Optimization - Exercise session 5
Duality (part 2)

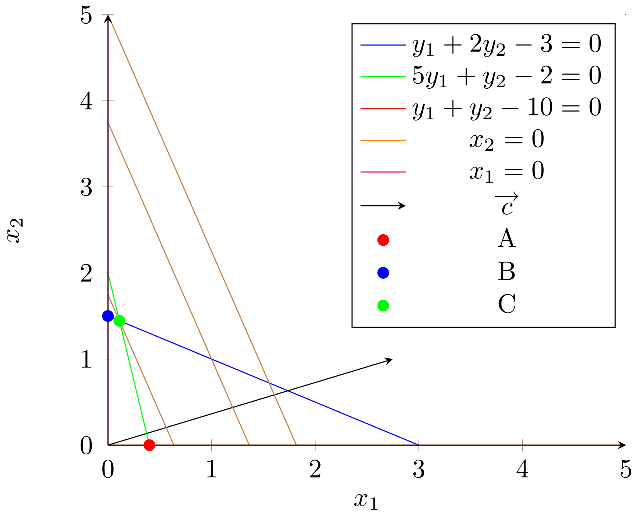
1. Find the dual of problems
1.1 $\min_x 3x_1 + 2x_2 + 10x_3$
such that

$$x_1 + 5x_2 + x_3 \geq 11,$$
$$2x_1 + x_2 + x_3 \geq 4,$$
$$x_1, x_2, x_3 \geq 0.$$

The dual
$\max_y 11y_1 + 4y_2$
such that

$$y_1 + 2y_2 \leq 3,$$
$$5y_1 + y_2 \leq 2,$$
$$y_1 + y_2 \leq 10,$$
$$y_1, y_2 \geq 0.$$

Let's solve the dual problem using the graphical method.



It is clear that the maximum is attained at the green point $(1/9, 13/9)$ and is equal to 7.
If x is an optimal solution of the primal and y is an optimal solution of the dual, then

$$(1/9, 13/9)\,(Ax - b) = 0$$

where

$$A = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$
$$b^T = (11, 4)$$

Consequently,

$$(1/9, 13/9) \begin{pmatrix} x_1 + 5x_2 + x_3 - 11 \\ 2x_1 + x_2 + x_3 - 4 \end{pmatrix} = 0$$

Hence,

$$\begin{gathered} x_1 + 5x_2 + x_3 = 11 \\ 2x_1 + x_2 + x_3 = 4 \\ 3x_1 + 2x_2 + 10x_3 = 7 \end{gathered}$$

Solution of this system of linear equations is $(1, 2, 0)$. Therefore, the optimal solution of the initial problem is $(1, 2, 0)$ and is equal to 7, basis $\{1, 2\}$ and

$$A_B = \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}.$$

What happens for b modified (optimality condition for reduced costs is not impacted, and the feasibility condition still holds for small $\Delta b = [-\epsilon_1; -\epsilon_2]$ with $\epsilon_i \geq 0$ (we study the case we buy vitamines from outside, hence, we reduce the constraints for the required vitamines obtained via food), formula for the new primal optimal solution, then if the basis does not change, the optimal dual solution neither).

Let's build the new objective value for the Primal (See slide 39, Lecture 5)

$$C_B^T(x_B + A_B^{-1}\Delta b) = f^\star + y_*^T \Delta b = 7 - 1/9 * \epsilon_1 - 13/9 * \epsilon_2$$

Now, we need to decide if buying synthetic vitamines make sense: we build the total cost (including the amount for buying synthetic vitamines):

$$f(\epsilon) = f^\star + y_*^T \Delta b + z_1 * \epsilon_1 + z_2 * \epsilon_2 = 7 - 1/9 * \epsilon_1 - 13/9 * \epsilon_2 + z_1 * \epsilon_1 + z_2 * \epsilon_2 =$$

$$= 7 - \epsilon_1(1/9 - z_1) - \epsilon_2(13/9 - z_2)$$

with z_1 and z_2 the unit prices of synthetic vitamines. We then conclude that it becomes interesting to buy

$$\text{vit A if} \quad z_1 < 1/9$$

$$\text{vit B if} \quad z_2 < 13/9$$

Giving the selling prices for these ($z_1 = 1$, and $z_2 = 1/2$), then we decide to buy only vit B and no vit A ($\epsilon_1 = 0$)

The remaining part is what value for ϵ_2? For that, we need to find bound for it, by ensuring that

$$x_b^\star(\epsilon_2) = A_B^{-1}(b + \Delta b) \geq 0$$

It normally gives $-18 \leq \epsilon_2 \leq 9/5$ (approx 1.8).

Then we can buy at max $\epsilon_2 = 9/5$

compute the new $x_B = [0; 11/5; 0]$,

and the new objective function value

$$(7 - 1/9 * 0 - 13/9 * 9/5 = 22/5),$$

and finally the total cost

$$= 7 - \epsilon_2 * (13/9 - z_2) = 5.3.$$

2. Consider the following problem:
maximize $x_1 + 2x_2 + x_3 + x_4$ s.t

$$2x_1 + x_2 + 5x_3 + x_4 \leq 8 + \delta,$$

$$2x_1 + 2x_2 + 4x_4 \leq 12,$$

$$3x_1 + x_2 + 2x_3 \leq 18,$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

The solution associated with the base (x_2, x_3, x_7) is optimal for $\delta = 0$ (x_7 is the slack variable of the 3rd constraint); the corresponding solution is given by

$$(x_1, x_2, x_3, x_4) = (0, 6, 0.4, 0).$$

For what values of δ does the base remain optimal?

Solution

maximize $x_1 + 2x_2 + x_3 + x_4$ s.t

$$2x_1 + x_2 + 5x_3 + x_4 + x_5 = 8 + \delta,$$

$$2x_1 + 2x_2 + 4x_4 + x_6 \leq 12,$$

$$3x_1 + x_2 + 2x_3 + x_7 \leq 18,$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

We have

$$A_B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

$$b = [8, 12, 18]^T,$$

$$\Delta b = [\delta, 0, 0]$$

The base remains unchanged as long as

$$A_B^{-1} b + A_B^{-1} \Delta b$$

remains positive.

Consequently,

$$A_B^{-1} b + A_B^{-1} \Delta b = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/5 & -1/10 & 0 \\ -2/5 & -3/10 & 1 \end{bmatrix} (b + \Delta b) =$$

$$= [6, 2/5 + \delta/5, 56/5 - (2\delta)/5] \geq 0$$

Hence,

$$-2 \leq \delta \leq 28.$$

3. Consider the primal-dual pair in standard form:
minimize $7y_1 + 15y_2 + 9y_3$

$$y_1 + y_2 \geq 3,$$
$$y_1 + 3y_2 + 2y_3 \geq 13,$$
$$2y_2 + 3y_3 \geq 13,$$
$$y_1, y_2, y_3 \geq 0.$$

maximize $3x_1 + 13x_2 + 13x_3$

$$x_1 + x_2 \leq 7,$$
$$x_1 + 3x_2 + 2x_3 \leq 15,$$
$$2x_2 + 3x_3 \leq 9,$$
$$x_1, x_2, x_3 \geq 0.$$

a)

$$A_B = [[1, 1, 0], [1, 3, 2], [0, 2, 3]];$$

$$(y^*)^T = c_b^T A_B^{-1} = [1, 2, 3].$$
$$(x^*)^T = b^T A_B^{-1} = [4, 3, 1].$$

b) How does the optimal objective evolve as the right-hand term of the second constraint decreases?
maximize $3x_1 + 13x_2 + 13x_3$

$$x_1 + x_2 \leq 7,$$
$$x_1 + 3x_2 + 2x_3 \leq 15 - \varepsilon,$$
$$2x_2 + 3x_3 \leq 9,$$
$$x_1, x_2, x_3 \geq 0.$$

$$c_b^T(x_B + A_B^{-1}[0, \varepsilon, 0]) = 64 - 2\varepsilon.$$

When 15 decreases by ε, the objective function decreases by 2ε. The optimal objective will decrease.

c) How many units can the term on the right-hand side of the first constraint vary without changing the optimal base?

The base remains unchanged as long as

$$A_B^{-1}b + A_B^{-1}\Delta b$$

remains positive.

Consequently,

$$A_B^{-1}b + A_B^{-1}\Delta b =$$
$$= [4 + (5\delta)/2, 3 - (3\delta)/2, 1 + \delta] \geq 0$$

Hence,

$$4 + (5\delta)/2 \geq 0 \quad \Rightarrow \delta \geq -8/5$$
$$3 - (3\delta)/2 \geq 0 \quad \Rightarrow \delta \leq 2$$
$$1 + \delta \geq 0 \quad \Rightarrow \delta \geq -1$$

Therefore,

$$-1 \leq \delta \leq 2.$$

(d) By how many units can the objective coefficient associated with x_1 be varied without modifying the optimal base?

$$A_B = [[1, 1, 0], [1, 3, 2], [0, 2, 3]];$$
$$c_N = [0, 0, 0]^T, c_B = [-3, -13, -13]^T$$
$$(c_N^T - C_B^T A_B^{-1} A_N) - (\Delta c_N^T + \Delta C_B^T A_B^{-1} A_N) = [1 - (5\delta)/2, 2 + (3\delta)/2, 3 - \delta] \geq 0.$$

Hence,

$$\delta <= 2/5, \quad \delta <= 3, \quad \delta >= -4/3.$$

Therefore, $\delta \in [-4/3; 2/5]$

(e) Would the base remain optimal if we added a new variable x_4 with objective coefficient $c_4 = 5$ and constraint vector $a_4 = [2, -1, 5]^T$?

$$c_4 - (y^*)^T a_4 = 5 - [1, 2, 3] \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = 5 - (2 - 2 + 15) = -10.$$

The solution does not remain optimal.

f) Find a solution to the problem obtained by adding the constraint $x_1 + x_2 + 2x_3 \leq 10$.

$$c_4 - (y^*)^T a_4 = 5 - [1, 2, 3] \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 5 - (1 - 2 + 6) = 0.$$

The solution remains optimal.