Optimization - Exercise session 6
Discrete optimization

1. Yves and Muriel want to share the main household tasks (shopping, cooking, washing up and cleaning) between them. Their efficiency in performing these tasks differs. Yves is quick to do the shopping and washing up, but Murielle lags behind him in cooking and cleaning.

	shopping	cooking	washing	cleaning
Yves	4.5	7.8	3.6	3.1
Murielle	4.9	7.2	4.3	2.9

The young couple want to share the tasks fairly (two tasks per person) and optimally (minimum total time). Formulate this problem as an integer optimization problem. Give a relaxation of this problem. Is the solution of the relaxed problem integer? What can you deduce from it?

Solution

Suppose that

$$x_{1j} = \begin{cases} 1, & \text{Yves takes j-th task} \\ \text{otherwise} \end{cases}$$

$$x_{2j} = \begin{cases} 1, & \text{Murielle takes j-th task} \\ \text{otherwise} \end{cases}$$

Then we have the following an integer optimization problem:
minimize

$$4.5x_{11} + 7.8x_{12} + 3.6x_{13} + 3.1x_{14} + 4.9x_{21} + 7.2x_22 + 4.3x_{23} + 2.9x_{24}$$

s.t
(two tasks per person:)

$$x_{12} + x_{12} + x_{13} + x_{14} = 2$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 2$$

(different tasks:)

$$x_{11} + x_{12} = 1$$

$$x_{12} + x_{22} = 1$$

$$x_{13} + x_{23} = 1$$

$$x_{14} + x_{24} = 1$$

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24} \in \{0, 1\}.$$

relaxed of the problem
minimize

$$4.5x_{11} + 7.8x_{12} + 3.6x_{13} + 3.1x_{14} + 4.9x_{21} + 7.2x_22 + 4.3x_{23} + 2.9x_{24}$$

s.t
(two tasks per person:)

$$x_{12} + x_{12} + x_{13} + x_{14} = 2$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 2$$

(different tasks:)

$$x_{11} + x_{12} = 1$$

$$x_{12} + x_{22} = 1$$

$$x_{13} + x_{23} = 1$$

$$x_{14} + x_{24} = 1$$

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24} \in [0, 1].$$

Since Yves is quick to do the shopping and washing up, but Murielle lags behind him in cooking and cleaning, then

$$x_{optimal} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. You have 5 objects with respective weights of 2, 4, 3, 3 and 2. The utilities are 9, 20, 14, 10 and 6 respectively. You wish to select a set of objects with a total weight of less than 10 and whose sum of utilities is maximal. Formulate the problem as an integer optimization problem. Apply the "branch and bound" algorithm to find the solution. Justify the steps in the algorithm.

Solution

Our gaol is maximize sum of utilities $9x_1 + 20x_2 + 14 + x_3 + 10x_4 + 6x_5$.
Since a total weight of less than 10, we get

$$2x_1 + 4x_2 + 3x_3 + 3x_4 + 2x_5 \leq 10.$$

Hence, we have the following integer optimization problem
maximize $9x_1 + 20x_2 + 14 + x_3 + 10x_4 + 6x_5$.
s.t.

$$2x_1 + 4x_2 + 3x_3 + 3x_4 + 2x_5 \leq 10.$$

$$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.$$

relaxed of the problem:
maximize $9x_1 + 20x_2 + 14 + x_3 + 10x_4 + 6x_5$.
s.t.

$$2x_1 + 4x_2 + 3x_3 + 3x_4 + 2x_5 \leq 10.$$

$$x_1, x_2, x_3, x_4, x_5 \in [0, 1].$$

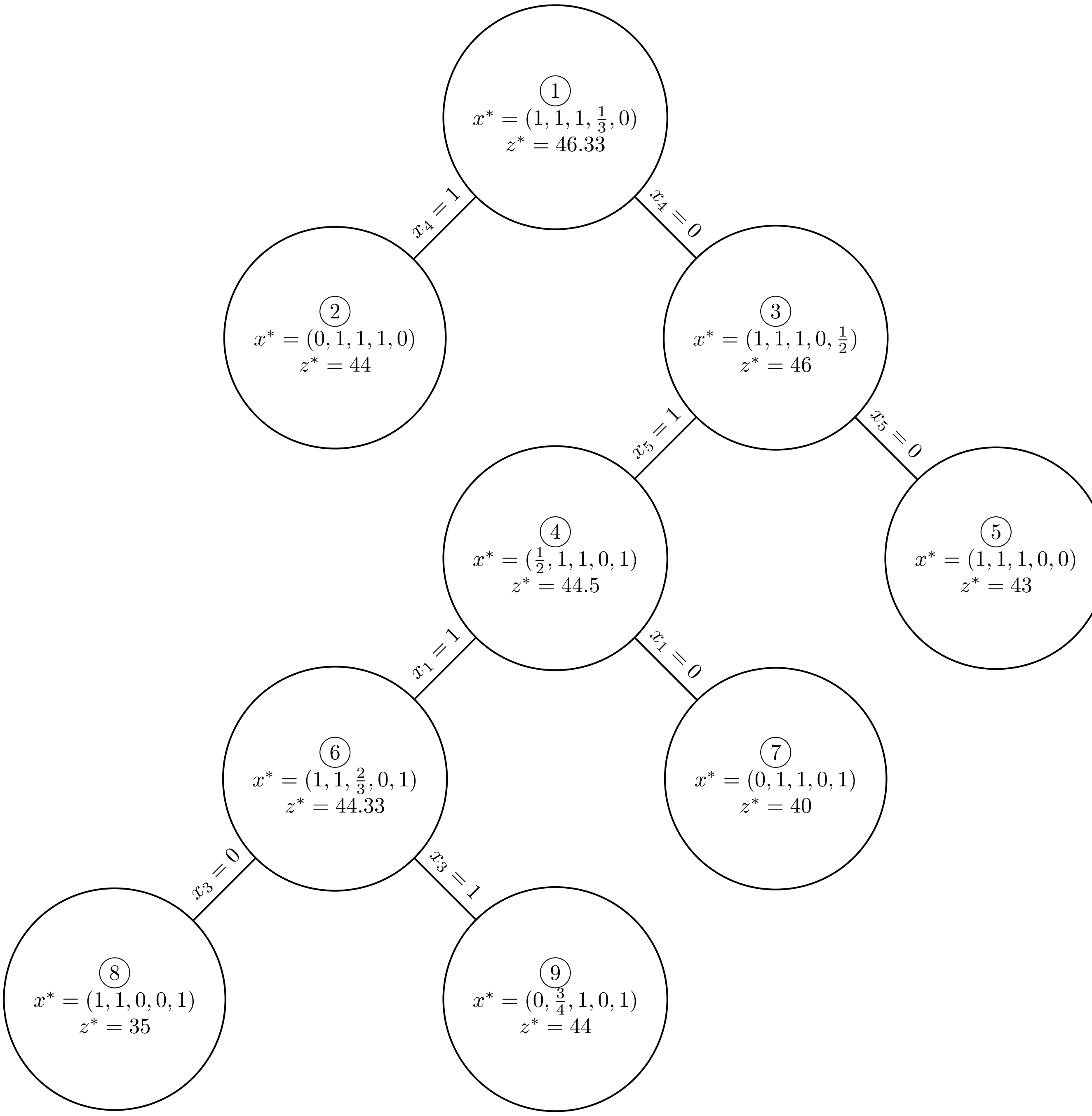
Lets calculate the following ratios:

x_1	x_2	x_3	x_4	x_5
9/2	20/4	14/3	10/3	6/2

If we sort these rations in descending order we obtain

x_2	x_3	x_1	x_4	x_5
5	4.666	4.5	3.333	3

Consequently, we will take variables in the order x_2, x_3, x_1, x_4 and x_5.
Due to the picture bellow we can conclude that the optimal value is 44 and the maximum is attained at $(0, 1, 1, 0)$


1
$x^* = (1, 1, 1, \frac{1}{3}, 0)$
$z^* = 46.33$
$x_4 = 1$
$x_4 = 0$
2
$x^* = (0, 1, 1, 1, 0)$
$z^* = 44$
3
$x^* = (1, 1, 1, 0, \frac{1}{2})$
$z^* = 46$
$x_5 = 1$
$x_5 = 0$
4
$x^* = (\frac{1}{2}, 1, 1, 0, 1)$
$z^* = 44.5$
5
$x^* = (1, 1, 1, 0, 0)$
$z^* = 43$
$x_1 = 1$
$x_1 = 0$
6
$x^* = (1, 1, \frac{2}{3}, 0, 1)$
$z^* = 44.33$
7
$x^* = (0, 1, 1, 0, 1)$
$z^* = 40$
$x_3 = 0$
$x_3 = 1$
8
$x^* = (1, 1, 0, 0, 1)$
$z^* = 35$
9
$x^* = (0, \frac{3}{4}, 1, 0, 1)$
$z^* = 44$

11. You have 96 Roubles at your disposal. You would like to buy beverages, available in various packagings.

Drink	A	B	C	D	E
Unit volume (liters)	15	16	17	18	20
Unit price (Rubles)	16	17	18	19	20

You want to buy as large a volume as possible, without your budget.

(a) Formulate this problem as an integer linear optimization optimization problem.

(b) How many packages of each type will you buy? ? Detail your reasoning.

Solution

maximize $15x_A + 16x_B + 17x_C + 18x_D + 20x_E$
s.t.

$$16x_A + 17x_B + 18x_C + 19x_D + 20x_E \leq 96$$

$$x_A, x_B, x_C, x_D, x_E \in \mathbb{N} \bigcup \{0\}.$$

The relaxed problem is maximize $15x_A + 16x_B + 17x_C + 18x_D + 20x_E$
s.t.

$$16x_A + 17x_B + 18x_C + 19x_D + 20x_E \leq 96$$

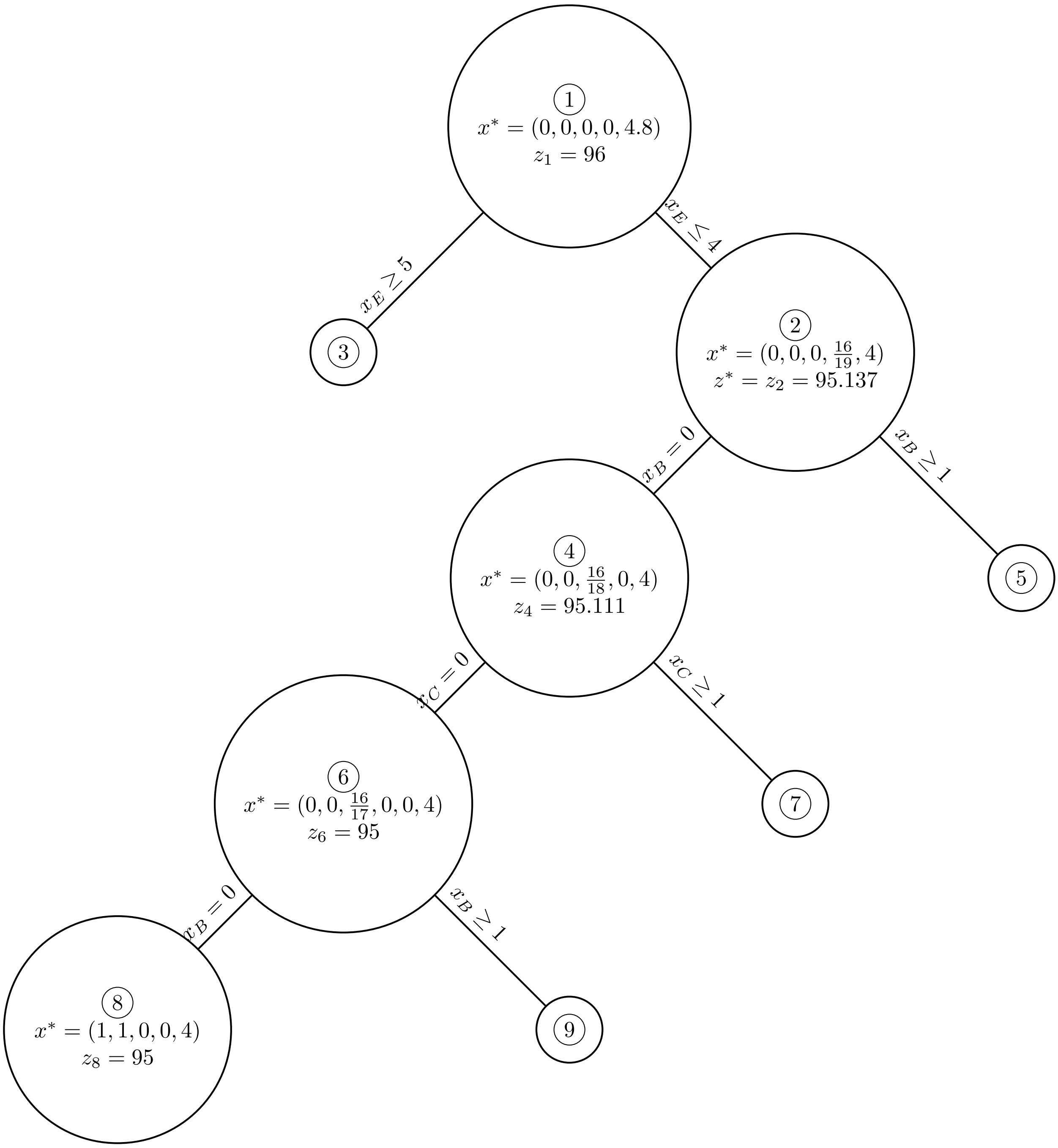
$$x_A, x_B, x_C, x_D, x_E \geq 0.$$

x_1	x_2	x_3	x_4	x_5
15/16	16/17	17/18	18/19	1

If we sort these rations in descending order we obtain

x_E	x_D	x_C	x_B	x_A
1	0.947368	0.944444	0.941176	0.9375

Using calculations below we have that optimal solution is $x^* = (1, 1, 0, 0, 4), z^* = 95$


1
$x^* = (0, 0, 0, 0, 4.8)$
$z_1 = 96$
$x_E \geq 5$
$x_E \leq 4$
3
2
$x^* = (0, 0, 0, \frac{16}{19}, 4)$
$z^* = z_2 = 95.137$
$x_B = 0$
$x_B \geq 1$
4
$x^* = (0, 0, \frac{16}{18}, 0, 4)$
$z_4 = 95.111$
5
$x_C = 0$
$x_C \geq 1$
6
$x^* = (0, 0, \frac{16}{17}, 0, 0, 4)$
$z_6 = 95$
7
$x_B = 0$
$x_B \geq 1$
8
$x^* = (1, 1, 0, 0, 4)$
$z_8 = 95$
9