Birth-Death Processes - (iii) The Polya Process

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- Here,

$$\lambda_n = \lambda\left[\frac{1+an}{1+\lambda at}\right]$$

and still the death component is zero i.e $\mu_n = 0$ with 'a' being an arbitrary constant.

- Therefore the difference differential equations are:

$$P_n'(t) = -\lambda\left[\frac{1+an}{1+\lambda at}\right]p_n(t) + \lambda\left[\frac{1+a(n-1)}{1+\lambda at}\right]p_{n-1}(t), \quad n \geq 1$$

$$P_0'(t) = \left[\frac{-\lambda}{1+\lambda at}\right]P_0(t), \quad n = 0$$

$$\therefore \sum_{n=1}^{\infty} P_n'(t)s^n = \lambda\Big[-\sum_{n=1}^{\infty}\Big(\frac{1+an}{1+\lambda at}\Big)P_n(t)s^n + \sum_{n=1}^{\infty}\Big(\frac{1+a(n-1)}{1+\lambda at}\Big)P_{n-1}(t)s^n\Big]$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \Big[\frac{\lambda}{1+\lambda at}\Big]\Big\{-\sum_{n=1}^{\infty}(1+an)P_n(t)s^n + \sum_{n=1}^{\infty}(1+a(n-1))P_{n-1}(t)s^n\Big\}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{ -\sum_{n=1}^{\infty} P_n(t)s^n - a\sum_{n=1}^{\infty} nP_n(t)s^n \right.$$
$$\left. + s\sum_{n=1}^{\infty} P_{n-1}(t)s^{n-1} + as\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^{n-1} \right\}$$

But

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$

so

$$\frac{\partial}{\partial t}G(s,t) = \sum_{n=0}^{\infty} P_n'(t)s^n, \quad \frac{\partial}{\partial s}G(s,t) = \sum_{n=1}^{\infty} nP_n(t)s^{n-1}$$

The Polya Process contd...

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\Big\{ -[G(s,t) - P_0(t)] - as\frac{\partial}{\partial s}G(s,t)$$
$$+ sG(s,t) + as^2\frac{\partial}{\partial s}G(s,t)\Big\}$$

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\Big\{ -G(s,t)[1-s]$$
$$+ \frac{\partial}{\partial s}G(s,t)(-as+as^2) - \left(\frac{1+\lambda at}{\lambda}\right)P_0'(t)\Big\}$$

Since

$$P_0'(t) = -\left[\frac{\lambda}{1+\lambda at}\right]P_0(t), \quad \implies P_0(t) = -\left[\frac{1+\lambda at}{\lambda}\right]P_0'(t)$$

The Polya Process contd...

$$\frac{\partial}{\partial t}G(s,t) - P_0'(t) + P_0'(t) = \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\}$$

$$\begin{aligned}
\Longrightarrow \ \frac{\partial}{\partial t}G(s,t) &= \left[\frac{\lambda}{1+\lambda at}\right]\left\{G(s,t)(s-1) + as\frac{\partial}{\partial s}G(s,t)(s-1)\right\} \\
&= \left[\frac{\lambda}{1+\lambda at}\right]\left\{(s-1)G(s,t) + as(s-1)\frac{\partial}{\partial s}G(s,t)\right\}
\end{aligned}$$

- Hence the *Lagrange's linear equations* are

$$\frac{\partial G(s,t)}{\partial t} + \left[\frac{\lambda}{1+\lambda at}\right] as(1-s)\frac{\partial G(s,t)}{\partial s} = -\left[\frac{\lambda}{1+\lambda at}\right](1-s)G(s,t)$$

which can also be expressed as

$$(1+\lambda at)\frac{\partial G(s,t)}{\partial t} + \lambda as(1-s)\frac{\partial G(s,t)}{\partial s} = -\lambda(1-s)G(s,t)$$

- The corresponding **auxiliary equations** are

$$\frac{\partial t}{1+\lambda at} = \frac{\partial s}{\lambda as(1-s)} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

The Polya Process contd...

- Consider

$$\frac{\partial t}{1+\lambda a t} = \frac{\partial s}{\lambda a s(1-s)}$$

- Taking integration on both sides gives

$$\int \frac{\partial t}{1+\lambda a t} = \int \frac{\partial s}{\lambda a s(1-s)}$$

$$\frac{1}{\lambda a} ln(1+\lambda a t) + \frac{1}{\lambda a} ln\ c_2 = \frac{1}{\lambda a} ln\left(\frac{s}{1-s}\right)$$

$$c_2 = \left(\frac{s}{1-s}\right)(1+\lambda a t)^{-1}$$

- Also from

$$\frac{\partial s}{\lambda as(1-s)} = \frac{\partial G(s,t)}{-\lambda(1-s)G(s,t)}$$

we get

$$\int \frac{\partial s}{as} = \int \frac{\partial G(s,t)}{-G(s,t)} \implies \frac{1}{a} ln\ s + ln\ c_1 = -ln\ G(s,t)$$

and

$$\implies c_1 = s^{\frac{1}{a}}[G(s,t)]$$

- The **general solution** is

$$c_1 = \Psi(c_2)$$

$$\implies s^{\frac{1}{a}} G(s,t) = \Psi\left[\frac{s}{1-s}(1+\lambda at)^{-1}\right]$$

The Polya Process contd...

- Using the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$G(s,0) = s^{-\frac{1}{a}} \Psi\left(\frac{s}{1-s}\right)$$

- But by definition

$$\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} P_n(t) s^n \\ &= P_0(t) + P_1(t)s + P_2(t)s^2 + \cdots \end{aligned}$$

$$\begin{aligned} G(s,0) &= P_0(0) + P_1(0)s + P_2(0)s^2 + \cdots \\ &= P_0(0) = 1 \end{aligned}$$

The Polya Process contd...

$$\implies s^{-\frac{1}{a}} \Psi\left(\frac{s}{1-s}\right) = 1$$

and therefore

$$\Psi\left(\frac{s}{1-s}\right) = s^{\frac{1}{a}}$$

which accordingly implies that

$$\Psi(w) = \left(\frac{w}{1+w}\right)^{\frac{1}{a}}$$

for any arbitrary w.

so

$$G(s,t) = s^{-\frac{1}{a}}\Psi\left[\left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}\right] = s^{-\frac{1}{a}}\Psi[w(1+\lambda at)^{-1}]$$

$$= s^{-\frac{1}{a}}\left[\frac{w(1+\lambda at)^{-1}}{1+w(1+\lambda at)^{-1}}\right]^{\frac{1}{a}}$$

$$= s^{-\frac{1}{a}}\left[\frac{\frac{s}{1-s}(1+\lambda at)^{-1}}{1+\frac{s}{1-s}(1+\lambda at)^{-1}}\right]^{\frac{1}{a}}$$

$$= s^{-\frac{1}{a}}s^{\frac{1}{a}}\left[\frac{1}{(1+\lambda at)(1-s)+s}\right]^{\frac{1}{a}}$$

$$= \left[\frac{1}{1+\lambda at-\lambda ats}\right]^{\frac{1}{a}}$$

and

$$
\begin{aligned}
G(s,t) &= (1+\lambda at)^{-\frac{1}{a}}\left[1-\frac{\lambda at}{1+\lambda at}s\right]^{-\frac{1}{a}} \\
&= (1+\lambda at)^{-\frac{1}{a}}\frac{[1+\lambda at-\lambda ats]^{-\frac{1}{a}}}{(1+\lambda at)^{-\frac{1}{a}}} \\
&= (1+\lambda at)^{-\frac{1}{a}}\left[1+\binom{-\frac{1}{a}}{1}\left(\frac{-\lambda at}{1+\lambda at}s\right)\right. \\
&\quad \left. +\cdots+\binom{-\frac{1}{a}}{n}\left(\frac{-\lambda at}{1+\lambda at}s\right)^{n}+\cdots\right]
\end{aligned}
$$

so that $P_n(t)$ is the coefficient of s^n in the above expansion for $G(s,t)$.

$$
\begin{aligned}
P_n(t) &= (1+\lambda at)^{-\frac{1}{a}}\binom{-\frac{1}{a}}{n}\left(\frac{-\lambda at}{1+\lambda at}\right)^{n} \\
&= (1+\lambda at)^{-\frac{1}{a}-n}\binom{-\frac{1}{a}}{n}(-\lambda at)^{n}
\end{aligned}
$$

Mean

$$G(s,t) = [(1+\lambda at) - \lambda ats]^{-\frac{1}{a}}$$

$$\begin{aligned} G'(s,t) &= \left(\frac{-1}{a}\right)[(1+\lambda at) - \lambda ats]^{-\frac{1}{a}-1}(-\lambda at) \\ &= \lambda t[(1+\lambda at) - \lambda ats]^{-\frac{1}{a}-1} \end{aligned}$$

$$\begin{aligned} E(n) &= G'(1,t) \\ &= \lambda t[1+\lambda at - \lambda at]^{-\frac{1}{a}-1} \\ &= \lambda t \end{aligned}$$

Variance

$$G''(s,t) = \lambda t(-\frac{1}{a} - 1)[(1+\lambda at) - \lambda ats]^{-\frac{1}{a}-2}(-\lambda at)$$

$$\begin{aligned} G''(1,t) &= \lambda t(-\frac{1}{a} - 1)(1)(-\lambda at) \\ &= \lambda t(\frac{-1-a}{a})(1)(-\lambda at) \\ &= (\lambda t)^2(1+a) \end{aligned}$$

$$\begin{aligned} Var(n) &= G''(1,t) + G'(1,t) - [G'(1,t)]^2 \\ &= (\lambda t)^2(1+a) + \lambda t - (\lambda t)^2 \\ &= (\lambda t)^2 + (\lambda)^2 a(t)^2 + \lambda t - (\lambda t)^2 \\ &= \lambda t(\lambda at + 1) \end{aligned}$$