Optimization – Exercise session 3
The Simplex Method

1. Let the optimization problem be

$$\begin{array}{lrcl} \min_x & -2x_1 - x_2 & & \\ \text{such that} & x_1 - x_2 & \leq & 2, \\ & x_1 + x_2 & \leq & 6, \\ & x_1, x_2 & \geq & 0. \end{array}$$

   Convert this problem into standard form and find a vertex for which $x_1 = x_2 = 0$. Solve the problem using the simplex method. Draw a graphical representation in terms of the variables x_1, x_2 and indicate the path followed by the method.

2. Consider the problem

$$\begin{array}{lrcl} \min_x & 20x_1 + \alpha x_2 + 12x_3 & & \\ \text{such that} & x_1 & \leq & 400, \\ & 2x_1 + \beta x_2 + x_3 & \leq & 1000, \\ & 2x_1 + \gamma x_2 + 3x_3 & \leq & 1600, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

   Propose, if possible, values for α, β and γ for which:

   (a) The optimal cost is finite and the optimal solution is unique.

   (b) The optimal cost is finite and there are infinitely many optimal solutions.

   (c) The optimal cost is unbounded (find a parameterization of x values among which there are solutions with arbitrarily low costs).

   (d) The polyhedron has a degenerate vertex.

3. Solve problems using the simplex algorithm

$$\begin{array}{lrcl} \max_x & 2x_1 + 3x_2 & & \\ \text{such that} & x_1 + 2x_2 & \leq & 4, \\ & x_1 + x_2 & = & 3, \\ & x_1, x_2 & \geq & 0. \end{array}$$

$$\begin{array}{lrcl} \max_x & 20x_1 + 16x_2 + 12x_3 & & \\ \text{such that} & x_1 & \leq & 400, \\ & 2x_1 + x_2 + x_3 & \leq & 1000, \\ & 2x_1 + 2x_2 + 3x_3 & \leq & 1600, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

$$\begin{array}{lrcl} \max_x & 4x_1 + 3x_2 + 6x_3 & & \\ \text{such that} & 3x_1 + x_2 + 3x_3 & \leq & 30, \\ & 2x_1 + 2x_2 + 3x_3 & \leq & 40, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

$$\begin{array}{lrcl} \max_x & -x_1 + 4x_2 & & \\ \text{such that} & -3x_1 + 4x_2 & \leq & 6, \\ & x_1 + 2x_2 & \leq & 4, \\ & x_2 & \geq & -3. \end{array}$$

4. Let's take one of the previous problems. A student has 100 hours' work available to study exams A, B and C. He expects to earn 1/5 points for course A, 2/5 points for course B and 3/5 points for course C for each hour worked on each course. Each exam is graded out of 20. The exercises in these courses account for half of the final score. His results for the exercises were communicated to him. He obtained 12/20 for A, 12.5/20 for B and 13.4/20 for C. The student must obtain an

overall mark of at least 10/20 for each course. All courses are weighted equally, and the student wishes to obtain the highest possible average.

Formulate this problem as a linear optimization problem and solve it. You can use the fact that the student benefits from using all 100 hours of work. Will the student earn a distinction ($\geq 14/20$ in average)?

5. A company produces goods A, B and C. The production of the goods requires the use of 4 machines. The production times and profits generated are shown in the table below.

	1	2	3	4	profit
A	1	3	1	2	6
B	6	1	3	3	6
C	3	3	2	4	6

If the production times available on machines 1, 2, 3 and 4 are 84, 42, 21 and 42, determine the quantity of goods to produce to maximize profit.

6. Solve by the simplex method using Bland's rule

$$\begin{array}{lll} \max_x & 10x_1 - 57x_2 - 9x_3 - 24x_4 & \\ \text{such that} & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0, \\ & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0, \\ & x_1 \leq 1, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

7. Propose a method for finding a vertex of the polyhedron

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &\geq 3, \\ -x_1 + x_2 + x_3 &\geq 5, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

8. Consider the optimization problem

$$\begin{array}{lll} \min_x & 20x_1 + \alpha x_2 + 12x_3 & \\ \text{such that} & x_1 \leq 4, \\ & 2x_1 - x_2 + x_3 \leq 10, \\ & 2x_1 + \beta x_2 + 3x_3 \leq 16, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Find an optimal solution using the simplex algorithm when $\alpha = -2$ and $\beta = 1$. Propose values for α and β for which the optimal cost is unbounded, and in this case propose a solution for which the optimal cost is less than -1000.

9. Consider the optimization problem

$$\begin{array}{lll} \max_x & \alpha x_1 + 16x_2 + 12x_3 & \\ \text{such that} & x_1 \leq 400, \\ & 2x_1 + x_2 + x_3 \leq 1000, \\ & 2x_1 + 2x_2 + 3x_3 \leq 1600, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Find an optimal solution using the simplex algorithm when $\alpha = 20$. Is there a value of α for which the optimal cost is unbounded? If so, propose a solution for which the cost is greater than 10000. If not, justify your answer.