Geometry

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1 Topological space

Definition. A **neighbourhood topology** is a function $\nu : T \to \mathscr{P}(T)$ mapping **points** (elements of T, from here on denoted by x or y) to **neighbourhoods** (subsets of T, from here on denoted by n or m). It must satisfy the following axioms.

i) A neighbourhood of a point must contain the point itself.

$$\forall x \forall n : n \in \nu(x) \to x \in n$$

ii) All supersets of a neighbourhood of a point are neighbourhoods of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land n \subseteq m) \to m \in \nu(x)$$

iii) The intersection of two neighbourhoods of the same point is a neighbourhood of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land m \in \nu(x)) \to n \cap m \in \nu(x)$$

iv) Any neighbourhood n of a point must contain another neighbourhood m of the same point, such that n is a neighbourhood of all points in m.

$$\forall x \forall n \exists m \forall y : (n \in \nu(x) \land m \subseteq n \land m \in \nu(x) \land y \in m) \to n \in \nu(y)$$

Definition. A **topological space** (T, ν) is a set T equipped with a neighbourhood topology ν.

Definition. $u \subseteq T$ is **open** if it is a neighbourhood of all the points it contains.

$$\forall x : x \in u \to u \in \nu(x)$$

Definition. $u \subseteq T$ is **closed** its complement $\complement(u)$ is open.

Definition. $u \subseteq T$ is **clopen** if it is both open and closed.

2 Curvature Tensor

Definition. The **Riemann curvature tensor** $R^{\rho}{}_{\sigma\mu\nu}$ and the **Cartan torsion tensor** $T^{\lambda}_{\mu\nu}$ are the tensors that satisfy the following equation for all vector fields ϕ.

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] \phi^{\rho} = R^{\rho}{}_{\sigma\mu\nu}\phi^{\sigma} + T_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi^{\rho}$$

Proposition 1. *In a holonomic basis the Riemann curvature tensor and Cartan torsion tensors have the following expressions in terms of the Christoffel symbols.*

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$$

Proof. We begin by computing the action of one of the firt term of the commutator onto a component of ϕ, recalling that $\nabla_\nu \phi^\rho$ is a type $(1,1)$ tensor.

$$\begin{aligned}
\nabla_\mu \nabla_\nu \phi^\rho &= \partial_\mu \nabla_\nu \phi^\rho + \Gamma^{\rho}_{\mu\lambda} \nabla_\nu \phi^\lambda - \Gamma^{\lambda}_{\nu\mu} \nabla_\lambda \phi^\rho \\
&= \partial_\mu \left(\partial_\nu \phi^\rho + \Gamma^{\rho}_{\nu\sigma} \phi^\sigma \right) + \Gamma^{\rho}_{\mu\lambda} \left(\partial_\nu \phi^\lambda + \Gamma^{\lambda}_{\nu\sigma} \phi^\sigma \right) - \Gamma^{\lambda}_{\nu\mu} \left(\partial_\lambda \phi^\rho + \Gamma^{\rho}_{\lambda\sigma} \phi^\sigma \right) \\
&= \partial_\mu \partial_\nu \phi^\rho + \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} \right) \phi^\sigma + \Gamma^{\rho}_{\nu\sigma} \partial_\mu \phi^\sigma + \Gamma^{\rho}_{\mu\lambda} \partial_\nu \phi^\lambda + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} \phi^\sigma - \Gamma^{\lambda}_{\nu\mu} \partial_\lambda \phi^\rho - \Gamma^{\lambda}_{\nu\mu} \Gamma^{\rho}_{\lambda\sigma} \phi^\sigma
\end{aligned}$$

In the fourth term, we replace the dummy index λ with σ in order to factor the expression.

$$\begin{aligned}
\nabla_\mu \nabla_\nu \phi^\rho &= \partial_\mu \partial_\nu \phi^\rho + \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} \right) \phi^\sigma + \Gamma^{\rho}_{\nu\sigma} \partial_\mu \phi^\sigma + \Gamma^{\rho}_{\mu\sigma} \partial_\nu \phi^\sigma + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} \phi^\sigma - \Gamma^{\lambda}_{\nu\mu} \partial_\lambda \phi^\rho - \Gamma^{\lambda}_{\nu\mu} \Gamma^{\rho}_{\lambda\sigma} \phi^\sigma \\
&= \left(\partial_\mu \partial_\nu - \Gamma^{\lambda}_{\nu\mu} \partial_\lambda \right) \phi^\rho + \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\nu\sigma} \partial_\mu + \Gamma^{\rho}_{\mu\sigma} \partial_\nu + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\mu} \Gamma^{\rho}_{\lambda\sigma} \right) \phi^\sigma
\end{aligned}$$

We obtain the second term of the commutator by simply switching ν with μ.

$$\nabla_\nu \nabla_\mu \phi^\rho = \left(\partial_\nu \partial_\mu - \Gamma^{\lambda}_{\mu\nu} \partial_\lambda \right) \phi^\rho + \left(\partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\sigma} \partial_\nu + \Gamma^{\rho}_{\nu\sigma} \partial_\mu + \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\sigma} \right) \phi^\sigma$$

We now take the difference of the two terms, remembering that $\left[\partial_\mu , \partial_\nu \right] = 0$.

$$\begin{aligned}
\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) \phi^\rho &= \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \right) \partial_\lambda \phi^\rho + \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \right) \phi^\sigma + \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \right) \Gamma^{\rho}_{\lambda\sigma} \phi^\sigma \\
&= \left(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \right) \phi^\sigma + \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \right) \nabla_\lambda \phi^\rho
\end{aligned}$$

Where we recognise the expressions for $R^{\rho}{}_{\sigma\mu\nu}$ and $T_{\mu\nu}{}^{\lambda}$