MASTER OF SCIENCE (STATISTICAL SCIENCES)

Solutions MFI 8202: STOCHASTIC PROCESSES _ Examination Jan-April, 2019

Question 1 (30 Marks)

a) Consider a steady state in performance measure of a simple queuing system (M/M/1) with arrival rate λ and service rate μ. Let $P_n(t)$ = **P** [n customers in the system at time t]. Show that

a. $\boldsymbol{\lambda P_{n-1} = \mu P_n}$

$\boldsymbol{P_n = \frac{\lambda}{\mu} P_{n-1} = \left(\frac{\lambda}{\mu}\right)^n P_0}$ **To find** P_0 **we solve** $\boldsymbol{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1}$ **Therefore** $\boldsymbol{P_0 = \left(1 - \left(\frac{\lambda}{\mu}\right)\right)}$

$$\boldsymbol{P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \left(\frac{\lambda}{\mu}\right)\right)}$$

b. **To solve** L_s **:The average number of entities in the system**

$$\boldsymbol{L_s = \sum_{n=0}^{\infty} n\, P_n = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \left(\frac{\lambda}{\mu}\right)\right) = \left(1 - \left(\frac{\lambda}{\mu}\right)\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n}$$

$$\boldsymbol{= \left(1 - \left(\frac{\lambda}{\mu}\right)\right) \frac{\left(\frac{\lambda}{\mu}\right)}{\left(1 - \left(\frac{\lambda}{\mu}\right)\right)^2} = \frac{\lambda}{\mu - \lambda}}$$

c. $\boldsymbol{Wq = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)}}$

(10 marks)

b) Clearly define the following terms as used in stochastic processes

i. Stochastic Process

A stochastic process is a collection or ensemble of random variables indexed by a variable *t*, usually representing time.

ii. Branching Process

A Branching process is a Markov process that models a population in which each individual in generation n, produces a random number of offsprings for generation n + 1.

iii. Martingale

A sequence {X_n, n = 0, 1, 2 ...} of random variables is said to be a martingale with respect to the filtration {F_n, n = 0, 1, 2,...} if

(i) Xn in F_n for every n = 0, 1, 2,...,

(ii) E|Xn| < 1 for every n = 0, 1, 2,..., and

(iii) E(Xn+1|Fn) = X_n for every n = 0, 1, 2,....

iv. Random Walk

Let {ϵj}, j> 1 be a sequence of i.i.d.r.v.'s with P(ϵj = 1) = p and P(ϵj = −1) = 1 − p, where
0 < p < 1. Let X_0 = 0 and let Xn =sum of ϵj for n > 1. The stochastic process {Xn : n > 0} is called a random walk.

(4 marks)

c) Suppose that $\{X_n : n > 1\}$ is a random walk with $X_0 = 0$ and probability p of a step to the right, find:

i. $Var(-18 + 5X_4)$

0+ $var(5X_4)$=25*4*4*p*q=400pq

ii. $Var(-200 + 2X_2 - 7X_3)$.

0+ $Var(2X_2 - 7X_3)$=$Var(2X_2-7(X_3-X_2))$=4*4*2*p*q+49*4*1*p*q

32pq+196pq=228pq

iii. $Var(5X_4 - 3X_5 + 3X_6)$.

25$var(X_4)$+9var (X_5-X_4) +9$var(X_6-X_5)$=25*4*4*p*q+9*4*1*pq+9*4*1*pq

=400pq+36pq+36pq=472pq

(6 marks)

d) Clearly distinguish the following terms as used in Martingale techniques

i. Super-martingale and sub-martingale

An (Ft)-adapted, real-valued process M is called a martingale (with respect to the filtration (Ft)) if
i) E|Mt| < ∞ for all t ∈ T;
ii) E(Mt | Fs) a.s. = Ms for all s ≤ t.
If property (ii) holds with '≥' (resp. '≤') instead of '=', then M is called a submartingale (resp. supermartingale).

ii. Filtration and σ- algebra

A family of σ-fields {Ft} is defined to be a filtration if Ft1 ⊂ Ft2 whenever $t_1 \leq t_2$ while collection of events. A class F of events must satisfy certain properties. These properties of the class F of events and their interpretations (in a parenthesis) are as follows:
(a) Ω ∈ F (Ω is the event "something happens").
(b) If A ∈ F then Ac ∈ F (if A is an event, "A does not happen" must also be an event).
(c) If A1, A2, . . . ∈ F then S∞ i=1 Ai ∈ F. (if one can speak of a sequence of events, then one can speak of "at least one of them occurs" as an event).
The class F of subsets is called a σ-algebra if it satisfies all these properties.

(4 marks)

e) Let ('Ω, A, P) be a probability space, **F** subset of A a sub-σ algebra and X a random variable with E (|X| <∞). Describe three properties of conditional probability E(X| **F**)

Any three of these four

i) Linearity. E[aX + Y |G] = aE[X|G] + E[Y |G].
ii) Monotonicity. If X1 ≤ X2 a.s, then E[X1|G] ≤ E[X2|G].
iii) Conditional Jensen's inequality. Let φ be a convex function and E[|X|], E[|φ(X)|] < ∞. Then φ(E[X|G]) ≤ E[φ(X)|G].
iv) Tower property. Suppose G1 ⊂ G2 ⊂ F. Then E[E[X|G1]|G2] = E[X|G1] and E[E[X|G2]|G1] = E[X|G1]. That is the smaller field wins.

(6 marks)

Question 2 (15 Marks)

a) Consider a homogeneous Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

Calculate the 3-step transition matrix.

$$\boldsymbol{P^2} = \begin{pmatrix} .5 & .1875 & .3125 \\ .5 & .375 & .125 \\ .3125 & .375 & .3125 \end{pmatrix}$$

$$\boldsymbol{P^3} = \begin{pmatrix} .453 & .328 & .219 \\ .328 & .1875 & .234 \\ .5 & .234 & .266 \end{pmatrix}$$

(3 Marks)

b) Clearly distinguish the following terms as used in Markov chains

i. Transient state and Recurrent state

A state 'i' is called Recurrent, if we go from that state to any other state 'j', then there is at least one path to return back to 'i'.

On the other hand, there will be at least one state 'j', to which we can go from state 'i', but can not return to 'i'. then state 'i' in this case Transient

ii. Periodicity and Ergodic Chain

A state in a Markov chain is periodic if the chain can return to the state only at multiples of some integer larger than 1; The period of a state i is the greatest common denominator (gcd) of all integers n > 0, for which $p_{ii}(n) > 0$. A state i is said to be ergodic if it is aperiodic and positive recurrent. In other words, a state i is ergodic if it is recurrent, has a period of 1, and has finite mean recurrence time. If all states in

an irreducible Markov chain are ergodic, then the chain is said to be ergodic

iii. Reducible chain and Irreducible chain

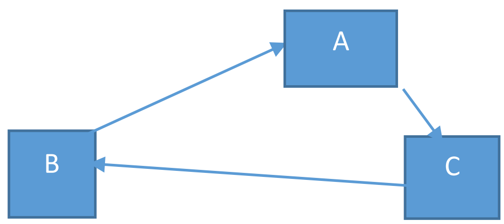
If all the states in the Markov Chain belong to one closed communicating class, then the chain is called an irreducible Markov chain. Irreducibility is a property of the chain. ... If all entries are nonzero, then the matrix is irreducible. Otherwise, it's reducible.

(6 marks)

c) Consider the Markov chain with three states, S={A, B, C}, that has the following transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

i. Draw the state transition diagram for this chain



(2 Marks)

ii. Is this chain irreducible? Explain

No. If all entries are nonzero, then the matrix is irreducible.

iii. Is this chain aperiodic? Explain

No; State x is aperiodic if d(x) = 1

(4 marks)

<u>Question 3 (15 marks)</u>

a) Describe the three main components of a queuing system

i. **Arrival process- Entities arrive to the system according to some arrival pattern. The arrivals are independent of preceding arrivals that is successive interarrival times are statistically independent and completely random.**

ii. **Queuing Discipline- Queue represents a certain number of customers waiting for service (of course the queue may be empty). Typically the customer being served is considered not to be in the queue.**

iii. **Service Mechanism- Service represents some activity that takes time and that the customers are waiting for. Again take it very generally. It may be a real service carried on persons or machines, but it may be a CPU time slice, connection created for a telephone call, being shot down for an enemy plane, etc. Typically a service takes random time.**

(3 marks)

b) On the average 70 customers arrive according to a Poisson process during a day to a small post office in a village. The service times are exponentially distributed with rate 10 clients per hour and the office operates 10 hours daily.

Find

i. The mean queue length,
ii. The probability that the number of waiting customer is greater than 3.
iii. The mean waiting time

(6 marks)

c) Clearly, explain 3 practical areas where queuing processes are applicable

Any three of the following with clear description of arrival pattern, queuing discipline and service mechanism

Design of a garage forecourt

Airports - runway layout, luggage collection, shops, passport control etc.

Hair dressers

Supermarkets

Restaurants

Manufacturing processes

Bus scheduling

Hospital appointment bookings

Printer queues

Minimising page faults in computing

(3 marks)

d) Describe Kendall queuing classification and the Little's law

Kendall's Notation for Queues A/B/C/D/E Shorthand notation where A, B, C, D, E describe the queue Applicable to a large number of simple queueing scenarios

A Inter-arrival time distribution

B Service time distribution

C Number of servers

D Maximum number of jobs that can be there in the system (waiting and in service)

Default ¥ for infinite number of waiting positions

E Queueing Discipline (FCFS, LCFS, SIRO etc.)

Default is FCFS

And Little's Result

N=λW

Nq=λWq

Result holds in general for virtually all types of queueing situations where

λ = Mean arrival rate of jobs that actually enter the system

(3 marks)

<u>Question 4 (15 Marks)</u>

a) Derive the Poisson process using the four Chapman-Kolmogorov postulates
We verify that p0(t) satisfy four chapman Kolmogorov postulates. Now suppose that is satisfied for a certain k. We have
P=k+1(t) = −λpk+1(t) + λpk(t) = −λpk+1(t) + λ e−λt(λt)kk!
P=k+1(t) + λpk+1(t) = λ e−λt(λt)k
k! p=k+1(t)eλt + λpk+1(t)eλt = λ e−λt(λt)k/k! eλt d/dt pk+1(t)eλt== λ
(λt)k
k! = λk+1 (t)k
k!
pk+1(t)eλt =
(λt)k+1(k + 1)!pk+1(t) =e−λt(λt)k+1(k + 1)!
2. Sufficient condition:
The proof will be complete when we show that
∗ P(Nt+h − Nt > 1) = o(h)
∗ P(Nt+h − Nt = 1) = λh + o(h)
We have
∗ P(Nt+h − Nt = 1) =e−λh(λh)11!= λh k=0
(−λh)k k! (−λh)k+1
k! = λh + o(h).∗P(Nt+h − Nt = 0) =
e−λh(λh)0 0! = e−λh = 1 − λh + o(h).
∗P(Nt+h − Nt > 1) = 1 − P(Nt+h − Nt = 1) − P(Nt+h − Nt = 1) = o(h).
Hence from part (a) show that the mean and the variance is λh

(10 Marks)

(5 marks)

<u>Question 5 (15 Marks)</u>

a) Suppose that $\{X_n : n > 1\}$ is a random walk with $X_0 = 0$ and probability p of a step to the right, find:

   a. $P\{X_6 = -2\}$
   b. $P\{X_5 = -1, X_7 = 2, \}$
   c. $P\{X_4 = 1, X_{10} = 4, X_{16} = 2\}$

Solution: (i)

P{X4 = −2} = P{Binom(4, p) = (4 + (−2))/2} =4C_1p1q3 = 4p1q3.

(ii) P{X3 = −1,X6 = 2, } = P{X3 = −1,X6 − X3 = 3, } = P{X3 = −1}P{X6 − X3 = 3} =3C_1p1q2*3C_3p3q0 = 3p4q2.

(iii) P{X5 = 1,X10 = 4,X16 = 2} = P{X5 = 1,X10 − X5 = 3,X10 = 4,X16 − X10 = −2} = P{X5 = 1}P{X10 − X5 = 3}P{X16 − X10 = −2} =5C_3p1q4*5C_4p4q1*6C_2p2q4 = 750p7q9.

(5 marks)

b) Find the limiting probability matrix and the value of n for that matrix

$$\begin{pmatrix} .3 & .5 & .2 \\ .4 & .4 & .2 \\ .6 & .1 & .3 \end{pmatrix}$$

(10 marks)