Optimisation

Lecture 12 - Duality

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Outline

Introduction

Starting point: a simple problem

- The dual problem, which we will formally define later on, can be motivated as a way to find *bounds* on a given optimization problem.
- We will begin with an example:

$$\begin{aligned} &\min_{x_1,x_2} x_1^2 + x_2^2 + 2x_1 \\ &\text{s.t. } x_1 + x_2 = 0 \end{aligned} \tag{1}$$

- Of course it is easy to solve : By eliminating x_2 via the relation $x_2 = -x_1$, the problem becomes 1 dimensional:

$$\min_{x_1} 2x_1^2 + 2x_1$$

- The unconstrained minimizer is $x_1 = -\frac{1}{2}$, hence $x^\star = (-\frac{1}{2}, \frac{1}{2})^T$ with an optimal value $f^\star = -\frac{1}{2}$.

A simple lower-bound

- **The small exercise we want to make here**: find lower bounds on the value of the problem by solving unconstrained problems.
- for instance by simply removing the single constraints:

$$\min_{x_1, x_2} x_1^2 + x_2^2 + 2x_1 \tag{2}$$

- It turns out that the optimal value of Problem (2) is a lower-bound on the value of Problem (1) ! $f^\star(\text{P }(2)) \leqslant f^\star(\text{P }(1))$ with (P (i)) denoting the Problem (i).
- **Check**: optimal solution of Problem (2) is attained at stationary point $x_1 = -1, x_2 = 0$ with an optimal value of -1 (indeed, it is a lower bound for $f^\star$).

Other lower-bounds

- To find other lower-bounds, we use the following trick: take a real number μ and consider the following problem, equivalent to Problem (1):

$$\begin{aligned} &\min_{x_1,x_2} x_1^2 + x_2^2 + 2x_1 + \mu(x_1+x_2) \\ &\text{s.t. } x_1 + x_2 = 0 \end{aligned} \tag{3}$$

- Now: eliminate the equality constraint and obtain the unconstrained problem:

$$\min_{x_1,x_2} x_1^2 + x_2^2 + 2x_1 + \mu(x_1+x_2) \tag{4}$$

We have for all $\mu \in \mathbb{R}$:

$$f^\star(\text{P }(4)) \leqslant f^\star(\text{P }(1))$$

The optimal solution of Problem (4) is attained at the stationary point $(x_1, x_2) = (-1 - \frac{\mu}{2}, -\frac{\mu}{2})$, and the corresponding optimal value, which we denote $q(\mu)$ is:

$$q(\mu) := f^\star(\text{P }(4)) = \left(-1 - \frac{\mu}{2}\right)^2 + \left(-\frac{\mu}{2}\right)^2 + 2\left(-1 - \frac{\mu}{2}\right) + \mu(-1-\mu) = -\frac{\mu^2}{2} - \mu - 1.$$

The largest (other) lower-bounds

- For example, $q(0) = -1$ is the lower bound obtained by Problem (2).
- What interests us the most is the best (i.e., largest), lower-bound obtained by this technique.
- The best lower-bound is the solution of the problem:

$$\max\{q(\mu) : \mu \in \mathbb{R}\} \tag{5}$$

- This problem will be called *the dual problem*, and by its construction, its optimal value is a lower-bound on the optimal value of the original problem, which we call the *primal problem*.
- Here: the optimal solution of the dual problem is attained at $\mu = -1$, and the corresponding optimal value of the dual is $-\frac{1}{2}$
- .. which exactly $f^\star$! meaning that the best lower bound obtained by the this technique is actually equal to the optimal value $f^\star$.
- Later we will refer this property as "strong duality".

The dual problem

Definition of the dual problem

Consider the general NL problem:

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leqslant 0, \quad i = 1, 2, ..., m, \\
& h_j(x) = 0, \quad j = 1, 2, ..., p,
\end{aligned} \tag{6}
$$

- where f, g_i, h_j are functions defined on $\mathbb{R}^n$ [1].
- Problem (6) will be referred to as the primal problem.
- The Lagrangian is

$$
L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) \quad (x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p) \tag{7}
$$

- The dual objective function $q : \mathbb{R}^m_+ \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty\}$ is defined to be

$$
q(\lambda, \mu) = \min_{x\in\mathbb{R}^n} L(x, \lambda, \mu). \tag{8}
$$

[1] we do not assume anything on the functions, not even assumed continuous

The Dual Problem

- the optimal value of the minimization problem (8) is not always finite; there are values of (λ, μ) for which $q(\lambda, \mu) = -\infty$.
- It is therefore natural to define the domain of the dual objective function as

$$\mathrm{dom}(q) = \{(\lambda, \mu) : q(\lambda, \mu) > -\infty\} \tag{9}$$

- The dual problem is given by

$$\begin{aligned} \max_{\lambda,\mu} \quad & q(\lambda, \mu) \\ \text{s.t.} \quad & (\lambda, \mu) \in \mathrm{dom}(q) \end{aligned} \tag{10}$$

Convexity of the Dual Problem

We begin by showing that the dual problem is always convex; it consists of maximizing a concave function over a convex feasible set.

Theorem

Consider problem (6) with f, g_i, h_j being functions defined on $\mathbb{R}^n$, and let q be the dual function defined in (10). Then

1. $\mathrm{dom}(q)$ is a convex set.
2. q is a concave function over $\mathrm{dom}(q)$.

Proof

- (1) Take $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathrm{dom}(q)$ and $\alpha \in [0, 1]$,then

$$\begin{aligned} \min_x L(x, \lambda_1, \mu_1) > -\infty \\ \min_x L(x, \lambda_2, \mu_2) > -\infty \end{aligned} \tag{11}$$

Proof Contd.

- Note that $L(x, \lambda, \mu)$ is affine function w.r.t λ, μ
- Therefore,

$$\begin{aligned} q(\alpha\lambda_1 + (1-\alpha)\lambda_2, \alpha\mu_1 + (1-\alpha)\mu_2) &= \min_x L(x, \alpha\lambda_1 + (1-\alpha)\lambda_2, \alpha\mu_1 + (1-\alpha)\mu_2) \\ &= \min_x [\alpha L(x, \lambda_1, \mu_1) + (1-\alpha) L(x, \lambda_2, \mu_2)] \\ &\geqslant \alpha \min_x L(x, \lambda_1, \mu_1) + (1-\alpha) \min_x L(x, \lambda_2, \mu_2) \\ &= \alpha q(\lambda_1, \mu_1) + (1-\alpha) q(\lambda_2, \mu_2) > -\infty \end{aligned} \tag{12}$$

  Hence, $\alpha(\lambda_1, \mu_1) + (1-\alpha)(\lambda_2, \mu_2) \in \mathrm{dom}(q)$, and the convexity of $\mathrm{dom}(q)$ is established.
- (b) $L(x, \lambda, \mu)$ is affine function w.r.t λ, μ, so, it is a concave function w.r.t. λ, μ.[2]
- Hence, since q is the minimum of concave functions, it must be concave.

[2]is affine if and only if it is convex and concave

The Weak Duality Theorem

- The first important result is closely connected to the motivation of the construction of the dual problem: the optimal dual value is a lower bound on the optimal primal value.
- This result is called the *weak duality theorem*, and unsurprisingly, its proof is rather simple.

The Weak Duality Theorem

Theorem

Consider the primal problem (6) and its dual problem (10). Then

$$q^\star \leqslant f^\star$$

where $f^\star, q^\star$ are the primal and dual optimal values respectively.

Proof.

- The feasible set of the primal problem is $S = \{x \in \mathbb{R}^n : g_i(x) \leqslant 0 \forall i, h_j(x) = 0 \forall j\}$
- Then for any $(\lambda, \mu) \in \mathrm{dom}(q)$ we have :

$$\begin{aligned} q(\lambda,\mu) &= \min_{x\in\mathbb{R}^n} L(x,\lambda,\mu) \leqslant \min_{x\in S} L(x,\lambda,\mu) \\ &= \min_{x\in S} \left(f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) \right) \\ &\leqslant \min_{x\in S} f(x) = f^\star \end{aligned}$$

Taking the maximum over $(\lambda, \mu) \in \mathrm{dom}(q)$, the result follows.

Example

- The weak duality theorem states that the dual optimal value is a lower bound on the primal optimal value.
- The example in the introduction illustrated that the lower bound can be tight.
- However, the lower bound does not have to be tight, and the next example shows that it can be totally uninformative ! (in attention is not paid to the domain of q).

Consider the problem

$$\min_{x_1,x_2} \quad x_1^2 - 3x_2^2 \text{ s.t. } x_1 = x_2^3 \tag{13}$$

Solved in class

Strong Duality in the Convex Case

Roadmap

- In the convex case we can prove under rather mild conditions that strong duality holds; that is, the primal and dual optimal values coincide.
- We need first the results from the non-linear Farkas lemma. This lemma will be the key in proving the strong duality.

The Nonlinear Farkas Lemma

Theorem

Let $f, g_1, g_2, \ldots, g_m$ be convex functions over $\mathbb{R}^n$. Assume that there exists $\hat{x} \in \mathbb{R}^n$ such that

$$g_1(\hat{x}) < 0, \ldots, g_m(\hat{x}) < 0 \tag{14}$$

Let $c \in \mathbb{R}$. Then the following two claims are equivalent:

1. the following implication holds:

$$x \in \mathbb{R}^n, g_i(x) \leqslant 0, i = 1, \ldots, m \rightarrow f(x) \geqslant c \tag{15}$$

2. there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geqslant 0$ such that

$$\min_x \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \geqslant c \tag{16}$$

Proof of 2 → 1

The implication 2 → 1 is straightforward. Indeed:

- Suppose that there exist $\lambda_1, \lambda_2, ..., \lambda_m \geqslant 0$ such that (16) holds, and let $x \in \mathbb{R}^n$ satisfy $g_i(x) \leqslant 0, i = 1, ..., m$.
- By (16) we have :

$$\begin{aligned} \min_x \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} &\geqslant c \\ \rightarrow f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) &\geqslant c \end{aligned} \tag{17}$$

Hence:

$$f(x) \geqslant c - \sum_{i=1}^{m} \lambda_i g_i(x) \geqslant c. \tag{18}$$

The implication 1 → 2 is much harder to prove and requires other results from the theory (such as the theorem of *Separation of Two Convex Sets*), and is out of scope of this humble introduction to duality.

Strong Duality of Convex Problems with Inequality Constraints

We are now able to show a strong duality result in the convex case under a Slater-type condition.

Theorem

Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leqslant 0, \quad i = 1, \dots, m \end{aligned} \tag{19}$$

where $f, g_i, i = 1, 2, \dots, m$ are convex functions over $\mathbb{R}^n$. Suppose that there exists $\hat{x} \in \mathbb{R}^n$ for which $g_i(\hat{x}) < 0, i = 1, 2, \dots, m$. If problem (19) has a finite optimal value, then

1. the optimal value of the dual problem is attained.
2. $f^\star = q^\star$

Proof of Strong Duality Theorem

- Since $f^\star > -\infty$ (it is finite by assumption) is the optimal value of problem (19), it follows that the following implication holds:

$$x \in \mathbb{R}^n, g_i(x) \leqslant 0 \forall i \rightarrow f(x) \geqslant f^\star$$

- By the nonlinear Farkas Lemma there exists $\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_m \geqslant 0$ such that

$$q(\tilde{\lambda}) = \min_x \{f(x) + \sum_{i=1}^{m} \tilde{\lambda}_i g_i(x)\} \geqslant f^\star$$

- By the weak duality theorem,

$$q^\star \geqslant q(\tilde{\lambda}) \geqslant f^\star \geqslant q^\star$$

- Hence $f^\star = q^\star$ and $\tilde{\lambda}$ is an optimal solution of the dual problem !

Note that similar results (again for convex problems) can be derived for problems including both inequalities and equalities constraints, but the theorem and the proof become much more technical.

Example

Consider the problem

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 - x_2 \\ \text{s.t.} \quad & x_2^2 \leqslant 0 \end{aligned} \tag{20}$$

Solved in class.

Complementary Slackness Conditions

We can also derive the complementary slackness conditions under the sole assumption that $q^\star = f^\star$ (without any convexity assumptions).

Theorem

Consider the optimization problem

$$\min_x f(x), \quad \text{s.t. } g_i(x) \leqslant 0, \quad i = 1, \ldots, m \tag{21}$$

and assume that $q^\star = f^\star$ where $q^\star$ is the optimal value of the dual problem. Let $x^\star, \lambda^\star$ be feasible solutions of the primal and dual problems. Then $x^\star, \lambda^\star$ are optimal solutions of the primal and dual problems iff

$$x^\star \in \operatorname{argmin}_x L(x, \lambda^\star) \tag{22}$$

$$\lambda_i^\star g_i(x^\star) = 0, i = 1, \ldots, m. \tag{23}$$

Proof

- We have:

$$q^\star = q(\lambda^\star) = \min_x L(x, \lambda^\star) \leqslant L(x^\star, \lambda^\star) = f(x^\star) + \sum_{i=1}^{m} \lambda_i^\star g_i(x^\star) \leqslant f(x^\star) = q^\star$$

  where the last inequality follows from the fact that $\lambda_i^\star \geqslant 0, g_i(x^\star) \leqslant 0$.
- Therefore, since $q^\star = f^\star$, all the inequalities in the above chain are satisfied and become equalities !
- It implies:
  1. $x^\star \in \operatorname{argmin}_x L(x, \lambda^\star)$, and
  2. $\sum_{i=1}^{m} \lambda_i^\star g_i(x^\star) = 0$ for all $i = 1, 2, ..., m$.

Examples

Does this general approach allows to recover the expression of the dual in L.P. ?

Consider the linear program in geometric form:

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \geqslant b \end{aligned} \tag{24}$$

where $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

- We assume that the problem is feasible, so Slater's condition holds and then strong duality holds !
- The Lagrangian function is (with $\lambda \geqslant 0$):

$$L(x, \lambda) = c^T x + \lambda^T (b - Ax) = (c^T - \lambda^T A)x + \lambda^T b,$$

- Dual objective function:

$$q(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda) = \min_{x \in \mathbb{R}^n} (c^T - \lambda^T A)x + \lambda^T b = \begin{cases} \lambda^T b, & \text{if } c^T = \lambda^T A \\ -\infty, & \text{else.} \end{cases}$$

- Hence: $\mathrm{dom}(q) = \{\lambda \in \mathbb{R}^m_+ | q(\lambda) > -\infty\} = \{\lambda \geqslant 0 | c^T = \lambda^T A\}$!

Does this general approach allows to recover the expression of the dual in L.P. ?

Recall that the dual problem is given by the general expression

$$\begin{aligned} \max_{\lambda} \quad & q(\lambda) \\ \text{s.t.} \quad & (\lambda) \in \operatorname{dom}(q) \end{aligned} \tag{25}$$

Here we then have:

$$\begin{aligned} \max_{\lambda} \quad & b^T\lambda \\ \text{s.t.} \quad & A^T\lambda = c \\ & \lambda \geqslant 0 \end{aligned} \tag{26}$$

Exactly the expression we have derived in Lecture 5 of the LP part of the course, see slide 23 !

Strictly Convex Quadratically Constrained Quadratic Program (QCQP)

We start with a famous instance: Given a polytope

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geqslant b\}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, we wish to compute the orthogonal projection of a given point y.

- There is no simple expression for the orthogonal projection onto polytopes, but using duality we show that a method finding the projection can be derived.
- For a given $y \in \mathbb{R}^n$, the problem can be written as:

$$\begin{aligned} \min_{x} \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & Ax \geqslant b \end{aligned} \tag{27}$$

- The Lagrangian function of this problem is ($\lambda \in \mathbb{R}^m_+$):

$$L(x, \lambda) = \|x - y\|_2^2 + 2\lambda^T(b - Ax)$$

note that a "2" is put in front of λ^T to simplify constants for the computation of $\text{argmin}_x L(x, \lambda)$.

Strictly Convex Quadratically Constrained Quadratic Program (QCQP)

- The minimizer of the Lagrangian is attained at the stationary point of the Lagrangian which is the solution to
$$\nabla_x L(x^\star, \lambda) = (x^\star - y) - A^T\lambda = 0$$
hence $x^\star = A^T\lambda + y$
- Substituting this value back into the Lagrangian we obtain that
$$\begin{aligned} q(\lambda) = \min_x L(x, \lambda) &= L(x^\star, \lambda) \\ &= \|A^T\lambda\|_2^2 + 2\lambda^T(b - A(A^T\lambda + y)) \\ &= \lambda^T AA^T\lambda - 2\lambda^T A^T A\lambda - 2\lambda^T(Ay - b) \\ &= -\lambda^T AA^T\lambda - 2(Ay - b)^T\lambda \end{aligned}$$
- The dual problem is $\max_\lambda q(\lambda) \quad \text{s.t. } \lambda \in \text{dom}(q)$, which is here:
$$\max_\lambda -\lambda^T AA^T\lambda - 2(Ay - b)^T\lambda \quad \text{s.t. } \lambda \geqslant 0 \tag{28}$$
$\rightarrow$ non-negative least squares problem (NNLS).

Strictly Convex Quadratically Constrained Quadratic Program (QCQP)

- We assume that $\mathcal{P}$ is nonempty, and under this assumption, strong duality holds.
- We can solve the dual problem by the projected gradient method: perform a gradient step and project back to non-negative orthant !
- Using a constant step size, it can be chosen as $\frac{1}{L} = 2\lambda_{max}(AA^T)$.
- The general step of the method would be then:

$$\lambda_{k+1} := [\lambda_k - \frac{2}{L}(AA^T\lambda_k + Ay - b)]_+$$

- Demo: consider the feasible set $\mathcal{P} = \{(x_1, x_2) | x_1 + x_2 \leqslant 1, x_1 \geqslant 0, x_2 \geqslant 0\}$
  This is the triangle in the plane with vertices $(0,0)^T, (0,1)^T, (1,0)^T$.
  We want to compute the orthogonal projection of $(2,-1)^T$ onto $\mathcal{P}$. see colab file

Strictly Convex Quadratically Constrained Quadratic Program (QCQP)

This specific problem can be generalized: consider the following general strictly convex quadratic program:

$$\begin{aligned} \min_{x} \quad & x^T Q x + 2 f^T x \\ \text{s.t.} \quad & Ax \geqslant b \end{aligned} \tag{29}$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $f \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

Exercise

- Build the the Lagrangian of the problem,
- Compute the minimizer of the Lagrangian and build the dual objective function,
- Solve the dual problem.

Important remark: in the case Q is positive semi-definite, the procedure has to be adapted and one trick is to decompose $Q = D^T D$ with D some matrix $D \in \mathbb{R}^{n \times n}$.

Conclusions

Summary

We have seen

- How to derive the best lower-bounds for general NL problems, building the *dual problem*.
- Its construction requires to follow some steps:
  1. Put the NL program in "standard" form (see Problem (6)).
  2. Build the *Lagrangian function*
  3. Compute the *dual objective function* $q(\lambda, \mu)$, with a strong attention to the domain of $q(\lambda, \mu)$!
  4. Build the *dual problem*: $\max_{\lambda,\mu} \quad q(\lambda, \mu)$ s.t. $(\lambda, \mu) \in \mathrm{dom}(q)$
- Several fundamental theoretical results:
  - the convexity of the dual problem.
  - The weak duality.
  - For convex problems, under Slater's condition, the strong duality.
  - Complementary Slackness Conditions under the sole assumption that $q^\star = f^\star$ (without convexity assumptions).

Preparations for the exam

- Review the lecture :).
- I **strongly** encourage to solve this exercise:
  Consider the problem of maximizing a linear function on an ellipse:

$$\min_{x} \quad c^T x \text{ s.t. } x^T Q x \leqslant 1 \tag{30}$$

  where $c \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ symmetric and positive definite.

  1. To which category of optimization problems does this problem belong? Justify
  2. What is the dual of this problem? (First write the Lagrangian, then the dual objective function and finally the dual problem).
  3. Does strong duality apply? Justify
  4. Solve the dual problem and deduce the optimal primal value.

Appendix A - Orthogonal Projection onto the Unit Simplex

A more advanced application is the following famous problem: Given a vector $y \in \mathbb{R}^n$, we would like to compute the orthogonal projection of the vector y onto $\Delta_n = \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = 1, x_i \geqslant 0 \forall i\}$. The corresponding optimization problem is

$$\begin{aligned} \min_x \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & e^T x = 1 \\ & x \geqslant 0 \end{aligned} \tag{31}$$

where $e = (1, ..., 1)^T \in \mathbb{R}^n$.

- We will associate a Lagrange multiplier $\lambda \in \mathbb{R}$ to the linear equality constraint $e^T x = 1$ and obtain the Lagrangian function

$$\begin{aligned} L(x, \lambda) = \|x - y\|_2^2 + 2\lambda(e^T x - 1) &= \|x\|_2^2 - 2(y - \lambda e)^T x + \|y\|_2^2 - 2\lambda \\ &= \sum_{i=1}^n (x_i^2 - 2(y_i - \lambda)x_i) + \|y\|_2^2 - 2\lambda \end{aligned}$$

Appendix A - Orthogonal Projection onto the Unit Simplex

- The arising problem is therefore separable with respect to the variables x_i and hence the optimal x_i is the solution to the one-dimensional problem

$$\min_{x_i \geqslant 0} (x_i^2 - 2(y_i - \lambda)x_i)$$

- The optimal x_i is

$$x_i = \begin{cases} y_i - \lambda, & y_i \geqslant \lambda \\ 0 & \text{else} \end{cases} = \max(0, y_i - \lambda) = [y_i - \lambda]_+$$

and the optimal value is $-[y_i - \lambda]_+^2$.

- The dual problem is therefore

$$\max_{\lambda \in \mathbb{R}} q(\lambda) := -\sum_{i=1}^{n} [y_i - \lambda]_+^2 - 2\lambda + \|y\|_2^2$$

Appendix A - Orthogonal Projection onto the Unit Simplex

- q is concave, differentiable and it can be shown [3] that there exists an optimal solution to the dual problem attained at a point $\lambda^\star$ in which

$$q'(\lambda^\star) = 0$$

- meaning that

$$\sum_{i=1}^{n} [y_i - \lambda^\star]_+ = 1$$

- To compute $\lambda^\star$, we need to solve the above equation, in order words find the root of the function $h(\lambda) = \sum_{i=1}^{n} [y_i - \lambda]_+ - 1$
- It is nonincreasing over $\mathbb{R}$ and is in fact strictly decreasing over $(-\infty, \max_i y_i]$.
- Moreover, $h(\lambda = y_{max}) = -1$, and $h(\lambda = y_{min} - \frac{2}{n}) = \sum_{i=1}^{n} y_i - n y_{min} + 2 - 1 > 0$!
- We can therefore invoke a bisection procedure to find the unique root $\lambda^\star$ of the function h over the interval $[y_{min} - \frac{2}{n}, y_{max}]$, and then define $x^\star = [y - \lambda^\star e]_+$ ▸ see colab file

[3] the reason comes from the fact we can show that $-q$ is a coercive and continuous function (since it is differentiable).

Goodbye, So Soon

THANKS FOR THE ATTENTION

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