Optimization - Exercise session 3
The Simplex Method

1. Let the optimization problem be
$\min_x \; -2x_1 - x_2$
such that

$$\begin{cases} x_1 - x_2 \leq 2, \\ x_1 + x_2 \leq 6, \\ x_1, x_2 \geq 0. \end{cases}$$

Convert this problem into standard form and find a vertex for which $x_1 = x_2 = 0$. Solve the problem using the simplex method. Draw a graphical representation in terms of the variables x_1, x_2 and indicate the path followed by the method.

Solution

To begin with we need to rewrite this problem in standard form. We have
$\min_x \; -2x_1 - x_2 + x_3 + 0x_4$
such that

$$\begin{cases} x_1 - x_2 + x_3 = 2, \\ x_1 + x_2 + x_4 = 6, \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

If $x_1 = x_2 = 0$ then a vertex is $(0, 0, 2, 6)$.
Using the previous equalities, we get

Basic	x_1	x_2	x_3	x_4	Solution
z	-2	-1	0	0	0
x_3	1	-1	1	0	2
x_4	1	1	0	1	6

The solution (0, 0, 2, 6) is the BFS associated with the basic variables x_3 and x_4 (the point A, see bellow).
The reduced costs associated with the variables x_1 and x_2 are negative.
Since the most negative number in z-line is -2, we choose to enter x_1 in the base.
We can't increase x_1 without limit, since we have to satisfy the constraints

$$\begin{cases} x_1 + x_3 = 2, \\ x_1 + x_4 = 6. \end{cases}$$

The first constraint is the most restrictive since $\frac{2}{1} \geq \frac{6}{1}$. It's x_3 that leaves the base.

Basic	x_1	x_2	x_3	x_4	Solution	Ratio
z	-2	-1	0	0	0	
x_3	1	-1	1	0	2	2/1
x_4	1	1	0	1	6	6/1

After elementary transformations on the rows, we obtain the canonical table

Basic	x_1	x_2	x_3	x_4	Solution
z	0	$-1 + 2(-1)$	$0 + 2 \cdot 1$	0	$0 - 2 \cdot 2$
x_1	1	-1	1	0	2
x_4	0	$1 - (-1)$	0-1	1	6-2

We have

Basic	x_1	x_2	x_3	x_4	Solution
z	0	-3	2	0	4
x_1	1	-1	1	0	2
x_4	0	2	-1	1	4

The solution (2, 0, 0, 4) is the new BFS associated with the basic variables x_1 and x_4 (the point B, see bellow).

The non-basic variables are x_2 and x_3.

The reduced costs associated with the variables x_2 is negative. The cost decreases if x_2 increases, increases decreases if x_3 increases.

We choose to enter x_2 in the base. We can't increase x_2 without limit, since we have to satisfy the constraints:

1. The first imposes $x_1 - x_2 = 2$.
2. and the second $2x_2 + x_4 = 4$.

The second constraint is the most restrictive. It's x_4 that leaves the base.

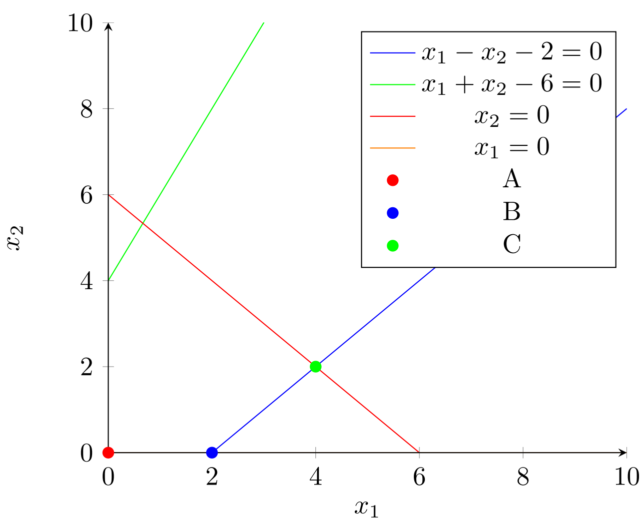
After elementary transformations on the rows, we obtain the canonical table

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$2 - 3/2$	$3/2$	$4 + 4 \cdot 3/2$
x_1	1	0	1-1/2	0 +2	$2 + 2 \cdot 4$
x_2	0	1	-1/2	1/2	4/2

We get

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	1/2	3/2	10
x_1	1	0	1/2	1/2	6
x_2	0	1	-1/2	1/2	2

The associated reduced costs are positive. Therefore, the solution is optimal. The minimum is attained at $(6, 2)$ and is equal to -10 (the point C, see bellow).



2. Consider the problem
$\min_x 20x_1 + \alpha x_2 + 12x_3$
such that

$$x_1 \leq 400,$$

$$2x_1 + \beta x_2 + x_3 \leq 1000,$$

$$2x_1 + \gamma x_2 + 3x_3 \leq 1600,$$

$$x_1, x_2, x_3 \geq 0.$$

Propose, if possible, values for α, β and γ for which:
(a) The optimal cost is finite and the optimal solution is unique.
(b) The optimal cost is finite and there are infinitely many optimal solutions.
(c) The optimal cost is unbounded (find a parameterization of x values among which there are solutions with arbitrarily low costs).
(d) The polyhedron has a degenerate vertex.

Solution

To begin with we need to rewrite this problem in standard form. We have

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
	20	α	12	0	0	0	z
x_4	1	0	0	1	0	0	400
x_5	2	β	1	0	1	0	1000
x_6	2	γ	3	0	0	1	1600

(a) If $\alpha \geq 0$ then the optimal cost is finite and the optimal solution is unique.
From the first row: we have $20x_1 + \alpha x_2 + 12x_3 = z$ and by conditions of the problem $x_1, x_2, x_3 \geq 0$. So the cost of another feasible solution can only be greater than or equal to 0 and the vertex (0,0,0,400,1000,1600) is optimal with cost 0.
(b) For $\alpha = 0, \beta = \gamma = 1$. For instance, the optimal costs is finite (z=0) and the BFS/vertex is optimal (0,0,0,400,1000,1600). If we pivot then we change of vertex but it will be optimal also. So everything between the current vertex and the next one is optimal. Here are infinitely many optimal solutions. You can do the pivoting for that case, you will get the new optimal vertex (0,1000,0,400, 0, 600).
(c) If $\alpha < 0$, $\beta \leq 0$, and $\gamma \leq 0$ then the optimal cost is unbounded The reduced cost of x_2 is negative.If x_2 increases and x_1, $x_3 \in \mathbb{R}$, the cost decreases.
(d)
If $\alpha < 0$ then the reduced cost associated with $x_2 = \lambda$ is negative, so we enter it in the base. If $x_1 = x_3 = 0$, then

$$x_4 = 400;$$

$$x_5 = 1000 - \beta\lambda,$$

$$x_6 = 1600 - \gamma\lambda.$$

If $\gamma = 1.6\beta$ then the second and the third constraints are activated in $\lambda = 1000/\beta$: the variables x_5 and x_6 are both candidates for leaving the base. Whichever variable we choose, the other variable will be zero at the next iteration, and we'll find ourselves in a degenerate vertex.

3. Solve problems using the simplex algorithm
3.1

$$\max_x 2x_1 + 3x_2$$

such that

$$x_1 + 2x_2 \leq 4,$$

$$x_1 + x_2 = 3,$$

$$x_1, x_2 \geq 0.$$

3.2

$$\max_x 20x_1 + 16x_2 + 12x_3$$

such that

$$x_1 \leq 400,$$

$$2x_1 + x_2 + x_3 \leq 1000,$$

$$2x_1 + 2x_2 + 3x_3 \leq 1600,$$

$$x_1, x_2, x_3 \geq 0.$$

Solution

To begin with we need to rewrite this problem in standard form. We have
3.1

$$\min_x -2x_1 - 3x_2$$

such that

$$x_1 + 2x_2 + x_3 = 4,$$

$$x_1 + x_2 = 3,$$

$$x_1, x_2 \geq 0.$$

This problems has the following canonical form

$$-x_2 = z + 6$$

such that

$$x_2 + x_3 = 1,$$

$$x_1 + x_2 = 3,$$

$$x_1, x_2 \geq 0.$$

Consequently, the basic solution $(1, 0, 3)$ is feasible and we have

Basic	x_1	x_2	x_3	Solution
z	0	-1	0	6
x_3	0	1	1	1
x_1	1	1	0	3

The solution (1, 0, 3) is the BFS associated with the basic variables x_1 and x_3.
The reduced costs associated with the variables x_2 is negative. Hence, we choose to enter x_2 in the base.
We can't increase x_2 without limit, since we have to satisfy the constraints

$$\begin{cases} x_2 + x_3 = 1, \\ x_1 + x_2 = 3. \end{cases}$$

The first constraint is the most restrictive since $\frac{1}{1} \geq \frac{3}{1}$. It's x_3 that leaves the base.

Basic	x_1	x_2	x_3	Solution
z	0	0	1	7
x_2	0	1	1	1
x_1	1	0	-1	2

The associated reduced costs are positive. Therefore, the solution is optimal. The maximum is attained at $(2, 1)$ and is equal to 7.

3.2 To begin with we need to rewrite this problem in standard form. We have

$$\min_{x} -20x_1 - 16x_2 - 12x_3$$

such that

$$x_1 + x_4 = 400,$$

$$2x_1 + x_2 + x_3 + x_5 = 1000,$$

$$2x_1 + 2x_2 + 3x_3 + x_6 = 1600,$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The starting basic solution is $(x_1, x_2, x_3, s_1, s_2, s_3) =$ (0,0,0,400, 1000,1600).

Iteration	Basic	x_1	x_2	x_3	x_4	x_5	x_6	*Solution*	Ration
0	z	$\boxed{-20}$	-16	-12	0	0	0	0	
x_1 enters	x_4	1	0	0	1	0	0	400	400/1
	x_5	2	1	1	0	1	0	1000	1000/2 =500
x_4 leaves	x_6	2	2	3	0	0	1	1600	1600/2 =800
1	z	0	$\boxed{-16}$	-12	20	0	0	8000	–
x_2 enters	x_1	1	0	0	1	0	0	400	–
	x_5	0	**1**	1	-2	1	1	200	200/**1**
x_5 leaves	x_6	0	**2**	3	-2	0	0	800	800/**2**
2	z	0	0	4	$\boxed{-12}$	16	0	11200	
x_4 enters	x_1	1	0	0	**1**	0	0	400	400/**1**
	x_2	0	1	1	-2	1	0	200	
x_6 leaves	x_6	0	0	1	**2**	-2	1	400	400/**2**
3	z	0	0	10	0	4	6	13600	
	x_1	1	0	-1/2	0	1	-1/2	200	
	x_2	0	1	2	0	-1	1	600	
	x_4	0	0	1/2	1	-1	1/2	200	

The associated reduced costs are positive. Therefore, the solution is optimal. The maximum is attained at $(200, 600, 0)$ and is equal to 13600.