GROMOV'S SIMPLICIAL NORM AND BOUNDED COHOMOLOGY

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Abstract. These are lecture notes for the course *Gromov's Simplicial Norm and Bounded Cohomology* in Spring 2022 at the University of Texas at Austin.

Contents

1. Introduction to Gromov's simplicial norm

One interesting topic in geometry and topology is to relate geometric quantities of a manifold to topological invariants. One typical problem asks about which manifolds admit Riemannian metrics with negative (positive, or non-positive) sectional (Ricci, or scalar) curvature.

Here we are interested in volumes of closed manifolds. Usually one needs a Riemannian metric to make sense of it, but is it possible to get a topological invariant out of it? The Mostow rigidity (and Gauss–Bonnet in dimension 2) implies that the volume of a hyperbolic closed manifold is determined by its topology. Gromov's *simplicial volume*, as a special case of the *simplicial norm*, is a way to define this invariant in a purely topological way.

Why should one be interested in such an invariant? The following basic problem is an example where one needs a topological notion of volume/area.

Problem 1.1. *Given two orientable connected closed surfaces S, S', what is the largest possible degree* $\deg(f)$ *of a continuous map $f : S \to S'$?*

As we will see below (Lemma 1.11), the simplicial volumes of S and S', denoted $\|S\|_1$ and $\|S'\|_1$, satisfy

$$\|S\|_1 \geq |\deg(f)| \cdot \|S'\|_1$$

for any continuous map f. Intuitively, S needs to have enough area to cover S' for $|\deg(f)|$ times. This provides an upper bound $\|S\|_1/\|S'\|_1$ when $\|S'\| > 0$, or equivalently when S' has genus at

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least two as we will prove. Moreover, the upper bound obtained this way is actually sharp, and in Section 1.4 we will exactly determine the set of all possibly degrees

$$\deg(S, S') := \{\deg(f) \mid f : S \to S'\}.$$

1.1. **The simplicial norm.** Fix $n \in \mathbb{Z}_{\geq 0}$. Given a topological space X, Gromov [Gro82] introduced a semi-norm $\|\cdot\|_1$ on the singular homology $H_n(X;\mathbb{R})$ for each n as a real vector space to measure the size of each homology class. Recall that $H_n(X;\mathbb{R})$ is the homology of the singular chain complex

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n_1} \xrightarrow{\partial_{n-1}} \cdots,$$

where $C_n(X;\mathbb{R})$ is the space of singular n-chains, namely the real vector space spanned by the set $S_n(X)$ of all singular n-simplices. As usual, we have the subspaces $B_n \subset Z_n \subset C_n$, where $Z_n := \ker \partial_n$ and $B_n := \operatorname{Im} \partial_{n+1}$ are the spaces of cycles and boundaries respectively. So by definition $H_n(X;\mathbb{R})$ is the quotient Z_n/B_n.

Given the standard basis $S_n(X)$, equip the space $C_n(X;\mathbb{R})$ with the ℓ^1-norm, i.e. $|c|_1 = \sum_{i=1}^k |\lambda_i|$ for any $c = \sum_{i=1}^k \lambda_i c_i$ expressed uniquely as a (finite) linear combination of basis elements $c_i \in S_n(X)$ with coefficients $\lambda_i \in \mathbb{R}$.

Definition 1.2 (Simplicial norm)**.** The restriction of this ℓ^1-norm to Z_n induces a semi-norm on its quotient $H_n(X;\mathbb{R})$, explicitly,

$$\|\sigma\|_1 := \inf_{[c]=\sigma} |c|_1,$$

where the infimum is taken over all cycles $c \in Z_n$ representing the homology class $\sigma \in H_n(X;\mathbb{R})$. This semi-norm is called *Gromov's simplicial norm.*

In words, $\|\sigma\|_1$ is the infimal number of simplices that we need to represent σ.

The following property is immediate from the definition but important.

Proposition 1.3 (Functorial)**.** *For any continuous map $f : X \to Y$, then the induced map $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is non-increasing with respect to the simplicial norm, i.e.*

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1$$

for any $\sigma \in H_n(X;\mathbb{R})$.

Proof. For any cycle $c = \sum_i \lambda_i c_i \in Z_n(X;\mathbb{R})$ representing σ, the cycle $f_*c = \sum \lambda_i f_* c_i = \sum_i \lambda_i (f \circ c_i)$ represents $f_*\sigma$. Hence by definition

$$\|f_*\sigma\|_1 \leq |f_*c|_1 \leq \sum_i |\lambda_i| = |c|_1.$$

Since c is arbitrary, taking infimum implies

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1.$$

□

Corollary 1.4 (Invariance)**.** *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is an isometric isomorphism (i.e. an isomorphism that is norm-preserving) with respect to the simplicial norm.*

More generally, if for a map $f : X \to Y$ there is $g : Y \to X$ such that $g_* f_*$ is the identity on $H_n(X;\mathbb{R})$, then f_* is an isometric embedding (i.e. injective and norm-preserving).

Proof. The first part easily follows from the second part by taking g to be a homotopy inverse of f.

For the second part, by functoriality of g and the fact that $g_* f_* = id$, we have $\|\sigma\|_1 = \|(g_* f_*)\sigma\|_1 \leq \|f_*\sigma\|_1$. Combining with the functoriality of f, we must have $\|\sigma\|_1 = \|f_*\sigma\|_1$ for any $\sigma \in H_n(X;\mathbb{R})$. Hence f_* is norm-preserving. Injectivity easily follows from the fact that $g_* f_* = id$. □

It is often convenient to consider cycles with rational coefficients since they can be scaled to integral cycles. We can always find a rational homology class arbitrarily close to a given homology class with respect to the simplicial norm; see the lemma below. This follows from the fact that B_n and Z_n are rational subspaces. Here a point $c \in C_n(X;\mathbb{R})$ is rational if $c \in C_n(X;\mathbb{Q})$, and an $\mathbb{R}$-linear subspace is rational if it has a basis consisting of rational points. Any point in a rational subspace V is a limit of rational points in V with respect to the norm $|\cdot|_1$ (think about it). Here B_n and Z_n are rational because the boundary maps $\partial_{k+1} : C_{k+1}(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ are rational linear, i.e. obtained from $C_{k+1}(X;\mathbb{Q}) \to C_k(X;\mathbb{Q})$ by tensoring with $\mathbb{R}$ over $\mathbb{Q}$.

Lemma 1.5. *If $\sigma \in H_n(X;\mathbb{Q})$, then $\|\sigma\|_1 = \inf |c|_1$ where the infimum is taken over all* rational *cycles $c = \sum \lambda_i c_i$ (i.e $\lambda_i \in \mathbb{Q}$ and $\partial c = 0$).*

For a general $\sigma \in H_n(X;\mathbb{R})$ and any $\epsilon > 0$, there is $\sigma' \in H_n(X;\mathbb{Q})$ with $\|\sigma - \sigma'\|_1 \leq \epsilon$.

Proof. For the first part, note that $B_n(X;\mathbb{Q})$ is dense in $B_n(X;\mathbb{R})$ with respect to the norm $|\cdot|_1$, since $B_n(X;\mathbb{R})$ is a rational subspace. As $\sigma \in H_n(X;\mathbb{Q})$, it can be represented by some rational cycle c. All other (resp. rational) cycles take the form $c+b$ with $b \in B_n(X;\mathbb{R})$ (resp. $b \in B_n(X;\mathbb{Q})$), so the result follows by density.

The second part is due to the density of $Z_n(X;\mathbb{Q})$ in $Z_n(X;\mathbb{R})$, which holds since $Z_n(X;\mathbb{R})$ is a rational subspace. □

Exercise 1.6. *Recall that $H_0(X;\mathbb{R})$ is isomorphic to the $\mathbb{R}$-vector space with basis corresponding to the path connected components of the space X. For any path component C and a point $c \in C$, thought of as a singular 0-simplex, we have a homology class $\sigma = [c]$. Show that $\|\sigma\|_1 = 1$.*

Remark 1.7. *If A is a subspace of X, then we can define a simplicial (semi-)norm similarly on the relative homology group $H_n(X, A;\mathbb{R})$. Here one can treat $H_n(X, A;\mathbb{R})$ as the homology of the chain complex $C_n(X, A) = C_n(X)/C_n(A)$ (with the induced differentials). These vector spaces are equipped with semi-norms induced from $C_n(X)$ and thus we can define an induced semi-norm on $H_n(X, A;\mathbb{R})$ as before. When A is empty, this agrees with our definition above.*

More generally, one can analogously define simplicial norm for any normed *chain complex; see* [Fri17].

Exercise 1.8. *Concretely, we can think of $H_n(X, A;\mathbb{R}) = Z_n(X, A)/B_n(X, A)$, where $B_n(X, A) = B_n(X) \cup C_n(A)$ and $Z_n(X, A) = \partial_n^{-1} C_{n-1}(A)$, with $C_i(A)$ treated naturally as a subspace of $C_i(X)$ for both $i = n-1, n$. Show that the semi-norm induced from this quotient agrees with the definition in the remark above.*

1.2. **The simplicial volume.** Now we specialize to measure the size of an oriented connected compact manifold M with (possibly empty) boundary ∂M. Let $n = \dim M$. The orientation picks out a generator $[M] \in H_n(M, \partial M;\mathbb{Z}) \cong \mathbb{Z}$, called the *fundamental class*. We think of it as a class in $H_n(M, \partial M;\mathbb{R}) \cong \mathbb{R}$ using the map $H_n(M, \partial M;\mathbb{Z}) \to H_n(M, \partial M;\mathbb{R})$ induced by the standard inclusion $\mathbb{Z} \to \mathbb{R}$. Concretely, if M has a triangulation, then the sum of all n-simplices with compatible orientation is a cycle representing the fundamental class.

Definition 1.9 (Simplicial volume)**.** The simplicial volume of M is $\|[M]\|_1$, which we often abbreviate as $\|M\|_1$. Note that the choice of orientation does not affect the simplicial volume.

If M is non-orientable, then M has an orientable double cover N, and we define $\|M\|_1 := \|N\|_1/2$. If M is disconnected, define $\|M\|_1$ as the sum of volumes of its components.

Exercise 1.10. *If M is orientable and closed, with finitely many components N_i. Show that $\sum_i \|N_i\|_1 = \|\sum_i [N_i]\|_1$, which explains the definition above for the disconnected case.*

Recall that, for any continuous map $f : M^n \to N^n$ between oriented connected closed (occ) manifolds, the degree $\deg(f)$ is the unique integer such that $f_*[M] = \deg(f) \cdot [N]$.

Lemma 1.11. *For any continuous map $f : M^n \to N^n$ between occ manifolds, we have*

$$|\deg(f)| \cdot \|N\|_1 \leq \|M\|_1.$$

Moreover, if f is a (finite) covering map, then equality holds.

Proof. The inequality follows from functoriality (Proposition 1.3) since $\|f_*[M]\|_1 = \|\deg(f)\cdot[N]\|_1 = |\deg(f)| \cdot \|[N]\|_1$.

Let $c = \sum_i \lambda_i c_i$ be a cycle representing the fundamental class $[N]$. Each map $c_i : \Delta^n \to N$ has $d := |\deg(f)|$ lifts $\tilde{c}_i^j$ to M, $j = 1, \cdots, d$. Then $\tilde{c} = \sum_i \sum_{j=1}^d \tilde{c}_i^j$ is a cycle and clearly $f_*[\tilde{c}] = d[c] = |\deg(f)| \cdot [N] = \pm f_*[M]$. Hence $[\tilde{c}] = \pm[M]$, and $\|M\|_1 \leq |\deg(f)| \cdot |c|_1$. Since c is arbitrary, minimizing its norm gives the reversed inequality we desire. $\square$

Corollary 1.12. *If an orientable closed connected manifold M admits a selfmap $f : M \to M$ with $|\deg(f)| > 1$, then $\|M\|_1 = 0$.*

Example 1.13.

(1) For any sphere S^n, $n \geq 1$, we have $\|S^n\|_1 = 0$.
(2) For the n-torus $T^n = (S^1)^n$, $n \geq 1$ we have $\|T^n\|_1 = 0$.
(3) More generally, if $M = S^1 \times N$ for a closed manifold N, then $\|M\|_1 = 0$.

These properties of the simplicial volume help us understand the simplicial norm of certain homology classes.

Lemma 1.14. *For $n \geq 1$, if a homology class $\sigma \in H_n(X; \mathbb{R})$ is represented by a sphere, i.e. there is a map $f : S^n \to X$ with $f_*[S^n] = \sigma$, then $\|\sigma\|_1 = 0$.*

Proof. By functoriality and the fact that spheres (of dimension at least one) have zero simplicial volume, $\|\sigma\|_1 = \|f_*[S^n]\|_1 \leq \|S^n\|_1 = 0$. Thus $\|\sigma\|_1 = 0$. $\square$

Corollary 1.15. *For any X, the simplicial norm $\|\cdot\|_1$ vanishes on $H_1(X; \mathbb{R})$.*

Proof. Basically, every 1-cycle is a bunch of circles and thus this should follow from Lemma 1.14. To make it precise, we use the approximation by rational cycles from Lemma 1.5 to reduce the problem to integral cycles, which is a standard trick in these topics.

By the second part of Lemma 1.5, it suffices to show that $\|\sigma\|_1 = 0$ for all rational homology classes $\sigma \in H_n(X; \mathbb{R})$. Any such σ is represented by some rational cycle c, and up to scaling, it suffices to consider the case where c is integral, i.e. $c = \sum_i n_i c_i$ for some $n_i \in \mathbb{Z} \setminus \{0\}$. Up to changing the orientation on c_i we may assume $n_i > 0$.

Now create n_i disjoint oriented segments for each c_i for all i. The fact that $\partial c = 0$ implies that we can pair the boundary points of these segments so that the endpoint of a segment s is always paired with the starting point of some segment s' so that the corresponding paths glue up in X respecting the orientations. The end result is a closed oriented 1-manifold, i.e. a disjoint union of finitely many oriented circles S_k^1 indexed by k. In other words, there is a map $\varphi : \sqcup_k S_k^1 \to X$ such that $\varphi_* \sum_k [S_k^1] = \sigma$. Hence by Lemma 1.14 and the triangle inequality,

$$\|\sigma\|_1 \leq \sum_k \|\varphi_*[S_k^1]\|_1 = 0,$$

so $\|\sigma\|_1 = 0$ as desired. $\square$

1.3. **Volumes of surfaces.** In this section we aim to obtain the first nontrivial examples. We have seen that the simplicial norm is boring on H_0 and vanishes on H_1. Interesting examples emerge in H_2. For orientable connected closed surfaces, we have seen in Example 1.13 that the simplicial volume vanishes when the genus is zero or one. For surfaces of higher genus, the simplicial volume is nonzero and is proportional to the Euler characteristic.

Theorem 1.16. *For any orientable connected closed surface S of genus at least two, we have $\|S\|_1 = -2\chi(S)$.*

Remark 1.17. *Note that by Gauss–Bonnet, for any hyperbolic metric, S has area $-2\pi\chi(S) = \pi\|S\|_1$, so the simplicial volume is proportional to the hyperbolic volume. The factor π is the area of the ideal hyperbolic triangle, or equivalently, the supremum of areas of all hyperbolic triangles (ideal or not). We will generalize this to higher dimension, which is referred to as Gromov's proportionality theorem.*

To combine the results for all genera, it is convenient to introduce the following χ^- notation.

Notation 1.18. For an orientable connected compact surface S, let $\chi^-(S) = \chi(S)$ if $\chi(S) \leq 0$ and let $\chi^-(S) = 0$ otherwise, i.e. we adjust $\chi(S)$ to 0 when S is a sphere or a disk. For a general orientable compact surface $S = \sqcup\Sigma_i$ with components, let $\chi^-(S) := \sum \chi^-(\Sigma_i)$. In other words, $\chi^-(S)$ is the Euler characteristic of S after deleting all components homeomorphic to spheres or disks.

Then the following theorem easily follows from Theorem 1.16 and the case of the sphere and torus.

Theorem 1.19. *For any orientable closed surface S, we have $\|S\|_1 = -2\chi^-(S)$.*

We will prove Theorem 1.16 by establishing inequalities in both directions, which involve two different kinds of ideas.

The strategy for proving $\|S\|_1 \leq -2\chi(S)$ is to construct a sequence of cycles representing the fundamental class $[S]$ approaching the optimal value. As we explained earlier, one concrete way to represent the fundamental class is to triangulate S and take the formal sum of triangles with compatible orientations.

Suppose S has a triangulation with v vertices, e edges and f faces, we know $\chi(S) = v - e + f$. The cycle described above has norm f.

Lemma 1.20. *We have $2e = 3f$, so $\chi(S) = v - \frac{f}{2}$ and $f = 2v - 2\chi(S)$. Hence $\|S\|_1 \leq 2v - 2\chi(S)$.*

Proof. Each triangle has 3 edges, each of which is shared by two triangles. □

So this is close to be optimal except for the error $2v$. The best one can do here is to take a triangulation with $v = 1$, which exists.

Exercise 1.21. *For any occ surface S, there is a triangulation with a single vertex.*

The bounded error can be remedied by taking finite covers.

Lemma 1.22. *If S has genus at least one, then $\|S\|_1 \leq -2\chi(S)$.*

Proof. For any such S and any $d \in \mathbb{Z}_+$, there is a degree d cover $f : S' \to S$. Then by taking a triangulation on S' with a single vertex, we have $\|S'\| \leq$ by Lemma 1.20. Note that both χ and $\|\cdot\|_1$ are multiplicative, i.e. $d\chi(S) = \chi(S')$ and $d\|S\|_1 = \|S'\|_1$ (by Lemma 1.11). Thus we obtain

$$\|S\|_1 = \frac{\|S'\|_1}{d} \leq \frac{2 - 2\chi(S')}{d} = \frac{2 - 2d\chi(S)}{d} = \frac{2}{d} - 2\chi(S).$$

Taking $d \to \infty$, we obtain the desired inequality. □

We considered above all possible ways of representing (resp. a multiple of) the fundamental class using triangulations (resp. of a finite cover).

The reversed inequality uses a technique called "straightening", which involves hyperbolic geometry and works in all dimensions. So we will prove it after a rush course on hyperbolic geometry.

1.4. Application: degrees of maps between surfaces.

2. Bounded cohomology

3. Quasimorphisms

4. More on Gromov's simplicial norm

5. Mostow's rigidity

6. Actions on the circle and the bounded Euler class

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