Optimization – Exercise session 2

Polyhedron geometry

An *polyhedron* is a subset of $\mathbf{R}^n$ which can be written as $\mathcal{P} = \{x \in \mathbf{R}^n | \ Ax \geq b\}$. A polyhedron is an intersection of a finite number of convex sets and is therefore a convex set.
Let $\mathcal{P}$ be a polyhedron of $\mathbf{R}^n$. The solution $x^\star$ in $\mathbf{R}^n$ is a *Basic Feasible Solution* of $\mathcal{P}$ (for polyedra, *vertex*, *extreme point* and *basic feasible solution* are synonymous) if $x^\star$ is a feasible solution ($x^\star \in \mathcal{P}$) and if there are n linearly independent constraints active in $x^\star$. A basic feasible solution $x^\star$ is *degenerated* if there are more than n active constraints in $x^\star$. Two basic feasible solutions are *adjacent* if they share $n-1$ active linearly independent constraints.

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1. Let the polyhedron be defined by the linear inequalities

$$\begin{array}{rcl} x_1 + 4x_2 - 2x_3 & = & 7, \\ 2x_2 - 3x_3 & \leq & 1, \\ x_2 & \geq & 2, \\ x_3 & \geq & 0. \end{array}$$

   Is the point $(1,2,1)$ a vertex of the polyhedron? And the point $(5,1/2,0)$?

2. Find the vertices of the polyhedron defined by

$$\begin{array}{rcl} -2x_1 + x_2 & \leq & 2, \\ -x_1 + x_2 & \leq & 3, \\ x_1 & \leq & 3, \\ x_1, x_2 & \geq & 0. \end{array}$$

3. Let the polyhedron be defined by the linear inequalities

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 & \leq & 8, \\ x_2 + 6x_3 & \leq & 12, \\ x_1 & \leq & 4, \\ x_2 & \leq & 6, \\ x_1, x_2, x_3 & \geq & 0. \end{array}$$

   Find which of the following points are vertices and determine which are degenerate: $(2,6,0)$, $(4,6,0)$, $(4,0,2)$. Are these vertices adjacent?

4. Find all the vertices of the polyhedron in standard form $\mathcal{P} = \{x \in \mathbf{R}^4 \mid Ax = b\ ,\ x \geq 0\}$ with

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   Determine adjacent vertices and degenerate vertices.

5. When does a half-space contain another half-space? Give conditions for which

$$\{x|a^T x \leq b\} \subseteq \{x|\tilde{a}^T x \leq \tilde{b}\}.$$

6. Which of the following sets are polyhedra?

   (a) $\mathcal{S} = \{x \in \mathbf{R}^n | x \geq 0, 1^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$ where $a_1, ..., a_n \in \mathbf{R}$ and $b_1, b_2 \in \mathbf{R}$.

   (b) $\mathcal{S} = \{x \in \mathbf{R}^n \mid x \geq 0, x^T y \leq 1$ for all y such that $\|y\| = 1\}$.

   (c) $\mathcal{S} = \{x \in \mathbf{R}^n \mid x \geq 0, x^T y \leq 1$ for all y such that $\sum_i |y_i| = 1\}$.

   (d) **More challenging:** $\mathcal{S} = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1, \ -1 \leq y_2 \leq 1\}$ for a_1, a_2 given and fixed.

   Where possible, express the set as $\{x \mid Ax \leq b\}$.

7. Suppose that $\{x \in \mathbf{R}^n \mid a_i^T x \geq b_i, i = 1, \ldots, m\}$ and $\{x \in \mathbf{R}^n \mid g_i^T x \geq h_i, i = 1, \ldots, k\}$ are two representations of the same polyhedron. Show that if the vectors $a_1, \ldots, a_m$ generate $\mathbf{R}^n$ then so do the vectors $g_1, \ldots, g_k$.

8. Find, if possible, a linear optimization problem that has finite optimal cost but no optimal vertex. If possible, find a problem in standard form with two variables that has this property. Justify your answer if this is not possible.

9. Consider the optimization problem

$$\begin{array}{ll} \text{minimze} & c^T x \\ & Ax \ \geq \ b \end{array}$$

   Assume that the polyhedron $\{x|Ax \geq b\}$ has at least one vertex and that the optimal cost is finite. You have a routine for solving linear systems of n equations with n unknowns. The routine warns if the determinant is zero and returns a solution if it is not. The routine uses $O(n^3)$ arithmetic operations. Propose a simple algorithm for solving the optimization problem that uses only the routine. Estimate the number of operations required to solve a problem with n variables and m constraints. (Note. We write $f(n) = O(g(n))$ if there are positive numbers n_0 and c for which $f(n) \leq cg(n)$ for all $n \geq n_0$).

10. The polyhedron of $\mathbf{R}^3$ defined by the linear inequalities

$$\begin{array}{rcl} 4x_1 - x_2 + x_3 & \leq & 7 \\ -x_1 + 3x_2 - x_3 & \leq & 9 \\ -x_1 + 6x_2 + 5x_3 & \leq & 5 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$$

   is bounded, has a vertex and contains a sphere of strictly positive radius. We want to find the radius of the largest sphere entirely contained in this polyhedron. Formulate this problem as an optimization problem and transform it into a linear optimization problem. Answer the following question without performing any calculations: Among the spheres of maximum radius, is there one that touches four of the planes that define the polyhedron? Justify your answer.

11. The following experimental points (x_i, y_i) are available: $(0,0),(1,3),(2,4)$ and $(4,2)$. We are looking for the equation of the line $y = ax + b$ that minimizes the error measure

$$\max_i |(ax_i + b) - y_i|.$$

Formulate this problem as a linear optimization problem. Let

$$z_* = \min_{a,b} \max_i |(ax_i + b) - y_i|,$$

for how many points (x_i, y_i) the equality $z_* = |(ax_i + b) - y_i|$ is obtained?

Consider the generalization of this problem to m points (x_i, y_i) and a polynomial $p(x)$ of degree n. The aim is to minimize the error measure

$$\max_i |p(x_i) - y_i|.$$

Let z_* be the optimal objective for this problem. Discuss the number of points (x_i, y_i) for which $z_* = |p(x_i) - y_i|$. Let's assume that this problem has several solutions. Among all the solutions to the problem, we are looking for the one for which the quantity

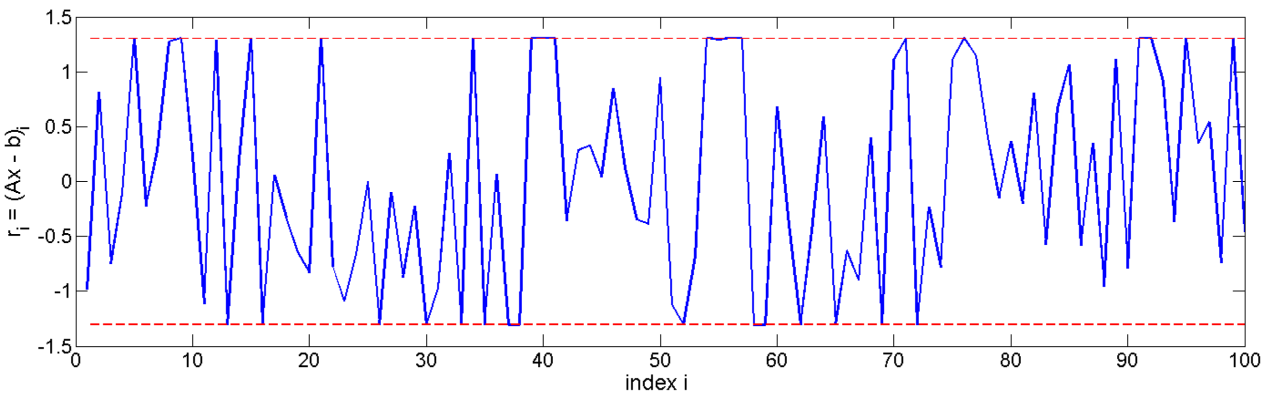
$$\sum_i |(ax_i + b) - y_i|$$

is minimum. How can we formulate this problem as a linear optimization problem?

12. Let $\|\cdot\|_\infty$ be the infinte norm ($\|x\|_\infty = \max_i |x_i|$). Let $A \in \mathbf{R}^{100\times n}$ and $b \in \mathbf{R}^{100}$. The number of columns in A is not known, but we know that the optimization problem

$$\min_x \|Ax - b\|_\infty$$

has a unique solution $x^* \in \mathbf{R}^n$ whose corresponding residual vector $Ax^* - b$ is shown below. What can we deduce from this graph for n? Justify your answer, giving details of your reasoning.



13. True or false? Justify your choices with a few lines, a counter-example or a drawing.

   (a) The union of two polyhedra is a polyhedron.
   (b) The convex hull of the union of nondisjoint polyhedra is also a polyhedron?
   (c) The empty set is a polyhedron.
   (d) Any P polyhedron can be written in geometric form $P = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{Ax} \geq \mathbf{b}\}$.
   (e) Any P polyhedron can be written in standard form $P = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
   (f) The set of optimal solutions to a linear optimization problem is a polyhedron.
   (g) A linear optimization problem may have exactly two optimal solutions.
   (h) In any optimal solution of a linear optimization problem with n variables, there are at least n active constraints.