Optimisation

Lecture 10 - Convexity

Fall semester - 2024

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Innopolis University
November 5, 2024

Outline

Convex Sets

Convex set

A set $X \subset \mathbb{R}^n$ is convex if and only if

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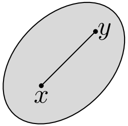

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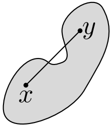

Non-convex set

Examples of convex sets :

- The sets $\emptyset, \mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_{++}$
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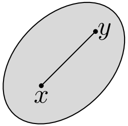

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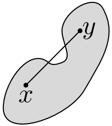

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- The half-spaces open or closed : $\{x \mid c^T x < b\}$ and $\{x \mid c^T x \leqslant b\}$, as well as hyper-plans $\{x \mid c^T x = b\}$

Domain defined by functional constraints

In optimization problems, the feasible domain/set X is often defined by functional constraints :

$$X = \{x \mid h_i(x) = 0 \text{ for } i \in \mathcal{E} \text{ and } h_i(x) \leqslant 0 \text{ for } i \in \mathcal{I}\}.$$

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We can demonstrate that if

- each function $h_i(x),\ i \in \mathcal{E}$ defining an equality is affine and
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(this condition is sufficient but not necessary)

Combining convex sets

If two sets $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$are convex,
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$$g(x) = Ax + b,$$

we can show that $g(X) = \{g(x) \mid x \in X\}$, the image of X by g, is convex.

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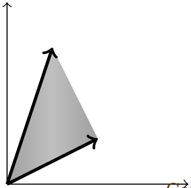
Many operations preserve convexity (including linear transformations, but also sums, etc.) but not the union of two sets.

Other convex sets: convex cones

For two points x_1 and x_2, we call

- *convex combination* :
  points defined by $\lambda_1 x_1 + \lambda_2 x_2$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$.



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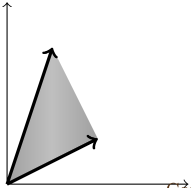

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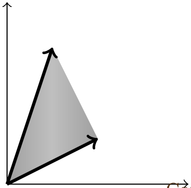
A set $K \in \mathbb{R}^n$ is called a *convex cone* if

1. for every $x \in K$ and $\lambda \geqslant 0$, we have $\lambda x \in K$,
2. for every $x_1, x_2 \in K$, we have $\lambda_1 x_1 + \lambda_2 x_2 \in K$ for all $\lambda_1, \lambda_2 \geqslant 0$.

or a convex cone is a set that contains all the conic combinations of the points in the set.



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- The set of nonnegative continuous functions is a convex cone.

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Convex function: several definitions

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Using the definition of *convex set*.

- Epigraph of f is convex

A function f is said to be **concave** if and only if $-f$ is convex.

Convex function: definition 1

A function f defined on a domain X is a convex function if and only if

- its domain X is a convex set and,
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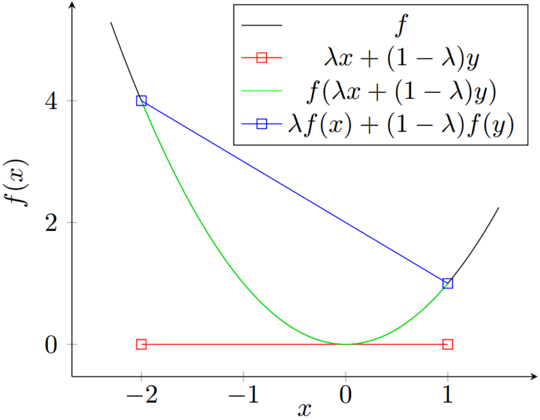
In other words, f is convex if it lies below the strings that underlie its graph :



Convex function: definition 2

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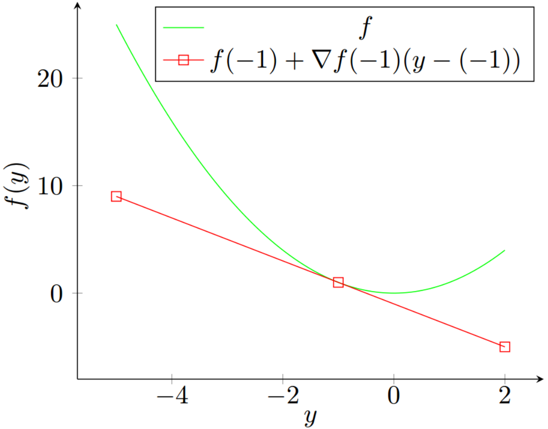
In other words, f lies above all its Taylor approximations of order 1 :



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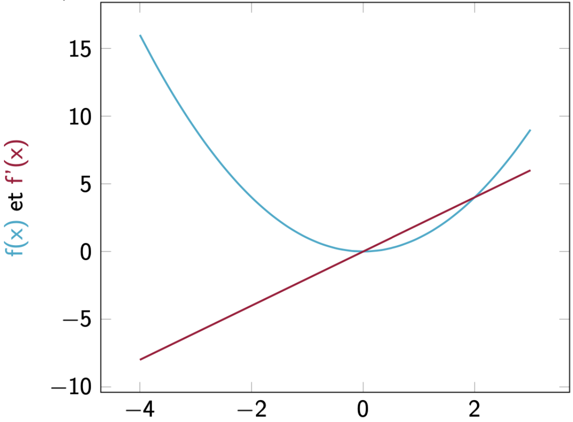
In other words, the derivative of f is monotonically increasing:



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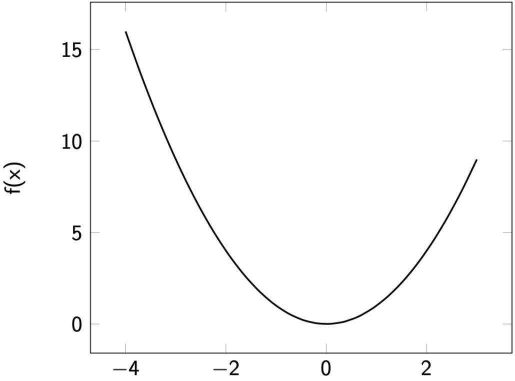
In other words, this means that the graph of f has positive curvature over its entire domain X :



Convex function: definition 5 (epigraph)

A function f defined on a domain X is a convex function if and only if

- the epigraph of f on X is a convex set.

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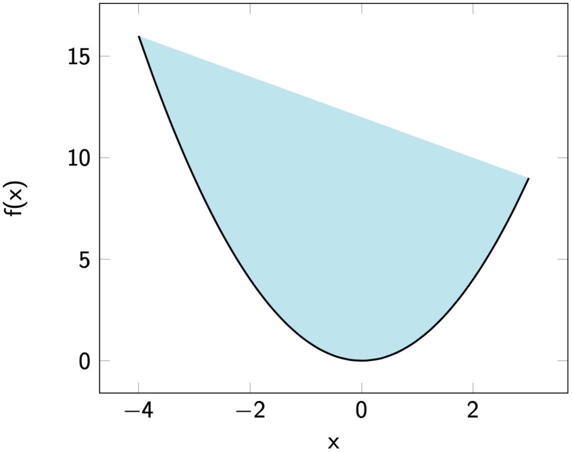

Examples of convex functions

- Examples of convex functions with one variable :
  - $f(x) = x^\alpha$ for $\alpha \geqslant 1$ or $\alpha \leqslant 0$ is convex over $\mathbb{R}_{++}$,
  - $f(x) = |x|^\alpha$ for $\alpha \geqslant 1$,
  - $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{R}$,
  - $f(x) = -\log x$ is convex over $\mathbb{R}_{++}$,
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- The constant, linear and affine functions are convex :
  - $f(x) = \alpha$
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- The norm function and its square are convex :
  - $f(x) = \|x\|$
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- The quadratic form $f(x) = x^T Q x$ is convex for $Q \succeq 0$

Operations that preserve convexity

Nonnegative linear combination.

- If f is a convex function, αf is also convex for $\alpha > 0$
- If f_1 and f_2 are convex, then their sum $f_1 + f_2$ is convex

These two results imply that $f = \alpha_1 f_1 + ... + \alpha_m f_m$
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The maximum (point by point).
If f_1 and f_2 are convex, then their pointwise maximum, defined by

$$f(x) = \max\{f_1(x), f_2(x)\}$$

is also convex.

Operations that preserve convexity

Composition with a linear transformation $g(x) = Ax + b$.
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Composite functions : $f(x) = h(g(x))$.
If $h(x)$ is *non-decreasing* convex and $g(x)$ convex, then $f(x)$ is convex.
If $h(x)$ is *non-increasing* convex and $g(x)$ concave, then $f(x)$ is convex.

The benefits of convexity in optimization

Convex optimization problem

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(by extension, a maximization problem will be said to be convex if, once converted into a minimization problem, we obtain a convex problem)

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- Convex quadratic optimization under equality constraints

$$\min_{x \in \mathbb{R}^n} x^T Q x + c^T x \text{ such that } Ax = b$$

(with $Q \succeq 0$)

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In the unconstrained case

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Application: 2-norm polynomial regression (least squares) :

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2.$$

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For a convex problem, suppose that the **Slater's condition** is satisfied, the KKT conditions become necessary and sufficient, **without having to consider (LICQ) !**
(for all feasible convex problems including only equality constraints, the KKT conditions are necessary and sufficient.)

Examples

Example of an exam question

Let

$$\mathcal{C} = \{(x, y) \mid x \leqslant y\} \text{ and } \mathcal{D} = \{(x, y) \mid (x - 3)^2 + y^2 \leqslant 2\}.$$

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- Checking KKT conditions is both necessary and sufficient.

Conclusions

Summary

We have seen:

- what is a convex set.
- What is a convex function f: 5 definitions.
- Example of convex functions, and the operations that preserve convexity.
- What is a convex optimization problem: min problem + convex objective function f + convex feasible set.
- Properties of a convex problem: *Any local minimum for this problem is also a global minimum.*, and *The set of its minima form a convex set.*
- The impact on OC's in the convex case:
  1. Unconstrained setting: the first-order optimality condition $\nabla f(x^*)$ becomes necessary and **sufficient**.
  2. Constrained setting: the KKT conditions become **sufficient** (but not necessary).
     Both sufficient and necessary if LICQ verified or Slater's condition is satisfied.

Preparations for the next lecture

- Review the lecture;
- Solve the example of exam question.

Goodbye, So Soon

THANKS FOR THE ATTENTION

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