (i.e. 2-dimensional submanifolds) and, more generally, the d-volume for d-dimensional submanifolds.

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Definition 13.55: Gram determinant

Suppose that $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map where with $m < n$ (equivalently a $n \times m$ matrix acting on column m-vectors by matricial multiplication).
We define the Gram determinant of L as the square root of the determinant of the $d \times d$ matrix LL^T, that is:

$$\sqrt{\det(L^T L)}$$

13.56. — If L is a $n \times m$ matrix we can essentially repeat the proof of the polar decomposition theorem starting from the nonnegative definite $m \times m$ matrix LL^T. Doing so, we find that $L = R_2 S R_1$ where R_1 is $m \times m$ orthogonal S is $m \times m$ diagonal with nonnegative entries. Now, R_2 is and $n \times d$ matrix satisfying $R_2^T R_2 = \mathrm{I}_n$. In other words, the columns of R_2 correspond to the first m vectors of an orthonomal basis of $\mathbb{R}^n$.

In other words any linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ can as a composition of:

1. A linear map $R_1 : \mathbb{R}^m \to \mathbb{R}^m$ that preserves the lengths of vectors (i.e., an Euclidean isometry).

2. A linear map $S : \mathbb{R}^m \to \mathbb{R}^m$ that is diagonal with nonnnegative entries (it gives the 'streching factors')

3. A linear map $R_2 : \mathbb{R}^m \to \mathbb{R}^n$ that preserves the lengths of vectors (i.e., an Euclidean isometry).

In this setting, the Gram determinant is the determinant of the ‘stretching matrix’ S.

Definition 13.57: m-volume of parametrized m-submanifold

Let ϕ be a parametrized m-dimensional submanifold: more precisely suppose that $V \subset \mathbb{R}^m$ open and $\phi \in C^1(V, \mathbb{R}^n)$ is such that $J\phi(x)$ has rank m for all $x \in V$— we define the m-volume of ϕ as

$$\mathrm{vol}_m(\phi) = \int_V \sqrt{\det(J\phi^T J\phi)}(x)\,dx$$

13.58. — Notice the integral $\int_V \sqrt{\det(J\phi^T J\phi)}(x)\,dx$ is always well-defined, at least in the improper sense.

Lemma 13.59: reparametrization invariance of m-volume

Assume that $\psi : U \to V$ is an any C^1 diffeomorphism. Then,

$$\mathrm{vol}_m(\phi) = \mathrm{vol}_m(\phi \circ \psi)$$

Proof. By the chain rule

$$J(\phi \circ \psi)(x) = J\phi(\psi(x))J\psi(x)$$

25 hence

$$\det\left(J\psi(x)^T J\phi(\psi(x))^T J\phi(\psi(x))J\psi(x)\right) = \det(J\psi(x)^T)\det\left(J\phi(\psi(x))^T J\phi(\psi(x))\right)\det(J\psi(x))$$

and the Gram determinant for $J(\phi \circ \psi)(x)$ equals

$$\sqrt{\det\big(J\psi(x)^T J\phi(\psi(x))^T}|\det J\psi(x)|dx.$$

But by the change of variables formula we obtain (recall $\psi(U) = V$):

$$\begin{aligned}\mathrm{vol}_m(\phi \circ \psi) &= \int_U \sqrt{\det\big(J\phi(\psi(x))^T J\phi(\psi(x))}|\det\psi(x)|\,dx \\ &= \int_{\psi(U)} \sqrt{\det\big(J\phi(y)^T J\phi(y)}\,dy = \mathrm{vol}_m(\phi).\end{aligned}$$

In this proof we are assuming that the integrals are well-defined in Riemann sense (as it will always be the case in practice). However, it is not difficult to show that a small modifiaction of this proofs applies when the integrals are defined in the improper sense. □

Exercise 13.60. — Prove that if F is a rigid motion (Euclidean isometry), then the d-dimensional volume is invariant under F, i.e., $\mathrm{vol}_d(F \circ \phi) = \mathrm{vol}_d(\phi)$. Additionally, discuss how the additivity and normalization properties can be reinterpreted in the context of the d-dimensional volume.

Exercise 13.61. — Show that the length of a curve is nothing but its vol_d for $d = 1$.

Interlude: Vector Product in $\mathbb{R}^3$

Recall that given two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, the cross product (also known as the vector product) is defined as

$$a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

25 It is customary to compute the cross product as the formal determinant

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\mathbf{i} = e_1$, $\mathbf{j} = e_2$, and $\mathbf{k} = e_3$.
This determinant can be computed using the rule of Sarrus as follows:

$$a \times b = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

We also have the following identity:

$$|a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2. \tag{13.11}$$

This identity can be established by direct computation:

$$|a \times b|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$|a|^2|b|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

25

$$(a \cdot b)^2 = (a_1b_1 + a_2b_2 + a_3b_3)^2.$$

Another useful and well-known identity involving the cross product (which can also be established by direct computation) is:

$$(a \times b) \cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This determinant gives the volume of the parallelepiped spanned by the vectors a, b, and c, and it is zero if and only if the vectors are coplanar.

From now on, and whenever no confusion between the linear map Df_x and its matrix $Jf(x)$ in the standard basis is possible, we will also use the (standard) notation $Df(x)$ to refer to the Jacobi matrix.

Lemma 13.62: Surface area

In the case $d = 2$ and $n = 3$, if $\phi : V \to \mathbb{R}^3$ then its ***area*** *can be computed as:*

$$\begin{aligned} A(\phi) := \mathrm{vol}_2(\phi) &= \int_U \sqrt{\langle \partial_1\phi, \partial_1\phi\rangle \langle \partial_2\phi, \partial_2\phi\rangle - \langle \partial_1\phi, \partial_2\phi\rangle^2}(x)\, dx_1 dx_2 \\ &= \int_U |\partial_1\phi \times \partial_2\phi|(x)\, dx_1 dx_2 \end{aligned}$$

where we integrate the Euclidean modulus of $\partial_1\phi \times \partial_2\phi : U \to \mathbb{R}^3$: the function that maps x to the cross (or vector) product of the two vector $\partial_1\phi(x)$ and $\partial_2\phi(x)$.

Proof. The Jacobi matrix $D\phi$ of ϕ at a point $x \in U$ is a 3×2 matrix given by:

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$$D\phi(x) = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \\ \partial_1\phi_3 & \partial_2\phi_3 \end{bmatrix}.$$

The transpose of $D\phi(x)$, denoted $D\phi^T$, is a 2×3 matrix:

$$D\phi^T = \begin{bmatrix} \partial_1\phi_1 & \partial_1\phi_2 & \partial_1\phi_3 \\ \partial_2\phi_1 & \partial_2\phi_2 & \partial_2\phi_3 \end{bmatrix}.$$

The Gram matrix is the product of $D\phi(x)^T$ and $D\phi(x)$:

$$D\phi^T D\phi = \begin{bmatrix} \partial_1\phi \cdot \partial_1\phi & \partial_1\phi \cdot \partial_2\phi \\ \partial_2\phi \cdot \partial_1\phi & \partial_2\phi \cdot \partial_2\phi \end{bmatrix}$$

which is a 2×2 matrix representing the inner products of the partial derivatives.

Hence, the determinant of this matrix is:

$$\det(D\phi(x)^T D\phi(x)) = (\partial_1\phi \cdot \partial_1\phi)(\partial_2\phi \cdot \partial_2\phi) - (\partial_1\phi \cdot \partial_2\phi)^2 = |\partial_1\phi|^2|\partial_2\phi|^2 - (\partial_1\phi \cdot \partial_2\phi)^2$$

But using the identity (13.11) the right hand side is equal to $|\partial_1\phi \times \partial_2\phi|^2$, and hence the lemma follows.

□

Example 13.63. — Consider the parameterization of a sphere of radius r in $\mathbb{R}^3$ given by:

$$\psi(\theta, \varphi) = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$ and $0 \leq \varphi < 2\pi$.

To compute the area of the sphere, we first find the partial derivatives of ψ:

$$\partial_\theta\psi = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix}$$

and

$$\partial_\varphi\psi = \begin{pmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix}.$$

The Gram matrix is given by

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$$\begin{bmatrix} \partial_\theta\psi \cdot \partial_\theta\psi & \partial_\theta\psi \cdot \partial_\varphi\psi \\ \partial_\varphi\psi \cdot \partial_\theta\psi & \partial_\varphi\psi \cdot \partial_\varphi\psi \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{bmatrix}$$

Hence the gram determinant (the square root of the determinant of the matrix above) is

$$r^2\sin\theta.$$

Finally, the area A of the sphere is given by:

$$A = \int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\,d\theta\,d\varphi = 4\pi r^2,$$

the expected result.

Definition 13.64: Support of a function

Let $U \subset \mathbb{R}^n$ an open set and $f : U \to \mathbb{R}$ a continuous function. We define the **support** of f, denoted $\operatorname{spt}(f) \subset \mathbb{R}^n$ as the set

$$\operatorname{spt}(f) = \overline{\{x \in U \mid f(x) \neq 0\}}.$$

If $\operatorname{spt}(f)$ is bounded and contained in U then we hay that f is **compactly supported in** U.

Lemma 13.65: Extension of compactly supported functions

If $U \subset \mathbb{R}^n$ is open $f : U \to \mathbb{R}$ is of class C^k for some $k \geq 0$ continuous and compactly supported in U, then the function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{for } x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$$

belongs to $C^k(\mathbb{R}^n)$.

Proof. It is clear that $\widetilde{f}$ has continuous k-th order partial derivatives inside U. And also in $\mathbb{R}^n \setminus \overline{U}$, since $\widetilde{f} \equiv 0$ in this open set.

Hence to prove that $\widetilde{f}$ is of class C^k we only need to show that it has continuous k-th order partial derivatives across the boundary ∂U. But since $\mathrm{spt}(f)$ is closed and contained in U, for every point in $x_\circ \in \partial U$ there is $r > 0$ such that $\tilde{f} \equiv 0$ $B_r(x_\circ)$. Hence, that $\widetilde{f}$ and all of its
26 partial derivatives (all zero) are indeed continuous across ∂U. □

After we discuss the notion of d-volume for d-submanifolds we can accordingly define integrals of compatly supported continous function with respect to the d-volume measure

Definition 13.66: integration with respect to the m-volume

Assume that $M \subset \mathbb{R}^n$ a parametrized m-dimensional submanifold: that is, suppose that there are $V \subset \mathbb{R}^m$ open, and $\phi \in C^1(V, \mathbb{R}^n)$, such that $J\phi(x)$ has rank m for all $x \in V$, such that $M = \phi(V)$. Given $f : M \to \mathbb{R}$ continuous (where M is endowed with the restriction of the Euclidean metric) and with $\mathrm{spt}(f \circ \phi) \subset V$ we define

$$\int_M f \,\mathrm{dvol}_m = \int_M f(p)\,\mathrm{dvol}_m(p) := \int_V f \circ \phi \sqrt{\det(J\phi^T J\phi)(x)}\,dx$$

13.67. — We notice $f \circ \phi : V \to \mathbb{R}$ is continuous and $\mathrm{spt}(f \circ \phi) \subset \phi$. This implies (exercise) that $(f \circ \phi)$ is Riemann integrable over V.

Lemma 13.68: Invariances of integral with respect to m-volume

The integral $\int_M f \,\mathrm{dvol}_m$ as in Definition 13.66 is invariant under reparametrization of M and Euclidean isometries.

More precisely: let $M = \phi(V)$ be C^1 parametrized m-submanifold, where $V \subset \mathbb{R}^m$ open, and for $f : M \to \mathbb{R}$ continuous such that $\operatorname{spt}(f \circ \phi) \subset V$,

- *If $\Psi : U \to V$ is a C^1 diffeomorphism then*

$$\begin{aligned} \int_M f \,\mathrm{dvol}_m &= \int_V f \circ \phi \sqrt{\det(J\phi^T J\phi)(x)}\, dx \\ &= \int_U f \circ \phi \circ \psi \sqrt{\det(J(\phi \circ \psi)^T J(\phi \circ \psi)(y)}\, dy \end{aligned}$$

- *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is an Euclidean isometry then*

$$\int_{F(M)} f \circ F^{-1} \,\mathrm{dvol}_m = \int_M f \,\mathrm{dvol}_m.$$

26 *Proof.* The invariance under reparametrizations is proven exactly as in Lemma 13.59.

For the invariance under Euclidean isometry, let $F(p) = Rp + a$ where R is orthogonal and $a \in \mathbb{R}^n$. Notice that $F \circ \phi$ is a parametrization of $F(M)$ and $F \circ \phi$ is a parametrization of $F(M)$ and $(f \circ F^{-1}) : F(M) \to \mathbb{R}$ is continuous (being the composition of continuous maps). Then,

$$\int_{F(M)} f \circ F^{-1} \,\mathrm{dvol}_m = \int_V f \circ F^{-1} \circ F \circ \phi \sqrt{\det(J(F \circ \phi)^T J(F \circ \phi))(x)}\, dx.$$

But since $F(p) = Rp + a$, we have $JF(p) = R$ for all p. Therefore, by the chain rule:

$$J(F \circ \phi)(x) = RJ\phi(x).$$

Thus

$$J(F \circ \phi)(x)^T J(F \circ \phi)(x) = J\phi(x)^T R^T R J\phi(x) = J\phi(x)^T J\phi(x)$$

and the lemma follows. □

We finish the section showing how to compute $(n-1)$-volumes and integrals in the important case of $(n-1)$-dimensional graphs in $\mathbb{R}^n$.

Lemma 13.69: Volume element on graphical $(n-1)$-submanifolds

Let $V \subset \mathbb{R}^{n-1}$ open an let $g \in C^1(V)$. Consider the graphical $(n-1)$-submanifold $M = \{x \in \mathbb{R}^n \times \mathbb{R} \mid x_n = g(x_1, \dots, x_{n-1})\}$ with graphical parametrization

$$\phi(x) = (x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1}))^T.$$

Then,

$$\begin{aligned}\int_M f \, d\mathrm{vol}_{\mathrm{n-1}} = \int_M f(p) \, d\mathrm{vol}_{\mathrm{n-1}}(p) = \\ = \int_V (f \circ \phi)(x_1, \dots, x_{n-1}) \sqrt{1 + |\nabla g(x_1, \dots, x_{n-1})|^2} \, dx_1 \cdots d_{x_{n-1}}.\end{aligned}$$

In particular, for $f \equiv 1$ we obtain

$$\mathrm{vol}_{n-1}(M) = \int_V \sqrt{1 + |\nabla g(x_1, \dots x_{n-1})|^2} dx.$$

Proof. The Jacobian matrix of the parametrization $\phi : \mathbb{R}^{n-1} \to M$ defined by $\phi(x) = (x, g(x))$ is:

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$$J\phi(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1}(x) & \frac{\partial g}{\partial x_2}(x) & \cdots & \frac{\partial g}{\partial x_{n-1}}(x) \end{bmatrix}.$$

Computing the Gram matrix $G = J\phi(x)^T J\phi(x)$ we obtain:

$$G = \begin{bmatrix} 1 + \left(\frac{\partial g}{\partial x_1}(x)\right)^2 & \frac{\partial g}{\partial x_1}(x)\frac{\partial g}{\partial x_2}(x) & \cdots & \frac{\partial g}{\partial x_1}(x)\frac{\partial g}{\partial x_{n-1}}(x) \\ \frac{\partial g}{\partial x_2}(x)\frac{\partial g}{\partial x_1}(x) & 1 + \left(\frac{\partial g}{\partial x_2}(x)\right)^2 & \cdots & \frac{\partial g}{\partial x_2}(x)\frac{\partial g}{\partial x_{n-1}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_{n-1}}(x)\frac{\partial g}{\partial x_1}(x) & \frac{\partial g}{\partial x_{n-1}}(x)\frac{\partial g}{\partial x_2}(x) & \cdots & 1 + \left(\frac{\partial g}{\partial x_{n-1}}(x)\right)^2 \end{bmatrix}.$$

This may seem involved, but we can easily compute the determinant of the Gram matrix G by using the following trick.

For x fixed, put

$$w := \begin{bmatrix} \frac{\partial g}{\partial x_1}(x) & \frac{\partial g}{\partial x_2}(x) & \cdots & \frac{\partial g}{\partial x_{n-1}}(x) \end{bmatrix}^T$$

and choose some orthogonal $(n-1) \times (n-1)$ matrix R such that $Rw = |w|e_1$ (for example, any rotation sending $w/|w|$ to e_1)

Notice that, since $w^T R^T = |w|e_1^T$, we have

$$J\phi(x)R^T = \begin{bmatrix} \mathrm{Id} \\ w^T \end{bmatrix} R^T = \begin{bmatrix} R^T \\ |w|e_1^T \end{bmatrix} = \left[\begin{array}{cccc} \multicolumn{4}{c}{R^T} \\ \hline |w| & 0 & \cdots & 0 \end{array}\right].$$

Hence, $RGR^T = RJ\phi(x)^T J\phi(x)R^T$ is computed as:

$$\left[\begin{array}{c|c} & |w| \\ R & 0 \\ & \vdots \\ & 0 \end{array}\right] \left[\begin{array}{cccc} \multicolumn{4}{c}{R^T} \\ \hline |w| & 0 & \cdots & 0 \end{array}\right] = \begin{bmatrix} 1+|w|^2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

where we used $RR^T = \mathrm{I}_{n-1}$.

Since orthogonal matrices have determinant one, $\det(R) = 1$), we have:

26

$$\det(G) = \det(RGR^T) = 1 + |w|^2 = 1 + |\nabla g(x)|^2.$$

Hence, the 'volume element' on the manifold M is:

$$d\mathrm{vol_{n-1}} = \sqrt{\det(G)}\,dx = \sqrt{1 + |\nabla g(x)|^2}\,dx,$$

completing the proof. □

13.7 From local to global: partition of unity on compact submanifolds

In the previous section we discussed how to compute the d-volume (or the integral of a function) on 'pieces' of d-dimensional submanifold that can be described using a single parametrization. Here, we will see how can we define integrals over arbitrary compact submanifolds $M \subset \mathbb{R}^n$, which may not admit one single global parametrization.

To do so, it is convenient to introduce the so-called partitions of unity.

Lemma 13.70: Partition of unity

Let $K \subset \mathbb{R}^n$ be a compact set $\mathcal{B} = \{B_{r_1}(p_1), B_{r_2}(p_2) \dots, B_{r_N}(p_N)\}$ some finite collection of open balls (of $\mathbb{R}^n$) covering K , i.e., $K \subset \bigcup_{\ell=1}^N B_{r_\ell}(p_\ell)$.
Then there exists a collection of functions $\eta_\ell : \mathbb{R}^n \to [0, \infty)$ of class C^∞ such that $\operatorname{spt}(\eta_\ell) \subset B_{r_\ell}(p_\ell)$ for each ℓ and

$$\sum_{\ell=1}^N \eta_\ell(x) = 1 \quad \textit{for all } x \in U,$$

where U is some open set containing K.

Proof. We first show that for some $\theta \in (0,1)$ $B_{\theta r_\ell}(p_\ell)$ stills a covers of K. Indeed, for $k \geq 1$ integer $\theta_k = 1 - 2^{-k}$ and let $\Omega_k := \cup_{\ell=1}^N B_{\theta_k r_\ell}(p_\ell)$. Observe that $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ and that

$$\bigcup_{k=1}^\infty \Omega_k = \cup_{\ell=1}^N B_{r_\ell}(p_\ell)$$

is an open cover of K. Hence, it has a finite subcover, implying that there exists $k_\circ$ such that

$$\Omega_{k_\circ} \supset K.$$

26 We can then take $\theta := \theta_{k_\circ}$.

Now, define, for $x \in \mathbb{R}^n$:

$$\Phi(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

$$\widetilde{\Phi}(x) = \begin{cases} \exp\left(\frac{-1}{|x|^2-1}\right) & \text{if } |x| > 1 \\ 0 & \text{if } |x| \leq 1, \end{cases}$$

and notice that $\Phi, \widetilde{\Phi} \in C^\infty(\mathbb{R}^n)$ and $\operatorname{spt}(\Phi) = \overline{B_1}$ and $\operatorname{spt}(\widetilde{\Phi}) = \mathbb{R}^n \setminus B_1$.

Define

$$\xi_\ell(x) := \Phi\Big(\frac{x - p_\ell}{\theta^{1/3} r_\ell}\Big)$$

$$\widetilde{\xi}_\ell(x) = \widetilde{\Phi}\Big(\frac{x - p_\ell}{\theta r_\ell}\Big) \qquad \text{and} \qquad \widetilde{\xi}(x) := \prod_{i=1}^N \widetilde{\xi}_\ell(x).$$

Notice that $\theta < \theta^{2/3} < \theta^{1/3} < 1$; that for $x \in \overline{B_{\theta^{2/3} r_\ell}(p_\ell)}$ we have

$$\xi_\ell(x) \geq \exp\left(\frac{-1}{1 - (\theta^{-1/3+2/3})^2}\right) > 0;$$

and that for $x \in \mathbb{R}^n \setminus \big(\cup_{1\le\ell\le N} B_{\theta^{2/3} r_\ell}(p_\ell)\big)$

$$\widetilde{\xi}(x) \geq \exp\left(\frac{-N}{(\theta^{-1+2/3})^2-1}\right) > 0.$$

On the other hand putting

$$U := \bigcup_{\ell=1}^{N} B_{\theta r_\ell}(p_\ell) \supset K$$

we have

26 $$\widetilde{\xi}_\ell = 0 \in B_{\theta_\ell}(p_\ell)$$

and hence their product vanishes in the union of the balls:

$$\widetilde{\xi} = 0 \in U.$$

Therefore,

$$\eta_\ell(x) := \frac{\xi_\ell(x)}{\sum_{\ell=1}^{N} \xi_\ell(x) + \widetilde{\xi}(x)}$$

are C^∞ functions (the denominator is always positive), and by construction $\eta_\ell \geq 0$, $\operatorname{spt}(\eta_i) = \overline{B_{\theta^{1/3} r_\ell}(x_i)} \subset B_{r_\ell}(x_i)$ and $\sum_{\ell=1}^{N} \eta_i = 1$ in $\cup_{1\le\ell\le N} B_{\theta r_\ell}(p_\ell) \supset K$ since $\widetilde{\xi}$ vanishes in this set. □

Definition 13.71: Partition of unity (for compact submanifolds of $\mathbb{R}^n$)

Let $M \subset \mathbb{R}^n$ be a compact C^k submanifold of dimension m, where $1 \le m < n$, and
27 $\mathcal{B} = \{B_{r_1}(p_1), B_{r_2}(p_2) \dots, B_{r_N}(p_N)\}$, with $p_\ell \in M$, some collection of open balls (of $\mathbb{R}^n$) covering M, i.e., with $M \subset \bigcup_{\ell=1}^{N} B_{r_\ell}(p_\ell)$.

We call any collection of N functions $\eta_\ell : \mathbb{R}^n \to [0,\infty)$ as in Lemma 13.70 (for $K = M$) a **partition of unity on M subordinated to $\mathcal{B}$**.

Definition 13.72: Graphical cover of a compact m-submanifold

Let $M \subset \mathbb{R}^n$ be a compact m-dimensional submanifold of class C^k, with $k \geq 1$. We call collection of N maps $\phi_\ell : V_\ell \to \mathbb{R}^n$, $1 \leq \ell \leq N$, of class C^k, a **graphical cover of** M if the following properties are satisfied:

1. For each ℓ, V_ℓ is an open set of $\mathbb{R}^m$, $r_\ell > 0$.

2. For each ℓ, there is an orthogonal $n \times n$ matrix R_ℓ such that $\phi_\ell : V_\ell \to \mathbb{R}^n$ is of the form
$$\phi_\ell(x_1, \dots x_m) = R_\ell(x_1, \dots, x_m, g_\ell(x_1, \dots, x_m))^T,$$
for some $g_\ell \in C^k(V_\ell, \mathbb{R}^{n-m})$.

3. For all ℓ, there is $p_\ell \in M$ and $r_\ell > 0$ such that
$$\phi_\ell(V_\ell) \cap B_{r_\ell}(p_\ell) = M \cap B_{r_\ell}(p_\ell) \quad \text{and} \quad B_{r_\ell}(p_\ell) \subset R_\ell(V_\ell \times \mathbb{R}^{n-m}).$$

4. The balls $B_{r_\ell}(p_\ell)$ cover M: i.e., $M \subset \bigcup_{\ell=1}^N B_{r_\ell}(p_\ell)$.

27 Proposition 13.73: Existence of graphical covers

Any compact m-dimensional C^k submanifold admits graphical covers.

Proof. Using Proposition 12.12 (4), for any given point $p_\circ \in M$ exists $V_{p_\circ} \subset \mathbb{R}^m$ open and $g_{p_\circ} \in C^1(V_{x_\circ}, \mathbb{R}^{n-m})$ and $R_{p_\circ}$, a permutation of the coordinates (which is in particular an orthogonal linear map), such that, for some open neighborhood U of $p_\circ$ (i.e., for some open set $U \subset \mathbb{R}^n$ containing $p_\circ$) we have
$$M \cap U_{p_\circ} = \phi_{p_\circ}(V_{p_\circ}),$$
where
$$\phi_{p_\circ}(x_1, \dots x_m) = R_{p_\circ}(x_1, \dots, x_m, g_\ell(x_1, \dots, x_m))^T$$

For each $p_\circ = \phi_{p_\circ}(y_\circ)$, pick $r_{p_\circ} > 0$ such that $B_{r_{p_\circ}}(p_\circ) \subset U$ and $B_{r_{p_\circ}}(y_\circ) \subset V_{p_\circ}$. (notice that the first is a ball of $\mathbb{R}^n$ while the second is a ball of $\mathbb{R}^m$)

Now, since M is compact and the collection of open balls $\left\{B_{r_{p_\circ}}(p_\circ)\right\}_{p_\circ \in M}$ covers M, there is a finite subcover. Let $p_1, p_2, \dots, p_N$ be the centers of the balls in the subcover. Then defining $r_\ell := r_{p_\ell}$, $V_\ell := V_{p_\ell}$ we obtain the wished graphical cover of M. □

Definition 13.74: Integral over a compact submanifold

Let $M \subset \mathbb{R}^n$ be an m-dimensional compact C^1 submanifold and $f : M \to \mathbb{R}$ a continuous function (where M is endowed with the restriction of the Euclidean metric).
Given a graphical cover $\phi_\ell : V_\ell \to \mathbb{R}^n$, $1 \le \ell \le N$, let $\eta_\ell : \mathbb{R}^n \to [0, \infty)$ be a partition of unity on M subordinated to the balls $\{B_{r_\ell}(p_\ell)\}_{1\le \ell \le N}$ associated to the graphical cover (see Definition 13.72). We define:

$$\begin{aligned}\int_M f\,\mathrm{dvol}_d &:= \sum_{\ell=1}^N \int_{M\cap B_\ell(x_\ell)} f\eta_\ell\,\mathrm{dvol}_d\\ &= \sum_{\ell=1}^N \int_{V_\ell} f(\phi_\ell(y))\eta_\ell(\phi_\ell(y))\sqrt{J\phi_\ell^T J\phi_\ell}\,(y)\,dy_1\dots dy_d.\end{aligned}$$

Lemma 13.75: Integral is independent of cover and partition of unity

Let $M \subset \mathbb{R}^n$ be an m-dimensional compact C^k submanifold and $f : M \to \mathbb{R}$ a continuous function.
Given two graphical covers $\phi_\ell : V_\ell \to \mathbb{R}^n$, $1 \le \ell \le N$, and $\tilde\phi_k : W_k \to \mathbb{R}^n$ of M, where $1 \le \ell \le N$ and $1 \le m \le \tilde N$; and corresponding partitions of unity η_ℓ, $\tilde\eta_k$. Then:

27

$$\sum_{\ell=1}^N \int_{V_\ell} (f\eta_\ell)\circ\phi_\ell \sqrt{J\phi_\ell^T J\phi_\ell} = \sum_{m=1}^{\tilde N} \int_{W_k} (f\tilde\eta_k)\circ\tilde\phi_k \sqrt{J\tilde\phi_k^T J\tilde\phi_k}$$

Proof. (Extra material) By definition of graphical cover we have

$$\phi_\ell(x_1,\dots x_d) = R_\ell(x_1,\dots,x_d,g_\ell(x_1,\dots,x_d))^T,$$

$$\tilde\phi_k(x_1,\dots x_d) = \tilde R_k(x_1,\dots,x_d,\tilde g_k(x_1,\dots,x_d))^T,$$

for suitable maps $g_\ell : V_\ell \to \mathbb{R}^{n-m}$, $\tilde g_k : W_k \to \mathbb{R}^{n-m}$ and orthogonal matrices R_ℓ, $\tilde R_k$.

Let $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ denote the standard projection onto the first d components:

$$(x_1,\dots,x_d,x_{d+1},\dots,x_n) \mapsto (x_1,\dots,x_d)$$

The key observation to prove this lemma is that, for all ℓ,

$$\phi_\ell^{-1}(x) = \pi_d(R_\ell^T x) \qquad \text{for all } x \in \phi_\ell(\overline{V}_\ell);$$

(recall that for orthogonal matrices $R^{-1} = R^T$), and similarly, for all m,

$$\tilde\phi_k^{-1}(x) = \pi_d(\tilde R_k^T x) \qquad \text{for all } x \in \tilde\phi_k(\overline{W}_k);$$

Notice that for every pair ℓ, k, define the open sets

$$V_{\ell,k} := \phi_\ell^{-1}(B_{s_k}(y_k)) \subset V_\ell \quad \text{and} \quad W_{k,\ell} := \tilde{\phi}_k^{-1}(B_{r_\ell}(x_\ell)) \subset W_k$$

and the 'transition' maps

$$\psi_{\ell,k} : V_{\ell,k} \to W_{k,\ell} \quad \text{and} \quad \psi_{k,\ell} : W_{k,\ell} \to V_{\ell,k}$$

respectively defined as

$$\psi_{k,\ell}(x) = \pi_d(\tilde{R}_k^T \phi_\ell(x)) \quad \text{and} \quad \psi_{\ell,k}(x) = \pi_d(R_\ell^T \tilde{\phi}_k(x)).$$

Notice that they are C^k diffeomorphisms. Indeed, on the one hand $\psi_{\ell,k}$ and $\psi_{k,\ell}$ are inverse to each other. On the other hand they are both C^k maps being the composition of a C^k maps, and a linear map.

Now, notice that

$$\sum_{\ell=1}^{N} \sum_{k=1}^{\tilde{N}} (\eta_\ell \tilde{\eta}_k)(x) = \left(\sum_{\ell=1}^{N} \eta_\ell(x) \right) \left(\sum_{m=1}^{\tilde{N}} \tilde{\eta}_k(x) \right) = 1 \quad \text{for all } x \in M$$

27 where each function $(\eta_\ell \tilde{\eta}_k) : \mathbb{R}^n \to [0, \infty)$ has compact support in $B_{r_\ell}(x_\ell) \cap B_{s_k}(y_\ell)$. As a consequence,

$$\operatorname{spt}\big((\eta_\ell \tilde{\eta}_k) \circ \phi_\ell\big) \subset V_{\ell,k}.$$

By these observations we can put:

$$\begin{aligned}\sum_{\ell=1}^{N} \int_{V_\ell} (f\eta_\ell) \circ \phi_\ell \sqrt{J\phi_\ell^T J\phi_\ell} &= \sum_{\ell=1}^{N} \int_{V_\ell} \left(\sum_{k=1}^{\tilde{N}} (f\eta_\ell \tilde{\eta}_k) \circ \phi_\ell \sqrt{J\phi_\ell^T J\phi_\ell} \right). \\ &= \sum_{\ell=1}^{N} \sum_{k=1}^{\tilde{N}} \int_{V_{\ell,k}} (f\eta_\ell \tilde{\eta}_k) \circ \phi_\ell \sqrt{J\phi_\ell^T J\phi_\ell}.\end{aligned}$$

But since

$$\phi_\ell = \tilde{\phi}_k \circ \psi_{\ell,k} \quad \text{in } V_{\ell,k},$$

and $\psi_{\ell,k}$ is a diffeomorphism, the invariace of the integral with respect to $d\mathrm{vol}_d$ under reparametrization (Lemmas 13.59 and 13.68) yields

$$\int_{V_{\ell,k}} (f\eta_\ell \tilde{\eta}_k) \circ \phi_\ell \sqrt{J\phi_\ell^T J\phi_\ell} = \int_{W_{k,\ell}} (f\eta_\ell \tilde{\eta}_k) \circ \tilde{\phi}_k \sqrt{J\tilde{\phi}_k^T J\tilde{\phi}_k}.$$

Then summing over all ℓ and k the lemma follows. □

Chapter 14

Global integral theorems

In this chapter, we deal with integral theorems for vector fields in $\mathbb{R}^n$, which are in some sense multidimensional generalizations of the fundamental theorem of integral and differential calculus.

14.1 The integration by parts formula

14.1.1 Bounded, smooth domains, exterior unit normal

Definition 14.1: Tangent and normal vectors

Let $M \subset \mathbb{R}^n$ by a m-dimensional C^1 submanifold. For a given point $p \in M$, we say that a vector $\tau \in \mathbb{R}^n$ is **tangent** to M at p if there exists a sequence $p_k \in M$ with $p_k \to p$ and $p_k \neq p$, and a sequence of positive number r_k converging to zero such that

$$\lim_k \frac{p_k - p}{r_k} \to \tau.$$

27 We say that a vector $\nu \in \mathbb{R}^n$ is **normal** (i.e., perpendicular) to M at p if

$$\tau \cdot \nu = 0 \quad \text{for all } \tau \text{ tangent to } M \text{ at } p.$$

Lemma 14.2: The tangent space: characterization for graphs

Let M be a m-dimensional submanifold of $\mathbb{R}^n$.
Assume that $\phi : V \to B_r(p_\circ)$ is a C^1 graphical parametrization of $M \cap B_r(p_\circ)$, for some $p_\circ \in M$ and let $p = \phi(y)$ be any point belongs to $M \cap B_r(p_\circ)$.
Then, $\tau \in \mathbb{R}^n$ is tangent to M at p if and only if τ belongs to the linear space $D\phi_y(\mathbb{R}^m) \subset \mathbb{R}^n$ (the range of the map $D\phi_y$).
In particular, the collection of all tangent vectors to M at given point $p \in M$ is a linear subspace of $\mathbb{R}^n$: the so-called the ***tangent space of*** *M at p.*

Proof. We have

$$\phi(y) = R(y, g(y))^T,$$

for some $g : V \to \mathbb{R}^{m-n}$ of class C^1.

Let $y \in V$ be the (unique) point mapped to p: that is $p = \phi(y)$

Let us show that first that if $\tau \in D\phi_y(\mathbb{R}^m)$ then τ is tangent to M at p. Indeed, let z be such that $D\phi_y(z) = \tau$. Then since ϕ is C^1, for any sequence $r_k > 0$ converging to 0 we have

$$\lim_{r_k \to 0} \frac{\phi(y + r_k z) - \phi(y)}{r_k} = D\phi_y(z)$$

Hence defining $p_k := \phi(y + r_k z) \to p$ we have shown

$$\lim_{r_k \to 0} \frac{p_k - p}{r_k} = D\phi_y(z) = \tau,$$

hence τ is tangent. This proves that every vector belonging to the image of $D\phi_y$ is tangent to M at p.

To prove the opposite implication, let $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ be the projection onto the first m-components $(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \mapsto (x_1, \dots, x_m)$

Notice that

$$(\pi_m \circ R^T)(\phi(y)) = y \quad \text{for all } y \in V.$$

27 Given any sequence $p_k \to p$ in $M \cap U$ such that

$$\lim_k \frac{p_k - p}{r_k} \to \tau,$$

for some $r_k > 0$ put $y_k := (\pi_m \circ R^T)(p_k)$.

Notice that $(\pi_m \circ R^T)$ is 1-Lipschitz, since it is the composition of orthogonal map and a projection. Therefore, the sequence

$$z_k := \frac{y_k - y}{r_k}$$

must be bounded, since

$$|z_k| = \frac{|y_k - y|}{r_k} \leq \frac{|p_k - p|}{r_k} \to |\tau| < \infty.$$

Hence z_k has accumulation points in $\mathbb{R}^n$. Let $z_{k_\ell} \to z \in \mathbb{R}^m$ as $\ell \to \infty$ be one of the accumulation points.

Then, using again that ϕ is C^1

$$\frac{p_k - p}{r_k} = \frac{\phi(y + r_k z_k) - \phi(y)}{r_k} = D\phi_y(z_k) + \frac{o(|r_k z_k|)}{r_k}.$$

Using this for $k = k_\ell$ and sending $\ell \to \infty$ we obtain

$$\tau = \lim_\ell \frac{p_{k_\ell} - p}{r_{k_\ell}} = \lim_\ell D_y\phi(z_{k_\ell}) = D\phi_y(z).$$

This proves that every tangent vector at p belongs to $D\phi_y(\mathbb{R}^m)$. □

Definition 14.3: Bounded C^k domain

Given $k \geq 1$, bounded open set $\Omega \subset \mathbb{R}^n$ is called **bounded C^k domain** if $M := \partial\Omega$ is a $(n-1)$-dimensional submanifold of class C^k.

Definition 14.4: Exterior and interior normal vectors

uppose that Ω is a bounded C^1 domain. A normal vector ν to $\partial\Omega$ at $x \in \partial\Omega$ is called **exterior** if

$$p + h\nu \in \mathbb{R}^n \setminus \overline{\Omega} \quad \text{for all } h > 0 \text{ sufficiently small.}$$

And it is called **interior** if

$$p + h\nu \in \mathbb{R}^n \subset \Omega \quad \text{for all } h > 0 \text{ sufficiently small.}$$

27 Proposition 14.5: Exterior unit normal for graphs

Suppose that Ω is a bounded C^1 domain. Put $y := (x_1, \ldots, x_{n-1})$ and assume that

$$\phi(y) = (y, g(y))^T,$$

where $V \subset \mathbb{R}^{n-1}$ is open and $g : V \to \mathbb{R}$, is a graphical C^1 parametrization of $M \cap B_r(p_\circ)$.
Suppose also that:

$$\Omega \cap B_r(p_\circ) = \{x = (y, x_n) \in \mathbb{R}^n \mid x_n < g(y)\} \cap B_r(p_\circ)$$

Then, for $p = (y, g(y))$, the vector

$$\nu(p) = \nu(y, g(y)) := \frac{(-\nabla g(y), 1)}{\sqrt{1 + |\nabla g(y)|^2}}$$

is normal, exterior, and has norm one.

By Lemma 14.2 tangent vectors at $p = (y, g(y))$ belong to the image of $D\phi_y$. In order words, are linear combinations of the column vectors in the matrix

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$$J\phi(y) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \partial_1 g(y) & \partial_2 g(y)(x) & \cdots & \partial_{n-1} g(y) \end{bmatrix}.$$

Proof. But

$$(-\partial_1 g(y), -\partial_2 g(y), \cdots, -\partial_{n-1} g(y), 1)^T$$

is has zero scalar product with each of this column vectors, so it is perpendicular.

Also, it is clear that ν has norm one, because we are dividing by the square root, that is precisely the norm of the vector in the numerator.

Finally to show that ν is points outwards notice that, for $h > 0$ small the point

$$(y - h\nabla g(y), g(y) + h)$$

does not belong to Ω because:

$$g(y - h\nabla g(y)) = g(y) + Dg_y(-\nabla g(y)h) + o(h) = g(y) - |\nabla g(y)|^2 h + o(h) < g(y) + h.$$

□

Definition 14.6: exterior unit normal map

If $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain, the continuous map $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ (where $\mathbb{S}^{n-1}$ is the unit $(n-1)$-sphere, i.e., $\partial B_1(0) \subset \mathbb{R}^n$) assigning to each point $p \in \partial\Omega$ the exterior normal vector at p is called **exterior unit normal map**.

28 14.7. — Notice that the fact that the exterior and interior unit normal are uniquely defined and are continuous follows from Proposition 14.5, since we showed that every bounded C^1 domain is locally graphical around each of its boundary points.

14.1.2 Integration by parts formula: special cases

An important special case of the integration by parts formula is the following:

Proposition 14.8: Special local integration by parts for graphs

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain. Suppose that $\Omega \cap B_r(p_\circ)$ is graphical: that is, there exists $R : \mathbb{R}^n \to \mathbb{R}^n$ orthogonal $V \subset \mathbb{R}^{n-1}$ open, and $g \in C^1(V, \mathbb{R})$ such that:

$$\Omega \cap B_r(p_\circ) = R\left\{(y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y \in V,\ x_n < g(y)\right\} \cap B_r(p_\circ).$$

Then for all $f \in C^1(\overline{\Omega \cap B_r(p_\circ)})$ with $\operatorname{spt}(f) \subset B_r(p_\circ) \subset R(V \times \mathbb{R})$ the following formula holds true for $w := Re_n$

$$\int_{\Omega \cap B_r(p_\circ)} \partial_w f \, d\mathrm{vol}_n = \int_{\partial\Omega \cap B_r(p_\circ)} f(p)(\nu(p) \cdot w) \, d\mathrm{vol}_{n-1}(p),$$

where $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ is the exterior unit normal map.

In the proof of Proposition 14.8 we will need the following variant of Fubini's theorem:

Lemma 14.9: Yet another variant of Fubini

Suppose that $\Omega \subset \mathbb{R}^n$ satisfies

$$\Omega \cap B_r(p_\circ) = \big\{(y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y \in V,\ x_n < g(y)\big\} \cap B_r(p_\circ).$$

Then for all $h \in C^0(\overline{\Omega \cap B_r(p_\circ)})$ with $\operatorname{spt}(h) \subset \overline{\Omega} \cap B_r(p_\circ) \subset \overline{\Omega} \cap (V \times \mathbb{R})$ we have

$$\int_{\Omega \cap B_r(p_\circ)} h(x)dx = \int_V \int_{-\infty}^{g(y)} h(y, x_n)dx_n dy,$$

where to define the integral on the right hand side we are considering h to be extended by zero in the complement of its support.

Proof. (extra material) Notice first that since the support of h is contained in the open ball $B_r(p_\circ)$ we f be zero and obtain a continuous function, still denoted h, defined in all the 'epigraph' of g, namely the set

$$\{(y, x_n) \in \mathbb{R}^n \mid x_n \le g(y)\}.$$

Now let us fix $C \in \mathbb{N}$ large enough so that

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$$\operatorname{spt}(h) \subset A := \{(y, x_n) \in \mathbb{R}^n \mid y \in V,\ -C \le x_n \le g(y)\}.$$

By definition of $\int_A h$, for each $\epsilon > 0$ there are dyadic step functions $h_{\text{low}} : \mathbb{R}^n \to R$ and $h_{\text{up}} : \mathbb{R}^n \to R$ such that

$$h_{\text{low}} \le h\mathbf{1}_A \le h_{\text{up}} \tag{14.1}$$

and

$$\int h_{\text{low}} \le \int_A h \le \int h_{\text{up}} \le \int h_{\text{low}} + \epsilon.$$

Consider $H_{\text{low}}, H_{\text{up}} : \mathbb{R}^{n-1} \to \mathbb{R}$, defined as

$$H_{\text{low}}(y) := \int_{\mathbb{R}} h_{\text{low}}(y, x_n)dx_n \quad \text{and} \quad H_{\text{up}}(y) := \int_{\mathbb{R}} h_{\text{up}}(y, x_n)dx_n$$

are notice that they are step functions in $\mathbb{R}^{n-1}$ (they are constant in every dyadic cube of the same pixel size as the one of h_{down} and h_{up}.

Also, from integrating (14.1) we obtain:

$$H_{\text{low}} \le H\mathbf{1}_V \le H_{\text{up}}$$

for $H : V \to \mathbb{R}$ defined as

$$H(y) := \int_{-C}^{g(y)} h(y, x_n)dx_n.$$

But notice also that, since h_{low} and h_{up} we have dyadic step functions

$$\int_{\mathbb{R}^{n-1}} H_{\text{low}} = \int_{\mathbb{R}^n} h_{\text{low}} \qquad \text{and} \qquad \int_{\mathbb{R}^{n-1}} H_{\text{up}} = \int_{\mathbb{R}^n} h_{\text{up}}$$

(because the integrals reduce to discrete sums).

Therefore we have shown that

$$\int_{\mathbb{R}^n} h_{\text{low}} = \int_{\mathbb{R}^{n-1}} H_{\text{low}} \leq \int_V H(y)dy \leq \int_{\mathbb{R}^{n-1}} H_{\text{up}} = \int_{\mathbb{R}^n} h_{\text{up}} \leq \int_{\mathbb{R}^n} h_{\text{low}} + \epsilon.$$

But then sending ϵ to zero we prove the lemma. □

We can now give the

Proof of Proposition 14.8. Notice first that it is enough to do the proof the in the 'special case' $R = \text{Id}$. Indeed, the general case simply follows from the 'special' one applied in 'a rotated Euclidan coordinate frame'.

More precisely we can apply the special case to $\widetilde{\Omega} = R^T(\Omega)$, $\widetilde{p}_\circ = R^T(p_\circ)$, $\widetilde{f} = f \circ R$, observing that :

- $\widetilde{\Omega} \cap B_r(\widetilde{p}_\circ) = R^T\big(\Omega \cap B_r(p_\circ)\big)$.
- $\partial_n \widetilde{f}(q) = D(f \circ R)_q(e_n) = (Df_{R(q)} \circ R)(e_n) = Df_{R(q)}(w) = \partial_w f(R(q))$, holds for all 28 $q \in R^T(\Omega \cap B_r(p_\circ))$.
- If $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ is the exterior normal vector for the domain Ω, then $\widetilde{\nu} = R^T \circ \nu \circ R$ is the exterior unit normal for $\widetilde{\Omega}$. Hence, $\tilde{\nu}(q) \cdot e_n = R^T\nu(R(q)) \cdot e_n = \nu(R(q)) \cdot Re_n = \nu(R(q)) \cdot w$.

Hence since the integrals are invariant under Euclidean isometries (by the Change of Variables Formula and Lemma 13.68), and R is an isometry, we obtain:

$$\int_{\Omega \cap B_r(p_\circ)} \partial_w f(p) d\text{vol}_n(q) = \int_{\widetilde{\Omega} \cap B_r(\widetilde{p}_\circ)} \partial_n \widetilde{f}(q) d\text{vol}_n(q)$$

and

$$\int_{\partial\Omega \cap B_r(p_\circ)} f(p)(\nu(p) \cdot w) d\text{vol}_{n-1}(p) = \int_{\partial\widetilde{\Omega} \cap B_r(\widetilde{p}_\circ)} \widetilde{f}(q)(\widetilde{\nu}(q)) d\text{vol}_{n-1}(q)$$

So from now on let us assume without loss of generality that $R = \text{Id}$. Applying Lemma 14.9 to the function $h = \partial_n f$ —conveniently extended by zero to the whole 'hypograph' of g, $\{(x_n, y) \in V \times \mathbb{R} \mid x_n \leq g(y),\ y \in V\}$—, we obtain:

$$\int_{\Omega \cap B_r(p_\circ)} \partial_n f(x) dx = \int_V \int_{-\infty}^{g(y)} \partial_n f(y, x_n) dx_n dy = \int_V f(y, g(y)) dy.$$

For the last equality we used the Fundamental Theorem of Calculus and that $f(y, -C) = 0$ for C large.

Hence, it only remains to show that

$$\int_V f(y,g(y))dy = \int_{\partial\Omega\cap B_r(p_\circ)} f(p)\nu_n(p)\,\mathrm{dvol}_{n-1}(p),$$

where $\nu_n = \nu\cdot e_n$ is the n-th component of the exterior normal vector.

But this follows from combining Lemma 13.69 and Proposition 14.5: indeed, using by Lemma 13.69 we have submanifold we have:

$$\int_{\partial\Omega\cap B_r(p_\circ)} f(p)\nu_n(p)\,\mathrm{dvol}_{n-1}(p) = \int_V f(y,g(y))\nu_n(y,g(y))\,\sqrt{1+|\nabla g(y)|^2}dy$$

but by Proposition 14.5

$$\nu_n(y,g(y))\,\sqrt{1+|\nabla g(y)|^2} = 1,$$

so the proposition follows □

Corollary 14.10: Local integration by parts for graphs

Let $\Omega\subset\mathbb{R}^n$ be a bounded C^1 domain. Suppose that $\Omega\cap B_r(p_\circ)$ is graphical: that is, there exists $R:\mathbb{R}^n\to\mathbb{R}^n$ orthogonal $V\subset\mathbb{R}^{n-1}$ open, and $g\in C^1(V,\mathbb{R})$ such that:

$$\Omega\cap B_r(p_\circ) = R\left\{(y,x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid y\in V,\ x_n<g(y)\right\}\cap B_r(p_\circ).$$

28

Then for all $f\in C^1(\overline{\Omega\cap B_r(p_\circ)})\subset R(V\times\mathbb{R})$ with $\mathrm{spt}(f)\subset B_r(p_\circ)$ and $w\in\mathbb{R}^n$ the following formula holds true:

$$\int_{\Omega\cap B_r(p_\circ)} \partial_w f\mathrm{dvol}_n = \int_{\partial\Omega\cap B_r(p_\circ)} f(p)(\nu(p)\cdot w)\,\mathrm{dvol}_{n-1}(p) \tag{14.2}$$

where $\nu:\partial\Omega\to\mathbb{S}^{n-1}$ is the exterior unit normal map.

Proof. The difference between Proposition 14.8 and Corollary 14.10 is that can take w to be any vector in $\mathbb{R}^n$, and not just Re_n.

However, we can deduce this from Proposition 14.8 with a ‘smart trick’.

Observe that the ‘integration by parts formula’ (14.2) is linear in w: if holds for $w=w_1$ and for $w=w_2$, then it also holds for $w=t_1w_1+t_2w_2$ where t_1,t_2 are real numbers. Hence, to establish (14.2) for all w it is enough to prove the formula for n linearly independent vectors.

To do so, fix $s\in(0,r)$ such that $\mathrm{spt}(f)\subset B_s(p_\circ)$ and let us show that there exists $\epsilon>0$ such that for any orthogonal linear transformation $O:\mathbb{R}^n\to\mathbb{R}^n$ satisfying $\|O-\mathrm{Id}\|_2<\epsilon$ (i.e., O is sufficiently close to the identity, where we measure the difference with the Hilbert-Schmidt norm) we have

$$\Omega\cap B_s(p_\circ) = OR\big(\{(y,x_n)\in\mathbb{R}^n\mid y\in W_O,\ x_n<g_O(y)\}\big)\cap B_s(p_\circ).$$

where $W_O \subset \mathbb{R}^{n-1}$ is a suitable open set and $g_O : W_O \to \mathbb{R}$ some C^1 map. In other words we are claiming that 'by continuity' $\partial\Omega \cap B_s(p_\circ)$ can be expressed a graph in slightly rotated *Euclidean coordinate frame*^{a}.

Indeed, put $W := \big\{y \in \mathbb{R}^{n-1} \mid (y, g(y)) \in B_s(p_\circ)\big\}$. W is an open set, beign the pre-image of an open set by a continuous map.

We first observe that, by Corollay 10.26, g is locally Lispchitz. Hence, the compact set $\overline{W} \subset V$ can be covered by finitely many open balls with g being Lipschitz in each of them. Hence, g is Lipchitz in the union of these balls, which is an open set containing $\overline{W}$.

Let Λ denote the Lipchitz constant: in particular we have

$$|g(y) - g(\bar{y})| \leq \Lambda|y - \bar{y}| \quad \text{for all } y, \bar{y} \in W.$$

Geometrically, this can be interpreted as follows: put

$$\Gamma := \partial\Omega \cap B_s(p_\circ),$$

a $(n-1)$ submanifold of $\mathbb{R}^n$. Fix $\Lambda' > \Lambda$, then for all $p = R(y, g(y)) \in \Gamma$ (i.e., for all $y \in W$) the cone

$$R\left\{(\bar{y}, \bar{x}_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |\bar{x}_n - g(y)| \geq \Lambda'|\bar{y} - y|\right\},$$

intersects with Γ only at the point p.

28 Hence for O sufficiently close to the identity, for all and p as above, the line

$$\{p + tORe_n \mid t \in \mathbb{R}\}$$

intersect Γ only at p (because this line lies inside the cone).

In other words, for each O sufficiently close to the identity, the 'projection'

$$\pi_{n-1} \circ R^T \circ O^T : \Gamma \to \mathbb{R}^{n-1}$$

is injective, where, $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection onto the first $n-1$ components —i.e., $\pi_{n-1} : (y, x_n) \mapsto y$. Put $W_O := \pi_{n-1} \circ R^T \circ O^T(W)$ and

$$\psi_O := (\pi_{n-1} \circ O^T \circ R^T \circ \phi) : W \to W_O.$$

We have show that ψ_O is a bijection for O sufficiently close to the identity. Let us show that W_O is open and that Ψ_O is a diffeomorphism.

Indeed, since

$$(\pi_{n-1}\big(R^T(y, g(y))^T\big) = y \quad \text{for all } y \in V.$$

we obtain that $\psi_O = \mathrm{Id}$ if $O = \mathrm{Id}$.

Since $g \in C^1(\overline{W})$, $D\psi_O$ is differentiable a given $y \in \overline{W}$. By the chain rule we have

$$D(\psi_O)_y = \pi_{n-1} \circ O^T \circ R^T \circ D\phi_y$$

Now since for $O = \mathrm{Id}$ we have $\psi_O = \mathrm{Id}$ —and thus $D(\psi_O)_y = \mathrm{Id}$—, by continuity (of the map $O \mapsto J\psi_O(y)$, for fixed y) there is $\epsilon > 0$ such that $D(\psi_O)_y$ is invertible, provided $\|O - \mathrm{Id}\| < \epsilon$, for $\epsilon > 0$ small enough. The fact that we can take $\epsilon > 0$ that works for all $y \in \overline{W}$ follows from the compactness of this set (similarly as in the proof of Proposition 9.78).

Then, using the Inverse function theorem we obtain that ψ is not only injective but also a diffeomorphism.

Finally, since

28 $$\int_{\Omega\cap B_r(p_\circ)} \partial_w f \mathrm{dvol}_n = \int_{\Omega\cap B_s(p_\circ)} \partial_w f$$

and

$$\int_{\partial\Omega\cap B_r(p_\circ)} f(\nu\cdot w)\,\mathrm{dvol}_{n-1} = \int_{\partial\Omega\cap B_s(p_\circ)} f(\nu\cdot w)\,\mathrm{dvol}_{n-1}$$

applytin Proposition 14.8 gives (with R replace by OR), we obtain the validity of (14.2) for all $w = ORe_n$, whenever O is orthogonal with $\|O - \mathrm{Id}\| < \epsilon$. But such set of w spans all of $\mathbb{R}^n$ and hence, by linearity, 14.2 holds for all $w \in \mathbb{R}^n$. This proves the corollary. □

[a] A Euclidean coordinate frame corresponds to the choice of a origin and an orthonormal basis. The coordinates of points in the Euclidean space can be expressed with respect to the new coordinate system and all the geometric properties remained unchanged.

14.1.3 The integration by parts formula and the divergence theorem

We can now give the following important result:

Theorem 14.11: Integration by parts

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and $f \in C^1(\overline{\Omega})$. Then, the following formula holds true for every vector $w \in \mathbb{R}^n$:

$$\int_\Omega \partial_w f \mathrm{dvol}_n = \int_{\partial\Omega} f(p)(\nu(p)\cdot w)\,\mathrm{dvol}_{n-1}(p),$$

29 *where $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ is the exterior unit normal map.*

We now give an immediate corollary that justifies the name ‘integration by parts formula’

Corollary 14.12: Integration by parts

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and let f and h be two function in $C^1(\overline{\Omega})$. Then, for every $i = 1, 2, \dots, n$

$$\int_\Omega \partial_i f\, h\,\mathrm{dvol}_n = -\int_\Omega f\,\partial_i h\,\mathrm{dvol}_n + \int_{\partial\Omega} f h\,\nu_i\,\mathrm{dvol}_{n-1},$$

where $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ is the exterior unit normal map.

Proof. Apply Theorem 14.11 to the product fg in the direction $w = e_i$ and use that $\partial_i(fh) = \partial_i fh + f\partial_i h$. □

Before giving the proof we need a simple preliminary result

Lemma 14.13: No boundary term case

Assume that $h \in C^1(\mathbb{R}^n)$ has compact support.
Then, for all $w \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \partial_w h(x)dx = \int_{\mathrm{spt}(h)} \partial_w h(x)dx = 0$$

Proof. Notice first that the support of $\partial_w h$ must be contained in the support of h (as if $h = 0$ in some open set then $\partial_w h = 0$ in the same open set).

Also, up to a rotation of the coordinates, it is enough to consider the case $w = e_n$.

Let $(-C, C)^n$ be a cube ontaining the support of h. Then, using Fubini's theorem:

$$\int_{(-C,C)^n} \partial_n h = \int_{(-C,C)^{n-1}} dx_1 \cdots d_{x_{n-1}} \int_{-C}^{C} \partial_n h(x)dx_n$$

But for each $(x_1, \ldots, x_{n-1})$

29 $$\int_{-C}^{C} \partial_n h(x)dx_n = h(x_1, \ldots, x_{n-1}, C) - h(x_1, \ldots, x_{n-1}, -C) = 0.$$

□

Proof of Theorem 14.11. Let $\phi_\ell : V_\ell \to B_{r_\ell}(p_\ell)$, $1 \le \ell \le N$, be a graphical cover of $M := \partial\Omega$ (see Definition 13.72 and Proposition 13.73) and $\eta_\ell : \mathbb{R}^n \to [0, \infty)$ be a partition of unity on M subordinated to $\{B_{r_\ell}(p_\ell)\}_{1\le \ell\le N}$.

(Recall that by definition of graphical cover $V_\ell \subset \mathbb{R}^n$ are open and we have

$$M \cap B_{r_\ell}(p_\ell) = \phi_\ell(V_\ell) \qquad \text{where} \qquad \phi_\ell(y) = R_\ell(y, g_\ell(y))^T,$$

$R_\ell; \mathbb{R}^n \to \mathbb{R}^n$ are orthogonal transformations and $g_\ell : V_\ell \to \mathbb{R}$ are C^1 maps.)

Also by definition of integral over a submanifold:

$$\int_{\partial\Omega} h\, \mathrm{dvol}_{n-1} = \sum_{\ell=1}^{N} \int_{\partial\Omega\cap B_\ell(x_\ell)} h\eta_\ell\, \mathrm{dvol}_{n-1},$$

for any continuous function $h : \partial\Omega \to \mathbb{R}$.

Define now $\widetilde{\eta} : \mathbb{R}^n \to \mathbb{R}$ as

$$\widetilde{\eta}(x) = \begin{cases} 1 - \sum_{\ell=1}^{N} \eta_\ell(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Notice that since η_ℓ is a partition of unity of $\partial\Omega$ we have $\sum_{\ell=1}^N \eta_\ell(x)$ in some open neighborhood of $\partial\Omega$ (in other words, in some open set containing $\partial\Omega$).

Then, it follows that $\widetilde{\eta}$ is of class C^∞ in all of $\mathbb{R}^n$, that the support of $\widetilde{\eta}$ is contained in Ω and that $\sum_{\ell=1}^N \eta_\ell + \widetilde{\eta} = 1$ in Ω. Hence,

$$\begin{aligned}\int_\Omega \partial_w f \, \mathrm{dvol}_n &= \sum_{\ell=1}^N \int_\Omega \partial_w(f\eta_\ell)\, \mathrm{dvol}_n + \int_\Omega \partial_w(f\widetilde{\eta})\, \mathrm{dvol}_n \\ &= \sum_{\ell=1}^N \int_{\Omega\cap B_{r_\ell}(p_\ell)} \partial_w(f\eta_\ell)\, \mathrm{dvol}_n + \int_\Omega \partial_w(f\widetilde{\eta})\mathrm{dvol}_n.\end{aligned}$$

Now, on the one hand, since the support of $(f\widetilde{\eta})$ a compact subset of Ω we can extend this function by zero outside of Ω obtaining a compactly supported $C^1(\mathbb{R}^n)$ function. But then Lemma 14.13 gives:

$$\int_\Omega \partial_w(f\widetilde{\eta})\mathrm{dvol}_n = 0$$

On the other hand, for each ℓ we can apply Corollary 14.10 and obtain

$$\int_{\Omega\cap B_{r_\ell}(p_\ell)} \partial_w(f\eta_\ell)\, \mathrm{dvol}_n = \int_{\partial\Omega\cap B_{r_\ell}(p_\ell)} f\eta_\ell(\nu\cdot w)\mathrm{dvol}_{n-1}.$$

29 Hence, after summing in ℓ we obtain the desired formula. □

We can now give the divergence theorem as a corollary

Definition 14.14: Vector Field in $\mathbb{R}^n$, divergence

Let $U \subset \mathbb{R}^n$ be open. A C^k **vector field** on U is a function $F : U \to \mathbb{R}^n$ of class C^k. If we write

$$F(x) = (F_1(x), \ldots, F_n(x)),$$

then functions $\{F_i\}$, from U to $\mathbb{R}^n$, are called the **components** of the vector field. Given a C^1 vector field F the **divergence** of F is defined as the map $\mathrm{div}(F) : U \to \mathbb{R}$ given by

$$\mathrm{div}F(x) := \sum_{i=1}^n \partial_i F_i(x).$$

We now prove an important corollary of the integration by parts formula:

Theorem 14.15: Divergence theorem

Let Ω be a bounded C^1 domain and $F : \overline{\Omega} \to \mathbb{R}$ some C^1 vector field. Then,

$$\int_\Omega \mathrm{div}F\, \mathrm{dvol}_n = \int_{\partial\Omega} F\cdot\nu\, \mathrm{dvol}_{n-1}$$

Proof. Applying Theorem 14.11 to $w = e_i$ and $f = F_i$ we obtain:

$$\int_\Omega \partial_i F_i \, d\mathrm{vol}_n = \int_{\partial\Omega} F_i(p)(\nu(p) \cdot e_i) \, d\mathrm{vol}_{n-1}(p).$$

Hence, summing over $i = 1, \ldots, n$ and noticing $\sum_i F_i(p)(\nu(p) \cdot e_i) = F(p) \cdot \nu(p)$ we conclude. □

14.1.4 Examples and application of the divergence theorem

Example 14.16 (Archimedes' Principle). — Consider a solid $\Omega \subset \mathbb{R}^3$ completely immersed in a cylindrical water tank. If the coordinate system is set so that $x_3 = 0$ coincides with the water's surface, then the pressure of the water P at a depth $-x_3$ is given by $P = -\rho g x_3$, where ρ is the density of the water and g is the acceleration due to gravity.

The vertical force acting of element of surface near $q \in \partial\Omega$ is proportional to the preasure, and points towards $-\nu(q)$. Hence, the total force the water exerts on the buyant can be computed as:

$$F = -\int_{\partial\Omega} P\nu \, dS$$

where ν is the outward pointing surface normal vector. Substituting the expression for pressure, we get: 29

$$F_i = \int_{\partial\Omega} \rho g x_3 \nu_i \, dS$$

By the divergence theorem, we can convert this surface integral into a volume integral:

$$F_i = \rho g \int_\Omega \mathrm{div}(x_3 e_i) \, dV.$$

Since $\mathrm{div}(x_3 e_i)$ equals 1 if $i = 3$ and 0 otherwise, we obtain $F_1 = F_2 = 0$ and

$$F_3 = \rho g \int_\Omega dV = \rho g V$$

where V is the volume of Ω.

This confirms that the buoyant force is equal to the weight of the displaced water, validating Archimedes' principle.

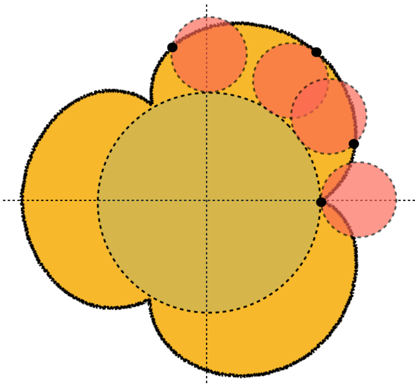
Example 14.17. — Consider a compact region B in $\mathbb{R}^2$ and the smooth vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = x$. Then, $\mathrm{div}(F) = 2$, and the divergence theorem yields

$$2\,\mathrm{vol}(B) = \int_B \mathrm{div}(F) dx = \int_{\partial B} F \cdot \nu dL.$$

if we can apply it to the region B. So, for example, if ∂B is parametrized by a suitable curve $\gamma : [a,b] \to \partial B$, then

$$\mathrm{vol}(B) = \frac{1}{2}\int_a^b \langle \gamma(t), \nu \circ \gamma(t)\rangle dt = \frac{1}{2}\int_a^b \gamma_1(t)\gamma_2'(t) - \gamma_2(t)\gamma_1'(t)dt.$$

As a concrete example, we calculate the area of the region B inside the curve obtained by unrolling a circle with radius $\frac{1}{m}$ on a circle with radius 1 for an integer $m \geq 1$, while tracking a point on the moving circle. This curve is called an **epicycloid**, and in the special case $m = 1$, it is also called a **cardioid**. The area enclosed by the epicycloid, denoted by B, is not smoothly bounded, but it is not difficult to show that the divergence theorem also holds for the area B.



29 We represent elements of $\mathbb{R}^2$ as column vectors. The epicycloid is described by the positively oriented closed path $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$$\gamma(t) = \frac{m+1}{m}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix} + \frac{1}{m}\begin{pmatrix}\cos((m+1)t)\\ \sin((m+1)t)\end{pmatrix}$$

The outward normal to γ is given by

$$n_\gamma(t) = \frac{m+1}{m}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix} + \frac{m+1}{m}\begin{pmatrix}\cos((m+1)t)\\ \sin((m+1)t)\end{pmatrix}$$

The scalar product $\langle \gamma(t), n_\gamma(t)\rangle$ is

$$\frac{(m+1)^2}{m^2} + \frac{m+1}{m^2} + a\cos(t)\cos((m+1)t) + a\sin(t)\sin((m+1)t)$$

for a constant a that we do not worry about. We obtain

$$\mathrm{vol}(B) = \frac{1}{2}\int_0^{2\pi} \langle \gamma(t), n_\gamma(t)\rangle dt = \pi\frac{(m+1)(m+2)}{m^2}$$

since the integral over $[0, 2\pi]$ of $\cos(t)\cos((m+1)t)$ and also $\sin(t)\sin((m+1)t)$ is zero.

14.1.5 Rotation and the Green's Theorem

Definition 14.18: Rotation – Curl

Let $F : U \to \mathbb{R}^2$ be a continuously differentiable vector field on an open set $U \subseteq \mathbb{R}^2$. The **vorticity**, **rotation**, or **curl** of F is the real-valued function defined by

$$\mathrm{rot} F(x) = \mathrm{curl} F(x) := \partial_1 F_2(x) - \partial_2 F_1(x), \qquad x \in U.$$

Theorem 14.19: Green's Theorem in the plane

Let $F : U \to \mathbb{R}^2$ be a continuously differentiable vector field on an open set $U \subseteq \mathbb{R}^2$. Then, for any compact C^1 domain $\Omega \subseteq U$,

$$\int_\Omega \mathrm{rot} F dx = \int_{\partial\Omega} F \cdot \tau \, dL,$$

with $\tau = \nu^\perp = (-\nu_2, \nu_1)$, where ν is the exterior unit normal.

Proof. We define the vector field $F^\perp : U \to \mathbb{R}^2$ by $F^\perp(x) = (-F_2(x), F_1(x))$. Then,

$$-\mathrm{div}(F^\perp) = -\partial_1 F_1^\perp - \partial_2 F_2^\perp = \partial_1 F_2 - \partial_2 F_1 = \mathrm{rot}(F).$$

29 But by the divergence theorem

$$\int_\Omega \mathrm{div}(F^\perp) dx = \int_{\partial\Omega} F^\perp \cdot \nu dL$$

but

$$F^\perp \cdot \nu = F_2\nu_1 - F_1\nu_2 = -F \cdot \nu^\perp = -F \cdot \tau.$$

So, the theorem follows. □

Applet 14.20 (Divergence and Rotation)**.** *This applet illustrates the concepts of irrotational and divergence-free, as well as the theorems of this section.*

14.21. — As an application of the divergence theorem in the plane, we discuss in the following exercises the **Jordan Curve Theorem**. The question that this theorem answers is essentially the following. Suppose you have drawn a *complicated* closed curve without self-intersections in the plane. Does this curve then divide the plane into an *inside* and an *outside*? How do you even define whether a point is in the inside or the outside? Although this is intuitively clear, proving it is surprisingly difficult. This problem was recognized by Camille Jordan (1838-1922). Jordan's original proof of the theorem (around 1882) uses a kind of polygonal approximation to the curve. It is problematic in several places and was doubted for several decades but is now considered essentially correct. The problem can also be formulated in higher dimensions. For example, one can ask whether a complicatedly deformed sphere in

$\mathbb{R}^3$ divides space $\mathbb{R}^3$ into a well-defined inside and outside. A positive answer to the general problem is provided by the **Jordan-Brouwer Separation Theorem**. A modern proof of this, which works in any dimension and does not require analysis, can be found in [Hat02] Proposition 2B.1, or in [Rot88], Theorem 6.35. Here, we formulate and prove the theorem in the case of smooth curves in $\mathbb{R}^2$.

Theorem 14.22: Jordan Curve Theorem

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a smooth, regular (i.e., $\gamma'(t) \neq 0$ for all t), simple (i.e., $\gamma|_{[0,1)}$ injective), closed path. Then, one can uniquely write the complement of $\Gamma = \gamma([0,1])$ as the disjoint union

$$\mathbb{R}^2 \setminus \Gamma = \operatorname{Inn}(\Gamma) \cup \operatorname{Out}(\Gamma), \qquad \operatorname{Inn}(\Gamma) \cap \operatorname{Out}(\Gamma) = \emptyset$$

where the interior $\operatorname{Inn}(\Gamma)$ is an open, bounded, connected subset and the exterior $\operatorname{Out}(\Gamma)$ is an open, unbounded, connected subset. Furthermore, $\partial \operatorname{Inn}(\gamma) = \partial \operatorname{Out}(\gamma) = \Gamma$.

14.23. — A path γ as in the theorem $\gamma : [0,1] \to \mathbb{R}^2$ is injective and smooth except for the equation $\gamma(0) = \gamma(1)$, and $\gamma'(t) \neq 0$ for all $t \in [0,1]$. In the following, we additionally assume that $\gamma'(0) = \gamma'(1)$, meaning the image $\Gamma = \gamma([0,1])$ has no kink at the point $\gamma(0) = \gamma(1)$. Thus, γ could be extended to a periodic, continuously differentiable function on $\mathbb{R}$. We define, 29 for $u \in \mathbb{R}^2 \setminus \Gamma$, the **rotation number** of γ around u as

$$I_\gamma(u) = \frac{1}{2\pi} \int_0^1 \frac{\langle \gamma(t) - u, \nu \circ \gamma(t) \rangle}{|\gamma(t) - u|^2} dt.$$

Exercise 14.24. — Consider the closed curve $\gamma : [0,1] \to \mathbb{R}^2$ for $r > 0$

$$\gamma(t) = (r\cos(2\pi t), r\sin(2\pi t))$$

parametrizing the boundary of the circular disk $B(0,r)$. Realize $I_\gamma(u)$ for $u \notin \partial B(0,r)$ as the flux integral of a vector field over $\partial B(0,r)$, and show that for all $u \notin \partial B(0,r)$, the rotation number is given by

$$I_\gamma(u) = \begin{cases} 1 & \text{if } u \in B(0,r) \\ 0 & \text{if } u \notin B(0,r) \end{cases}$$

For circles, the rotation number is thus able to decide whether a point is inside or outside the circle. Does it work for a rectangle? For an ellipse?

Exercise 14.25. — Let $\gamma : [a,b] \to \mathbb{R}^2$ be a smooth, regular, simple, closed path. Show that I_γ is locally constant.

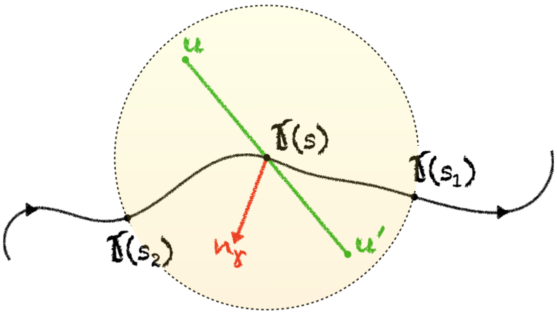
14.26. — Exercise 14.25 shows, in particular, that I_γ vanishes outside a sufficiently large ball. In fact, $I_\gamma(u)$ approaches zero as $|u| \to \infty$, since in the definition of $I_\gamma(u)$, the norm of u appears to the power of 2 in the denominator and to the power of 1 in the numerator. If $B(0,r)$ is a ball containing Γ, then I_γ must be constant on $\mathbb{R}^2 \setminus B(0,r)$ due to Exercise 14.25, and at the same time, it must go to zero for increasing radii, proving the claim. The essential idea to prove Jordan's Curve Theorem is now to examine what happens to the rotation number $I_\gamma(u)$ when u crosses the curve Γ. Using the divergence theorem, one can show that the rotation number changes by ± 1 when crossing Γ, depending on the direction in which the normal vector of γ points.

Exercise 14.27. — Let $u, u' \in \mathbb{R}^2 \setminus \gamma([a,b])$, such that the line segment from u to u' intersects the trace of γ at exactly one point $\gamma(s)$. Show that

$$I_\gamma(u') - I_\gamma(u) = \operatorname{sgn}(\langle u - u', \nu(\gamma(s)) \rangle)$$

Note that the trace of γ is intersected at exactly one point. Thus, the statement can be reduced to the following illustration:

29



Now, replace the path segment of γ between the times s_1, s_2 chosen sufficiently close to s with either of the circular segments as shown in the image.

Exercise 14.28. — Show that there is at least one point $u_0 \in \mathbb{R}^2 \setminus \Gamma$ with $I_\gamma(u_0) = 1$ or $I_\gamma(u_0) = -1$.

Exercise 14.29. — Show that the sets

$$\operatorname{Inn}(\gamma) = \{u \in \mathbb{R}^2 \setminus \Gamma \mid I_\gamma(u) \neq 0\} \quad \text{and} \quad \operatorname{Out}(\gamma) = \{u \in \mathbb{R}^2 \setminus \Gamma \mid I_\gamma(u) = 0\}$$

are open and path-connected.

Exercise 14.30. — Prove Jordan's Curve Theorem 14.22.

14.2 Line integrals and the Poincaré Lemma

14.2.1 The work of a Vector Field along a path

Definition 14.31: Piecewise C^k path

Let $U \subset \mathbb{R}^n$ be an open set and $k \geq 1$ an integer. A continuous map $\gamma : [0,1] \to U \subset \mathbb{R}^n$ is called **piecewise C^k path** if there exist finitely many points $0 = t_0 < t_1 < t_2 < \ldots < t_N = 1$ in $[0,1]$ such that for all $k = 1, 2, \ldots, N$, the restriction $\gamma|_{[t_{k-1}, t_k]}$ is belongs to $C^k([t_{k-1}, t_k], \mathbb{R}^n)$.

Definition 14.32: Work of a field along a path (line integral)

Let $U \subset \mathbb{R}^n$ be an open subset, and let $F : U \to \mathbb{R}^n$ be a continuous vector field and $\gamma : [a,b] \to U$ a C^1 map (also called 'path'). We define the **work of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$** as

$$\int_\gamma F \cdot d\gamma := \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt.$$

If γ is piecewise continuously differentiable with respect to a partition $a = t_0 < t_1 < \cdots < t_N = b$, the integral is interpreted as the sum of integrals over intervals $[t_{k-1}, t_k]$.

30 In Physics literature, one often finds notations like:

$$\int_\gamma F \cdot d\gamma = \int_\gamma \vec{F} \cdot d\vec{r}.$$

where $\vec{r}$ is the position vector.

Lemma 14.33: Reparametrization invariance of line integrals

Let $U \subset \mathbb{R}^n$ be an open subset, $f : U \to \mathbb{R}^n$ be a continuous vector field, let $\gamma : [a,b] \to \mathbb{R}^d$ be a continuously differentiable path and let $\psi \colon [0,1] \to [a,b]$ be a C^1 function such that $\psi(0) = a, \psi(1) = b$. Then

$$\int_\gamma F = \int_{\gamma \circ \psi} F.$$

Proof. This is a consequence of the change of variable formula (the one for one variable seen in Analysis I) and the chain rule:

$$\begin{aligned} \int_{\gamma\circ\psi} F \cdot d(\gamma \circ \psi) &= \int_0^1 F(\gamma(\psi(t))) \cdot (\gamma \circ \psi)'(t) dt = \int_0^1 F(\gamma(\psi(t))) \cdot \gamma'(\psi(t))\psi'(t) dt \\ &= \int_a^b F(\gamma(s)) \cdot \gamma'(s) ds = \int_\gamma F \cdot d\gamma. \end{aligned}$$

□

Definition 14.34: Potential of a Vector Field

Let $U \subset \mathbb{R}^n$ be open, and $F : U \to \mathbb{R}^n$ be a continuous vector field. A continuously differentiable function $f : U \to \mathbb{R}$ is called a **potential** for F, if $\nabla f = F$ holds. That is to say

$$\partial_i f(x) = F_i(x) \text{ for all } x \in U, 1 \leq i \leq n.$$

If F admits a potential f in U, we say that F is **conservative in U**.

In the next two propositions we show that a vector field F admits a potential if and only if the value of $\gamma \mapsto \int_\gamma F$ depends only on the endpoints of γ.

Proposition 14.35: work along a path as difference of potential

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Let $U \subset \mathbb{R}^n$ be open, and let $F : U \to \mathbb{R}^n$ be a continuous vector field. Suppose there exists a potential $f : U \to \mathbb{R}$ for F. Then

$$\int_\gamma F = f(\gamma(1)) - f(\gamma(0))$$

for any C^1 path $\gamma : [0,1] \to U$.

Proof. If $\gamma : [0,1] \to U$ is a continuously differentiable path, then for $t \in [0,1]$, $F(\gamma(t)) = \operatorname{grad} f(\gamma(t)) = Df(\gamma(t))$, and thus

$$\int_\gamma F \cdot d\gamma = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)dt = \int_0^1 Df(\gamma(t))(\gamma'(t))dt = \int_0^1 (f \circ \gamma)'(t)dt = f(\gamma(1)) - f(\gamma(0))$$

by the chain rule. If γ is only piecewise continuously differentiable with respect to a partition $0 = s_0 < s_1 < \ldots < s_N = 1$, the calculation can be applied to the subintervals $[s_{k-1}, s_k]$. This leads to a telescoping sum where all terms except $f(\gamma(1)) - f(\gamma(0))$ cancel. □

Example 14.36. — Consider the vector field F on $\mathbb{R}^2$ defined by $F(x,y) = (-y,x)$ and calculate the integral of F along different paths from $(0,0)$ to $(1,1)$.

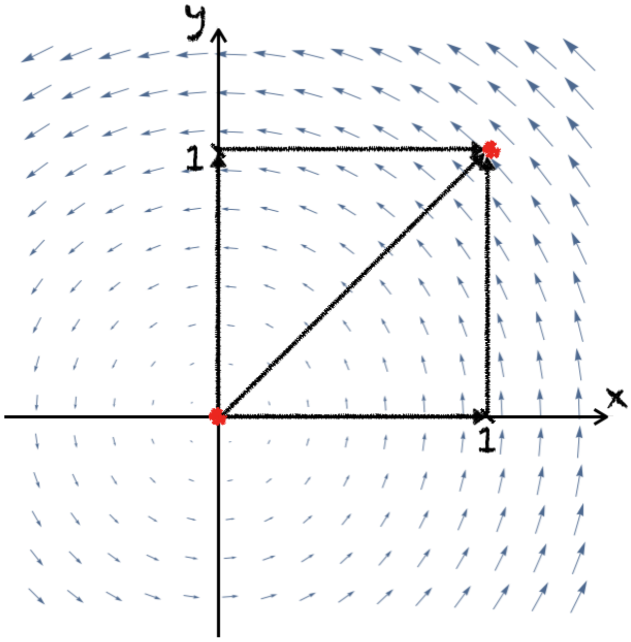

Figure 14.1: The vector field F - vector lengths are scaled by a factor of 0.1.

Let γ_0, γ_1, and $\gamma_2 : [0,1] \to \mathbb{R}^2$ be the paths from $(0,0)$ to $(1,1)$ given by $\gamma_0(t) = (t,t)$ and

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$$\gamma_1(t) = \begin{cases} (2t, 0) & \text{if } t \in [0, \frac{1}{2}] \\ (1, 2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \qquad \gamma_2(t) = \begin{cases} (0, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ (2t-1, 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Then,

$$\int_{\gamma_0} F \cdot d\gamma_0 = \int_0^1 \langle F(\gamma_0(t)), \gamma_0'(t) \rangle \, dt = \int_0^1 \langle (-t, t), (1,1) \rangle \, dt = 0$$

$$\int_{\gamma_1} F \cdot d\gamma_1 = \int_0^{\frac{1}{2}} \langle (0, 2t), (2, 0) \rangle \, dt + \int_{\frac{1}{2}}^1 \langle (1-2t, 1), (0, 2) \rangle \, dt = 1$$

$$\int_{\gamma_2} F \cdot d\gamma_2 = \int_0^{\frac{1}{2}} \langle (-2t, 0), (0, 2) \rangle \, dt + \int_{\frac{1}{2}}^1 \langle (-1, 2t-1), (2, 0) \rangle \, dt = -1$$

We see that the work performed $\int_\gamma F$ depends on the chosen path γ. If one moves perpendicular to the vector field, no work is done. If one moves with the vector field, positive work is done, and if one moves against the vector field, negative work is done.

From these calculations, it follows in particular that the vector field F does not possess a potential.

Exercise 14.37. — A **loop** in an open subset $U \subset \mathbb{R}^n$ is a path $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1)$. Show that a continuous vector field $F : U \to \mathbb{R}^n$ is conservative if and only if, for every piecewise continuously differentiable loop γ in U,

$$\int_\gamma F = 0$$

holds.

Lemma 14.38: Integrability conditions

*Let $U \subset \mathbb{R}^n$ be open, and let F be a C^1 conservative vector field on U, with components $F_1, \dots, F_n$. Then we necessarily have the **integrability conditions**:*

$$\partial_j F_k = \partial_k F_j$$

for all pairs $j, k \in \{1, \dots, n\}$.

Proof. If $\nabla f = F$. Then, for $j, k \in \{1, \dots, n\}$, we have

$$\partial_j F_k = \partial_j \partial_k f = \partial_k \partial_j f = \partial_k F_j. \tag{14.3}$$

We are allowed to commute second derivatives because f is $C^2(U)$, since $\nabla f = F \in C^1(U, \mathbb{R}^n)$, hence Schwarz Lemma applies. □

30 The following example shows that the integrability conditions 14.38 are necessary but not generally sufficient for F to have a potential.

Example 14.39. — Let $U = \mathbb{R}^2 \setminus \{0\}$, and consider the vector field $F : U \to \mathbb{R}^2$ given by

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

for $(x, y) \in U$. A direct calculation shows

$$\partial_1 F_2(x, y) = \partial_x \left(\frac{x}{x^2 + y^2} \right) = \frac{-x^2 + y^2}{(x^2 + y^2)^2} = \partial_y \left(\frac{-y}{x^2 + y^2} \right) = \partial_2 F_1(x, y)$$

thus satisfying the integrability conditions (14.3) throughout U. However, F is not conservative. Let $\gamma : [0, 1] \to U$ be the continuously differentiable loop defined by

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$$

which rotates once counterclockwise around the unit circle. Then,

$$\int_\gamma F \cdot d\gamma = 2\pi \int_0^1 \langle (-\sin(2\pi t), \cos(2\pi t)) \,, (-\sin(2\pi t), \cos(2\pi t)) \rangle dt = 2\pi,$$

even though γ is a closed path with $\gamma(0) = \gamma(1) = (1, 0)$.

Applet 14.40 (Integrability Conditions)**.** *What different values for the path integral can you obtain when considering closed paths? Why does the value of the path integral usually not change, but sometimes does when you move the middle three points?*

14.2.2 The Poincaré Lemma

In order to state the main theorem of this section we need to introduce the concept of *homotopy of curves*.

Definition 14.41: Homotopy, simple connected subset of $\mathbb{R}^n$

Let $U \subset \mathbb{R}^n$ set γ_0 and γ_1 be to continuous maps from $[0,1] \to U$ with the same initial point $x_0 = \gamma_0(0) = \gamma_1(0)$ and the same endpoint $x_1 = \gamma_0(1) = \gamma_1(1)$.
A **homotopy** from γ_0 to γ_1 within U is a continuous function $H : [0,1] \times [0,1] \to U$ with the following properties:

$$H(0,t) = \gamma_0(t), \quad H(1,t) = \gamma_1(t) \quad \text{and} \quad H(s,0) = x_0, \quad H(s,1) = x_1$$

for all $t \in [0,1]$ and all $s \in [0,1]$. We say $\boldsymbol{\gamma_1}$ **is homotopic to** $\boldsymbol{\gamma_0}$ if there exists a homotopy from γ_0 to γ_1.
A subset U of $\mathbb{R}^n$ is called **simply connected** if for any two given paths in U with the same end points are homotopic.

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14.42. — Let H be a homotopy from γ_0 to γ_1 as in the definition. For each fixed $s \in [0,1]$, the function $\gamma_s : t \mapsto H(s,t)$ is a path from x_0 to x_1. For $s = 0$ and $s = 1$, we obtain the given paths γ_0 and γ_1. This way, we can view the homotopy H as a parametrized family of paths depending continuously on the parameter $s \in [0,1]$.

14.43. — Heuristically, a connected topological space X is called **simply connected** if every path γ from x_0 to x_1 in X can be continuously deformed into any other given path from x_0 to x_1.

Theorem 14.44: Poincaré Lemma

Let $U \subset \mathbb{R}^n$ be open, and let $F : U \to \mathbb{R}^n$ be a continuously differentiable vector field that satisfies the integrability conditions

$$\partial_k F_j = \partial_j F_k \tag{14.4}$$

for all $j, k \in \{1, \ldots, n\}$. Let $\gamma_0 : [0,1] \to U$ and $\gamma_1 : [0,1] \to U$ be piecewise C^1 paths with the same initial point x_0 and the same endpoint x_1. If γ_0 and γ_1 are homotopic, then

$$\int_{\gamma_0} F \cdot d\gamma_0 = \int_{\gamma_1} F \cdot d\gamma_1.$$

Theorem 14.44 is an example of a so-called **global integration theorem** because, it has something to do with the global nature of the domain U. This is evident in the following important

Before proving Theorem 14.44 we show a simpler proof under the additional assumption that U is convex. 30

Lemma 14.45: Poincaré Lemma in convex domains

Let $U \subset \mathbb{R}^n$ be open and convex, and let $F : U \to \mathbb{R}^n$ be a continuously differentiable vector field that satisfies the integrability conditions (14.4)*. Then F is conservative in U.*

Proof. The necessity of the integrability conditions was already proven in Corollary 14.38. For the converse, assume without loss of generality that $0 \in U$. We use the path integral of F along the straight line from 0 to $x \in U$ to define a function $F : U \to \mathbb{R}$ by

$$f(x) := \int_0^1 F(tx) \cdot x \, dt$$

for $x \in U$, and let us prove that f is a potential of F.

Indeed, fix $j \in \{1, \dots, n\}$ and consider, as a preparation for the computation of $\partial_j f$, for $h \in \mathbb{R}^n$

$$\begin{aligned}\partial_j(F \cdot x) &= \partial_j \left(\sum_{k=1}^n F_k(tx) x_k \right) \\ &= \sum_{k=1}^n \partial_j F_k(tx) t \, x_k + F_j(tx) = \sum_{k=1}^n \partial_k F_j(tx) t \, x_k + F_j(tx) \\ &= \partial_t(tF_j(tx)).\end{aligned}$$

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where we used the, integrability conditions.

Hence, using Theorem 13.41, for $x \in U$ we can compute $\partial_j f(x)$ as

$$\partial_j f(x) = \partial_j \int_0^1 F(tx) \cdot x \, dt = \int_0^1 \partial_t(tF_j(tx))dt = \big[tF_j(tx)\big]_{t=0}^{t=1} = F_j(x). \tag{14.5}$$

Thus, $F = \nabla f$, and f is C^2. □

Exercise 14.46. — For which values of $\lambda \in \mathbb{R}$ is the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = \big(\lambda x \exp(y), (y + 1 + x^2) \exp(y)\big)$$

conservative? Determine a potential for f for these values.

Proof of Theorem 14.44. Suppose that γ_0 and γ_1 are two given piecewise C^1 homotopic paths joining x_0 and x_1 and let $H : [0,1]^2 \to U$ be an homotopy, i.e. $H(s,t)$ is continuous and $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$, for all $t \in [0,1]$.

The map H is continuous in the compact set $[0,1]^2$ so it is uniformly continuous. Also the image of H, $K := H([0,1]^2)$ is a compact subset of U.

By the previous considerations there exist $\delta > 0$ and $\epsilon > 0$ such that for all $x = H(s,t) \in K$, we have $B_\delta(x) \subset U$ and $\bar{x} = H(\bar{s}, \bar{t}) \in B_\delta(x)$ for all $(s,t), (\bar{s}, \bar{t}) \in [0,1]^2$ with $|\bar{s} - s| < \epsilon$ and $|\bar{t} - t| < \epsilon$.

Decrease ϵ if necessary so that $K = 1/\epsilon$ is an integer, and consider the 'mesh'

$$s_l := \epsilon l, \qquad t_m := \epsilon l, \quad 0 \le l, m \le K.$$

For every $0 \le l, m \le K$ let

$$x_{l,m} := H(s_l, t_m).$$

and let $f_{l,m} : B_\delta(x_{l,m}) \to \mathbb{R}$ be the a potential of F restricted to $B_\delta(x_{l,m})$, which exists thanks to Lemma 14.45. (We can choose for concreteness the potential satisfying $f_{l,m}(x_{l,m}) = 0$, although this will not play any role in the proof).

Let us show that for all $0 \le l \le K-1$ and $0 \le l, m \le K-1$ we have

31

$$\begin{aligned} f_{l+1,m}(x_{l+1,m+1}) - f_{l+1,m}(x_{l+1,m}) = &+ f_{l,m+1}(x_{l+1,m+1}) - f_{l,m+1}(x_{l,m+1}) \\ &+ f_{l,m}(x_{l,m+1}) - f_{l,m}(x_{l,m}) \\ &- f_{l,m}(x_{l+1,m}) + f_{l,m}(x_{l,m}) \end{aligned} \tag{14.6}$$

Indeed, it follows from noticing first that

$$f_{l+1,m}(x_{l+1,m+1}) - f_{l+1,m}(x_{l+1,m}) = f_{l,m}(x_{l+1,m+1}) - f_{l,m}(x_{l+1,m})$$

because the functions $f_{l+1,m}$, $f_{l,m}$ are both potentials for X in $B_\delta(x_{\ell+1,m}) \cap B_\delta(x_{\ell,m})$ and the two points $x_{l+1,m+1}$ and $x_{l+1,m+1}$ and $x_{l+1,m}$ belong two the intersection of the two balls.

Similarly,

$$f_{l,m+1}(x_{l+1,m+1}) - f_{l,m+1}(x_{l,m+1}) = f_{l,m}(x_{l+1,m+1}) - f_{l,m}(x_{l,m+1}).$$

But then (14.6) follows from the trivial identity

$$\begin{aligned} f_{l,m}(x_{l+1,m+1}) - f_{l,m}(x_{l+1,m}) = &+ f_{l,m}(x_{l+1,m+1}) - f_{l,m+1}(x_{l,m}) \\ &+ f_{l,m}(x_{l,m+1}) - f_{l,m}(x_{l,m}) \\ &- f_{l,m}(x_{l+1,m}) + f_{l,m}(x_{l,m}) \end{aligned}$$

Finally define, for $0 \leq l \leq K$

$$S(l) := \sum_{m=0}^{K-1} \big(f_{l,m}(x_{l,m+1}) - f_{l,m}(x_{l,m})\big)$$

Notice that:

$$\int_{\gamma_0} F \cdot d\gamma_0 = \sum_{m=0}^{K-1} \big(f_{0,m}(x_{0,m+1}) - f_{0,m}(x_{0,m})\big) = S(0)$$

and

$$\int_{\gamma_1} F \cdot d\gamma_1 = \sum_{m=0}^{K-1} \big(f_{K,m}(x_{K,m+1}) - f_{K,m}(x_{K,m})\big) = S(K)$$

But summing (14.6) over all $0 \leq m \leq K-1$, for fixed $0 \leq l \leq K-1$, we obtain:

$$\sum_{m=0}^{K-1} \big(f_{l+1,m}(x_{l+1,m+1}) - f_{l+1,m}(x_{l+1,m})\big) = \sum_{m=0}^{K-1} \big(f_{l+1,m}(x_{l,m+1}) - f_{l,m}((x_{l,m}))\big) \tag{14.7}$$

since all the rest of terms cancel (because it is a telescopic sum and the first and last terms are also zero using $x_{\ell,0} = x_{\ell+1,0} = \gamma_0(t)$ and $x_{\ell,K} = x_{\ell+1,K} = \gamma_1(t)$).

31 This proves $S(l+1) = S(l)$ for all $0 \leq l \leq K-1$ and hence

$$\int_{\gamma_0} F \cdot d\gamma_0 = S(0) = S(K) = \int_{\gamma_1} F \cdot d\gamma_1.$$

□

Exercise 14.47. — Let $U \subset \mathbb{R}^n$ be open and simply connected. Show that continuously differentiable vector field on U is conservative if and only if it satisfies the integrability conditions (14.4).

14.3 Stokes' theorem in 3D, and a word on differential forms

Definition 14.48. — Let $F : U \to \mathbb{R}^3$ be a continuously differentiable vector field on an open set $U \subseteq \mathbb{R}^3$. The **vorticity**, **rotation**, or **curl** of F is the vector field on U defined by

$$\operatorname{rot} F = \operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

Definition 14.49: Smoothly bounded domain in orientable C^k surface (extra material)

Let $M \subset \mathbb{R}^3$ be a C^k surface (i.e. a 2-dimensional submanifold).
We say that M is **orientable** if there exist a continuous map $\nu : \overline{M} \to \mathbb{S}^2$ that such that $\nu(p)$ is normal to M at p for all $p \in M$. Such map ν is called an **orientation** of M.
A subset $\Omega \subset M$ is called **relatively open** if there is $U \subset \mathbb{R}^3$ open such that $\Omega = M \cap U$.
A relatively open subset $\Omega \subset M$ is called **smoothly bounded domain** if $\overline{\Omega}$ is compact and $\Gamma := \overline{\Omega} \setminus \Omega \subset M \subset \mathbb{R}^3$ is a C^k 1-dimensional submanifold. Then Γ is called the **boundary** of Ω.
Given an orientation $\nu : M \to \mathbb{S}^2$. For each $p \in \Gamma$ we will put $\tau(p)$ the unit tangent vector to Γ at p such that $\nu(p) \times \tau(p)$ “points inwards to Ω”: more precisely, there is a sequence $p_k \in \Omega$ such that $\frac{p_k - p}{|p_k - p|} \to \nu(p) \times \tau(p)$.

Theorem 14.50: Stokes’ for smooth smoothly bounded domains (extra material)

Let F be a continuously differentiable vector field on an open subset $U \subseteq \mathbb{R}^3$. Let $M \subset U \subset \mathbb{R}^3$ be an orientable C^1 surface and $\Omega \subset M$ a smoothly bounded domain with boundary Γ. Then, 31

$$\int_\Omega \langle \operatorname{rot} F, \nu \rangle \, dS = \int_\Gamma \langle F, \tau \rangle dS,$$

where $\nu : M \to \mathbb{S}^2$ is an orientation of M and $\tau : \Gamma \to \mathbb{S}^2$ a unit tangent vector to Γ such that that $\nu \times \tau$ points inwards to Ω.

To illustrate the cornerstone idea of the proof without introducing superfluous technical complications we will work in the simplified framework of immersed disks. The general case can be done similarly using a partition of unity.

Definition 14.51: Immersed disk

The set $\mathbb{D} : \left\{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 \right\}$ is called the **unit disk**.
We call any map $\psi : \overline{\mathbb{D}} \to \mathbb{R}^3$ of class C^1 and such that $D\psi_x$ has maximal rank for all $x \in \overline{\mathbb{D}}$ an **immersion of the disk** or, simply, **immersed disk**.

14.52. — If $\psi : \overline{\mathbb{D}} \to \mathbb{R}^3$ is an immersed disk the continuous two vectors $\partial_1 \psi(x)$ and $\partial_2 \psi(x)$ are linearly independent and span the plane tangent to the surface $\psi(\overline{\mathbb{D}})$ at the point $\psi(x)$. Hence, the map $\widetilde{\nu} : \mathbb{D} \to \mathbb{S}^2 \subset \mathbb{R}^3$ defined as

$$\widetilde{\nu}(x) := \frac{\partial_1 \psi(x) \times \partial_1 \psi(x)}{\left| \partial_1 \psi(x) \times \partial_1 \psi(x) \right|}$$

gives a unit normal vector to the surface the surface $\psi(\overline{\mathbb{D}})$ at the surface's point $\psi(x)$.

Stoke's theorem for immersed disks reads as follows.

Theorem 14.53: Stokes' for immersed disks

Let $U \subset \mathbb{R}^3$ be an open set and $F : U \to \mathbb{R}^3$ a C^1 map (a vector field). Let $\psi : \overline{\mathbb{D}} \to U$ be an immersion of the disk and put

$$\gamma(t) := (\cos t, \sin t) \quad \textit{and} \quad \Gamma(t) := \psi(\gamma(t))$$

31 *Then*

$$\begin{aligned} \int_{\psi(\mathbb{D})} \langle \operatorname{rot} F, \nu \rangle dS &:= \int_{\psi(\mathbb{D})} \langle (\operatorname{rot} F) \circ \psi, \widetilde{\nu} \rangle \left| \partial_1 \psi \times \partial_2 \psi \right| dx_1 dx_2 \\ &= \int_0^{2\pi} \langle (F \circ \Gamma)(t), \Gamma'(t) \rangle \, dt = \int_\Gamma F \cdot d\Gamma. \end{aligned}$$

In words: the flux of the curl of F across an immersed disk equals the circulation of F along the boundary of the immersed disk.

Before proving the theorem we need to the following

Interlude: Skew symmetric matrices and cross product in $\mathbb{R}^3$

Suppose that A is a skew symmetric (i.e. $A^T = -A$) 3×3 matrix.
Then A is of the form:

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

for some $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$.
Notice that for any vector $b = (b_1, b_2, b_3)^T \in \mathbb{R}^3$ we have:

$$Ab = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_2 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \omega_2 b_3 - \omega_3 b_2 \\ \omega_3 b_1 - \omega_1 b_3 \\ \omega_1 b_2 - \omega_2 b_1 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{14.8}$$

31 This crucially used, for example, classical rigid body dynamics. The orientation of a rigid body is described by a orthogonal 3×3 matrix $R(t)$ that varies smoothly with time. Then, differentiation the identity $R(t)R(t)^T = \mathrm{Id}$ with respect to t one obtains that $A = R'R^T$ is skew-symmetric. The associated vector $\omega := (\alpha, \beta, \gamma)$, which varies with time, is the so-called angular velocity of the rigid body. It describes the infinitesimal change in its orientation. The angulart velocity $\omega(t)$ and its derivative with respect to t, $\omega'(t)$, appears in Euler's equations of rigid body dynamics (conveniently multiplied by the inertia matrix I of the body). Namely,

$$\mathrm{I}\omega' + \omega \times (\mathrm{I}\omega) = \text{applied torques}.$$

(For example, for a body in free fall under the gravitational field the applied torques are just zero.)

Proof of Theorem 14.53. Let us define the following C^1 vector field G on $\overline{\mathbb{D}}$:

$$G(x) = \begin{pmatrix} \langle F \circ \psi(x), \partial_1 \psi(x) \rangle \\ \langle F \circ \psi(x), \partial_2 \psi(x) \rangle \end{pmatrix},$$

for $x = (x_1, x_2) \in \mathbb{D}$.

We will apply the 2D version of Stokes theorem (Green's theorem 14.19, which is essentially
32 same as the divergence theorem) to G. But in order to do so we need to express the 2D rotation of G on V in terms of the 3D rotation of F. We claim that

$$\mathrm{rot}(G) = \langle \mathrm{rot}(F) \circ \psi, \partial_1 \psi \times \partial_2 \psi \rangle$$

holds. To see this, we calculate (using the chain rule):

$$\mathrm{rot}(G) = \partial_1 G_2 - \partial_2 G_1$$

$$\begin{aligned}
&= \partial_1\langle F\circ\psi, \partial_2\psi\rangle - \partial_2\langle F\circ\psi, \partial_1\psi\rangle \\
&= \langle (DF\circ\psi)\partial_1\psi, \partial_2\psi\rangle + \langle F\circ\psi, \partial_1\partial_2\psi\rangle - \langle (DF\circ\psi)\partial_2\psi, \partial_1\psi\rangle - \langle F\circ\psi, \partial_1\partial_2\psi\rangle \\
&= \langle (DF\circ\psi)\partial_1\psi, \partial_2\psi\rangle - \langle (DF\circ\psi)\partial_2\psi, \partial_1\psi\rangle \\
&= \langle ((DF - DF^T)\circ\psi)\partial_1\psi, \partial_2\psi\rangle,
\end{aligned}$$

Now we notice that the skew symmetric matrix $DF - DF^T$ can be put as

$$DF - DF^T = \begin{pmatrix} 0 & \partial_2 F_1 - \partial_1 F_2 & \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 & 0 & \partial_3 F_2 - \partial_2 F_3 \\ \partial_1 F_3 - \partial_3 F_1 & \partial_2 F_3 - \partial_3 F_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

for $(\omega_1, \omega_2, \omega_3)^T = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)^T = \mathrm{rot}(F)$.

Hence, using (14.8) we obtain:

$$\mathrm{rot}(G) = \langle ((DF - DF^T)\circ\psi)\partial_1\psi, \partial_2\psi\rangle = \langle (\mathrm{rot}\, F\circ\psi)\times\partial_1\psi, \partial_2\psi\rangle = \langle \partial_1\psi\times\partial_2\psi, (\mathrm{rot}\, F\circ\psi)\rangle.$$

Here, we used the simple identity;

$$\langle a\times b, c\rangle = \langle b\times c, a\rangle = \langle c\times a, b\rangle = \det(a\,|\,b\,|\,c),$$

for all vectors a, b, c in $\mathbb{R}^3$.

32 Hence, using Green's theorem 14.19 and the chain rule for the derivative of $\psi\circ\gamma$, we obtain

$$\begin{aligned}
\int_\Omega \langle \mathrm{rot} F, \nu\rangle\, dS = \int_{\psi(\mathbb{D})} \langle \mathrm{rot} F\circ\psi, \frac{\partial_1\psi\times\partial_2\psi}{|\partial_1\psi\times\partial_2\psi|}\rangle\, |\partial_1\psi\times\partial_2\psi|\, dx_1 dx_2 = \int_{\mathbb{D}} \mathrm{rot} G dx \\
&= \int_0^{2\pi} \langle G\circ\gamma, \gamma'\rangle dt \\
&= \int_0^{2\pi} \langle \begin{pmatrix} \langle F\circ\psi\circ\gamma, \partial_1\psi\circ\gamma\rangle \\ \langle F\circ\psi\circ\gamma, \partial_2\psi\circ\gamma\rangle \end{pmatrix}, \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix}\rangle\, dt \\
&= \int_0^{2\pi} \sum_{j=1}^3 (F_j\circ\psi\circ\gamma)\Big((\partial_1\psi_j\circ\gamma)\gamma_1' + (\partial_2\psi_j\circ\gamma)\gamma_2'\Big) dt \\
&= \int_0^{2\pi} \langle F\circ\psi\circ\gamma, (\psi\circ\gamma)'\rangle dt = \int_\Gamma F\cdot d\Gamma,
\end{aligned}$$

as claimed. □

EXAMPLE 14.54. — Let γ be the concatenation of the four paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4 : [0,1]\to\mathbb{R}^3$ given by

$$\begin{aligned}
\gamma_1(t) &= (t, 0, t^2), \\
\gamma_2(t) &= (1, t, 1 - t^2), \\
\gamma_3(t) &= (1 - t, 1, (1 - t)^2 - 1),
\end{aligned}$$

32

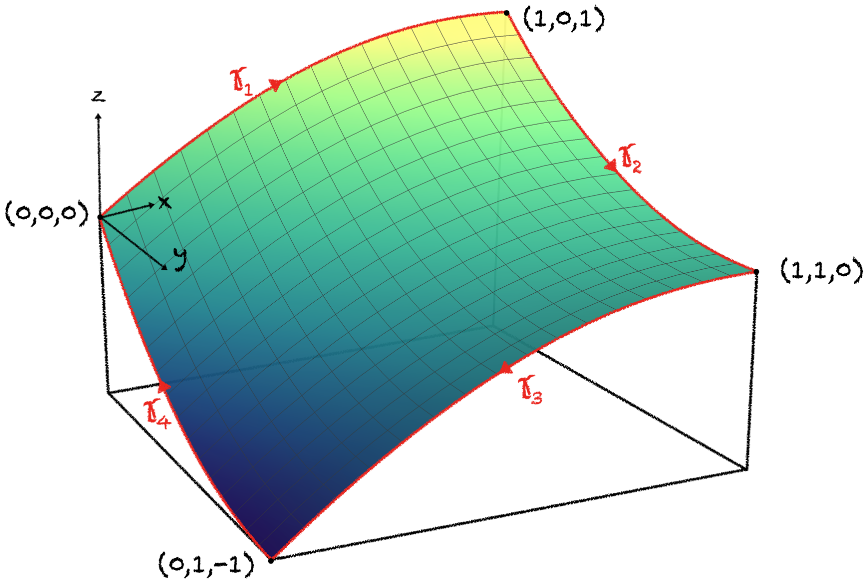
$$\gamma_4(t) = (0, 1-t, -(1-t)^2)).$$

The first two components of γ describe a parameterization of the unit square $[0,1]^2$, and the third component, over a point (x,y), is at height $z = x^2 - y^2$. Thus, γ parametrizes the boundary of the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x, y \leq 1,\ \ z = x^2 - y^2\}$$

which we can understand as the graph of the function $f : [0,1]^2 \to \mathbb{R}$ given by $f(x,y) = x^2 - y^2$. We can parametrize S with a single chart $\psi : [0,1]^2 \to \mathbb{R}^3$, given by $\psi(x,y) = (x, y, x^2 - y^2)$. Calculating in advance:

$$\partial_x\psi(x,y) = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \quad \partial_y\psi(x,y) = \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix}, \quad \partial_x\psi \times \partial_y\psi = \begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix}.$$



Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x,y,z) = \begin{pmatrix} yz + \cos(x) \\ xz + \sin(y) \\ 2xy \end{pmatrix}.$$

The path integral $\int_\gamma F dt$ can be calculated directly from the definition, where it is expressed as the sum of four path integrals. In this case, however, the latter are relatively complicated to compute. The rotation of the vector field F is, however, of a simple form. Indeed, it holds that

$$\mathrm{rot}F(x,y,z) = \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}$$

for all $(x, y, z) \in \mathbb{R}^3$. According to Stokes' theorem, we have

$$\int_\gamma F dt = \int_S \mathrm{rot}(F) dn = \int_0^1 \int_0^1 \langle \mathrm{rot}(F), \partial_x \psi \wedge \partial_y \psi \rangle dx dy$$
$$= \int_0^1 \int_0^1 (-2x^2 - 2y^2) dx dy = -\tfrac{4}{3}.$$

Exercise 14.55. — Calculate, using Stokes' theorem, the path integral $\int_\gamma F dt$, where the vector field F on $\mathbb{R}^3$ and the path $\gamma : [0, 1] \to \mathbb{R}^3$ are given by

$$F(x, y, z) = \begin{pmatrix} 2xz\cos(x^2) \\ z \\ \sin(x^2) \end{pmatrix} \quad \text{and} \quad \gamma(t) = \begin{pmatrix} t \\ t^2 \\ (t)^2 - (t^2)^2 \end{pmatrix}$$

respectively. As a hint, note that γ takes values in the surface $z = x^2 - y^2$.

Exercise 14.56. — Let f be a twice continuously differentiable function, and F a twice
32 continuously differentiable vector field on an open set $U \subseteq \mathbb{R}^3$. Show that

$$\mathrm{rot}(\mathrm{grad}(f)) = 0 \quad \text{and} \quad \mathrm{div}(\mathrm{rot}(F)) = 0$$

hold.

Exercise 14.57. — Given the smooth vector field $F(x, y, z) = (yz, x^2, 1)$ on $\mathbb{R}^3$ and the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ 0 < z < 1\}$$

compute the flux integral $\int_M \mathrm{rot}F \cdot \nu dS$ directly and using Stokes' theorem.

Exercise 14.58. — Determine real parameters α, β, γ such that the vector field

$$(x, y, z) = (x + 2y + \alpha z, \beta x - 3y - z, 4x + \gamma y + 2z)$$

in $\mathbb{R}^3$ becomes irrotational, i.e., $\mathrm{rot}(F) = 0$. Determine, after choosing these parameters, a potential for F.

Exercise 14.59. — Let $0 < h < 1$, and let S be the part of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ satisfying $z > -h$. Consider the vector field $F(x, y, z) = (-y, x, 0)$. Calculate the flux

$\int_S \operatorname{rot}(F)dn$ outwardly directly, once with Stokes' theorem, and once with Gauss' theorem.

Exercise 14.60. — Consider on $U = \mathbb{R}^3 \setminus (\{(0,0)\} \times \mathbb{R})$ the vector field

$$F(x,y,z) = \left(\frac{2(xz+y)}{x^2+y^2}, \frac{2(yz-x)}{x^2+y^2}, \log\left(x^2+y^2\right)\right)$$

and the curves $\gamma_1, \gamma_2 : [0, 2\pi] \to U$ given by

$$\gamma_1(t) = (\cos(t), \sin(t), 0), \quad \text{and} \quad \gamma_2(t) = (2\cos(t), 2\sin(t), 2)$$

Calculate the difference $\int_{\gamma_1} Fdt - \int_{\gamma_2} Fdt$ using Stokes' theorem. Is F conservative?

Exercise 14.61. — The **Laplace operator** on $\mathbb{R}^3$ is

$$\Delta : C^2(\mathbb{R}^3, \mathbb{R}) \to C^0(\mathbb{R}^3, \mathbb{R}) \quad \Delta f(x) = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} f(x)$$

given, or alternatively by $\Delta f = \operatorname{div}(\operatorname{grad} f)$. Prove the identity

$$\operatorname{rot}(\varphi f) = (\operatorname{grad}\varphi) \times F$$

32 for continuously differentiable vector fields $F : \mathbb{R}^3 \to \mathbb{R}^3$ and C^2 functions $\varphi : \mathbb{R}^3 \to \mathbb{R}$.

Exercise 14.62. — Let $A \in \mathrm{GL}_3(\mathbb{R})$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field. Define the vector field $F_A := A \circ F \circ A^{-1}$. Prove that $\operatorname{div}(F_A) = \operatorname{div}(F) \circ A^{-1}$ holds, and prove that

$$\operatorname{rot}(F_A) = \operatorname{rot}(F) \circ A^{-1}$$

if $A \in \mathrm{SO}(3, \mathbb{R})$.

Exercise 14.63. — In this exercise, we want to show that a solution u of the wave equation can be assigned a natural energy $E(t)$ that remains constant over time. Let $B \subseteq \mathbb{R}^3$ be a smoothly bounded region. Let $f, g : U \to \mathbb{R}$ be twice continuously differentiable functions defined on an open neighborhood U of B. Prove the **Green's formula**

$$\int_B \langle \operatorname{grad} f, \nabla g \rangle dx = \int_{\partial B} g\,(\operatorname{grad} f) dn - \int_B g\,(\Delta f) dx$$

Now consider a twice differentiable function $u : \mathbb{R} \times U \to \mathbb{R}$, where u is a function in three "space variables" $p = (x, y, z)$ and one "time variable" t. Write

$$\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$$

for the Laplace operator applied to u with respect to the spatial variables. Assuming u satisfies the wave equation

$$\begin{cases} \partial_t^2 u(t,p) = \Delta u(t,p) & \text{for all } (t,p) \in \mathbb{R} \times B \\ u(t,p) = 0 & \text{for all } (t,p) \in \mathbb{R} \times \partial B \end{cases}$$

Show using Green's formula that the energy function $E : \mathbb{R} \to \mathbb{R}_{>0}$ given by

$$E(t) = \int_B \Big((\partial_t u(t,p))^2 + (\partial_x u(t,p))^2 + (\partial_y u(t,p))^2 + (\partial_z u(t,p))^2 \Big) dxdydz$$

is constant for $t \in \mathbb{R}$.

14.4 A glimpse into differential forms

In this section, abusing notation, we write $D\phi_y$ to denote both the differential and the its associated (Jacobi) matrix.

Out most fundamental guiding principle when we defined surface integrals was the invariance under reparamentrization. Let $M \subset \mathbb{R}^n$ is an m-dimensional submanifold, and $\phi : V \to M$ a regular parametrization (i.e. $D\phi_x$ has maximal rank for all $x \in V$). Take any positive diffeomorphism (i.e., $\psi : W \to V$ with $\det(D\psi_y) > 0$ for all $y \in W$. Then, for all
32 (integrable) $h : M \to \mathbb{R}$, we have:

$$\int_M h d\mathrm{vol}_m = \int_V h \circ \phi \sqrt{\det D\phi^T D\phi} dx = \int_W h \circ \phi \circ \psi \circ \sqrt{\det D(\phi \circ \psi)^T D(\phi \circ \psi)} dy.$$

It then is natural to abstract this and ask: what other 'integrals over parametrized submanifolds' are invariant under reparametrization?

In what follows $\mathbb{R}^{k \times l}$ denotes the space of $k \times l$ matrices, also denoted $\mathrm{Mat}_{k,l}(\mathbb{R})$, corresponding to linear maps $\mathbb{R}^l \to \mathbb{R}^k$

More precisely, assume $U \subset \mathbb{R}^n$ is open and $M \subset U$. We would like to find functions $\omega : U \times \mathbb{R}^{n \times m} \to \mathbb{R}$, $\omega : (p, A) \mapsto \omega_p(A) = \omega(p, A)$, such that

$$\int_V \omega_{\phi(x)}(D\phi_x)\, dx = \int_W \omega_{\phi \circ \psi(y)}(D(\phi \circ \psi)_y)\, dy, \tag{14.9}$$

for every reparametrization.

This invariance is key if we want to to meaningfully define

$$\int_M \omega := \int_V \omega_{\phi(x)}(D\phi_x)\, dx.$$

Here, abusing notation, we are denoting by $D\Phi_x$ the Jacobi matrix $J\Phi(x)$.

Notice that $\omega_p(A) = h(p)\sqrt{\det(A^T A)}$ satisfies it. But are there other possibilities?

Recalling that the invariant under reparametrization ultimately boils down to the change of variables formula of $\mathbb{R}^m$ it is natural to consider the the determinant of a certain square matrix obtained from A by some transformation.

Using the change of variables formula, it is not too complicated to show that the condition ω must satisfy in order to (14.9) to hold is:

$$\omega_p(AB) = \omega_p(A)\text{det}(B), \qquad \text{for all } A \in \mathbb{R}^{n\times m} \text{ and } B \in \mathbb{R}^{m\times m}.$$

Arguably, the simplest possible transformation achieving this is multiplication by a $m \times n$ matrix: Given $\Theta : U \to \mathbb{R}^{m\times n}$ (smooth), $\Theta : p \mapsto \Theta_p = \Theta(p)$, put

$$\omega_p(A) = \text{det}(\Theta_p A). \tag{14.10}$$

Notice that when $A = D\phi_x$ (a $n \times m$) the matrix $\Theta_p A$ will be $m \times m$ and hence we can compute the determinant of this square matrix.

Notice also that if two maps ω_1 and ω_2 satisfy (14.9) then for any pair of (smooth) functions $f_1, f_2 : U \to \mathbb{R}$ the linear combination

$$\omega(p, A) = f_1(p)\omega_1(p, A) + f_2(p)\omega_2(p, A)$$

also satisfies (14.9)

32 Hence it is natural to consider the vector space of all ω of finite combinations form

$$\omega(p, A) = \sum_{\ell=1}^{N} f_k(p)\omega_\ell(p, A) \quad \text{where } \omega_\ell(p, A) = \text{det}(\Theta_{\ell,p} A). \tag{14.11}$$

However, there is a lot of redundancy in the representations (14.11). To find 'canonical' unique representations let us consider the set of all increasing multiindexes of length m

$$\mathcal{I}_m := \{I = (i_1, i_2, \dots, i_m) \mid 1 \le i_1 < i_2 < \cdots < i_m \le n\},$$

and let $\vartheta_I : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map (or matrix) defined for all $v \in \mathbb{R}^n$ by

$$\vartheta_I v := \sum_{k=1}^{m} v_{i_m} e_m, \qquad \text{or in other words} \qquad \vartheta_I : (v_1, \dots, v_n) \mapsto (v_{i_1}, \dots, v_{i_m}).$$

Using the multilinearity of the determinant one can show (it is a not so easy exercise of linear algebra) that every ω of the form (14.11) can be uniquely written as

$$\omega(p, A) = \sum_{I \in \mathcal{I}_m} f_I(p)\text{det}(\vartheta_I A) \tag{14.12}$$

where $f_I : U \to \mathbb{R}$ are (smooth) functions.

In the differential geometry literature, the map $A \mapsto \det(\vartheta_I A)$ is commonly denoted by

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}.$$

As a mathematical object, the maps $\omega : U \times \mathbb{R}^{n\times m} \to \mathbb{R}$ of the form (14.12) are called **differential m-forms** in U.

As we have seen the most fundamental property of m-forms is the fact that it is meaningful to integrate them over a submanifold, in the same that every reparametrization gives the same result.

Also, let $\Psi : \widetilde{U} \to U$ is a diffeomorphism, and let $\widetilde{M} := \Psi^{-1}(M) \subset \widetilde{U}$.

Then it is an simple exercise (again using the change of variable formula) to prove that the so called **pull-back** form $\Psi^*\omega$ defined as

$$(\Psi^*\omega)_q(A) = \omega_{\Psi(q)}(D\Psi_q A)$$

is a m-form in $\widetilde{U}$ satisfying

$$\int_{\widetilde{M}} \Psi^*\omega = \int_{\Psi(\widetilde{M})} \omega$$

for any $\widetilde{M} \subset \widetilde{U}$ smooth m-submanifold (i.e. $M = \Psi(\widetilde{M}) \subset U$ smooth m-manifold).

However, what makes differential forms really useful mathematical objects is the so-called **exterior differential**, an opertor, denoted d, which acts on m-forms transforming them
32 into $(m+1)$-forms. Toghether with the crucial (and nontrivial) fact that d commutes with pull-backs.

For ω of the form

$$\omega(p, A) = \sum_{I\in\mathcal{I}_m} f_I(p)\det(\vartheta_I A)$$

and $B : \mathbb{R}^{n\times(m+1)}$ one defines

$$d\omega(p, B) = \sum_{I\in\mathcal{I}_m} \det\left(\begin{pmatrix} \vartheta_I \\ \hline Df_I(p) \end{pmatrix} B\right),$$

where $Df_I(p) = \nabla f_I(p)^T$ denotes the Jacobi 'matrix' (a row vector).

Within this language one has the generalized Stoke's theorem: Assume that $U \subset \mathbb{R}^n$ and that ω is a m-form in U. Suppose that $M \subset U$ is an oriented $(m+1)$-submanifold M with boundary.

We will not define this notion precisely. Let us just say the following:

- That M is oriented means that that there is a consistent (and continuous) notion of positivity for basis of the tangent spaces for all points of M. (A orientation corresponds to choosing a continuous unit normal in the cases of surfaces in $\mathbb{R}^3$.)
- That M is a $(m+1)$-submanifold M with boundary means that the set $\overline{M} \setminus M \subset \mathbb{R}^n$, is m-submanifold, which is denoted ∂M. When M is oriented with boundary, one can

see that ∂M naturally 'inherits' an orientation from that of M. (For example, in the Stokes' theorem for surfaces in $\mathbb{R}^3$ we consider immersed disks. Then the boundary is a closed curve, and this curve is oriented depending on the chosen normal vector.)

32 Within the previous setup we have the following 'Generalized Stokes' Theorem':

$$\int_M d\omega = \int_{\partial M} \omega.$$

This statentement is the generalization Stokes' theorem to submanifolds of arbitrary dimension m in ambient spaces of arbitrary dimension n (with $0 < m < n$).

Chapter 15

Ordinary differential equations

Differential equations play a central role in science, technology and economics. In mathematics, the study of differential equations is a very broad subject area in its own right. Parts of it are for example the numerical solution of differential equations, geometric theory of differential equations, functional analysis, or the study of special classes of nonlinear equations.

This chapter is a continuation of Chapter 8 (in Analysis I).

15.1 Systems of Differential Equations

15.1.1 Linear Differential Equations

15.1. — In this section, we fixed and open, non-empty interval $I \subseteq \mathbb{R}$ and $t_\circ \in I$.

Let $d \in \mathbb{N}$. A **linear ODE with constant coefficients** of order m is an equation of the following type:

$$y^{(m)} + a_{d-1}y^{(m-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t), \tag{15.1}$$

where $a_0, a_1 \ldots, a_{m-1}$ belong to $\mathbb{R}$ (or, possibly, to $\mathbb{C}$). For simplicity we will assume $f : I \to \mathbb{R}$ (or $\mathbb{C}$) to be of class C^∞

Here, as it is standard, we denote $' = \frac{d}{dt}$,

33 $$y^{(k)} = \frac{d^k}{dt^k} y = (\tfrac{d}{dt})^k y,$$

in other words, we have the recursive definition:

$$y^{(0)} = y, \quad y^{(k)} = (y^{(k-1)})'.$$

Our goal will be to find solutions $y = y(t)$ to the equation (15.1). When the coefficients a_i are real, we will typically look for functions $y : I \to \mathbb{R}$ solving (15.1). However, as will soon discover, it will be convenient to consider also complex-valued solutions $y : I \to \mathbb{C}$, as this will lead to a simpler and more unified theory (e.g. the two solutions $\cos t$ and $\sin t$ of $f'' + f = 0$ arise as the real and imaginary parts of the complex solution e^{it}). As we will see, the need to consider complex-valued solutions will be intimately connected with the fact that polynomials —even with those real coefficients— may have complex roots.

15.2. — The **initial conditions** for the **initial value problem** are given:

$$y^{(m-1)}(t_\circ) = w_{m-1}, \quad \dots \quad, \quad y'(t_\circ) = w_1, \quad y(t_\circ) = w_0, \tag{15.2}$$

where $w_0, w_1, \dots, w_{m-1} \in \mathbb{R}$.

15.3. — **Structure of the solution set**: The set of all solutions of (15.1) has the form $y_{\text{part}} + V$ where y_{part} is a particular solution and V is the set solution of the **homogeneous equation**: that is (15.1) with $f(t)$ replaced by 0:

$$Lz := z^{(d)} + a_{d-1}z^{(d-1)} + \cdots + a_2y'' + a_1z' + a_0z = 0, \tag{15.3}$$

Indeed, notice that the set V of solutions $z = z(t)$ to $Lz = 0$ is a $\mathbb{C}$-vector space because if $z_1(t)$ and $z_2(t)$ are two solutions then $c_1z_1(t) + c_2z_2(t)$ is also a solution for all $c_1, c_2 \in \mathbb{C}$.

15.4. — To find solutions of the homogeneous differential equation $Lz = 0$ we consider the Ansatz $z(x) = \exp(\alpha x)$ for $\alpha \in \mathbb{C}$. Then $z^{(\ell)} = \alpha^\ell z$ and

33

$$Lz = (\alpha^{(m)} + a_{d-1}\alpha^{(m-1)} + \cdots + a_2y'' + a_1\alpha + a_0)z$$

Hence, $z(t) = \exp(\alpha t)$ is a solution of the differential equation $Lz = 0$ if and only if $\alpha \in \mathbb{C}$ is a root of the so-called **characteristic polynomial** of the operator L

$$p(X) = X^m + a_{m-1}X^{m-1} + \ldots + a_1X + a_0 \in \mathbb{R}[X] \quad (\text{or } \mathbb{C}[X])$$

A useful notation is

$$L = p(\tfrac{d}{dt}) = (\tfrac{d}{dt})^m + a_{m-1}(\tfrac{d}{dt})^{m-1} + \ldots + a_1(\tfrac{d}{dt}) + a_0.$$

This must be interpreted as an identity of **linear operators**, i.e. —in this context— $\mathbb{C}$-linear maps $C^\infty(I, \mathbb{C}) \to C^\infty(I, \mathbb{C})$. In other words, operators act (linearly) on $C^\infty(I, \mathbb{C})$

If α is a real root of $p(X)$, then $z(t) = \exp(\alpha t)$ provides a real-valued solution. For real coefficients and a complex root $\alpha = \beta + \gamma i \in \mathbb{C}$ with $\beta, \gamma \in \mathbb{R}$, $\overline{\alpha} = \beta - \gamma i$ is also a root of $p(X)$. Thus, all linear combinations $c_1 \exp(\alpha t) + c_2 \exp(\overline{\alpha} t)$, and in particular

$$\frac{\exp(\alpha t) + \exp(\overline{\alpha} t)}{2} = \exp(\beta t)\cos(\gamma t), \quad \frac{\exp(\alpha t) - \exp(\overline{\alpha} t)}{2i} = \exp(\beta x)\sin(\gamma t)$$

are solutions of $Lz = 0$.

Notice that $\exp(\alpha t)$ and $\exp(\overline{\alpha} x)$ span the same $\mathbb{C}$-subspace of solutions to $Lz = 0$ as $\exp(\beta t)\cos(\gamma x)$ and $\exp(\beta x)\sin(\gamma t)$. In the following general discussion, we will no longer perform this transition from exponential functions to products of exponential functions and trigonometric functions, except in concrete examples with a physical background where we are interested in real-valued solutions. If $p(X)$ has exactly d distinct roots over $\mathbb{C}$, then the above discussion yields d linearly independent solutions over $\mathbb{R}$.

Exercise 15.5. — Let $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ be pairwise distinct complex numbers and $I \subseteq \mathbb{R}$ a non-empty, open interval. Show that the complex-valued functions $z_k \in C^\infty(I)$

$$z_k(t) = \exp(\alpha_k t)$$

for $k = 1, \ldots, ,$ are linearly independent over $\mathbb{C}$.

15.6. — We now consider the case where the characteristic polynomial of the differential operator L has a multiple root. To do this, we examine linear combinations of functions of the form

$$z(t) = t^n \exp(\alpha t),$$

for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$. Notice that

$$\tfrac{d}{dt}(t^n e^{\alpha t}) = n t^{n-1} e^{\alpha t} + \alpha t^n e^{\alpha t} \tag{15.4}$$

33

Proposition 15.7: Solutions of poly-exp form

Let $p \in \mathbb{C}[X]$ be a polynomial of degree $m \geq 1$, factored as

$$p(X) = \prod_{j=1}^{k} (X - \alpha_j)^{m_j}$$

where we assume that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $L = p(d/dt)$ be the differential operator with characteristic polynomial p. Then, the associated homogeneous differential equation $Lz = 0$ has m linearly independent solutions $z(t) = t^n \exp(\alpha_j t)$ for $0 \leq n < m_j$ and $1 \leq j \leq k$.

Proof. By assumption, the polynomial p has degree $m = \sum_{j=1}^{k} m_j$, and we have indeed provided d functions by Proposition 15.8 that are linearly independent. For a fixed j and $n \geq 0$, we have

$$(\tfrac{d}{dt} - \alpha_j)(t^n e^{\alpha_j t}) = t^{n-1} e^{\alpha_j t} + \alpha_j t^n e^{\alpha_j t} - \alpha_j t^n e^{\alpha_j t} = n t^{n-1} e^{\alpha_j t}.$$

For $0 \leq n < m_j$, using this computation and induction yields

$$(\tfrac{d}{dt} - \alpha_j)^{m_j}(t^n e^{\alpha_j t}) = 0.$$

Therefore,

$$p(\tfrac{d}{dt})(t^n e^{\alpha_j t}) = \left(\prod_{i\neq j}(\tfrac{d}{dt} - \alpha_i)^{m_i}\right)(\tfrac{d}{dt} - \alpha_j)^{m_j}(t^n e^{\alpha_j t}) = 0,$$

which completes the proof. □

Proposition 15.8: Linear independence of poly-exp functions

The functions $t \mapsto t^n \exp(\alpha t)$, $\alpha \in \mathbb{C}$ $n \in \mathbb{N}$ are linearly independent (within the $\mathbb{C}$-vector space of C^∞ functions $I \to \mathbb{C}$).

Proof. Assume by contradiction that there exist pairwise distinct $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ and nonzero polynomials $q_1(X), \dots, q_N(X) \in \mathbb{C}[X]$ such that

$$\sum_{k=1}^{N} q_k(t)\exp(\alpha_k t) = 0 \qquad \text{for all } t \in I. \tag{15.5}$$

Among all such relations, consider one for which $N \geq 1$ is minimal. If $N = 1$, multiply (15.5) by $\exp(-\alpha_1 x)$ to obtain the equation $q_1(t) = 0$ for all $t \in I$, contradicting $q_1(X) \neq 0$. Thus, assume $N > 1$. Multiply (15.5) by $\exp(-\alpha_N)$ to get

33

$$\sum_{k=1}^{N-1} q_k(t)\exp((\alpha_k - \alpha_N)t) + q_N(t) = 0$$

for all $x \in I$. Now, and differentiate with respect to t exactly $\deg(q_N) + 1$ times. The term $q_N(x)$ vanishes, and according to (15.4), $q_k(x)\exp((\alpha_k - \alpha_N)t)$ turns into an expression of the form $p_k(t)\exp((\alpha_k - \alpha_N)t)$, where $p_k(X) \in \mathbb{C}[X]$ has the same degree as q_k. Thus,

$$\sum_{k=1}^{N-1} p_k(t)\exp((\alpha_k - \alpha_N)t) = 0$$

for all $t \in I$, contradicting the minimality of N. □

Exercise 15.9. — For $\alpha \in \mathbb{C}$ and $N \in \mathbb{N}$, show that the linear operator $\frac{d}{dt}$ the linear space generated by

$$\exp(\alpha t),\ x\exp(\alpha t),\ \frac{x^2}{2!}\exp(\alpha t),\ \frac{t^3}{3!}\exp(\alpha x), \dots, \frac{t^N}{N!}\exp(\alpha t),$$

to itself.

Compute the matrix of the linear operator with respect to this basis (of the considered subspace). Generalize this for multiple exponents $\alpha_1, \dots, \alpha_k$ and notice the Jordan normal form of obtained matrices.

15.10. — The next natural step is to show that, actually, the solutions obtained in Proposition 15.7 are *all* solutions to the homogeneous equation $Lz = 0$.

For this we need the following

Proposition 15.11: The initial value problem for linear constant coefficient ODE

Let $p \in \mathbb{C}[X]$ be a polynomial of degree $m \geq 1$ and put $L = p(\frac{d}{dt})$, i.e. the operator with characteristic polynomial p. For every given set of initial conditions

$$z^{(m-1)}(t_\circ) = w_{m-1}, \quad \dots \quad , \quad z'(t_\circ) = w_1, \quad y(t_\circ) = w_0, \tag{15.6}$$

where $w_0, w_1, \dots, w_{d-1} \in \mathbb{C}$.

There is a unique solution of $Lz = 0$ which satisfies them.

Moreover, this solution is spanned by the poly-exp solutions obtained in Proposition 15.7

To prove the proposition we will need the following

Lemma 15.12: zero in - zero out

Let $p \in \mathbb{C}[X]$ be a polynomial of degree $m \geq 1$ and put $L = p(\frac{d}{dt})$. If $z : I \to \mathbb{C}$ of class 33 C^m satisfies $Lz = 0$ in and (15.6) with $w_0 = w_1 = \cdots = w_{m-1} = 0$ then $z(t) = 0$ for all $t \in I$.

Lemma 15.13: Grönwall's inequality

Suppose that $u \in C^1([a, b])$ satisifes

$$u'(t) \leq \beta(t)u(t) \quad \textit{for all } t \in [a, b],$$

where $\beta \in C^0([a, b])$ is some given function.

Then,

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)ds\right)$$

Proof. Let $v(t) := \exp\left(\int_a^t \beta(s)ds\right) > 0$. Then, $v'(t) = \beta(t)v(t)$.

Now, consider the C^1 function:

$$f(t) := \frac{u(t)}{v(t)}$$

and notice that

$$f'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{v(t)^2} \leq \frac{\beta(t)u(t)v(t) - u(t)\beta(t)v(t)}{v(t)^2} = 0$$

Hence (using e.g. the mean value theorem) the function $f : [a, b] \to \mathbb{R}$ is decreasing which gives

$$\frac{u(t)}{v(t)} \leq \frac{u(a)}{v(a)} = 1.$$

Hence, $u(t) \leq v(t)$ for all $t \in [a, b]$ as claimed. □

We can now give the

Proof of Lemma 15.12. Consider the auxiliary function

$$u(t) := (z(t))^2 + (z'(t))^2 + \cdots + (z^{(m-1)}(t))^2$$

Using the ODE $p(\frac{d}{dt})z = 0$ we obtain $|z^{(m)}| \leq C \sum_{i=0}^{m-1} |z^{(i)}|$ where $C = \max_{1 \leq i \leq m-1} |a_i|$. Therefore,

$$\begin{aligned} u'(t) &= 2 \sum_{i=0}^{m-2} z^{(i)}(t) z^{(i+1)}(t) + 2 z^{(m-1)}(t) z^{(m)}(t) \\ &\leq 2 \sqrt{\sum_{i=0}^{m-2} (z^{(i)}(t))^2} \sqrt{\sum_{i=1}^{m-1} (z^{(i)}(t))^2} + C |z^{(m-1)}(t)| \sum_{i=0}^{m-1} |z^{(i)}| \\ &\leq 2(1+C) \sum_{i=0}^{m-1} |z^{(i)}|^2 = 2(1+C) u(t). \end{aligned}$$

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Therefore Grömwall's inequality gives $u(t) \leq u(t_\circ) e^{(1+C)t}$. So, if the initial conditions are $w_0 = w_1 = \cdots = w_{m-1} = 0$ we have $u(t_\circ) = 0$ and hence $0 \leq u(t) \leq u(t_\circ) = 0$ holds for all $t \geq t_\circ$. In particular since $(z(t))^2 \leq u(t)$ we conclude that $z \equiv 0$ for $t \geq t_\circ$.

To prove the same for $t \leq t_\circ$ we can use essentially the same argument, up to reversing time.

Consider $\widetilde{u}(t) = (z(-t))^2 + (z'(-t))^2 + \cdots + (z^{(m-1)}(-t))^2$. The same argument as before (with a few extra minus signs here and there which do not harm the estimate) gives

$$\widetilde{u}'(t) \leq 2(1+C)\widetilde{u}(t),$$

and hence, by Gömwall we have $0 \leq \widetilde{u}(t) \leq \widetilde{u}(-t_\circ) = 0$ for $t \geq -t_\circ$ which implies $z \equiv 0$ for $t \leq t_\circ$. □

Proof of Proposition 15.12. Let V denote the linear subspace of $C^m(I, \mathbb{C})$ of all solutions of $Lz = 0$. Consider the linear evaluation map $E : V \to \mathbb{C}^m$ defined by

$$E : z \mapsto (z(t_\circ), z'(t_\circ), \ldots, z^{(m-1)}(t_\circ)).$$

By Lemma 15.12 the map E has trivial kernel (i.e., $z(t) \equiv 0$ is the only element of the kernel). Hence, $\dim(V) \leq \dim(\mathbb{C}^m) = m$.

But by Propostion 15.7 the dimension of V is at least m, since the m different ‘poly-exp’ functions from Proposition 15.7 belong to V and are linearly independent (by Proposition 15.8).

33 This proves that E is a linear isomorphism between V and $\mathbb{C}^m$. In other words there is a one-to-one linear correspondence between solution to the homogeneous equation and given initial conditions.

In particular, every solution is of a (complex) linear combination of the ‘poly-exp functions constructed in 15.7. □

15.14. — We now turn to the inhomogeneous problem. For a polynomial $p(X) \in \mathbb{C}[X]$ and a forcing function $f(t)$ of ‘poly-exponential’ form

$$f(t) = q(t)e^{\alpha t}$$

for some $q(X) \in \mathbb{C}[X]$ and $\alpha \in \mathbb{C}$, there is a simple procedure to find a particular solution y_{part} to the inhomogeneous differential equation $p(\frac{d}{dt})y = f$.

1. If $f(t) = q(t)e^{\alpha t}$ for a polynomial q of degree n and $\alpha \in \mathbb{C}$ with $p(\alpha) \neq 0$, then use the Ansatz $y_{\mathrm{part}}(t) = Q(t)e^{\alpha t}$, where $Q(X)$ is a polynomial of degree n with coefficients yet to be determined. Now, calculate $p(\frac{d}{dt})y_{\mathrm{part}}$ and determine coefficients so that $p(\frac{d}{dt})y_{\mathrm{part}} = f$ holds.

2. If $f(x) = q(x)e^{\alpha x}$ for a polynomial $q(X)$ of degree n and $\alpha \in \mathbb{C}$ with $p(\alpha) = 0$, then repeat the above procedure, but with the Ansatz $y_{\mathrm{part}}(t) = Q(t)t^m e^{\alpha t}$, where m indicates the multiplicity of the root α of $p(X)$. 34

3. For every f that linear combination of ‘poly-exp’ functions, one can apply the above procedure to summands of the form $q(x)e^{\alpha x}$ in g and then add the resulting solution functions. The simple, but crucial, making this possible is the so-called:

   Superposition principle: If $y_\ell : I \to \mathbb{C}$ solve

   $$Ly_\ell = f_\ell \quad \text{with} \quad y_\ell^{(i)}(t_\circ) = w_{i,\ell} \quad \text{for } 0 \leq i \leq m-1 \text{ and } 1 \leq \ell \leq N,$$

   then the linear combination $y = \sum_{\ell=1}^N c_\ell y_\ell$, where $c_\ell \in \mathbb{C}$ solves

   $$Ly = \sum_{\ell=1}^N c_\ell f_\ell \quad \text{with} \quad y^{(i)}(t_\circ) = \sum_{\ell=1}^N c_\ell w_\ell.$$

Example 15.15. — (Pure resonance of ‘pumped’ harmonic oscillator)

Consider the forced harmonic oscillator $y : [0, \infty) \to \mathbb{R}$

$$\begin{cases} y''(t) + ky(t) = \sin(\omega t) & \text{for } t > 0 \\ y(0) = y'(0) = 0 \end{cases}$$

The general solution to the homogeneous equation $z''(t) + kz(t) = 0$ is

$$z(t) = C_1 \cos(\sqrt{k}t) + C_2 \sin(\sqrt{k}t).$$

For the particular solution, we plug

$$y_{\text{part}}(t) = A \sin(\omega t) + B \cos(\omega t).$$

Substituting $y_{\text{part}}(t)$ into the non-homogeneous equation gives

$$-A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) + k(A \sin(\omega t) + B \cos(\omega t)) = \sin(\omega t).$$

Equating coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, we get

$$A(k - \omega^2) = 1 \quad \text{and} \quad B(k - \omega^2) = 0.$$

34 Thus, $B = 0$ and $A = \frac{1}{k-\omega^2}$ (assuming $k \neq \omega^2$). Therefore, the particular solution is

$$y_{\text{part}}(t) = \frac{\sin(\omega t)}{k - \omega^2}.$$

A general solution is of the form

$$y(t) = z(t) + y_{\text{part}}(t) = C_1 \cos(\sqrt{k}t) + C_2 \sin(\sqrt{k}t) + \frac{\sin(\omega t)}{k - \omega^2}.$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$,

$$y(0) = C_1 = 0,$$

$$y'(t) = -C_1 \sin(\sqrt{k}t) + C_2\sqrt{k} \cos(\sqrt{k}t) + \frac{\omega \cos(\omega t)}{k - \omega^2},$$

$$y'(0) = C_2\sqrt{k} + \frac{\omega}{k - \omega^2} = 0.$$

Solving for C_2,

$$C_2 = -\frac{\omega}{\sqrt{k}(k - \omega^2)}.$$

Thus, the complete solution for $\omega \neq \sqrt{k}$ is

$$y(t) = -\frac{\omega}{\sqrt{k}(k - \omega^2)} \sin(\sqrt{k}t) + \frac{\sin(\omega t)}{k - \omega^2} = \frac{1}{(\sqrt{k} + \omega)} \left(\frac{\sin(\sqrt{k}t)}{\sqrt{k}} - \frac{\sin(\sqrt{k}t) - \sin(\omega t)}{\sqrt{k} - \omega}. \right)$$

In the case of resonance, i.e. in the limit $(\sqrt{k}-\omega) \to 0$ we obtain:

$$y(t) := \frac{1}{2\sqrt{k}} \left(\frac{\sin(\sqrt{k}t)}{\sqrt{k}} - t\cos(\sqrt{k}t) \right).$$

This shows that the amplitude grows linearly with time when the driving frequency matches the natural frequency of the system, leading to 'resonance'.

There is fascinating scientific "mythology" associated with resonance: from opera singers shattering glasses by hitting the right pitch to bridges collapsing when troops march in unison.

Exercise 15.16. — Illustrate the above procedure with an example and show that it always leads to a solution.

Exercise 15.17. —

1. Determine the general solution on $I = \mathbb{R}$ of the differential equation $y'' - 4y' + 4y = \sin t$. 34
2. Determine the solutions on $I = \mathbb{R}$ of the initial value problem

$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 3.$$

3. Find a solution on the interval $I = (0, \infty)$ of the initial value problem

$$y' - \left(\frac{4}{t} + 1\right) y = t^4, \quad y(1) = 1.$$

[Hint: consider (multiplication by integrating factor) $u(t) := e^{H(t)}y(t)$, with H satisfying $H'(t) = \frac{4}{t} + 1$.]

15.18. — Lastly, we give *Duhamel's principle* allowing to represent *any* forced solution as a "sum", actually integral, of *pulses*.

Proposition 15.19: Duhamel's representation formula

Given a linear constant coefficient differential operator L of order $m \geq 0$, ***pulse at zero*** *to $P_0 : [0, \infty) \to \mathbb{R}$ to be the solution to the homogeneous equation*

$$LP_0 = 0 \text{ with } P_0^{(m-1)}(0) = 1, P_0^{(m-2)}(0) = \cdots = P_0'(0) = P_0(0) = 0.$$

Then, given $I \subset \mathbb{R}$ open interval with $t_\circ \in I$ and $g : I \to \mathbb{R}$ continuous the integral formula

$$y(t) = \int_{t_\circ}^{t} P_0(t-s) f(s) ds \tag{15.7}$$

gives the (unique) solution of

$$Ly = f(t) \quad \text{with } y^{(m-1)}(t_\circ) = \cdots = y'(t_\circ) = y(t_\circ) = 0.$$

34 In the following proof we need to use the following

Exercise 15.20. — Let $F : [a, b] \to \mathbb{R}$ be defined by the integral

$$F(t) = \int_a^t h(t, s) ds,$$

where $h : [a, b]^2 \to \mathbb{R}$. Assume that both h and $\partial_t h$ are (uniformly) continuous with respect to both variables s and t. Prove that, for all $t \in (a, b)$,

$$F'(t) = h(t, t) + \int_a^t \partial_t h(t, s) ds.$$

Hint: prove and use the identity

$$\frac{F(t+\delta) - F(t)}{\delta} = \frac{1}{\delta} \int_t^{t+\delta} h(t+\delta, s) ds + \int_a^t \frac{h(t+\delta, s) - h(t, s)}{\delta} ds.$$

Proof of Propostion 15.19. Differentiating (15.7) we obtain:

$$y'(t) = P_0(0) f(t) + \int_{t_\circ}^{t} P_0'(t-s) f(s) ds = \int_0^t P_0'(t-s),$$

where we used the assumption $P_0(0) = 0$

35 Differentiating again we obtain

$$y''(t) = P_0'(0) f(t) + \int_0^t P_0''(t-s) f(s) ds = \int_{t_\circ}^{t} P_0''(t-s),$$

where we used $P_0'(0) = 0$.

Differentiating again and again (and using $P_0^{(i)}(0) = 0$ for $0 \leq i \leq m-1$ obtain:

$$(\tfrac{d}{dt})^i y(t) = y^i(t) = \int_0^t P_0^{(i)}(t-s)f(s)ds \quad \text{for } 0 \leq i \leq m-1.$$

This shows in particular that

$$y^{(m-1)}(t_\circ) = \cdots = y'(t_\circ) = y(t_\circ) = 0.$$

It remains to show that $Ly(t) = f(t)$.

The key observation now is that when we differentiate the identity obtained above for $i = m-1$ we obtain

$$(\tfrac{d}{dt})^m y(t) = y^m(t) = P_0^{(m-1)} f(t) + \int_{t_\circ}^t P_0^{(m)}(t-s)f(s)ds = f(t) + \int_{t_\circ}^t P_0^{(m)}(t-s)f(s)ds$$

since now $P_0^{(m-1)}(0)$ equals 1 (and not 0 as in the previous cases)

Therefore if $L = \sum_{i=0}^m a_i(\frac{d}{dt})^i$ (with $a_m = 1$), using that $LP_0 = 0$ we have :

$$\begin{aligned} Ly(t) &= \sum_{i=0}^m a_i(\tfrac{d}{dt})^i y(t) = \sum_{i=0}^m a_i \int_{t_\circ}^t P_0^{(i)}(t-s)f(s)ds + a_m f(t) \\ &= \int_{t_\circ}^t LP_0(t-s)f(s)ds + f(t) = f(t). \end{aligned}$$

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□

15.1.2 General Systems of First Order Differential Equations

We generalize the concept of a differential equation to vector-valued functions in a real variable or, equivalently, to systems of differential equations with multiple unknown functions in one variable.

15.21. — Let $I \subset \mathbb{R}$ be an interval, $U \subset I \times \mathbb{R}^n$ open, and $F : U \to \mathbb{R}^n$ a continuous function. A differential equation of the form

$$u'(t) = F(t, u(t)) \tag{15.8}$$

for an unknown function u is called a **n-dimensional system of first-order differential equations**. The domain of definition for a solution u may only be a subinterval of I. For (15.8) to make sense, it must hold that $(t, u(t)) \in U$ for all t in the domain of u. To $(t_0, x_0) \in U$, we refer to the equations

$$u'(t) = F(t, u(t)), \quad u(t_0) = x_0$$

as an **initial value problem** for the **initial value** x_0 at $t_0 \in I$. The differential equation (15.8) is called **autonomous** if F is constant with respect to the variable t. We often interpret t as time, x as position, and F as a time-dependent **vector field**. The vector field indicates the direction and speed of the sought path $t \mapsto u(t)$ at $(t, x) \in U$.

15.22. — We only discuss first-order systems of differential equations. Due to the following construction, this is not a real restriction. Let $f : U \to \mathbb{R}^k$ be a continuous function on an open subset $U \subseteq \mathbb{R}^{mk+1}$. We can transform the m-th order differential equation

$$v^{(m)}(t) = f\big(t, v(t), v'(t), \dots, v^{(m-1)}(t)\big) \tag{15.9}$$

into a first-order differential equation system by considering a function $u = (u_0, u_1, \dots, u_{m-1})$ with values in $\mathbb{R}^n$ and the system

$$\begin{cases} u_0'(t) & = u_1(t) \\ u_1'(t) & = u_2(t) \\ \vdots & \\ u_{m-1}'(t) & = f(t, u_0(t), \dots, u_{m-1}(t)) \end{cases} \tag{15.10}$$

If we define

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$$F(t, x_0, \dots, x_{m-1}) = (x_1, x_2, \dots, x_{m-1}, f(t, x)),$$

we can summarize the system as $u'(t) = F(t, u(t))$. Via $v(t) = u_0(t)$, equations (15.9) and (15.10) are equivalent. The function $F : U \to \mathbb{R}^m$ "inherits" most properties of f. For example, if f is differentiable, then F is differentiable as well, and the same applies to Lipschitz continuity, which is important in Picard-Lindelöf's Theorem 15.34. The same procedure, slightly adjusted, provides an equivalent first-order differential equation system for a higher-order differential equation in vector-valued functions. The same discussion applies to the transfer of initial values.

15.23. — As a simple example, consider the case of a linear first-order differential equation system with constant coefficients. Such a system is given by

$$u' = Au$$

where u represents a function with values in $\mathbb{R}^n$, and A is a given $n \times n$ matrix with real —or complex— coefficients. To describe the solutions of such a system, we use the **matrix exponential function**. The matrix exponential function $\exp : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ is defined by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

.

Notice that the series defining each entry of the matrix $\exp(A)$ is absolutely convergent since

$$\sum_{k=0}^{\infty} \frac{1}{k!}\|A^k\|_2 \leq \sum_{k=0}^{\infty} \frac{1}{k!}\|A\|^k \leq \exp(\|A\|_2)$$

For commuting matrices A and B, $\exp(A+B) = \exp(A)\exp(B)$ holds, and for invertible matrices S, $\exp(SAS^{-1}) = S\exp(A)S^{-1}$. To calculate a matrix exponential $\exp(A)$, one brings A into Jordan normal form, $A = S(D+N)S^{-1}$, where D is diagonal and N is nilpotent
35 and commutes with D. (Indeed, in Jordan's normal form we have $Ne_i = e_{i+1}$ we have κ_i is either 0 or 1. Hence $DNe_i = D(\kappa_i e_{i+1}) = \kappa_i d_{i+1,i+1} e_{i+1} = N(\kappa_i d_{i,i} e_i) = NDe_i$, since when $\kappa_i = 1$ we have $d_{i+1,i+1} = d_{i,i}$.)

The exponential of the diagonal matrix D with diagonal coefficients d_{ii} is diagonal with coefficients $\exp(d_{ii})$, and the exponential of the nilpotent matrix N is calculated as a finite sum. Finally, we have

$$\exp(A) = S\exp(D)\exp(N)S^{-1},$$

since D and N commute.

Proposition 15.24: Matricial solution to linear ODE

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, $x_0 \in \mathbb{R}^n$, and $t_0 \in \mathbb{R}$. The initial value problem for differentiable functions $u : \mathbb{R} \to \mathbb{R}^n$

$$u' = Au, \quad u(t_0) = x_0$$

has the uniquely determined solution $u(t) = \exp(A(t-t_0))x_0$.

Proof. We begin by determining the derivative of the mapping $t \mapsto \exp(At)$. Notice first that:

$$\lim_{h\to 0} \frac{\exp(Ah) - \exp(0)}{h} = \lim_{h\to 0} \sum_{n=1}^{\infty} \frac{1}{n!} A^n h^{n-1} = A.$$

36 But then:

$$\begin{aligned} \frac{\partial}{\partial t}\big(\exp(At)\big) &= \lim_{h\to 0} \frac{\exp(A(t+h)) - \exp(At)}{h} = \\ &= \lim_{h\to 0} \frac{\exp(Ah) - \exp(0)}{h} \exp(At) = A\exp(At). \end{aligned}$$

For the function $u(t) = \exp(A(t-t_0))x_0$, we have $u(t_0) = x_0$ and

$$u'(t) = A\exp(A(t-t_0))x_0 = Au(t)$$

as claimed. To show uniqueness, assume that on an interval $I \subseteq \mathbb{R}$ with $t_0 \in I$, we have $v : I \to \mathbb{C}^n$ of class C^1 solving the initial value problem. Consider the function $t \mapsto \exp(-A(t-t_0))v(t)$ on I and calculate its derivative using the product rule. It holds

$$\begin{aligned}\frac{\partial}{\partial t}\big(\exp(-A(t-t_0))v(t)\big) &= -A\exp(-A(t-t_0))v(t)+\exp(-A(t-t_0))v'(t)\\ &= -A\exp(-A(t-t_0))v(t)+\exp(-A(t-t_0))Av(t)=0\end{aligned}$$

since A and $\exp(-A(t-t_0))$ commute. Therefore, the mapping $t \mapsto \exp(-A(t-t_0))v(t)$ is constant and thus equal to its value $v(t_0)=x_0$ at t_0.

Multiplying the obtained identity $\exp(-A(t-t_0))v(t)=x_0$ by $\exp(A(t-t_0))$ on both sides we obtain $v(t)=\exp(A(t-t_0))x_0$ for all $t \in I$, as claimed. This proves the uniequ □

15.25. — Due to the construction from Section 15.22, Proposition 15.24 also provides a solution method for linear differential equations of m-th order with constant coefficients. If we transform such a differential equation $Lv=0$, for example,

$$v^{(m)}+a_{m-1}v^{(m-1)}+\ldots+a_0v^{(0)}=0$$

for $a_0,\ldots,a_{m-1}\in\mathbb{C}$ into a first-order differential equation system, we obtain the equation $u'=Au$ with

$$A=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 1\\ -a_0 & -a_1 & \cdots & -a_{m-2} & -a_{m-1}\end{pmatrix}.$$

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This matrix is the companion matrix of the characteristic polynomial of the above differential equation. The characteristic polynomial of the matrix A above is precisely the characteristic polynomial $X^m+a_{m-1}X^{m-1}+\ldots+a_1X+a_0$ of the differential operator L.

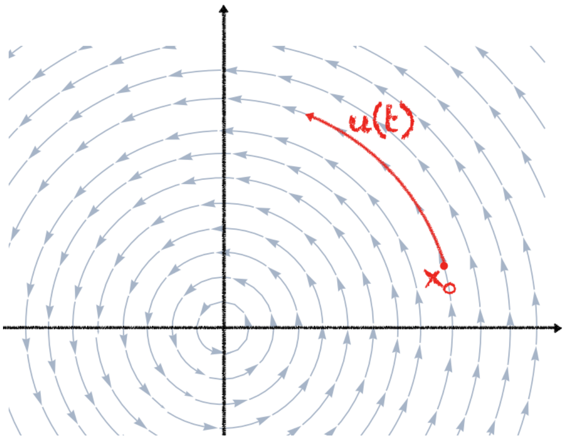
Exercise 15.26. — Let $m,n,k\in\mathbb{N}$, and let $A:\mathbb{R}\to\mathrm{Mat}_{m,n}(\mathbb{C})$ and $B:\mathbb{R}\to\in\mathrm{Mat}_{n,k}(\mathbb{C})$ be differentiable functions. Show that $AB:\mathbb{R}\to\mathrm{Mat}_{m,k}(\mathbb{C})$ defined by $AB(t)=A(t)B(t)$ is also differentiable with derivative $(AB)'=A'B+AB'$. Where was this used in the above proof?

15.1.3 Examples of Autonomous Differential Equation Systems

Example 15.27. — Consider the differential equation system in two unknown real-valued functions u_1 and u_2 given by $u_1'=-u_2$ and $u_2'=u_1$. This can also be written as $u'=F(t,u(t))$ with

$$F(t,x)=Ax=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}x \qquad u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

Due to the time-independence of F, this differential equation system is autonomous. If we interpret the vector field F as the flow of a medium in the plane, F describes a laminar flow that rotates positively with constant angular velocity around the origin. A solution u of the differential equation $u' = Au$ describes the path that a particle travels in this flow. To uniquely determine this path, we need to know the position of the particle at time $t_0 = 0$, that is, we must specify an initial condition $u(0) = x_0$ for the differential equation.



The uniquely determined solution u to the initial value problem $u' = Au$ with any initial value $x_0 = u(0) \in \mathbb{R}^2$ is given by

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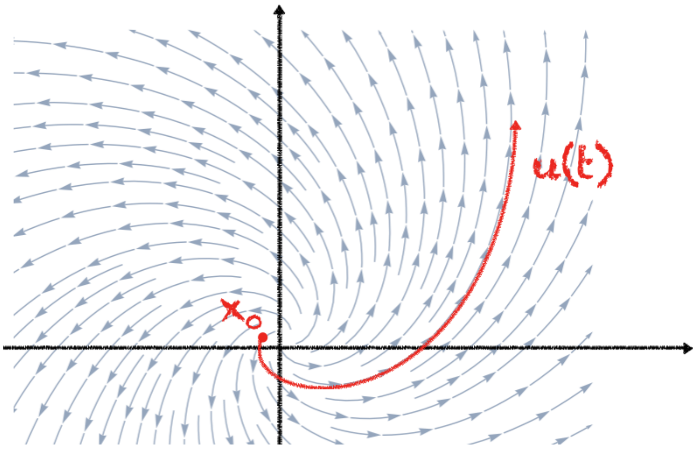
$$u(t) = \exp\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t\right) x_0 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} x_0$$

confirming the physical considerations.

Example 15.28. — Consider the autonomous differential equation system in two unknown real-valued functions u_1 and u_2 given by $u_1' = u_1 - u_2$ and $u_2' = u_1 + u_2$, which can be succinctly written as

$$u' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} u.$$

We represent the vector field or its flow again in a graphic.



The uniquely determined solution x to the initial value problem with any initial value $u(0) = x_0 \in \mathbb{R}^2$ is given by the spiral path

$$\begin{aligned} u(t) &= \exp\left(\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} t\right) x_0 = \exp\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} t\right) \exp\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} t\right) x_0 \\ &= \exp(t) \begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix} x_0. \end{aligned}$$

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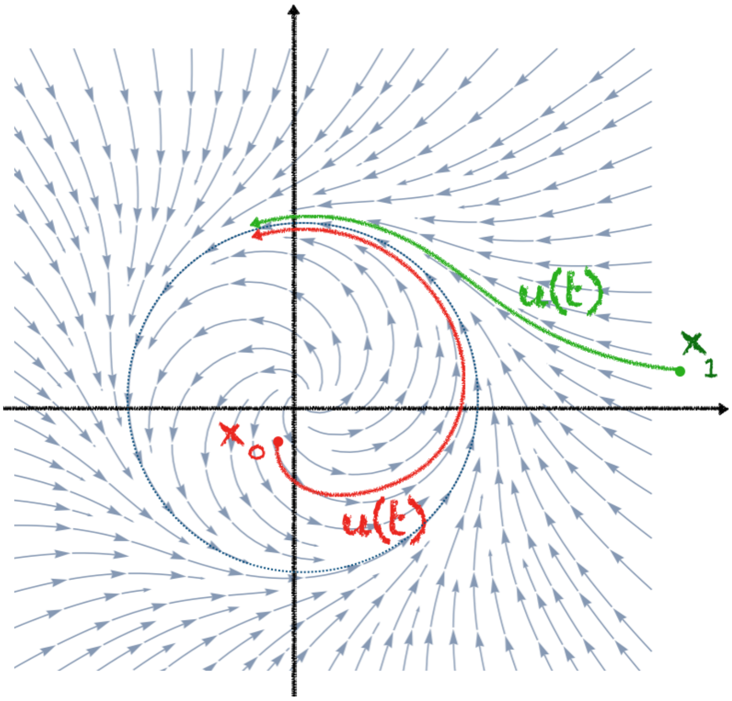
Example 15.29. — Consider the nonlinear autonomous differential equation system

$$\begin{aligned} u' &= u - v - (u^2 + v^2)u \\ v' &= u + v - (u^2 + v^2)v \end{aligned} \tag{15.11}$$

that we can write briefly as $(u', v') = F(u, v)$, along with an initial condition $(u(0), v(0)) = (x_0, y_0)$. For the initial value $(x_0, y_0) = (0, 0)$, we have the constant solution $(u, v) = (0, 0)$. To justify that this is indeed the only solution for the initial value $(0, 0)$, we can use the Picard-Lindelöf theorem 15.34. Now, assume that the initial value (x_0, y_0) is not $(0, 0)$. The corresponding time-independent vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x, y) = \begin{pmatrix} x - y - (x^2 + y^2)x \\ x + y - (x^2 + y^2)y \end{pmatrix}$$

can be represented as shown in the following image.



Hoping to simplify the problem, we introduce polar coordinates. That is, we write $r(t)^2 = u(t)^2 + v(t)^2$ and

$$u(t) = r(t)\cos(\varphi(t)), \quad v(t) = r(t)\sin(\varphi(t))$$

for a suitable function φ. Calculating,

$$(r^2)' = (u^2+v^2)' = 2xx'+2yy' = 2\big(u^2-uv-(u^2+v^2)u^2\big)+2\big(uv+v^2-(u^2+v^2)v^2\big) = 2(r^2-r^4),$$

and thus, since $(r^2)' = 2rr'$, we obtain the initial value problem

$$r' = r - r^3, \quad r(0) = \sqrt{x_0^2 + y_0^2}$$

for the real-valued function r. A solution can be found using separation of variables, resulting in

$$r(t) = \frac{\exp(t)}{\sqrt{A + \exp(2t)}}, \quad A = \frac{1}{x_0^2 + y_0^2} - 1$$

for the constant $A \in (-1, \infty)$ adapted to the initial value $r(0)$.

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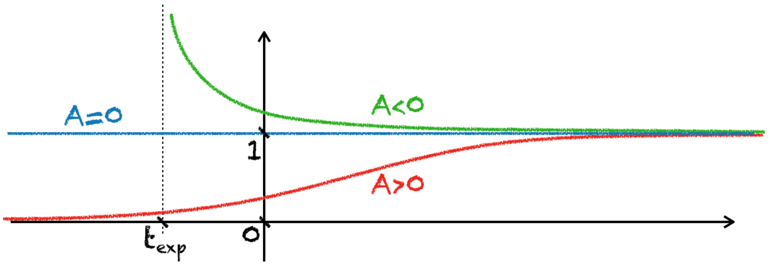

The function r is defined for all times $t \geq 0$, and also for all $t \leq 0$ if $A \geq 0$, i.e., if the initial point (x_0, y_0) lies on or inside the unit circle. For $A < 0$, there is a time $t_{\text{exp}} = \frac{1}{2}\log(-A)$ at which the function r "explodes". It holds $\lim_{t\to\infty} r(t) = 1$ regardless of the initial value, as one could already guess from the illustration of the vector field. To determine the angle φ, we calculate using the chain rule and the already known relationship $r' = (1 - r^2)r$

$$u' = r'\cos(\varphi) - r\varphi'\sin(\varphi) = u - r^2u - \varphi'v.$$

On the other hand, $u' = u - v - r^2u$ according to (15.11), from which $u\varphi' = u$ follows. Similarly, we show $v\varphi' = v$. Thus, either $u = v = 0$, corresponding to the constant solution, or $\varphi' = 1$ and therefore $\varphi(t) = t + B$ for a constant $B \in \mathbb{R}$ chosen appropriately for the initial value. In summary,

$$u(t) = \frac{\exp(t)\cos(B+t)}{\sqrt{A + \exp(2t)}}, \quad u(t) = \frac{\exp(t)\sin(B+t)}{\sqrt{A + \exp(2t)}}$$

for suitable constants $A > -1$ and B. As one could suspect from the representation of the vector field, the solutions form spirals approaching the unit circle.

Applet 15.30 (Various Differential Equations and Their Solutions)**.** *Illustration of solutions to various differential equations, each exhibiting different behaviors.*

15.2 Spaces of continuous functions and existence of solutions to ODE

Even relatively simple initial value problems do not always have a uniquely determined solution. However, if we impose a weak condition on the vector field F for a differential equation system

$$u'(t) = F(t, u(t))$$

we obtain both existence and uniqueness. This is one of the most fundamental theorems in ODE. Here we essentially follow Lindelöf's work [Lin1894].

Before stating and proving this fundamental result we give some preliminary results on the space of bounded continuous functions, which will be essencial to prove the existence of solutions to ODE.

15.2.1 The space of bounded continuous functions with values in $\mathbb{R}^n$

If (X, d) is a metric space (in our application to ODE we will have $X = I \subset \mathbb{R}$ an open interval with the standard distance given by the absolute value). We denote by $C(X, \mathbb{R}^n)$ the vector space of continuous functions defined on X. We denote by $C_b(X, \mathbb{R}^m)$ the vector space of continuous functions which are also bounded, that is, $\sup_{x\in X} |f| < +\infty$. By

36 Proposition 15.31: The normed space of bounded continuous functions is complete

For all $f \in C_b(X, \mathbb{R}^n)$ set $\|f\| := \sup_{x\in X} |f(x)|$. Then $(C_b(X, \mathbb{R}^n), \|\cdot\|)$ is a complete normed vector space.
Furthermore, $f_k \to f$ in this space if and only if the functions (f_k) converge uniformly to f, meaning that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in X, \forall k \geq N : |f(x) - f_k(x)| < \varepsilon.$$

The norm $\|\cdot\|_\infty$ is called the **supremum norm**.

Proof. First of all, $\|f\|_\infty < \infty$ by assumption. Let us check the properties of the norm

- **Zero norm implies zero function:** If $\|f\| = 0$, then $|f(x)| = 0$ for all $x \in X$, that is $f(x) = 0$ for all $x \in X$. Hence f is the zero element of the vector space $C_b(X, \mathbb{R}^n)$.
- **Homogeneity:** For any $\lambda \in \mathbb{R}, x \in X$, we have $|\lambda f(x)| = |\lambda||f(x)|$. Taking the sup over all $x \in X$, we get $\|\lambda f\| = |\lambda|\|f\|$.
- **Triangle inequality:** For any $f, g \in C_b(X, \mathbb{R})$ and $x \in X$ we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|.$$

Taking the sup over all $x \in X$, we get $\|f+g\| \leq \|f\| + \|g\|$.

Now, let $(f_k)_{k=0}^{\infty}$ be a Cauchy sequence in $C_b(X, \mathbb{R}^n)$ with respect to $\|\cdot\|$. For each $x \in X$, the sequence $(f_k(x))_{k=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^n$, and therefore converges to some limit $y_x \in \mathbb{R}^n$. Define the function $f : X \to \mathbb{R}^n$ by $f(x)\colon x \mapsto y_x$, we claim that $f \in C_b(X, \mathbb{R})$ and that $(f_k)_{k=0}^{\infty}$ converges to f uniformly.

- **Continuity of f:** Fix $x_0 \in X$. Since $(f_n)_{n=0}^{\infty}$ is a Cauchy sequence (with respect to $\|\cdot\|$), given $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $k, \ell \geq N_\epsilon$ and all $x \in X$, we have $|f_k(x) - f_\ell(x)| < \epsilon/4$. Hence, for all $x \in X$

  $$|f(x) - f_{N_\epsilon}(x)| < \epsilon/4.$$

  Then for any given $x \in X$, $\epsilon > 0$, and sequence $x_k \to x$ we have

  $$\begin{aligned} |f(x_k) - f(x)| &\leq |f(x_k) - f_{N_\epsilon}(x_k)| + |f_{N_\epsilon}(x_k) - f_{N_\epsilon}(x)| + |f_{N_\epsilon}(x) - f(x)| \\ &\leq \frac{\epsilon}{2} + |f_{N_\epsilon}(x_k) - f_{N_\epsilon}(x)|. \end{aligned}$$

  But f_{N_ϵ} is continuous, $|f_{N_\epsilon}(x_k) - f_{N_\epsilon}(x)|$ will eventually become less than $\epsilon/2$. This proves that the limit function f is continous.

  36 In particular, for $k \geq N$, we have

  $$|f_k(x) - f_N(x)| < \epsilon$$

  for all $x \in X$. Fix $k \geq N$. Since $|f_k(x_0) - f_N(x_0)| < \epsilon$ for all $x_0 \in X$, the sequence $(f_k(x))_{k=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges to some limit $y_{x_0} \in \mathbb{R}$. This shows that the function $f : X \to \mathbb{R}^n$ is indeed continuous.

- **Uniform convergence:** Let $\epsilon > 0$. Since $(f_k)_{k=0}^{\infty}$ is a Cauchy sequence with respect to $\|\cdot\|$, there exists $N \in \mathbb{N}$ such that for all $k, \ell \geq N$ and all $x \in X$, we have $|f_k(x) - f_\ell(x)| < \epsilon$. Fix $k \geq N$. Then, for all $x \in X$, we have

  $$|f(x) - f_k(x)| = \lim_{\ell \to \infty} |f_\ell(x) - f_k(x)| \leq \epsilon.$$

  This shows that $(f_k)_{k=0}^{\infty}$ converges uniformly to f.

- **Boundedness:** Taking $\varepsilon = 1$ in the uniform convergence definition, we find $f_N \in C_b$ such that

  $$|f(x)| \leq |f_N(x)| + |f(x) - f_N(x)| \leq \|f_N\| + 1,$$

  taking the supremum for $x \in X$ we find that f is bounded.

Conversely, suppose $(f_n)_{n=0}^{\infty}$ converges uniformly to f. Let $\epsilon > 0$. Since $(f_n)_{n=0}^{\infty}$ converges uniformly to f, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$, we have $|f_n(x) -$

$f(x)| < \epsilon$. This implies that

$$\sup\{|f_n(x) - f(x)| \mid x \in X\} = \|f_n - f\| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $(f_n)_{n=0}^\infty$ converges to f with respect to $\|\cdot\|$. This
36 completes the proof. □

Exercise 15.32. — Show that if $f_j \to f$ uniformly and $x_j \to x$, then $f_j(x_j) \to f(x)$.

By Weierstrass theorem (Corollary 9.79), $C(X, \mathbb{R}^m) = C_b(X, \mathbb{R}^m)$, whenever X is a compact metric space.

Theorem 15.33: Sequential Ascoli–Arzelà

Let $K \subset \mathbb{R}^m$ be a compact set and let $(f_k)_{k=0}^\infty$ be a bounded sequence in $C(K, \mathbb{R})$. Assume that $(f_k)_{k=0}^\infty$ is ***equi-continuous****, meaning that*

$$\forall \epsilon > 0, \exists \delta > 0, \forall k \in \mathbb{N}, \big(|x - y| < \delta \implies |f_k(x) - f_k(y)| < \epsilon\big).$$

Then $(f_k)_{k=0}^\infty$ has a subsequence that converges uniformly to some $f \in C(K, \mathbb{R}^m)$.

Proof. -*Step 1.* By the Bolzano-Weierstrass theorem, the sequence $(f_k(x))_{k=0}^\infty$ for each $x \in K$ has a convergent subsequence because $\|f_k\|_\infty = \sup_K |f_k|$ is uniformly bounded. Thus, for each fixed $x \in K$ the $(f_k(x))_{k=0}^\infty$ admits convergent subsequences.

-*Step 2.* Using a diagonal argument, we can extract a subsequence $(f_{k_j})_{j=0}^\infty\}$ such that $(f_{j\ell}(y))_{j=0}^\infty\}$ converges for all x in a countable *dense* subset of K.

Indeed, Let $Y = \{y_\ell\}_{\ell \in \mathbb{N}}$ be a coutable subset of K such that $\overline{Y} = K$ (excercise: prove
38 that since K is compact there exists countable subsets Y with $\overline{Y} = K$).

The goal now is to find a subsequence $(f_{k_j})_{j=0}^\infty$ such that $f_{n_{k_j}}(x_\ell)$ converges for each $\ell \geq 0$. To do so we start from $\ell = 0$ and choose $k_j^0 = h^0(j)$ with $h^0 : \mathbb{N} \to \mathbb{N}$ increasing, such that $\big(f_{k_j^0}(y_0)\big)_{j=0}^\infty$ converges. Next, we take $\ell = 1$ and choose $k_j^1 = h^1 \circ h^0(j)$ with $h^1 : \mathbb{N} \to \mathbb{N}$ increasing such that $\big(f_{k_j^1}(y_1)\big)_{j=0}^\infty$ converges. Next, for $\ell = 2$ choose $k_j^2 = h^2 \circ h^1 \circ h^0(j)$, with $h^2 : \mathbb{N} \to \mathbb{N}$ increasing, such that $\big(f_{k_j^2}(y_2)\big)_{j=0}^\infty$ converges.

Repeating this procedure we obtain, for each $\ell \geq 1$ a subsequence k_j^ℓ of k_j^{l-1} such that $\big(f_{k_j^\ell}(y_\ell)\big)_{j=0}^\infty$ converges.

The diagonal subsequence is then constructed by defining $k_j := k_j^j$. Notice that for each given $\ell \in \mathbb{N}$, the $(k_j)_{j \geq \ell}$ is by construction subsequence of $(k_j^\ell)_{j \geq \ell}$ and hence $\big(f_{k_j}(y_\ell)\big)_{j=0}^\infty$ converges, as we wanted to prove.

-*Step 3.* Let us show that $(f_{k_j})_{j=0}^\infty$ converges in $C(K, \mathbb{R}^n)$. Take $\epsilon > 0$. By equi-continuity, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ and $|x - y| \leq \delta$, $|f_k(x) - f_k(y)| \leq \epsilon/3$. Since K is compact, it can be covered by finitely balls of radius δ centered at points in the dense subset $Y = \{y_\ell\}$.

Let $\{y_{\ell_1}, y_{\ell_2}, \dots, y_{\ell_N}\}$ be the centers of such a finite cover. For large enough j, j', $|f_{k_j}(x_{\ell_i}) - f_{k_{j'}}(x_{\ell_i})| < \epsilon/3$ for each $i = 1, 2 \dots, N$.

Finally, for any given $x \in K$, there exists y_{ℓ_i} such that $d(x, y_{\ell_i}) < \delta$. Therefore,

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$$|f_{k_j}(x) - f_{k_{j'}}(x)| \leq |f_{k_j}(x) - f_{k_j}(y_{\ell_i})| + |f_{k_j}(y_{\ell_i}) - f_{k_{j'}}(y_{\ell_i})| + |f_{k_{j'}}(y_{\ell_i}) - f_{k_{j'}}(x)| < \epsilon.$$

This proves that $(f_{k_j})_{j=0}^{\infty}$ is a Cauchy sequence in $C(K, \mathbb{R}^n)$. Hence, by Proposition 15.31 the sequence converges in $C(K, \mathbb{R}^n)$. □

15.2.2 The Cauchy-Lipschitz or Picard-Lindelöf Theorem

Theorem 15.34: Cauchy-Lipschitz/Picard-Lindelöf

Let $n \geq 1$, $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be open, $(t_0, x_0) \in U$, and $F : U \to \mathbb{R}^n$ be continuous. Assume that F is "locally Lipschitz continuous in space", meaning that for all $(t_1, x_1) \in U$, there exist $\varepsilon > 0$ and $L \geq 0$, such that for all $(t, x_2), (t, x_3) \in B\big((t_1, x_1), \varepsilon\big) \cap U$, the estimate

$$|F(t, x_2) - F(t, x_3)| \leq L|x_2 - x_3|$$

holds. Then, there exists a (not necessarily bounded) interval $I = I_{\max} = (a, b) \subseteq \mathbb{R}$ with $t_0 \in I$ and a differentiable function $u : I \to \mathbb{R}^d$ with the following properties:

1. *$(t, u(t)) \in U$ for all $t \in I$, and u is a solution to the initial value problem*

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$$\begin{cases} u'(t) = F(t, u(t)) & \text{for all } t \in I \\ u(t_0) = x_0 \end{cases}$$

2. *For any other solution $v : J \to \mathbb{R}^d$ of the same initial value problem defined on an open interval J with $t_0 \in J$, $J \subseteq I$, and $u|_J = v$.*
3. *The limits $\lim_{t \to a} (t, u(t))$ and $\lim_{t \to b} (t, u(t))$ do not exist, or do not belong to U.*

15.35. — Informally, Picard-Lindelöf's theorem states that every initial value problem for a vector field locally Lipschitz continuous in space has a unique solution. Furthermore, this solution can either be extended indefinitely ($b = \infty$) or until it "explodes", meaning it leaves the domain U of the differential equation in finite time. Refer to Exercise 15.40 for a sharper formulation of this claim. The hypothesis that F is locally Lipschitz continuous in space cannot be omitted, as shown in Example 15.41, but is usually fulfilled in practice. In Exercise 15.37, we verify that continuously differentiable functions are locally Lipschitz continuous. In particular, the above theorem can be applied to continuously differentiable vector fields F.

36 The proof of the theorem proceeds in two steps. First, we show, using the Banach fixed-point theorem, the existence and uniqueness of the solution to the initial value problem locally, i.e., in a small neighborhood of (t_0, x_0). The precise statement for this is Proposition 15.44. In a second step, we then piece together local solutions to form a solution on a maximal interval.

37 **Proposition 15.36: Local existence and uniqueness via Picard**

Let $r > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, *and consider*

$$F : (t_0 - r, t_0 + r) \times B(x_0, r) \to \mathbb{R}^n$$

a continuous function. Suppose there exist constants $C \geq 1$ *and* $L > 0$ *such that*

$$|F(t,x)| \leq C \quad \textit{and} \quad |F(t,x_1) - F(t,x_2)| \leq L|x_1 - x_2|$$

for all $t \in (t_0 - r, t_0 + r)$ *and* $x, x_1, x_2 \in B(x_0, r)$. *Then, for any* $\delta > 0$ *with* $\delta < \min\{\frac{r}{2C}, \frac{1}{2L}\}$, *there exists a unique differentiable function* $u : [t_0 - \delta, t_0 + \delta] \to B(x_0, r)$ *satisfying*

$$\begin{cases} u(t_0) = x_0, \\ u'(t) = F(t, u(t)) \quad \textit{for all } t \in (t_0 - \delta, t_0 + \delta) \end{cases} \tag{15.12}$$

Proof. Choose $\delta > 0$ as in the proposition and write $I = (t_0 - \delta, t_0 + \delta)$. According to the Fundamental Theorem of Differential and Integral Calculus, applied to each component of u_i of u, the initial value problem in (15.12) is equivalent to the integral equation

$$u(t) = x_0 + \int_{t_0}^{t} F(s, u(s))ds \tag{15.13}$$

for all $t \in I$. The equation (15.13) is interpreted componentwise, that is, $u_i(t) = x_{0,i} + \int_{t_0}^{t} F_i(s, u(s))ds$ for all $i = 1, \ldots, n$.

To interpret this equation as a fixed-point equation, we define a suitable complete metric space V and a suitable Lipschitz contraction $T : V \to V$.

Choose $r_0 \in (\frac{r}{2}, r)$. The space of continuous functions $C(I, \mathbb{R}^n)$ equipped with the supremum norm $\|\cdot\|$ is complete, as stated in Proposition 15.31. The subset

$$V = \{u \in C(I, \mathbb{R}^n) \mid |u(t) - x_0| \leq r_0\}$$

is the closed ball in $C(I, \mathbb{R}^n)$ with radius r_0 and center the constant function with value x_0. In particular, V is complete as a closed subset of a complete metric space. The mapping $T : V \to C(I, \mathbb{R}^n)$ defined by

$$(Tu)(t) = x_0 + \int_{t_0}^{t} F(s, u(s))ds$$

is called the **Picard map**. Fixed points of T are precisely functions that satisfy (15.13). To prove the proposition, it suffices, according to the Banach Fixed-Point Theorem 9.58, to show that T restricts to a Lipschitz contraction on V. We note that a continuous function $u : I \to B(x_0, r)$ attains its extreme values, and thus, it takes values in the closed ball $B(x_0, r_0)$ for some suitable $r_0 \in (0, r)$.

For all $u \in V$ and $t \in I$,

$$|Tu(t) - x_0| = \left|\int_{t_0}^{t} F(s, u(s))ds\right| \leq \int_{t_0}^{t} |F(s, u(s))|\, ds \leq C|t - t_0| \leq C\delta \leq \frac{r}{2} \leq r_0$$

where the absolute value in the third expression is necessary for the case $t < t_0$. This shows that the Picard map T restricts to $T : V \to V$ as claimed. Let $u_1, u_2 \in V$. Then,

$$\begin{aligned} |Tu_1(t) - Tu_2(t)| &= \left|\int_{t_0}^{t} F(s, u_1(s)) - F(s, u_2(s))ds\right| \\ &\leq \int_{t_0}^{t} |F(s, u_1(s)) - F(s, u_2(s))|ds \\ &\leq \int_{t_0}^{t} L|u_1(s) - u_2(s)|ds \\ &\leq L|t - t_0| \sup_{s \in (t_0 - \delta, t_0 + \delta)} |(u_1 - u_2)(s)| \\ &\leq L\delta\|u_1 - u_2\| < \frac{1}{2}\|u_1 - u_2\| \end{aligned}$$

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for all $t \in I$. Thus, T is a Lipschitz contraction with Lipschitz constant $\frac{1}{2}$. □

Proof of Theorem 15.34. First, we prove the *uniqueness*. Let $u_1 : I_1 \to \mathbb{R}^d$ and $u_2 : I_2 \to \mathbb{R}^d$ be two solutions of the initial value problem

$$u'(t) = F(t, u(t)), \qquad u(t_0) = x_0 \tag{15.14}$$

on open intervals $I_1, I_2 \subseteq \mathbb{R}$ with $t_0 \in I := I_1 \cap I_2 = (\alpha, \beta)$. We claim that $u_1(t) = u_2(t)$ for all $t \in I$. To show this, we consider the subset

$$S = \big\{t \in I \mid u_1(t) = u_2(t)\big\}$$

of I. The complement $I \setminus S$ is open since u_1 and u_2 are continuous. Also, S is not empty since $t_0 \in S$. For each $t_1 \in S$, both u_1 and u_2 solve the initial value problem

$$u'(t) = F(t, u(t)), \qquad u(t_1) = x_1$$

for the common value $x_1 = u_1(t_1) = u_2(t_1)$. Thus, according to Proposition 15.44), there exists a $\delta > 0$ such that $u_1(t) = u_2(t)$ for all $t \in (t_1 - \delta, t_1 + \delta)$. Therefore, $S \subseteq I$ is also open, and it follows that $S = I$ as claimed.

Now we turn to the *existence* of a maximal solution. For this, we consider the set $\mathcal{J}$ of all pairs (J, v) consisting of an open interval $J \subseteq \mathbb{R}$ with $t_0 \in J$ and a differentiable function $v : J \to \mathbb{R}^n$ that solves the initial value problem in the theorem. According to Proposition 15.44, $\mathcal{J}$ is non-empty. We set

$$I = \bigcup_{(J,v)\in\mathcal{J}} J$$

and define $u : I \to \mathbb{R}^n$ by $u(t) = v(t)$ if $t \in J$ for some $(J, v) \in \mathcal{J}$. Due to the uniqueness property in the first part of the proof, this is well-defined, the set I is an open interval, and u is a solution to the initial value problem in the theorem.

The proof of the claimed behavior of the maximal solution (I, u) near a and b remains. If $b = \infty$, there is nothing to show. We assume that $b = \sup I < \infty$ and the limit

$$\lim_{t\to b}(t, u(t)) = (b, x_b) \in U$$

exists, leading to a contradiction. According to Proposition 15.44, there exists a solution w to the initial value problem

$$w(b) = x_b, \qquad w'(t) = F(t, v(t)),$$

defined on an interval $J = (b - \delta, b + \delta)$ for some $\delta > 0$. We use this solution to define a solution to the original initial value problem. Let $v : (a, b + \delta) \to \mathbb{R}^d$ be given by

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$$v(t) = \begin{cases} u(t) & \text{if } t \in (a, b) \\ w(t) & \text{if } t \in [b, b + \delta) \end{cases}$$

We claim that v also solves the initial value problem (15.14), which contradicts the definition of b. Since u and w are continuous, and by definition

$$\lim_{t\to b} u(t) = x_b = v(b) = \lim_{t\to b} v(t)$$

holds, v is continuous. It holds $v(t) = F(t, v(t))$ for all $t \in (a, b + \delta)$, except possibly at $t = b$. It remains to show that v is differentiable at b and $v(b) = F(b, v(b))$ suffices. The right-hand derivative of v at b satisfies

$$\lim_{h\to 0} \frac{v(b + h) - v(b)}{h} = w'(b) = F(b, w(b)) = F(b, v(b))$$

as desired. The left-hand derivative requires a small argument based on the continuity of $t \mapsto f(t, u(t))$, see Exercise 15.38, and then proceeds analogously. As mentioned, this contradiction to the definition of b shows that u has the claimed behavior for $t \to b$. The proof of the behavior for $t \to a$ is analogous. □

Exercise 15.37. — Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ be continuous, such that the partial derivatives $\partial_{x_k} f$ for $k \in \{1, \ldots, n\}$ exist and are continuous on U. Show that F is *locally Lipschitz* and thus satisfies the conditions of Theorem 15.34.

EXERCISE 15.38. — Let $t_0 < b$ be real numbers, and $u : [t_0, b] \to \mathbb{R}^n$ be differentiable on $[t_0, b)$ such that $u'(t) = f(t)$ for a continuous function $f : [t_0, b] \to \mathbb{R}^n$ for all points $t \in [t_0, b)$. Then, the left-hand derivative $u'(b) = f(b)$ exists.

EXERCISE 15.39. — Let $U \subset \mathbb{R}^n$ be open, and let $F : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Show that every solution to the differential equation $u'(t) = F(t, u(t))$ is smooth.

EXERCISE 15.40 (Challenge). — Let U and F be as in the Picard-Lindelöf theorem, and let $u : (a, b) \to \mathbb{R}^d$ be the corresponding maximal solution to an initial value problem for $(t_0, x_0) \in U$ and suppose that $b < \infty$. As a strengthening of Theorem 15.34, show that for every compact set $K \subseteq U$, there exist time points $\beta \in (a, b)$ such that $(t, u(t)) \notin K$ for all $t \in (\beta, b)$. In this sense, when the solution is not defined for arbitrarily large time, then it must leave every compact subset of U.

15.2.3 Examples

37 EXAMPLE 15.41. — We demonstrate, using a simple example, the necessity locally Lipschitz in space hypothesis on the vector field F in the Picard-Lindelöf theorem. This hypotesis is required to guaratee uniqueness.

Indeed, Consider the initial value problem

$$u' = Au^\alpha, \qquad u(0) = 0,$$

with $\alpha \in (0, 1)$. Observe that $u(t) \equiv 0$ is a trivial solution. However, the Ansatz $u(t) = t^\gamma$ satisfies

$$u'(t) = \gamma t^{\gamma-1} = \gamma (t^\gamma)^{\frac{\gamma-1}{\gamma}} = \gamma (u(t))^{\frac{\gamma-1}{\gamma}} = Au(t)^\alpha,$$

is another solution of the same initial value problem when $1 - 1/\gamma = \alpha$ (that is, $\gamma := \frac{1}{1-\alpha}$) and $A = \gamma$.

EXERCISE 15.42 (Attractor). — Let F be a Lipschitz continuous vector field on an open subset $U \subseteq \mathbb{R}^2$, with $F(0) = 0$. Then, the constant function with value 0 is a solution to the autonomous differential equation $u'(t) = F(u(t))$ with the initial value $u(0) = 0$. We are interested in the behavior of solutions to an initial value $u(0) = x$ near 0 in different situations. Suppose that

$$F(x) = Ax + o(|x|) \qquad \text{as } x \to 0$$

for a matrix $A \in \mathrm{Mat}_2(\mathbb{R})$. Show that in the cases

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

for negative $\lambda_1, \lambda_2, \lambda$, there exists a neighborhood U around the origin such that the following holds: If u is a solution to the initial value problem

$$u'(t) = F(u(t)), \qquad u(0) = x_0$$

with $x_0 \in U$, then $\lim_{t\to\infty} u(t) = 0$. Generalize this to higher dimensions and all matrices whose eigenvalues have negative real parts.

37 Exercise 15.43. — Let $U = B(0,2) \subseteq \mathbb{R}^2$, and let $F : U \to \mathbb{R}^2$ be given by

$$F(x) = \begin{pmatrix} -x_2 + x_1^3 g(|x|^2) \\ x_1 + x_2^3 g(|x|^2) \end{pmatrix},$$

where $g : [0,4] \to \mathbb{R}$ is Lipschitz continuous. Suppose $\alpha < \beta$ in $[0,4)$ with $g(\alpha) = g(\beta) = 0$ and $g(r) > 0$ for all $r \in (\alpha, \beta)$. Show that every solution to the initial value problem $u'(t) = F(u(t))$ with initial value $u(0) = x_0 \in U$ and $|x_0|^2 \in (\alpha, \beta)$ is defined on the entire $\mathbb{R}$ and approaches the solution

$$u_\beta(t) = \sqrt{\beta} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

while rotating counterclockwise around the origin. If instead $g(r) < 0$ for all $r \in (\alpha, \beta)$, the behavior is analogous, approaching the solution $u_\alpha(t) = \sqrt{\alpha}\left(\begin{smallmatrix} \cos(t) \\ \sin(t) \end{smallmatrix}\right)$ from the outside.

The Picard's fixed point methods gives a existence and uniqueness of solution for small time around points (t_0, x_0) where F is locally Lipchitz. There are, however, other methods to prove 38 the existence of solutions, which do not give uniqueness. We explain next Peano's theorem, establishing a local existence of solution result applying to merely continuous F. It's proof 'alla Tonelli' makes use of Ascoli-Arzelà.

15.3 Peano existence theorem

Proposition 15.44: Peano's existence result

Let $r > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and consider

$$F : [t_0, t_0 + r) \times B(x_0, r) \to \mathbb{R}^n$$

a continuous function. Suppose there exist constants $C \geq 1$ such that

$$|F(t,x)| \leq C$$

for all $(t,x) \in [t_0, t_0 + r) \times B(x_0, r)$. Then, for any $\delta > 0$ with $\delta < \frac{r}{2C}$, there exists a differentiable function $u : [t_0, t_0 + \delta] \to B(x_0, r)$ satisfying

$$\begin{cases} u(t_0) = x_0, \\ u'(t) = F(t, u(t)) \quad \text{for all } t \in [t_0, t_0 + \delta] \end{cases} \tag{15.15}$$

Proof. Given $\tau \in (0, \delta)$ (tiny) we considered the 'delayed' problem

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$$\begin{cases} u_\tau(t) = x_0, t \in [0, \tau] \\ u_\tau'(t) = F(t, u_\tau(t - \tau)) \quad \text{for all } t \in [t_0, t_0 + \delta) \end{cases} \tag{15.16}$$

We notice that problem (15.16) admits a unique solution. Indeed, this can be proved by induction integrating the equation over small intervals of length τ.

Notice that by definition then $u_\tau(t) = x_0$ if $t \in [0, \tau]$. now supposed that we have showed that (15.16) admits a unique solution for $t \in [0, k\tau]$ with $k \geq 1$ integer. Then for any $t \in [k\tau, (k+1)\tau]$ we can 'explicitly' solve

$$u_\tau(t) = u_\tau(k\tau) + \int_{k\tau}^{t} F(s, u_\tau(s - \tau))ds.$$

The key observation that gives unique solvability of the delayed equation the value of $u'(t)$ for $t \in [k\tau, (k+1)\tau]$, depends only on $F(s, u_\tau(t - \tau))$ which was computed in the previous time step $[(k-1)\tau, k\tau]$.

In other words, by introducing a delay, we are transforming our equation 'in implicit form'

$$u'(t) = F(t, u(t)), \quad t \in [k\tau, (k+1)\tau],$$

where u is both at the left and the right hand of sides of the equation, into an explicit differential equation of the type

$$u'(t) = \widetilde{F}(t), \quad t \in [k\tau, (k+1)\tau],$$

where $\tilde{F}(t)$ is known.

The previous solution can be continued up $t = t_0 + \delta$. Indeed, since $F(t, x) \le C$ we have

$$|u_\tau'(t)| \le C \tag{15.17}$$

which gives

$$|u_\tau(t) - x_0| = C(t - t_0) \le C\delta < r/2. \tag{15.18}$$

Notice also that, for $t \in [t_0, t_0 + \delta]$, u_τ solves the integral formulation of (15.16)

$$\begin{cases} u_\tau(t) = x_0, t \in [0, \tau] \\ u_\tau(t) = x_0 + \int_{t_0}^{t} F(s, u_\tau(s - \tau)\, ds \quad \text{for all } t \in [t_0 + \tau, t_0 + \delta] \end{cases} \tag{15.19}$$

Now take some sequence $(\tau_k)_{k=0}^{\infty}$ such that $\tau_k \to 0$. For example $\tau_k := 1/k$.

Let us show that the sequence $(u_{1/k})_{k=0}^{\infty}$, which belongs to $C([t_0, t_0 + \delta], \mathbb{R}^n)$ admits a converging subsequence (with the sup norm $\|\cdot\|_\infty$)

Indeed, by (15.17) we have

$$|u_{1/k}(s) - u_{1/k}(t)| \le C|s - t| \quad \text{for all } s, t \in [t_0, t_0 + \delta]$$

And by (15.18) we have

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$$\|u_{1/k}\|_\infty \le |x_0| + r/2$$

Therefore the sequence is bounded and equi-continuous and hence by the Arzelà-Ascoli theorem admits a converging subsequence $u_{1/k_j} \to u$ (in $C([t_0, t_0 + \delta], \mathbb{R}^n)$ as $j \to \infty$.

It only remains to show that u is a solution of (15.15). But this follows noticing that the uniform convergence and equi-continuity can be used to pass to the limit the integral formulation (15.19) (for $\kappa = 1/k_j$) to obtain

$$u(t) = x_0 + \int_{t_0}^{t} F(s, u(s)\, ds \quad \text{for all } t \in [t_0, t_0 + \delta].$$

The details of this passage to the limit are left as an exercise. [*Hint:* using that F is uniformly continuous on $[t_0, t_0 + r/2] \times \overline{B(x_0, r/2)}$ show that

$$\max_{s \in [t_0, t_0 + \delta]} \big|F(s, u_{1/k_j}(s - 1/k_j)) - F(s, u(s))\big| \to 0 \quad \text{as } j \to \infty.]$$

□

Exercise 15.45. — Assuming $F(t, x)$ Lipchitz with respect to x show that a local uniqueness of solution result for the initial value problem can be shown as consequence of Grönwal's inequality. Combined with Peano's existence result this gives an alternative proof of Proposition 15.44.

[*Hint:* Prove that if u, v are two solutions with the same initial condition x_0 then $w(t) := |u(t) - v(t)|^2$ satisfies $w'(t) \leq 2Lw(t)$, where L is the Lipchitz constant of F (with respect to x).]

15.4 Differentiability with respect to initial conditions

In what follows we assume that $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ is convex is open and convex in space, meaning that if (t, x_1) and (t, x_2) belong to Ω then also $(t, (1-\lambda)x_1 + \lambda x_2)$ belongs to Ω for all $\lambda \in [0, 1]$.

Let us assume that $x_0 \in \Omega$ and that F and all its spatial derivatives $\partial_{x_i} F$ are continuous in Ω.

We would like to analyze the dependence of the solution $u(t) = u(t; x_0)$ of the initial value problem

$$\begin{cases} u(t_0) = x_0, \\ u'(t) = F(t, u(t)) \quad \text{for all } t \in [t_0, t_1]. \end{cases} \tag{15.20}$$

on the initial condition x_0. (Implicitly, here we assume that $u(t) \in \Omega$ for all $t \in [t_0, t_1]$.)

More precisely, we will study the differentiability of the map $(t, x_0) \mapsto u(t; x_0)$ with respect to the variable x_0.

To do so, let us fix $v_0 \in \mathbb{R}^n$ vector and let, for $h > 0$ small $u_h(t) = u(t; x_0 + hv_0)$. Define

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$$w_h(t) = w_h(t; v_0) = \frac{u_h(t) - u_0(t)}{h}.$$

We have

$$w_h'(t) = \frac{u_h'(t) - u_0'(t)}{h} = \frac{F(t, u_h) - F(t, u_0)}{h} = \frac{F(t, u_0 + hw_h) - F(t, u_0)}{h}$$

Let us define the (convex in space) δ-tube

$$\mathcal{T}_\delta := \{(x, t) \mid |x - u_0(t)| \leq \delta,\ t \in [t_0, t_1]\}.$$

Notice that F is L-Lipchitz for some constant L.

Then, for h sufficiently small we have

$$|w_h'| \leq \frac{\big|F(t, u_0 + hw_h) - F(t, u_0)\big|}{h} \leq L|w_h|,$$

provided h taken sufficiently small.

Hence,

$$\begin{aligned} (|w_h|^2)' &= \langle w_h, w_h' \rangle + \langle w_h', w_h \rangle \\ &= 2\langle w_h', w_h \rangle \leq 2|w_h'||w_h| \leq 2L|w_h|^2 \end{aligned}$$

Since $|w_h|^2(0) = |v_0|^2$, Grönwall's inequality gives

$$|w_h(t)|^2 \leq e^{2Lt}|h|^2|v_0|^2 \quad \Leftrightarrow \quad |w_h(t)| \leq e^{Lt}|hv_0|.$$

This proves the uniform bound

$$|w_h(t)| \leq e^{L(t_1-t_0)}|hv_0|,$$

valid for all $t \in [t_0, t_1]$. (Notice that, in turn, this quantifies how small we must take h to ensure that $u(t) + hw_h(t)$ belongs to the $\mathcal{T}_\delta$ for all t: namely $e^{L(t_1-t_0)}|hv_0| < \delta$.)

Finally, since $\partial_{x_i}F$ are uniformly continuous in in $\mathcal{T}_\delta$ we obtain

$$\left|F(t, u_0 + y) - F(t, u_0) - \sum_{i=1}^{n} \partial_{x_i}F(t, u_0)y_i\right| \leq o(|y|).$$

Therefore,

$$\left|w_h'(t) - \sum_{i=1}^{n} \partial_{x_i}F(t, u_0(t))w_{h,i}(t)\right| = \frac{o(|h|)}{|h|}; \tag{15.21}$$

with initial condition $w_h(t_0) = v_0$.

Take any sequence $h = h_k \to 0$. By Arzelà-Ascoli $w_{h_k} : [t_0, t_1] \to \mathbb{R}^n$ has a uniformly converging subsequence, to a certain limit w. Passing to the limit the integral formulation of
38 (15.21) we obtain that this subsequential limit w solves the linear ODE

$$w'(t) = \sum_{i=1}^{n} \partial_{x_i}F(t, u_0(t))w_i(t) \quad \Leftrightarrow \quad w'(t) = A(t)w(t),$$

where $A(t) = J_xF(t, u_0(t))$; with initial condition $w(t_0) = v_0$.

Since this solution is unique, we have proven that every sequence has a subsequence converging to the same function w, which yields (see 9.23) the existence of the limit

$$\lim_{h\to 0} w_h = w.$$

Moreover, notice that since w solves a linear ODE the dependence of $w(t) = w(t; v_0)$ with respect to v_0 is linear (see Exercise 15.46 below).

In particular this shows that

$$u(t; x_0 + hv_0) - u(t; x_0) - w(t; v_0)h = o(h),$$

where $v_0 \mapsto w(t, v_0)$ is a linear map.

In other words, $u(t; x_0)$ is continuously differentiable with respect to the initial condition x_0.

Exercise 15.46. — Let $A : [t_0, t_1] \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a continuous funtion and, for given $v_0 \in \mathbb{R}^n$, let $w(t; v_0)$ denote the solution to

$$w'(t) = A(t)w(t) \quad t \in [t_0, t_1] \quad \text{with initial condition } w(t_0) = v_0.$$

38 Then for all $v_0, \tilde{v}_0 \in \mathbb{R}^n$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$ we have

$$w(t; \lambda v_0 + \tilde{\lambda}\tilde{v}_0) = \lambda w(t, v_0) + \tilde{\lambda} w(t, v_0).$$

[*Hint:* Prove that a linear combination of solutions is a solution to the ODE and verify it satisfies the correct initial condition.]

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