Optimization - Exercise session 4
Duality (part 1)

1. Find the dual of problems
1.1 $\min_x 3x_1 - 9x_2$
such that

$$2x_1 - x_2 + x_3 \geq 3,$$
$$x_1 - x_2 - 3x_3 \geq 2,$$
$$x_1, x_2, x_3 \geq 0.$$

The dual
$\max_x 3y_1 + 2y_2$
such that

$$2y_1 + y_2 \leq 3,$$
$$-y_1 - y_2 \leq -9,$$
$$y_1 - 3y_2 \leq 0,$$

$$y_1, y_2 \geq 0.$$

1.2 $\min_x 3x_1 - 9x_2$
such that

$$4x_1 - x_2 + x_3 = 3,$$
$$x_1 - x_2 - 6x_3 \geq 2,$$
$$x_1 - 4x_2 - 6x_3 \leq 8,$$
$$x_1, x_2 \geq 0.$$

The dual is
$\max_y 3y_1 + 2y_2 - 8y_3$
such that

$$4y_1 + y_2 - y_3 \leq 3,$$
$$-y_1 - y_2 + 4y_3 \leq -9,$$
$$y_1 - 6y_2 + 6y_3 = 0,$$

$$y_2, y_3 \geq 0.$$

1.3 $\min_x c^T x$ such that

$$a_1^T x = b_1,$$
$$a_2^T x \geq b_2,$$
$$a_3^T x \leq b_3,$$
$$x \geq 0.$$

The dual is
$\max_x b_1 y_1 + b_2 y_2 - b_3 y_3$
such that

$$a_{1,1} y_1 + a_{2,1} y_2 + a_{3,1} y_3 \leq c_1,$$
$$a_{1,2} y_1 + a_{2,2} y_2 + a_{3,2} y_3 \leq c_2,$$
$$\ldots$$
$$a_{1,n} y_1 + a_{2,n} y_2 + a_{3,n} y_3 \leq c_n,$$
$$y_2, y_3 \geq 0.$$

1.4 $\min_{x \in \mathbb{R}^3} c^T x$
such that

$$a^T x = b,$$
$$x_1 free$$

,

$$x_2 \geq 0,$$

$$x_3 \leq 0.$$

This problem is equivalent to
$\min_{x\in\mathbb{R}^3} c_1x_1^+ - c_1x_1^- + c_2x_2 - c_3x_3'$
such that

$$a_1x_1^+ - a_1x_1^- + a_2x_2 - a_3x_3 = b,$$

$$x_1^+, x_1^-, x_2, x_3' \geq 0.$$

$$A = \begin{pmatrix} a_1 & -a_1 & a_2 & -a_3 \end{pmatrix}$$

The dual is
$\max_{y\in\mathbb{R}^3} by_1$
such that

$$a_1y_1 \leq c_1,$$

$$-a_1y_1 \leq -c_1$$

$$a_2y_1 \leq c_2,$$

$$-a_3y_1 \leq -c_3,$$

$$y_1 - free.$$

Hence,
$\max_{y\in\mathbb{R}^3} by_1$
such that

$$a_1y_1 = c_1,$$

$$a_2y_1 \leq c_2$$

$$a_3y_1 \geq c_3,$$

$$y_1 - free.$$

2. Find a dual formulation of problems
2.1 $\min_x c^T x$
such that

$$Ax = b,$$
$$x \geq a.(a \geq 0)$$

2.2
$\min_x c^T x$
such that

$$b_1 \leq Ax \leq b_2,$$
$$x \geq 0.$$

Solution

2.1 This problem is equivalent to
$\min_{x'} c^T x'$
such that

$$Ax' = b - Aa,$$
$$x' \geq 0,$$

where $x' = x - a$.
Hence, the dual is
$\max_a (b - Aa)^T y$
such that

$$A^T y \leq c,$$
$$y \in \mathbb{R}.$$

2.2 This problem is equivalent to
$\min_x c^T x$
such that

$$Ax \geq b_1,$$
$$-Ax \geq -b_2,$$
$$x \geq 0.$$

Hence, the dual is
$\max_{y,y'} b_1^T y - b_2^T y'$
such that

$$A^T y - A^T y' \leq c,$$
$$y \geq 0.$$

where $c = [c', c'']$.

3. How to solve the problem $\min_x 50x_1 + 25x_2$
such that

$$x_1 + 3x_2 \geq 8,$$
$$3x_1 + 4x_2 \geq 19,$$
$$3x_1 + x_2 \geq 7,$$
$$x_1, x_2 \geq 0,$$

without an initialization phase?

Solution

To begin with we need to write the dual. We have
$\max_y 8y_1 + 19y_2 + 7y_3$
such that

$$y_1 + 3y_2 + 3y_3 \leq 50,$$
$$3y_1 + 4y_2 + y_3 \leq 25,$$
$$y_1, y_2, y_3 \geq 0.$$

Let's transform into a min problem in standard form. We obtain
$\min_y -8y_1 - 19y_2 - 7y_3$
such that

$$y_1 + 3y_2 + 3y_3 + y_4 = 50,$$
$$3y_1 + 4y_2 + y_3 + y_5 = 25,$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0.$$

Use simplex method, we get
Based on the optimal dual solution, derive the optimal primal solution with strong duality and complementary slackness
If $y_1 = y_2 = y_3 0$ then a vertex is $(0, 0, 0, 50, 25)$.
Using the previous equalities, we get

Basic	y_1	y_2	y_3	y_4	y_5	Solution
z	-8	-19	-7	0	0	0
y_4	1	3	3	1	0	50
y_5	3	4	1	0	1	25

The solution (0, 0, 0, 50, 25) is the BFS associated with the basic variables x_4 and x_5.
The reduced costs associated with the variables x_1, x_2 and x_3 are negative.
Since the most negative number in z-line is -19, we choose to enter x_2 in the base.
We can't increase x_2 without limit, since we have to satisfy the constraints

$$\begin{cases} 3x_2 + x_4 = 50, \\ 4x_2 + x_5 = 25. \end{cases}$$

The second constraint is the most restrictive since $\frac{50}{3} \geq \frac{25}{4}$. It's x_5 that leaves the base.
Iteration 1.

Basic	y_1	y_2	y_3	y_4	y_5	Solution
z	25/4	0	-9/4	0	19/4	475/4
y_4	-5/4	0	9/4	1	-3/4	125/4
y_2	3/4	1	1/4	0	1/4	25/4

Iteration 2.
The solution (0, 25/4, 125/4, 0, 0) is the new BFS associated with the basic variables y_2 and y_4.
The non-basic variables are y_1, y_3 and y_5.

The reduced costs associated with the variables y_3 is negative. The cost decreases if y_3 increases.
We choose to enter y_3 in the base. We can't increase x_3 without limit, since we have to satisfy the constraints:
1. The first imposes $9/4y_1 + y_4 = 125/4$.
2. and the second $x_2 + 1/4x_3 = 25/4$.
The first constraint is the most restrictive since $125/9 \geq 25$. It's y_3 that leaves the base.
After elementary transformations on the rows, we obtain the canonical table

Basic	y_1	y_2	y_3	y_4	y_5	Solution
z	5	0	0	1	4	150
y_3	-5/9	0	1	4/9	-1/3	125/9
y_2	8/9	1	0	-1/9	1/3	25/9

The associated reduced costs are positive. Therefore, the solution is optimal. The minimum is attained at $(0, 25/9, 125/9)$ and is equal to -150.
If x is an optimal solution of the primal and y is an optimal solution of the dual, then

$$(0, 25/9, 125/9)\,(Ax - b) = 0$$

where

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \\ 3 & 1 \end{pmatrix}$$

$$b^T = (8, 19, 7)$$

Consequently,

$$(0, 25/9, 125/9) \begin{pmatrix} x_1 + 3x_2 - 8 \\ 3x_1 + 4x_2 - 19 \\ 3x_1 + x_2 - 7 \end{pmatrix} = 0$$

Hence,

$$3x_1 + 4x_2 = 19$$
$$3x_1 + x_2 = 7$$

Therefore, the optimal solution of the initial problem is $(1, 4)$ and by strong duality the maximum is equal to 150.

6. Consider the following problem
$\min_x 2x_1 + 9x_2 + 3x_3$
such that

$$-2x_1 + 2x_2 + x_3 \geq 1,$$

$$x_1 + 4x_2 - x_3 \geq 1,$$

$$x_1, x_2, x_3 \geq 0.$$

Find the dual of this problem and solve it graphically. Use complementary slackness conditions to obtain a solution of the primal.

Solution

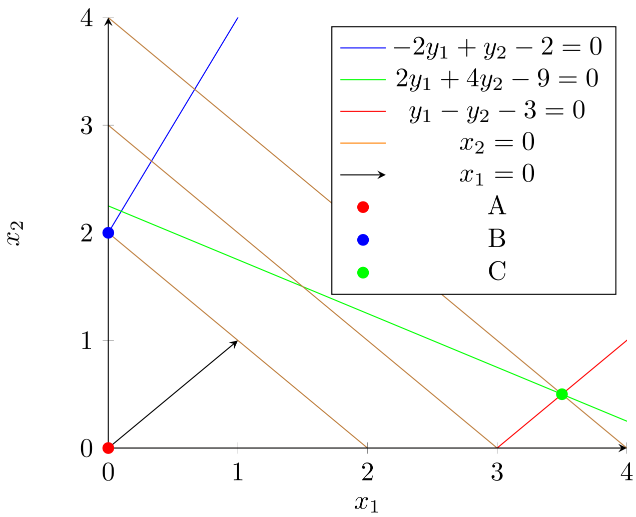
The dual is
$\max_y y_1 + y_2$
such that

$$-2y_1 + y_2 \leq 2,$$

$$2y_1 + 4y_2 \leq 9,$$

$$y_1 - y_2 \leq 3,$$

$$y_1, y_2 \geq 0.$$



It is clear that the maximum is attained at the green point $(7/2, 1/2)$ and is equal to 4.
If x is an optimal solution of the primal and y is an optimal solution of the dual, then

$$(7/2, 1/2)\,(Ax - b) = 0$$

where

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 1 & 4 & -1 \end{pmatrix}$$

$$b^T = (1, 1)$$

Consequently,

$$(7/2, 1/2) \begin{pmatrix} -2x_1 + 2x_2 + x_3 - 1 \\ x_1 + 4x_2 - x_3 - 1 \end{pmatrix} = 0$$

Hence,

$$\begin{gathered} -2x_1 + 2x_2 + x_3 = 1 \\ x_1 + 4x_2 - x_3 = 1 \\ 2x_1 + 9x_2 + 3x_3 = 4 \end{gathered}$$

Solution of this system of linear equations is $(0, 1/3, 1/3)$. Therefore, the optimal solution of the initial problem is $(0, 1/3, 1/3)$ and is equal to 4.