Nonlinear programming – Exercise session 1
Optimality conditions in the unconstrained case

1. Consider the function $f(x_1, x_2) = (x_1 - 1)^2 + 4(x_2 - 1)^3 - 3x_2^2$. Find the stationary points of this function and determine their nature. Does the function have a global minimum? a global maximum?

2. For the following optimization problems, (i) compute the gradient and the Hessian matrix of the objective function, (ii) identify the stationary points, (iii) eliminate those that do not satisfy the necessary conditions for optimality, and (iv) identify those that satisfy the sufficient conditions for optimality.

   (a) $\min x_1^1 + x_2^2$.

   (b) $\min \frac{1}{3}x_1^3 + x_2^3 - x_1 - x_2$.

   (c) $\min x^2 + \frac{1}{x - \frac{3}{2}}$.

   (d) $\min x_1^6 - 3x_1^4x_2^2 + 3x_1^2x_2^4 - x_2^6$.

3. Consider the function
$$f(x_1, x_2) = (x_1 - 1)^2 + \lambda({x_1}^2 - x_2)^2$$
for $\lambda \in \mathbb{R}$. Find the stationary points of f and discuss their nature in terms of λ.

4. Determining the local minima for each of the functions below, justifying carefully, and identifying which of these are global and which are strict :
$$\min_{x \in \mathbb{R}} f(x).$$

   (a) $f(x) = |x + 2| + |x - 3|$.

   (b) $f(x) = x \sin x + \cos x$.

   (c) $f(x) = \begin{cases} x^2 - 4x + 2 \text{ when } x < 1 \\ x^3 - 5x + 3 \text{ when } x \geq 1 \end{cases}$.

   (d) $f(x) = \max\{x^2 - 3x + 3, 3x^2 + x - 3\}$.

5. Consider the quadratic function $f(x) = \frac{1}{2}x^TQx - c^Tx$ with Q symmetrical. Under what condition does this function have a stationary point? A local minimum? A local maximum? A stationary point but no local minimum or maximum?

6. If possible, find a function f of two variables and a point point x^* that maximizes f and for which $\nabla^2 f(x^*) \succcurlyeq 0$. Same question with $\nabla^2 f(x^*) \succ 0$.

7. Demonstrate the following assertions by finding a counterexample with f a function of $\mathbb{R}^2 \to \mathbb{R}$ and $x^* = (1, 1)$.

   (a) The condition '$\nabla f(x^*) = 0$' is not sufficient for x^* to be a minimum.

   (b) The condition '$\nabla^2 f(x^*) \succcurlyeq 0$ and $\nabla f(x^*) = 0$' is not sufficient for x^* to be a minimum.

   (c) The condition '$\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$' is not necessary for x^* to be a minimum.

   (d) The condition '$\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$' is not necessary for x^* to be a strict minimum.