Optimization - Exercise session 7
Discrete optimization

1. Consider the function $f(x_1,x_2) = (x_1-1)^2+4(x_2-1)^3-3x_2^2$. Find the stationary points of this function and determine their nature. Does the function have a global minimum? a global maximum?

Solution

A stationary point (or critical point) of a function f is a point x^* satisfying the first-order optimality condition:

$$\nabla f(x_1,x_2) = (0,0).$$

Consequently,

$$\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} 2(x_1-1) = 0 \\ 12(x_2-1)^2 - 6x_2 = 0 \end{cases}.$$

Hence, $x_1 = 1$ and

$$12x_2^2 + 12 - 24x_2 - 6x_2 = 0 \quad \Rightarrow$$

$$12x_2^2 - 30x_2 + 12 = 0 \quad \Rightarrow$$

$$x_{2,1} = 2;$$

$$x_{2,2} = \frac{1}{2};$$

Therefore, we get two stationary points $x^{'} = (1,2)$ and $x^{''} = (1,\frac{1}{2})$.

To determine their nature let's check second-order optimality condition. Obviously,

$$\nabla^2 f(x_1,x_2) = \begin{pmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{pmatrix}$$

and

$$\nabla^2 f(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 18 \end{pmatrix}$$

Consequently, $\lambda_1 = 2$ and $\lambda_2 = 18$. Hence, $\nabla^2 f(1,2) \succ 0$. Therefore, $(1,2)$ is local minimum.

$$\nabla^2 f(1,1/2) = \begin{pmatrix} 2 & 0 \\ 0 & -18 \end{pmatrix}$$

Consequently, $\lambda_1 = 2$ and $\lambda_2 = -18 < 0$. Therefore, $(1,1/2)$ is a saddle point.

f is unbounded, since

$$\lim_{x_2 \to -\infty} (x_1-1)^2 + 4(x_2-1)^3 - 3x_2^2 = -\infty$$

$$\lim_{x_2 \to +\infty} (x_1-1)^2 + 4(x_2-1)^3 - 3x_2^2 = +\infty$$

Therefore, we have local minimum and don't have local maximum.

2. For the following optimization problems, (i) compute the gradient and the Hessian matrix of the objective function, (ii) identify the stationary points, (iii) eliminate those that do not satisfy the necessary conditions for optimality, and (iv) identify those that satisfy the sufficient conditions for optimality.

(a) $\min x_1^2 + x_2^2$.

(b) $\min 1/3x_1^3 + x_2^3 - x_1 - x_2$.

(c) $\min x^2 + \frac{1}{x-3/2}$.

(d) $\min x_1^6 - 3x_1^4x_2^2 + 3x_1^2x_2^4 - x_2^6$.

Solution

(a)

(i)

$$\nabla f = (2x_1, 2x_2).$$

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(ii)

$$\nabla f = (2x_1, 2x_2) = (0,0) \Leftrightarrow (x_1, x_2) = (0,0)$$

(iii)

$$\nabla^2 f \succeq 0$$

(iv)

$$\lambda_1 = \lambda_2 = 2 > 0$$

(b)

(i)

$$\nabla f = (x_1^2 - 1, 2x_2^2 - 1).$$

$$\nabla^2 f(0,0) = \begin{pmatrix} 2x_1 & 0 \\ 0 & 6x_2 \end{pmatrix}$$

(ii)

$$\nabla f = (0,0) \Leftrightarrow \begin{cases} x_1^2 - 0 & \Rightarrow x_1 = \pm 1 \\ 3x_2^2 - 1 = 0 & \Rightarrow x_2 = \pm\sqrt{3}/3 \end{cases}$$

Hence, we have the following critical points:

$$(1, \sqrt{3}/3), \quad (1, -\sqrt{3}/3), \quad (-1, \sqrt{3}/3), \quad (-1, -\sqrt{3}3)$$

(iii)

$$\nabla^2 f(1, -\sqrt{3}/3), \quad \nabla^2 f(-1, \sqrt{3}/3),$$

are semi Definite.

(iv)

$$\nabla^2 f(1, \sqrt{3}/3)$$

is positive Definite.

$$\nabla^2 f(-1, -\sqrt{3}/3)$$

is negative definite.

(c) $f(x) = x^2 + \frac{1}{x-3/2}$.

(i)

$$f'(x) = 2x - \frac{1}{(x-3/2)^2}, \quad f''(x) = 2 + \frac{1}{(x-3/2)^3}$$

(ii)

$$f'(x) = 0 \Leftrightarrow$$

$$2x - \frac{1}{(x-3/2)^2} = 0 \Leftrightarrow$$

$$2x\frac{1}{(x-3/2)^2}=1 \Leftrightarrow$$

$$2x^3-6x^2+9/2x-1=0 \Leftrightarrow$$

$$\frac{1}{2}(x-2)(2x-1)^2=0 \Leftrightarrow$$

$$x=2 \quad \text{or} \quad x=\frac{1}{2}$$

(iii)

$$f''(2) \geq 0, \quad f'(1/2) \geq 0 \Rightarrow$$

2 and 1/2 satisfy the necessary conditions for optimality.

(iv)

$$f''(2)=0, \quad f'(1/2)=0 \Rightarrow$$

2 do not satisfy the sufficient conditions for optimality.

(d)

$$f(x_1,x_2)=x_1^6-3x_1^4x_2^2+3x_1^2x_2^4-x_2^6$$

(i)

$$\nabla f(x_1,x_2)=(6x_1^5-12x_1^3x_2^2+6x_1x_2^4,-6x_1^4x_2+12x_1^2x_2^3-6x_2^5)$$

$$\nabla^2 f(x_1,x_2)=\begin{pmatrix} 30x_1^4-36x_1^2x_2^2+6x_2^4 & -24x_1^3x_2+24x_1x_2 \\ -24x_1^3x_2+24x_1x_2^3 & -30x_2^4+36x_1^2x_2^2-6x_1^4 \end{pmatrix}$$

(ii)

$$\nabla f(x_1,x_2)=0 \Leftrightarrow$$

$$\begin{cases} 6x_1^5-12x_1^3x_2^2+6x_1x_2^4=0 \\ -6x_1^4x_2+12x_1^2x_2^3-6x_2^5=0 \end{cases} \Rightarrow$$

$$\begin{cases} x_1(x_1^4-2x_1^2x_2^2+x_2^4)=0 \\ x_2(x_1^4-2x_1^2x_2^3+x_2^4=0 \end{cases} \Rightarrow$$

$$\begin{cases} x_1=0 \quad \text{or} \quad (x_1^2-x_2^2)^2=0 \\ x_2=0 \quad \text{or} \quad (x_1^2-x_2^2)^2=0 \end{cases} \Rightarrow$$

$\{(x_1,x_2):x_1=x_2\}\bigcup\{(x_1,x_2):x_1=-x_2\}$ are stationary points.

(iii) If $x_1=x_2$ then

$$\nabla^2 f(x_1,x_2)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $x_1=-x_2$ then

$$\nabla^2 f(x_1,x_2)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \; . \end{pmatrix}$$

Hence, $\{(x_1,x_2):x_1=x_2\}\bigcup\{(x_1,x_2):x_1=-x_2\}$ satisfy the necessary conditions for optimality.

(iv)

$\{(x_1,x_2):x_1=x_2\}\bigcup\{(x_1,x_2):x_1=-x_2\}$ don't satisfy the sufficient conditions for optimality.

3. Consider the function

$$f(x_1, x_2) = (x_1 - 1)^2 + \lambda(x_1^2 - x_2)^2$$

for λ. Find the stationary points of f and discuss their nature in terms of λ.

Solution

$$\nabla f(x_1, x_2) = 0 \Rightarrow \begin{array}{l} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{array} \Rightarrow$$

$$\left\{ \begin{array}{c} 2(x_1 - 1) + 4\lambda x_1(x_1^2 - x_2) = 0 \\ -2\lambda(x_1^2 - x_2) = 0 \end{array} \right. \qquad \Rightarrow$$

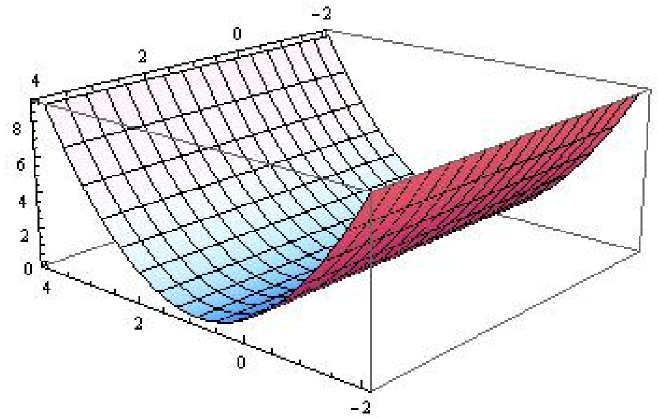
If $\lambda = 0$ then $2(x_1 - 1) = 0$. Hence, $x_1 = 1$ for any x_2. Therefore, $(1, x_2)$ are stationary points.
If $\lambda \neq 0$ then $x_1^2 - x_2 = 0$. Hence, $x_1 = 1$. Therefore, $(1, 1)$ is stationary points.
We have

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 2 + 12\lambda x_1^2 - 4\lambda x_2 & -4\lambda x_1 \\ -4\lambda x_1 & 2\lambda \end{pmatrix}.$$

If $\lambda = 0$, then

$$\nabla^2 f(1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $(1, x_2)$ satisfy the necessary conditions for optimality. Moreover, $(1, x_2)$ is minimum of $f(x_1, x_2) = (x_1 - 1)^2$ since



If $\lambda \neq 0$ then

$$\begin{pmatrix} 2 + 8\lambda & -4\lambda \\ -4\lambda & 2\lambda \end{pmatrix}.$$

Hence,

$$\det(\nabla^2 f(1, 1)) = 4\lambda$$

$$tr(\nabla^2 f(1, 1)) = 2 + 10\lambda$$

If $\lambda < 0$ then $(1, 1)$ is a saddle point.
If $\lambda > 0$ then $(1, 1)$ is a local minimum.
Since $f(1, 1) \leq f(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^n$, $(1, 1)$ is a global minimum.

4. Determining the local minima for each of the functions below, justifying carefully, and identifying which of these are global and which are strict :

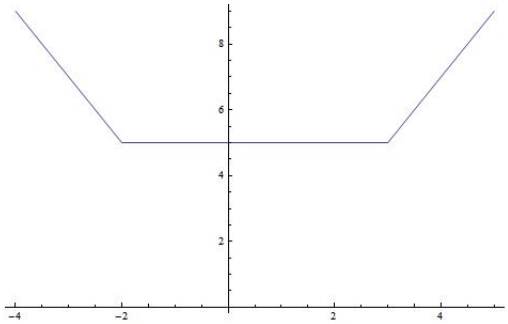
$$\min_{x} \in \mathbb{R} f(x).$$

(a) $f(x) = |x+2| + |x-3|$.
(b) $f(x) = x\sin x + \cos x$.
(c) $f(x) = \begin{cases} x^2 - 4x + 2, & \text{when} \quad x < 1; \\ x^3 - 5x + 3 & \text{when} \quad x \geq 1. \end{cases}$
(d) $f(x) = \max\{x^2 - 3x + 3, 3x^2 + x - 3\}$.

Solution

(a)

$$f(x) = \begin{cases} -2x + 1, & \text{if} \quad x \leq -2 \\ 5, & \text{if} \quad -2 < x \leq 3 \\ 2x - 1 & \text{if} \quad x \geq 3 \end{cases}$$



Global minimums $x^* \in [2,3]$ and $f(x^*) = 5$.
(b)

$$f'(x) = x\cos x + \sin x - \sin x = x\cos x = 0.$$

$$x = 0 \quad \text{or} \quad x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}.$$

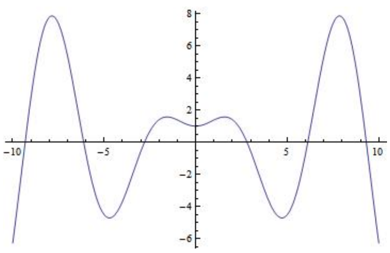
$$f''(x) = \cos x - x\sin x.$$

$$f''(0) = 1 > 0$$

Hence, $x = 0$ is a local minimum.

$$f''\left(\frac{\pi}{2} + k\pi\right) = -\left(\frac{\pi}{2} + k\pi\right)(-1)^k = \left(\frac{\pi}{2} + k\pi\right)(-1)^{k+1} > 0$$

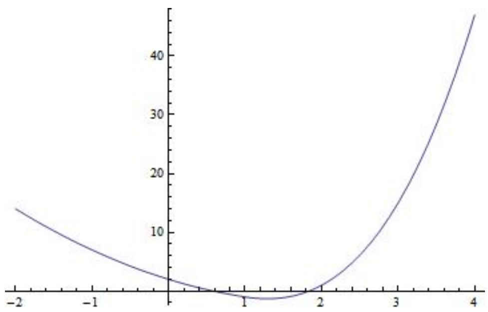
If k is odd then $f''\left(\frac{\pi}{2} + k\pi\right) < 0$. Hence, local maximum.
If k is even then $f''\left(\frac{\pi}{2} + k\pi\right) < 0$. Hence, local maximum.



Since $f(\pi) = -1 < f(0) = 1$ and

$$f\left(\frac{\pi}{2} + k\pi\right) = \frac{\pi}{2} + k\pi < \left(\frac{\pi}{3} + 2k\pi\right) = \left(\frac{\pi}{2} + k\pi\right)\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\pi\sqrt{3}}{6} + k\sqrt{3}\pi + \frac{1}{2},$$

f is unbounded and there are no global minimum and maximum.
(c)



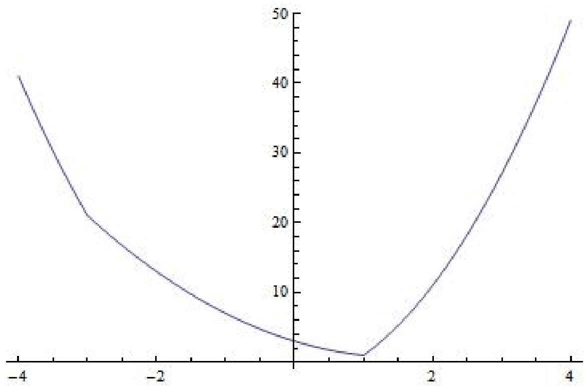
$$0 = f'(x) = \begin{cases} 2x - 4, & \text{if} \quad x < 1 \\ 3x^2 - 5 & \text{if} \quad x \geq 1 \end{cases}$$

Hence, $x = 2$ and $x = \sqrt{\frac{5}{3}}$. Since $2 > 1$ we will not consider this point.
Taking into account that

$$f''(x) = \begin{cases} 2, & \text{if} \quad x < 1 \\ 6x & \text{if} \quad x \geq 1 \end{cases}$$

and $f\left(\sqrt{\frac{5}{3}}\right) > 0$, we get $\sqrt{\frac{5}{3}}$ is a local minimum.

Using that f is decreasing for $x < 1$ and increasing for $x > 1$ we can conclude that $\sqrt{\frac{5}{3}}$ is a global minimum.
(d)



It easy to show that

$$f(x) = \begin{cases} 3x^2 + x - 3, & \text{if} \quad x \in (-\infty, -3) \bigcup (1, +\infty) \\ x^2 - 3x + 3, & \text{if} \quad x \in [-3, 1] \end{cases}$$

We have used that $x^2 - 3x + 3 - 3x^2 - x + 3 = -2x^2 - 4x + 6 = -2(x+3)(x-1)$. We get

$$f'(x) = \begin{cases} 6x + 1, & \text{if} \quad x \in (-\infty, -3) \bigcup (1, +\infty) \\ 2x - 3, & \text{if} \quad x \in [-3, 1] \end{cases}$$

and

$$\begin{cases} f'(x) > 0, & \text{if} \quad x \in (-\infty, -3) \\ f'(x) < 0, & \text{if} \quad x \in (1, +\infty) \\ f'(x) < 0, & \text{if} \quad x \in [-3, 1] \end{cases}$$

Therefore, the point 1 is a global minimum.

5. Consider the quadratic function $f(x) = \frac{1}{2}x^T Q x - c^T x$ with Q symmetrical. Under what condition does this function have a stationary point? A local minimum? A local maximum? A stationary point but no local minimum or maximum?

Solution

It is ease to see that

$$\nabla f(x) = QX - c = 0.$$

Consequently, if $\nabla f(x^*) = 0$ then $Qx^* = c$. We can find x^* if Q is invertible.

x^* is local minimum.

6. If possible, find a function f of two variables and a point point x that maximizes f and for which $\nabla^2 f(x^*) \succeq 0$. Same question with $\nabla^2 f(x^*) \succ 0$.

Solution

(a)

$$f(x_1, x_2) = -x_1^4 - x_2^4$$

(0,0) is a maximum point and

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0.$$

(b)
If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimum of f. Contradiction.

7. Demonstrate the following assertions by finding a counterexample with f a function of $\mathbb{R}^2 \to \mathbb{R}$ and $x^* = (1,1)$.

(a) The condition $\nabla f(x^*) = 0$ is not sufficient for x^* to be a minimum.

(b) The condition $\nabla^2 f(x^*) \succeq 0$ and $\nabla f(x^*) = 0$ is not sufficient for x^* to be a minimum.

(c) The condition $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$ is not necessary for x^* to be a minimum.

(d) The condition $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$ is not necessary for x^* to be a strict minimum.

Solution

Suppose that

$$f(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2.$$

$$\nabla f(x_1, x_2) = \begin{pmatrix} -2(x_1 - 1) \\ -2(x_2 - 1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously,

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and $\lambda_1 = \lambda_2 = -2$.

Hence, $(1,1)$ is a local maximum.

(b)

Suppose that

$$f(x_1, x_2) = -(x_1 - 1)^4 - (x_2 - 1)^4.$$

$$\nabla f(x_1, x_2) = \begin{pmatrix} -4(x_1 - 1)^3 \\ -4(x_2 - 1)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously,

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0.$$

It is easy to see that $f(x_1, x_2) < f(1,1)$ for all $(x_1, x_2) \neq (1,1)$. Consequently, $(1,1)$ is a local maximum (global).

(c)

$$f(x_1, x_2) = 0 \quad \forall x_1, x_2.$$

$(1,1)$ is a minimum, even without $\nabla^2 f(x^*) \succ 0$.

(d)

Suppose that

$$f(x_1, x_2) = (x_1 - 1)^4 + (x_2 - 1)^4.$$

$$\nabla f(x_1, x_2) = \begin{pmatrix} 4(x_1 - 1)^3 \\ 4(x_2 - 1)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently,

$$x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously,

$$\nabla^2 f(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0.$$

It is easy to see that $f(x_1, x_2) > 0 = f(1,1)$ for all $(x_1, x_2) \neq (1,1)$. Consequently, $(1,1)$ is a local minimum (global), even without $\nabla^2 f(x^*) \succ 0$.