Gravity

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1 Classical Laws of Motion

In classical mechanics time and space are treated as two distinct concepts. Events are identified by their position in some frame of reference, represented by a vector $\boldsymbol{x}$, and the time at which they occur, represented by a scalar t.
The simplest formulation of CM is based on the concept of **force**. A force $\boldsymbol{F}$ is a vector quantity representing an interaction between two bodies.

N0 **Principle of Superposition:** the net force on a body is the vector sum of the individual forces acting on it.

N1 **Law of Inertia:** an inertial frame of reference is one where, in the absence of a net force, bodies are in uniform linear motion.

N2 **Law of Acceleration:** in an inertial frame of reference the acceleration of a body is directly proportional to the net force acting on it.

N3 **Law of Reaction:** a body exerting a force on another is itself subject to a force equal in magnitude and opposite in direction produced by the second body.

This formulation of mechanics is (unduly) credited to Newton because it was first published as a unified theory in the *Principia*, despite **N1** being already known to Galileo and **N3** being a consequence of the Cartesian principle of conservation of momentum. **N0** was omitted by Newton and most sources refer to this formulation of mechanics as Newton's *three* laws as a result.
Law **N2** can be used to introduce the concept of **inertial mass** m, which is taken as the proportionality constant between force and acceleration. The law can then be stated as

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}} \tag{N2}$$

2 Galilean relativity

In general the concept of relativity implies that the laws of physics must be invariant under certain classes of transformations. The first step towards relativity is postulating that space and time must be homogeneous and isotropic.

$\mathbf{H_s}$ **Homogeneity of Space:** the laws of physics are invariant under spatial translations.

$\mathbf{H_t}$ **Homogeneity of Time:** the laws of physics are invariant under time translations.

$\mathbf{I_s}$ **Isotropy of Space:** the laws of physics are invariant under spatial rotation and reflections.

$\mathrm{I_t}$ Isotropy of Time: the laws of physics are invariant under time reversal.

This means that the symmetry group of physical laws must be, at least, the combination of the one dimensional Euclidean group E(1) acting on the time coordinate and of the three dimensional Euclidean group E(3) acting on the spatial coordinates. We name this **Euclidean spacetime** E(1, 3).

$$\mathrm{E}(1,3) \cong \mathrm{E}(1) \rtimes \mathrm{E}(3)$$

Often reflections and time reversal are ignored, restricting us the special subgroup. It is helpful to observe that SE(1) is isomorphic to the group of one dimensional translations T(1).

$$\mathrm{SE}(1,3) \cong \mathrm{SE}(1) \rtimes \mathrm{SE}(3) \cong \mathrm{T}(1) \rtimes \mathrm{SE}(3)$$

Galilean relativity involves adding two other principles to those mentioned above.

SR Special Principle of Relativity: the laws of physics are the same in all frames of reference in uniform linear motion with respect to one another.

$\mathrm{G_t}$ Principle of Galilean Time: the rate at which time flows is the same in all frame of reference.

This last principle was thoroughly tested by Galileo using water clocks. Unfortunately we now know it to be incorrect, although it holds approximately for frames or reference moving with respect to each other at speeds much smaller than the speed of light. The transformations that satisfy these two principles are known as **Galilean boosts** and take the following form

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t \\ t' = t \end{cases} \tag{GTr}$$

where $\boldsymbol{v}$ is a parameter of the transformation, representing the velocity of the relative motion. It is fairly simple to prove that Galilean boosts form a group, which we shall denote by $\widetilde{\mathrm{B}}(3)$. In doing so, one obtains the well-known formula for the addition of classical velocities.

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \tag{CV$_+$}$$

The group of symmetries that classical mechanics must adhere to, the combination of Galilean boosts and Euclidean spacetime, is known as the **Galilean group** Gal(3).

$$\mathrm{Gal}(3) = \widetilde{\mathrm{B}}(3) \rtimes \mathrm{E}(1,3)$$

3 Galilean gravity

The simplest theory of gravity in our possession is due to Galileo and simply states that acceleration due to gravity is constant, regardless of the size, velocity or position of a body. In hindsight, Galilean gravity can be thought of as gravity in the limit where Earth is and indefinitely extended plane with a uniform mass distribution.

$$\ddot{\boldsymbol{x}} = \boldsymbol{g}$$

This theory, although excessively simple, lays the foundation of a fundamental principle.

$\mathrm{EP_w}$ (Weak) Equivalence Principle: all bodies subject to a gravitational field will undergo the same acceleration if they are located at the same coordinates.

This principle is backed up by all experimental evidence to date.

4 Hookean gravity

Generally credited to Newton, most of the fundamental work on this theory was actually due to Hooke. The basic principle of the theory is that every body exerts a gravitational force on every other body. This force pulls the objects towards each other and its intensity is:

- directly proportional to the **active gravitational mass** $\widetilde{m}^{(a)}$ of the body exerting the force,
- directly proportional to the **passive gravitational mass** $\widetilde{m}^{(p)}$ of the body subject to the force,
- inversely proportional to the square of their distance.

This can be expressed as the following formula

$$F_{1\to 2} = G\frac{\widetilde{m}_1^{(a)}\widetilde{m}_2^{(p)}}{\|\boldsymbol{r}_{1\to 2}\|^2}$$

where $F_{1\to 2}$ is the intensity of the force exerted by body 1 on body 2 and $\boldsymbol{r}_{1\to 2} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ is the relative position of body 2 with respect to body 1. G is a universal constant know as the **gravitational constant** or **Cavendish constant**.
By applying **N3** we require that $F_{1\to 2} = F_{2\to 1}$. This can be manipulated to obtain that the ratio between active and passive gravitational mass must be constant for all bodies.

$$\frac{\widetilde{m}_1^{(a)}}{\widetilde{m}_1^{(p)}} = \frac{\widetilde{m}_2^{(a)}}{\widetilde{m}_2^{(p)}}$$

Therefore we can take this constant to be unity and drop the distinction between the two types of gravitational mass. Our law then becomes

$$F_{1\to 2} = G\frac{\widetilde{m}_1\widetilde{m}_2}{\|\boldsymbol{r}_{1\to 2}\|^2}$$

We follow up by applying **N2** to the second body and finding its acceleration $\ddot{r}_2$.

$$\ddot{r}_2 = G\frac{\widetilde{m}_1}{R^2}\frac{\widetilde{m}_2}{m_2}$$

By virtue of the $\mathbf{EP_w}$ this acceleration must not depend upon any properties of the body itself, the ratio between gravitational and inertial mass must therefore be a constant. This constant can also be taken to be unity, removing any and all practical distinctions between the various types of mass. We thus obtain the **Law of Universal Gravitation** in its most commonly used forms.

$$F_{1\to 2} = G\frac{m_1 m_2}{\|\boldsymbol{r}_{1\to 2}\|^2} \qquad \boldsymbol{F}_{1\to 2} = -G\frac{m_1 m_2}{\|\boldsymbol{r}_{1\to 2}\|^3}\boldsymbol{r}_{1\to 2} \tag{UG}$$