Stochastic Processes

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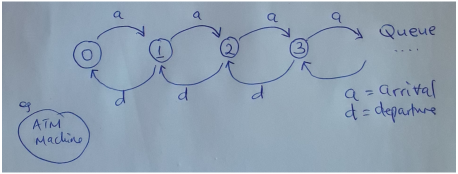
- Stochastic is a Greek word which means *randomness* or *guesswork*.
- "Randomness in process": A process is a sequence of events where each step follows from the last after a random choice e.g.

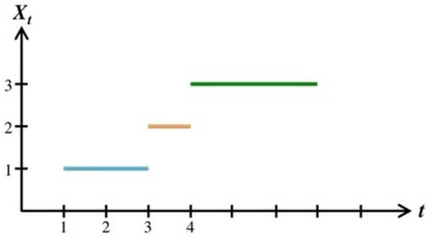


- A stochastic process is a collection of random variables $\{X_t, t \in T\}$. The set T is the index set (parameter set). t usually denotes time, i.e. at every time t in the set T, a random number $X(t)$ is observed.
- The random variables take values in a common *state space* S, and are defined on a common *sample space* Ω , i.e. the state space, S, is the set of real values that $X(t)$ can take.
- When the index set is countable the process is called **discrete-time**. Examples include $T = \mathbb{Z}^+$, $T = \{0, 1, 2, \cdots, N\}$, etc.
- When the index set contains an interval of $\mathbb{R}$ the process is called **continuous-time**. Examples include $T = \{\mathbb{R}\}$, $T = [a, b]$, etc.
- To highlight how they depend on the model of randomness, a stochastic process may be written as $X_t(\omega)$ or $X(t)(\omega)$, where $t \in T$ and $\omega \in \Omega$.

- For each fixed $\omega \in \Omega$, $X_t(\omega)$ corresponds to a function defined on T that is called a *sample path or a stochastic realization of the process* .
- Most of the time, the variable ω is omitted when speaking of a stochastic process.
- In many stochastic processes, the index set T often represents time, and we refer to X_t as the state of the process at time t, where $t \in T$.

Brief overview on Stochastic Processes contd...

- Any realization or sample path is also called a *sample function of a stochastic process*. For example, if events are occurring randomly in time and X_t represents the number of events that occur in $[0, t]$, as shown in the figure below, then a sample path corresponds to the initial event that occurs at time $t = 1$, the second event occurs at $t = 3$, and the third event at $t = 4$. There are no events occurring anywhere else.



A sample path of X_t which is the number of events in $[0, t]$.

Brief overview on Stochastic Processes contd...

- The main elements that distinguish stochastic processes are the state space S, the index set T , and the dependence relations among the random variables X_t.
- **State space S** is the space in which the possible values of each X_t lie, i.e
  1. If the state space is countable, such as $S = (0, 1, 2, 3, \cdots)$, we refer to the process as an *integer-valued* or a **discrete-state stochastic process**.
  2. If $S = (-\infty, \infty)$ is a real-valued space, the process is called a *real-valued stochastic process*.
  3. If S is *Euclidean k space*, then it is a $k-$ *vector* stochastic process.

- **Index set T** characterizes stochastic processes into:
  - (i) If T is a *countable set (or a set of integers)* representing specific time points, the stochastic process is said to be a **discrete-time** process. For instance, $\{X_t, t = 0, 1, 2, 3, \cdots\}$ is a discrete-time stochastic process indexed by non-negative integers.
  - (ii) If T is an *interval of the real line*, the stochastic process is said to be **continuous-time** process.
- To differentiate between discrete-time and continuous-time processes, the notation may be altered slightly, writing $X(t)$ (for continuous-time process) rather than X_t. Hence, $\{X(t), 0 \leq t < \infty\}$ is a continuous-time stochastic process indexed by the non-negative real numbers.

- The values of X_t may be *1-dimensional, 2-dimensional, or general m dimensions*. Two major examples of stochastic processes are the **Brownian motion** and **Poisson process**.
- Stochastic processes can be categorized into *classical* and *modern* classifications. Classical stochastic processes are characterized by different dependence relationships among random variables viz:

  1. **Process with stationary independent increments**:
- If the real-valued random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent for all choices of $t_1, t_2, \cdots, t_n$ for $t_i \geq 0$, satisfying $t_1 < t_2 < \cdots < t_n$, then X_t is said to be a process with **independent increments**.

- If the index set contains a smallest index t_0, it is also assumed that $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.
- If the index set is discrete, i.e. $T = (0, 1, 2, \cdots)$, then a process with independent increments reduces to a sequence of independent random variables $Z_0 = X_0, Z_i = X_i - X_{i-1}$ for $i = 1, 2, 3, \cdots$
- If the distribution of the increments

$$X(t_1 + h) - X(t_1) \tag{1}$$

depends only on the length of the interval h and not on the time t_1, the process is said to have **stationary increments**.
- For a process with stationary increments, the distribution of $X(t_1 + h) - X(t_1)$ is the same as the distribution of $X(t_2 + h) - X(t_2)$, regardless of the values of t_1, t_2 and h.

Brief overview on Stochastic Processes contd...

- If a process $\{X_t, t \in T\}$, where T is either continuous or discrete has stationary independent increments and a finite mean, then the expectation of X_t is given by

$$E(X_t) = m_0 + m_1 t \tag{2}$$

where $m_0 = E(X_0)$ and $m_1 = E(X_1) - m_0$. A similar assertion holds for the variance i.e.

$$\sigma^2_{X_t} = \sigma^2_0 + \sigma^2_1(t) \tag{3}$$

where

$$\sigma^2_0 = E((X_0 - m_0)^2), \;\; \sigma^2_1 = E((X_1 - m_1)^2) - \sigma^2_0$$

- As can be seen later, both the Brownian motion and Poisson process have stationary independent increments.

2. Martingales

- Suppose X is a random variable measuring the outcome of some random experiment. If one knows nothing about the outcome of the experiment, then the best guess for the value of X would be the expectation, $E(X)$.
- Conditional expectation concerns making the best guess for X given some but not all information about the outcome.
- Let $\{X_t\}$ be a real-valued stochastic process with discrete or continuous index set T. The process $\{X_t\}$ is called a *martingale* if

$$E(|X_t|) < \infty, \quad for\ t = 0, 1, 2, \cdots \tag{4}$$

and

$$E(X_{t_{n+1}}|X_{t_0} = a_0, X_{t_1} = a_1, \cdots, X_{t_n} = a_n) = a_n \qquad (5)$$

for any $t_0 < t_1 < \cdots < t_{n+1}$ and for all values $a_0, a_1, \cdots, a_n$, states that the expected value of $X_{t_{n+1}}$ given the past and present values of $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ equals the present value of X_{t_n}.

- A sequence of random variables having this property is called a **martingale**.
- We may re-write equation (**??**) as

$$E(X_{t_{n+1}}|X_{t_0}, X_{t_1}, \cdots, X_{t_n}) = X_{t_n}$$

and taking expectations on both sides,

$$E(E(X_{t_{n+1}}|X_{t_0}, X_{t_1}, \cdots, X_{t_n})) = E(X_{t_n})$$

using the law of total probability in the form

$$E(E(X_{t_{n+1}}|X_{t_1}, X_{t_2}, \cdots, X_{t_n})) = E(X_{t_{n+1}})$$

gives us

$$E(X_{t_{n+1}}) = E(X_{t_n})$$

and consequently, a martingale has constant mean

$$E(X_{t_0}) = E(X_{t_k}) = E(X_{t_n}), \quad 0 \leq k \leq n$$

- It can be verified that the process $X_n = Z_1 + Z_2 + \cdots + Z_n$ for $n = 1, 2, 3, \cdots$ is a discrete-time martingale if the Z_i are independent and have means zero.
- Similarly, if X_t for $0 \leq t < \infty$ has independent increments whose means are zero, then $\{X_t\}$ is a **continuous-time martingale**.

3. Markov processes

- A *Markov process* is a process with the property that, given the present state X_{t_n}, the future state $X_{t_{n+1}}$ depends only on the present state X_{t_n}, not on the past states $X_{t_1}, X_{t_2}, \cdots, X_{t_{n-1}}$.
- However, if our knowledge of the present state of the process is imprecise, then the probability of some future behavior will be altered by additional information relating to the past behavior of the system.
- A process is said to be Markov if

$$P\{a < X_{t_{n+1}} \leq b / X_{t_n} = x_n, \cdots, X_{t_2} = x_2, X_{t_1} = x_1\} \\ = P\{a < X_{t_{n+1}} \leq b / X_{t_n} = x_n\}$$

whenever $t_1 < t_2 < \cdots < t_n < t$

- A Markov process having a countable or denumerable state space is called a **Markov chain**. A *Markov process* for which all sample functions $\{X_t, 0 \leq t < \infty\}$ are continuous functions is called a **diffusion process**. The Poisson process is a continuous-time Markov chain and Brownian motion is a **diffusion process**.

4. **Stationary Processes**

- A stochastic process $\{X_t\}$ for $t \in T$, where T is either discrete or continuous, is said to be **strictly stationary** if the joint distribution functions of the families of random variables $(X_{t_1+h}, X_{t_2+h}, \cdots, X_{t_n+h})$ and $(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$ are the same for all $h > 0$ and arbitrary selections $t_1, t_2, \cdots, t_n$ from T.
- This condition asserts that the process is in probabilistic equilibrium and that the particular times of the process being examined are of no relevance.
- A stochastic process $\{X_t, t \in T\}$, is said to be **wide sense stationary** or **co-variance stationary** if it possesses finite second moments and if $Cov(X_t, X_{t+h}) = E(X_t X_{t+h}) - E(X_t)E(X_{t+h})$ depends only on h for all $t \in T$.

5. Renewal processes

- Let $T_1, T_2, \cdots$ be *i.i.d*, non-negative random variables with distribution function $F(x) = P\{T_i \leq x\}$. The random variables T_i can be considered to be lifetimes of a component or as the times between occurrences of some events: the first event is placed in operation at time zero; it fails at time T_1 and is immediately replaced by a new event which then fails at time $T_1 + T_2$, and so on, thus motivating the name **"renewal process"**.
- The time of the n^{th} renewal is $S_n = T_1 + T_2 + \cdots + T_n$
- A renewal counting process N_t counts the number of renewals in the interval $[0, t]$, $N_t = n$ for $S_n \leq t < S_{n+1}, n = 0, 1, 2, \cdots$.
- The Poisson process with parameter λ is a renewal counting process with $\mu = \frac{1}{\lambda}$.

6. **Point processes**

- Let S be a set in n-dimensional space and let $\mathbb{A}$ be a family of subsets of S.
- A **point process** is a stochastic process indexed by the sets $A \in \mathbb{A}$ and having the set $\{0, 1, 2, \cdots \infty\}$ of non-negative integers as its state space.

Remark: *Modern classification of stochastic processes* depends on whether the random variables or the index sets are discrete or continuous i.e.:

1. Stochastic processes whose time and random variables are discrete-valued, e.g. discrete-time Markov chain models and discrete-time branching processes.
2. Stochastic processes whose time is continuous but the random variables are discrete-valued e.g. continuous-time Markov chain models.
3. Stochastic processes whose random variables are continuous but the time is discrete-valued.
4. Stochastic processes whose both time and random variables are continuous-valued, e.g continuous-time and continuous-state Markov processes. These models are also referred to as diffusion processes, where the stochastic realization is a solution of a stochastic differential equation.

- **A stochastic process** can therefore be defined as a sequence of random variables in a system. In studying stochastic processes it's important to review generating functions.