Gravity

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1 Introduction to Classical Mechanics

In Classical Mechanis (CM) time and space are treated as two distinct concepts. Events are identified by their position in some frame of reference, represented by a vector $\boldsymbol{r}$, and the time at which they occur, represented by a scalar t.
The simplest formulation of CM can be summarised in three laws and is based on the concept of **force**. A force $\boldsymbol{F}$ is a vector quantity representing an interaction between two bodies. This formulation is (unduly) credited to Newton because it was first published as a unified theory in the *Principia*.

N0 Principle of Superposition: the net force on a body is the vector sum of the individual forces acting on it.

N1 Law of Inertia: an inertial frame of reference is one where, in the absence of external interactions, bodies are in uniform linear motion.

N2 Law of Acceleration: in an inertial frame of reference the acceleration of a body is directly proportional to the net force acting on it.

N3 Law of Reaction: when a body exerts a force on another it is subject to a force equal in magnitude and opposite in direction produced by the second body.

Law **N2** can be used to introduce the concept of **inertial mass** m, which is taken as the proportionality constant between force and acceleration. The law can then be stated with the following equation.

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} \tag{N2}$$

2 Galilean gravity

The simplest theory of gravity in our possession is due to Galileo and simply states that acceleration due to gravity is constant, regardless of the size, velocity or position of a body. In hindsight, Galilean gravity can be thought of as gravity in the limit where Earth is and indefinitely extended plane with a uniform mass distribution.

$$\ddot{\boldsymbol{r}} = \boldsymbol{g}$$

This theory, although excessively simple, lays the foundation of a fundamental principle.

EP Equivalence Principle: all bodies subject to a gravitational field will undergo the same acceleration if they are located at the same coordinates in space and time.

This principle is backed up by all experimental evidence to date.

3 Hookean gravity

Generally credited to Newton, most of the fundamental work on this theory was actually due to Hooke. The basic principle of the theory is that every body exerts a gravitational force on every other body. This force pulls the objects towards each other and its intensity is:

- directly proportional to the **active gravitational mass** $\widetilde{m}^{(a)}$ of the body exerting the force,
- directly proportional to the **passive gravitational mass** $\widetilde{m}^{(p)}$ of the body subject to the force,
- inversely proportional to the square of their distance.

This can be expressed as the following formula

$$F_{1\to 2} = G\frac{\widetilde{m}_1^{(a)}\widetilde{m}_2^{(p)}}{\|\boldsymbol{r}_{1\to 2}\|^2}$$

where $F_{1\to 2}$ is the intensity of the force exerted by body 1 on body 2 and $\boldsymbol{r}_{1\to 2} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the relative position of body 2 with respect to body 1. G is a universal constant know as the **gravitational constant** or **Cavendish constant**.
By applying **N3** we require that $F_{1\to 2} = F_{2\to 1}$. This can be manipulated to obtain that the ratio between active and passive gravitational mass must be constant for all bodies.

$$\frac{\widetilde{m}_1^{(a)}}{\widetilde{m}_1^{(p)}} = \frac{\widetilde{m}_2^{(a)}}{\widetilde{m}_2^{(p)}}$$

Therefore we can take this constant to be unity and drop the distinction between the two types of gravitational mass. Our law then becomes

$$F_{1\to 2} = G\frac{\widetilde{m}_1\widetilde{m}_2}{\|\boldsymbol{r}_{1\to 2}\|^2}$$

We follow up by applying **N2** to the second body and finding its acceleration $\ddot{r}_2$.

$$\ddot{r}_2 = G\frac{\widetilde{m}_1}{R^2}\frac{\widetilde{m}_2}{m_2}$$

By virtue of the **EP** this acceleration must not depend upon any properties of the body itself, the ratio between gravitational and inertial mass must therefore be a constant. This constant can also be taken to be unity, removing any and all practical distinctions between the various types of mass. We thus obtain the **Law of Universal Gravitation** in its most commonly used forms.

$$F_{1\to 2} = G\frac{m_1 m_2}{\|\boldsymbol{r}_{1\to 2}\|^2} \qquad \boldsymbol{F}_{1\to 2} = -G\frac{m_1 m_2}{\|\boldsymbol{r}_{1\to 2}\|^3}\boldsymbol{r}_{1\to 2} \tag{UG}$$