Geometry

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1 Topological space

Definition. A **neighbourhood topology** on a set $\mathfrak{T}$ is a function $\nu : \mathfrak{T} \to \mathscr{P}(\mathfrak{T})$ mapping **points** (elements of $\mathfrak{T}$, from here on denoted by x or y) to **neighbourhoods** (subsets of $\mathfrak{T}$, from here on denoted by n or m). It must satisfy the following axioms.

i) A neighbourhood of a point must contain the point itself.

$$\forall x \forall n : n \in \nu(x) \to x \in n$$

ii) All supersets of a neighbourhood of a point are neighbourhoods of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land n \subseteq m) \to m \in \nu(x)$$

iii) The intersection of two neighbourhoods of the same point is a neighbourhood of that point.

$$\forall x \forall n \forall m : (n \in \nu(x) \land m \in \nu(x)) \to n \cap m \in \nu(x)$$

iv) Any neighbourhood n of a point must contain another neighbourhood m of the same point, such that n is a neighbourhood of all points in m.

$$\forall x \forall n \exists m \forall y : (n \in \nu(x) \land m \subseteq n \land m \in \nu(x) \land y \in m) \to n \in \nu(y)$$

Definition. A **topological space** $(\mathfrak{T}, \nu)$ is a set $\mathfrak{T}$ equipped with a neighbourhood topology ν.

Definition. $u \subseteq \mathfrak{T}$ is **open** if it is a neighbourhood of all the points it contains.

$$\forall x : x \in u \to u \in \nu(x)$$

Definition. $u \subseteq \mathfrak{T}$ is **closed** its complement $\mathfrak{T} \setminus u$ is open.

Definition. $u \subseteq \mathfrak{T}$ is **clopen** if it is both open and closed.

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically indistinguishable** if they have exactly the same neighbourhoods.

$$\forall n : n \in \nu(x) \leftrightarrow n \in \nu(y)$$

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically distinguishable** if there is at least one neighbourhood of one of the points that isn't a neighbourhood of the other.

$$\exists n : n \in \nu(x) \veebar n \in \nu(y)$$

Proposition 1. *Topological indistinguishability is an equivalence relation.*

Proof. Trivial, the necessary properties are inherited from the biconditional.

Corollay 1.1. *Topological distinguishability is an apartness relation.*

Proof. The negation of an equivalence relation is an apartness relation.

2 Curvature Tensor

Definition. The **Riemann curvature tensor** $R^{\rho}{}_{\sigma\mu\nu}$ and the **Cartan torsion tensor** $T^{\lambda}_{\mu\nu}$ are the tensors that satisfy the following equation for all vector fields ϕ.

$$\left[\nabla_\mu , \nabla_\nu\right] \phi^\rho = R^{\rho}{}_{\sigma\mu\nu}\phi^\sigma + T_{\mu\nu}{}^{\lambda}\nabla_\lambda\phi^\rho$$

Proposition 2. *In a holonomic basis the Riemann curvature tensor and Cartan torsion tensors have the following expressions in terms of the Christoffel symbols.*

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$T_{\mu\nu}{}^{\lambda} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$$

Proof. We begin by computing the action of one of the firt term of the commutator onto a component of ϕ, recalling that $\nabla_\nu\phi^\rho$ is a type $(1,1)$ tensor.

$$\begin{aligned}
\nabla_\mu\nabla_\nu\phi^\rho &= \partial_\mu\nabla_\nu\phi^\rho + \Gamma^\rho_{\mu\lambda}\nabla_\nu\phi^\lambda - \Gamma^\lambda_{\nu\mu}\nabla_\lambda\phi^\rho \\
&= \partial_\mu\left(\partial_\nu\phi^\rho + \Gamma^\rho_{\nu\sigma}\phi^\sigma\right) + \Gamma^\rho_{\mu\lambda}\left(\partial_\nu\phi^\lambda + \Gamma^\lambda_{\nu\sigma}\phi^\sigma\right) - \Gamma^\lambda_{\nu\mu}\left(\partial_\lambda\phi^\rho + \Gamma^\rho_{\lambda\sigma}\phi^\sigma\right) \\
&= \partial_\mu\partial_\nu\phi^\rho + \left(\partial_\mu\Gamma^\rho_{\nu\sigma}\right)\phi^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu\phi^\sigma + \Gamma^\rho_{\mu\lambda}\partial_\nu\phi^\lambda + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\phi^\sigma - \Gamma^\lambda_{\nu\mu}\partial_\lambda\phi^\rho - \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma}\phi^\sigma
\end{aligned}$$

In the fourth term, we replace the dummy index λ with σ in order to factor the expression.

$$\begin{aligned}
\nabla_\mu\nabla_\nu\phi^\rho &= \partial_\mu\partial_\nu\phi^\rho + \left(\partial_\mu\Gamma^\rho_{\nu\sigma}\right)\phi^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu\phi^\sigma + \Gamma^\rho_{\mu\sigma}\partial_\nu\phi^\sigma + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\phi^\sigma - \Gamma^\lambda_{\nu\mu}\partial_\lambda\phi^\rho - \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma}\phi^\sigma \\
&= \left(\partial_\mu\partial_\nu - \Gamma^\lambda_{\nu\mu}\partial_\lambda\right)\phi^\rho + \left(\partial_\mu\Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\nu\sigma}\partial_\mu + \Gamma^\rho_{\mu\sigma}\partial_\nu + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma}\right)\phi^\sigma
\end{aligned}$$

We obtain the second term of the commutator by simply switching ν with μ.

$$\nabla_\nu\nabla_\mu\phi^\rho = \left(\partial_\nu\partial_\mu - \Gamma^\lambda_{\mu\nu}\partial_\lambda\right)\phi^\rho + \left(\partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\partial_\nu + \Gamma^\rho_{\nu\sigma}\partial_\mu + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}\right)\phi^\sigma$$

We now take the difference of the two terms, remembering that $\left[\partial_\mu , \partial_\nu\right] = 0$.

$$\begin{aligned}
\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)\phi^\rho &= \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\partial_\lambda\phi^\rho + \left(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)\phi^\sigma + \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\Gamma^\rho_{\lambda\sigma}\phi^\sigma \\
&= \left(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)\phi^\sigma + \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)\nabla_\lambda\phi^\rho
\end{aligned}$$

Where we recognise the expressions for $R^{\rho}{}_{\sigma\mu\nu}$ and $T_{\mu\nu}{}^{\lambda}$.