Birth-Death Processes

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- A birth-death model is a continuous-time stochastic process that is mostly used to study how the number of individuals in a population change through time e.g *inception* & *extinction*, from *good* to *obsolescence*, *giving birth* & *dying* etc.
- Let $Z(t)$ represents the population size at time t and $P_n(t)$ be the probability that the population is of size n at a time , t.

$$P_n(t) = Prob[Z(t) = n]$$

- Let Δt represent a small interval of time over which this population is being studied. For simplicity of this study, we make the following assumptions on the time interval, Δt.
  (i) the probability that a birth occurs within the time interval, Δt from a population of size n is $\lambda_n + O(\Delta t)$;
  (ii) the probability that from a population of size n, a death occurs within the time interval Δt is $\mu_n(t) + O(\Delta t)$, that is

$\lambda_n(t) \implies$ *Birth*
$\mu_n(t) \implies$ *Death*
$\Delta t \implies$ small change in time interval

- From **(i)** and **(ii)**, $\lambda_n(t)$ can also be written as $\lambda_n \Delta(t)$ and $\mu_n(t)$ can also be written as $\mu_n \Delta(t)$.
  - (iii) probability that there is no change is $1 - (\lambda_n + \mu_n)\Delta t + O\Delta t$
  - (iv) probability that there is more than a birth or a death in a time interval Δt is negligible.
- With the assumptions **(i)** to **(iv)**, the problem now is to build the model from $P_n(t + \Delta t)$

$$P_n(t+\Delta t) = P_r\Big[Z(t) = n-1, Z(\Delta t) = 1\Big]$$

- A similar way to express this will be

$$P_n(t+\Delta t) = P_r\Big[Z(t) = n+1, Z(\Delta t) = -1\Big]$$

$$P_n(t+\Delta t) = P_r\Big[Z(t) = n, Z(\Delta t) = 0\Big]$$

$$\begin{aligned}\therefore \;\; P_n(t+\Delta t) = P_r\Big[Z(t) = n-1, Z(\Delta t) = 1\Big] \\ + P_r\Big[Z(t) = n+1, Z(\Delta t) = -1\Big] \\ + P_r\Big[Z(t) = n, Z(\Delta t) = 0\Big]\end{aligned}$$

- The conditional probabilities may be given as

$$\begin{aligned}P_n(t) = {} & P_r\Big(Z(\Delta t) = 1/Z(t) = n-1\Big) \\ & \times P_r\Big((Z(t) = n+1) + P_r\Big(Z(\Delta t) = -1/Z(t) = n+1\Big)\Big) \\ & \times P_r\Big((Z(t) = n+1) + P_r\Big(Z(\Delta t) = 0/Z(t) = n\Big)\Big) \\ & \times P_r(Z(t) = n)\ \{from\ Baye's\ Theorem\}\end{aligned}$$

- Thus we've the model

$$\begin{aligned}P_n(t) = {} & \Big[\lambda_{n-1}(\Delta t) + O(\Delta t)\Big].P_{n-1}(t) \\ & + \Big[\mu_{n+1}(\Delta t) + O(\Delta t)\Big].P_{n+1}(t) \\ & + \Big[1 - (\lambda_n + \mu_n)(\Delta t) + O(\Delta t)\Big].P_n(t)\end{aligned}$$

Birth-Death Processes contd...

- Now, $P_n'(t) = \frac{d}{dt}P_n(t)$ and from the first principles of differentiation we've

$$P_n'(t) = \lim_{\Delta t \to 0}\left[\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t}\right]$$

$$\begin{aligned}\therefore\ P_n'(t) = \lim_{\Delta t \to 0}\Bigg\{&\Big[\lambda_{n-1}(\Delta t) + O(\Delta t)\Big].P_{n-1}(t)\\ &+ \Big[\mu_{n+1}(\Delta t) + O(\Delta t)\Big].P_{n+1}(t)\\ &+ \Big[1 - (\lambda_n + \mu_n)(\Delta t) + O(\Delta t)\Big].P_n(t)\Bigg\}\end{aligned}$$

Birth-Death Processes contd...

$$P'_n(t) = -\left(\lambda_n + \mu_n\right)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \;\; n \geq 1 \tag{1}$$

- This is called an epidemiological process.
- Since

  $$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$$

  and when there is no birth, $n = 0$, thus

  $$P'_0(t) = -\mu_0 P_0(t) + \mu_1 P_1(t) \tag{2}$$

- Equations (1) and (2) are called **difference differential equations** *(d.d.e)*

- One is expected to solve the **o.d.e** for special cases of λ_n and μ_n along with initial conditions.
- One way is to use the $p.g.f$ technique together with ordinary differential equations

Solutions of Linear partial differential equations (p.d.e)

- We suppose that there exists the equation

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \tag{3}$$

- The equation (3) is called the *Lagrange's linear equation* where **P, Q, R** are functions of X, Y and Z. It can be shown that equation (3) is equivalent to

$$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R} \tag{4}$$

- Equation (4) is called an **auxiliary set of equations**. From equation (4) we can have

$$\frac{\partial x}{P} = \frac{\partial y}{Q}, \quad \frac{\partial y}{Q} = \frac{\partial z}{R}$$

and proceed to solve the two resulting simultaneous equations.

- Let $U(X, Y) = \textit{constant}$ and $V(X, Y)$ also a constant be any two solutions of the auxiliary equations (4), then a general solution to equation (3) is $\psi(U, V) = 0$ or $U = \Psi(V)$. In most of our problems we shall use $U = \Psi(V)$ with appropriate conditions.

Pure Birth Process

- The general pure birth process is obtained from the birth-death process by putting $\mu_n = 0$. Thus we have

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \; n \geq 1$$

$$P_0'(t) = -\mu_0 P_0(t) + \mu_1 P_1(t), \; n = 0$$

- Putting $\mu_n = 0$ in the above two equations yields:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t), \quad n \geq 1$$

$$P_0'(t) = -\lambda_0 P_0(t), \quad n = 0$$

- Usually if $\lambda_n = \lambda$, we've the *Poisson process*.
- When $\lambda_n = n\lambda$, we've the *simple birth process* or the *Yule-Furry process* while the *Polya process* is given when $\lambda_n = \lambda\left(\frac{1+an}{1+\lambda at}\right)$

- When $\lambda_n = \lambda$ and $\mu_n = 0$. Thus the difference differential equations are $P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), \;\; n \geq 1$. From here we get

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \;\; n \geq 1 \qquad (5)$$

and

$$P_0'(t) = -\lambda P_0(t), \quad n = 0$$

- Define $G(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$ to be the generated $p.g.f$.
- This means $\frac{\partial}{\partial s}[G(s,t)] = \sum_{n=1}^{\infty} P_n(t) s^{n-1}$ which means that $\frac{\partial}{\partial s}[G(s,t)] = \sum_{n=0}^{\infty} P_n'(t) s^n$, where $P_n'(t)$ is from the general model.
- To introduce the $p.g.f$ $G(s,t)$ and its derivatives in the difference differential equations (5), multiply the equations by s^n and sum the result over n.

Poisson Process contd...

$$\sum_{n=1}^{\infty} P_n'(t)s^n = -\lambda \sum_{n=1}^{\infty} P_n(t)s^n + \lambda \sum_{n=1}^{\infty} P_{n-1}(t)s^n$$

Now

$$\begin{aligned}
\frac{\partial}{\partial t}(G(s,t)) - P_0'(t) &= -\lambda\Big[G(s,t) - s\sum_{n=1}^{\infty} P_{n-1}(t)s^{n-1}\Big] - P_0'(t) \\
\frac{\partial}{\partial t}(G(s,t)) - P_0'(t) &= -\lambda\Big[G(s,t) - sG(s,t)\Big] + \lambda P_0(t) \\
&= -\lambda G(s,t) + \lambda s G(s,t) + \lambda P_0(t) \\
&= -\lambda G(s,t)[1-s] + \lambda P_0(t) \\
\frac{\partial}{\partial t}(G(s,t)) + \lambda P_0(t) - \lambda P_0(t) &= -\lambda G(s,t)(1-s) \\
\frac{\partial}{\partial t}(G(s,t)) &= -\lambda G(s,t)(1-s) \\
&= -\lambda(1-s)G(s,t)
\end{aligned}$$

- Therefore, the Lagrange's linear equations are

$$\frac{\partial}{\partial t}G(s,t) + 0\frac{\partial}{\partial t}(G(s,t)) = -\lambda(1-s)G(s,t)$$

and the corresponding auxiliary equations are

$$\frac{\partial t}{1} = \frac{\partial s}{0} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

- From $\frac{\partial t}{1} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$, integrating both sides gives

$$\int \frac{\partial t}{1} = \int \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

$$t = \frac{1}{-\lambda(1-s)} lnG(s,t) + c$$

$$-\lambda t(1-s) = lnG(s,t) + c$$

$$\exp\{-\lambda t(1-s) - c\} = G(s,t), \quad i.e$$

$$G(s,t) = e^{-\lambda t(1-s)} e^{k}, \quad k = -c$$

OR

$$t + c = \frac{1}{-\lambda(1-s)} lnG(s,t)$$

$$-\lambda t(1-s) + c = lnG(s,t)$$

$$e^{-\lambda t(1-s)} e^{c} = G(s,t)$$

Recall:$e^{lnx} = x$

Poisson Process contd...

- Suppose the initial conditions are

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

  Then $G(s, 0) = e^k$

- But
  $G(s, t) = \sum_{n=0}^{\infty} P_n(t)s^n = P_0(t)s^0 + P_1(t)s^1 + P_2(t)s^2 + \cdots$
  and $G(s, 0) = P_0(0) + P_1(0)s^1 + P_2(0)s^2 + \cdots$
  $\implies P_0(0) = 1$
  $\therefore \quad e^k = G(s, 0) = 1$, hence

$$\begin{aligned} G(s, t) &= e^{-\lambda t(1-s)} \\ &= e^{-\lambda t} e^{\lambda ts} \end{aligned}$$

$$G(s, t) = e^{-\lambda t}\left[\frac{(\lambda ts)^0}{0!} + \frac{(\lambda ts)^1}{1!} + \frac{(\lambda ts)^2}{2!} + \cdots\right]$$

- $P_n(t)$ is the coefficient of s^n in the expansion of $G(s,t)$ which is $\frac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $n = 0, 1, 2, \cdots$ which is a Poisson distribution with parameter λt.

QUIZ 1:

Show that expected value = variance $=\lambda t$

Exercise:

Develop the $G(s,t)$ of the simple birth process i.e. $\lambda_n = n\lambda$.

Methods of Obtaining $E(n)$ and $Var(n)$

1. $$E(n) = G'(1,t)$$
$$Var(n) = G''(1,t) + G'(1,t) - [G'(1,t)]^2$$

2. $$E(n) = \sum_n n p_n(t)$$
$$Var(n) = E(n^2) - [E(n)]^2$$

3. Feller's method:
$$E(n) = M_1(t),\ M_1(t) = \sum_{n=0}^{\infty} nP_n(t)$$
$$Var(n) = M_2(t) - (M_1(t))^2,\ M_2(t) = \sum_{n=0}^{\infty} n^2 P_n(t)$$

solution (a) using $p.g.f$ **technique**

$$\begin{aligned}
G(s,t) &= e^{-\lambda t(1-s)} \\
\frac{\partial}{\partial s}G(s,t) &= G'(s,t) = \lambda t\ e^{-\lambda t(1-s)} \\
E(n) &= G'(1,t) \\
&= \lambda t\ e^{-\lambda t(1-1)} \\
&= \lambda t
\end{aligned}$$

$$G''(s,t) = (\lambda t)^2\ e^{-\lambda t(1-s)}$$

$$G''(1,t) = (\lambda t)^2$$

$$\begin{aligned}
Var(n) &= G''(1,t) + G'(1,t) - [G'(1,t)]^2 \\
&= (\lambda t)^2 + \lambda t - (\lambda t)^2 \\
&= \lambda t
\end{aligned}$$

solution (b) using Feller's Method

Define

$$M_1(t) = \sum_{n=0}^{\infty} nP_n(t) \Leftrightarrow M_1'(t) = \sum_{n=0}^{\infty} nP_n'(t)$$

$$M_2(t) = \sum_{n=0}^{\infty} n^2 P_n(t) \Leftrightarrow M_2'(t) = \sum_{n=0}^{\infty} n^2 P_n'(t)$$

$$M_3(t) = \sum_{n=0}^{\infty} n^3 P_n(t) \Leftrightarrow M_3'(t) = \sum_{n=0}^{\infty} n^3 P_n'(t)$$

$E(n)$ using Feller's Method contd...

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t),\ n \geq 1$$

$$\sum_{n=1}^{\infty} nP_n'(t) = -\lambda \sum_{n=1}^{\infty} nP_n(t) + \lambda \sum_{n=1}^{\infty} nP_{n-1}(t),\ n \geq 1$$

$$M_1'(t) = -\lambda M_1(t) + \lambda \sum_{n=1}^{\infty} ([n-1]+1) P_{n-1}(t)$$

$$M_1'(t) = -\lambda M_1(t) + \lambda \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) + \lambda \sum_{n=1}^{\infty} P_{n-1}(t)$$

$$M_1'(t) = -\lambda M_1(t) + \lambda M_1(t) + \lambda \implies M_1'(t) = \lambda$$

$$\int M_1'(t)dt = \int \lambda dt \implies M_1(t) = \lambda t + C$$

$$M_1(0) = C, \;\; But \;\; M_1(t) = \sum_{n=0}^{\infty} nP_n(t)$$

$$\therefore M_1(0) = 0P_0(0) + 1P_1(0) + 2P_2(0) + 3P_3(0) + \cdots = 0 \implies C = 0$$

$$M_1(t) = \lambda t = E(n)$$

$Var(n)$ using Feller's Method contd...

$$M_2'(t) = \sum_{n=1}^{\infty} n^2 P_n'(t)$$

$$\sum_{n=1}^{\infty} n^2 P_n'(t) = -\lambda \sum_{n=1}^{\infty} n^2 P_n(t) + \lambda \sum_{n=1}^{\infty} n^2 P_{n-1}(t)$$

$$M_2'(t) = -\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} \left[(n-1)^2 + 2(n-1) + 1 \right] P_{n-1}(t)$$

$$M_2'(t) = -\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} (n-1)^2 P_{n-1}(t) + 2\lambda \sum_{n=1}^{\infty} (n-1) P_{n-1}(t)$$

$$+ \lambda \sum_{n=1}^{\infty} P_{n-1}(t)$$

$$M_2'(t) = -\lambda M_2(t) + \lambda M_2(t) + 2\lambda M_1(t) + \lambda$$

$$M_2'(t) = 2\lambda M_1(t) + \lambda$$

$$M_2'(t) = 2\lambda(\lambda t) + \lambda, \;\; since \;\; M_1(t) = \lambda t$$

$$\therefore M_2'(t) = 2\lambda^2 t + \lambda$$

$$\int M_2'(t)dt = \int (2\lambda^2 t + \lambda)dt$$

$$M_2(t) = \lambda^2 t^2 + \lambda t + C \implies M_2(0) = C$$

$Var(n)$ using Feller's Method contd...

But by definition

$$\begin{aligned} M_2(t) &= \sum_{n=0}^{\infty} n^2 P_n(t) \\ &= 0^2 P_0(t) + 1^2 P_1(t) + 2^2 P_2(t) + \cdots \end{aligned}$$

$$\begin{aligned} M_2(0) &= 0^2 P_0(0) + 1^2 P_1(0) + 2^2 P_2(0) + \cdots \\ &= 0 \implies C = 0 \end{aligned}$$

$$\therefore M_2(t) = \lambda^2 t^2 + \lambda t$$

Hence

$$\begin{aligned} Var(n) &= M_2(t) - (M_1(t))^2 \\ &= \lambda^2 t^2 + \lambda t - (\lambda t)^2 \\ &= \lambda t \end{aligned}$$

From

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \;\; n \geq 1$$

$$P_0'(t) = -\mu_0 P_0(t) + \mu_1 P_1(t), \;\; n = 0$$

- Putting $\mu_n = 0, \; \lambda_n = n\lambda, \; \lambda_{n-1} = \lambda(n-1)$

$$P_n'(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t), \quad n \geq 1$$

$$P_0'(t) = 0, \quad n = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} P_n'(t)s^n = -\lambda \sum_{n=1}^{\infty} nP_n(t)s^n + \lambda \sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^n$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = -\lambda s \sum_{n=1}^{\infty} nP_n(t)s^{n-1} + \lambda s^2 \sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^{n-2}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = -\lambda s\frac{\partial G(s,t)}{\partial s} + \lambda s^2\frac{\partial G(s,t)}{\partial s}$$
$$= -\lambda s(1-s)\frac{\partial}{\partial s}G(s,t)$$

- The Lagrange's linear equation is

$$\frac{\partial G(s,t)}{\partial t} + \lambda s(1-s)\frac{\partial G(s,t)}{\partial s} = 0$$

- Auxiliary equations:

$$\frac{\partial t}{1} = \frac{\partial s}{\lambda s(1-s)} = \frac{\partial G(s,t)}{0}$$

- From $\frac{\partial t}{1} = \frac{\partial G(s,t)}{0}$, we get
  $\int 0 \partial t = \int \partial G(s,t) \implies c_1 = G(s,t)$

Yule-Furry Process contd...

- From $\frac{\partial t}{1} = \frac{\partial s}{\lambda s(1-s)}$, we get

$$\int \lambda \partial t = \int \frac{\partial s}{s(1-s)} \implies \lambda t + c_2 = ln(\frac{s}{1-s})$$

leading to

$$\frac{s}{1-s} = e^{\lambda t + c_2} = e^{\lambda t} e^{c_2} \implies c_2 = \frac{s}{1-s} e^{-\lambda t}$$

- The general solution is: $c_1 = \Psi(c_2)$

$$G(s,t) = \Psi\left[\frac{s}{1-s} e^{-\lambda t}\right]$$

$$G(s,0) = \Psi\left[\frac{s}{1-s}\right]$$

- But

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$
$$= P_0(t)s^0 + P_1(t)s + P_2(t)s^2 + \cdots$$

- Suppose the initial condition is

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} G(s,0) &= P_0(0) + P_1(0)s + P_2(0)s^2 + \cdots \\ &= P_1(0)s \\ &= s \implies \Psi\left[\frac{s}{1-s}\right] = s \end{aligned}$$

Yule-Furry Process contd...

- Let

$$w = \frac{s}{1-s} \implies s = \frac{w}{1+w}$$
$$\implies \Psi(w) = \frac{w}{1+w}$$

$$\begin{aligned} G(s,t) &= \Psi\left(\frac{s}{1-s}e^{-\lambda t}\right) \\ &= \Psi(we^{-\lambda t}) \\ &= \frac{we^{-\lambda t}}{1+we^{-\lambda t}} \end{aligned}$$

Yule-Furry Process contd...

$$\therefore G(s,t) = \frac{\frac{s}{1-s}e^{-\lambda t}}{1+\frac{s}{1-s}e^{-\lambda t}}$$

$$= \frac{se^{-\lambda t}}{1-s+se^{-\lambda t}}$$

$$= \frac{se^{-\lambda t}}{1-(1-e^{-\lambda t})s}$$

$$= se^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1}$$

$$= se^{-\lambda t}\Big[1+(1-e^{-\lambda t})s+(1-e^{-\lambda t})^2s^2+\cdots$$

$$+(1-e^{-\lambda t})^{n-1}s^{n-1}+\cdots\Big]$$

Recall: $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

- $\therefore P_n(t)$ is the coefficient of s^n in the expansion of $G(s,t)$ i.e

$$P_n(t) = e^{-\lambda t}(1-e^{-\lambda t})^{n-1}, \quad n = 1,2,\cdots$$

which is a **geometric distribution**.

Yule-Furry Process contd...

$$G(s,t) = se^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1}$$

$$\begin{aligned} G'(s,t) &= e^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1} \\ &\quad + se^{-\lambda t}(-1)[1-e^{-\lambda t}]\left[1-(1-e^{-\lambda t})s\right]^{-2} \end{aligned}$$

$\therefore$

$$\begin{aligned} E(n) = G'(1,t) &= e^{-\lambda t}e^{\lambda t} + e^{-\lambda t}(1-e^{-\lambda t})e^{2\lambda t} \\ &= 1 + e^{\lambda t}(1-e^{-\lambda t}) \\ &= 1 + e^{\lambda t} - 1 \\ &= e^{\lambda t} \end{aligned}$$

$$G''(s,t) = e^{-\lambda t}(1-e^{-\lambda t})[1-(1-e^{-\lambda t})s]^{-2}$$
$$+ e^{-\lambda t}(1-e^{-\lambda t})\Big\{[1-(1-e^{-\lambda t})s]^{-2}$$
$$+ 2(1-e^{-\lambda t})s[1-(1-e^{-\lambda t})s]^{-3}\Big\}$$

$$G''(1,t) = e^{-\lambda t}(1-e^{-\lambda t})[1-(1-e^{-\lambda t})]^{-2}$$
$$+ e^{-\lambda t}(1-e^{-\lambda t})\Big\{[1-(1-e^{-\lambda t})]^{-2}$$
$$+ 2(1-e^{-\lambda t})[1-(1-e^{-\lambda t})]^{-3}\Big\}$$
$$= e^{\lambda t} - 1 + (e^{-\lambda t} - e^{-2\lambda t})\Big\{e^{-2\lambda t} + 2e^{3\lambda t} - 2e^{2\lambda t}\Big\}$$
$$= e^{\lambda t} - 1 + (e^{-\lambda t} - e^{-2\lambda t})\Big\{2e^{3\lambda t} - e^{\lambda t}\Big\}$$
$$= e^{\lambda t} - 1 + 2e^{2\lambda t} - e^{\lambda t} - 2e^{\lambda t} + 1$$
$$= 2e^{2\lambda t} - 2e^{\lambda t}$$

Yule-Furry Process contd...

$$
\begin{aligned}
Var(n) &= G''(1,t) + G'(1,t) - [G'(1,t)]^2 \\
&= 2e^{2\lambda t} - 2e^{\lambda t} + e^{\lambda t} - [e^{\lambda t}]^2 \\
&= 2e^{2\lambda t} - 2e^{\lambda t} + e^{\lambda t} - e^{2\lambda t} \\
&= e^{2\lambda t} - e^{\lambda t} \\
&= e^{\lambda t}(e^{\lambda t} - 1)
\end{aligned}
$$

(iii) The Polya Process

- Here,

$$\lambda_n = \lambda\left[\frac{1+an}{1+\lambda at}\right]$$

and still the death component is zero i.e $\mu_n = 0$ with 'a' being an arbitrary constant.

- Therefore the difference differential equations are:

$$P_n'(t) = -\lambda\left[\frac{1+an}{1+\lambda at}\right]p_n(t) + \lambda\left[\frac{1+a(n-1)}{1+\lambda at}\right]p_{n-1}(t), \quad n \geq 1$$

$$P_0'(t) = \left[\frac{-\lambda}{1+\lambda at}\right]P_0(t), \quad n = 0$$

$$\therefore \sum_{n=1}^{\infty} P_n'(t)s^n = \lambda\Big[-\sum_{n=1}^{\infty}\Big(\frac{1+an}{1+\lambda at}\Big)P_n(t)s^n + \sum_{n=1}^{\infty}\Big(\frac{1+a(n-1)}{1+\lambda at}\Big)P_{n-1}(t)s^n\Big]$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \Big[\frac{\lambda}{1+\lambda at}\Big]\Big\{-\sum_{n=1}^{\infty}(1+an)P_n(t)s^n + \sum_{n=1}^{\infty}(1+a(n-1))P_{n-1}(t)s^n\Big\}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda a t}\right]\left\{ -\sum_{n=1}^{\infty} P_n(t)s^n - a\sum_{n=1}^{\infty} nP_n(t)s^n \right.$$
$$\left. + s\sum_{n=1}^{\infty} P_{n-1}(t)s^{n-1} + as\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^{n-1} \right\}$$

But

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$

so

$$\frac{\partial}{\partial t}G(s,t) = \sum_{n=0}^{\infty} P_n'(t)s^n, \quad \frac{\partial}{\partial s}G(s,t) = \sum_{n=1}^{\infty} nP_n(t)s^{n-1}$$

The Polya Process contd...

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda a t}\right]\Big\{-[G(s,t) - P_0(t)] - as\frac{\partial}{\partial s}G(s,t)$$
$$+ sG(s,t) + as^2\frac{\partial}{\partial s}G(s,t)\Big\}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda a t}\right]\Big\{-G(s,t)[1-s]$$
$$+ \frac{\partial}{\partial s}G(s,t)(-as + as^2) - \left(\frac{1+\lambda a t}{\lambda}\right)P_0'(t)\Big\}$$

Since

$$P_0'(t) = -\left[\frac{\lambda}{1+\lambda a t}\right]P_0(t), \quad \Longrightarrow \quad P_0(t) = -\left[\frac{1+\lambda a t}{\lambda}\right]P_0'(t)$$

The Polya Process contd...

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) + P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\}$$

$$\begin{aligned}\implies \frac{\partial}{\partial t}G(s,t) &= \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\} \\ &= \left[\frac{\lambda}{1+\lambda at}\right]\left\{(s-1)G(s,t) + as(s-1)\frac{\partial}{\partial s}G(s,t)\right\}\end{aligned}$$

- Hence the *Lagrange's linear equations* are

$$\frac{\partial G(s,t)}{\partial t} + \left[\frac{\lambda}{1+\lambda at}\right] as(1-s)\frac{\partial G(s,t)}{\partial s} = -\left[\frac{\lambda}{1+\lambda at}\right](1-s)G(s,t)$$

which can also be expressed as

$$(1+\lambda at)\frac{\partial G(s,t)}{\partial t} + \lambda as(1-s)\frac{\partial G(s,t)}{\partial s} = -\lambda(1-s)G(s,t)$$

- The corresponding **auxiliary equations** are

$$\frac{\partial t}{1+\lambda at} = \frac{\partial s}{\lambda as(1-s)} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

- Consider

$$\frac{\partial t}{1+\lambda at} = \frac{\partial s}{\lambda as(1-s)}$$

- Taking integration on both sides gives

$$\int \frac{\partial t}{1+\lambda at} = \int \frac{\partial s}{\lambda as(1-s)}$$

$$\frac{1}{\lambda a} ln(1+\lambda at) + \frac{1}{\lambda a} ln\ c_2 = \frac{1}{\lambda a} ln\left(\frac{s}{1-s}\right)$$

$$c_2 = \left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}$$

- Also from

$$\frac{\partial s}{\lambda a s(1-s)} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

we get

$$\int \frac{\partial s}{as} = \int \frac{\partial G(s,t)}{-G(s,t)} \implies \frac{1}{a} ln\ s + ln\ c_1 = -ln\ G(s,t)$$

and

$$\implies c_1 = s^{\frac{1}{a}}[G(s,t)]$$

- The **general solution** is

$$c_1 = \Psi(c_2)$$
$$\implies s^{\frac{1}{a}} G(s,t) = \Psi\left[\frac{s}{1-s}(1+\lambda a t)^{-1}\right]$$

The Polya Process contd...

- Using the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$G(s, 0) = s^{-\frac{1}{a}} \Psi\left(\frac{s}{1-s}\right)$$

- But by definition

$$\begin{aligned} G(s, t) &= \sum_{n=0}^{\infty} P_n(t) s^n \\ &= P_0(t) + P_1(t)s + P_2(t)s^2 + \cdots \end{aligned}$$

$$\begin{aligned} G(s, 0) &= P_0(0) + P_1(0)s + P_2(0)s^2 + \cdots \\ &= P_0(0) = 1 \end{aligned}$$

$$\implies s^{-\frac{1}{a}}\Psi\left(\frac{s}{1-s}\right) = 1$$

and therefore

$$\Psi\left(\frac{s}{1-s}\right) = s^{\frac{1}{a}}$$

which accordingly implies that

$$\Psi(w) = \left(\frac{w}{1+w}\right)^{\frac{1}{a}}$$

for any arbitrary w.

so

$$
\begin{aligned}
G(s,t) &= s^{-\frac{1}{a}}\Psi\left[\left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}\right] = s^{-\frac{1}{a}}\Psi[w(1+\lambda at)^{-1}] \\
&= s^{-\frac{1}{a}}\left[\frac{w(1+\lambda at)^{-1}}{1+w(1+\lambda at)^{-1}}\right]^{\frac{1}{a}} \\
&= s^{-\frac{1}{a}}\left[\frac{\frac{s}{1-s}(1+\lambda at)^{-1}}{1+\frac{s}{1-s}(1+\lambda at)^{-1}}\right]^{\frac{1}{a}} \\
&= s^{-\frac{1}{a}}s^{\frac{1}{a}}\left[\frac{1}{(1+\lambda at)(1-s)+s}\right]^{\frac{1}{a}} \\
&= \left[\frac{1}{1+\lambda at-\lambda ats}\right]^{\frac{1}{a}}
\end{aligned}
$$

and

$$\begin{aligned}
G(s,t) &= (1+\lambda at)^{-\frac{1}{a}}\left[1-\frac{\lambda at}{1+\lambda at}s\right]^{-\frac{1}{a}} \\
&= (1+\lambda at)^{-\frac{1}{a}}\frac{[1+\lambda at-\lambda ats]^{-\frac{1}{a}}}{(1+\lambda at)^{-\frac{1}{a}}} \\
&= (1+\lambda at)^{-\frac{1}{a}}\left[1+\binom{-\frac{1}{a}}{1}\left(\frac{-\lambda at}{1+\lambda at}s\right)\right. \\
&\quad \left.+\cdots+\binom{-\frac{1}{a}}{n}\left(\frac{-\lambda at}{1+\lambda at}s\right)^n+\cdots\right]
\end{aligned}$$

so that $P_n(t)$ is the coefficient of s^n in the above expansion for $G(s,t)$.

$$\begin{aligned}
P_n(t) &= (1+\lambda at)^{-\frac{1}{a}}\binom{-\frac{1}{a}}{n}\left(\frac{-\lambda at}{1+\lambda at}\right)^n \\
&= (1+\lambda at)^{-\frac{1}{a}-n}\binom{-\frac{1}{a}}{n}(-\lambda at)^n
\end{aligned}$$

Mean

$$G(s,t) = [(1+\lambda at) - \lambda ats]^{-\frac{1}{a}}$$

$$\begin{aligned} G'(s,t) &= \left(\frac{-1}{a}\right)[(1+\lambda at) - \lambda ats]^{-\frac{1}{a}-1}(-\lambda at) \\ &= \lambda t[(1+\lambda at) - \lambda ats]^{-\frac{1}{a}-1} \end{aligned}$$

$$\begin{aligned} E(n) &= G'(1,t) \\ &= \lambda t[1+\lambda at - \lambda at]^{-\frac{1}{a}-1} \\ &= \lambda t \end{aligned}$$

Variance

$$G''(s,t) = \lambda t(-\frac{1}{a} - 1)[(1 + \lambda at) - \lambda ats]^{-\frac{1}{a}-2}(-\lambda at)$$

$$\begin{aligned} G''(1,t) &= \lambda t(-\frac{1}{a} - 1)(1)(-\lambda at) \\ &= \lambda t(\frac{-1-a}{a})(1)(-\lambda at) \\ &= (\lambda t)^2(1+a) \end{aligned}$$

$$\begin{aligned} Var(n) &= G''(1,t) + G'(1,t) - [G'(1,t)]^2 \\ &= (\lambda t)^2(1+a) + \lambda t - (\lambda t)^2 \\ &= (\lambda t)^2 + (\lambda)^2 a(t)^2 + \lambda t - (\lambda t)^2 \\ &= \lambda t(\lambda at + 1) \end{aligned}$$

(a) Simple Birth-Death Process

- Here, $\lambda_n = n\lambda$ and $\mu_n = n\mu$
- The difference differential equations are

$$P_n'(t) = -n(\lambda+\mu)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \;\; n \geq 1$$

and $$P_0'(t) = \mu P_1(t), \quad n = 0$$

$$\sum_{n=1}^{\infty} P_n'(t)s^n = -(\lambda+\mu)\sum_{n=1}^{\infty} nP_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^n + \mu\sum_{n=1}^{\infty}(n+1)P_{n+1}(t)s^n$$

$$\frac{\partial G(s,t)}{\partial t} - P_0'(t) = -(\lambda+\mu)s\frac{\partial G(s,t)}{\partial s} + \lambda s^2\frac{\partial G(s,t)}{\partial s} + \left(\mu\frac{\partial G(s,t)}{\partial s} - \mu P_1(t)\right)$$

$$\frac{\partial G(s,t)}{\partial t} = (1-s)(\mu-\lambda s)\frac{\partial G(s,t)}{\partial s}$$

- The *Lagrange's linear equation* is

$$\frac{\partial G(s,t)}{\partial t} - (1-s)(\mu-\lambda s)\frac{\partial G(s,t)}{\partial s} = 0$$

- The *auxiliary equations are*

$$\frac{\partial t}{1} = \frac{\partial s}{-(1-s)(\mu-\lambda s)} = \frac{\partial G(s,t)}{0}$$

From

$$\frac{\partial t}{1} = \frac{\partial G(s,t)}{0}$$

we have

$$\int \partial G(s,t) = \int 0 \partial t \implies G(s,t) = c_1$$

From

$$\frac{\partial t}{1} = \frac{\partial s}{-(1-s)(\mu - \lambda s)} \implies \int \frac{\partial t}{1} = \int \frac{\partial s}{-(1-s)(\mu - \lambda s)}$$

$$t + c = \frac{1}{\mu - \lambda} ln\left(\frac{\mu - \lambda s}{1 - s}\right)$$

$$\frac{\mu - \lambda s}{1 - s} = e^{(\mu-\lambda)t} e^{(\mu-\lambda)c}$$

$$\left(\frac{\mu - \lambda s}{1 - s}\right) e^{-(\mu-\lambda)t} = e^{(\mu-\lambda)c} = c_2$$

- The general solution is

$$c_1 = \Psi(c_2), \quad i.e$$

$$G(s,t) = \Psi\left[\left(\frac{\mu - \lambda s}{1 - s}\right) e^{-(\mu-\lambda)t}\right]$$

- Using the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

Simple Birth-Death Process contd...

$$G(s,0) = \Psi\left(\frac{\mu - \lambda s}{1-s}\right)$$

But

$$\begin{aligned} G(s,0) &= P_0(0) + P_1(0)s + P_2(0)s^2 + \cdots \\ &= P_1(0)s = s \quad since \quad P_1(0) = 1 \end{aligned}$$

$$\Psi\left(\frac{\mu - \lambda s}{1-s}\right) = s$$

Let

$$w = \frac{\mu - \lambda s}{1-s} \Leftrightarrow s = \frac{\mu - w}{\lambda - w}$$

$$\Psi(w) = \frac{\mu - w}{\lambda - w}$$

Simple Birth-Death Process contd...

$$
\begin{aligned}
G(s,t) &= \Psi(w\ e^{-(\mu-\lambda)t}) \\
&= \frac{\mu - w\ e^{-(\mu-\lambda)t}}{\lambda - w\ e^{-(\mu-\lambda)t}} \\
&= \frac{\mu - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}}{\lambda - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}} \\
&= \frac{\mu(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}}{\lambda(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}} \\
&= \frac{\mu(1-e^{-(\mu-\lambda)t}) - (\mu-\lambda e^{-(\mu-\lambda)t})s}{[\lambda-\mu e^{-(\mu-\lambda)t}] - \lambda[1-e^{-(\mu-\lambda)t}]s}
\end{aligned}
$$

$$
G(s,t) = \frac{\frac{\mu(1-e^{-(\mu-\lambda)t})}{\lambda-\mu\ e^{-(\mu-\lambda)t}} - \frac{\mu-\lambda e^{-(\mu-\lambda)t}}{\lambda-\mu e^{-(\mu-\lambda)t}} s}{1 - \lambda\left[\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right] s}
$$

Simple Birth-Death Process contd...

$$
\begin{aligned}
P_n(t) &= Coefficient\ of\ s^n \\
&= \left[\frac{\mu(1-e^{-(\mu-\lambda)t})}{\lambda-\mu\ e^{-(\mu-\lambda)t}}\right]\left[\lambda\left(\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right)\right]^n \\
&- \left[\left(\frac{\mu-\lambda e^{-(\mu-\lambda)t}}{\lambda-\mu e^{-(\mu-\lambda)t}}\right)\right]\left[\lambda\left(\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right)\right]^{n-1}
\end{aligned}
$$

(b) The Zero Growth Rate

- Here $\lambda_n = n\lambda$, $\mu_n = n\mu$ and $\lambda = \mu$
- The difference-differential equations are

$$P_n'(t) = -2\lambda n P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\lambda P_{n+1}(t), \;\; n \geq 1$$

$$P_0'(t) = \lambda P_1(t), \quad n = 0$$

$$\sum_{n=1}^{\infty} P_n'(t)s^n = -2\lambda \sum_{n=1}^{\infty} nP_n(t)s^n + \lambda \sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^n + \lambda \sum_{n=1}^{\infty}(n+1)P_{n+1}(t)s^n$$

$$\frac{\partial G(s,t)}{\partial t} - P_0'(t) = -2\lambda s \frac{\partial G(s,t)}{\partial s} + \lambda s^2 \frac{\partial G(s,t)}{\partial s} + \lambda \left[\frac{\partial G(s,t)}{\partial s} - P_1(t) \right]$$

$$\frac{\partial G(s,t)}{\partial t} = \lambda(1-s)^2 \frac{\partial G(s,t)}{\partial s}, \; \textit{since} \; P_0'(t) = \lambda P_1(t)$$

- The *Lagrange's Linear equation* is

$$\frac{\partial G(s,t)}{\partial t} - \lambda(1-s)^2 \frac{\partial G(s,t)}{\partial s} = 0$$

- The auxiliary equations are

$$\frac{\partial t}{1} = \frac{\partial s}{-\lambda(1-s)^2} = \frac{\partial G(s,t)}{0}$$

The Zero Growth Rate contd...

$$\int \partial G(s,t) = \int 0 \partial t \implies G(s,t) = c_1$$

$$\int -\lambda \partial t = \int \frac{\partial s}{(1-s)^2} = -\lambda t + c = \frac{1}{1-s}$$

$$c_2 = \frac{1}{1-s} + \lambda t$$

- The general solution is

$$c_1 = \Psi(c_2) \implies G(s,t) = \Psi\left[\frac{1}{1-s} + \lambda t\right]$$

The Zero Growth Rate contd...

- The initial conditions are

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$G(s,0) = P_1(s) = s$$

$$G(s,0) = \Psi\left(\frac{1}{1-s}\right) = s$$

- Let

$$w = \frac{1}{1-s} \Leftrightarrow w - ws = 1 \implies s = \frac{w-1}{w}$$

$$\Psi\left(\frac{1}{1-s}\right) = s$$

$$\therefore \Psi(w) = \frac{w-1}{w}$$

The Zero Growth Rate contd...

$$
\begin{aligned}
\therefore G(s,t) &= \Psi(w+\lambda t) \\
&= \frac{w+\lambda t-1}{w+\lambda t} \\
&= \frac{\frac{1}{1-s}+\lambda t-1}{\frac{1}{1-s}+\lambda t} \\
&= \frac{1+\lambda t-\lambda ts-1+s}{1+\lambda t-\lambda ts} \\
&= \frac{\lambda t+(1-\lambda t)s}{1+\lambda t-\lambda ts} \\
&= \frac{\frac{\lambda t}{1+\lambda t}+\left(\frac{1-\lambda t}{1+\lambda t}\right)s}{1-\frac{\lambda t}{1+\lambda t}s}
\end{aligned}
$$

The Zero Growth Rate contd...

$$G(s,t) = \left[\frac{\lambda t}{1+\lambda t} + \left(\frac{1-\lambda t}{1+\lambda t}\right)s\right]\left[1 - \frac{\lambda t}{1+\lambda t}s\right]^{-1}$$

Recall: $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ so that $G(s,t)$ becomes

$$G(s,t) = \left[\frac{\lambda t}{1+\lambda t} + \left(\frac{1-\lambda t}{1+\lambda t}\right)s\right]\left[1 + \frac{\lambda t}{1+\lambda t}s + \cdots \right.$$
$$\left. + \left(\frac{\lambda t}{1+\lambda t}s\right)^{n-1} + \left(\frac{\lambda t}{1+\lambda t}s\right)^{n}\right]$$

$$
\begin{aligned}
P_n(t) &= coefficient \;\; of \;\; s^n \\
&= \left(\frac{\lambda t}{1+\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)^n + \left(\frac{1-\lambda t}{1+\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)^{n-1} \\
&= \frac{(\lambda t)^{n+1}}{(1+\lambda t)^{n+1}} + (1-\lambda t)\frac{(\lambda t)^{n-1}}{(1+\lambda t)^n} \\
&= \frac{(\lambda t)^{n+1} + (1-\lambda t)(1+\lambda t)(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}} \\
&= \frac{(\lambda t)^{n+1} + (1-(\lambda t)^2)(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}} \\
&= \frac{(\lambda t)^{n+1} + (\lambda t)^{n-1} - (\lambda t)^{n+1}}{(1+\lambda t)^{n+1}} \\
&= \frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}
\end{aligned}
$$

The Zero Growth Rate contd...

- Differentiating $G(s,t)$ *w.r.t* s yields

$$G'(s,t) = \left(\frac{1-\lambda t}{1+\lambda t}\right)\left[1 - \frac{\lambda ts}{1+\lambda t}\right]^{-1} + \left[\frac{\lambda t + (1-\lambda t)s}{1+\lambda t}\right]\left(\frac{\lambda t}{1+\lambda t}\right)\left[1 - \frac{\lambda ts}{1+\lambda t}\right]^{-2}$$

$$\begin{aligned}E(n) = G'(1,t) &= \left(\frac{1-\lambda t}{1+\lambda t}\right)\left(\frac{1}{1+\lambda t}\right)^{-1} \\ &+ \left(\frac{1}{1+\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)\left(\frac{1}{1+\lambda t}\right)^{-2} \\ &= 1 - \lambda t + \lambda t \\ &= 1\end{aligned}$$

The Zero Growth Rate contd...

$$\begin{aligned}G''(s,t) &= \left(\frac{1-\lambda t}{1+\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)\left[1-\frac{\lambda ts}{1+\lambda t}\right]^{-2} \\ &+ \frac{\lambda t}{1+\lambda t}\left\{\left(\frac{1-\lambda t}{1+\lambda t}\right)\left[1-\frac{\lambda ts}{1+\lambda t}\right]^{-2}\right. \\ &+ \left.\left[\frac{\lambda t+(1-\lambda t)s}{1+\lambda t}\right]\left(\frac{2\lambda t}{1+\lambda t}\right)\left[1-\frac{\lambda ts}{1+\lambda t}\right]^{-3}\right\}\end{aligned}$$

$$\begin{aligned}G''(1,t) &= \left(\frac{1-\lambda t}{1+\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)(1+\lambda t)^2 \\ &+ \left(\frac{\lambda t}{1+\lambda t}\right)\left\{\left(\frac{1-\lambda t}{1+\lambda t}\right)(1+\lambda t)^2+\left(\frac{2\lambda t}{(1+\lambda t)^2}\right)(1+\lambda t)^3\right\} \\ &= (1-\lambda t)\lambda t+\lambda t+\lambda t(1-\lambda t)+2(\lambda t)^2 = 2\lambda t\end{aligned}$$

$$\begin{aligned}Var(n) &= G''(1,t)+G'(1,t)-[G'(1,t)]^2 \\ &= 2\lambda t+1-1 = 2\lambda t\end{aligned}$$

Feller's Method:

$$M_1(t) = \sum_{n=1}^{\infty} nP_n(t)$$

$$M_2(t) = \sum_{n=1}^{\infty} n^2 P_n(t)$$

$$P_n'(t) = -2\lambda nP_n(t) + \lambda(n-1)P_{n-1}(t) + \lambda(n+1)P_{n+1}(t), \;\; n \geq 1$$

$$\sum_{n=1}^{\infty} nP_n'(t) = -2\lambda \sum_{n=1}^{\infty} n^2 P_n(t) + \lambda \sum_{n=1}^{\infty} n(n-1)P_{n-1}(t)$$
$$+ \lambda \sum_{n=1}^{\infty} n(n+1)P_{n+1}(t), \;\; n \geq 1$$

The Zero Growth Rate contd...

$$M_1'(t) = -2\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} (n-1+1)(n-1)P_{n-1}(t)$$
$$+ \lambda \sum_{n=1}^{\infty} (n+1-1)(n+1)P_{n+1}(t)$$
$$= -2\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} (n-1)^2 P_{n-1}(t) + \lambda \sum_{n=1}^{\infty} (n-1)P_{n-1}(t)$$
$$+ \lambda \sum_{n=1}^{\infty} (n+1)^2 P_{n+1}(t) - \lambda \sum_{n=1}^{\infty} (n+1)P_{n+1}(t)$$
$$= -2\lambda M_2(t) + \lambda M_2(t) + \lambda M_1(t) + \lambda[M_2(t) - P_1(t)]$$
$$- \lambda[M_1(t) - P_1(t)]$$

$$M_1'(t) = -2\lambda M_2(t) + \lambda M_2(t) + \lambda M_1(t) + \lambda M_2(t) - \lambda P_1(t) - \lambda M_1(t) + \lambda P_1(t) \implies M_1'(t) = 0$$

$$\int M_1'(t) = \int 0dt \implies M_1(t) = c \Leftrightarrow M_1(0) = c$$

But

$$M_1(t) = \sum_{n=0}^{\infty} nP_n(t) = 0P_0(t) + 1P_1(t) + 2P_2(t) + \cdots$$

$$M_1(0) = 0P_0(0) + 1P_1(0) + 2P_0(t) + \cdots = 1 \implies c = 1$$

Hence

$$M_1(t) = 1 = E(n)$$

The Zero Growth Rate contd...

QUIZ:

Find $M_2'(t)$ hence find $Var(n)$.