Optimisation

Lecture 5 - Duality

Fall semester - 2024

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Outline

Introduction

Duality

- A linear optimization problem always has a *companion* problem, a **dual** problem, for which the role of variables and constraints is *reversed*.
- The initial problem is called the primal problem.
- For
  1. every variable in the primal problem, there is a constraint in the dual, and
  2. for every constraint in the primal problem, there is a variable in the dual.

Mixing (or dietary) problems

- This week, your sports coach (or dietician or dentist) recommends that you consume 10 units of vitamin A, 8 units of vitamin B and 7 units of vitamin C.
- All you want to eat are apples and bananas, whose vitamin content and price per unit is:

	Vit. A	Vit. B	Vit. C	price
Apples	2	1	1	4
Bananas	1	2	1	3
Total to be consumed	10	8	7	

Table: Data (per unit).

How do you bound the optimum value?

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\begin{aligned}
\min_{x_1,x_2} \quad & f(x) = 4x_1 + 3x_2 \\
\text{such that} \quad & 2x_1 + x_2 \geqslant 10, \\
& x_1 + 2x_2 \geqslant 8, \\
& x_1 + x_2 \geqslant 7, \\
& x_1, \quad x_2 \geqslant 0.
\end{aligned}
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- Let $\mathcal{P}$ be the feasible domain of this problem.

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- Let $\mathcal{P}$ be the feasible domain of this problem.
- Without solving the problem, how to bound the optimal value

$$f^\star = \inf_{x \in \mathcal{P}} f(x) \ ?$$

Upper-bound

- Any feasible solution provides an upper bound on the optimal value !

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- Indeed, by definition, for all $y \in \mathcal{P}$, we have

$$f(y) \geqslant f^{\star} = \inf_{x \in \mathcal{P}} f(x).$$

Upper-bound : Example

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- (8,0) is feasible $\Rightarrow f^\star \leqslant 32$. At most, you'll need to spend 32 Rubles to assimilate the required vitamins.

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- (8,0) is feasible $\Rightarrow f^\star \leqslant 32$. At most, you'll need to spend 32 Rubles to assimilate the required vitamins.
- (0,10) is feasible $\Rightarrow f^\star \leqslant 30$.
- (3,4) is feasible $\Rightarrow f^\star \leqslant 24$.

Lower-bound

- Any linear combination of equalities and inequalities (with positive weights for inequalities) provides an inequality (or equality) valid.

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Since $f(x) = 4x_1 + 3x_2$ and $x_1, x_2 \geqslant 0$, we have for all $x \in \mathcal{P}$

$$f(x) = 4x_1 + 3x_2 \geqslant 4x_1 + 2x_2 \geqslant 20.$$

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$\Rightarrow$ Any feasible solution must have a cost of at least 20.
$\Rightarrow f^\star \geqslant 20.$

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$\Rightarrow f^\star \geqslant 21.$

Without solving the problem, we already know that the feasible solution (3,4) (whose cost is 24) costs us at worst 3 Rubles, too much.

The Dual

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For all $y_1, y_2, y_3 \geqslant 0$, we have

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\begin{array}{rcl}
\big(2x_1 + x_2 \geqslant 10\big) & . & y_1 \\
+\big(x_1 + 2x_2 \geqslant 8\big) & . & y_2 \\
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$$y_1\big(2x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(x_1 + x_2) \geqslant 10y_1 + 8y_2 + 7y_3$$

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$$\iff x_1\big(2y_1 + y_2 + y_3) + x_2(y_1 + 2y_2 + y_3\big) \geqslant 10y_1 + 8y_2 + 7y_3$$

The dual

If we impose $2y_1 + y_2 + y_3 \leqslant 4$ and $y_1 + 2y_2 + y_3 \leqslant 3$, then

$$\begin{aligned} f(x) &= 4x_1 + 3x_2 \\ &\geqslant \underbrace{x_1}_{\geqslant 0} \underbrace{(2y_1 + y_2 + y_3)}_{\leqslant 4} + \underbrace{x_2}_{\geqslant 0} \underbrace{(y_1 + 2y_2 + y_3)}_{\leqslant 3} \\ &\geqslant 10y_1 + 8y_2 + 7y_3 = g(y). \end{aligned}$$

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We want to maximize this lower bound:

$$\begin{aligned} \max_{y_1,y_2,y_3} \quad & g(y) = 10y_1 + 8y_2 + 7y_3 \\ \text{such that} \quad & 2y_1 + y_2 + y_3 \leqslant 4, \\ & y_1 + 2y_2 + y_3 \leqslant 3, \\ & y_1, y_2, y_3 \geqslant 0. \end{aligned} \tag{Dual}$$

We will denote the feasible domain/set of the dual
$\mathcal{D} = \{y \in \mathbb{R}^3 \mid 2y_1 + y_2 + y_3 \leqslant 4, y_1 + 2y_2 + y_3 \leqslant 3 \text{ and } y_1, y_2, y_3 \geqslant 0\}$.

The dual

- We have just demonstrated that for all $x \in \mathcal{P}$ and for all $y \in \mathcal{D}$, we have

$$f(x) \geqslant g(y).$$

- In particular, for all $y \in \mathcal{D}$,

$$f^\star = \inf_{x \in \mathcal{P}} f(x) \geqslant g(y).$$

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  Any feasible solution of the dual provides a lower bound for the optimal value of the primal.

- The <u>tightest</u> lower bound is the optimal solution of the dual:

$$f^\star = \inf_{x \in \mathcal{P}} f(x) \geqslant \sup_{y \in \mathcal{D}} g(y) = g^\star.$$

Primal - Dual

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\min_x & f(x) = 4x_1 + 3x_2 \\
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\quad \geqslant \quad
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\text{such that} & 2y_1 + y_2 + y_3 \leqslant 4 \\
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- For example, the dual solution $y = (1, 0, 2)$ has a cost of 24: $f^\star \geqslant 24$.

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- For example, the dual solution $y = (1, 0, 2)$ has a cost of 24: $f^\star \geqslant 24$.
  This involves taking the first inequality once and the third twice:

$$f(x) = 4x_1 + 3x_2 = 1 * (2x_1 + x_2) + 2 * (x_1 + x_2) \geqslant 10 + 2 * 7 = 24.$$

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- The solution $y = (1, 0, 2)$ of the dual **certifies** that $x = (3, 4)$ is optimal.

Generalisation and Weak Duality

Weak Duality

$$\begin{array}{cccccc} \min_x & f(x) = c^T x & \color{red}{\geqslant} & \max_y & g(y) = b^T y \\ \text{such that} & Ax \geqslant b & & \text{such that} & A^T y \leqslant c \\ & x \geqslant 0 & & & y \geqslant 0. \end{array}$$

Weak Duality

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Theorem: Weak Duality

For all $x \in \mathcal{P} = \{x \mid Ax \geqslant b, x \geqslant 0\}$ and for all $y \in \mathcal{D} = \{x \mid A^T y \leqslant c, y \geqslant 0\}$,

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Proof

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Proof We have

$$f(x) = c^T x \quad \underbrace{\geqslant}_{\substack{x \geqslant 0 \\ c \geqslant A^T y}} \quad \left(A^T y\right)^T x = y^T (Ax) \quad \underbrace{\geqslant}_{\substack{y \geqslant 0 \\ Ax \geqslant b}} \quad y^T b = b^T y = g(y).$$

Optimality certificate

$$\begin{array}{ccccc} \min_x & f(x) = c^T x & \geqslant & \max_y & g(y) = b^T y \\ \text{such that} & Ax \geqslant b & & \text{such that} & A^T y \leqslant c \\ & x \geqslant 0 & & & y \geqslant 0. \end{array}$$

Corollary

If we find $x^\star \in \mathcal{P} = \{x \mid Ax \geqslant b, x \geqslant 0\}$ and $y^\star \in \mathcal{D} = \{x \mid A^T y \leqslant c, y \geqslant 0\}$ such that

$$f(x^\star) = c^T x^\star \quad = \quad b^T y^\star = g(y^\star),$$

then $x^\star$ is an optimal solution of the primal, and $y^\star$ is an optimal solution of the dual.

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Proof - Part 1
By weak duality: For every $x \in \mathcal{P}$ and for every $y \in \mathcal{D}$

$$c^T x \geqslant b^T y.$$

In particular,

$$c^T x \geqslant b^T y^\star = c^T x^\star \quad \text{for all } x \in \mathcal{P},$$

and therefore $x^\star$ is an optimal solution of the primal.

Optimality certificate

Proof - Part 2
In the same way,

$$b^T y \leqslant c^T x^\star = b^T y^\star \quad \text{for all } y \in \mathcal{D},$$

and $y^\star$ is an optimal solution of the dual.

Dual of a standard form

What is the dual of

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{such that} \quad & Ax = b, \\ & x \geqslant 0? \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

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- For each equality $a_i^T x = b_i$ $i = 1, 2, \ldots, m$, we introduce the dual variable y_i and take the linear combinations of the primal's equalities:

$$(A^T y)^T x = \sum_{i=1}^m y_i a_i^T x = \sum_{i=1}^m y_i b_i = b^T y.$$

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$$(A^T y)^T x = \sum_{i=1}^{m} y_i a_i^T x = \sum_{i=1}^{m} y_i b_i = b^T y.$$

- We then have:

$$A^T y \leqslant c \quad \Rightarrow \quad c^T x \underbrace{\geqslant}_{\substack{x \geqslant 0 \\ c \geqslant A^T y}} (A^T y)^T x = y^T (Ax) \underbrace{=}_{Ax = b} y^T b = b^T y.$$

Dual of a standard form

Primal:

$$\begin{array}{ll} \min_x & c^T x \\ \text{such that} & Ax = b, \\ & x \geqslant 0. \end{array} \quad \geqslant \quad \begin{array}{ll} \max_y & b^T y \\ \text{such that} & A^T y \leqslant c. \end{array}$$

Dual:

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- We have

$$A^T y = c \quad \Rightarrow \quad c^T x \underbrace{=}_{c = A^T y} (A^T y)^T x = y^T (Ax) \underbrace{\geqslant}_{\substack{y \geqslant 0 \\ Ax \geqslant b}} y^T b = b^T y.$$

Dual of a geometric form

$$\begin{array}{lll} \text{Primal:} & & \\ \min_x & c^T x \\ \text{such that} & Ax \geqslant b. \end{array} \quad \geqslant \quad \begin{array}{lll} \text{Dual:} & & \\ \max_y & b^T y \\ \text{such that} & A^T y = c, \\ & y \geqslant 0. \end{array}$$

The dual of any primal problem

$$
\begin{aligned}
\min_{x=(x_1,x_2)} \quad & c_1^T x_1 + c_2^T x_2 \\
\text{such that} \quad & A_1 x_1 + A_2 x_2 \geqslant b_1, \qquad \text{(Primal)} \\
& A_3 x_1 + A_4 x_2 = b_2, \\
& x_1 \geqslant 0, x_2 \text{ free.}
\end{aligned}
$$

$$
\begin{aligned}
\max_{y=(y_1,y_2)} \quad & b_1^T y_1 + b_2^T y_2 \\
\text{such that} \quad & A_1^T y_1 + A_3^T y_2 \leqslant c_1, \qquad \text{(Dual)} \\
& A_2^T y_1 + A_4^T y_2 = c_2, \\
& y_1 \geqslant 0, y_2 \text{ free.}
\end{aligned}
$$

Exercises.

1. Prove the weak duality for this primal-dual pair.
2. Prove that the dual of the dual is the primal.

Strong Duality

Strong Duality

Theorem: Strong Duality

If there exists an optimal solution $x^\star$ of the primal, then there exists an optimal solution $y^\star$ of the dual such that

$$c^T x^\star = b^T y^\star.$$

Strong Duality

Theorem: Strong Duality

If there exists an optimal solution $x^\star$ of the primal, then there exists an optimal solution $y^\star$ of the dual such that
$$c^T x^\star = b^T y^\star.$$

Proof Consider the standard form:

$$\begin{array}{cc} \min_x & c^T x \\ \text{such that} & Ax = b, \\ & x \geqslant 0. \end{array} \quad \geqslant \quad \begin{array}{cc} \max_y & b^T y \\ \text{such that} & A^T y \leqslant c. \end{array}$$

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Let $x^\star$ be an optimal vertex (which always exists in standard form: why?). Reordering the variables, we obtain
$$x^\star = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$$
where $x_B = A_B^{-1} b$ are the basic variables and $x_N = 0$ are the non-basic variables. Also note $A = [A_B, A_N]$ and $c = [c_B, c_N]$.

Strong Duality

Proof - continued

Since $x^\star$ is optimal, reduced costs $c_N^T - c_B^T A_B^{-1} A_N$ are positive:

$$c_N^T \geqslant c_B^T A_B^{-1} A_N.$$

Let's define $(y^\star)^T = c_B^T A_B^{-1}$ and show that (1) $y^\star$ is a feasible solution of the dual, and (2) $c^T x^\star = b^T y^\star$. By weak duality, this will complete the proof.

1. $y^\star$ is feasible for the dual since

$$(y^\star)^T A = (y^\star)^T \Big[A_B, A_N\Big] = \Big[(y^\star)^T A_B, (y^\star)^T A_N\Big] = \Big[c_B^T, c_B^T A_B^{-1} A_N\Big] \leqslant \Big[c_B^T, c_N^T\Big] = c^T.$$

2. We have

$$(y^\star)^T b = c_B^T A_B^{-1} b = c_B^T x_B = c^T x^\star.$$

Primal-dual solution cases

Dual / Primal	Finite Optimum	Unbounded	$\varnothing$
Finite Optimum			
Unbounded			
$\varnothing$			

$\varnothing$ = No feasible solution.

Primal-dual solution cases

Dual / Primal	Finite Optimum	Unbounded	$\emptyset$
Finite Optimum	OK	/	/
Unbounded	/		
$\emptyset$	/		

$\emptyset$ = No feasible solution. / = Cannot happen.

By Strong Duality.

Primal-dual solution cases

Dual / Primal	Finite Optimum	Unbounded	$\varnothing$
Finite Optimum	OK	/	/
Unbounded	/	/	OK
$\varnothing$	/	OK	

$\varnothing$ = No feasible solution. / = Cannot happen.

By Strong Duality.

By Weak Duality: if the primal is unbounded, the dual has no feasible solution. If the dual is unbounded, the primal has no feasible solution.

Primal-dual solution cases

Dual / Primal	Finite Optimum	Unbounded	$\emptyset$
Finite Optimum	OK	/	/
Unbounded	/	/	OK
$\emptyset$	/	OK	OK*

$\emptyset$ = No feasible solution. / = Cannot happen.

By Strong Duality.

By Weak Duality: if the primal is unbounded, the dual has no feasible solution. If the dual is unbounded, the primal has no feasible solution.

Example*: $\max_x \; 2x_1 - x_2$ such that $x_1 - x_2 = 1, -x_1 + x_2 = -2, x_1 \geqslant 0$ and $x_2 \geqslant 0$ is impossible, and its dual $\min_y \; y_1 - 2y_2$ such that $y_1 - y_2 \geqslant 2, -y_1 + y_2 \geqslant -1$ neither.

Complementary Slackness Conditions

Main result

Consider the pair

$$\begin{array}{ccccc} \min_x & c^T x & \textcolor{red}{\geqslant} & \max_y & b^T y \\ \text{such that} & Ax \geqslant b, & & \text{such that} & y \geqslant 0, \\ & x \geqslant 0. & & & A^T y \leqslant c. \end{array}$$

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- There is interdependence between the $x \geqslant 0$ constraints of the primal and the $A^T y \leqslant c$ constraints of the dual.
- At optimal solutions, at least one of the constraints is tight.

Proposition

If x is an optimal solution of the primal and y is an optimal solution of the dual, then

$$x^T(c - A^T y) = 0 \quad \text{and} \quad y^T(Ax - b) = 0.$$

In other words:

1. For all i, either $x_i = 0$, or $c_i - {a_i}^T y = 0$,
2. For all j, either $y_j = 0$, or $a_j^T x - b_j = 0$.

Main result

Proof. We will only demonstrate the first case. The other is obtained by duality. If x and y are optimal solutions, by strong duality,

$$c^T x = b^T y.$$

Moreover, since $c \geqslant A^T y$,

$$c^T x \geqslant (A^T y)^T x \geqslant b^T y.$$

This implies $c^T x = (A^T y)^T x$ or equivalently $x^T (c - A^T y) = 0$.

Example

$$
\begin{array}{cc}
\min_x & 4x_1 + 3x_2 \\
\text{such that} & 2x_1 + x_2 \geqslant 10, \\
 & x_1 + 2x_2 \geqslant 8, \\
 & x_1 + x_2 \geqslant 7, \\
 & x_1 \geqslant 0, \\
 & x_2 \geqslant 0.
\end{array}
\quad {\color{red}\geqslant} \quad
\begin{array}{cc}
\max_y & 10y_1 + 8y_2 + 7y_3 \\
\text{such that} & y_1 \geqslant 0, \\
 & y_2 \geqslant 0, \\
 & y_3 \geqslant 0, \\
 & 2y_1 + y_2 + y_3 \leqslant 4, \\
 & y_1 + 2y_2 + y_3 \leqslant 3.
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- The pair of solutions (3,4) and (1,0,2) satisfy the complementarity slackness conditions.

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\begin{array}{llcll}
\min_x & 4x_1 + 3x_2 & & \max_y & 10y_1 + 8y_2 + 7y_3 \\
\text{such that} & 2x_1 + x_2 \geqslant 10, & & \text{such that} & y_1 \geqslant 0, \\
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- The pair of solutions (3,4) and (1,0,2) satisfy the complementarity slackness conditions.
- This is quite logical: an inequality that is not active at optimality in the primal is not used to generate the largest lower bound (its weight in the 'best' linear combination is its associated dual variable).

Sensitivity analysis

Interpretation of the dual variables

$$
\begin{array}{llcll}
\min_x & f(x) = 4x_1 + 3x_2 & & \max_y & g(y) = 10y_1 + 8y_2 + 7y_3 \\
\text{such that} & 2x_1 + x_2 \geqslant 10 & & \text{such that} & 2y_1 + y_2 + y_3 \leqslant 4 \\
 & x_1 + 2x_2 \geqslant 8 & {\color{red}\geqslant} & & y_1 + 2y_2 + y_3 \leqslant 3 \\
 & x_1 + x_2 \geqslant 7 & & & y_1, y_2, y_3 \geqslant 0. \\
 & x_1, \quad x_2 \geqslant 0. & & &
\end{array}
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- The optimal primal solution is (3,4) (purchase of 3 units of apples at 4 Rubles, and 4 units of bananas at 3 Rubles).

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- The optimal primal solution is (3,4) (purchase of 3 units of apples at 4 Rubles, and 4 units of bananas at 3 Rubles).
- As a reminder, the constraints correspond to the quantity of vitamins A, B and C required (respectively).
- What do the optimal dual variables (1,0,2) correspond to?

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- A salesman wants to sell you vitamin A: what price would you be willing to pay?

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- Suppose you buy ϵ vitamin A at zRuble/unit, the primal-dual pair becomes

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\min_x & 4x_1 + 3x_2 \\
\text{such that} & 2x_1 + \ \ x_2 \geqslant 10 - \epsilon \\
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& x_1 + \ \ x_2 \geqslant 7 \\
& x_1, \quad x_2 \geqslant 0.
\end{array}
\quad \geqslant \quad
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\max_y & (10 - \epsilon)y_1 + 8y_2 + 7y_3 \\
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- Let's denote $x^\star(\epsilon)$ the optimal solution of the primal.

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- Let's denote $x^\star(\epsilon)$ the optimal solution of the primal.
- Constraints of the dual are unchanged: solution (1,0,2) remains feasible.
- By the Weak Duality:

$$c^T x^\star(\epsilon) \geqslant (10-\epsilon) * 1 + 7 * 2 = 24 - \epsilon.$$

Interpretation of the dual variables

- Your new cost $f(\epsilon)$ satisfies

$$f(\epsilon) = c^T x^\star(\epsilon) + z\epsilon \geqslant 24 - \epsilon + z\epsilon = 24 + \epsilon(z-1).$$

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$$f(\epsilon) = c^T x^\star(\epsilon) + z\epsilon \geqslant 24 - \epsilon + z\epsilon = 24 + \epsilon(z-1).$$

- There's no point in buying if vitamin A costs more than 1 Ruble !
- The variable y_1 associated with the constraint on the amount of vitamin A to be assimilated actually represents the marginal value of vitamin A. (This requires the use of strong duality; see later).
- For example, $y_2 = 0$ and you'll never want to buy vitamin B (you're already consuming too much with the current optimal solution).

Sensitivity analysis - the starting point

Consider the primal-dual pair in standard form:

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\begin{array}{llcll}
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- Suppose we have a base

$$x^\star = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \qquad \text{where } A = [A_B, A_N] \text{ and } c = [c_B, c_N],$$

  feasible i.e. $x_B = A_B^{-1} b \geqslant 0$ are the basic variables and $x_N = 0$ the non-basic variables, and optimal i.e. $c_N^T - c_B^T A_B^{-1} A_N \geqslant 0$.
- The corresponding dual solution is given by $(y^\star)^T = c_B^T A_B^{-1}$.

Sensitivity analysis - change in b

- When the resource vector changes from b to $b + \Delta b$, the optimality condition $c_N^T - c_B^T A_B^{-1} A_N \geqslant 0$ remains satisfied.

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$$A_B^{-1}(b + \Delta b) = A_B^{-1} b + A_B^{-1} \Delta b \geqslant 0.$$

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- If $A_B^{-1}b > 0$ (non-degenerate vertex), this condition is satisfied for any Δb sufficiently small.

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- The new optimal solution is $(A_B^{-1}(b + \Delta b), 0) = (x_B + A_B^{-1} \Delta b, 0)$ (the optimal base doesn't change).

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- The new optimal solution is $(A_B^{-1}(b + \Delta b), 0) = (x_B + A_B^{-1} \Delta b, 0)$ (the optimal base doesn't change).
- This implies that the optimal dual solution $y^\star$ does not change for Δb sufficiently small. (recall that $(y^\star)^T = C_B^T A_B^{-1}$)

Sensitivity analysis - change in b

- In the optimal solution associated with $b + \Delta b$, the objective function takes the value

$$c_B^T(x_B + A_B^{-1}\Delta b) = c_B^T x_B + c_B^T A_B^{-1}\Delta b = f^\star + \Delta f = f^\star + (y^\star)^T \Delta b,$$

  a change of $(y^\star)^T \Delta b$.
- The optimal solution of the dual $y^\star$ gives the sensitivity of the optimal objective to a variation in b:
  When b increases by Δb, the objective function increases by $(y^\star)^T \Delta b$.

Example

$$
\begin{array}{cc}
\min_{x,s} & 4x_1 + 3x_2 \\
\text{such that} & 2x_1 + x_2 - s_1 = 10 - \epsilon \\
& x_1 + 2x_2 - s_2 = 8 \\
& x_1 + x_2 - s_3 = 7 \\
& x, s \geqslant 0.
\end{array}
\quad \geqslant \quad
\begin{array}{cc}
\max_{y} & (10 - \epsilon)y_1 + 8y_2 + 7y_3 \\
\text{such that} & 2y_1 + y_2 + y_3 \leqslant 4 \\
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\end{array}
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- We obtain

$$\Delta b = \begin{pmatrix} -\epsilon \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad A_B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In particular, the base variables $(x_1, x_2, s_2) = A_B^{-1} b = (3, 4, 3) > 0$ (non-degenerate vertex).

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$$

  In particular, the base variables $(x_1, x_2, s_2) = A_B^{-1} b = (3, 4, 3) > 0$ (non-degenerate vertex).
- The base remains unchanged as long as $A_B^{-1} b + A_B^{-1} \Delta b$ remains positive.
- **For example**, for $\epsilon = 1$, we have $(x_1, x_2, s_2) = (2, 5, 4)$ and the optimal quantity of apples and bananas to buy is 2 and 5 respectively, for a total price of 23.

Example

- Moreover, the dual optimal solution (2,0,1) does not change and we have

$$c^T x^\star(\epsilon) = c^T x^\star + (y^\star)^T \Delta b = c^T x^\star - 2\epsilon,$$

  for any sufficiently small ϵ.

- *Remark.* More precisely, the base remains unchanged as long as $-3 \leqslant \epsilon \leqslant 3$.
  Exercise: prove this statement :)

Sensitivity analysis - change in c

- When the objective vector passes from c to $c + \Delta c$, the feasibility condition $A_B^{-1} b \geqslant 0$ remains satisfied.

Sensitivity analysis - change in c

- When the objective vector passes from c to $c + \Delta c$, the feasibility condition $A_B^{-1} b \geqslant 0$ remains satisfied.
- The optimality condition remains satisfied if

$$
\begin{aligned}
&(c_N + \Delta c_N)^T - (c_B + \Delta c_B)^T A_B^{-1} A_N \\
&\qquad = \underbrace{(c_N^T - c_B^T A_B^{-1} A_N)}_{\geqslant 0} - (\Delta c_N^T + \Delta c_B^T A_B^{-1} A_N) \geqslant 0.
\end{aligned}
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\end{aligned}
$$

- If $c_N^T - c_B^T A_B^{-1} A_N > 0$ (non-degenerate vertex), this condition is satisfied for any Δc sufficiently small.

Example

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\begin{array}{cc}
\min_{x,s} & (4-\epsilon)x_1 + 3x_2 \\
\text{such that} & 2x_1 + x_2 - s_1 = 10 \\
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& x, s \geqslant 0.
\end{array}
\quad \geqslant \quad
\begin{array}{cc}
\max_{y} & 10y_1 + 8y_2 + 7y_3 \\
\text{such that} & 2y_1 + y_2 + y_3 \leqslant (4-\epsilon) \\
& y_1 + 2y_2 + y_3 \leqslant 3 \\
& y \geqslant 0.
\end{array}
$$

- The primal solution $(x^\star, s^\star) = (3,4,0,3,0)$ remains optimal if

$$(c_N^T - c_B^T A_B^{-1} A_N) - (\Delta c_N^T + \Delta c_B^T A_B^{-1} A_N) = (1,2) - (-\epsilon, \epsilon) \geqslant 0$$

i.e. if $-1 \leqslant \epsilon \leqslant 2$.

- For information, we have:

$$A_B^{-1} A_N = \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ 1 & -3 \end{pmatrix}.$$

Sensitivity analysis - new variable

- We introduce a new variable x_{n+1} with objective coefficient c_{n+1} and associated vector a_{n+1}.

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- The base remains feasible (A_B and b do not change).
- As the new variable is **added to the non-basic variables**, the solution remains optimal if the input added to the reduced costs $c_N^T - c_B^T A_B^{-1} A_N$ corresponding to the new variable remains positive, i.e. if

$$c_{n+1} - (y^\star)^T a_{n+1} \geqslant 0.$$

Example

- Suppose we have a new fruit to eat, say oranges (cost: 6, vitamins A: 1, B: 1, C: 2). The optimal solution is to buy 3 apples and 4 bananas:

$$6 - (1, 0, 2) * (1, 1, 2)^T = 1 \geqslant 0.$$

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- Suppose we have a new fruit to eat, say oranges (cost: 6, vitamins A: 1, B: 1, C: 2). The optimal solution is to buy 3 apples and 4 bananas:

$$6 - (1, 0, 2) * (1, 1, 2)^T = 1 \geqslant 0.$$

- On the other hand, if the orange costs less than 5, you'll have to buy some :)

Example: Optimal distribution of resources between competing activities

- Three resources A, B and C are used to obtain two products P and T. Two transformations are possible

$$A + 3C \to P \quad \text{and} \quad 2B + 2C \to T.$$

- Resources are available in limited quantities and resource C is shared. There are 4 units of A, 12 of B and 18 of C. Products P and T yield profits, respectively, 3 and 5 per unit produced.
- How many units of P and T must be produced to maximize profit?

Example: Optimal distribution of resources between competing activities

Modeling

$$
\begin{array}{cc}
\max_x & 3x_1 + 5x_2 \\
\text{such that} & x_1 \leqslant 4, \\
 & 2x_2 \leqslant 12, \\
 & 3x_1 + 2x_2 \leqslant 18, \\
 & x_1, x_2 \geqslant 0.
\end{array}
$$

- What is the dual of this problem?

Example: Optimal distribution of resources between competing activities

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- What is the dual of this problem?
- What do dual variables represent?
- What happens if I reduce the quantity of resource A from 4 to 3?
- What happens if I increase the resource quantity B from 12 to 13?
- Would you be willing to pay 2 Rubles, for an additional resource unit C?

Dual simplex algorithm

Dual simplex algorithm

Definition

The dual simplex algorithm is the simplex algorithm applied to the dual.

Recall:

- A base is *feasible* if $x_B = A_B^{-1} b \geqslant 0$.
- The base is *optimal* if $c_N^T - c_B^T A_B^{-1} A_N \geqslant 0$, or if $y^T = c_B^T A_B^{-1}$ is such that $A^T y \leqslant c$.

The primal simplex algorithm : starts from a BFS, or a vertex, ($A_B^{-1} b \geqslant 0$) and moves from vertex to vertex until a vertex is obtained which satisfies the optimality condition .

The dual simplex algorithm: starts from a feasible dual vertex ($c_N^T - c_B^T A_B^{-1} A_N \geqslant 0$) and moves from adjacent vertex to adjacent vertex until an optimal vertex is reached ($A_B^{-1} b \geqslant 0$).

Intuitively, we have the following interpretation in terms of the primal: the dual simplex algorithm starts from an optimal primal "vertex" ($c_N^T - c_B^T A_B^{-1} A_N \geqslant 0$) and moves from optimal "vertex" to optimal "vertex" until it reaches a feasible vertex ($A_B^{-1} b \geqslant 0$).

Adding a constraint to the primal

- If we add an inequality constraint to the primal (very useful for "branch and bound", see next lecture), the corresponding dual solution remains feasible (taking the new variable to zero) and basic.
- Indeed, let the following be the dual problem before we add the new variable:

$$\max_{y \in \mathbb{R}^m} b^T y \qquad \text{such that} \qquad A^T y \leqslant b,$$

and let $y^\star$ be an optimal vertex.

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  and let $y^\star$ be an optimal vertex.
- We add the constraint $d^T x \geqslant e$ to the primal and the corresponding dual variable $z \geqslant 0$. The dual becomes:

$$\max_{y \in \mathbb{R}^m, z \in \mathbb{R}} b^T y + e\, z \quad \text{such that} \quad (A^T\, d) \begin{pmatrix} y \\ z \end{pmatrix} = A^T y + d\, z \leqslant c \text{ and } z \geqslant 0.$$

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- The $\binom{y^*}{0}$ solution remains feasible (since $A^T y^* + d\, z = A^T y^* \leqslant c$) and basic since m constraints $A^T y^* \leqslant c$ are tight (y^* is a vertex) and so is the $z \geqslant 0$ constraint (and they are linearly independent).

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  $\Rightarrow$ The dual algorithm can easily be restarted from this solution.

Conclusions

Summary

We have seen

- How to find:
  - an **upper-bound:** any feasible solution x gives this,
  - a **lower-bound**: Any linear combination of equalities and inequalities (with positive weights for inequalities) such that $\leqslant c^T x$.
- The **Dual** problem: maximize a general lower-bound.
  Any feasible solution of the dual provides a lower bound for the optimal value of the primal
- **Weak Duality**: for any feasible x and y, we have $c^T x \geqslant b^T y$.
  and important *Corollary* on the *optimality certificate.*
- Dual of a primal in standard form (y free) and in geometric form ($y \geqslant 0$).
- **Strong Duality**: if $x^\star$ (resp. $y^\star$) exists $\Rightarrow$ $y^\star$ (resp. $x^\star$) exists such that $c^T x^\star = b^T y^\star$.
- **Complementary Slackness Conditions**
- **Sensitivity** analysis: $b \to b + \Delta b$, $c \to c + \Delta c$ and a new variable.
- Dual Simplex Algorithm.

Preparations for the next lecture

- Review the lecture :); many important results and notions have been introduced.
- Solve the last example dedicated to Optimal distribution of resources between competing activities.

Goodbye, So Soon

THANKS FOR THE ATTENTION

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