Optimization – Exercise session 1

1 Linear programs

Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x_1, \ldots, x_n) \\ \text{such that} \quad & x = (x_1, \ldots, x_n) \in \Omega. \end{aligned}$$

The scalars $x_1, \ldots, x_n \in \mathbb{R}$ are the *decision variables*, f is the *objective function* (a minimization problem is often referred to as a cost function, and a maximization problem as a profit function), and Ω is the *feasible set.* An element of $\mathbb{R}^n$ is a *solution*, it is a *feasible solution* if $x \in \Omega$. If the feasible solution $x^\star$ is such that $f(x^\star) \leq f(x)$ for all other feasible solutions x, the solution $x^\star$ is optimal. The optimum objective is then given by $f(x^\star)$. When it exists, the optimal cost is unique. On the other hand, there may be many optimal solutions, and the set of optimal solutions may be unbounded. If for any K there exists a feasible solution $x \in \Omega$ such that $f(x) \leq K$, the cost is said to be *unbounded.* The cost is also said to be *equal to* $-\infty$.

---

1. A student has 100 working hours to study for exams A, B and C. He expects to earn 1/5 points for course A, 2/5 points for course B and 3/5 for course C per hour of work and per course. Each exam is graded out of 20. The exercises in these courses account for half of the final score. His results for the exercises were communicated to him. He obtained 12/20 for A, 12.5/20 for B and 13.4/20 for C. The student must obtain an overall mark of at least 10/20 for each course. All courses have the same weighting and the student wants to obtain the highest possible average. Formulate this problem as a linear program.

2. A company produces goods A, B and C. The production of the goods requires the use of 4 machines. The production times and profits are shown in the following table

	1	2	3	4	profit
A	1	3	1	2	6
B	6	1	3	3	6
C	3	3	2	4	6

   The production times available on machines 1, 2, 3 and 4 are 84, 42, 21 and 42, and the company is seeking to maximize its profit. Formulate this problem as a linear optimization problem.

3. In Moscow there are I districts, J schools and G levels of teaching (for instance: kindergarten, primary school, higher school, etc). School j has a capacity of c_{jg} pupils for level g. In district i, the number of pupils required for level g is equal to s_{ig}. Finally, the distance from school j to district i is equal to d_{ij}. We ask you to assign the pupils to the schools in such a way as to minimize the total distance covered by all the pupils. Formulate this problem as a linear optimization problem (or linear program).

4. Solve the following problem geometrically

$$\begin{aligned} \max_x \quad & 12x_1 + 16x_2 \\ \text{such that} \quad & x_1 + x_2 \leq 150, \\ & 1/2x_1 + 3x_2 \leq 200, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The objective function changes to $17x_1 + 16x_2$. What happens now? Is the set of optimal solutions always finite?

5. Let's consider the linear optimization problem

$$\begin{aligned} \min_x \quad & c_1x_1 + c_2x_2 + c_3x_3 \\ & x_1 + x_2 \geq 1, \\ & x_1 + 2x_2 \leq 3, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Represent the set of feasible solutions. Find the optimal cost and the optimal solutions for the following values of c: $c = (-1, 0, 1)$, $c = (0, 1, 0)$ and $c = (0, 0, -1)$.

6. Find, if possible, linear optimization problems which

   (a) possess a finite optimal cost and exactly one optimal solution.

   (b) possess exactly two optimal solutions.

   (c) have an infinite optimal cost.

   (d) have a finite optimal cost and an infinite, bounded set of optimal solutions.

   (e) have a finite optimal cost and an unbounded set of optimal solutions.

7. Let's consider the linear optimization problem

$$\begin{aligned} \min_x \quad & c^T x \\ \text{such that} \quad & Ax \geq b. \end{aligned}$$

How does the optimum objective evolve if a constraint is added?

More challenging exercises :

8. A subset V of $\mathbb{R}^n$ is convex if $x, y \in V \Rightarrow \lambda x + (1 - \lambda)y \in V$ for all $0 < \lambda < 1$. Prove that the set of optimal solutions to a linear optimization problem is a convex set.

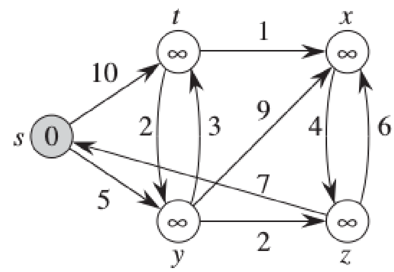
9. Consider a directed graph given by a set of nodes V and edges E. The edge $(i, j) \in E$ has a maximum capacity of h_{ij} and has a unit cost of c_{ij}. In certain nodes $i \in V$ there is a quantity b_i entering ($b_i > 0$) or leaving ($b_i < 0$). We assume that incoming and outgoing quantities balance out, $\sum_i b_i = 0$. We're looking for a feasible flow with minimum cost.

   (a) Formulate this problem as a linear optimization problem.

   (b) Is the solution to such a problem always unique?

(c) Show how to find the shortest path between two nodes in a graph using the solution of a minimum-cost flow problem.

Let us consider an example to solve with *linprog* from ***SciPy***



Compute the shortest path between s and x by solving the associated Minimum Cost Flow Problem.

2 Canonical forms

Geometric form

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & c^T x \\ & Ax \geq b. \end{array}$$

Standard form

$$\begin{array}{llll} \min_{x\in\mathbb{R}^n} & c^T x & & \\ & Ax & = & b, \\ & x & \geq & 0. \end{array}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

---

1. Write the following linear program in geometric form

$$\begin{array}{lrcl} \max_x & -x_1 + 8x_2 & & \\ \text{such that} & x_1 + x_2 & = & 1, \\ & x_1 + 2x_2 & \leq & 3, \\ & x_1 & \geq & 0. \end{array}$$

2. Write the following linear program in standard form

$$\begin{array}{lrcl} \min_x & -x_1 + 8x_2 & & \\ \text{such that} & x_1 + x_2 & \geq & 1, \\ & x_1 + 2x_2 & \leq & 3, \\ & x_1 & \geq & 0. \end{array}$$

3. Write the following linear program in standard form

$$\begin{array}{lrcr} \min_x & x_1 - 5x_2 - 7x_3 & & \\ \text{such that} & 5x_1 - 2x_2 + 6x_3 & \geq & 5, \\ & 3x_1 + 4x_2 - 9x_3 & = & 3, \\ & 7x_1 + 3x_2 + 5x_3 & \leq & 9, \\ & x_1 & \geq & -2. \end{array}$$

4. Let the optimization problem be

$$\begin{array}{lrcl} \min_x & 2x_1 + 3\,|x_2 - 10| & & \\ & |x_1 + 2| + |x_2| & \leq & 5. \end{array}$$

Formulate this problem as a linear optimization problem.

5. Can a linear optimization problem always be written as $\min c^T x$ under the constraint $Ax = b$? If so, demonstrate, if not, justify.

More challenging exercises :

6. A rocket moves along a straight trajectory. Let x_t, v_t and a_t be the position, velocity and acceleration of the rocket at time t. Discretizing time and considering a unit increment, find expressions for x_{t+1} and v_{t+1} as a function of x_t, v_t and a_t. We assume that the acceleration a_t is under our control. Furthermore, the rocket is at rest at the origin at time $t = 0$ ($x_0 = 0$ and $v_0 = 0$). We wish to launch the rocket to a soft landing at time T at a distance of one unit from its starting point ($x_T = 1$ and $v_T = 0$). Furthermore, we wish to achieve this objective while minimizing total fuel consumption $\sum_{t=0}^{T-1} |a_t|$. Formulate this problem as a linear optimization problem.

7. The components of a microchip are placed on a square chip $\{(x, y) \mid -1 \leq x, y \leq 1\}$. Each component has several terminals which are connected to terminals on other components, or to input/output terminals located on the chip perimeter. The connections and positions of the input/output terminals are fixed; the only variables are the (x_i, y_i) coordinates of the n components. Connections between components are specified as follows. We define the vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and we construct a matrix $A \in \mathbb{R}^{m \times n}$ and two vectors $b_x \in \mathbb{R}^m, b_y \in \mathbb{R}^m$. Each row of A and each entry of b_x and b_y describes a connection. For each $i = 1, \ldots, m$, we can distinguish two cases, depending on whether the i line of A describes a connection between two components, or between a component and an input/output terminal.

- If i describes a connection between two components j and k (with $j < k$), then

$$a_{il} = \begin{cases} 1 & \text{if } l = j, \\ -1 & \text{if } l = k, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and } b_{x,i} = 0, b_{y,i} = 0$$

  and we have $a_i^T x - b_{x,i} = x_j - x_k$ and $a_i^T y - b_{y,i} = y_j - y_k$.

- If i describes a connection between a component j and an input/output terminal with coordinate $(\bar{x}, \bar{y})$, then

  $$a_{il} = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } b_{x,i} = \bar{x}, b_{y,i} = \bar{y}$$

  and we have $a_i^T x - b_{x,i} = x_j - \bar{x}$ and $a_i^T y - b_{y,i} = y_j - \bar{y}$.

The problem is to determine the component coordinates that minimize the greatest Manhattan distance between two connected components, or between a component and a terminal. (Note: The Manhattan distance between points of coordinates (x_0, y_0) and (x_1, y_1) is given by $|x_1 - x_0| + |y_1 - y_0|$.) Formulate this problem as a linear optimization problem.