Review on Generating Functions

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- It is usually useful to employ the powerful method of generating functions when studying random variables that assume integral values (e.g. the Poisson, Binomial, Geometric e.t.c.)
- Most of the *stochastic processes* that we come across involve integral-valued random variables and therefore we could use generating functions in their studies.

- **Definition 1**: Let $a_0, a_1, a_2, \cdots$ be a sequence of real numbers. Using a random variable S, one may define a function

$$A(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$$
$$= \sum_{k=0}^{\infty} a_k s^k \tag{1}$$

- If the power of the series converges in some interval say $-s_0 \leq S \leq s_r$, then $A(s)$ is called a *generating function* of the real numbers $a_0, a_1, a_2, \cdots$. The variable S itself has no specific significance.

Now,

$$\frac{dA(s)}{ds} = a_1 + 2a_2 s + 3a_3 s^2 + 4a_4 s^3 + \cdots + ka_k s^{k-1} + \cdots$$

$$\frac{dA(s)}{ds}|_{s=0} = a_1$$

$$\frac{d^2A(s)}{ds^2} = 2a_2 + 6a_3 s + 12a_4 s^2 + \cdots + k(k-1)a_k s^{k-2} + \cdots$$

$$\frac{d^2A(s)}{ds^2}|_{s=0} = 2a_2 \implies a_2 = \frac{1}{2!}\frac{d^2A(s)}{ds^2}|_{s=0}$$

$$\frac{d^3A(s)}{ds^3}|_{s=0} = 3!a_3 \implies a_3 = \frac{1}{3!}\frac{d^3A(s)}{ds^3}|_{s=0}$$

$$\vdots$$

$$\frac{d^kA(s)}{ds^k}|_{s=0} = k!a_k \implies a_k = \frac{1}{k!}\frac{d^kA(s)}{ds^k}|_{s=0}$$

Example 1

Consider $\{1\}$, determine the generating function of the series.

Solution

- Let $a_k = \{1\}$

$$A(s) = \sum_{k=0}^{\infty} a_k s^k$$
$$= \sum_{k=0}^{\infty} 1 \times s^k$$
$$= s^0 + s^1 + s^2 + \cdots \quad \textit{Geometric Progression}$$

- The sum of a $G.P$ to infinity is defined by $\frac{a}{1-r}$,

$$\frac{a}{1-r} = \frac{s^0}{1-s} = \frac{1}{1-s}$$

Example 2

Determine the generating functions of :

(i) $a_k = \{0, 0, 0, 1, 1, 1, \cdots\}$
(ii) $a_k = \{1, 1, 1, \cdots\}$
(iii) $a_k = \{\frac{1}{k!}\}$

Solution (i)

$A(s) = \sum_{k=0}^{\infty} a_k s^k$,
Given that $a_k = \{0, 0, 0, 1, 1, 1, \cdots\}$

$$\begin{aligned} A(s) &= 0s^0 + 0s^1 + 0s^2 + 1s^3 + 1s^4 + 1s^5 + \cdots \\ &= s^3 + s^4 + s^5 + \cdots \end{aligned}$$

which is a $G.P$ with $a = s^3$, $r = \frac{s^4}{s^5} = s$

$$\begin{aligned} s_\infty &= \frac{a}{1-r}, \ r \neq 1 \\ &= \frac{s^3}{1-s} \end{aligned}$$

Solution (ii)

- Given that

$$a_k = \{1, 1, 1, \cdots\}, \quad \forall k \implies a_0 = 1, a_1 = 1, a_2 = 1, \cdots$$

- Then the generating function is given by

$$\begin{aligned} A(s) &= \sum_{k=0}^{\infty} a_k s^k \\ &= a_0 + a_1 s + a_2 s^2 + \cdots \\ &= 1 + s + s^2 + s^3 + \cdots \\ &= \frac{1}{1-s}, \quad |s| < 1 \end{aligned}$$

Solution (iii)

$a_k = \{\frac{1}{k!}\}$

$$\begin{aligned} A(s) &= \sum_{k=0}^{\infty} a_k s^k \\ &= \sum_{k=0}^{\infty} \{\frac{1}{k!}\} s^k \\ &= \frac{1}{0!}s^0 + \frac{1}{1!}s^1 + \frac{1}{2!}s^2 + \frac{1}{3!}s^3 + \cdots \\ &= 1 + s^1 + \frac{1}{2}s^2 + \frac{1}{3}s^3 + \cdots \end{aligned}$$

Recall: $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^s = \frac{s^0}{0!} + \frac{s^1}{1!} + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots$$

$$A(s) = e^s$$

- **Definition 2**: Let X be a random variable which assumes non-negative integral values $0, 1, 2, \cdots$ and that

$$P(X = k) = P_k, \quad k = 0, 1, 2, \cdots, \sum_k P_k = 1$$

- If we take a_k to be P_k in equation (1), then the corresponding generating function of the probability, $P(s)$ of the sequence $P_0, P_1, P_2, \cdots$ is called the **probability generating function (p.g.f)** of the random variable X. We have

$$\begin{aligned} P(s) &= \sum_{k=0}^{\infty} P_k s^k = \sum_{k=0}^{\infty} P(X = k) s^k \\ &= E(s^X) \end{aligned}$$

- Where $E(s^X)$ is the expected value of the random variable X which is a function of X. This series converges for $-1 \leq s \leq 1$.
- Next,

$$P'(s) = \frac{dP(s)}{ds} = \sum_{k=1}^{\infty} kP_k s^{k-1} \quad and$$

$$P''(s) = \sum_{k=2}^{\infty} k(k-1)P_k s^{k-2}$$

- Now,

$$E(X) = \sum_{k=1}^{\infty} kP_k = P'(1)$$

$$E[X(X-1)] = \sum_{k} k(k-1)P_k = P''(1)$$

$$
\begin{aligned}
\therefore Var(X) &= E(X^2) - [E(X)]^2 \\
&= E[X(X-1)] + E(X) - [E(X)]^2 \\
&= P''(1) + P'(1) - [P'(1)]^2
\end{aligned}
$$

Remark:
When $s = e^t$, then $P(e^t) = E(e^{tx})$ which is the **moment generating function** of X.

Example 1: Binomial Distribution

Let $X \sim Binom(n, p)$ i.e

$P(X = k) = P_k \binom{n}{k} p^k q^{n-k}, k = 0, 1, 2, \cdots, q + p = 1$ The $p.g.f$ of X is

$$\begin{aligned} P(s) &= \sum_{k=0}^{\infty} P_k s^k \\ &= \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k} s^k \\ &= (q + ps)^n \end{aligned}$$

Mean of X

$$P'(s) = n(q + ps)^{n-1}p$$

$$\therefore E(X) = P'(1) = n(q + p)^{n-1}p = np \quad since \quad q + p = 1$$

Variance of X

$$P''(s) = n(n-1)(q + ps)^{n-2}p^2$$

$$\therefore P''(1) = n(n-1)(q + p)^{n-2}p^2 = n(n-1)p^2$$

$$\begin{aligned} Var(X) &= P''(1) + P'(1) - (P'(1))^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p) = npq \end{aligned}$$

Example 2

Let X be a *r.v* with *p.g.f* $P(s)$. Find the *p.g.f* of $Y = mX + n$, where m and n are integers and $m \neq 0$.

solution

Let $P_Y(s)$ be the *p.g.f* of Y. Then

$$\begin{aligned} P_Y(s) &= E(s^Y) = E(s^{mX+n}) \\ &= E(s^{mX}s^n) \\ &= s^n E(s^{mX}) \\ &= s^n P(s^m) \quad \textit{since} \quad P(s) = E(s^X) \end{aligned}$$

Remarks

1. We can obtain the probabilities P_k from the *p.g.f*. P_k can be found from $P(s)$ by applying differentiation, i.e. $P_k = \frac{1}{k!}\left[\frac{d^k P(s)}{ds^k}\Big|_{s=0}\right]$
2. P_k is also given by the coefficient of s^k in the expansion of $P(s)$ as a power series in s.

Generating Function of Poisson Distribution

- A Poisson distribution with a parameter λ is defined as

$$P(x) = \begin{cases} \frac{e^{-\lambda}}{x!}\lambda^x, x = 1, 2, 3, \cdots \\ 0, \qquad \text{otherwise} \end{cases}$$

- The generating function is given by
$A(s) = \sum_{k=0}^{\infty} a_k P_k$, where $P_k = P(x)$, therefore
$A(s) = \sum s^k P_k$

$$\begin{aligned} A(s) &= \sum s^k P_k \\ &= \sum s^k \frac{e^{-\lambda}}{x!}\lambda^x, \;\; where \;\; k = x \end{aligned}$$

- Thus we've

$$A(s) = \sum_{x=1}^{\infty} s^x \frac{e^{-\lambda}}{x!}\lambda^x,$$

G.F of Poisson Distribution contd...

$$
\begin{aligned}
A(s) &= s^1 \frac{e^{-\lambda}}{1!}\lambda^1 + s^2 \frac{e^{-\lambda}}{2!}\lambda^2 + s^3 \frac{e^{-\lambda}}{3!}\lambda^3 + \cdots \\
&= se^{-\lambda}\lambda + \frac{1}{2}s^2 e^{-\lambda}\lambda^2 + \frac{1}{6}s^3 e^{-\lambda}\lambda^3 + \cdots \\
&= \lambda s\left(e^{-\lambda} + \frac{1}{2}s\lambda e^{-\lambda} + \frac{1}{6}s^2\lambda^2 e^{-\lambda+\cdots}\right) \\
&= \lambda e^{-\lambda} s\left(1 + \frac{1}{2}s\lambda + \frac{1}{6}s^2\lambda^2 + \cdots\right) \\
&= e^{-\lambda}\left[\frac{(s\lambda)^1}{1!} + \frac{(s\lambda)^2}{2!} + \frac{(s\lambda)^3}{3!} + \cdots\right]
\end{aligned}
$$

which is an exponential sum; $\sum \frac{1}{k!} = e^k$

$$A(s) = e^{-\lambda}e^{s\lambda} = e^{\lambda(s-1)}$$

1. Mean

Recall: Let $P(s)$ be the generating function of $f(x)$, then

$$E(X) = \frac{d}{ds}P(s)|_{s=1}$$

$$P(s) = e^{\lambda(s-1)}$$

$$\begin{aligned} P'(s) &= \frac{d}{ds}e^{-\lambda}e^{\lambda s} \\ &= \lambda e^{\lambda(s-1)} \end{aligned}$$

$$\begin{aligned} Mean,\ E(x) &= \frac{d}{ds}P(s)|_{s=1} = P'(1) \\ &= \lambda e^{\lambda(1-1)} \\ &= \lambda \end{aligned}$$

Variance of Poisson Distribution using *P.g.f*

$$\begin{aligned} P''(s) &= \frac{d}{ds} P'(s)|_{s=1} \\ &= \lambda^2 e^{\lambda(s-1)} \end{aligned}$$

$$\begin{aligned} P''(1) &= \lambda^2 e^{\lambda(1-1)} \\ &= \lambda^2 e^0 \\ &= \lambda^2 \\ &= E(X^2) \end{aligned}$$

$$\begin{aligned} Var(X) &= P''(1) + P'(1) - (P'(1))^2 \\ &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda \end{aligned}$$

Sums of Random Variables...(a) Fixed Number

- Let X and Y be two independent non-negative integral-valued r.vs with probability distributions

$$a_k = P(X = k), \quad b_j = P(Y = j)$$

- The sum $Z = X + Y$ is a r.v.
- Let the $p.d.f$s of X and Y be $P_X(S)$ and $P_Y(S)$.
- Let $P_Z(S)$ be the $p.g.f$ of Z. Then

$$\begin{aligned} P_Z(S) &= E(s^Z) = E(s^{X+Y}) = E(s^X s^Y) \\ &= E(s^X)E(s^Y) \ \textit{since } X \textit{ and } Y \textit{ are independent} \\ &= P_X(S)P_Y(S) \end{aligned}$$

This result says that the $p.g.f$ of the sum of 2 independent r.vs X and Y is the **product** of the $p.g.f$ of X and that Y.

- The result extends to the case when we've n independent r.vs $X_1, X_2, \cdots X_n$ i.e. the $p.g.f$ of $s_n = X_1 + X_2 + \cdots + X_n$ is the product of the $p.g.f$s of the individual r.vs.
- If the r.vs $X_1, X_2, \cdots X_n$ are **i.i.d** each with $p.gf$. $P(s)$, then the $p.g.f$ of

$$s_n = X_1 + X_2 + \cdots + X_n \quad is \quad [P(s)]^n$$

Example: Let X and Y be independent Poisson r.v's with parameters λ and μ respectively so that their $p.g.fs$ are $e^{-\lambda(1-s)}$ and $e^{-\mu(1-s)}$. Find the $p.g.f$ of $Z = X + Y$.

Solution

Recall: Given G_Z where $Z = X + Y$, then the value of their $p.g.f$ is the *product* of individual $p.g.f\,s$. Thus,

$$G_Z(s) = G_X(s).G_Y(s)$$

Similarly, if $Z = X - Y$. Then, the joint probability is obtained as

$$G_Z(s) = \frac{G_X(s)}{G_Y(s)}$$

$\therefore$ for $Z = X + Y$

$$\begin{aligned} G_Z(s) &= G_X(s).G_Y(s) \\ &= e^{-\lambda(1-s)} \times e^{-\mu(1-s)} \\ &= e^{-\lambda(1-s)-\mu(1-s)} \\ &= e^{(1-s)(-\lambda-\mu)} \\ &= e^{-(\lambda+\mu)(1-s)} \end{aligned}$$

- The sum of a Poisson distribution is also a Poisson distribution i.e. the sums of Poisson distribution with parameters λ and μ is also a Poisson distribution with parameter $\lambda + \mu$.
- For n independent Poisson r.vs, $s_n = X_1 + X_2 + \cdots + X_n$, s_n will be a Poisson r.v with parameter $\sum_{i=1}^{n} \lambda_{in}$.

- Let $s_N = X_1 + X_2 + \cdots + X_N$, where N is a random number. **Theorem**: Let $X_i, i = 1, 2, 3, \cdots$ be $i.i.d$ r.vs with $P(X_i = k) = P_k$ and $p.g.f$

$$P(s) = \sum_{k=0}^{\infty} P_k s^k, \quad i = 1, 2, 3, \cdots$$

- Let $s_N = X_1 + X_2 + \cdots + X_N$, where N is a random number independent of the $X_i^{'}$s.
- Let the distribution of N be given by $P(N = n) = g_n$ and $p.g.f$ of N be $G(s) = \sum_n g_n s^n$. Then the $p.g.f$, $H(s)$ of s_N is given by the compound function, $G(P(s))$, i.e.

$$H(s) = \sum_j P(s_N = j)s^j = G(P(s))$$

$$Next, \quad H'(s) = P'(s)G'(P(s))$$

$$\begin{aligned}\therefore E(s_N) &= H'(1) = P'(1)G'(P(1)) \\ &= P'(1)G'(1) \\ &= E(X)E(N)\end{aligned}$$

Example: Suppose N has a Poisson distribution with mean, λ and $X_i, i = 1, 2, \cdots, N$ be $i.i.d$ with $p.g.f$ $P(s)$. Find the $p.g.f$ of $s_N = X_1 + X_2 + \cdots + X_N$.

solution: The $p.g.f$ of s_N is $H(s) = G(P(s))$.

- The $p.g.f$ of a Poisson r.v with mean λ is given by

$$G(s) = e^{-\lambda(1-s)}, \quad \therefore H(s) = e^{-\lambda(1-P(s))}$$

$$E(s_N) = E(s_i)E(N) = \lambda E(X)$$

- The distribution having a generating function of the form $\exp\{\lambda(P(s) - 1)\}$ where $P(s)$ is itself a generating function is called a *compound Poisson distribution*.

Bivariate P.g.fs

- Suppose we've a distribution with two r.v's X and Y whose joint probability distribution is given by $P(X = j, Y = k) = P_{jk}$ for $j = 0, 1, 2, \cdots$; $k = 0, 1, 2, \cdots$ and that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_{jk} = 1$$

- Then the joint $p.d.f$ of X and Y will be

$$P(s_1, s_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_{jk} s_1^j s_2^k$$

- Assuming convergence for some values of s_1 and s_2, i.e. $|s_1| < 1$ and $|s_2| < 1$, then

$$P(s_1, s_2) = E\left(s^X s^Y\right)$$

Bivariate P.g.fs contd...

- For $s_1 = s_2$, then $P(s,s) = E\left(s^{X+Y}\right)$
- Further if X and Y are *mutually independent* then

$$P(s_1, s_2) = E(s_1^X)E(s_2^Y)$$

- If, however, X and Y are *identically independently distributed* with a common $p.g.f$ say, $G(s)$ then

$$\begin{aligned} P(s_1, s_2) &= E(s_1^X)E(s_2^Y), \; i.e. \; X = Y \\ &= E(s_1^X)E(s_2^X) \end{aligned}$$

$$\begin{aligned} \therefore P(s_1, s_2) &= G(s).G(s) \\ &= [G(s)]^2 \end{aligned}$$

Joint Distributions: A review

- Let X and Y be two r.v's with joint probability given by

$$P(s_x s_Y) = E(s_X s_Y)$$

- Accordingly the $p.g.f$ of X is given by

$$P(s_1) = E(X) = E(s_1^X) = P(s, 1)$$

- Similarly, the $p.g.f$ of Y is given by

$$P(s_2) = E(s_2^Y) = P(1, s)$$

These are called **marginal distributions**.

- Alternatively, the marginal distributions maybe given as

$$P(X = j, Y = k) = P_{jk}, \ \ where\ j = 0, 1, 2, \cdots ; \ k = 0, 1, 2, \cdots$$

Joint Distributions: A review contd...

- To set up the marginal distributions, we've for X:

$$P(X=j)=\sum_{k=0}^{\infty} P_{jk}, \ \ where\ j=0,1,2,\cdots$$

While the marginal distribution of Y is

$$P(Y=k)=\sum_{j=0}^{\infty} P_{jk}, \ \ where\ k=0,1,2,\cdots$$

Joint Distributions: A review contd...

$$
\begin{aligned}
P(s_1) &= \sum_{j=0}^{\infty} s^j P(X = j) \\
&= \sum_{j=0}^{\infty} s_1^j \sum_{k=0}^{\infty} P(X = j, Y = k) \\
&= \sum_{j=0}^{\infty} s_1^j \sum_{j=0}^{\infty} P_{jk} \\
&= \sum_{j=0}^{\infty} s_1^j P(X = j)
\end{aligned}
$$

P.g.f of Conditional Distributions

- The probability distribution of X given $Y = k$ is given by

$$P_r\{X = j/Y = k\} = P_r\frac{\{X = j, Y = k\}}{P_r(Y = k)}$$

where $P_r(Y = k) > 0$

- The $p.g.f$ of the conditional distribution of X given $Y = k$ is defined by

$$\begin{aligned} P_{X/Y}(s_1) &= \sum_j s_1^j P_r\{X = j/Y = k\} \\ &= \sum_j s_1^j \frac{P_r\{X = j, Y = k\}}{P_r(Y = k)} \end{aligned}$$

- Similarly

$$P_{Y/X}(s_2) = \sum_k s_2^k \frac{P_r\{X = j, Y = k\}}{P_r(X = j)}$$

$E(X)$, $Var(X)$ and *Covariances* of Bivariate Distributions

$$P(s_1, s_2) = \sum_j \sum_k P_{jk} s_1^j s_2^k$$

$$P(s_1) = P(s_1, 1)$$

$$P'(s_1) = \frac{\partial P(s_1, 1)}{\partial s_1}$$

$$E(X) = P'(1)$$

$$E(Y) = P'(1) = \frac{\partial P(1, s_2)}{\partial s_2}|_{s_2=1}$$

The second moment is

$$E(Y(Y-1)) = \frac{\partial^2 P(1, s_2)}{\partial s_2^2}|_{s_2=1}$$

$E(X)$, $Var(X)$ and $Covariances$ of Bivariate Distributions

$$E(X(X-1)) = \frac{\partial^2 P(s_1, 1)}{\partial s_1^2}\Big|_{s_1=1}$$

$$E(XY) = \frac{\partial^2 P(s_1, s_2)}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=1}$$

$$Cov(X, Y) = \delta_{XY} = E(XY) - E(X)E(Y)$$

$$\rho_{XY} = \frac{Cov(X, Y)}{\delta_X \delta_Y}$$

Example: The joint $p.d.f$ of X and Y is

$$P_{jk} = P_r\{X = j, Y = k\} = q^{j+k}p^2,\ j, k = 0, 1, 2, \cdots,\quad q + p = 1$$

Solution

$$\begin{aligned} P(s_1, s_2) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P_{jk} s_1^j s_2^k \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q^{j+k} p^2 s_1^j s_2^k \\ &= p^2 \sum_{j=0}^{\infty}(qs_1)^j \sum_{k=0}^{\infty}(qs_2)^k \\ &= p^2(1 + qs_1 + (qs_1)^2 + \cdots)(1 + qs_2 + (qs_2)^2 + \cdots) \end{aligned}$$

Example contd...

$$P(s_1, s_2) = p^2\left(\frac{1}{1-qs_1}\right)\left(\frac{1}{1-qs_2}\right) = \frac{p^2}{(1-qs_1)(1-qs_2)}$$

<u>p.g.f of X</u>

$$P(s_1) = P(s_1, 1) = \frac{p^2}{(1-qs_1)(1-q)} = \frac{p^2}{(1-qs_1)p} = \frac{p}{(1-qs_1)}$$

$$P'(s_1) = P(s_1, 1) = \frac{pq}{(1-qs_1)^2}$$

$$E(X) = p'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Example contd...

$$p''(s_1) = \frac{2pq^2}{(1 - qs_1)^3}$$

$$p''(1) = \frac{2pq^2}{(1 - q)^3} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$\begin{aligned} Var(X) &= p''(1) + p'(1) - [p'(1)]^2 \\ &= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} \\ &= \frac{q^2}{p^2} + \frac{q}{p} \\ &= \frac{q}{p^2} \end{aligned}$$

<u>p.g.f of Y</u>

$$P(s_2) = P(1, s_2) = \frac{p^2}{(1-q)(1-qs_2)} = \frac{p^2}{(1-qs_2)p} = \frac{p}{(1-qs_2)}$$

$$P'(s_2) = \frac{pq}{(1-qs_2)^2}$$

$$E(Y) = p'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Example contd...

$$p''(s_2) = \frac{2pq^2}{(1-qs_2)^3}$$

$$p''(1) = \frac{2pq^2}{(1-q)^3} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$\begin{aligned}
Var(Y) &= p''(1) + p'(1) - [p'(1)]^2 \\
&= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} \\
&= \frac{q^2}{p^2} + \frac{q}{p} \\
&= \frac{q}{p^2}
\end{aligned}$$

Example contd...

$$P(s_1, s_2) = \frac{p^2}{(1 - qs_1)(1 - qs_2)}$$

$$\frac{\partial P(s_1, s_2)}{\partial s_1} = \frac{p^2 q}{(1 - qs_1)^2(1 - qs_2)}$$

$$\frac{\partial^2 P(s_1, s_2)}{\partial s_1 \partial s_2} = \frac{p^2 q^2}{(1 - qs_1)^2(1 - qs_2)^2}$$

$$\begin{aligned} E(XY) &= \frac{\partial^2 P(s_1, s_2)}{\partial s_1 \partial s_2}\Big|_{s_1 = s_2 = 1} \\ &= \frac{p^2 q^2}{(1 - q)^2(1 - q)^2} = \frac{q^2}{p^2} \end{aligned}$$

Example contd...

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{q^2}{p^2} - \left(\frac{q}{p}\right).\left(\frac{q}{p}\right) = 0$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\delta_X \delta_Y} = 0$$

Tailed or Cumulative Probabilities...(1) Upper Tail

- Let X have a $p.d.f$

$$P_r\{X = k\} = p_k, \quad k = 0, 1, 2, \cdots \quad with \ p.g.f$$

$$P(s) = \sum_{k=0}^{\infty} p_k s^k, \ \ and$$

$$q_k = P\{X > k\} = p_{k+1} + p_{k+2} + \cdots, k = 0, 1, 2, \cdots$$

with generating function given by

$$\phi(s) = \sum_{k=0}^{\infty} q_k s^k, where \ \sum_k q_k \neq 1$$

Required: Express $\phi(s)$ in terms of $P(s)$

- The generating function of the sequence q_k is given by

$$
\begin{aligned}
\phi(s) &= \sum_{k=0}^{\infty} q_k s^k \\
&= q_0 + q_1 s + q_2 s^2 + q_3 s^3 + \cdots \\
&= (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + p_4 + \cdots)s \\
&+ (p_3 + p_4 + p_5 + \cdots)s^2 + (p_4 + p_5 + \cdots)s^3 + \cdots \\
&= p_1 + p_2(1 + s) + p_3(1 + s + s^2) + p_4(1 + s + s^2 + s^3) + \cdots \\
&= p_1 + p_2\left(\frac{1 - s^2}{1 - s}\right) + p_3\left(\frac{1 - s^3}{1 - s}\right) + p_4\left(\frac{1 - s^4}{1 - s}\right) + \cdots \\
&= \frac{p_1(1 - s) + p_2(1 - s^2) + p_3(1 - s^3) + p_4(1 - s^4) + \cdots}{(1 - s)}
\end{aligned}
$$

$$\phi(s) = \frac{(p_1 + p_2 + p_3 + p_4 + \cdots) - (p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \cdots)}{(1-s)}$$

$$= \frac{(p_0 + p_1 + p_2 + p_3 + p_4 + \cdots) - (p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \cdots)}{(1-s)}$$

$$\phi(s) = \frac{1 - p(s)}{(1-s)}$$

Alternatively, we could express $\phi(s)$ in terms of $p(s)$ as follows:

$$q_k = p_{k+1} + p_{k+2} + p_{k+3} + \cdots$$

$$q_{k-1} = p_k + p_{k+1} + p_{k+2} + p_{k+3} \cdots$$

$$\implies q_{k-1} - q_k = p_k$$

Multiplying by s^k and summing over k gives

$$\sum_{k=1}^{\infty} q_{k-1} s^k - \sum_{k=1}^{\infty} q_k s^k = \sum_{k=1}^{\infty} p_k s^k$$

But

$$\phi(s) = \sum_{k=0}^{\infty} q_k s^k \quad and \quad p(s) = \sum_{k=0}^{\infty} p_k s^k$$

Upper tail contd...

$$s\sum_{k=1}^{\infty} q_{k-1}s^{k-1} - \sum_{k=1}^{\infty} q_k s^k = \sum_{k=1}^{\infty} p_k s^k$$

$$s\phi(s) - [\phi(s) - q_0] = p(s) - p_0$$

$$\phi(s)[s-1] = p(s) - [p_0 + q_0]$$

$$\phi(s)[s-1] = p(s) - 1]$$

$$\therefore \phi(s) = \frac{p(s)-1}{s-1} \implies \phi(s) = \frac{1-p(s)}{1-s}, \ |s| < 1$$

Quiz

Let

$$q_k = P_r\{X \leq k\}, k = 0, 1, 2, \cdots$$

and

$$p_k = P_r\{X = k\}, k = 0, 1, 2, \cdots$$

Define

$$p(s) = \sum_{k=0}^{\infty} p_k s^k$$

and

$$\phi(s) = \sum_{k=0}^{\infty} q_k s^k$$

Express $\phi(s)$ in terms of $p(s)$.