CE7453: Photogrammetric Computer Vision

Lecture 6

Geometric Transformation
RANSAC Algorithm
Panorama

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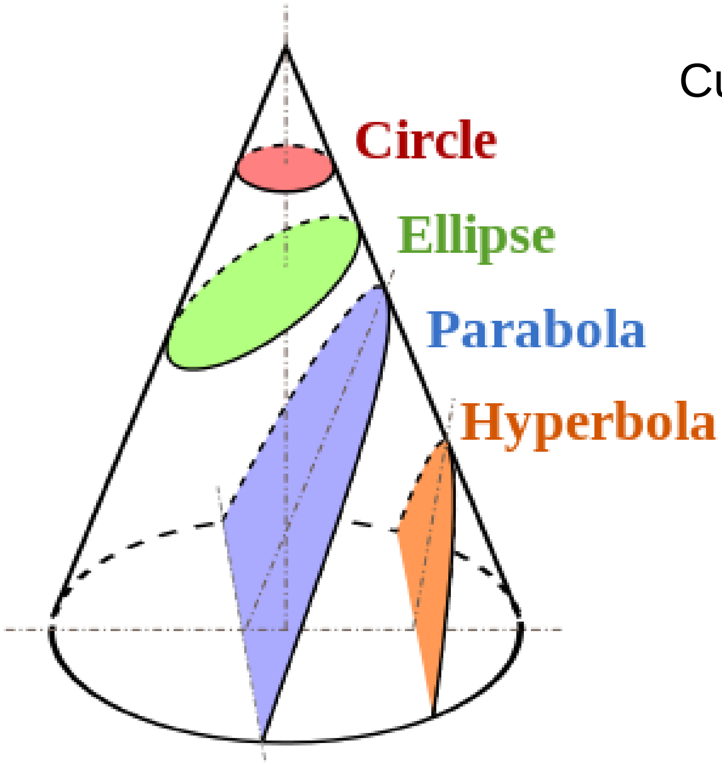
How projective distortion are presented in a mathematical sense, particularly in the homogenous coordinate system?

Conics



Wikipedia

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a:b:c:d:e:f\}$

Conics – Cont.

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

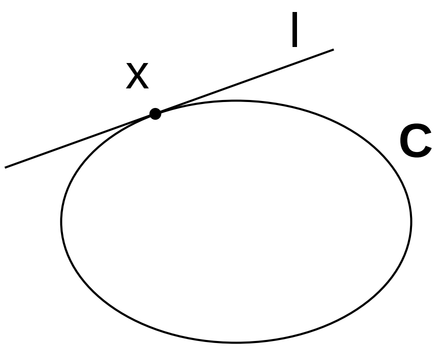
stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

The conics are the NULL vector of this matrix (5X6), this shows that the conics can be obtained uniquely up to a scale in homogenous coordinate system.

Conics – Cont.

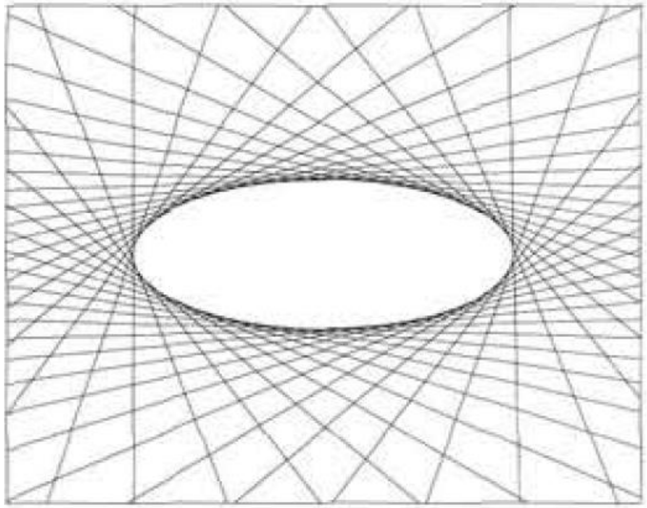
- The line l tangent to **C** at point x on **C** is given by l=**C**x



- $\mathrm{l}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = \mathbf{0}$ **-** this line pass over **x**
- If not a tangent line, there exist another point **y** that **l** intersect with, being $\mathbf{y}^{\mathrm{T}}\mathbf{C}\mathbf{y}$=0**,** since **y** also on the line ,then 0 = $\mathrm{l}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{y} = \mathbf{0}$, $(\boldsymbol{x}+\alpha\boldsymbol{y})^{T}\boldsymbol{C}(\boldsymbol{x}+\alpha\boldsymbol{y}) = \mathbf{0}$ for any α, the whole line is on the conic, this degenerate the conic

Conics – Cont.

- Dual Conics – we define a conic through five points, we can also define a conic through lines – Duality of homogenous coordinate system.



A line satisfies $\boldsymbol{l}^T\boldsymbol{C}^*\boldsymbol{l} = 0$ is tangent to the conic $\boldsymbol{C}$
Where $\boldsymbol{C}^*$ is the adjoint matrix (up to a scale of the inverse, being det(C)).

To understand:

$\boldsymbol{l} = \boldsymbol{C}\mathbf{x},\ \mathbf{x} = \mathbf{C}^{-1}\boldsymbol{l}$**, plug in to** $\mathbf{x}^T\mathbf{C}\mathbf{x}=0$

Conics

- Degenerated Conics: The conic is not a full rank, with one less it can be written as:

$$C = \mathrm{l}^{\mathrm{T}}\mathrm{m} + \mathrm{m}^{\mathrm{T}}\mathrm{l}$$

To lines in the matrix, and the third line is parallel to one of then.

Prove it is not full rank?

- none-zero NULL vector $\boldsymbol{l} \times \boldsymbol{m}$

Projective Transformations

P^2 : Projective Plane – 3D homogenous coordinates

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1)$,$h(x_2)$,$h(x_3)$ do.

Theorem:

A mapping h:$P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 represented by a vector x it is true that $h(\mathrm{x})=\mathbf{H}\mathrm{x}$

Definition: Projective transformation

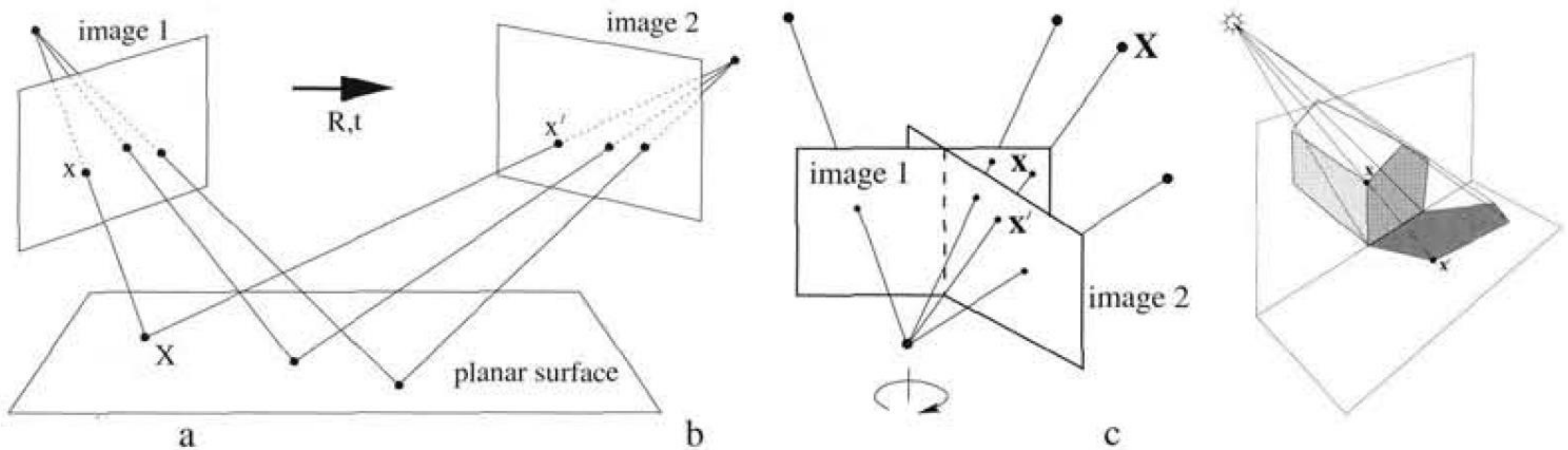
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

8DOF

projectivity=collineation=projective transformation=homography

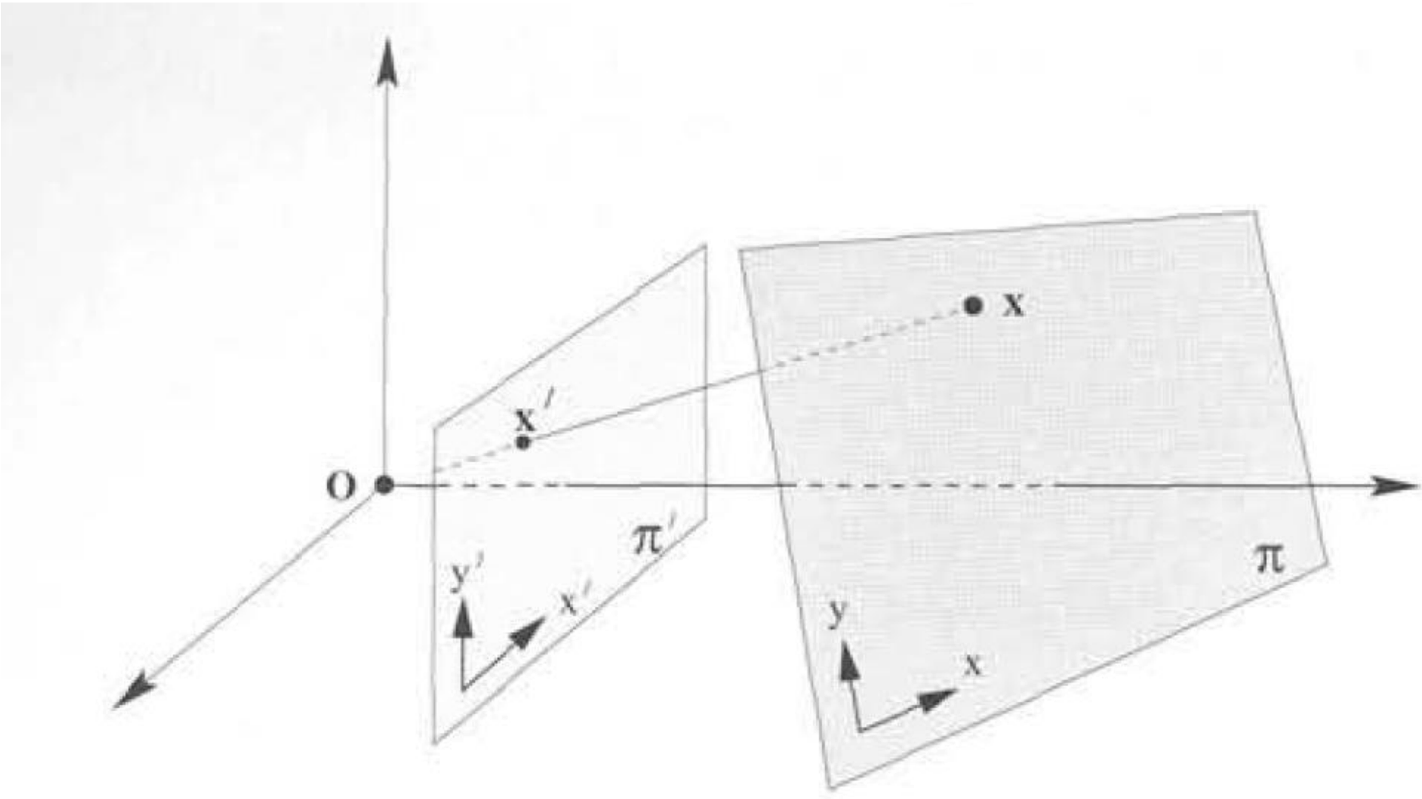
Projectivity

Examples:



Composite of Projectivities is still a Projectivity

Perspectivity



Mapping is defined in a central perspective Projection

Is the composite of perspectivity always being perspectivity?

Transformation of lines and Conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$(\mathbf{Hx})^{\mathbf{T}}\mathbf{l}'=0$

$$\mathrm{l}' = \mathbf{H}^{-\mathrm{T}}\,\mathrm{l}$$

Transformation for conics

$(\mathbf{Hx})^{\mathrm{T}}\boldsymbol{C}'\mathbf{Hx} = 0$

$$\mathbf{C}' = \mathbf{H}^{-\mathrm{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$(\mathbf{H}^{-\mathrm{T}}\mathbf{l})^{\mathbf{T}}\,\mathbf{C}'^{*}\mathbf{H}^{-\mathrm{T}}\mathbf{l} = \mathbf{0}$

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathrm{T}}$$

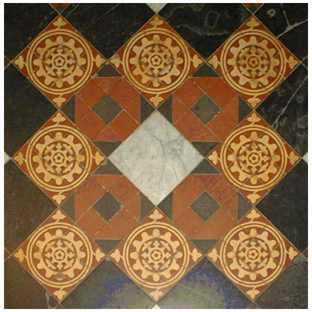
Hierarchy of Perspective Transformation

Projective linear group

- Affine group (last row (0,0,1))
  - Euclidean group (upper left 2x2 orthogonal)
    - Oriented Euclidean group (upper left 2x2 det 1)

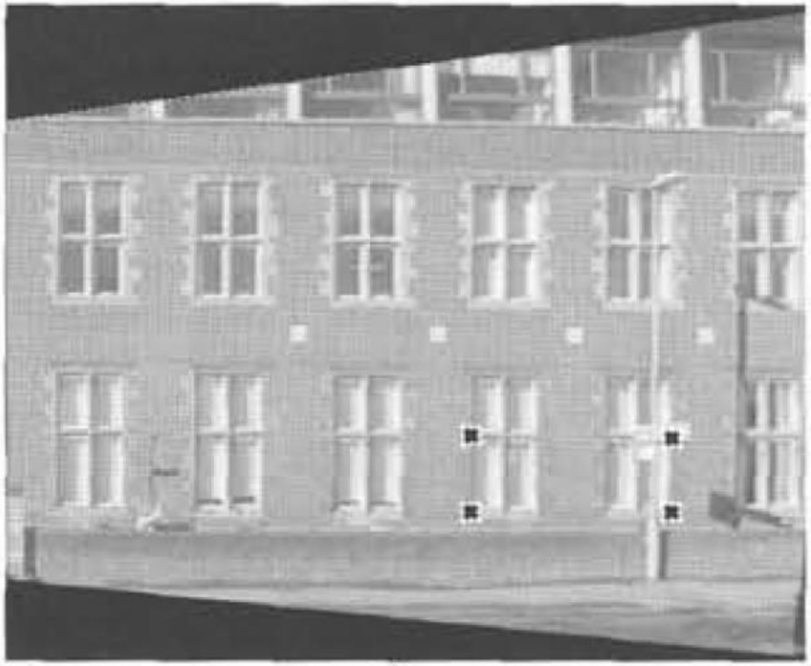
Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged



Removing Perspectives

a



b

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}, \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}.$$

$$\begin{aligned} x'\left(h_{31}x + h_{32}y + h_{33}\right) &= h_{11}x + h_{12}y + h_{13} \\ y'\left(h_{31}x + h_{32}y + h_{33}\right) &= h_{21}x + h_{22}y + h_{23}. \end{aligned}$$

Linear with respect to h_{ij} up to a scalar, needing four points eight equations to solve.

Class I: Isometries

(*iso*=same, *metric*=measure)

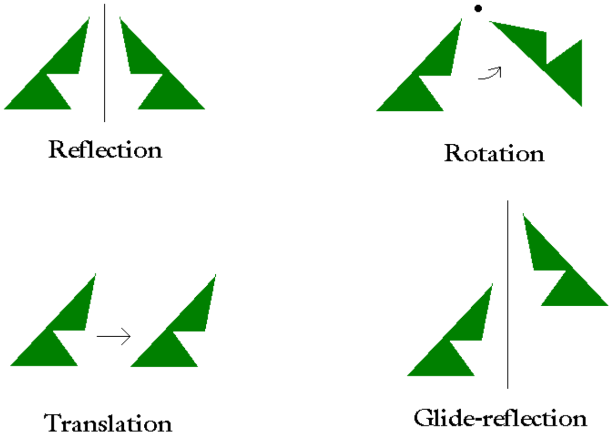
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

E.g. Keep you camera plane parallel, perspective center unchanged, only rotate the plane

Or flip the plane (orientation reversing)

orientation preserving: $\varepsilon = 1$
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\, \mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$



3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

(*isometry + scale*)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\, \mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also know as *equi-form* (shape preserving)

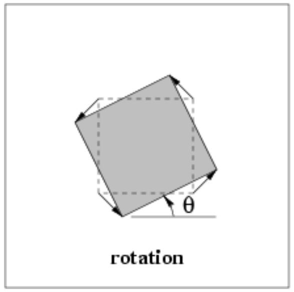
metric structure = structure up to similarity (in literature)

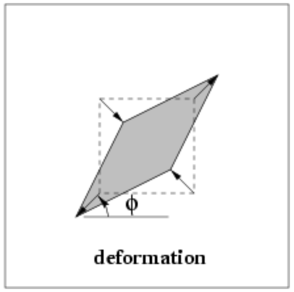
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A \, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

E.g. Shared pixel size in different directions

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

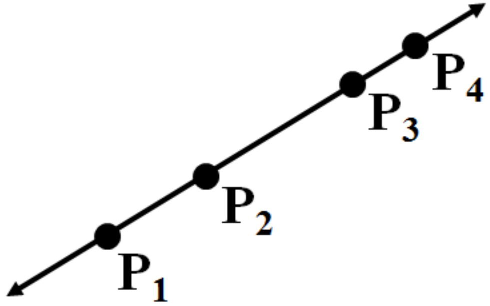
Class VI: Projective transformations

$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = (v_1, v_2)^\mathsf{T}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line (ratio of ratio)



$$\frac{\|\mathbf{P}_3 - \mathbf{P}_1\|\,\|\mathbf{P}_4 - \mathbf{P}_2\|}{\|\mathbf{P}_3 - \mathbf{P}_2\|\,\|\mathbf{P}_4 - \mathbf{P}_1\|} \qquad \mathbf{P}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

Perspective Distortions

Similarity

Affine

Perspective

More advanced Invariants

Ideal line $\mathrm{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ preserves at infinity under affine transformation $\mathrm{H_A} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Circular points $\mathrm{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ *and* $\mathrm{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$ preserves under the similarity transformation

$$\begin{aligned} \mathbf{I}' &= \mathrm{H_S}\mathbf{I} \\ &= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &= se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^{\mathsf{T}}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

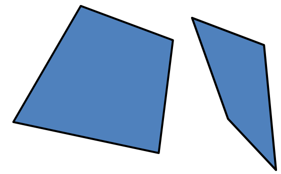
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

An overview

Projective
8dof

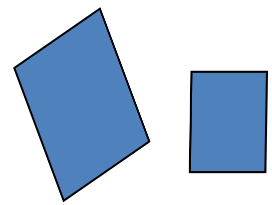
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

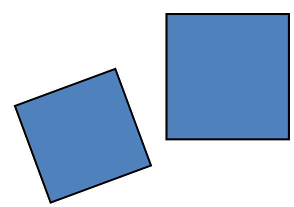
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity $\mathbf{l}_\infty$

Similarity
4dof

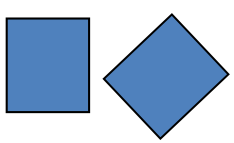
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

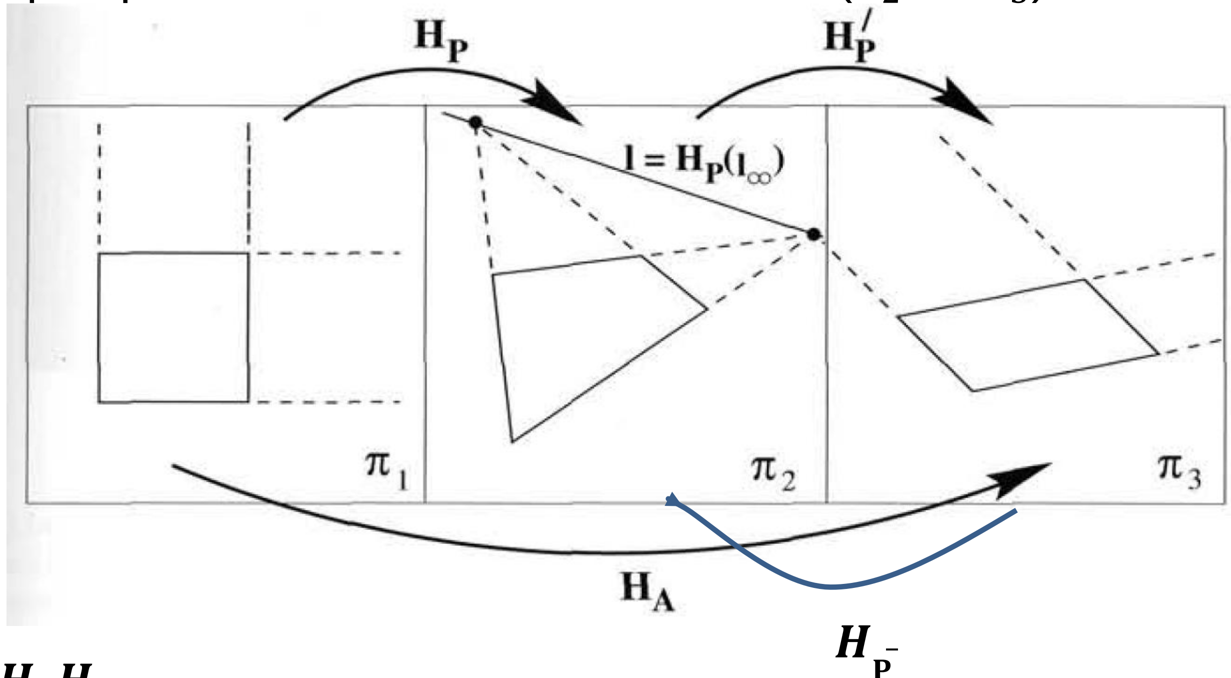
Number of invariants

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration minus the number of degrees of freedom of the transformation

e.g. configuration of 4 points in general position has 8 dof (2/pt) and so 4 similarity, 2 affinity and zero projective invariants

Affinely Recovery of Perspective Distortion

Affine transformation preserve parallel lines, given parallel lines in the image how to recover the perspective distortion to an affine one ($\pi_2 \rightarrow \pi_3$) ?



$$\boldsymbol{H}_P = \boldsymbol{H}_A \boldsymbol{H}_{\bar{\mathbf{P}}}$$

$\boldsymbol{H}_{\bar{\mathbf{P}}}$ distorts parallelism, the intersection of the two parallel lines under the projective transformation is call vanishing points, lying on the lines at infinity l_∞

Affinely Recovery of Perspective Distortion – Cont.

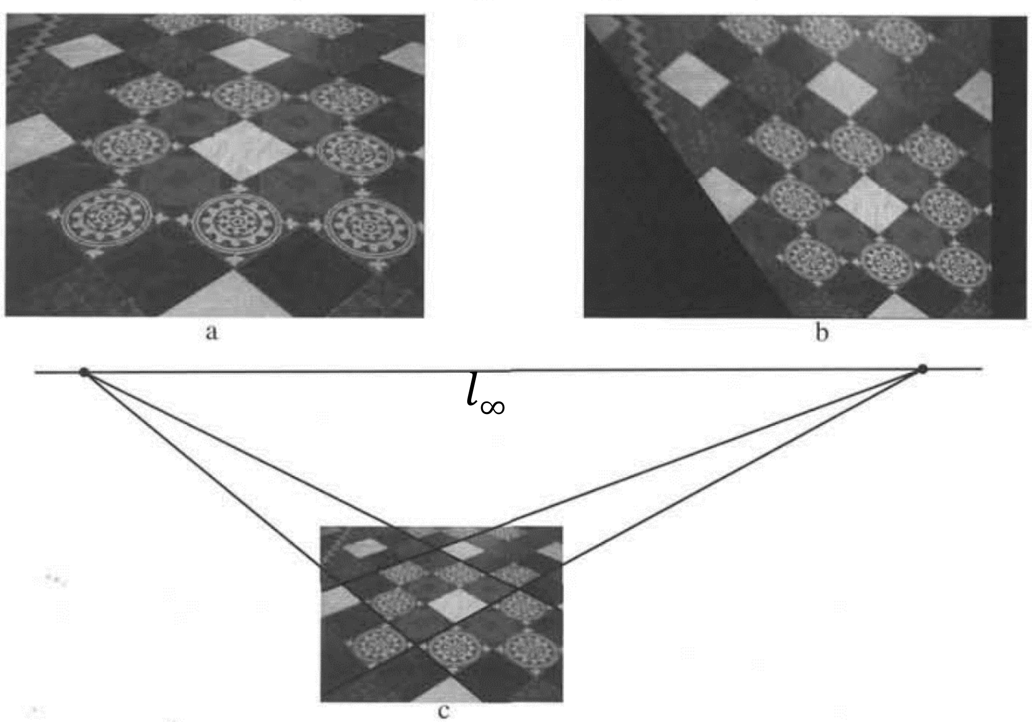
Recall that $\boldsymbol{H_A} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ preserves the line at the infinity. $\boldsymbol{H}_{\bar{\mathbf{P}}}$ distorted it by projecting l_∞ to the perspective plane, with the form of:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

We can then get $\boldsymbol{H_A} = \boldsymbol{H}_{\bar{\mathbf{P}}}^{-1}\boldsymbol{H_P}$, where $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ is the line of infinity at the perspectively distorted image

Affinely Recovery of Perspective Distortion – Cont.

- $\boldsymbol{H}_{\mathbf{P}}^{-1}$ can be then applied over the projectively distorted image and get the affinely distorted image – parallel lines.



1. Find the $l_\infty = \begin{bmatrix} l_1 \\ l_2 \\ 3 \end{bmatrix}$ by connecting two vanishing points
2. Write $\boldsymbol{H}_{\mathbf{P}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$
3. Apply $\boldsymbol{H}_{\mathbf{P}}^{-1}$ to the whole image

Additional correction

- Correct from affine to similarity, using the invariants of the similarities (two circular points)

More details see book Multi-view geometry section 2.7.5

Projective Transformation in 3D

2.4 The hierarchy of transformations 59

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 2.5).
Similarity 7 dof	$\begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 2.6).
Euclidean 6 dof	$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Volume.

All direct extension of 2D projective transformation

Estimating Projective Transformation

- Given a set of points, how to estimate the homography? – refer to slide 13 in this lecture.

- Minimal solution – four correspondences, homography has 8 DOF.

Direct Linear Transform (DLT)

$$\mathbf{x}'_i \propto \mathbf{H}\mathbf{x}_i \qquad \mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = 0 \qquad \mathbf{x}'_i = (x'_i, y'_i, w'_i)^\mathsf{T}$$

$$\mathbf{H}\mathbf{x}_i = \left(\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i, \mathbf{h}^{2\mathsf{T}}\mathbf{x}_i, \mathbf{h}^{3\mathsf{T}}\mathbf{x}_i\right)^\mathsf{T}$$

$\mathbf{h}^{j\mathsf{T}}$ is the jth row of H

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i - w'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i - x'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i - y'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \end{pmatrix}$$

$$\begin{bmatrix} 0^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \\ -y'_i\mathbf{x}_i^\mathsf{T} & x'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

$$\mathbf{A}_i\mathbf{h} = 0$$

DLT – Cont.

- Equations are linear in $\mathbf{h}$, $\mathbf{A}_i\mathbf{h} = 0$
- Only 2 out of 3 are linearly independent (indeed, 2 eq/pt)

$$\begin{bmatrix} 0^\top & -w'_i\mathbf{x}_i^\top & y'_i\mathbf{x}_i^\top \\ w'_i\mathbf{x}_i^\top & 0^\top & -x'_i\mathbf{x}_i^\top \\ -y'_i\mathbf{x}_i^\top & x'_i\mathbf{x}_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

(only drop third row if $w_i' \neq 0$)

DLT – Cont.

- Solving for H from 4 points (each point provide two independent equations of h)

$$\mathbf{Ah} = 0$$

- size A is 8x9 or 12x9, but rank 8
- Trivial solution is h=0
- 1-D null-space yields solution of interest
  - pick for example the one with $\|\mathbf{h}\| = 1$
- No exact solution because of inexact measurement, i.e. “noise”
  - Minimize $\|\mathbf{Ah}\|$ with constraint $\|\mathbf{h}\| = 1$

DLT – Cont.

<u>Objective</u>

Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix **H** such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

<u>Algorithm</u>

(i) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ compute $\mathbf{A}_i$. Usually only two first rows needed.

(ii) Assemble n 2x9 matrices $\mathbf{A}_i$ into a single $2n$x9 matrix **A**

(iii) Obtain SVD of **A**. Solution for **h** is last column of **V**

(iv) Determine **H** from **h**

Geometric Distance

DLT minimizes $\|\mathbf{Ah}\|$

$\mathbf{e} = \mathbf{Ah}$ residual vector

$\mathbf{e}_i$ partial vector for each ($\mathbf{x}_i \leftrightarrow \mathbf{x}_i$')

$$d_{\mathrm{alg}}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i)^2 = \|\mathbf{e}_i\|^2 = \left\| \begin{bmatrix} 0^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & -y'_i\mathbf{x}_i^\mathsf{T} \\ -w'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \end{bmatrix} \mathbf{h} \right\|^2$$

The *algebraic distance* between $\mathbf{x}_1$, $\mathbf{x}_2$:

$$d_{\mathrm{alg}}(\mathbf{x}_1, \mathbf{x}_2)^2 = a_1^2 + a_2^2 \text{ where } \mathbf{a} = (a_1, a_2, a_3)^\mathsf{T} = \mathbf{x}_1 \times \mathbf{x}_2$$

$$\sum_i d_{\mathrm{alg}}(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i)^2 = \sum_i \|\mathbf{e}_i\|^2 = \|\mathbf{Ah}\|^2 = \|\mathbf{e}\|^2$$

Not geometrically/statistically meaningful, but given good normalization it works fine and is very fast (use for initialization)

Geometric Distance – Cont.

How do you know you get a good estimation under noisy input for estimation?

$\mathbf{x}$ measured coordinates
$\hat{\mathbf{x}}$ estimated coordinates

$d(.,.)$ Euclidean distance (in image)

x_i , x_i' – pair of observations, may contain noise

- Error in one image

$$\hat{\mathbf{H}} = \underset{\mathrm{H}}{\operatorname{argmin}} \sum_i d\left(\mathbf{x}_i', \mathbf{H}\mathbf{x}_i\right)^2$$

- Symmetric transfer error

$$\hat{\mathbf{H}} = \underset{\mathrm{H}}{\operatorname{argmin}} \sum_i d\left(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}_i'\right)^2 + d\left(\mathbf{x}_i', \mathbf{H}\mathbf{x}_i\right)^2$$

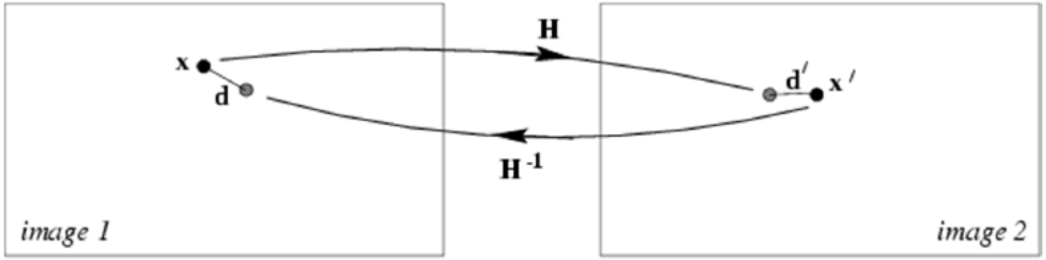
- Reprojection error

$$\left(\hat{\mathbf{H}}, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_i'\right) = \underset{\mathrm{H},\hat{\mathrm{x}}_i,\hat{\mathrm{x}}_i'}{\operatorname{argmin}} \sum_i d\left(\mathbf{x}_i, \hat{\mathbf{x}}_i\right)^2 + d\left(\mathbf{x}_i', \hat{\mathbf{x}}_i'\right)^2$$

$$\text{subject to } \hat{\mathbf{x}}_i' = \hat{\mathbf{H}}\hat{\mathbf{x}}_i$$

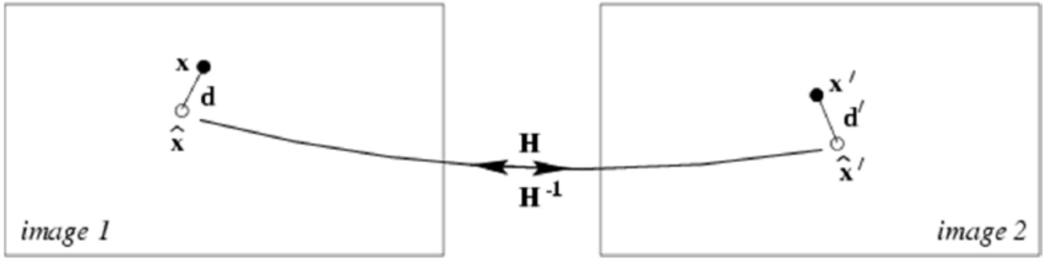
Geometric Distance – Cont.

Symmetric geometric error



$$d\left(\mathbf{x}, \mathbf{H}^{-1}\mathbf{x}'\right)^2 + d\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right)^2$$

Reprojection error

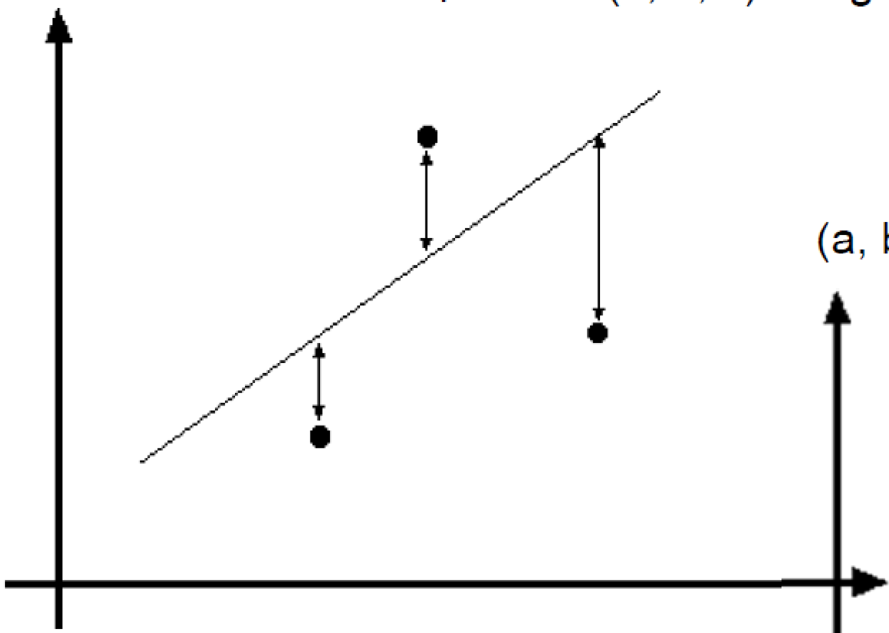


$$d\left(\mathbf{x}, \hat{\mathbf{x}}\right)^2 + d\left(\mathbf{x}', \hat{\mathbf{x}}'\right)^2$$

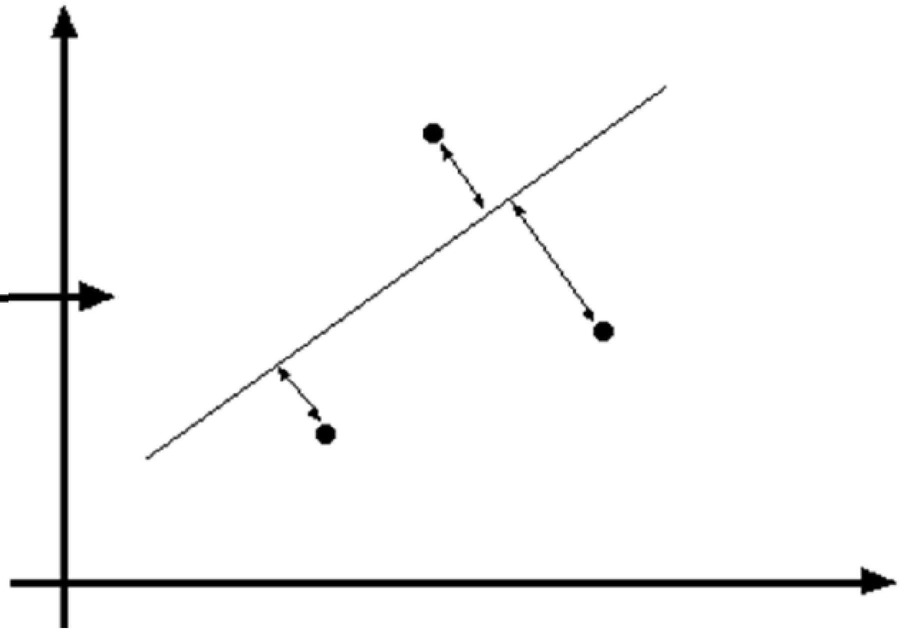
Algebraic and Geometric Distance

In the case of line fitting

standard least-squares $(a, b, c) = \arg\min \sum_i |ax_i + by_i + c|^2$



$(a, b, c) = \arg\min \sum_i |ax_i + by_i + c|^2/|a^2 + b^2|$



Algebraic and Geometric Distance – Cont.

Error in one image

$$\mathbf{x}'_i = (x'_i, y'_i, w'_i)^\mathsf{T} \quad \hat{\mathbf{x}}'_i = (\hat{x}'_i, \hat{y}'_i, \hat{w}'_i)^\mathsf{T} = \mathbf{H}\mathbf{x} = \left(\mathbf{h}^{1^\mathsf{T}}\mathbf{x}_i, \mathbf{h}^{2^\mathsf{T}}\mathbf{x}_i, \mathbf{h}^{3^\mathsf{T}}\mathbf{x}_i\right)^\mathsf{T}$$

$$\begin{bmatrix} 0^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} \qquad \mathbf{A}_i\mathbf{h} = \mathbf{e}_i = \begin{pmatrix} y'_i\hat{w}'_i - w'_i\hat{y}'_i \\ w'_i\hat{x}'_i - x'_i\hat{w}'_i \end{pmatrix}$$

$$d_{\mathrm{alg}}(\mathrm{x}'_\mathrm{i}, \hat{\mathrm{x}}'_\mathrm{i})^2 = (y'_i\hat{w}'_i - w'_i\hat{y}'_i)^2 + (w'_i\hat{x}'_i - x'_i\hat{w}'_i)^2$$

$$d(\mathrm{x}'_\mathrm{i}, \hat{\mathrm{x}}'_\mathrm{i})^2 = \left((y'_i / w'_i - \hat{y}'_i / \hat{w}'_i)^2 + (\hat{x}'_i / \hat{w}'_i - x'_i / w'_i)^2\right)^{1/2}$$

$$= d_{\mathrm{alg}}(\mathrm{x}'_\mathrm{i}, \hat{\mathrm{x}}'_\mathrm{i}) / w'_i\hat{w}'_i$$

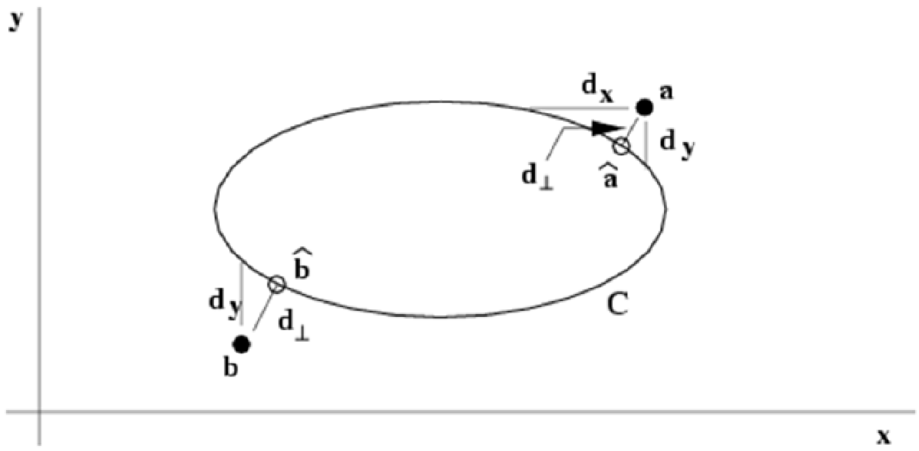
Typically $w'_i = 1$, but not $\hat{w}'_i = \mathbf{h}^{3^T}\mathbf{x}_\mathrm{i}$, except for affine transforms

Geometric Error

- Estimating homography ~ fit surface ν_H to points $\mathbf{X}=(x,y,x',y')^T$ in $\mathbf{R}^4$. Points on the surface should satisfy the constrain exactly.
- For each point $\mathbf{X}_i=(x_i,y_i,x_i',y_i')$, find a closest point $\hat{\mathbf{X}}_i$ on the surface ν_H
- Minimize the distance

$$d_\perp(\mathbf{X}_i,\nu_H)^2 = \left\|\mathbf{X}_i - \hat{\mathbf{X}}_i\right\|^2 = (x_i-\hat{x}_i)^2 + (y_i-\hat{y}_i)^2 + (x_i'-\hat{x}_i')^2 + (y_i'-\hat{y}_i')^2$$
$$= d(\mathrm{x}_i,\hat{\mathrm{x}}_i)^2 + d(\mathrm{x}_i',\hat{\mathrm{x}}_i')^2$$

- Analog to conic fitting (a 2D conic is easier to visualize)



$$d_{\mathrm{alg}}(\mathrm{x},\mathrm{C})^2 = \mathrm{x}^T\mathrm{C}\mathrm{x}$$
$$d_{\mathrm{gmt}}(\mathrm{x},\mathrm{C})^2 = \mathrm{d}_x^2 + \mathrm{d}_y^2$$
$$d_\perp(\mathrm{x},\mathrm{C})^2$$

Symmetry geometric error works for the point a, but not at the point b.

Sampson Error

- The reprojection error:

Mminimizes the geometric error $\left\|\mathbf{X}-\hat{\mathbf{X}}\right\|^2$, where $\hat{\mathbf{X}}$ is the closest point on the variety $\mathcal{V}_{\mathrm{H}}$ to the measurement $\mathbf{X}$.

Why difficult?

We need to find a $\hat{\mathbf{X}}$ for each $\mathbf{X}$ under each variety $\mathcal{V}_{\mathrm{H}}$

- Sampson error: 1st order approximation of $\hat{\mathbf{X}}$

define the cost function (the simple algebraic error) $\mathbf{C}_{\mathrm{H}}(\mathbf{X}) = \mathbf{A}\mathbf{h} = \begin{bmatrix} 0^{\mathsf{T}} & -w_i'\mathbf{x}^{\mathsf{T}} & y_i'\mathbf{x}^{\mathsf{T}} \\ w_i'\mathbf{x}^{\mathsf{T}} & 0^{\mathsf{T}} & -x_i'\mathbf{x}^{\mathsf{T}} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix}$

then, for any point $\hat{\mathbf{X}}$ on $\mathcal{V}_{\mathrm{H}}$, $\mathbf{C}_{\mathrm{H}}(\hat{\mathbf{X}}) = 0$

let $\boldsymbol{\delta}_{\mathrm{X}} = \hat{\mathbf{X}} - \mathbf{X}$ $\quad\quad \mathbf{C}_{\mathrm{H}}(\mathbf{X}+\boldsymbol{\delta}_{\mathrm{X}}) = \mathbf{C}_{\mathrm{H}}(\mathbf{X}) + \dfrac{\partial \mathbf{C}_{\mathrm{H}}}{\partial \mathbf{X}}\boldsymbol{\delta}_{\mathrm{X}} \approx 0$

Hence, $\mathbf{J}\boldsymbol{\delta}_{\mathrm{X}} = -\mathbf{e}$ with $\mathbf{J} = \dfrac{\partial \mathbf{C}_{\mathrm{H}}}{\partial \mathbf{X}}$

Sampson Error

- The reprojection error:

  Mminimizes the geometric error $\left\|\mathbf{X}-\hat{\mathbf{X}}\right\|^2$, where $\hat{\mathbf{X}}$ is the closest point on the variety $\mathcal{V}_H$ to the measurement $\mathbf{X}$

- Sampson error: 1st order approximation of $\hat{\mathbf{X}}$

- Original objective: find $\hat{\mathbf{X}}$ that minimizes $\left\|\mathbf{X}-\hat{\mathbf{X}}\right\|$ subject to $\mathbf{C}_H\left(\hat{\mathbf{X}}\right)=0$

  New equivalent objective: Find the vector $\boldsymbol{\delta}_X$ that minimizes $\left\|\boldsymbol{\delta}_X\right\|$ subject to $\mathbf{J}\boldsymbol{\delta}_X = -\mathbf{e}$

$$\left\|\boldsymbol{\delta}_X\right\|^2 = \boldsymbol{\delta}_X^T\boldsymbol{\delta}_X = \mathbf{e}^T\left(\mathbf{J}\mathbf{J}^T\right)^{-1}\mathbf{e} \qquad \text{(Sampson error)}$$

Sampson Error – Cont.

Objective function:

$$\|\boldsymbol{\delta}_{\mathrm{X}}\|^2 = \mathbf{e}^{\mathrm{T}}\left(\mathbf{J}\mathbf{J}^{\mathrm{T}}\right)^{-1}\mathbf{e}$$

A few points

(i) For a 2D homography $\mathbf{X}=(x,y,x',y')$

(ii) $\mathbf{e} = \mathbf{C}_{\mathrm{H}}(\mathbf{X})$ is the algebraic error vector

(iii) $\mathbf{J} = \partial \mathbf{C}_{\mathrm{H}} / \partial \mathbf{X}$ is a 2x4 matrix, e.g.

$$\mathbf{J}_{11} = \partial\left(-w_i'\mathbf{x}_i{}^{\mathrm{T}}\mathbf{h}^2 + y_i'\mathbf{x}_i{}^{\mathrm{T}}\mathbf{h}^3\right)/\partial x = -w_i'h_{21} + y_i'h_{31}$$

(iv) Similar to algebraic error $\|\mathbf{e}\|^2 = \mathbf{e}^{\mathrm{T}}\mathbf{e}$. In fact, same as Mahalanobis distance $\|\mathbf{e}\|^2_{\mathbf{J}\mathbf{J}^{\mathrm{T}}}$

(v) Must be summed for all points $\sum \mathbf{e}_i^{\mathrm{T}}\left(\mathbf{J}_i\mathbf{J}_i^{\mathrm{T}}\right)^{-1}\mathbf{e}_i$

(vi) Close to geometric error, but much fewer unknowns

Least Squares Solution

- The mosaic problem eventually becomes a non-linear least square problem

$$\arg\min_{\mathrm{H}} \sum_i d\left(\mathbf{x}_i', \mathbf{H}\mathbf{x}_i\right)^2$$

  - d is the algebraic, geometric or re-projection error
- Minimizing a general non-linear function

$$\underset{\mathrm{P}}{\operatorname{argmin}} \left\|\mathrm{X} - f(\mathrm{P})\right\|$$

- General methods
  - Newton iteration
  - Levenberg-Marquardt

Least Squares Solution

- F(x) being X-f(P), solution similar to the least squares image matching.

$$F(x,y,\boldsymbol{H}) \approx F(x,y,\boldsymbol{H}^0) + \boldsymbol{\mathcal{J}}\left(F(x,y,\boldsymbol{H}^0)\right)^T \cdot \Delta\boldsymbol{H} = 0$$

$$\boldsymbol{\mathcal{J}}(F(x,y,\boldsymbol{H}^0)) = [\frac{\partial F}{\partial a_1},\frac{\partial F}{\partial a_2},\frac{\partial F}{\partial a_3},\frac{\partial F}{\partial b_1},\frac{\partial F}{\partial b_2},\frac{\partial F}{\partial b_3},\frac{\partial F}{\partial k_1},\frac{\partial F}{\partial k_2}]_{|\boldsymbol{B^0}}^{T}$$

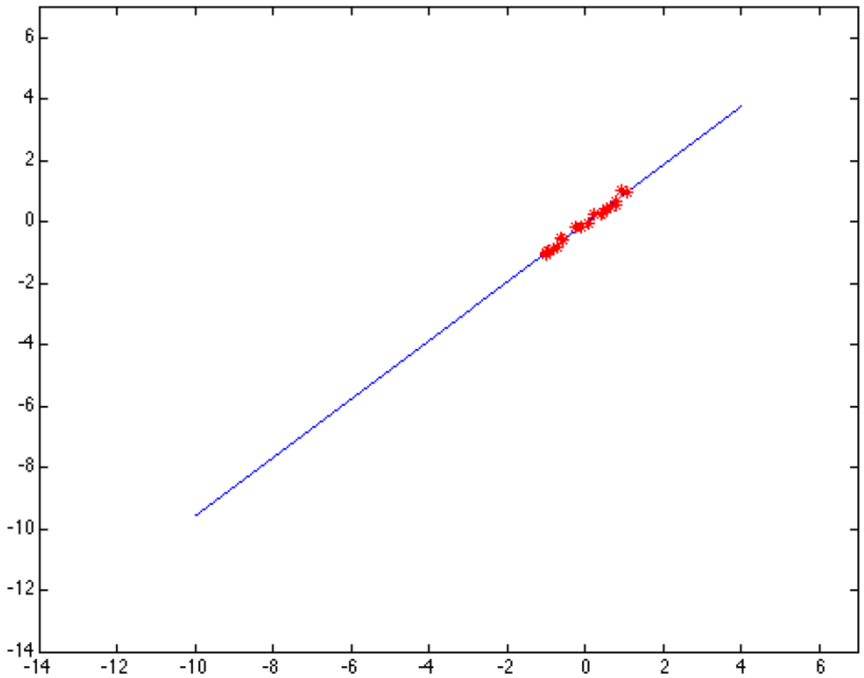
$$\Delta\boldsymbol{H} = [\Delta h_{11}, \Delta h_{12}, \Delta h_{13}, \Delta h_{21}, \Delta h_{22}, \Delta h_{23}, \Delta h_{31}, \Delta h_{32}, \Delta h_{33}]^{\boldsymbol{T}}$$

$$\Delta\,\boldsymbol{H} = \left[\sum_{x,y}^{M,N} \boldsymbol{\mathcal{J}}\left(F(x,y,\boldsymbol{H}^0)\right) \boldsymbol{\mathcal{J}}\left(F(x,y,\boldsymbol{H}^0)\right)^T\right]^{-\mathbf{1}} \left[-\sum_{x,y}^{M,N} F(x,y,\boldsymbol{H}^0)\boldsymbol{\mathcal{J}}\left(F(x,y,\boldsymbol{H}^0)\right)\right]$$

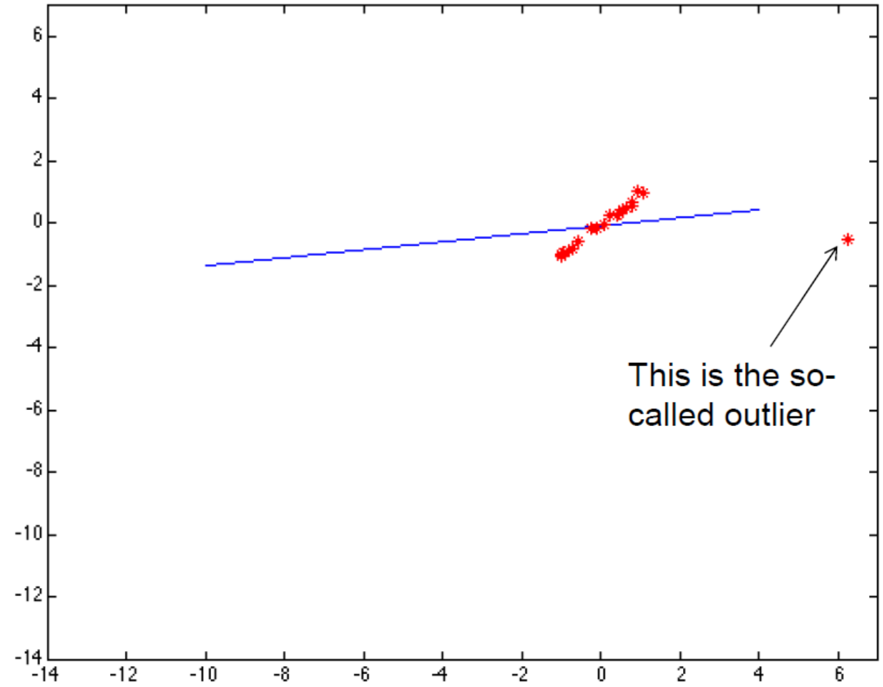
Problems of Parameter Estimation

Noise / outlier of the observations

standard least-squares: Line = arg min $\sum_i \text{Dist}^2(\text{Line}, x_i)$



A single outlier could 'drag' the line away….



Problems of Parameter Estimation – Cont.

- Solutions:

Weight least squares: weight observations far away

M-estimator: Square nearby, threshold far away

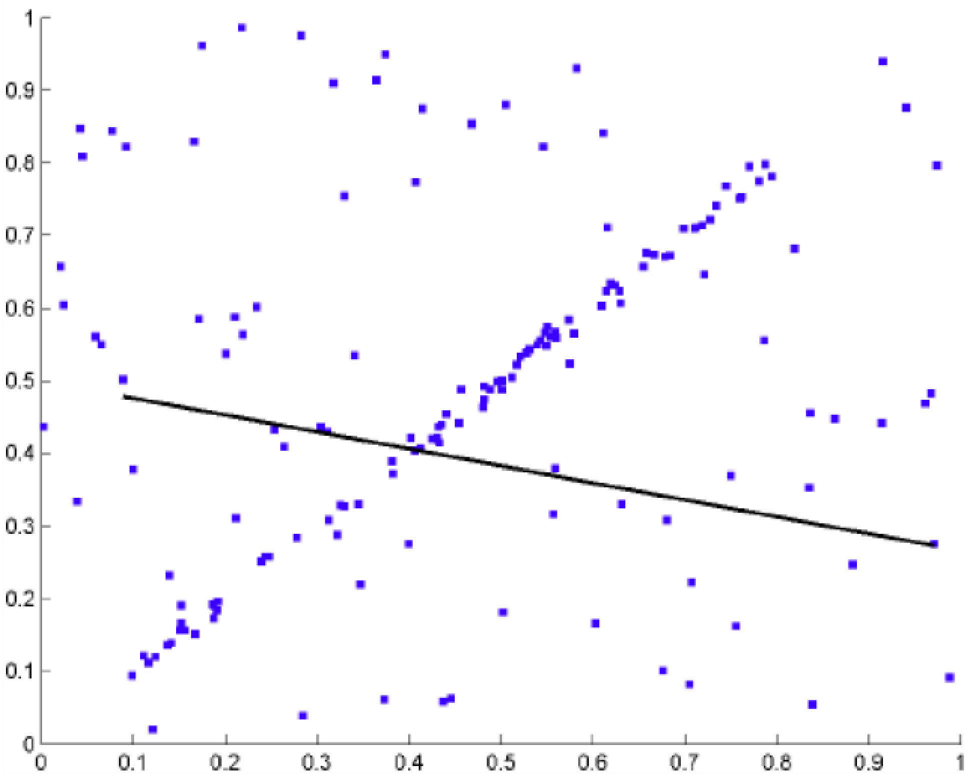
Ransac: RANdom SAmpling Concensus: try lucks with a number of minimal solutions using randomly selected observations, and choose the one with the best fit.

RANSAC

- Choose a small subset of samples at random
- Fit to that subset
- Anything that is close to the result is inlier; all others are outliers
- Do this many times and choose the best

- Issues
  - How many times?
    - Often enough that we are likely to have a good line
  - How big a subset?
    - Smallest possible
  - How to decide inlier and outlier according to a fit?
    - Depends on the problem
  - How to select the best result from all iterations?
    - Choose the one with the largest number of inliers

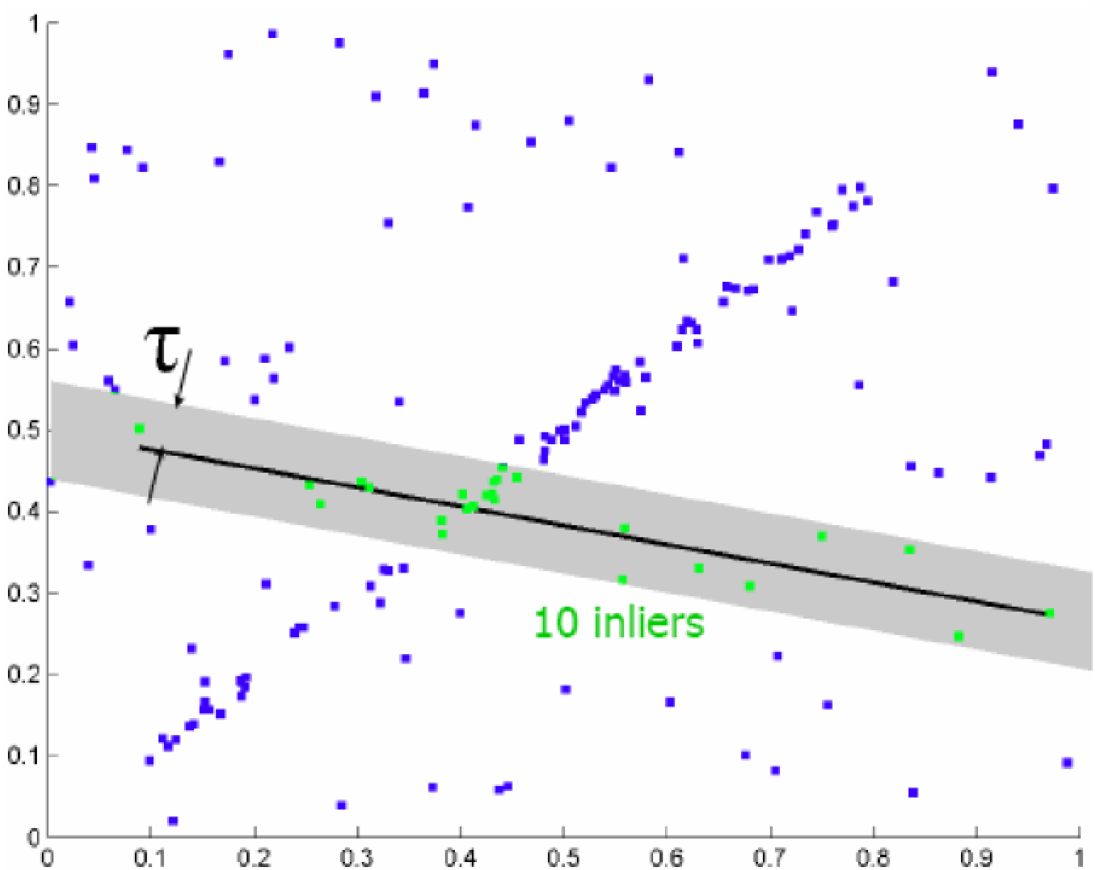
RANSAC – Cont.



1. sample randomly two points, get a line

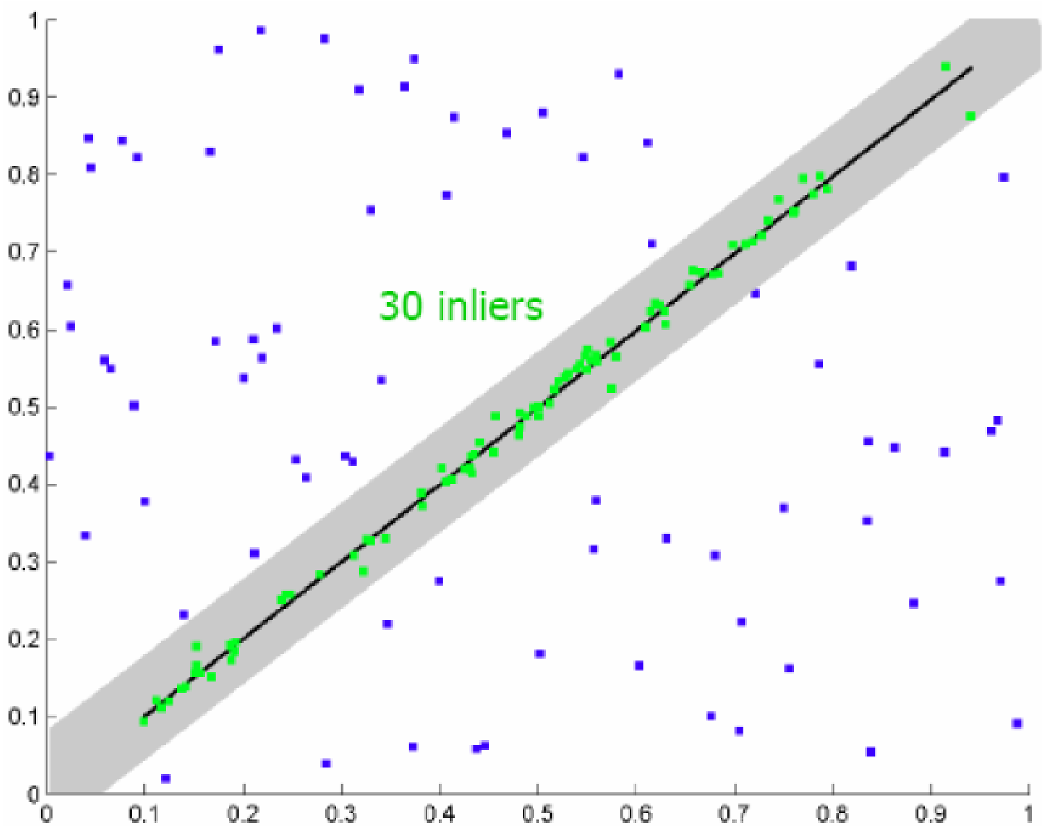
RANSAC

RANSAC – Cont.



1. sample randomly two points, get a line
2. count inliers for threshold T

RANSAC – Cont.



1. sample randomly two points, get a line
2. count inliers for threshold T
3. repeat N times and select model with most inliers

RANSAC

RANSAC – Cont.

How Many Iterations ?

- Suppose each time s samples are selected to fit a model
- Suppose e is the percentage of outlier
- Suppose N iterations have been run
- What is the probability p that at least one set of sample is free from outlier?
  - The probability q that a set has no outlier $q = (1-e)^s$
  - The probability (1-p) that all sets have outliers

$$1-p = (1-q)^N = \left(1-(1-e)^s\right)^N$$

- Decide the confidence probability p (e.g. 0.99), then determine N

$$N = \log(1-p) / \log\left(1-(1-e)^s\right)$$

RANSAC – Cont.

- How Many iterations – Cont.
  - Suppose each time s samples are selected to fit a model
  - Suppose e is the percentage of outlier
  - Decide the confidence probability p (e.g. 0.99), then determine N

$$N = \log(1-p) / \log\left(1-(1-e)^s\right)$$

	proportion of outliers e						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

RANSAC – Cont.

- Determining N adaptively

  e is often unknown a priori, so pick worst case, e.g. 50%, and adapt if more inliers are found, e.g. 80% would yield e=0.2

  – $N=\infty$, *sample_count* =0
  – While *N* >*sample_count* repeat
    - Choose a sample and count the number of inliers
    - Set e=1-(number of inliers)/(total number of points)
    - Recompute *N* from *e*
    - Increment the *sample_count* by 1
  – Terminate

RANSAC Algorithm

Objective

Robust fit of a model to a data set S which contains outliers.

Algorithm

(i) Randomly select a sample of s data points from S and instantiate the model from this subset.

(ii) Determine the set of data points S_i which are within a distance threshold t of the model. The set S_i is the consensus set of the sample and defines the inliers of S.

(iii) If the size of S_i (the number of inliers) is greater than some threshold T, re-estimate the model using all the points in S_i and terminate.

(iv) If the size of S_i is less than T, select a new subset and repeat the above.

(v) After N trials the largest consensus set S_i is selected, and the model is re-estimated using all the points in the subset S_i.

Panoramic Stitching – Assignment 2

Mosaics: Stitching Images Together

virtual wide-angle camera

Panorama

- We assume the images are taken from the same perspective center, then it forms perspectivity / projectivities among pairs of the images,

- The per-pixel relationship can be estimated by projective transformation / Homographies.

Panorama – Cont.

- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its optical center
  - Compute transformation between second image and first
  - Transform the second image to overlap with the first
  - Blend the two together to create a mosaic
  - If there are more images, repeat

Issues?

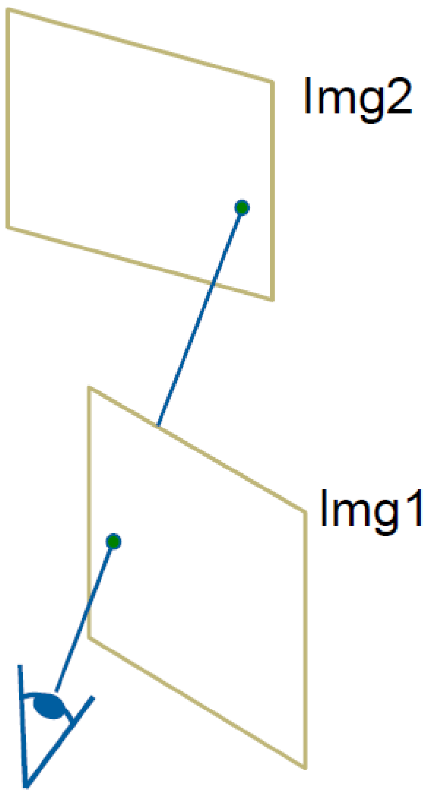
1. How do we find corresponding points? - Refer to Lecture 3
2. What if the points contain outliers? – RANSAC !!!

Panorama – Cont.

Homography Recap

- Perspective projection of a plane
  - Lots of names for this:
    - **homography**, texture-map, colineation, planar projective map
  - Modeled as a 2D warp using homogeneous coordinates

$$\underbrace{w\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}}_{\mathbf{x'}} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{x}}$$



- To apply a homography H
  - Compute **x**' = **Hx** (regular matrix multiply)
  - Convert **x**' from homogeneous to image coordinates
    - divide by w (third) coordinate

Panorama – Cont.

x' = Hx ➔ $$w\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Find the homography H given a set of p and p' pairs
- How many correspondences are needed?
- Can set scale factor *i*=1. So, there are 8 unkowns, 4 pairs are needed
- Set up a system of linear equations:

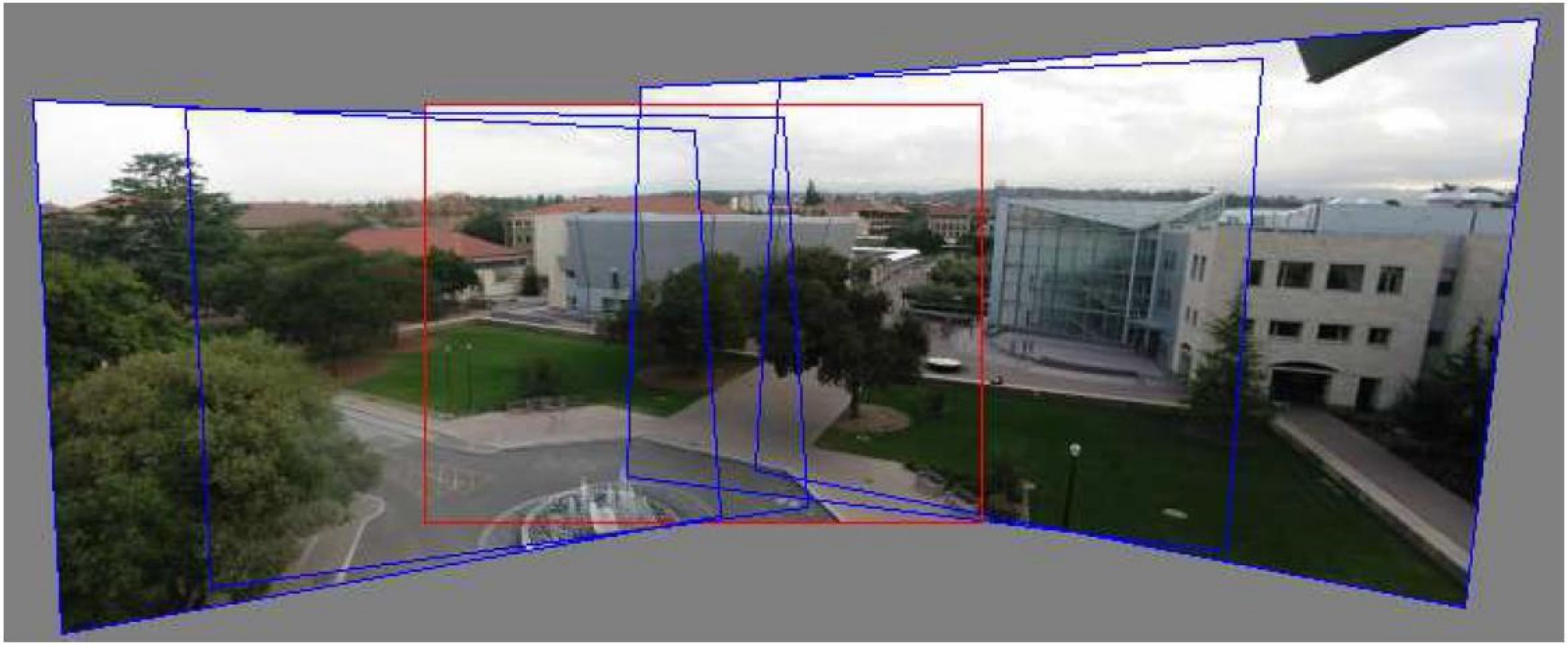
Ah = b

h is the vector of unknowns, h = $[a,b,c,d,e,f,g,h]^T$

- Need at least 8 eqs, but the more the better…
- Solve for h. If overconstrained, solve using least-squares:

$$\min\|Ah - b\|^2 \qquad \text{i.e. } A^T Ah = A^T b$$

Panorama – Cont.

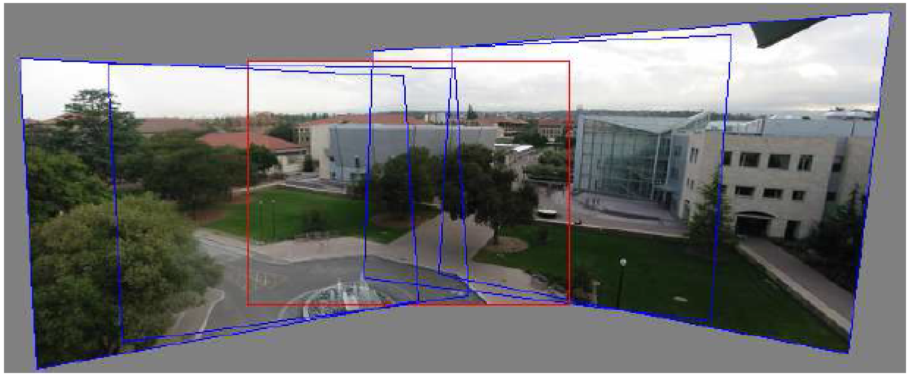


1. Pick one image (red)
2. Warp the other images towards it (usually, one by one)
3. blend

Panorama – Cont.

- Panorama = reprojection
- 3D rotation ➔ homography
  - Using homogeneous coordinates to represent pixel position
- Use feature correspondence
- Solve least square problem
  - Set of linear equations
- Warp all images to a reference one
- Use your favorite blending

E.g. Take the average



Questions?

Next class

- Two-view geometry: Relative orientation, Co-planarity, Fundemental/essential matrix
- Five-point algorithm – direct solution / iterative & rigorous solution.