Full Name: ______________

Variant 1

Duration: 180 Minutes

Innopolis University
Optimization
Fall 2024
Midterm
26.10.2024

You need to perform this exam alone and without the use of any equipment or book, apart from this booklet, simple calculator and pens. Please do not consult any person or use any equipment, otherwise you will be disqualified from this exam at the very first attempt.
Good Luck!

Grade Table (for teacher use only)

Question:	1	2	3	4	Total
Points:	25	25	25	25	100
Score:					

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1. *Theory.* True or false ? Justify.
   (a) (4 points) A polyhedron given in the standard form $\mathcal{P} = \{x \mid Ax = b, x \geq 0\}$ always has a vertex.
   (b) (4 points) A linear optimization problem can have an optimal solution without an optimal vertex.
   (c) (4 points) The point $(x_1, x_2, x_3) = (1, 1, 1)$ is a vertex of the polyhedron defined by the following inequality
   $$x_1 + x_2 + x_3 \leq 4, x_1 + x_2 \leq 2, x_1 \leq 1, \text{ and } x_2 \leq 1.$$
   (d) (4 points) Let's consider an n-dimensional polyhedron defined by m inequality constraints. If you have a vertex of this polyhedron, then the simplex method can solve any linear program over this polyhedron with a maximum number of $C_m^n = \frac{m!}{n!(m-n)!}$ pivots of the simplex array.
   (e) (4 points) A linear optimization problem can have exactly two optimal solutions.
   (f) (5 points) After each iteration of the simplex algorithm, the value of the objective function decreases strictly.

Solution:

Question a: if the polyhedron $\mathcal{P}$ is **non-empty**, then this polyhedron is a subset of the non-negative orthant. The latter does not include any straight line, so is for $\mathcal{P}$. Therefore, $\mathcal{P}$ has one vertex by the proposition: *Polyhedra that have a vertex are exactly those that do not contain a straight line.*
Question b: True: Example: $\min_{x,y} x + y$ such that $x + y \geq 0$ has no optimal vertex (no vertex at all) and the optimal value is 0. (For there to be an optimal vertex, in addition to a bounded optimal value, the polyhedron must have at least one vertex, cf. fundamental theorem).
Question c: False, the point $(x_1, x_2, x_3) = (1, 1, 1)$ tightens three constraints but these are not linearly independent.
Question d: Assuming $m \geq n$, then true (it is the case since we have a first vertex): Since an n-dimensional polyhedron defined by means of m inequality constraints has at most $\binom{m}{n}$ vertices, and the simplex method passes from vertex to adjacent vertex by improving the objective function, there will be at most $\binom{m}{n} - 1$ pivots (since we assume that we start from a vertex), provided that there is no cycling, which can be guaranteed using Bland's rule, for example.
Question e: False, say we have two optimal solutions $x^\star$ and $y^\star$ such that $f^\star = c^T x^\star = c^T y^\star$, then any convex combination $z^\star = \lambda x^\star + (1 - \lambda) y^\star$ with $0 \leq \lambda \leq 1$ is also optimal, indeed:

$c^T z^\star = c^T(\lambda x^\star + (1-\lambda)y^\star) = \lambda c^T x^\star + c^T y^\star - \lambda c^T y^\star = \lambda f^\star + f^\star - \lambda f^\star = f^\star$. (so if a linear program has more than one solution, it has infinitely many).
Question f: provided that there is no cycling, which can be guaranteed using Bland's rule, then true. Otherwise, false: In the case of a degenerate vertex, a pivot can be made by staying on the same vertex, so the objective function doesn't change; see course notes for an example.

2. *Mixing problem.* Your sports trainer recommends 16 units of vitamin A and 15 units of vitamin B. You want to eat only apples and bananas, whose vitamin quantities and price per unit are given in the table below 1 here-under.

	Vit. A	Vit. B	price
apples	4	1	2
bananas	3	2	1
Total to be consumed	16	15	

Table 1: Data (per unit).

Your objective is to reach the recommended quantities at minimum cost.

(a) (5 points) Model this problem as a linear optimization problem where the first variable (x_1) corresponds to the number of units of apples to buy and the second (x_2) to the number of units of bananas to buy. Write the problem in standard form with two slack variables, x_3 (first constraint) and x_4 (second constraint).

(b) (5 points) *Vertex and calculation of an initial solution.* Check that the point (15,0,44,0) corresponds to a vertex of the polyhedron in standard form. Justify. Put the simplex table into (x_1, x_3) canonical form.

(c) (10 points) Solve this problem **using the simplex method** from the basic feasible solution calculated in (b). How many units of apples and bananas will you consume? Justify.

(d) (5 points) Is the optimal solution you have found unique? Please justify.

Solution:

(a) minimize

$2x_1 + x_2$

s.t.

$$4x_1 + 3x_2 \geq 16$$
$$x_1 + 2x_2 \geq 15$$
$$x_1, x_2 \geq 0$$

Standard form:

minimize

$2x_1 + x_2$

s.t.

$$4x_1 + 3x_2 - x_3 = 16$$
$$x_1 + 2x_2 - x_4 = 15$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(b)

$$4 \cdot 15 + 3 \cdot 0 - 1 \cdot 44 = 16 = 16$$
$$1 \cdot 15 + 3 \cdot 0 - 1 \cdot 0 = 15 = 15$$

(x_1, x_3) canonical form:

$$2x_1 + x_2 = z$$
$$4x_1 + 3x_2 - x_3 = 16$$
$$x_1 + 2x_2 - x_4 = 15$$

$$-3x_2 + 2x_4 = z - 30$$
$$5x_2 - 4x_4 + x_3 = 44$$
$$x_1 + 2x_2 - x_4 = 15$$

(c)

Basic	x_1	x_2	x_3	x_4	Solution
z	0	-3	0	2	30
x_3	0	5	1	-4	44
x_1	1	2	0	-1	15

Basic	x_1	x_2	x_3	x_4	Solution
z	3/2	0	0	1/2	15/2
x_3	- 5/2	0	1	-1.5	44+75/2
x_2	1/2	1	0	-1/2	15/2

0 – Apple

15/2 – Banana

(d) – Unique

3. *Duality.*

Consider the following linear optimization problem

$$\begin{array}{lcccccc} \max\limits_{x_1,x_2} & x_1 & + & 2x_2 & & \\ \text{such that} & 2x_1 & + & 3x_2 & = & 6, \\ & 2x_1 & + & x_2 & \geq & 4, \\ & x_1 & + & 2x_2 & \leq & 4, \\ & x_1 \geq 0, & & x_2 \text{ free.} & & \end{array}$$

(a) (5 points) Write the dual of this problem.

(b) (8 points) **For this problem**, state the weak duality theorem, and prove it.

(c) (12 points) Assume that the optimal solution of the primal is $x^* = (3/2, 1)$, and use complementary slackness conditions to calculate the optimal solution of the dual.

Can you now prove, via the dual optimal solution, that $x^* = (3/2, 1)$ is indeed optimal for the primal? Justify your answer.

Solution:

Solution of this problem was given in class.

4. *Integer Programming*

Consider the following integer linear problem

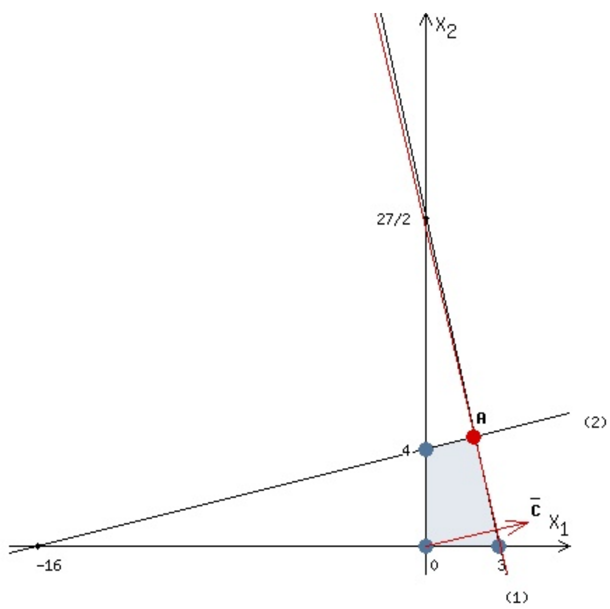
$$\begin{aligned} \max_{x_1,x_2} \quad & 17x_1 + 4x_2 \\ \text{such that} \quad & 9x_1 + 2x_2 \le 27, \\ & -x_1 + 4x_2 \le 16, \\ & x_1, x_2 \ge 0 \text{ and integers.} \end{aligned}$$

(a) (10 points) Graphically represent this problem, and graphically solve the relaxed problem.

(b) (15 points) Using graphical representation to solve sub-problems, solve this problem via branch and bound. Draw the search tree completely, and justify your reasoning accurately and completely.

Plan B: If you can't solve the problem in this way, can you nevertheless find the optimal solution, and prove that it is indeed optimal without explicitly calculating all admissible solutions?

Solution: (A)



$$x_1 = 2$$
$$x_2 = 9/2$$
$$F_{\max} = 52$$