Optimisation – Exercise Sessions 4 and 5

1 Duality I: formulation, complementary slackness, solution

Geometric form

$$\begin{array}{lll} \text{minimize} & c^T x & \\ & Ax & \geq b. \quad \text{(primal)} \end{array} \qquad \begin{array}{lll} \text{maximize} & b^T y & \\ & A^T y & = c, \quad \text{(dual)} \\ & y & \geq 0. \end{array}$$

Standard form

$$\begin{array}{lll} \text{minimize} & c^T x & \\ & Ax & = b \quad \text{(primal)} \\ & x & \geq 0 \end{array} \qquad \begin{array}{llll} \text{maximize} & b^T y & & \\ & A^T y & \leq & c \quad \text{(dual)} \end{array}$$

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1. Find the dual of problems

$$\begin{array}{llll} \min_x & 3x_1 - 9x_2 & & \\ \text{such that} & 2x_1 - x_2 + x_3 & \geq & 3, \\ & x_1 - x_2 - 3x_3 & \geq & 2, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

$$\begin{array}{llll} \min_x & 3x_1 - 9x_2 & & \\ \text{such that} & 4x_1 - x_2 + x_3 & = & 3, \\ & x_1 - x_2 - 6x_3 & \geq & 2, \\ & x_1 - 4x_2 - 6x_3 & \leq & 8, \\ & x_1, x_2 & \geq & 0. \end{array}$$

$$\begin{array}{llll} \min_x & c^T x & & \\ \text{such that} & a_1^T x & = & b_1, \\ & a_2^T x & \geq & b_2, \\ & a_3^T x & \leq & b_3, \\ & x & \geq & 0. \end{array}$$

$$\begin{array}{llll} \min_{x \in \mathbb{R}^3} & c^T x & & \\ \text{such that} & a^T x & = & b, \\ & x_1 & & \text{free}, \\ & x_2 & \geq & 0, \\ & x_3 & \leq & 0. \end{array}$$

2. Find a dual formulation of problems

$$\begin{array}{llll} \min_x & c^T x & & \\ \text{such that} & Ax & = & b, \\ & x & \geq & a. \quad (a \geq 0) \end{array}$$

$$\begin{array}{llllll} \min_x & c^T x & & & & \\ \text{such that} & b_1 & \leq & Ax & \leq & b_2, \\ & x & \geq & 0. & & \end{array}$$

3. How to solve the problem

$$\begin{array}{llll} \min_x & 50x_1 + 25x_2 & & \\ \text{such that} & x_1 + 3x_2 & \geq & 8, \\ & 3x_1 + 4x_2 & \geq & 19, \\ & 3x_1 + x_2 & \geq & 7, \\ & x_1, x_2 & \geq & 0, \end{array}$$

without an initialization phase?

4. Let be the problem

$$\begin{array}{lll} \min_x & 2x_1 - x_2 & \\ \text{such that} & 2x_1 - x_2 - x_3 & \geq 3, \\ & x_1 - x_2 + x_3 & \geq 2, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

What is the dual of this problem? Propose as precise upper and lower bounds as possible for the optimal objective of the primal. Do the same for the dual.

5. The optimal objective of the problem

$$\begin{array}{lll} \min_x & 7x_1 + 10x_2 & \\ \text{such that} & 2x_1 + 3x_2 & \geq 10, \\ & 3x_1 + 4x_2 & \geq 19, \\ & x_1 + 2x_2 & \geq 9, \\ & x_1, x_2 & \geq 0, \end{array}$$

is equal to $z_* = 47$. Is the solution $(0, 2, 1)$ an optimal feasible solution of the dual?

6. Consider the following problem

$$\begin{array}{lll} \min_x & 2x_1 + 9x_2 + 3x_3 & \\ \text{such that} & -2x_1 + 2x_2 + x_3 & \geq 1, \\ & x_1 + 4x_2 - x_3 & \geq 1, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

Find the dual of this problem and solve it graphically. Use complementary slackness conditions to obtain a solution of the primal.

7. Consider the following problem

$$\begin{array}{lll} \min_x & 5x_1 - 3x_2 & \\ \text{such that} & 2x_1 - x_2 + 4x_3 & \leq 4, \\ & x_1 + x_2 + 2x_3 & \leq 5, \\ & 2x_1 - x_2 + x_3 & \geq 1, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

The vertex associated with the basis (x_1, x_2, x_3) is an optimal vertex of this problem. What is the dual of this problem? Find a solution to the dual problem.

8. Consider the following linear optimization problem:

$$\begin{array}{lrll} \min_x & \sum_{i=1}^n c_i x_i & & \\ & \sum_{i=1}^n a_i x_i & = & b, \\ & x_i & \geq & 0 \quad i = 1, \ldots, n \end{array}$$

Note that this problem has only one constraint.

   (a) Using an elementary argument, find a simple condition for the existence of a feasible solution.

   (b) Assuming that the optimal cost is finite, propose a method for obtaining an optimal solution and justify your answer. (Hint: to answer this question, you may wish to refer to the fundamental theory of linear optimization).

   (c) Under what conditions does the problem have several solutions?

9. Let the optimization problem be

$$\begin{aligned} \max_x \quad & x_1 + x_2 + x_3, \\ & 2x_1 + x_3 \leq 2, \\ & 2x_2 + x_3 \leq 2, \\ & x_i \geq 0. \end{aligned}$$

   (a) List the vertices of this polyhedron. Is one of these vertices optimal?
   (b) What are (all) the optimal solutions?
   (c) Write down the dual of this problem, solve it, and relate it to the solution of the primal.

10. Let the optimization problem be

$$\begin{aligned} \min_{x_i} \quad & 3x_1 - x_2 + x_3, \\ & -x_1 - x_2 + x_3 \geq 1, \\ & x_1 - x_2 + \alpha x_3 = 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

    where $0.25 \leq \alpha \leq 4$ is a parameter.

   (a) What is the dual of this problem?
   (b) Solve the dual as a function of the parameter α using a graphical method.
   (c) Deduce solutions from the primal.

11. True or false? Justify your choices with a few lines, a counter-example or a drawing. Be specific enough to convince people that you can't guess the answer, but don't provide a formal, detailed justification.

   (a) The dual of a linear optimization problem always exists.
   (b) If a linear optimization problem has a feasible solution, then so does its dual, and the optimal costs are equal.
   (c) If a linear optimization problem admits a finite optimal cost, then so does its dual.

More challenging exercise

12. A company sells 20 tons of its production to 5 buyers (see prices paid in the table below). For logistical reasons:

   - Buyer A has a preferential agreement and buys at least 2 tonnes, under all circumstances.
   - Buyer B buys a maximum of 2 tonnes.
   - buyer C buys at least 2 tonnes or nothing at all (this constraint may be difficult to formulate. If you are stuck, wait for the integer programming lecture).
   - Buyer D buys a maximum of 5 tonnes.
   - buyer E buys a quantity that does not differ by more than 3 tonnes from the quantity bought by D.

The prices paid by the various buyers are in thousands of rubles per tonne:

Buyer	A	B	C	D	E
Price paid	20	50	20	25	15

(a) Formulate this problem as an optimization problem. Is this problem linear? If not, reformulate it as a linear optimization problem.

(b) Formulate the problem of maximizing the quantity sold to buyer C, from the set of optimal solutions to the problem formulated in (a).

(c) Solve the problem in (a), assuming that buyer C buys nothing.

2 Duality II: interpretation, post-optimal analysis

1. Consider the following dietary problem: The idea is to buy fruit, vegetables and meat at a minimum to get enough vitamins A and B. For a healthy diet, you need to consume 11 units of vitamin A and 4 units of vitamin B. The nutritional (per unit of weight) are given in the table below. table below:

	vegetables	Fruits	meat
Vitamin A	1	5	1
Vitamin B	2	1	1

The costs per unit weight of food are 3 (vegetables), 2 (fruit) and 10 (meat). Model this problem and solve it.

Next, consider the problem of a pharmaceutical company that artificially synthesizes vitamin A and B and sells the pure vitamin to the dietician. The company seeks to determine the selling prices of the vitamins that will ensure it takes over the entire market while maximizing its profit. Model this problem and solve it. Which of the two vitamins is more expensive?

Against all odds, the company decides to market vitamin A and B at prices of 1 and 0.5. How does the dietician compose the mix?

2. Consider the following problem:

$$\begin{array}{lll} \text{maximize} & x_1 + 2x_2 + x_3 + x_4 & \\ & 2x_1 + x_2 + 5x_3 + x_4 & \leq \ 8 + \delta, \\ & 2x_1 + 2x_2 + 4x_4 & \leq \ 12, \\ & 3x_1 + x_2 + 2x_3 & \leq \ 18, \\ & x_1, x_2, x_3, x_4 & \geq \ 0. \end{array}$$

The solution associated with the base (x_2, x_3, x_7) is optimal for $\delta = 0$ (x_7 is the slack variable of the 3rd constraint); the corresponding solution is given by $(x_1, x_2, x_3, x_4) = (0, 6, 0.4, 0)$. For what values of δ does the base remain optimal?

3. Consider the following problem:

$$\begin{array}{lll} \text{maximize} & 3x_1 + 13x_2 + 13x_3 & \\ & x_1 + x_2 & \leq \ 7, \\ & x_1 + 3x_2 + 2x_3 & \leq \ 15, \\ & 2x_2 + 3x_3 & \leq \ 9, \\ & x_1, x_2, x_3 & \geq \ 0. \end{array}$$

An optimal basis is given by (x_1, x_2, x_3). For this basis we obtain

$$A_B^{-1} = \begin{bmatrix} 5/2 & -3/2 & 1 \\ -3/2 & 3/2 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Give a solution of the primal and a solution of the dual.

(b) How does the optimal objective evolve as the right-hand term of the second constraint decreases?

(c) How many units can the term on the right-hand side of the first constraint vary without changing the optimal base?

(d) By how many units can the objective coefficient associated with x_1 be varied without modifying the optimal base?

(e) Would the base remain optimal if we added a new variable x_4 with objective coefficient 5 and constraint vector $(2, -1, 5)$?

(f) Find a solution to the problem obtained by adding the constraint $x_1 - x_2 + 2x_3 \leq 10$.

4. A textile company produces three goods in quantities x_1, x_2, x_3. The production schedule for the next month satisfies the following constraints

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 12, \\ 2x_1 + 4x_2 + x_3 &\leq 10, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The first constraint results from technical limitations. The second constraint expresses a limitation due to the quantity of cotton available on the market. The profits associated with products x_1, x_2, x_3 are 2, 3 and 3.

How does the profit evolve when the quantity of cotton available increases from 10 to $10 + \epsilon$ for $\epsilon > 0$? and when it increases from 10 to 12?

Replace the second constraint by $2x_1 + 4x_2 + x_3 \leq c$ and use a geometric argument to find the evolution of profit as a function of c.

5. A Russian university has a maximum of 5000 places for students. It recruits both Russian and foreign students. The university has 440 professors. There is at least one teacher for every 12 Russians and one for every 10 foreigners. The university has 2,800 dormitory places, and guarantees that at least 40% of Russian students and 80 foreign students will find a place in a university room. The university receives a subsidy of \$2,000 per Russian student and a minimum of \$3,000 per foreign student. It is assumed that the university seeks to maximize its profit. You are asked to answer the following questions.

(a) Formulate the profit maximization problem as a linear optimization problem. Solve the problem.

(b) Find the dual formulation of this problem. Let y_* be an optimal solution of the dual. Propose an interpretation for the different components of y_*.

(c) Is it to the university's advantage to recruit additional teachers at \$10,000 per teacher per year?

(d) The university hires additional professors at \$8,000 per year. How many teachers does it benefit from hiring?

6. A company wants to produce a new alloy with a minimum of 30 % copper and 20 % zinc. The company has the following alloys at its disposal:

alloy	% of copper	% of zinc	% other metals	price ($/kilo)
1	66	22	12	33
2	20	10	70	20
3	45	45	10	30
4	20	50	30	40
5	0	0	100	0

Note that the company offers a copper- and zinc-free alloy free of charge.

The aim is to find the proportions of these alloys that need to be mixed to produce the new alloy at minimum cost. You are asked to answer the following questions.

(a) Formulate the cost minimization problem as a linear optimization problem.

(b) Find the dual formulation of this problem. Let y_* be an optimal solution of the dual. Propose an interpretation for the different components of y_*. Solve this dual problem and find its solution y_*.

(c) It is assumed that the minimum percentage of copper required increases slightly (from 30 % to (30+δ) % with δ small). How does the optimum production cost of the new alloy evolve? Be as precise as possible.

(d) What is the optimal production cost of the initial problem?

7. A saver invests $1,000. He can choose between three investments A, B and C. The expected and guaranteed guaranteed for these investments are as follows (for 1 Dollar):

	expected	guaranteed
A	1.4	0.9
B	1.2	1.2
C	1.6	0.5

The investor has promised to invest at least $600 in B and C. He also wants a total interest of at least 5% and is looking for the distribution of his investment that maximizes his expected gain.

(a) Formulate this problem as a linear optimization problem and solve it. To find the solution, you can, if you wish, choose the initial solution of investing $1,000 in B.

(b) The saver chooses to invest *at most* 1000 Dollars. Formulate this problem as a linear optimization problem. Is the optimal solution you obtain for this modified version different from that obtained in point a?

(c) Write down the dual of the problem obtained in point a and propose an interpretation for the optimal dual variables.