Optimisation

Lecture 3 - Geometry

Fall semester - 2024

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Outline

Polyhedron geometry

Polyhedron geometry

- Linear optimization problems can be solved by the simplex method.
- The simplex method is an algebraic method based on geometric concepts.
- In this course, we analyze the link between the algebraic description of a polyhedron and its geometry.

Polyhedron

A $\mathbb{R}^n$ polyhedron is a subset of $\mathbb{R}^n$ that can be written as a intersection of a finite number of half-spaces of $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid a_i^T x \geqslant b_i \text{ for } i = 1, 2, \ldots, m\} = \{x \in \mathbb{R}^n \mid Ax \geqslant b\}.$$

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A bounded polyhedron is called a *polytope* (see slide 5 example).

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- The set of solutions of a system of linear equalities and inequalities

$$a_i^T x \leqslant b_i \text{ for } i \in \mathcal{I}, a_i^T x = b_i \text{ for } i \in \mathcal{E}.$$

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- A polyhedron in geometric form $\{x \mid Ax \geqslant b\}$.

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Example

Two types of telephones are produced: the 1st type requires 4 units of resource A and 3 of C, the 2nd type 2 units of resource B and 2 of C. The selling price is respectively 300 Rubles, and 500 Rubles, and there are 4 units of A, 12 of B and 18 of C.

$$\begin{aligned} \max_{x_1,x_2} \quad 3x_1 + 5x_2 \quad \text{such that} \quad & x_1 \leqslant 4, \\ & 2x_2 \leqslant 12, \\ & 3x_1 + 2x_2 \leqslant 18, \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Example

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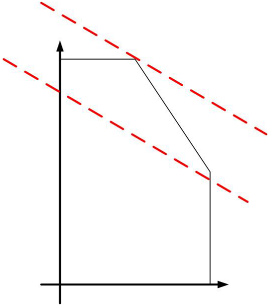

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- The problem is feasible but unbounded ($f^* = -\infty$), then $\mathcal{P}^* = \emptyset$.

Fundamental theorem

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If a linear optimization problem has finite optimal cost and the polyhedron $\mathcal{P}$ has a vertex, then there is a vertex of $\mathcal{P}$ that is optimal (see demonstration later).

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If a linear optimization problem has finite optimal cost and the polyhedron $\mathcal{P}$ has a vertex, then there is a vertex of $\mathcal{P}$ that is optimal (see demonstration later).

How do you describe and find the vertices of a polyhedron?

Polyhedra and representations

- The same polyhedron can be obtained using different representations.
- A distinction is made between properties relating to a polyhedron and properties relating to its representations.

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Polyhedra described by constraints

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$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x \geqslant \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

are identical.

Basic Feasible Solution

Active (or tight) constraints

Let be the polyhedron defined by

$$\begin{aligned}\mathcal{P} = \{x \in \mathbb{R}^n \mid a_i^T x &\geqslant b_i \text{ for } i \in \mathcal{I}, \\ a_i^T x &= b_i \text{ for } i \in \mathcal{E}\}.\end{aligned}$$

If the point x^* is such that $a_k^T x^* = b_k$ for a given index k, we say that the corresponding constraint is active (or tight) in x^*.

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In particular, the equality constraints $(i \in \mathcal{E})$ are all active at a point of the polyhedron.

Example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -1 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \\ 0 \end{pmatrix}$$

- At (0,1),

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- At (0,1), the constraints 1, 3, 4 and 5 are active.

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- At (0,1/2), the constraints 1 and 5 are active.
- At (1,0), the constraints 2 and 3 are active.

Linearly independent constraints

Constraints of the polyhedron defined by

$$\begin{aligned} \mathcal{P} = \{x \mid a_i^T x &\geqslant b_i \text{ for } i \in \mathcal{I}, \\ a_i^T x &= b_i \text{ for } i \in \mathcal{E}\}. \end{aligned}$$

are linearly independent if the corresponding vectors a_i are.

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- Constraints 1 and 3 are linearly independent.
- Constraints 1 and 5 are not linearly independent.

Basic feasible solution

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The solution $x^* \in \mathbb{R}^n$ is a basic feasible solution of $\mathcal{P}$ if $x^* \in \mathcal{P}$ and if there are n linearly independent constraints active in x^*.

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The solution $x^* \in \mathbb{R}^n$ is a basic feasible solution of $\mathcal{P}$ if $x^* \in \mathcal{P}$ and if there are n linearly independent constraints active in x^*.

A basic feasible solution x^* is degenerate if the number of active constraints in x^* is greater than n.

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- At $(1, 0)$ constraints 2 and 3 are active; $(1, 0)$ is a non-degenerate basic feasible solution.
- At $(2, 0)$ constraints 2 and 4 are active. Constraints 2 and 4 are linearly independent but $(2, 0)$ does not belong to $\mathcal{P}$. The point $(2, 0)$ is not a basic feasible solution

Degeneracy

Polyhedra described by constraints

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The degenerate nature of a basic feasible solution generally depends on the representation.

Adjacent Basic Feasible Solutions

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- Let x_1 and x_2 be two basic feasible solutions of $\mathcal{P}$.
- These two solutions are adjacent if there are $n-1$ linearly independent constraints active at both x_1 and x_2.

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- At (1, 0) constraints 2 and 3 are active; (1, 0) is a non-degenerate basic feasible solution.
- (0, 1) and (1, 0) are adjacent basic feasible solutions: constraint 3 is active in (0,1) and (1,0).

Extreme points, Vertices and BFS

Extreme points and vertices

- The point $x \in \mathcal{P}$ is an extreme point of the polyhedron $\mathcal{P}$ if it cannot be expressed as a *convex combination* of other points of $\mathcal{P}$, i.e., if there are no two different points $y, z \in \mathcal{P}$ from x and a scalar $0 \leqslant \lambda \leqslant 1$ such that $x = \lambda y + (1-\lambda) z$.
- The point $x \in \mathcal{P}$ is an vertex of the polyhedron $\mathcal{P}$ if x is separable from $\mathcal{P}$ by a hyperplane, that is, if there exists a vector c such that $c^T x < c^T y$ for all $y \in \mathcal{P}$, $y \neq x$.

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These definitions are geometrical, and do not depend on the representations chosen to describe the polyhedra (on the other hand, they retain meaning for any subset of $\mathbb{R}^n$).

Vertices, extreme points and basic feasible solutions

Theorem

Let $\mathcal{P}$ be a polyhedron and $x \in \mathcal{P}$. The following conditions are equivalent

- x is a vertex.
- x is an extreme point.
- x is a basic feasible solution.

Proof. See the proof of Theorem 2.3 from *Introduction to Linear Optimization*, Dimitri Bertsimas and John Tsitsiklis, Athena Scientific, 1997.

Example

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} x \geqslant \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}$$

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- (1,1) is a basic feasible solution. Constraints 2 and 4 are active at (1,1) and they are linearly independent.

Polyhedra without vertices

Not all polyhedra have vertices.

Examples.

- The hyperplane $\{x \in \mathbb{R}^n \mid a^T x = b\}$ (for $n \geqslant 2$).

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- The slice $\{x \in \mathbb{R}^n \mid b_1 \leqslant a^T x \leqslant b_2\}$ (for $n \geqslant 2$).

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- Half-space $\{x \in \mathbb{R}^n \mid a^T x \geqslant b\}$ (for $n \geqslant 2$).
- The slice $\{x \in \mathbb{R}^n \mid b_1 \leqslant a^T x \leqslant b_2\}$ (for $n \geqslant 2$).
- A polyhedron given in geometric form $\{x \in \mathbb{R}^n \mid Ax \geqslant b\}$ never has vertices when the matrix A has fewer than n rows.

Vertex calculation/enumeration

Vertex calculation: Geometric form

Let $\mathcal{P} = \{x \mid Ax \geqslant b\}$ be a $\mathbb{R}^n$ polyhedron. To find the vertices of the polyhedron, we write

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix} x \geqslant \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

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and list the square sub-matrices of A with n rows and the corresponding sub-vectors of b.

$$\tilde{A} = \begin{pmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_n}^T \end{pmatrix} \qquad \text{and} \qquad \tilde{b} = \begin{pmatrix} b_{i_1} \\ b_{i_2} \\ \vdots \\ b_{i_n} \end{pmatrix}.$$

If $\operatorname{rank}(\tilde{A}) = n$, we compute $x^* = \tilde{A}^{-1}\tilde{b}$. The polyhedron has n linearly independent constraints active at x^*. The point x^* is a vertex of $\mathcal{P}$ if $Ax^* \geqslant b$.

Example

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} x \geqslant \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}$$

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Pairs of adjacent vertices are (1,1)-(0,3) and (1,1)-(3,0).

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- **Hence**: To obtain a basic admissible solution, we need to tighten $n - m$ additional constraints among the n constraints $x_i \geqslant 0$.
- The choice of these variables is not arbitrary, since the resulting set of tight constraints must be a set of linearly independent constraints.

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Suppose the constraints are those corresponding to the last $n - m$ variables. We then have

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \qquad \text{where } x_B \in \mathbb{R}^m, x_N \in \mathbb{R}^{n-m}.$$

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The variables x_B are the basic variables, the variables x_N are the non-basic variables and the system $Ax = b$ is written as

$$\begin{pmatrix} A_B & A_N \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = A_B x_B + A_N x_N = b.$$

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If A_B is invertible, a solution of the system $Ax = b$ is given by $x_B^* = A_B^{-1} b$ and $x_N^* = 0$. This solution is such that

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If $A_B^{-1} b \geqslant 0$, this solution is also feasible and is therefore a vertex.

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To summarize, to find the vertices of the polyhedron $\mathcal{P}$:

1. choose a basis of m variables for which the associated basis matrix A_B is invertible, these are the basic variables.

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If one of the components of x_B^* is zero, the basic feasible solution is degenerate.

Example

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad x \geqslant 0.$$

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  and (1,1,0,0) is a vertex.

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and (8/3,0,-1/3,0) is not a vertex (non feasible).

Example

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Example

- $\{3, 4\}$: $x_1 = x_2 = 0$ and

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There are 4 vertices (non degenerate).

Brutus

A first algorithm (*Brutus*)

Consider the linear optimization problem

$$\min_x \quad c^T x \quad \text{such that} \quad x \in \mathcal{P} = \{x \mid Ax \geqslant b\}.$$

By the *fundamental theorem*, we know that if the optimal value is bounded and $\mathcal{P}$ contains a vertex, then an optimal vertex exists.

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Brute-force algorithm.

1. List all vertices of $\mathcal{P}$: $x_1^*, x_2^*, \ldots, x_k^*$.

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If the optimal value f^* is bounded:

$$c^T x_i^* \leqslant c^T x_j^* \text{ for all } 1 \leqslant j \leqslant k \quad \Rightarrow \quad x_i^* \text{ is optimal }.$$

Number of vertices

The polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \geqslant b\}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, can potentially have

$$C_m^n = \left(\begin{array}{c} m \\ n \end{array} \right) = \frac{m!}{n!(m-n)!}.$$

It's finite, but it's a lot.

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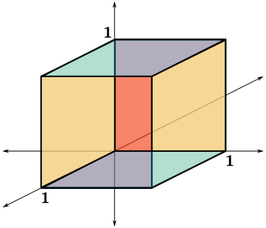
For example, a polyhedron with 10 variables ($n = 10$) and 30 constraints ($m = 30$) has at most 30 million vertices. With 50 constraints, we arrive at 10 billion vertices. ...

A polyhedron with many vertices

The unit cube:

$$\{x \in \mathbb{R}^n \mid 0 \leqslant x_i \leqslant 1 \ \text{ for } \ 1 \leqslant i \leqslant n\}$$



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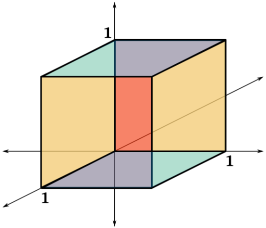

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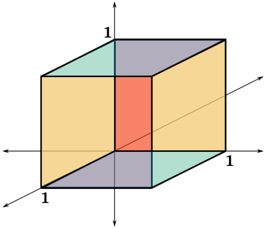

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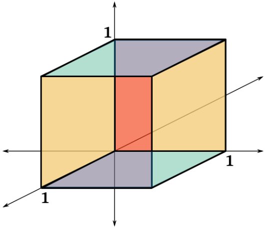

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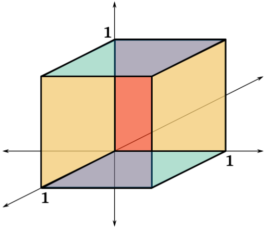

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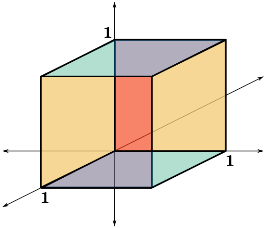

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- We could move from adjacent vertex to adjacent vertex. This is the simplex method.

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- There are $n - m$ constraints $x_i \geqslant 0$ tight in x^* and $n - m$ constraints $x_i \geqslant 0$ tight in x^{**}.
- The basic admissible solutions are adjacent if there are $n - m - 1$ constraints $x_i \geqslant 0$ that are tight at both x^* and x^{**}.
- *This condition is satisfied if and only if the basic feasible x^* and x^{**} have $m - 1$ base variables in common.* **Let us demonstrate this on the board**

Example

The vertices of the polyhedron defined by

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad x \geqslant 0,$$

are $(1, 1, 0, 0)$, $(2, 0, 0, 1)$, $(0, 8/5, 1/5, 0)$, $(0, 0, 1, 4)$.

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The vertex (1, 1, 0, 0) is adjacent to (2, 0, 0, 1) and to (0, 8/5, 1/5, 0) but is not adjacent to (0, 0, 1, 4).

Fundamental Theory

Existence of vertices

A polyhedron $\mathcal{P}$ contains a straight line if there exists an point x_0 and a non-zero vector d such that $x_0 + \lambda d \in \mathcal{P}$ for all $\lambda \in \mathbb{R}$.

Proposition

Polyhedra that have a vertex are exactly those that do not contain a straight line.

Proof Part 1 - necessary condition : if $\mathcal{P}$ does not contain any straight line $\Rightarrow$ $\mathcal{P}$ has a vertex. Let $x_0 \in \mathcal{P}$ and $\mathcal{I}$ the set of indices for which $a_i^T x = b_i$. If there are n linearly independent vectors in the set $\{a_i | i \in \mathcal{I}\}$, hence x_0 is a vertex and the proposition is proved.

Suppose x_0 is not a vertex. Hence, there exists a direction $d \in \mathbb{R}^n$ such that $a_i^T d = 0$ [2] for all $i \in \mathcal{I}$. Let $x = x_0 + \lambda d$ the equation of a straight line. By assumption, there is no straight line in $\mathcal{P}$, hence, there exists $j \notin \mathcal{I}$ and $\lambda^\star$ such that $a_j^T(x_0 + \lambda^\star d) = b_j$.[3] Furthermore, a_j is not linear combination of a_i for all $i \in \mathcal{I}$ since d is orthogonal to a_i for all $i \in \mathcal{I}$ and not to a_j ! Then, the point $x_1 = x_0 + \lambda^\star d$ tightens a new lin. ind. constraint. Step by step, we finally reach a vertex.

[2] imagine, you are on a facet, and you slide along a direction d until you reach something :)

[3] before that $a_j^T x_0 > b_j$ for all $j \notin \mathcal{I}$

Existence of vertices

Proof Part 2 - sufficient condition : if $\mathcal{P}$ has a vertex $\Rightarrow$ $\mathcal{P}$ does not contain any straight line.

- Let x_0 be the vertex of $\mathcal{P}$. There are therefore n linearly independent active constraints at x_0. Let $a_i^T x_0 = b_i$ for all $i = 1, ..., n$.
- Let us suppose there is a straight line $x = x_0 + \lambda d$ in $\mathcal{P}$, hence we have: $a_i^T(x_0 + \lambda d) \geqslant b_i$ for all i.
- For the $i = 1, ..., n$, we have $a_i^T(x_0 + \lambda d) = a_i^T x_0 + \lambda a_i^T d = b_i + \lambda a_i^T d \geqslant b_i$. Hence, we have $\lambda a_i^T d \geqslant 0$ for all $i = 1, ..., n$ and for all $\lambda \in \mathbb{R}$.
- Since these have to hold for any λ, it implies that $a_i^T d = 0$ for $i = 1, ..., n$.
- Finally, since these a_i are linearly independent by assumption, it necessarily implies that $d = 0$.[4]
- Hence, there is not straight line in $\mathcal{P}$.

[4] is it possible to find a vector $d \in \mathbb{R}^n$ orthogonal to n linearly independent vectors in $\mathbb{R}^n$? No :)

Presence of vertices

Proposition

A polyhedron given in geometric form $\mathcal{P} = \{x \mid Ax \geqslant b\}$ has a vertex if and only if the equation $Ad = 0$ has no other solution than the trivial solution $d = 0$.

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Proposition

A non-empty polyhedron given in the standard form $\mathcal{P} = \{x \mid Ax = b, x \geqslant 0\}$ always has a vertex.

Proof:The nonnegative orthant contains no straight lines, and any polyhedron in standard form is entirely contained in the nonnegative orthant.

The fundamental theorem

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Consider the problem of minimizing a linear function on a polyhedron $\mathcal{P}$. If the optimal cost is finite and the polyhedron has a vertex, then there is a vertex of the polyhedron that is optimal.

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Proof: Let

$$\min_{x} \quad c^T x \quad \text{such that} \quad x \in \mathcal{P} = \{x \mid Ax \geqslant b\}$$

with f^* the optimal cost so that the set of optimal solutions is given by

$$\mathcal{P}^* = \{x \mid Ax \geqslant b, c^T x = f^*\} \subseteq \mathcal{P}.$$

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$$\mathcal{P}^* = \{x \mid Ax \geqslant b, c^T x = f^*\} \subseteq \mathcal{P}.$$

<u>Key Step 1</u> - Existence of a vertex in $\mathcal{P}^*$:
Since $\mathcal{P}$ has a vertex, it does not contain a right-hand side, which implies that $\mathcal{P}^*$ does not contain one either. $\mathcal{P}^*$ therefore contains an optimal vertex x^*.

The fundamental theorem

Key Step 2 - This vertex of $\mathcal{P}^*$ is also a vertex of $\mathcal{P}$:
Let's assume by contradiction that x^* is not a vertex of $\mathcal{P}$.
Therefore: there exists $y, z \in \mathcal{P}$ $(y, z \neq x^*)$ and $\lambda \in [0,1]$ such that $x^* = \lambda y + (1-\lambda)z$ [5]
Since x^* is optimal:

$$c^T x^* \leqslant c^T y \text{ and } c^T x^* \leqslant c^T z \tag{1}$$

Moreover we have

$$c^T x^* = \lambda c^T y + (1-\lambda) c^T z^6 \tag{2}$$

Combining Equations (1) and (2), we necessary have

$$c^T y = c^T z = c^T x^* \Rightarrow y, z \in \mathcal{P}^*.$$

This is a contradiction since x^* is a vertex of $\mathcal{P}^{*7}$, therefore x^* is also a vertex of $\mathcal{P}$.

[5] Recall that, by step 1, we have $x^* \in \mathcal{P}^* \in \mathcal{P}$ and we assume that it is not a vertex, hence not *an extreme point* neither.

[6] we assumed that $x^* = \lambda y + (1-\lambda)z$, then we can multiply on both sides by c^T.

[7] hence an extreme point, hence cannot be expressed as convex combination of other points.

Number of tight/active constraints

- According to the fundamental theorem: a linear program with *finite optimal cost* on a polyhedron with *at least one vertex* has an optimal vertex.
- At a vertex, we tighten as many constraints as there are variables.
- **Therefore**: under the conditions stated above, there is always an optimal solution that satisfies as many constraints as there are variables.

Example: Chebyshev center of a polyhedron

We are looking for the largest sphere entirely contained in the polyhedron

$$\{x \in \mathbb{R}^n \mid a_i^T x \geqslant b_i, i = 1, 2, \ldots, m\}.$$

[8] for example, this will always be the case if the polyhedron is bounded - and not empty - i.e. it's a polytope.

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$$\{x \in \mathbb{R}^n \mid a_i^T x \geqslant b_i, i = 1, 2, \ldots, m\}.$$

We can assume $||a_i||_2 = 1$ for all i, and the problem is written as follows

$$\max_{x,t} \quad t \quad \text{such that} \quad a_i^T x \geqslant b_i + t, i = 1, 2, \ldots, m,$$

where x is the center of the sphere and t is the smallest distance from x to the hyperplanes defining the polyhedron (the radius).

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where x is the center of the sphere and t is the smallest distance from x to the hyperplanes defining the polyhedron (the radius).

- If the optimal cost is finite and the admissible set has a vertex[8]
- **then** then there is an optimal solution that tightens $n + 1$ constraints (one vertex).
  Why not n ?

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Example: Chebyshev center of a polyhedron

- For example, a polygon ($n = 2$) will always have a Chebyshev center touching at least three segments. [9]

[9] Note that in the case of a rectangle, for example, there are centers touching only two of them.

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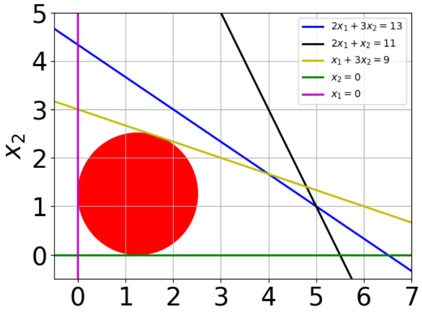

see colab file - Example 3

[9] Note that in the case of a rectangle, for example, there are centers touching only two of them.

Conclusions

Summary

We have seen

- Examples of *polyhedra*: hyperplance, slice, polyhedron in both equational/standard and geometric forms.
- What do we mean by *active/tight* and *linearly independent* constraints.
- What is a *Basic Feasible Solution* (BFS): a feasible solution at which there are n linearly independent active constraints.
- The notion of *adjacent* BFS: $n-1$ linearly independent constraints active at both.
- Equivalency between *vertices*, *extreme points* and *BFS*, and independent on the representations chosen for the polyhedra.
- Vertex calculation for both forms.
- A first algorithm (Brutus) and its limitation, and hence the idea of looking at *adjacent* vertices.
- Very$^{(\text{very})}$ important theoretical results:
  1. Existence of vertices (no straight lines $\leftrightarrow$ a vertex)
  2. Presence of vertices for both forms.
  3. The fundamental theorem.

Preparations for the next lecture

- Review the lecture :); many important results and notions have been introduced.
- Understand/master the demos (this is a tip for the mid-term exam, hear it).
- Potential helpers:
  1. An introduction to convex sets: ▸ Section 1.1
  2. Equivalency between *vertices*, *extreme points* and *BFS* ▸ Chapter 2 - Theorem 2.3 - pages 50-52

Goodbye, So Soon

THANKS FOR THE ATTENTION

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