Solution to Quiz 1

The integral we are trying to approximately solve in this task is

$$I = \int_0^2 \int_0^2 ... \int_0^2 \exp\left(-\sum_{i=1}^{d} x_i\right) dx_1 dx_2 ... dx_d. \tag{1}$$

Let the function $g(\boldsymbol{x})$ be given by

$$g(\boldsymbol{x}) = \exp\left(-\sum_{i=1}^{d} x_i\right). \tag{2}$$

Let $\boldsymbol{X} = [X_1, X_2, ...X_d]$ be a random vector of length d, where each element is generated i.i.d. according to the uniform distribution

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

We now perform the following experiment: We generate n outcomes of the random vector $\boldsymbol{X}$, denoted by $\boldsymbol{x}_1$, $\boldsymbol{x}_2$,...,$\boldsymbol{x}_n$, plug-in these outcomes into the function $g(\boldsymbol{x})$, perform a simple arithmetic mean of the obtained results, and denote the result of this arithmetic mean as I_a. Thereby, I_a is given by

$$I_a = \frac{1}{n} \sum_{k=1}^{n} g(\boldsymbol{x}_k). \tag{4}$$

Each time that we perform the above experiment in an i.i.d. fashion, we will obtain a different value for I_a. Hence, we can consider that I_a is itself a random variable, with mean value given by

$$E\{I_a\} = \frac{1}{n} \sum_{k=1}^{n} E\{g(\boldsymbol{X})\} = E\{g(\boldsymbol{X})\} = \frac{1}{2^d} \int_0^2 \int_0^2 ... \int_0^2 \exp\left(-\sum_{i=1}^{d} x_i\right) dx_1 dx_2 ... dx_d. \tag{5}$$

We see that $E\{I_a\}$ is 2^d times smaller than the integral that we are trying to solve in this Quiz, given by (1). Hence, we can approximately solve the integral in (1) if the outcome of the experiment, I_a, defined in (4), is multiplied by 2^d. The mean squared error that we will make by our approximation is then given by

$$MSE = E\left\{\left(2^d I_a - I\right)^2\right\} = 2^{2d} E\left\{\left(I_a - \frac{1}{2^d} I\right)^2\right\} = 2^{2d} \frac{1}{n} VAR(I_a), \tag{6}$$

where the last equality has been proved in the lecture notes. On the other hand,

$$VAR(I_a) = E\{I_a^2\} - E^2\{I_a\}, \tag{7}$$

where it is easy to obtain that

$$E\{I_a^2\} = \frac{1}{2^{2d}}\left(1 - e^{-4}\right)^d \tag{8}$$

and

$$E^2\{I_a\} = \frac{1}{2^{2d}}\left(1 - e^{-2}\right)^{2d}. \tag{9}$$

Plugging (9) and (8) into (7), and then plugging (7) into (6), we obtain the MSE as

$$MSE = 2^{2d}\frac{1}{n}\frac{1}{2^{2d}}\left(\left(1 - e^{-4}\right)^d - \left(1 - e^{-2}\right)^{2d}\right) = \frac{1}{n}\left(\left(1 - e^{-4}\right)^d - \left(1 - e^{-2}\right)^{2d}\right) \tag{10}$$

Plugging $d = 49$ and $n = 100$ in (10), we obtain the desired numerical value of the MSE as 0.004.