Optimisation

Lecture 1 - Introduction

Fall semester - 2024

Dr. Eng. Valentin Leplat

Innopolis University

August 27, 2024

Outline

Information on the course

General information on the course

- **Instructor**: Valentin Leplat
- **Teaching Assistant**: Ramil Nasibullin
- **Contacting the Instructors**:
  1. Students are strongly encouraged to contact the instructors and the teaching assistants any time they face problems with the course:
     - via email: v.leplat@innopolis.ru
     - via Telegram: @vleplat (from 9:00 am to 3:00 pm you may text).
     - Meeting in my office: Wednesday 11:00 - 12:00, Office 401.
  2. For simple and short questions-answers: use the Telegram channel or Moodle.

Prerequisite courses

- Calculus
- Linear Algebra
- Numerical Methods

Resources and supporting materials

- D. Bertsimas and John N. Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific Series in Optimization and Neural Computation, 6, 1997. ▸ Chapters 1 to 5
- Amir Beck. *Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB*. SIAM, 2014. ▸ Full pdf

Course Outline

1. Part 1 - Linear Programming:
   - Formulation(s), geometry, simplex method, duality and integer programming (branch and bound).
2. Part 2 - Nonlinear Programming:
   - formulation, optimality conditions, convexity, local methods (descent direction), duality, Alternating Direction Method of Multipliers, etc

This course is relatively new, hence it may evolve a bit along iterations :).

Course Outline

Course activities and grading breakdown:

Activity Type	**Percentage of the overall course grade**
Midterm	30
Final exam	30
Project in group	30
Attendance	10
Total	100

Project info:

- *Theme*: Sending an encrypted message on a channel with sparse noise.
- *When*: the project's description will be provided in week 2.

Remark: additional project on non-linear programming possible on request for motivated students with previous background - grade obtained can be used to avoid a question during final exam.

Course Outline

Course grading range:

Grade	**Range**	**Comment**
A. Excellent	90-100	
B. Good	70-89	
C. Satisfactory	60-69	
D. Fail	0-59	If your attendance is $\leqslant 50\%$, then "D".

How to get the highest grade?

1. Attend classes:
   - Lectures
   - Labs
2. Solve quick exercises/assignments (mentioned during lectures) on your own
3. Read books
4. Work in team on the project
5. Contact instructors when you have (serious :)) questions

Timeline

Week no	**Date**	**Topics**	**Labs**
1	August 27th	Introduction	✗
2	September 3rd	Formulations	✓
3	September 10th	Geometry of polyhedra	✓
4	September 17th	Simplex algorithm	✓
5	September 24th	Duality	✓
6	October 1st	Integer programming	✓
7	October 8th	Dikin's Method + Nonlinear programming: introduction	✗
8	October 15th	*Mid-term exam*	✗
9	October 22nd	Optimality conditions	✓
10	October 29th	Convexity	✓
11	November 5th	First-order methods	✓
12	November 12th	Duality	✓
13	November 19th	Alternating Direction Method of Multipliers	✓
14	November 26th[1]	Introduction to Stochastic Optimization	✓
15	$\geqslant$ December 10th	*Final exam*	✗

[1] Project(s) deadline

More info on Projects

1. **Build** your team: communicate by emails to the instructors before September 10th. the participants of each group
   Dear Instructor, Mr. Black and I (Ms. Brown) have agreed to work together on the linear programming project.
   Response: You will receive a unique ID group.

2. **What**: report + code

3. **When**: $\leqslant$ November 26th

4. **How**: upload a zip file on Moodle Platform named "project_linearprog_group_ID" *and* send a notification email to instructors.

5. **Monitoring**: Right after mid-term exam, feedback's will be gathered via a google form :

   5.1 *Estimation of your percentage of advancement*: from 0 to 100%.
   5.2 *Level of confidence in the accuracy of the tasks performed*: from 0 to 100%.
   5.3 *Level of difficulty*: too easy, adapted, challenging but ok, or too hard.

   If you want to share a feedback before or "after" ($\leqslant$ November 12th), feel free to do so at the end of labs or lectures.

General Introduction

Opening Remark and Credit

About more than 264 years ago.....

> *"Nothing takes place in the world whose meaning is not that of some maximum or minimum."*
>
> Leonhard Euler (1707-1783)

Why ?

Help to make the **best** decision.

$$\left\{ \begin{array}{cc} \text{Decision :} & \text{vector of variables } x \\ \text{Best :} & \text{objective function } f(x) \\ \text{Constraints :} & \text{feasible set } \mathcal{X} \end{array} \right\} \rightarrow \text{Optimization}$$

$$\min_{x \in \mathcal{X}} f(x)$$

- Many applications in practice
- Efficient methods in practice
- Modelling and resolution of large-scale problems

Notations and formulation

Minimization of function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ over a feasible set $\mathcal{X}$ writes as:

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{such that} \quad & x \in \mathcal{X} \end{aligned}$$

- The feasible set is usually defined with functional constraints:

$$\mathcal{X} = \{x \in \mathbb{R}^n | h_i(x) = 0 \quad \forall i \in \mathcal{E}, \quad h_j(x) \geqslant 0 \quad \forall j \in \mathcal{I}\}$$

- The constraints h_i, h_j are functions $h_i, h_j : \mathbb{R}^n \rightarrow \mathbb{R}$:
  - $h_i(x) = 0$ is an equality constraint
  - $h_j(x) \geqslant 0$ is an inequality constraint
- $\mathcal{E}$ and $\mathcal{I}$ are set of indices.

Notations and formulation

The general optimization problem can be therefore written as follows:

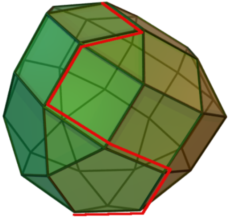
$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{such that} \quad & h_i(x) = 0 \quad \forall i \in \mathcal{E} \\ & h_j(x) \geqslant 0 \quad \forall j \in \mathcal{I} \end{aligned}$$

Particular case: all the functions are linear !

- If all the functions $f(.), h_i(.), h_j(.)$ are linear, the general problem can be simplified as:

$$\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{such that} & Ax \geqslant b \end{array}$$

- The constraints $Ax \geqslant b$ defines a feasible set that is a polyhedron (a polytope if bounded and non-empty).
- The objective function $c^T x$ forms a translating hyperplane in space.



In most cases: an optimal solution will be one of the vertices of the polyhedron

Particular case: all the functions are linear !

Why the linear programming ?

- Because many problems can be modelled as linear programs
- Because there is a very efficient algorithm (the simplex algorithm) for solving these problems
- Because these problems have a very rich structure (properties of optimality, duality)

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Why studying non-linear programming ?

- Because some problems are impossible to model linearly
- Because the simplex algorithm for linear programming is inapplicable to non-linear problems

Few examples

Outline

Example 1 - Optimal distribution of resources

- Three resources: A (wood), B (nails) and C (fabrics) are used to obtain two products, that are T (tables) and L (beds).
- We need:
  1. two units of wood to build a table, three for a bed,
  2. two units of nails for a table, and one for a bed,
  3. one unit of fabric for a table, and three for a bed.
- We have at disposal 13 units of A, 11 of B and 9 of C.
- T and L products earn 300 and 400 Rubles per unit produced, respectively.
- **Question**: How many units of L and T must be produced to maximize profit?

Example 1 - Formulation

1. **Choice of variables**: x_1 units of T, x_2 units of L.
2. **Constraints**:

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$$\begin{aligned} 2x_1 + 3x_2 &\leqslant 13, && \text{(resource A)} \\ 2x_1 + x_2 &\leqslant 11, && \text{(resource B)} \\ x_1 + 3x_2 &\leqslant 9, && \text{(resource C)} \end{aligned}$$

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x_1, x_2 &\geqslant 0.
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$$x_1, x_2 \geqslant 0.$$

3. **Objective function**:

$$\max_{x_1, x_2} \quad 300x_1 + 400x_2.$$

Example 1 - Geometrical Interpretation

The pairs (x_1, x_2) for which

$$2x_1 + 3x_2 \leqslant 13,$$

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$$x_1 + 3x_2 \leqslant 9, \text{ and}$$

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- For a given scalar z, the set $\{x = (x_1, x_2) \in \mathbb{R}^2 \mid 300x_1 + 400x_2 = z\}$ is a perpendicular line to vector $c = (3, 4)$.
- The straight lines obtained for different values of z are parallel.
- An optimal solution is obtained by translating this line as far as possible. The optimum solution is obtained at a vertex of the polyhedron.

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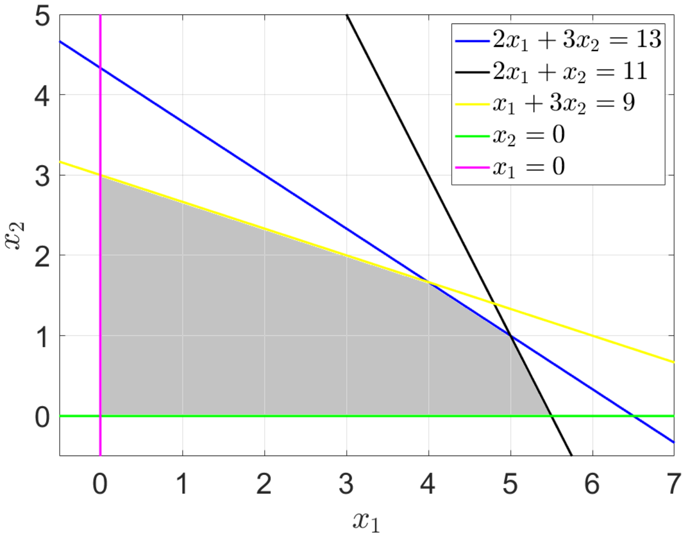
Solution: 1 units of L and 5 units of T for a profit of 1900. Colab File

Example 1 - Graphics



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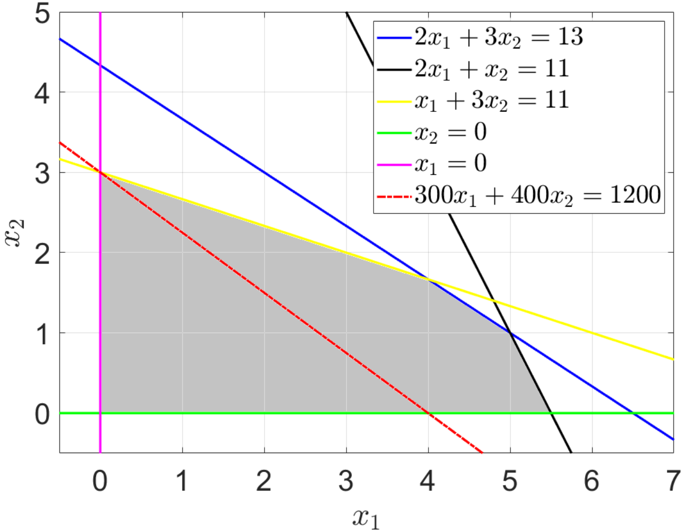

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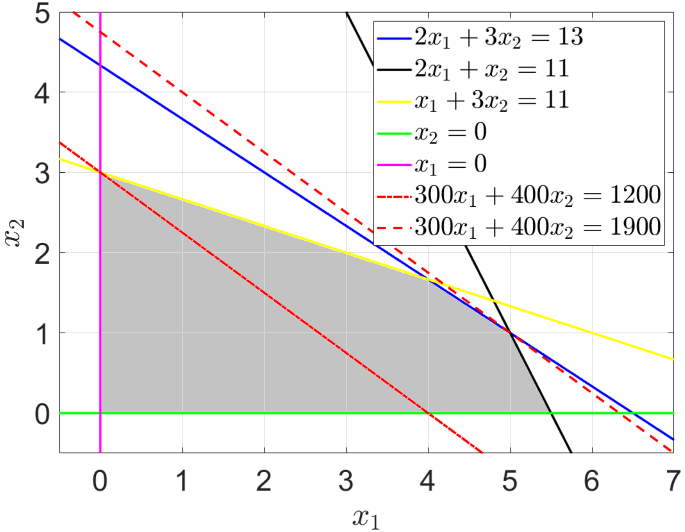

Example 2: Problem of mixing (or dietetics)

- This week, your sports coach (or dietician or dentist) recommends that you consume 10 units of vitamin A, 8 units of vitamin B and 7 units of vitamin C.
- All you want to eat are apples and bananas, whose vitamin content and price per unit is :

	Vit. A	Vit. B	Vit. C	Price
Apples	2	1	1	4
Bananas	1	2	1	3
Total to be consumed	10	8	7	

Table: Data (per unit).

Example 2: Formulation

1. **Choice of variables**:

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3. **Objective function**: total cost is minimum

$$\min_{x_1, x_2} \quad 4x_1 + 3x_2.$$

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\begin{aligned}
\min_{x_1,x_2} \quad & 4x_1 + 3x_2 \\
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- Can you solve this problem geometrically?
- What if there are many more foods (i.e. variables) and vitamins to obtain (i.e. constraints)?
- Answering this question is one of the main objectives of this course.

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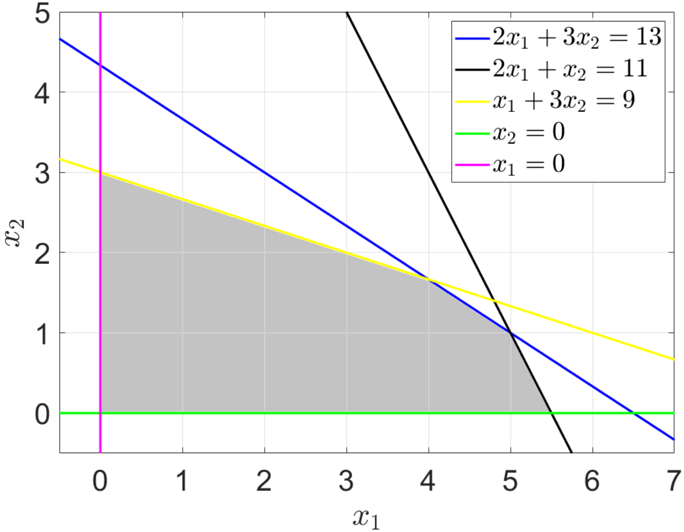

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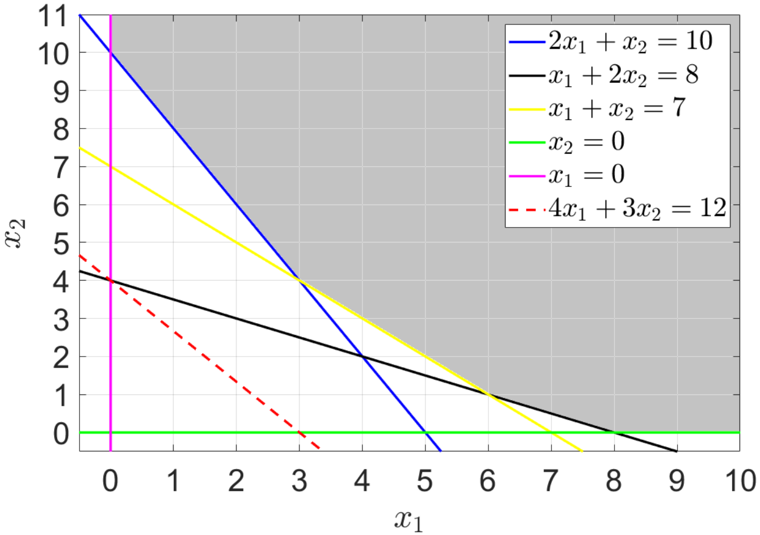

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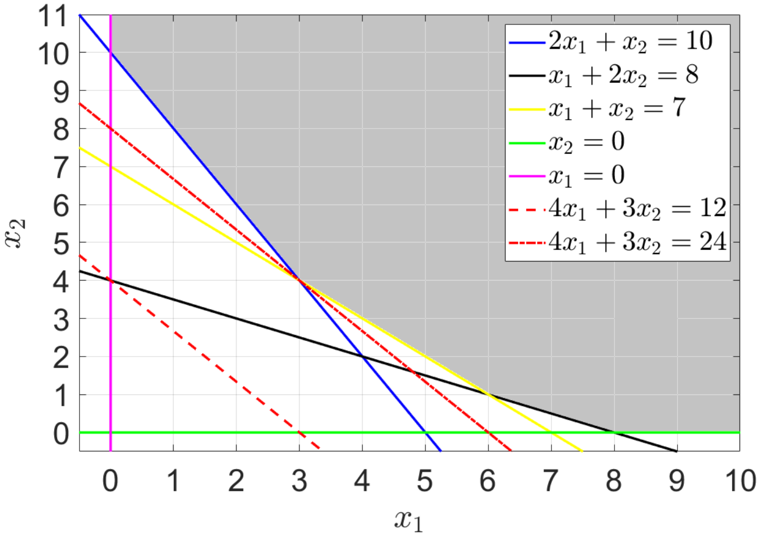

Example 3: Transportation problems

- A product is transported from m origins to n destinations.
- The product is available in quantities $a_1, a_2, \ldots, a_m$ at origins and the demands at destinations are $b_1, b_2, \ldots, b_n$.
- From origin i to destination j is c_{ij} (for example, the distance between i and j).

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Our goal: *we want to determine the quantities of product to be transported from i to j so as to satisfy demand while minimizing total transport costs.*

Example 3: Formulation

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3. **Objective function**: total cost is minimum

$$\min_{x} \quad \sum_{i,j} c_{ij} x_{ij}.$$

Question. What is a necessary and sufficient condition for the problem to be solvable?

Applications

- Planning, management and scheduling
  Production, schedules, crew composition, etc.
- Design and conception
  Sizing and optimization of structures and networks
- Economy and finance
  Portfolio selection, balance calculation
- Location and transport
  Relocation of depots, integrated circuits, tours
- Data analysis, machine learning
  Recommendation systems (Netflix, Amazon, etc.), image analysis (e.g. segmentation), automatic document classification, clustering, etc.
- And many more $\cdots$

The two faces of optimization

1. Modeling
   Translating the problem into mathematical language
   (more delicate than it looks)

$$\Updownarrow$$

Formulation of an optimization problem

2. Solving
   Development and implementation of efficient resolution algorithms in theory and practice.

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Close relationship:

- Formulating models/programs that can be solved
- Developing methods applicable to realistic models/programs.

The two faces of optimization

Premature optimization is the root of all evil.

– Donald Ervin Knuth,
The Art of Computer Programming.

Optimization: Taxonomy and Terminology

Outline

Optimization

- Optimization problems often have thousands of variables and constraints. They rarely have an analytical solution.
- We're looking for optimization algorithms that are fast, easy to implement, require little computation time and memory, are not sensitive to rounding errors, are guaranteed to converge, and allow post-optimal analysis (ideally!).
- Modeling is a crucial aspect of optimization.
- **A model is just a model**. We live in a world of satisfying approximation. We don't always seek to optimize exactly, but often to optimize satisfactorily.
- Each model has its own resolution method. The more precise the model class, the more efficient the method used.
- In general, compromise between model quality, complexity and resolution method (exact or approximate resolution - heuristics).

Taxonomy (hierarchy/types of optimization problems)

$$\min_{x=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n} \; f(x_1,x_2,\ldots,x_n)$$

in the feasible set, that is $(x_1,x_2,\ldots,x_n)^T \in \mathcal{D}$.

- **Variables**: continuous, discrete, binary, etc.

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But also…

Models: Multi-criteria optimization, stochastic models, temporal models, etc.

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Change category: sometimes possible via reformulation

Terminology : feasibility

Problems in finite dimensions

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$$\min_{x \in \mathbb{R}^n} f(x) \text{ such that } x \in \mathcal{D}$$

Terminology : feasibility

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$$\min_{x \in \mathbb{R}^n} f(x) \text{ such that } x \in \mathcal{D}$$

- Any point x belonging to the feasible set is called feasible solution.
- When $\mathcal{D} \neq \emptyset$, the problem is said to be possible or feasible.
- When $\mathcal{D} = \emptyset$, the problem is said to be impossible or infeasible.

Terminology : optimal value

$$\min_{x \in \mathbb{R}^n} f(x) \text{ such that } x \in \mathcal{D}.$$

- The optimal value of the problem, denoted $f^\star$, is the infimum of objective function values for feasible solutions, i.e.

$$f^\star = \inf \{f(x) \mid x \in \mathcal{D}\}.$$

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- When $f^\star$ is finite, the problem is said to be bounded .
- When $f^\star = -\infty$, the problem is called unbounded.
- When the problem is impossible, we conventionally set $f^\star = +\infty$ (worst possible value for a minimum).

Terminology : Solvability

$$\min_{x\in\mathbb{R}^n} f(x) \text{ such that } x \in \mathcal{D} \quad \text{et} \quad f^\star = \inf \{f(x) \mid x \in \mathcal{D}\}.$$

- An optimal solution, denoted $x^\star$, is a feasible solution that possesses the optimal value, i.e.

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- A problem that has (at least) one optimal solution is said to be solvable, otherwise it is called unsolvable.
- An impossible or unbounded problem is never solvable, but there are also possible, bounded and unsolvable, for instance:

$$\min \frac{1}{x} \text{ such that } x > 0 \text{ gives } f^\star = 0 \text{ but is unsolvable...}$$

Example

$$\begin{aligned}
\min_{x_1,x_2} \quad & c_1x_1 + c_2x_2 = c^T x \\
\text{tel que} \quad & -x_1 + x_2 \leqslant 1, \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

Optimal solutions for different objective functions :

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- $c = (0,1)$. Infinite, unbounded set of optimal solutions: $\{(x_1,0) | 0 \leqslant x_1\}$.
- $c = (-1,-1)$. Unbounded optimal value $(f^\star = -\infty)$. $(x_1 = x_2 \to +\infty)$

Terminology : problem types

$$\min_{x \in \mathbb{R}^n} f(x) \text{ such that } x \in \mathcal{D} \quad \text{and} \quad f^\star = \inf \{f(x) \mid x \in \mathcal{D}\}.$$

- Without loss of generality, we can consider only the minimization.
- If maximization, we have equivalence between optimal solutions

$$x^* \text{ optimal for } \max_{x \in \mathcal{D} \subseteq \mathbb{R}^n} f(x) \quad \Leftrightarrow \quad x^* \text{ optimal for } \min_{x \in \mathcal{D} \subseteq \mathbb{R}^n} -f(x),$$

and for optimal values (with a double minus sign):

$$\sup\{f(x) \mid x \in \mathcal{D} \subseteq \mathbb{R}^n\} = -\inf\{-f(x) \mid x \in \mathcal{D} \subseteq \mathbb{R}^n\}.$$

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- The problem of finding an admissible point (without an objective function) is a special case of an optimization problem, and can be formally expressed using a constant (or zero) objective function.

  $$\min_{x \in \mathbb{R}^n} 0 \text{ such that } x \in \mathcal{D}.$$

Description of the feasible set - linear problems

$$\min_{x \in \mathbb{R}^n} f(x) \text{ such that } h_i(x) = 0 \text{ pour } i \in \mathcal{E} \text{ and } h_i(x) \geqslant 0 \text{ pour } i \in \mathcal{I}$$

is called a linear optimization problem when all functions are linear or affine:

- Objective function: $f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ (constant term is useless)
- Constraints : $h_i(x) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n - b_i$

hence the equivalent formulation

$$\min_{x \in \mathbb{R}^n} c^T x \text{ such that } a_i^T x = b_i \text{ for } i \in \mathcal{E} \text{ and } a_i^T x \geqslant b_i \text{ for } i \in \mathcal{I},$$

with column vectors $c = (c_1 \ c_2 \ \cdots \ c_n)^T$, and $a_i = (a_{i1} \ a_{i2} \ \cdots \ a_{in})^T$ defined for all $i \in \mathcal{E} \cup \mathcal{I}$.

Summary

We have seen

- Notations and general formulation of an optimization problem (or programs),
- The special case of linear problems: all the functions are linear.
- A short geometrical interpretation of some linear problems.
- The two faces of optimization: Modeling and Solving
- Taxonomy and Terminology in optimization.

Preparations for the next lecture

- Master the terminology.
- Solve Example 2 with a linear solver from CVXPY[2] (take inspiration from the first example).

[2] open source Python-embedded modeling language for convex optimization problems

Goodbye, So Soon

THANKS FOR THE ATTENTION

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