# The Linear Birth-Death Processes

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- Here, $\lambda_n = n\lambda$ and $\mu_n = n\mu$
- The difference differential equations are

$$P_n'(t) = -n(\lambda+\mu)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \;\; n \geq 1$$

and $$P_0'(t) = \mu P_1(t), \quad n = 0$$

$$\sum_{n=1}^{\infty} P_n'(t)s^n = -(\lambda+\mu)\sum_{n=1}^{\infty} nP_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)s^n + \mu\sum_{n=1}^{\infty}(n+1)P_{n+1}(t)s^n$$

$$\frac{\partial G(s,t)}{\partial t} - P_0'(t) = -(\lambda+\mu)s\frac{\partial G(s,t)}{\partial s} + \lambda s^2\frac{\partial G(s,t)}{\partial s} + \mu\Big(\frac{\partial G(s,t)}{\partial s} - \mu P_0(t)\Big)$$

$$\frac{\partial G(s,t)}{\partial t} = (1-s)(\mu-\lambda s)\frac{\partial G(s,t)}{\partial s}$$

- The *Lagrange's linear equation* is

$$\frac{\partial G(s,t)}{\partial t} - (1-s)(\mu-\lambda s)\frac{\partial G(s,t)}{\partial s} = 0$$

- The *auxiliary equations are*

$$\frac{\partial t}{1} = \frac{\partial s}{-(1-s)(\mu-\lambda s)} = \frac{\partial G(s,t)}{0}$$

From

$$\frac{\partial t}{1} = \frac{\partial G(s,t)}{0}$$

we have

$$\int \frac{\partial t}{1} = \int \frac{\partial G(s,t)}{0}$$

$$\int \partial G(s,t) = \int 0 \partial t \implies G(s,t) = c_1$$

From

$$\frac{\partial t}{1} = \frac{\partial s}{-(1-s)(\mu - \lambda s)}$$

$$t + c = \frac{1}{\mu - \lambda} ln\left(\frac{\mu - \lambda s}{1 - s}\right)$$

# Simple Birth-Death Process contd...

$$\frac{\mu - \lambda s}{1 - s} = e^{(\mu-\lambda)t} + e^{(\mu-\lambda)c}$$

$$\left(\frac{\mu - \lambda s}{1 - s}\right) e^{-(\mu-\lambda)t} = e^{(\mu-\lambda)c} = c_2$$

- The general solution is

$$c_1 = \Psi(c_2), \quad i.e$$

$$G(s,t) = \Psi\left[\left(\frac{\mu - \lambda s}{1 - s}\right) e^{-(\mu-\lambda)t}\right]$$

- Using the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

# Simple Birth-Death Process contd...

$$G(s,0) = \Psi\left(\frac{\mu - \lambda s}{1-s}\right)$$

But

$$\begin{aligned} G(s,0) &= P_0(0) + P_1(0)s + P_2(0)s^2 + \cdots \\ &= P_1(0)s = s \quad since \quad P_1(0) = 1 \end{aligned}$$

$$\Psi\left(\frac{\mu - \lambda s}{1-s}\right) = s$$

Let

$$w = \frac{\mu - \lambda s}{1-s} \Leftrightarrow s = \frac{\mu - w}{\lambda - w}$$

$$\Psi(w) = \frac{\mu - w}{\lambda - w}$$

# Simple Birth-Death Process contd...

$$
\begin{aligned}
G(s,t) &= \Psi(w\ e^{-(\mu-\lambda)t}) \\
&= \frac{\mu - w\ e^{-(\mu-\lambda)t}}{\lambda - w\ e^{-(\mu-\lambda)t}} \\
&= \frac{\mu - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}}{\lambda - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}} \\
&= \frac{\mu(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}}{\lambda(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}} \\
&= \frac{\mu(1-e^{-(\mu-\lambda)t}) - (\mu-\lambda e^{-(\mu-\lambda)t})s}{[\lambda-\mu e^{-(\mu-\lambda)t}] - \lambda[1-e^{-(\mu-\lambda)t}]s}
\end{aligned}
$$

$$
G(s,t) = \frac{\frac{\mu(1-e^{-(\mu-\lambda)t})}{\lambda-\mu\ e^{-(\mu-\lambda)t}} - \frac{\mu-\lambda e^{-(\mu-\lambda)t}}{\lambda-\mu e^{-(\mu-\lambda)t}} s}{1 - \lambda\left[\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right] s}
$$

# Simple Birth-Death Process contd...

$$
\begin{aligned}
P_n(t) &= \textit{Coefficient of } s^n \\
&= \left[\frac{\mu(1-e^{-(\mu-\lambda)t})}{\lambda-\mu\ e^{-(\mu-\lambda)t}}\right]\left[\lambda\left(\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right)\right]^n \\
&- \left[\left(\frac{\mu-\lambda e^{-(\mu-\lambda)t}}{\lambda-\mu e^{-(\mu-\lambda)t}}\right)\right]\left[\lambda\left(\frac{1-e^{-(\mu-\lambda)t}}{\lambda-e^{-(\mu-\lambda)t}}\right)\right]^{n-1}
\end{aligned}
$$