Correction Q3 : Duality

P1

a) We first reformulate the problem as a minimization problem

$$\min_{x_1, x_2} \; -x_1 - 2x_2$$

$$\begin{aligned}
\text{s.t.} \quad & 2x_1 + 3x_2 = 6 && (1)\\
& 2x_1 + x_2 \geq 4 && (2)\\
& -x_1 - 2x_2 \geq -4 && (3)\\
& x_1 \geq 0, \; x_2 \text{ free}
\end{aligned}$$

We associate to eq. and ineq. constraints the following dual variables:

- y_1 free since (1) is an eq. cons.
- $y_2, y_3 \geq 0$ since (2) and (3) are ineq. cons.

We obtain the following dual $\quad$ P2

$$\max_{y_1, y_2, y_3} \quad 6y_1 + 4y_2 - 4y_3$$

$$\text{s.t.} \quad 2y_1 + 2y_2 - y_3 \leq -1 \quad (4)$$

$$3y_1 + 2y_3 - 2y_3 = -2 \quad (5)$$

$$y_1 \text{ free}, \; y_2, y_3 \geq 0$$

Note that primal variables x_1 and x_2 are respectively associated to constraints (4) and (5). Since $x_1 \geq 0 \Rightarrow$ (4) is ineq., and since x_2 is free $\Rightarrow$ (5) is an equality.

Correction Q3 : Duality

b) For any $x \in P$ (primal feasible) and $y \in D$ (dual feasible), the weak duality holds, that is

$$c^T x \geq b^T y$$

$$\iff -x_1 - 2x_2 \geq 6y_1 + 4y_2 - 4y_3$$

Let us prove it for our <u>problem</u> !

We have:

$$-x_1 - 2x_2 \geq (2 \cdot y_1 + 2y_2 - y_3)\, x_1 + (3y_1 + y_2 - 2y_3)\, x_2$$

<u>Since</u> : $-1 \cdot x_1 \geq (2y_1 + 2y_2 - y_3)\, x_1$ with $\underline{x_1 \geq 0}$!

(using ineq. (4))

and $-2 \cdot x_2 = (3y_1 + y_2 - 2y_3) \cdot x_2$
for all x_2 (using eq. (5))

We continue:

$$-x_1 - 2x_2 \geqslant (2y_1 + 2y_2 - y_3)x_1 + (3y_1 + y_2 - 2y_3)x_2$$

$$= \underset{\text{free}}{y_1}\underbrace{(2x_1 + 3x_2)}_{= 6 \text{ per } (1)} + \underset{\geqslant 0}{y_2}\underbrace{(2x_1 + x_2)}_{\geqslant 4 \text{ per } (2)}$$

$$+ \underset{\geqslant 0}{y_3}\underbrace{(-x_1 - 2x_2)}_{\geqslant -4 \text{ per } (3)}$$

then $\geqslant 6y_1 + 4y_2 - 4y_3$ ok!

<u>Correction Q 3 : Duality</u>

$\hookrightarrow$ Let us write the C.S. conditions for <u>our</u> pair Primal - Dual :

$$\left(-1-2y_1^*-2y_2^*+y_3^*\right)\cdot x_1^* = 0 \quad (6)$$

$$\left(2x_1^*+x_2^*-4\right)\cdot y_2^* = 0 \quad (7)$$

$$\left(-x_1^*-2x_2^*+4\right) y_3^* = 0 \quad (8)$$

Assuming $x_1^* = 3/2$ and $x_2^* = 1$, we have

$\rightarrow (7) : \underbrace{\left(\cancel{2}\cdot 3/\cancel{2} + 1 - 4\right)}_{=0}\, y_2^* = 0$

$= 0$ then $y_2^* > 0$

$\rightarrow (8) : \underbrace{\left(-3/2 - 2 + 4\right)}_{=1/2 \neq 0}\cdot y_3^* = 0$

$= 1/2 \neq 0 \Rightarrow \underline{y_3^* = 0}$!

P6

$\rightarrow (6): x_1^* = 3/2 > 0$

$$\Rightarrow -1 - 2y_1^* - 2y_2^* + y_3^* = 0 \;!$$

To compute y_1^* and y_2^*, we need an additional equation; let us consider (5) since y_1^*, y_2^*, y_3^* has to be feasible.

We have the following system:

$$\begin{cases} -1 - 2y_1^* - 2y_2^* + y_3^* = 0 \\ y_3^* = 0 \\ 3y_1^* + y_2^* - 2y_3^* = -2 \end{cases}$$

We solve and obtain: $\begin{cases} y_1^* = -3/4 \\ y_2^* = 1/4 \\ y_3^* = 0 \end{cases}$

Correction Q 3 : Duality

To show that $x^* = (3/2, 1)$ was indeed the optimal solution for the primal, we use the strong duality: If for any $x^* \in P$ and $y^* \in D$, we have

$$c^T x^* = b^T y^*$$

then x^* and y^* are respectively optimal solutions for the primal and the dual. Let us check this:

$\to$ $x^* = (3/2, 1)$ is primal feasible

$y^* = (-3/4, 1/4, 0)$ is dual feasible

P8

and

$c^T x^* = -3/2 - 2 = -7/2$

$b^T y^* = 6\left(-\frac{3}{4}\right) + 4\left(\frac{1}{4}\right) + 0 = -\frac{7}{2}$

$= !$

$\Longrightarrow$ x^* and y^* are optimal!