Geometry

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1 Topological space

A topological space is defined as the pair (T, v) where T is a set and $v : T \to \mathscr{P}(T)$ is a function, called **neighbourhood topology**, which has to satisfy the following axioms. The elements of T are called **points** and, given a point x the elements of $v(x)$ are called **neighbourhoods** of x.

TS.1 A neighbourhood of a point must contain the point itself.

TS.2 All supersets of a neighbourhood of a point are neighbourhoods of that point.

TS.3 The intersection of two neighbourhoods of the same point is a neighbourhood of that point.

TS.4 Any neighbourhood n of a point must contain another neighbourhood m of the same point, such that n is a neighbourhood of all points in m.

We can restate the axioms in formal terms. For brevity it is implied that $x, y \in T$ and $n, m \in \mathscr{P}(T)$.

TS.1 $\forall x \forall n : n \in v(x) \to x \in n$

TS.2 $\forall x \forall n \forall m : (n \in v(x) \land n \subseteq m) \to m \in v(x)$

TS.3 $\forall x \forall n \forall m : (n \in v(x) \land m \in v(x)) \to n \cap m \in v(x)$

TS.4 $\forall x \forall n \exists m \forall y : (n \in v(x) \land m \subseteq n \land m \in v(x) \land y \in m) \to n \in v(y)$

If (T, v) is a topological space, we say that $u \in \mathscr{P}(T)$ is **open** if it is a neighbourhood of all the points it contains. Restating this in formal terms, u is open if:

$$\forall x : x \in u \to u \in v(x)$$

We say that $u \in \mathscr{P}(T)$ is **closed** if its complement $\complement(u)$ is open. It is important to remark that the two definitions are not mutually exclusive, a set could be both open and closed, in which case it's usually called a clopen set.