Geometry

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1 Topological space

Definition. A **neighbourhood topology** on a set $\mathfrak{T}$ is a function $\nu : \mathfrak{T} \to \mathscr{P}(\mathfrak{T})$ mapping **points** (elements of $\mathfrak{T}$, from here on denoted by x, y or z) to **neighbourhoods** (subsets of $\mathfrak{T}$, from here on denoted by n or m). It must satisfy the following axioms.

i) A neighbourhood of a point must contain the point itself.

$$\forall x \forall n : n \in \nu(x) \to x \in n$$

ii) All supersets of a neighbourhood of a point are neighbourhoods of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land n \subseteq m) \to m \in \nu(x)$$

iii) The intersection of two neighbourhoods of the same point is a neighbourhood of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land m \in \nu(x)) \to n \cap m \in \nu(x)$$

iv) Any neighbourhood n of a point must contain another neighbourhood m of the same point, such that n is a neighbourhood of all points in m.

$$\forall x \forall n \exists m \forall y : (n \in \nu(x) \land m \subseteq n \land m \in \nu(x) \land y \in m) \to n \in \nu(y)$$

Definition. A **topological space** $(\mathfrak{T}, \nu)$ is a set $\mathfrak{T}$ equipped with a neighbourhood topology ν.

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically indistinguishable** if they have exactly the same neighbourhoods.

$$x \equiv y \leftrightarrow \forall n : n \in \nu(x) \leftrightarrow n \in \nu(y)$$

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically distinguishable** if they are not topologically indistinguishable, i.e. if there is at least one neighbourhood of one of the points that isn't a neighbourhood of the other.

$$x \not\equiv y \leftrightarrow \exists n : n \in \nu(x) \veebar n \in \nu(y)$$

Proposition 1. *Topological indistinguishability is an equivalence relation, i.e.* $(\mathfrak{T}, \equiv)$ *is a setoid.*

Proof. Trivial, the necessary properties are inherited from the biconditional.

Corollay 1.1. *Topological distinguishability is an apartness relation, i.e.* $(\mathfrak{T}, \not\equiv)$ *is a constructive setoid.*

Proof. The dual of an equivalence relation is an apartness relation.

Definition. A point x is a **limit point** for u if each of its neighbourhoods also contains at least one point of u distinguishable from x itself.

$$\mathrm{Lim}(x, u) \leftrightarrow \forall n \exists y : n \in \nu(x) \rightarrow (y \in n \wedge y \in u \wedge x \not\equiv y)$$

Definition. A point x is an **isolated point** in u if it has at least one neighbourhood that contains no other points in u except for those indistinguishable from x itself.

$$\mathrm{Iso}(x, u) \leftrightarrow \exists n \forall y : n \in \nu(x) \wedge (y \in n \rightarrow (y \notin u \veebar x \equiv y))$$

Proposition 2. *All points in a set are either limit points or isolated points.*

$$x \in u \rightarrow \mathrm{Lim}(x, u) \veebar \mathrm{Iso}(x, u)$$

Definition. A point x is an **interior point** for u if it has at least one neighbourhood that is entirely contained in u. The point must therefore belong to the set.

$$\mathrm{Int}(x, u) \leftrightarrow \exists n \forall y : (n \in \nu(x) \wedge y \in n) \rightarrow y \in u$$

Definition. A point x is a **boundary point** for u if each of its neighbourhoods contains at least one point belonging to u and at least one point not belonging to u.

$$\mathrm{Bnd}(x, u) \leftrightarrow \forall n \exists y \exists z : n \in \nu(x) \wedge y \in n \wedge y \in u \wedge z \in n \wedge z \notin u$$

Definition. A point x is an **exterior point** for u if it has at least one neighbourhood that is entirely contained in u. The point must therefore not belong to the set.

$$\mathrm{Ext}(x, u) \leftrightarrow \exists n \forall y : (n \in \nu(x) \wedge y \in n) \rightarrow y \notin u$$

Definition. u is **open** if it is a neighbourhood of all the points it contains.

$$\mathrm{Open}(u) \leftrightarrow \forall x : x \in u \rightarrow u \in \nu(x)$$

Definition. u is **closed** its complement $\mathfrak{X} \setminus u$ is open.

Observation 1. Despite what the choice of terminology might suggest, the definitions of open and closed are not mutually exclusive. A set that is both open and closed is said to be **clopen**.

2 Curvature Tensor

Definition. The **Riemann curvature tensor** $R^{\rho}{}_{\sigma\mu\nu}$ and the **Cartan torsion tensor** $T^{\lambda}_{\mu\nu}$ are the tensors that satisfy the following equation for all vector fields ϕ.

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] \phi^{\rho} = R^{\rho}{}_{\sigma\mu\nu}\phi^{\sigma} + T_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi^{\rho}$$

Proposition 3. *In a holonomic basis the Riemann curvature tensor and Cartan torsion tensors have the following expressions in terms of the Christoffel symbols.*

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$$

Proof. We begin by computing the action of one of the firt term of the commutator onto a component of ϕ, recalling that $\nabla_\nu \phi^\rho$ is a type $(1, 1)$ tensor.

$$\begin{aligned}
\nabla_\mu \nabla_\nu \phi^\rho &= \partial_\mu \nabla_\nu \phi^\rho + \Gamma^\rho_{\mu\lambda} \nabla_\nu \phi^\lambda - \Gamma^\lambda_{\nu\mu} \nabla_\lambda \phi^\rho \\
&= \partial_\mu \left(\partial_\nu \phi^\rho + \Gamma^\rho_{\nu\sigma} \phi^\sigma \right) + \Gamma^\rho_{\mu\lambda} \left(\partial_\nu \phi^\lambda + \Gamma^\lambda_{\nu\sigma} \phi^\sigma \right) - \Gamma^\lambda_{\nu\mu} \left(\partial_\lambda \phi^\rho + \Gamma^\rho_{\lambda\sigma} \phi^\sigma \right) \\
&= \partial_\mu \partial_\nu \phi^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma} \right) \phi^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu \phi^\sigma + \Gamma^\rho_{\mu\lambda} \partial_\nu \phi^\lambda + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} \phi^\sigma - \Gamma^\lambda_{\nu\mu} \partial_\lambda \phi^\rho - \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} \phi^\sigma
\end{aligned}$$

In the fourth term, we replace the dummy index λ with σ in order to factor the expression.

$$\begin{aligned}
\nabla_\mu \nabla_\nu \phi^\rho &= \partial_\mu \partial_\nu \phi^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma} \right) \phi^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu \phi^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu \phi^\sigma + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} \phi^\sigma - \Gamma^\lambda_{\nu\mu} \partial_\lambda \phi^\rho - \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} \phi^\sigma \\
&= \left(\partial_\mu \partial_\nu - \Gamma^\lambda_{\nu\mu} \partial_\lambda \right) \phi^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\nu\sigma} \partial_\mu + \Gamma^\rho_{\mu\sigma} \partial_\nu + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} \right) \phi^\sigma
\end{aligned}$$

We obtain the second term of the commutator by simply switching ν with μ.

$$\nabla_\nu \nabla_\mu \phi^\rho = \left(\partial_\nu \partial_\mu - \Gamma^\lambda_{\mu\nu} \partial_\lambda \right) \phi^\rho + \left(\partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma} \partial_\nu + \Gamma^\rho_{\nu\sigma} \partial_\mu + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} \right) \phi^\sigma$$

We now take the difference of the two terms, remembering that $\left[\partial_\mu , \partial_\nu \right] = 0$.

$$\begin{aligned}
\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) \phi^\rho &= \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \partial_\lambda \phi^\rho + \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) \phi^\sigma + \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \Gamma^\rho_{\lambda\sigma} \phi^\sigma \\
&= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) \phi^\sigma + \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \nabla_\lambda \phi^\rho
\end{aligned}$$

Where we recognise the expressions for $R^\rho{}_{\sigma\mu\nu}$ and $T_{\mu\nu}{}^\lambda$.