

Gravity

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1 Introduction to Classical Mechanics

In Classical Mechanics (CM) time and space are treated as two distinct concepts. Events are identified by their position in some frame of reference, represented by a vector \mathbf{x} , and the time at which they occur, represented by a scalar t .

The simplest formulation of CM is based on the concept of **force**. A force \mathbf{F} is a vector quantity representing an interaction between two bodies.

N0 Principle of Superposition: the net force on a body is the vector sum of the individual forces acting on it.

N1 Law of Inertia: an inertial frame of reference is one where, in the absence of a net force, bodies are in uniform linear motion.

N2 Law of Acceleration: in an inertial frame of reference the acceleration of a body is directly proportional to the net force acting on it.

N3 Law of Reaction: a body exerting a force on another is itself subject to a force equal in magnitude and opposite in direction produced by the second body.

This formulation of mechanics is (unduly) credited to Newton because it was first published as a unified theory in the *Principia*, despite **N1** being already known to Galileo and **N3** being a consequence of the Cartesian principle of conservation of momentum. **N0** was omitted by Newton and most sources refer to this formulation of mechanics as Newton's *three* laws as a result.

Law **N2** can be used to introduce the concept of **inertial mass** m , which is taken as the proportionality constant between force and acceleration. The law can then be stated as

$$\mathbf{F} = m \ddot{\mathbf{x}} \quad (\text{N2})$$

2 Galilean gravity

The simplest theory of gravity in our possession is due to Galileo and simply states that acceleration due to gravity is constant, regardless of the size, velocity or position of a body. In hindsight, Galilean gravity can be thought of as gravity in the limit where Earth is an indefinitely extended plane with a uniform mass distribution.

$$\ddot{\mathbf{x}} = \mathbf{g}$$

This theory, although excessively simple, lays the foundation of a fundamental principle.

EP_w (Weak) Equivalence Principle: all bodies subject to a gravitational field will undergo the same acceleration if they are located at the same coordinates.

This principle is backed up by all experimental evidence to date.

3 Hookean gravity

Generally credited to Newton, most of the fundamental work on this theory was actually due to Hooke. The basic principle of the theory is that every body exerts a gravitational force on every other body. This force pulls the objects towards each other and its intensity is:

- directly proportional to the **active gravitational mass** $\tilde{m}^{(a)}$ of the body exerting the force,
- directly proportional to the **passive gravitational mass** $\tilde{m}^{(p)}$ of the body subject to the force,
- inversely proportional to the square of their distance.

This can be expressed as the following formula

$$F_{1 \rightarrow 2} = G \frac{\tilde{m}_1^{(a)} \tilde{m}_2^{(p)}}{\|\mathbf{r}_{1 \rightarrow 2}\|^2}$$

where $F_{1 \rightarrow 2}$ is the intensity of the force exerted by body 1 on body 2 and $\mathbf{r}_{1 \rightarrow 2} = \mathbf{x}_2 - \mathbf{x}_1$ is the relative position of body 2 with respect to body 1. G is a universal constant known as the **gravitational constant** or **Cavendish constant**.

By applying **N3** we require that $F_{1 \rightarrow 2} = F_{2 \rightarrow 1}$. This can be manipulated to obtain that the ratio between active and passive gravitational mass must be constant for all bodies.

$$\frac{\tilde{m}_1^{(a)}}{\tilde{m}_1^{(p)}} = \frac{\tilde{m}_2^{(a)}}{\tilde{m}_2^{(p)}}$$

Therefore we can take this constant to be unity and drop the distinction between the two types of gravitational mass. Our law then becomes

$$F_{1 \rightarrow 2} = G \frac{\tilde{m}_1 \tilde{m}_2}{\|\mathbf{r}_{1 \rightarrow 2}\|^2}$$

We follow up by applying **N2** to the second body and finding its acceleration $\ddot{\mathbf{r}}_2$.

$$\ddot{\mathbf{r}}_2 = G \frac{\tilde{m}_1}{R^2} \frac{\tilde{m}_2}{m_2} \mathbf{r}_{1 \rightarrow 2}$$

By virtue of the **EP_w** this acceleration must not depend upon any properties of the body itself, the ratio between gravitational and inertial mass must therefore be a constant. This constant can also be taken to be unity, removing any and all practical distinctions between the various types of mass. We thus obtain the **Law of Universal Gravitation** in its most commonly used forms.

$$F_{1 \rightarrow 2} = G \frac{m_1 m_2}{\|\mathbf{r}_{1 \rightarrow 2}\|^2} \quad F_{1 \rightarrow 2} = -G \frac{m_1 m_2}{\|\mathbf{r}_{1 \rightarrow 2}\|^3} \mathbf{r}_{1 \rightarrow 2} \quad (\text{UG})$$

4 Galilean relativity

In general the concept of relativity implies that the laws of physics must be invariant under certain classes of transformations. The first step towards relativity is postulating that space and time must be homogeneous and isotropic.

H_s Homogeneity of Space: the laws of physics are invariant under spatial translations.

H_t Homogeneity of Time: the laws of physics are invariant under time translations.

I_s Isotropy of Space: the laws of physics are invariant under spatial rotation and reflections.

I_t Isotropy of Time: the laws of physics are invariant under time reversal.

This means that the symmetry group of physical laws must be, at least, the combination of the one dimensional Euclidean group $E(1)$ for the time coordinate and of the three dimensional Euclidean group $E(3)$ for the spatial coordinates. For the sake of brevity we denote this by $E(1, 3)$.

$$E(1, 3) \cong E(1) \rtimes E(3)$$

Often reflections and time reversal are ignored, restricting us the special subgroup. It is helpful to observe that $SE(1)$ is isomorphic to the group of one dimensional translations $T(1)$.

$$SE(1, 3) \cong SE(1) \rtimes SE(3) \cong T(1) \rtimes SE(3)$$

Galilean relativity involves adding another principle to the ones already mentioned above.

SR_w (Weak) Special Principle of Relativity: the laws of motion are the same in all frames of reference in uniform linear motion with respect to one another.

An important remark is that such wording of the principle only refers to the laws of *motion*, and therefore to acceleration. It does not ensure that *all* laws of physics are the same.