Nonlinear programming – Exercise session 2
Optimality conditions in the constrained case

1. Among the points on the parabola with equation $y = \frac{1}{5}(x-3)^2$, we want to determine which is closest to the point $(x, y) = (3, 2)$, which we formulate as the following optimization problem

$$\min f(x, y) = (x-3)^2 + (y-2)^2 \text{ such that } (x-3)^2 = 5y$$

   (a) Why is the *square* of the distance to the point $(3, 2)$ used as the objective function?
   (b) Find all the points satisfying the (necessary) first-order optimality condition (taking care to check the gradient independence condition), and derive the solution to the problem.
   Answer again, this time using the parabola with equation $y = \frac{1}{3}(x-3)^2$.
   (c) Could this problem be solved by eliminating x using the constraint and then solving the resulting unconstrained problem? And by eliminating y ?

2. On the $\mathbb{R}^2$ domain defined by the equations

$$\begin{cases} x_1 \geq -\frac{3}{2} \\ x_1 \geq x_2 \\ x_1^2 + x_2^2 \leq 4 \end{cases},$$

   find *all* local minima (with justification) of the functions
   (a) $f(x) = (x_1 - 1)^2$
   (b) $f(x) = -((x_1 - 1)^2 + (x_2 - 1)^2)$
   (c) $f(x) = (x_1 - 1)^2 + 2(x_2 - 1)^2$.

3. Consider the optimization problem

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^3 \geq x_2 \\ & x_2 \geq 0 \end{aligned}$$

   (a) What is the optimal solution to this problem?
   (b) Write and solve the KKT conditions for this problem. Comment.

4. (a) Prove that if x^* is a point satisfying the KKT conditions and the gradient independence condition, then the corresponding vector of Lagrange multipliers λ^* is unique.
   (b) Show that the reformulation of an equality $g(x) = 0$ in the equivalent form $g(x)^2 = 0$ is not a good idea from the point of view of optimal conditions.

5. **Optimality conditions for some standard problems.**

   Derive, for each of the problem classes listed below, the conditions for first-order optimality. Specify any necessary assumptions.

   When can these conditions be simplified? When can they be solved analytically? Where possible, formulate the conditions obtained in vector or matrix form.

   These problem classes are defined on the basis of the following data:

- c is a vector of $\mathbb{R}^n$,
- b, l and u are vectors of $\mathbb{R}^m$ (and we suppose $l < u$)
- A is a matrix of size $m \times n$ whose lines are named $\{a_i^T\}_{1 \le i \le m}$
- Q is a symmetric square matrix of size n.

The functions f and g are defined everywhere on $\mathbb{R}^n$ and the n functions h_1, h_2, …, h_n are defined on $\mathbb{R}$; all are assumed to be sufficiently differentiable. Inequalities between vectors are to be interpreted component by component.

(a) Unconstrained quadratic optimization

$$\min x^2 + 4xy + 5y^2 + 3x - 5y \text{ then in general } \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x$$

and quadratic optimization under linear equality constraints

$$\min x^2+4xy+5y^2+3x-5y \text{ such that } x-y=1 \text{ then in general } \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x \text{ such that } Ax = b$$

(where $Ax = b$ is equivalent to m equality constraints $a_i^T x = b_i$ for $1 \le i \le m$).

(b) Linear optimization on the boundary of a ball of radius R and center u.

$$\min 3x + 4y \text{ such that } ||(x, y) - (2, -1)|| = 5 \text{ then in general } \min_{x \in \mathbb{R}^n} c^T x \text{ such that } ||x - u|| = R$$

then on the whole ball

$$\min 3x + 4y \text{ such that } ||(x, y) - (2, -1)|| \le 5 \text{ then in general } \min_{x \in \mathbb{R}^n} c^T x \text{ such that } ||x - u|| \le R$$

(a reformulation may be necessary).

(c) Non-linear optimization under linear equality constraints

$$\min -x^2 - y^3 - z^2 \text{ such that } x + y = 1 \text{ et } y + z = 1 \text{ then in general } \min_{x \in \mathbb{R}^n} f(x) \text{ such that } Ax = b\ .$$

Can you give a geometrical interpretation of this optimal condition, referring to the gradient of f and the kernel of the matrix A (defined as $\ker A = \{x \mid Ax = 0\}$)?

(d) Classic linear optimization (i.e. on a polytope)

$$\min 3x + 4y \text{ such that } 2x + y \ge 3, x + 2y \ge 3 \text{ et } 2x + 2y \le 5$$

$$\text{then in general } \min_{x \in \mathbb{R}^n} c^T x \text{ such that } Ax \ge b\ .$$

(e) Non-linear optimization on positive variables

$$\min_{x \in \mathbb{R}^n} (x - 1)^3 + y^2 - y \text{ such that } x \ge 0 \text{ et } y \ge 0 \text{ then in general } \min_{x \in \mathbb{R}^n} f(x) \text{ such that } x \ge 0\ .$$

Rewrite the conditions as follows : $x_i = 0 \Rightarrow \ldots$ and $x_i > 0 \Rightarrow \ldots$

(f) Linear optimization on an ellipsoid defined by a Q matrix (assumed positive definite)

$$\min 3x + 4y \text{ such that } x^2 + 2xy + 2y^2 \le 1 \text{ then in general } \min_{x \in \mathbb{R}^n} c^T x \text{ such that } x^T Q x \le 1$$

Assuming $Q = L^T L$ with L an invertible matrix (which is always possible if Q is positive definite), establish an equivalence between this problem and one of the previous ones.