Lecture 12: Correlation

Content

- Covariance
- Correlation (aka Pearson product-moment correlation coefficient)
- Relationship between Pearson correlation and linear regression

Covariance

- To proceed further with our analysis we will use the concept of **covariance**, which we have already seen
- It expresses the degree in which the variation of a random variable is connected to the variation of another random variable
- It is defined as follows:
  - Given two random variables X and Y

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

Covariance – graphically

Source : *https://www.geeksforgeeks.org/mathematics-covariance-and-correlation*

About the covariance - 1

- We notice that:
  - The covariance of a random variable with itself is the variance:

$$Cov(X, X) = Var(X)$$

  - There is a similar property as for the variance

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

since:

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] =$$

$$= E(XY) - E(XE(Y)) - E(E(X)Y) + E(E(X)E(Y)) =$$

$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) =$$

$$= E(XY) - E(X)E(Y)$$

QED.

About the covariance - 2

- The covariance is symmetric:

$$Cov(X,Y) = Cov(Y,X)$$

- The covariance is linear with respect to multiplications by constants:

$$(\forall a,b \in \mathbb{R}) \;\; Cov(aX,bY) = abCov(X,Y)$$

- If $e \sim N(0,\sigma)$, $Cov(X,e) = 0$

$$Cov(X,e) = E(Xe) - E(X)E(e)$$

Since X and e are independent

$$E(Xe) = E(X)E(e)$$

Moreover, $e \sim N(0,\sigma)$

$$E(e) = 0$$

QED.

Pearson Correlation Coefficient

- AKA Pearson product-moment correlation coefficient or just correlation coefficient
- It expresses the linear correlation between two random variables
- It is defined as follows:
  - Given two random variables X and Y

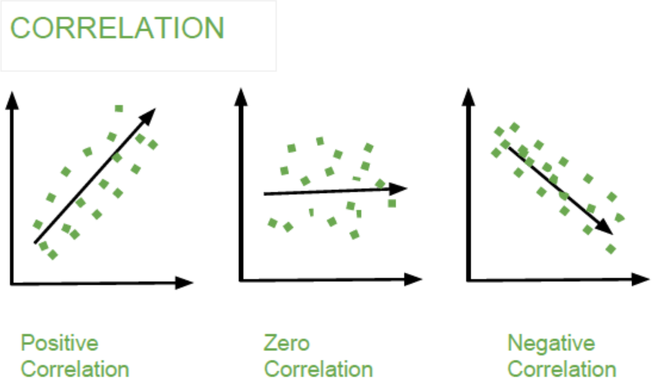
$$r_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

  - Where:

$$\sigma_Z = \sqrt{Var(Z)}$$

For the time being we intentionally ignore the difference between sample and population.

Pearson Correlation Coefficient – graphically



Source : `https://www.geeksforgeeks.org/mathematics-covariance-and-correlation`

About the Pearson Correlation Coefficient

- The Pearson correlation coefficient is also often expressed as:

$$r_{X,Y} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2 \sum_{i=1}^{n}(y_i - \overline{y})^2}}$$

- It is symmetric: $r_{X,Y} = r_{Y,X}$
- It is invariant with respect to multiplications by, and additions of constants: $(\forall a, b, c, d \in \mathbb{R}, b \neq 0, d \neq 0) \;\; r_{X,Y} = r_{(a+bX),(c+dY)}$
- It ranges from -1 to 1: $-1 \leq r_{X,Y} \leq 1$ $r_{X,Y} = 1$ means perfect linear relationship (all points lie on a monotonically increasing line)
  $r_{X,Y} = -1$ means perfect opposite linear relationship (all points lie on a monotonically decreasing line)
  $r_{X,Y} = 0$ means no linear relationship between X and Y

Back to Linear Regression (1/2)

- We now focus our attention to the case of the case of the linear regression
- Suppose we have two phenomena that we want to measure, X and Y
- Let us assume
  - that there is a linear relationship between them
  - that I can express the data I collect as:

$$\boldsymbol{y} = \boldsymbol{\theta}_0 + X\boldsymbol{\theta}_1 + \boldsymbol{\epsilon}$$

  - where ϵ is a stationary gaussian process $N(0, \sigma^2)$
- We know the solution that minimizes the square error

Back to Linear Regression (2/2)

- From this solution we have extracted the coefficient of determination
$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$$
- Where:
  - $SS_{res} = \sum_i (y_i - \hat{y}_i)^2$
  - SS_{res} is the distance between the reality and the 1-degree best approximation, that is, the OLS model
- and
  - $SS_{tot} = \sum_i (y_i - \overline{y})^2$
  - SS_{tot} is the distance between the reality and the 0-degree best approximation, that is, the mean
- I want to know the relationship between R^2 and the correlation coefficient between X and Y, $r_{X,Y}$

Our goal – understanding R and r

- We focus on 1D
- We are now going to prove a fundamental point.
- Under the assumption that the noise is gaussian and centered in 0, in a linear regression:

$$R^2 = r^2_{X,Y}$$

$R^2 = r^2_{X,Y}$ (1/4)

- Since

$$\hat{y} = \theta_0 + \theta_1 x$$

- we have from above (see page 7) that:

$$r_{X,Y} = r_{\hat{Y},Y}$$

- We define now the explained sum of squares (ESS)
  - $ESS = \sum_i (\hat{y}_i - \overline{y})^2$
  - ESS is the additional knowledge we get on the random variable using a polynomial of degree 1 vs. using a polynomial of degree 0
- We will now prove that **under our hypotheses**:

$$ESS + SS_{res} = SS_{tot}$$

- We start from:

$$(y_i - \overline{y}) = (y_i - \hat{y}_i) + (\hat{y}_i - \overline{y})$$

- which we square:

$$(y_i - \overline{y})^2 = (y_i - \hat{y}_i)^2 + 2(y_i - \hat{y}_i)(\hat{y}_i - \overline{y}) + (\hat{y}_i - \overline{y})^2$$

- and then we sum:

$$\sum_i (y_i - \overline{y})^2 = \sum_i (y_i - \hat{y}_i)^2 + \sum_i 2(y_i - \hat{y}_i)(\hat{y}_i - \overline{y}) + \sum_i (\hat{y}_i - \overline{y})^2$$

Source with modifications:
https://en.wikipedia.org/wiki/Explained_sum_of_squares

- Now we focus on:

$$\sum_i 2(y_i - \hat{y}_i)(\hat{y}_i - \overline{y}) = 2\sum_i (y_i - \hat{y}_i)(\hat{y}_i - \overline{y})$$

- and we want to prove that it is 0, that is $\sum_i (y_i - \hat{y}_i)(\hat{y}_i - \overline{y}) = 0$; considering:

$$y_i = \hat{y}_i + \epsilon_i$$

$$E(y_i) = E(\hat{y}_i + \epsilon_i) = E(\hat{y}_i) + E(\epsilon_i) = E(\hat{y}_i)$$

because ϵ is a stationary gaussian process $N(0, \sigma^2)$

Source with modifications:

`https://en.wikipedia.org/wiki/Explained_sum_of_squares`

$[R^2 = r^2_{X,Y}]$ – $ESS + SS_{res} = SS_{tot}$ (3/6)

- We can build a system:

$$\begin{cases} \hat{y}_i = \theta_0 + \theta_1 x_i \\ \overline{y} = \theta_0 + \theta_1 \overline{x} \end{cases}$$

- from which we deduce by subtraction:

$$\hat{y}_i - \overline{y} = \theta_1 (x_i - \overline{x})$$

- remembering that:

$$\theta_1 = \frac{Cov(x,y)}{Var(x)} = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n} (x_i - \overline{x})^2}$$

Source with modifications:
https://en.wikipedia.org/wiki/Explained_sum_of_squares

- So:

$$\sum_i (y_i - \hat{y}_i)(\hat{y}_i - \overline{y}) = \sum_i (y_i - \hat{y}_i)(\theta_1(x_i - \overline{x})) =$$

$$= \theta_1 \sum_i (y_i - \hat{y}_i)(x_i - \overline{x})$$

- Now, let's consider that:

$$(y_i - \hat{y}_i) = y_i - \hat{y}_i + \overline{y} - \overline{y} = (y_i - \overline{y}) - (\hat{y}_i - \overline{y}) =$$

$$= (y_i - \overline{y}) - \theta_1(x_i - \overline{x})$$

- Substituting $(y_i - \hat{y}_i)$ above we get:

$$\theta_1 \sum_i (y_i - \hat{y}_i)(x_i - \overline{x}) = \theta_1 \sum_i [(y_i - \overline{y}) - \theta_1(x_i - \overline{x})](x_i - \overline{x})$$

Source with modifications:
https://en.wikipedia.org/wiki/Explained_sum_of_squares

- We can conclude:

$$\theta_1 \sum_i [(y_i - \overline{y}) - \theta_1(x_i - \overline{x})](x_i - \overline{x}) =$$

$$= \theta_1 [\sum_i (y_i - \overline{y})(x_i - \overline{x}) - \sum_i \theta_1 (x_i - \overline{x})(x_i - \overline{x})] =$$

$$= \theta_1 [\sum_i (y_i - \overline{y})(x_i - \overline{x}) - \sum_i \frac{\sum_j (x_j - \overline{x})(y_j - \overline{y})}{\sum_j (x_j - \overline{x})^2} (x_i - \overline{x})^2] =$$

Source with modifications:
https://en.wikipedia.org/wiki/Explained_sum_of_squares

- And simplifying what is in [•]:

$$\sum_i (y_i - \overline{y})(x_i - \overline{x}) - \sum_i \frac{\sum_j (x_j - \overline{x})(y_j - \overline{y})}{\sum_j (x_j - \overline{x})^2}(x_i - \overline{x})^2 =$$

$$= \sum_i (y_i - \overline{y})(x_i - \overline{x}) - \sum_j (x_j - \overline{x})(y_j - \overline{y}) \sum_i \frac{(x_i - \overline{x})^2}{\sum_j (x_j - \overline{x})^2} =$$

$$= \sum_i (x_i - \overline{x})(y_i - \overline{y}) - \sum_j (x_j - \overline{x})(y_j - \overline{y}) \frac{\sum_i (x_i - \overline{x})^2}{\sum_j (x_j - \overline{x})^2} =$$

$$= \sum_i (x_i - \overline{x})(y_i - \overline{y}) - \sum_j (x_j - \overline{x})(y_j - \overline{y}) = 0$$

QED.

Source with modifications:

https://en.wikipedia.org/wiki/Explained_sum_of_squares

- Now we know that, under the assumption to deal with a Gaussian noise centered in 0 we have:

$$ESS + SS_{res} = SS_{tot}$$

- Under this hypothesis we have:

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} = \frac{SS_{tot} - SS_{res}}{SS_{tot}} = \frac{ESS}{SS_{tot}}$$

$R^2 = r^2_{X,Y}$ (3/4)

- We now consider the square of $r_{X,Y} = r_{\hat{Y},Y}$

$$r^2_{\hat{Y},Y} = \left(\frac{Cov(\hat{Y},Y)}{\sqrt{Var(Y)Var(\hat{Y})}}\right)^2 = \frac{Cov(\hat{Y},Y)Cov(\hat{Y},Y)}{Var(Y)Var(\hat{Y})} =$$

$$= \frac{Cov(\hat{Y},\hat{Y}+\epsilon)Cov(\hat{Y},\hat{Y}+\epsilon)}{Var(Y)Var(\hat{Y})} =$$

$$= \frac{(Cov(\hat{Y},\hat{Y})+Cov(\hat{Y},\epsilon))(Cov(\hat{Y},\hat{Y})+Cov(\hat{Y},\epsilon))}{Var(Y)Var(\hat{Y})} =$$

$$= \frac{Cov(\hat{Y},\hat{Y})Cov(\hat{Y},\hat{Y})}{Var(Y)Var(\hat{Y})}$$

Source with modifications:

https://economictheoryblog.com/2014/11/05/proof/

- But we know that $Cov(\hat{Y}, \hat{Y}) = Var(\hat{Y})$, therefore we get that

$$r^2_{X,Y} = \frac{Var(\hat{Y})Var(\hat{Y})}{Var(Y)Var(\hat{Y})} = \frac{Var(\hat{Y})}{Var(Y)} =$$

$$= \frac{\frac{\sum_i (\hat{y_i} - \bar{\hat{y}})^2}{n}}{\frac{\sum_i (y_i - \bar{y})^2}{n}} = \frac{\sum_i (\hat{y_i} - \bar{\hat{y}})^2}{\sum_i (y_i - \bar{y})^2} = \frac{ESS}{SS_{tot}}$$

since we have already proven that $\bar{y} = \bar{\hat{y}}$

QED

Source with modifications:

https://economictheoryblog.com/2014/11/05/proof/

Comment on $R^2 = r_{X,Y}^2$

- This is a major result
- It is the center of our subsequent investigation, in the case of normality of error we can model, interconnect, and understand relationships in an easy way
- The next question is on how the slope of the regression line (θ_1) relates to the correlation coefficient $r_{X,Y}$

$r_{X,Y}$ and θ_1

- We know that:

$$\theta_1 = \frac{Cov(X,Y)}{Var(X)}$$

- And that:

$$r_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

- Therefore:

$$\theta_1 Var(X) = r_{X,Y} \sigma_X \sigma_Y$$

- We can then conclude that:

$$\theta_1 = \frac{\sigma_X \sigma_Y}{Var(X)} r_{X,Y}$$

$$r_{X,Y} = \frac{Var(X)}{\sigma_X \sigma_Y} \theta_1$$

Comment on $r_{X,Y} \sim \theta_1$

- $r_{X,Y}$ and θ_1 are therefore directly and monotonically proportional
- It means that a positive relationships implies a positive slope and viceversa

General remark

- Right now we work with samples of larger populations of data
- We measure properties of samples, like mean, standard deviation, covariance, correlation coefficient
- All these properties are also random variable and have a distribution
- Our question is therefore, what kind of distribution is the one of the correlation coefficient
- Knowing its distribution allows us to understand the relationships existing between the variables it connect