Geometry

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1 Topological space

Definition. A **filter** on a poset $(\mathfrak{S}, \preceq)$ is a subset $\varphi \subseteq \mathfrak{S}$ that satisfies the following axioms.

i) It is non-empty.

$$\exists x : x \in \varphi$$

ii) It is upward closed, i.e. it contains all elements greater than its elements.

$$\forall x \forall y : (x \in \varphi \wedge x \preceq y) \rightarrow (y \in \varphi)$$

iii) It is downward directed, i.e. every pair of elements must have a lower bound.

$$\forall x \forall y \exists z : (x \in \varphi \wedge y \in \varphi) \rightarrow (z \in \varphi \wedge z \preceq y \wedge z \preceq y)$$

Definition. A **filter of subsets** on a set $\mathfrak{S}$ is a filter on the poset $(\mathscr{P}(\mathfrak{S}), \subseteq)$.

Observation. For future convenience, we denote the set of all filters of subsets on $\mathfrak{S}$ as $\mathscr{F}(\mathfrak{S})$.

Definition. A **neighbourhood topology** on a set $\mathfrak{T}$ is a function $\nu : \mathfrak{T} \rightarrow \mathscr{F}(\mathfrak{T})$ mapping **points** (elements of $\mathfrak{T}$, from here on denoted by x, y or z) to filters of subsets on $\mathfrak{T}$. The elements of the filter $\nu(x)$ are called **neighbourhoods** of x. The function ν must satisfy the following axioms.

i) All neighbourhoods of a point must contain the point itself.

$$\forall x \forall n : n \in \nu(x) \rightarrow x \in n$$

ii) Any neighbourhood n of a point must contain at least another neighbourhood m of the same point, such that n is a neighbourhood of all points in m.

$$\forall x \forall n \exists m \forall y : n \in \nu(x) \rightarrow (m \in \nu(x) \wedge m \subseteq n \wedge (y \in m \rightarrow n \in \nu(y)))$$

Definition. A **topological space** $(\mathfrak{T}, \nu)$ is a set $\mathfrak{T}$ equipped with a neighbourhood topology ν.

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically indistinguishable** if they have exactly the same neighbourhoods.

$$x \equiv y \leftrightarrow \forall n : n \in \nu(x) \leftrightarrow n \in \nu(y)$$

Definition. Two points $x, y \in \mathfrak{T}$ are said to be **topologically distinguishable** if they are not topologically indistinguishable, i.e. if there is at least one neighbourhood of one of the points that isn't a neighbourhood of the other.

$$x \not\equiv y \leftrightarrow \exists n : n \in \nu(x) \veebar n \in \nu(y)$$

Proposition 1. *Topological indistinguishability is an equivalence relation, i.e.* $(\mathfrak{T}, \equiv)$ *is a setoid.*

Proof. Trivial, the necessary properties are inherited from the biconditional.

Corollay 1.1. *Topological distinguishability is an apartness relation, i.e.* $(\mathfrak{T}, \not\equiv)$ *is a constructive setoid.*

Proof. The dual of an equivalence relation is an apartness relation.

Definition. A point x is a **limit point** for u if each of its neighbourhoods also contains at least one point of u distinguishable from x itself.

$$\mathrm{Lim}(x, u) \leftrightarrow \forall n \exists y : n \in \nu(x) \to (y \in n \land y \in u \land x \not\equiv y)$$

Definition. A point x is an **isolated point** in u if it has at least one neighbourhood that contains no other points in u except for those indistinguishable from x itself.

$$\mathrm{Iso}(x, u) \leftrightarrow \exists n \forall y : n \in \nu(x) \land (y \in n \to (y \notin u \veebar x \equiv y))$$

Proposition 2. *All points in a set are either limit points or isolated points.*

$$x \in u \to \mathrm{Lim}(x, u) \veebar \mathrm{Iso}(x, u)$$

Definition. A point x is an **interior point** for u if it has at least one neighbourhood that is entirely contained in u. The point must therefore belong to the set.

$$\mathrm{Int}(x, u) \leftrightarrow \exists n \forall y : (n \in \nu(x) \land y \in n) \to y \in u$$

Definition. A point x is a **boundary point** for u if each of its neighbourhoods contains at least one point belonging to u and at least one point not belonging to u.

$$\mathrm{Bnd}(x, u) \leftrightarrow \forall n \exists y \exists z : n \in \nu(x) \land y \in n \land y \in u \land z \in n \land z \notin u$$

Definition. A point x is an **exterior point** for u if it has at least one neighbourhood that is entirely contained in u. The point must therefore not belong to the set.

$$\mathrm{Ext}(x, u) \leftrightarrow \exists n \forall y : (n \in \nu(x) \land y \in n) \to y \notin u$$

Definition. u is **open** if it is a neighbourhood of all the points it contains.

$$\mathrm{Open}(u) \leftrightarrow \forall x : x \in u \to u \in \nu(x)$$

Definition. u is **closed** its complement $\mathfrak{T} \setminus u$ is open.

Observation. Despite what the choice of terminology might suggest, the definitions of open and closed are not mutually exclusive. A set that is both open and closed is said to be **clopen**.

2 Curvature Tensor

Definition. The **Riemann curvature tensor** $R^{\rho}{}_{\sigma\mu\nu}$ and the **Cartan torsion tensor** $T^{\lambda}_{\mu\nu}$ are the tensors that satisfy the following equation for all vector fields φ.

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] \varphi^{\rho} = R^{\rho}{}_{\sigma\mu\nu}\varphi^{\sigma} + T_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\varphi^{\rho}$$

Proposition 3. *In a holonomic basis the Riemann curvature tensor and Cartan torsion tensors have the following expressions in terms of the Christoffel symbols.*

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$$

Proof. We begin by computing the action of one of the firt term of the commutator onto a component of φ, recalling that $\nabla_{\nu}\varphi^{\rho}$ is a type $(1,1)$ tensor.

$$\begin{aligned}
\nabla_{\mu}\nabla_{\nu}\varphi^{\rho} &= \partial_{\mu}\nabla_{\nu}\varphi^{\rho} + \Gamma^{\rho}_{\mu\lambda}\nabla_{\nu}\varphi^{\lambda} - \Gamma^{\lambda}_{\nu\mu}\nabla_{\lambda}\varphi^{\rho} \\
&= \partial_{\mu}\left(\partial_{\nu}\varphi^{\rho} + \Gamma^{\rho}_{\nu\sigma}\varphi^{\sigma}\right) + \Gamma^{\rho}_{\mu\lambda}\left(\partial_{\nu}\varphi^{\lambda} + \Gamma^{\lambda}_{\nu\sigma}\varphi^{\sigma}\right) - \Gamma^{\lambda}_{\nu\mu}\left(\partial_{\lambda}\varphi^{\rho} + \Gamma^{\rho}_{\lambda\sigma}\varphi^{\sigma}\right) \\
&= \partial_{\mu}\partial_{\nu}\varphi^{\rho} + \left(\partial_{\mu}\Gamma^{\rho}_{\nu\sigma}\right)\varphi^{\sigma} + \Gamma^{\rho}_{\nu\sigma}\partial_{\mu}\varphi^{\sigma} + \Gamma^{\rho}_{\mu\lambda}\partial_{\nu}\varphi^{\lambda} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}\varphi^{\sigma} - \Gamma^{\lambda}_{\nu\mu}\partial_{\lambda}\varphi^{\rho} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\lambda\sigma}\varphi^{\sigma}
\end{aligned}$$

In the fourth term, we replace the dummy index λ with σ in order to factor the expression.

$$\begin{aligned}
\nabla_{\mu}\nabla_{\nu}\varphi^{\rho} &= \partial_{\mu}\partial_{\nu}\varphi^{\rho} + \left(\partial_{\mu}\Gamma^{\rho}_{\nu\sigma}\right)\varphi^{\sigma} + \Gamma^{\rho}_{\nu\sigma}\partial_{\mu}\varphi^{\sigma} + \Gamma^{\rho}_{\mu\sigma}\partial_{\nu}\varphi^{\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}\varphi^{\sigma} - \Gamma^{\lambda}_{\nu\mu}\partial_{\lambda}\varphi^{\rho} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\lambda\sigma}\varphi^{\sigma} \\
&= \left(\partial_{\mu}\partial_{\nu} - \Gamma^{\lambda}_{\nu\mu}\partial_{\lambda}\right)\varphi^{\rho} + \left(\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\nu\sigma}\partial_{\mu} + \Gamma^{\rho}_{\mu\sigma}\partial_{\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\lambda\sigma}\right)\varphi^{\sigma}
\end{aligned}$$

We obtain the second term of the commutator by simply switching ν with μ.

$$\nabla_{\nu}\nabla_{\mu}\varphi^{\rho} = \left(\partial_{\nu}\partial_{\mu} - \Gamma^{\lambda}_{\mu\nu}\partial_{\lambda}\right)\varphi^{\rho} + \left(\partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\sigma}\partial_{\nu} + \Gamma^{\rho}_{\nu\sigma}\partial_{\mu} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\right)\varphi^{\sigma}$$

We now take the difference of the two terms, remembering that $\left[\partial_{\mu}, \partial_{\nu}\right] = 0$.

$$\begin{aligned}
\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right)\varphi^{\rho} &= \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)\partial_{\lambda}\varphi^{\rho} + \left(\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right)\varphi^{\sigma} + \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)\Gamma^{\rho}_{\lambda\sigma}\varphi^{\sigma} \\
&= \left(\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right)\varphi^{\sigma} + \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)\nabla_{\lambda}\varphi^{\rho}
\end{aligned}$$

Where we recognise the expressions for $R^{\rho}{}_{\sigma\mu\nu}$ and $T_{\mu\nu}{}^{\lambda}$.