Optimization - Exercise session 3
Convexity

1.Prove the convexity of the optimization problems below (using the criteria seen in class, or other reasoning if necessary):
(a) $\min |x| - \log(xy)$ such that $x^2 + y^2 \leq 1$ and $x, y > 0$;
(b) $\min e^x + ey + ez$ such that $x + y \geq 1$ and $y + z = 2$;
(c) $\max \sqrt{x} + y$ such that $x \geq 1$ and $y^2 \leq x \leq 2$;
(d) $\min \max\{3x^2 + 4y, 3x + 4y^2\}$ such that $1 \leq x^2 + y^2 \leq 2$ and $x + y \geq \sqrt{2}$.

Solution

(a) Minimization problem (right)
It is clear that $|x|$ is convex.
Let $g(x) = -\log(xy)$ and $D = \{(x, y) \in \mathbb{R}^n : x^2 + y^2 \leq 1, x, y > 0\}$. Since

$$\nabla^2 g(x) = \begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \succeq 0 \quad \forall x, y \in D,$$

$g(x)$ is convex. Consequently, f is convex as linear combination of two convex functions.
D is a convex set.

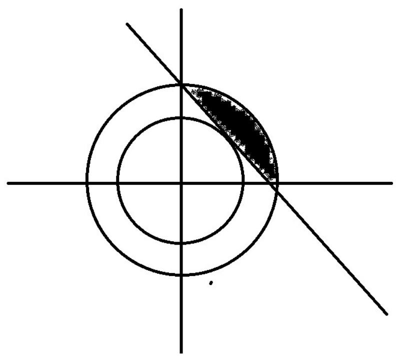
(b)
- Minimization problem (right)
- f is convex as linear combination of three convex functions.
- D is a convex set.

(c) We can rewrite the initial problem in the following form
$\min -\sqrt{x} - y$ such that $x \geq 1$ and $y^2 \leq x \leq 2$;
Suppose that $g(x) =$ and $D = \{(x, y) \in \mathbb{R}^2 : x \geq 1, \quad y^2 \leq x, \quad x \leq 2\}$.
- Minimization problem (right)
$x - 1$ and $2 - x$ are linear functions.
- f is convex as linear combination of 2 convex functions.
- $D = \{(x, y) \in \mathbb{R}^2 : x - 1 \geq 0, \quad 2 - x \geq 0, \quad x - y^2 \geq 0\}$ is a convex set.
$(x, y) \mapsto x - 1$ – linear
$(x, y) \mapsto 2 - x$ – linear
$g : (x, y) \mapsto x - y^2$ is concave since

$$-\nabla^2 g = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \succeq 0,$$

Consequently, D is convex.
$f(x, y) = -y - \sqrt{x}$ is convex since
$(x, y) \mapsto -\sqrt{x}$ is convex ($\sqrt{x}$ is concave).
$(x, y) \mapsto -y$ is linear

(d) - Minimization problem (right)
- $g(x, y) = 3x^2 + 4y$ is a convex function, $h(x, y) = 3x + 4y^2$ is a convex function. It is known that max-operation preserves convexity. Consequently, $\max\{g(x, my), g(x, y)\}$ is convex.
- $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2, \quad x + y \geq \sqrt{2}$ is convex:



2. A linear optimization problem is a problem defined solely using linear or affine functions (objective, equalities and inequalities), and can be written, for example

$$\min c^T x$$

such that

$$\begin{cases} Ax = b \\ Tx \leq u \end{cases}$$

Show that such a linear optimization problem is a convex problem. What do you deduce about the structure of the set of optimal solutions ?

Solution

- Minimization problem (right)
- $D = \{(x, y) \in \mathbb{R}^n : Ax = b, \quad 2 - x \geq 0, \quad Tx \leq u\}$ is a convex set.
$x \mapsto Ax - b$ – affine
$x \mapsto Tx - u$ – convex
- Suppose x and y are optimal solutions and $\lambda \in [0, 1]$. Then

$$c^T(\lambda x + (1 - \lambda)y) = \lambda c^T x + (1 - \lambda)c^T y = \lambda f^* + (1 - \lambda)f^* = f^*.$$

Consequently, the set of optimal solutions is convex.

3. Review the optimization problems introduced in the previous session (exercises 1, 5(a)-(f)) and, for each of them, determine whether it is a convex optimization problem. When the definition of the problem depends on unspecified data (vectors, matrices, functions), specify under which conditions on these data we can guarantee that the problem is convex. For problems for which you have explicit solutions of the optimal conditions, check that the properties guaranteed by convexity are satisfied.

Solution

We have the following problem

$$\min f(x,y),$$

such that

$$(x-3)^2 = 5y,$$

where $f(x,y) = (x-3)^2 + (y-2)^2$.

- Minimization problem (right)
- Since

$$\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succeq 0,$$

f is convex.

- $(x,y) \mapsto (x-3)^2 - 5y$ is non convex.

(5_a)

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + c^Tx$$

- Minimization problem (right)
- $D = \mathbb{R}^2$ is convex
- If $f(x) = \frac{1}{2}x^TQx + c^Tx$ then

$$\nabla^2 f = Q.$$

Consequently, f is convex if $\nabla^2 f = Q \succeq 0$.

If we have

$$\min x^2 + 4xy + 5y^2 + 3x - 5y$$

then

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

and

$$c = \begin{matrix} 3 \\ 5 \end{matrix}.$$

Since $tr(Q) = 6 > 0$ and $\det(Q) = 1 > 0$ then Q is positive definite. Hence, our problems is convex.

Let's consider the following problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + c^Tx$$

such that

$$Ax = b$$

- Minimization problem (right)
- $x \mapsto Ax - b$ – affine
- $Q \succeq 0$ then $f(x)$ is convex.

(5_b)

$$\min_{x\in\mathbb{R}^n} c^Tx$$

such that

$$\|x-u\| = R$$

- Minimization problem (right)
- $x \mapsto Ax - b$ – affine
- c^Tx is linear
- $\|x-u\| = R$ is non convex.

$$\min_{x\in\mathbb{R}^n} c^T x$$

such that

$$\|x-u\| \leq R$$

- Minimization problem (right)
- $x \mapsto Ax - b$ – affine
- $c^T x$ is linear
- $\|x-u\| \leq R$ is a closed ball. Consequently, the set convex.

Therefore, our problem is convex.

(5_c)

$$\min_{x\in\mathbb{R}^n} f(x)$$

such that

$$Ax = b$$

- Minimization problem (right)
- $x \mapsto Ax - b$ – affine

The problem is convex if f is a convex function.

(5_e) Non-linear optimization on positive variables

$$\min f(x)$$

such that $x \geq 0$.
- Minimization problem (right)
- $x \geq$ is $\mathbb{R}_{++}$ (convex)

The problem is convex if f is a convex function.

4. **Little house on the prairie.** On a map assimilated to the $\mathbb{R}^2$ plane are represented a meadow, of equation

$$\mathcal{P} = (x, y)|x^2 + (x + y)^2 \leq 18$$

as well as a river, of equation $\mathcal{R} = \{(x, y)|x + 2y + 12 = 0\}$ and a fence, of equation $\mathcal{C} = (x, y)|x = 2$.

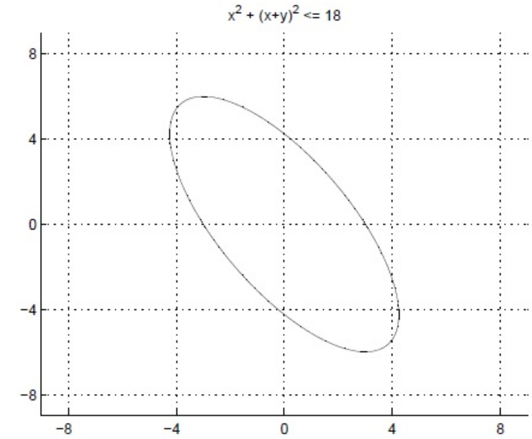
We want to build a small house and determine its (x, y) coordinates so that

. the house is in the meadow $\mathcal{P}$

. the house is on the left of the fence $\mathcal{C}$ (i.e. in the region where $x \leq 2$)

. the house is located as close as possible to the river $\mathcal{R}$

(a) Graphically sketch the sets $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$ appearing on the map. The following (exact) representation of the meadow can be used as a guide:



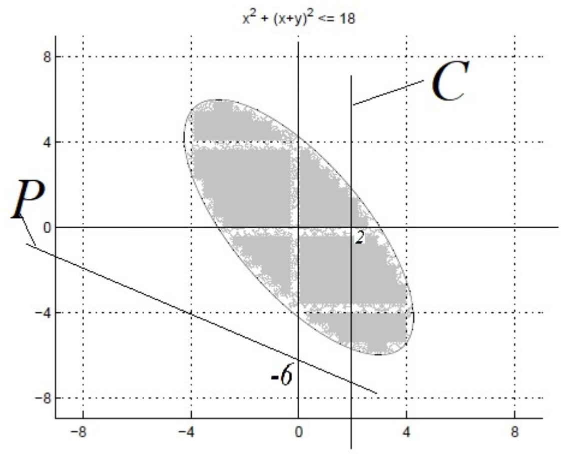
(b) Formulate the problem of locating the little house as an optimization problem (advice: pay particular attention to the simplicity of the objective function).

(c) Write down the KKT conditions for this optimization problem.

(d) Using these conditions, determine the optimal location for the house (we can assume that the condition of linear independence of the gradients of the active constraints is always satisfied).

Solution

(a)



(b) the house is located as close as possible to the river $\mathcal{R}$. Consequently, we should minimize a distance $d((x, y), \mathcal{P})$ from a point (x, y) to a line $ax + by + c = 0$. We have

$$d((x, y), \mathcal{P}) = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}.$$

Consequently,

$$\min \frac{|x + 2y + 12|}{\sqrt{5}}.$$

such that

$$x^2 + (x+y)^2 \le 18, \quad x \le 2.$$

Using the Graphically sketch we can conclude that $x + 2y + 12 \ge 0$ and it is enough to consider the following problem:

$$\min_{x,y} x + 2y.$$

such that

$$x^2 + (x+y)^2 \le 18, \quad x \le 2.$$

(c) KKT condition:

$$L(x, y, \lambda_1, \lambda_2) = x + 2y - \lambda_1(18 - x^2 - (x+y)^2) - \lambda_2(2 - x)$$

$$\begin{cases} \nabla_x L = 0 \\ \lambda_1(18 - x^2 - (x+y)^2) \\ \lambda_2(2-x) \\ \lambda_1, \lambda_2 \ge 0 \\ 2 - x \ge 0 \\ 18 - x^2 - (x+y)^2 \ge 0 \end{cases}$$

Consequently,

$$\begin{cases} 1 + \lambda_1(4x + 2y) + \lambda_2 = 0 & (1) \\ 1 + \lambda_1(x+y) = 0 & (2) \\ \lambda_1(18 - x^2 - (x+y)^2) & (3) \\ \lambda_2(2-x) & (4) \\ \lambda_1, \lambda_2 \ge 0 & (5) \\ 2 - x \ge 0 & (6) \\ 18 - x^2 - (x+y)^2 \ge 0 & (7) \end{cases}$$

* $\lambda_1 = 0, \lambda_2 = 0$ – impossible

* $\lambda_1 = 0, \lambda_2 > 0$ – impossible

* $\lambda_1 > 0, \lambda_2 = 0$:
Using (1) and (2) we get

$$x + y = 4x + 2y.$$

Consequently,

$$y = -3x$$

Using (3) we obtain

$$18 - x^2 - (x+y)^2 = 0.$$

Consequently,

$$x = \pm\sqrt{\frac{18}{5}}.$$

Hence,

$$\left(\sqrt{\frac{18}{5}}, -3\sqrt{\frac{18}{5}}\right), \quad \left(-\sqrt{\frac{18}{5}}, 3\sqrt{\frac{18}{5}}\right)$$

Therefor, if $\lambda_1 > 0$ and $\lambda_2 = 0$ we get

$$\left(\sqrt{\frac{18}{5}}, -3\sqrt{\frac{18}{5}}\right).$$

* $\lambda_1 > 0, \lambda_2 > 0$:
Using (3) and (4) we have

$$18 - x^2 - (x+y)^2 = 0, \quad x = 2.$$

Consequently,

$$14 = (y+2)^2$$

and

$$y = -2 \pm \sqrt{14}.$$

Hence,

$$(2, -2-\sqrt{14}).$$

Consequently, using (2) we get

$$\lambda_1 = \frac{1}{\sqrt{14}},$$

and using (1) we have

$$1 + \frac{1}{\sqrt{14}}(8 - 4 - 2\sqrt{14}) + \lambda_2 = 0.$$

Therefore, $\lambda_2 < 0$.

Obviously, the feasible set is convex, the initial function is linear. Consequently, we have a convex optimization problem. Therefore,

$$\left(\sqrt{\frac{18}{5}}, -3\sqrt{\frac{18}{5}}\right).$$

is a global minimum.

5. Example of an exam question We have the following problem to solve:

$$\max_{x,y} xy$$

such that $x \geq 0$, $y \geq 0$ and $x + y^2 \leq 2$

(a) determine whether it is a convex optimization problem. (advice: consider an equivalent min problem).

(b) Write down the KKT conditions for this optimization problem.

Solution

Let's consider the following equivalent problem

$$\min_{x,y} -xy$$

such that $x \geq 0$, $y \geq 0$ and $x + y^2 \leq 2$.

If $f(x,y) = -xy$ then

$$\nabla^2 f = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \preceq 0.$$

Consequently, the function is concave and our problem is not convex.

It is clear $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y^2 \leq 2\}$ is bounded and closed. Hence, D is compact.

(b) KKT conditions:

$$L(x, y, \lambda_1, \lambda_2) = -xy - \lambda_1 x - \lambda_2 y - \lambda_2(2 - x^2 - y^2),$$

$$\begin{cases} -y - \lambda_1 + \lambda_3 = 0 & (1) \\ -x - \lambda_2 + 2\lambda_3 y = 0 & (2) \\ \lambda_1 x \geq 0 & (3) \\ \lambda_2 y \geq 0 & (4) \\ \lambda_3(2 - x^2 - y^2) \geq 0 & (5) \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 & (6) \\ x \geq 0, y \geq 0, 2 - x^2 - y^2 \geq 0 & (7) \end{cases}.$$

* $\lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \Longrightarrow \quad x = y = 0.$

* $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 > 0.$
Using (1) we get $y = \lambda_3$.
Using (2) we have $-x + 2y^2 = 0$.
Using (5) we obtain $2 - x - y^2 = 0$.
Consequently, $3y^2 = 2 \Longrightarrow y = \pm\sqrt{\frac{2}{3}}$. We know that $y > 0$.
We have

$$\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$$

* $\lambda_1 = 0,\ \lambda_2 > 0,\ \lambda_3 = 0.$
Using (4) we get $y = 0$.
Using (2) we obtain $x = -\lambda_2$ (impossible).

* $\lambda_1 = 0,\ \lambda_2 > 0,\ \lambda_3 > 0.$
Using (1) we get $y = \lambda_3$. Consequently, using (7) we have $y = 0$ (impossible).

* $\lambda_1 > 0,\ \lambda_2 = 0,\ \lambda_3 = 0.$
Using (3) we get $x = 0$. Consequently, using (1) we have $y + \lambda_1 = 0$ (impossible, $\lambda_1 > 0$ and $y \geq 0$).

* $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 = 0.$
Using (3) and (4) we get $x = y = 0$. Consequently, using (2) we have $\lambda_2 = 0$ (impossible).

* $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0$ (impossible).

* $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 > 0$ (impossible).

Consequently, we have

$$(0,0), \quad \left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right).$$