Stochastic Processes

Assignment 2

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Question 1

It is known that Bush Babies always have identical quadruplets (four offspring). Each of the 4 little Bush Babies has a 1/3 chance of becoming a doctor,a lawyer or a scientist, independently of its 3 siblings. A doctor Bush Baby will reproduce further with probability 2/3 , a lawyer with probability 1/2 and a scientist with probability 1/4, again, independently of everything else. If it reproduces at all, a Bush Baby reproduces only once in its life, and then leaves the Bush Baby scene. (For the purposes of this problem assume that Bush Babies reproduce asexually.) Let us call the Bush Babies who have offspring fertile.

(i) What is the distribution of the number of fertile offspring? Write down its generating function:

solution

Taking into consideration the 3 possible professions (a doctor,a lawyer or a scientist), each *Bush Baby* is fertile with probability p, where;

$$p = \mathbb{P}[Fertile] = \mathbb{P}[Fertile|Lawyer]\mathbb{P}[Lawyer]+\mathbb{P}[Fertile|Doctor]\mathbb{P}[Doctor]+\mathbb{P}[Fertile|Scientist]\mathbb{P}[Scientist]$$

$$= \frac{1}{2} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{17}{36}$$

From the above; the number of fertile offspring is binomial with $n = 4$ and $p = \frac{17}{36}$.Therefore, the generating function of this distribution is;

$$P(s) = (\frac{19}{36} + \frac{17}{36}s)^4$$

.

(ii) What is the generating function for the number of great-grandchildren a *Bush Baby* will have? What is its expectation? (Note: do not expand powers of sums).

solution To get the number of great-grandchildren, we first compute the generating function $Q(s)$ of the number of fertile grandchildren.

This is simply given by the composition of P with itself as shown below;

$$Q(s) = P(P(s)) = (\frac{19}{36} + \frac{17}{36}(\frac{19}{36} + \frac{17}{36}s)^4)^4$$

.

The number of great-grandchildren is the number of fertile grandchildren multiplied by 4. Therefore, its generating function is given by;

$$Q(s) = P(P(s^4)) = \left(\frac{19}{36} + \frac{17}{36}\left(\frac{19}{36} + \frac{17}{36}s^4\right)^4\right)^4.$$

To compute the expectation, we need to evaluate $Q'(1)$:

$$Q'(s) = (P(P(s^4)))' = P'(P(s^4))P'(s^4)4s^3$$

and

$$P'(s) = 4\left[\frac{17}{36}\left(\frac{19}{36} + \frac{17}{36}s\right)^3\right] = \frac{17}{9}$$

so that

$$Q'(1) = 4P'(1)P'(1) = \frac{1156}{81}.$$

(iii) Let the *Bush Baby* population be modeled by a branching process, and let's suppose that it starts from exactly one individual at time 0. Is it certain that the population will go extinct sooner or later?

solution

We need to consider the population of fertile Bush Babies. Its offspring distribution has generating function $P(s) = (\frac{19}{36} + \frac{17}{36}s)^4$, so the population will go extinct with certainty if and only if the extinction probability is 1, i.e., if $s = 1$ is the smallest solution of the extinction equation $s = P(s)$. We know, however, that $P'(1) = 5 \times \frac{17}{36} = \frac{85}{36} > 1$, so there exists a positive solution to $P(s) = s$ which is smaller than 1. Therefore, it is not certain that the population will become extinct sooner or later.

Question 2

Consider the difference-differential equations of the pure birth process given by:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t), n \geq 1$$
$$P_0'(t) = -\lambda_0 P_0(t), n = 0$$

(a) substituting $\lambda = \lambda(\frac{1+an}{1+\lambda at})$ in difference-differential equation of the pure birth process show that the probability that the population of size n at time t, $P_n(t)$ for the process is given by

$$P_n(t) = (1 + \lambda at)^{-\frac{1}{a}} \begin{pmatrix} -\frac{1}{a} \\ n \end{pmatrix} \left(\frac{-\lambda at}{1 + \lambda at}\right)^n$$

Use the initial condition:

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

solution

In Polya Process;

$$\lambda_n = \lambda\left[\frac{1 + an}{1 + \lambda at}\right] \text{ for } n = 0, 1, 2, 3......$$

Replacing λ_n, the difference differential equations will be:

$$P_n'(t) = -\lambda\left[\frac{1+an}{1+\lambda at}\right]P_n(t) + \lambda\left[\frac{1+a(n-1)}{1+\lambda at}\right]P_{n-1}(t),\ n \geq 1$$

$$P'(t) = \left[\frac{-\lambda}{1+\lambda at}\right]P_0(t), n = 0$$

Therefore,

$$\sum_{n=1}^{\infty} P_n'(t)s^n = \lambda\left[-\sum_{n=1}^{\infty}\left(\frac{1+an}{1+\lambda at}\right)P_n(t)s^n + \sum_{n=1}^{\infty}\left(\frac{1+a(n-1)}{1+\lambda at}\right)P_{n-1}(t)s^n\right]$$

Since define $\frac{\partial}{\partial t}G(s,t) = \sum_{n=1}^{\infty} P_n'(t)s^n$, we will have:

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{-\sum_{n=0}^{\infty}(1+an)P_n(t)s^n + \sum_{n=1}^{\infty}(1+a(n-1))P_{n-1}(t)s^n\right\}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{-\sum_{n=0}^{\infty}P_n(t)s^n - a\sum_{n=1}^{\infty}nP_n(t)s^n + s\sum_{n-1}^{\infty}P_{n-1}(t)s^{n-1} + as\sum_{n-1}^{\infty}(n-1)P_{n-1}(t)s^{n-1}\right\}$$

But:

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$

So:

$$\frac{\partial}{\partial t}G(t)s = \sum_{n=0}^{\infty} P_n'(t)s^n, \quad \frac{\partial}{\partial s}G(s,t) = \sum_{n=1}^{\infty} nP_n(t)s^{n-1}$$

$$\frac{\partial}{\partial s}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{-[G(s,t) - P_0(t)] - as\frac{\partial}{\partial s}G(s,t) + sG(s,t) + as^2\frac{\partial}{\partial s}G(s,t)\right\}$$

$$\frac{\partial}{\partial s}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{-G(s,t)[1-s] + \frac{\partial}{\partial s}G(s,t)(-as+as^2) - \left(\frac{1+\lambda at}{\lambda}\right)P_0'(t)\right\}$$

Since:

$$P_0'(t) = -\left[\frac{\lambda}{1+\lambda at}\right]P_0(t),\implies P_0(t) = -\left[\frac{1+\lambda at}{\lambda}\right]P_0'(t)$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) + P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\}$$

$$\implies \frac{\partial}{\partial t}G(s,t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\}$$

$$= \left[\frac{\lambda}{1+\lambda at}\right]\left\{(s-1)G(s,t) + as(s-1)\frac{\partial}{\partial s}G(s,t)\right\}$$

Hence the Lagrange's linear equations will be:

$$\frac{\partial G(s,t)}{\partial t} + \frac{\lambda}{1+\lambda at}as(1-s)\frac{\partial G(s,t)}{\partial s} = -\left[\frac{\lambda}{1+\lambda at}\right](1-s)G(s,t)$$

Which can be expressed as:

$$(1+\lambda at)\frac{\partial G(s,t)}{\partial t}+\lambda as(1-s)\frac{\partial G(s,t)}{\partial s}=-\lambda(1-s)G(s,t)$$

The corresponding auxiliary equations are:

$$\frac{\partial t}{1+\lambda at}=\frac{\partial s}{\lambda as(1-s)}=\frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

Consider:

$$\frac{\partial t}{1+\lambda at}=\frac{\partial s}{\lambda as(1-s)}$$

Taking integration for both sides gives, we will have:

$$\int\frac{\partial t}{1+\lambda at}=\int\frac{\partial s}{\lambda as(1-s)}\ \frac{1}{\lambda a}ln(1+\lambda at)+\frac{1}{\lambda a}ln\ c_2=\frac{1}{\lambda a}ln\ \left(\frac{s}{1-s}\right)\ c_2=\left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}$$

Also from:

$$\frac{\partial s}{\lambda as(1-s)}=\frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

We get:

$$\int\frac{\partial s}{as}=\int\frac{\partial G(s.t)}{-G(s,t)}\Longrightarrow\frac{1}{a}ln\ s+ln\ c_1=\ -ln\ G(s,t)$$

and

$$\Longrightarrow c_1=s^{\frac{1}{2}}[G(s,t)]$$

The general solution is:

$$c_1=\psi(c_2)\Longrightarrow s^{\frac{1}{2}}[G(s,t)=\psi\left[\frac{s}{1-s}(1+\lambda at)^{-1}\right]$$

Using the initial condition given as:

$$P_n(0)=\begin{cases}1, & n=0\\ 0, & \text{otherwise}\end{cases}\ G(s,0)=s^{-\frac{1}{2}}\psi\left(\frac{s}{1-s}\right)$$

By definition:

$$G(s,t)=\sum_{n=0}^{\infty}P_n(t)s^n=P_0(t)+P_1(t)s+P_2(t)s^2+\ldots G(s,0)=P_0(0)+P_1(0)s+P_2(0)s^2+\ldots=P_0(0)=1$$

$$\Longrightarrow s^{-\frac{1}{2}}\psi\left(\frac{s}{1-s}\right)=1$$

And therefore,

$$\psi\left(\frac{s}{1-s}\right)=s^{-\frac{1}{2}}$$

Which accordingly implies that

$$\psi(w)=\left(\frac{w}{1+w}\right)^{\frac{1}{2}}$$

For any arbitrary w.

So:

$$G(s,t)=s^{-\frac{1}{2}}\psi\left[\left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}\right]=s^{-\frac{1}{2}}\psi[w(1+\lambda at)^{-1}]$$

$$= s^{-\frac{1}{2}} \left[\frac{w(1+\lambda at)^{-1}}{1+w(1+\lambda at)^{-1}} \right]^{\frac{1}{2}}$$

$$= s^{-\frac{1}{2}} \left[\frac{\frac{s}{1-s}(1+\lambda at)^{-1}}{1+\frac{s}{1-s}(1+\lambda at)^{-1}} \right]^{\frac{1}{2}}$$

$$= s^{-\frac{1}{2}} s^{-\frac{1}{2}} \left[\frac{1}{(1+\lambda at)(1-s)+s} \right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1+\lambda at - \lambda ats} \right]^{\frac{1}{2}}$$

and:

$$G(s,t) = (1+\lambda at)^{\frac{1}{2}} \left[1 - \frac{\lambda at}{1+\lambda at} \right]^{-\frac{1}{2}}$$

$$= (1+\lambda at)^{\frac{1}{2}} \frac{[1+\lambda at - \lambda ats]^{-\frac{1}{2}}}{(1+\lambda at)^{-\frac{1}{2}}}$$

$$= (1+\lambda at)^{-\frac{1}{2}} \left[1 + \begin{pmatrix} -\frac{1}{a} \\ 1 \end{pmatrix} \left(\frac{-\lambda at}{1+\lambda at} s \right) + \ldots + \begin{pmatrix} -\frac{1}{a} \\ n \end{pmatrix} \left(\frac{-\lambda at}{1+\lambda at} s \right)^n + \ldots \right]$$

So that $P_n(t)$ is the coefficient of s^n in the above expansion for G(s,t)

$$P_n(t) = (1+\lambda at)^{-\frac{1}{a}} \begin{pmatrix} -\frac{1}{a} \\ n \end{pmatrix} \left(\frac{-\lambda at}{1+\lambda at} \right)^n$$

(b) Obtain the mean and variance of the process using Feller's method.

solution

Mean

We define:

$$M_1(t) = \sum_{n=1}^{\infty} n P_n(t) \Longrightarrow M_1'(t) = \sum_{n=1}^{\infty} n P_n'(t)$$

$$M_2(t) = \sum_{n=1}^{\infty} n^2 P_n(t) \Longrightarrow M_2'(t) = \sum_{n=1}^{\infty} n^2 P_n'(t)$$

The difference differential equation is;

$$P_n'(t) = -\lambda \left[\frac{1+an}{1+\lambda at} \right] P_n(t) + \lambda \left[\frac{1+a(n-1)}{1+\lambda at} \right] P_{n-1}(t), \ n \geq 1$$

Multiply $P_n'(t)$ by n and sum the results over n

$$\sum_{n=1}^{\infty} P_n'(t) = -\lambda \left[\frac{1+an}{1+\lambda at} \right] n P_n(t) + \lambda \left[\frac{1+a(n-1)}{1+\lambda at} \right] n P_{n-1}(t)$$

$$= \frac{-\lambda}{1+\lambda at} \sum_{n=1}^{\infty} (1+an) n P(t) + \frac{\lambda}{1+\lambda at} \sum_{n=1}^{\infty} (1-a(n-1)) n P_{n-1}(t)$$

$$= \frac{-\lambda}{1+\lambda at}\left[\sum_{n=1}^{\infty} nP_n(t) + a\sum_{n=1}^{\infty} n^2 P_n(t)\right] + \frac{\lambda}{1+\lambda at}\left[\sum_{n=1}^{\infty} nP_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)nP_{n-1}(t)\right]$$

From definition of moments, we will have;

$$= \left[\frac{-\lambda}{1+\lambda at}\right]\{M_1(t) + aM_2(t)\} + \left[\frac{\lambda}{1+\lambda at}\right]\left\{\sum_{n=1}^{\infty} nP_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)nP_{n-1}(t)\right\}$$

To simplify the second part of the equation we introducing a form of One(1) as follows:

$$= \frac{\lambda}{1+\lambda at}\left\{-M_1(t) - aM_2(t) + \sum_{n=1}^{\infty}(n-1+1)P_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)(n-1+1)nP_{n-1}(t)\right\}$$

$$= \frac{\lambda}{1+\lambda at}\left\{-M_1(t) - aM_2(t) + \sum_{n=1}^{\infty}(n-1)P_{n-1}(t) + \sum_{n=1}^{\infty} P_{n-1} + a\sum_{n=1}^{\infty}(n-1)^2 P_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)\right\}$$

$$= \frac{\lambda}{1+\lambda at}\left[1 + aM_1(t)\right] = \frac{\lambda}{1+\lambda at} + \frac{a\lambda}{1+\lambda at}M_1(t)$$

But:

$$\sum_{n=1}^{\infty} nP_n'(t) = M_1'(t)$$

$$M_1'(t)\frac{\lambda}{1+\lambda at} + \frac{a\lambda}{1+\lambda at}M_1(t) \Longrightarrow M_1'(t) - \frac{\lambda}{1+\lambda at}M_1(t) = \frac{\lambda}{1+\lambda at}\text{Equation 1}$$

To proceed to integration, we consider the integrating factor:

$$e^{\int -\frac{\lambda}{1+\lambda at}dt}$$

$$\text{let u} = 1+\lambda at \Longrightarrow \frac{du}{dt} = a\lambda \Longrightarrow dt = \frac{du}{a\lambda}$$

$$\int -\frac{\lambda}{1+\lambda at}dt = -\int \frac{a\lambda}{u} \times \frac{du}{a\lambda} = -\int \frac{1}{u}du = -lnu = ln(1+\lambda at)^-1$$

$$\text{Integrating factor} = e^{ln(1+\lambda at)^-1} = (1+\lambda at) = \frac{\lambda}{1+\lambda at}$$

We multiply Equation 1 with the integrating factor

$$\frac{1}{1+\lambda at} \times M_1'(t) - \frac{\lambda}{1+\lambda at} \times \frac{a\lambda}{1+\lambda at}M_1(t) = \frac{1}{1+\lambda at} \times \frac{\lambda}{1+\lambda at}$$

$$= \frac{1}{1+\lambda at} \times M_1'(t) - \frac{a\lambda}{(1+\lambda at)^2}M_1(t) = \frac{\lambda}{(1+\lambda at)^2}$$

Equivalently

$$\frac{d}{dt}\left\{\frac{1}{1+\lambda at}\right\} \times M_1(t) = \frac{\lambda}{(1+\lambda at)^2}$$

Integrating both sides, we will have:

$$\left\{\frac{1}{1+\lambda at}\right\} \times \int \frac{\lambda}{(1+\lambda at)^2}dt$$

$$\text{let u} = 1 + \lambda at \Rightarrow \frac{du}{dt} = a\lambda \Rightarrow dt = \frac{du}{a\lambda}$$

Therefore:

$$\int \frac{\lambda}{(1+\lambda at)^2}dt = \lambda \int \frac{1}{u^2} \times \frac{du}{a\lambda} = \frac{1}{a}\int \frac{1}{u^2}du = \frac{1}{a}\int u^2 du = \frac{1}{a} \times \frac{u^{-2+1}}{-2+1} + c = -\frac{1}{a} \times u^-1 + c$$

It becomes:

$$\frac{M_1(t)}{1+\lambda at} = -\frac{1}{a(1+a\lambda t)} + c = -\frac{1}{a}(1+a\lambda t)^-1 + c$$

$$= \frac{M_1(t)}{1+\lambda at} = -\frac{1}{a}\Big[\frac{1}{1+a\lambda t}\Big] + c$$

$$\text{We assume that at t} = 0, X(0) = n \Rightarrow p_n(0) = 1$$

$

$$\text{When t=0, We have} M_1(0) = -\frac{1}{a} + c$$

$$\text{By definition} M_1(0) = \sum_{n=1}^{\infty} nP_n(0) = 0 \times p_0(0) + 1 \times p_1(0) + 2p_2(0) + \ldots + np_n(0) + \ldots = n$$

Therefore:

$$n = -\frac{1}{a} + c \Rightarrow c = n + \frac{1}{a}$$

It becomes:

$$\frac{M_1(t)}{1+\lambda at} = -\frac{1}{a}\left(\frac{1}{1+a\lambda t}\right) + \left(n + \frac{1}{a}\right)$$

$$M_1(t) = -\frac{1}{a} + \left(n + \frac{1}{a}\right)(1+a\lambda t)$$

$$= -\frac{1}{a} + \left\{n + n\lambda at + \frac{1}{a} + \lambda t\right\}$$

$$= -\frac{1}{a} + n + n\lambda at + \frac{1}{a} + \lambda t$$

$$n + n\lambda at + \lambda t$$

$$= n(1+\lambda at) + \lambda t$$

Therefore:

$$M_1(t) = n(1+\lambda at) + \lambda t \Rightarrow E[X(t)] = n(1+\lambda at) + \lambda t$$

For a special case:

$$\text{When} \quad n = 0, M_1(t) = \lambda t$$

Variance

In finding the variance, we multiply the difference differential equation by n^2 and sum over n as below:

$$\sum_{n=1}^{\infty} n^2 p'(t) = -\frac{\lambda}{1+\lambda at}\sum_{n=1}^{\infty}(1+an)n^2 p_n(t) + \frac{\lambda}{1+\lambda at}\sum_{n=1}^{\infty}[1+a(n-1)n^2 p_{n-1}(t)]$$

Since $M_2'(t)=\sum_{n=1}^{\infty}n^2P_n'(t)$ and $M_3'(t)=\sum_{n=1}^{\infty}n^3P_n'(t)$, we have;

$$M_2'(t)=-\frac{\lambda}{1+\lambda at}\{M_2(t)+aM_3(t)\}+\frac{\lambda}{1+\lambda at}\left\{\sum_{n=1}^{\infty}n^2p_{n-1}(t)+a\sum_{n=1}^{\infty}(n-1)n^2p_{n-1}(t)\right\}$$

$$=-\frac{\lambda}{1+\lambda at}\{M_2(t)+aM_3(t)\}+\frac{\lambda}{1+\lambda at}\left\{\sum_{n=1}^{\infty}(n-1+1)^2p_{n-1}(t)+a\sum_{n=1}^{\infty}(n-1)(n-1+1)^2p_{n-1}(t)\right\}$$

$$=-\frac{\lambda}{1+\lambda at}\{M_2(t)+aM_3(t)\}+\frac{\lambda}{1+\lambda at}\left\{M_2(t)+2M_1(t)+1+aM_3(t)+2aM_2(t)+aM_1(t)\right\}$$

$$=\frac{\lambda}{1+\lambda at}\left\{(2+a)M_1(t)+1+2aM_2(t)\right\}$$

Therefore,

$$M_2'(t)=\frac{\lambda}{1+\lambda at}\left\{(2+a)M_1(t)+1+2aM_2(t)\right\}$$

For a special case when $t=0, X(0)=0$ and we have $M_1(t)=\lambda t$

Thus,

$$M_2'(t)-\frac{2a\lambda}{1+\lambda at}M_2(t)=\frac{\lambda}{1+\lambda at}\left\{(2+a)M_1(t)+1\right\}$$

$$=\frac{\lambda}{1+\lambda at}\left\{(2+a)\lambda t+1\right\}$$

Equivalently, we will have;

$$M_2'(t)-\frac{2a\lambda}{1+\lambda at}M_2(t)=\frac{\lambda}{1+\lambda at}\left\{(2+a)\lambda t+1\right\}\ldots(1)$$

In integration, we consider the integration factor;

$$I=e^{\int\frac{2a\lambda}{1+\lambda at}dt}=e^{-2\int\frac{d}{dt}\ln(1+\lambda at)dt}=e^{-2ln(1+\lambda at)}=e^{ln(1+\lambda at)^{-2}}$$

Therefore,

$$I=(1+\lambda at)^{-2}$$

Multiplying the integrating factor with equation $\ldots(1)$ we get

$$(1+\lambda at)^{-2}M_2'(t)-\frac{2a\lambda}{(1+\lambda at)^3}\{(2+a)\lambda t+1\}$$

$$\implies\frac{d}{dt}\left[(1+\lambda at)^{-2}M_2'(t)\right]=\frac{\lambda}{(1+\lambda at)^3}\{(2+a)\lambda t+1\}$$

Integrating both sides w.r.t t, we have

$$(1+\lambda at)^{-2}M_2'(t) = (2+a)\lambda^2 \int \frac{t}{(1+\lambda at)^3}dt + \lambda \int \frac{dt}{(1+\lambda at)^3}$$

Let $u = (1+2\lambda at) \implies du = \lambda a dt$ and $u-1 = \lambda at \implies t = \frac{u-1}{\lambda a}$

Therefore,

$$(1+\lambda at)^{-2}M_2'(t) = (2+a)\lambda^2 \int \frac{u-1}{\lambda a u^3} \times \frac{du}{\lambda a} + \lambda \int \frac{1}{u^3} \times \frac{dt}{\lambda a} + c$$

$$= \frac{(2+a)}{a^2} \int \left[\frac{1}{u^2} - \frac{1}{u^3}\right] du + \frac{1}{a} \int \frac{1}{u^3} + c$$

$$(1+\lambda at)^{-2}M_2'(t) = \frac{(2+a)}{a^2} \left[\frac{u^{-2+1}}{-1} - \frac{u^{-3+1}}{-2}\right] + \frac{1}{a}\left[\frac{u^{-3+1}}{-2} + c\right]$$

$$= \frac{(2+a)}{a^2}\left[\frac{1}{u} - \frac{1}{-2u^2}\right] - \frac{1}{2au^2} + c$$

$$= \left(\frac{2}{a^2} + \frac{1}{a}\right)\left(-\frac{1}{u} - \frac{1}{2u^2} - \frac{1}{2au^2} + c\right)$$

$$= \frac{2}{a^2 u} + \frac{1}{a^2 u^2} - \frac{1}{au} + \frac{1}{2au^2} - \frac{1}{2au^2} + c$$

$$= \frac{2}{a^2 u} + \frac{1}{a^2 u^2} - \frac{1}{au} + c$$

Therefore,

$$M_2(t) = (1+\lambda at)^2 \left[\frac{2}{a^2 u} + \frac{1}{a^2 u^2} - \frac{1}{au}\right] + c(1+\lambda at)^2$$

$$= u^2\left[-\frac{2}{a^2 u} + \frac{1}{a^2 u^2} - \frac{1}{au}\right] + cu^2$$

$$= -\frac{2u}{a^2} + \frac{1}{a^2} - \frac{u}{a} + cu^2$$

$$= \left(\frac{-2u + 1 - ua}{a^2}\right) + cu^2$$

$$= \left(\frac{1 - u(2+a)}{a^2}\right) + cu^2$$

$$= \frac{1 - (1+\lambda at)(2+a)}{a^2} + c(1+\lambda at)^2$$

$$= \frac{1 - [2 + a + 2\lambda at + \lambda a^2 t]}{a^2} + c(1+\lambda at)^2$$

$$= \frac{1 - 2 - a - 2\lambda at - \lambda a^2 t}{a^2} + c(1+\lambda at)^2$$

$$= \frac{2u}{a^2} + \frac{1}{a^2} - \frac{u}{a} + cu^2$$

$$\left(\frac{2u+1-ua}{a^2}\right)+cu^2$$

Now,

$$M_2(t)=\frac{1-u(2+a)}{a^2}+cu^2$$
$$=\frac{1-(1+\lambda at)(2+a)}{a^2}+c(1+\lambda at)^2$$

$$=\frac{1-[2+a+2\lambda at+\lambda a^2t]}{a^2}+c(1+\lambda at)^2$$

$$=\frac{1-2-a-2\lambda at-\lambda a^2t}{a^2}+c(1+\lambda at)^2$$
$$=c(1+\lambda at)^2-\frac{(1+a)-\lambda at(2+a)}{a^2}$$

$$=c(1+\lambda at)^2-\frac{(1+a)-\lambda at(1+1+a)}{a^2}$$

$$=c(1+\lambda at)^2-\frac{(1+a)-\lambda at-\lambda at(1+a)}{a^2}$$

$$=c(1+\lambda at)^2-\left[\frac{(1+a)+\lambda at(1+a)}{a^2}\right]+\frac{\lambda t}{a}$$

Therefore,

$$M_2(t)=c(1+\lambda at)^2-\frac{(1+a)(1+\lambda at)}{a^2}+\frac{\lambda t}{a}\ldots(2)$$

Substituting $t=0$ in equation $\ldots(2)$ we have

$$M_2(0)=c-\frac{(1+a)}{a^2}$$

From definition,

$$M_2(0)=\sum_{n=1}^{\infty}n^2p_n(0) \text{ since } p_0(0)=1 \text{ and } p_n(0)=0 \text{ for } n\neq 0. \text{ Therefore,}$$

$$0=c-\left(\frac{1+a}{a^2}\right)$$

It follows that

$$M_2(t)=\frac{(1+a)}{a^2}(1+\lambda at)^2-\frac{(1+a)(1+\lambda at)}{a^2}+\frac{\lambda t}{a}$$

Simplifying the above equation, we have

$$M_2(t)=\frac{(1+a)(1+\lambda at)}{a^2}(1+\lambda at-1)+\frac{\lambda t}{a}$$

$$= \frac{(1+a)(1+\lambda at)(\lambda at)}{a^2} - \frac{\lambda t}{a} *$$
$$= \frac{(1+a)(1+\lambda at)(\lambda t)}{a^2} - \frac{\lambda t}{a} a$$
$$= \frac{\lambda t}{a} \left[(1+a)(1+\lambda at) - 1\right]$$
$$= \frac{\lambda t}{a} \left[(1 + \lambda at + a + \lambda a^2 t - 1\right]$$

Therefore,

$$M_2(t) = \frac{\lambda t}{a} \left[\lambda at + a + \lambda a^2 t\right]$$

$$= \lambda t \left[\lambda t + 1 + \lambda at\right]$$

Thus,

$$\text{Variance} = M_2(t) - [M_1(t)]^2$$

$$= \lambda t[\lambda t + 1 + \lambda at] - (\lambda t)^2$$
$$= \lambda t[\lambda t + 1 + \lambda at - \lambda t]$$
$$= \lambda t[1 + \lambda t]$$