GROMOV'S SIMPLICIAL NORM AND BOUNDED COHOMOLOGY

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Abstract. These are lecture notes for the course *Gromov's Simplicial Norm and Bounded Cohomology* in Spring 2022 at the University of Texas at Austin.

Contents

1. Introduction to Gromov's simplicial norm

One interesting topic in geometry and topology is to relate geometric quantities of a manifold to topological invariants. One typical problem asks about which manifolds admit Riemannian metrics with negative (positive, or non-positive) sectional (Ricci, or scalar) curvature.

Here we are interested in volumes of closed manifolds. Usually one needs a Riemannian metric to make sense of it, but is it possible to get a topological invariant out of it? The Mostow rigidity (and Gauss–Bonnet in dimension 2) implies that the volume of a hyperbolic closed manifold is determined by its topology. Gromov's *simplicial volume*, as a special case of the *simplicial norm*, is a way to define this invariant in a purely topological way.

Why should one be interested in such an invariant? The following basic problem is an example where one needs a topological notion of volume/area.

Problem 1.1. *Given two orientable connected closed surfaces S, S', what is the largest possible degree* $\deg(f)$ *of a continuous map $f : S \to S'$?*

As we will see below (Lemma 1.11), the simplicial volumes of S and S', denoted $\|S\|_1$ and $\|S'\|_1$, satisfy

$$\|S\|_1 \geq |\deg(f)| \cdot \|S'\|_1$$

Date: January 28, 2022.

for any continuous map f. Intuitively, S needs to have enough area to cover S' for $|\deg(f)|$ times. This provides an upper bound $\|S\|_1/\|S'\|_1$ when $\|S'\| > 0$, or equivalently when S' has genus at least two as we will prove. Moreover, the upper bound obtained this way is actually sharp, and in Section 1.6 we will exactly determine the set of all possibly degrees

$$\deg(S, S') := \{\deg(f) \mid f : S \to S'\}.$$

1.1. **The simplicial norm.** Fix $n \in \mathbb{Z}_{\geq 0}$. Given a topological space X, Gromov [Gro82] introduced a semi-norm $\|\cdot\|_1$ on the singular homology $H_n(X;\mathbb{R})$ for each n as a real vector space to measure the size of each homology class. Recall that $H_n(X;\mathbb{R})$ is the homology of the singular chain complex

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n_1} \xrightarrow{\partial_{n-1}} \cdots,$$

where $C_n(X;\mathbb{R})$ is the space of singular n-chains, namely the real vector space spanned by the set $S_n(X)$ of all singular n-simplices. As usual, we have the subspaces $B_n \subset Z_n \subset C_n$, where $Z_n := \ker \partial_n$ and $B_n := \operatorname{Im} \partial_{n+1}$ are the spaces of cycles and boundaries respectively. So by definition $H_n(X;\mathbb{R})$ is the quotient Z_n/B_n.

Given the standard basis $S_n(X)$, equip the space $C_n(X;\mathbb{R})$ with the ℓ^1-norm, i.e. $|c|_1 = \sum_{i=1}^k |\lambda_i|$ for any $c = \sum_{i=1}^k \lambda_i c_i$ expressed uniquely as a (finite) linear combination of basis elements $c_i \in S_n(X)$ with coefficients $\lambda_i \in \mathbb{R}$.

Definition 1.2 (Simplicial norm)**.** The restriction of this ℓ^1-norm to Z_n induces a semi-norm on its quotient $H_n(X;\mathbb{R})$, explicitly,

$$\|\sigma\|_1 := \inf_{[c]=\sigma} |c|_1,$$

where the infimum is taken over all cycles $c \in Z_n$ representing the homology class $\sigma \in H_n(X;\mathbb{R})$. This semi-norm is called *Gromov's simplicial norm.*

In words, $\|\sigma\|_1$ is the infimal number of simplices that we need to represent σ.

The following property is immediate from the definition but important.

Proposition 1.3 (Functorial)**.** *For any continuous map $f : X \to Y$, then the induced map $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is non-increasing with respect to the simplicial norm, i.e.*

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1$$

for any $\sigma \in H_n(X;\mathbb{R})$.

Proof. For any cycle $c = \sum_i \lambda_i c_i \in Z_n(X;\mathbb{R})$ representing σ, the cycle $f_*c = \sum \lambda_i f_* c_i = \sum_i \lambda_i (f \circ c_i)$ represents $f_*\sigma$. Hence by definition

$$\|f_*\sigma\|_1 \leq |f_*c|_1 \leq \sum_i |\lambda_i| = |c|_1.$$

Since c is arbitrary, taking infimum implies

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1.$$

□

Corollary 1.4 (Invariance)**.** *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is an isometric isomorphism (i.e. an isomorphism that is norm-preserving) with respect to the simplicial norm.*

More generally, if for a map $f : X \to Y$ there is $g : Y \to X$ such that g_*f_* is the identity on $H_n(X;\mathbb{R})$, then f_* is an isometric embedding (i.e. injective and norm-preserving).

Proof. The first part easily follows from the second part by taking g to be a homotopy inverse of f.

For the second part, by functoriality of g and the fact that $g_*f_* = id$, we have $\|\sigma\|_1 = \|(g_*f_*)\sigma\|_1 \leq \|f_*\sigma\|_1$. Combining with the functoriality of f, we must have $\|\sigma\|_1 = \|f_*\sigma\|_1$ for any $\sigma \in H_n(X;\mathbb{R})$. Hence f_* is norm-preserving. Injectivity easily follows from the fact that $g_*f_* = id$. □

It is often convenient to consider cycles with rational coefficients since they can be scaled to integral cycles. We can always find a rational homology class arbitrarily close to a given homology class with respect to the simplicial norm; see the lemma below. This follows from the fact that B_n and Z_n are rational subspaces. Here a point $c \in C_n(X;\mathbb{R})$ is rational if $c \in C_n(X;\mathbb{Q})$, and an $\mathbb{R}$-linear subspace is rational if it has a basis consisting of rational points. Any point in a rational subspace V is a limit of rational points in V with respect to the norm $|\cdot|_1$ (think about it). Here B_n and Z_n are rational because the boundary maps $\partial_{k+1} : C_{k+1}(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ are rational linear, i.e. obtained from $C_{k+1}(X;\mathbb{Q}) \to C_k(X;\mathbb{Q})$ by tensoring with $\mathbb{R}$ over $\mathbb{Q}$.

Lemma 1.5. *If $\sigma \in H_n(X;\mathbb{Q})$, then $\|\sigma\|_1 = \inf |c|_1$ where the infimum is taken over all* rational *cycles $c = \sum \lambda_i c_i$ (i.e $\lambda_i \in \mathbb{Q}$ and $\partial c = 0$).*

For a general $\sigma \in H_n(X;\mathbb{R})$ and any $\epsilon > 0$, there is $\sigma' \in H_n(X;\mathbb{Q})$ with $\|\sigma - \sigma'\|_1 \leq \epsilon$.

Proof. For the first part, note that $B_n(X;\mathbb{Q})$ is dense in $B_n(X;\mathbb{R})$ with respect to the norm $|\cdot|_1$, since $B_n(X;\mathbb{R})$ is a rational subspace. As $\sigma \in H_n(X;\mathbb{Q})$, it can be represented by some rational cycle c. All other (resp. rational) cycles take the form $c+b$ with $b \in B_n(X;\mathbb{R})$ (resp. $b \in B_n(X;\mathbb{Q})$), so the result follows by density.

The second part is due to the density of $Z_n(X;\mathbb{Q})$ in $Z_n(X;\mathbb{R})$, which holds since $Z_n(X;\mathbb{R})$ is a rational subspace. □

Exercise 1.6. *Recall that $H_0(X;\mathbb{R})$ is isomorphic to the $\mathbb{R}$-vector space with basis corresponding to the path connected components of the space X. For any path component C and a point $c \in C$, thought of as a singular 0-simplex, we have a homology class $\sigma = [c]$. Show that $\|\sigma\|_1 = 1$.*

Remark 1.7. *If A is a subspace of X, then we can define a simplicial (semi-)norm similarly on the relative homology group $H_n(X, A;\mathbb{R})$. Here one can treat $H_n(X, A;\mathbb{R})$ as the homology of the chain complex $C_n(X, A) = C_n(X)/C_n(A)$ (with the induced differentials). These vector spaces are equipped with semi-norms induced from $C_n(X)$ and thus we can define an induced semi-norm on $H_n(X, A;\mathbb{R})$ as before. When A is empty, this agrees with our definition above.*

More generally, one can analogously define simplicial norm for any normed *chain complex; see* [Fri17].

Exercise 1.8. *Concretely, we can think of $H_n(X, A;\mathbb{R}) = Z_n(X, A)/B_n(X, A)$, where $B_n(X, A) = B_n(X) \cup C_n(A)$ and $Z_n(X, A) = \partial_n^{-1} C_{n-1}(A)$, with $C_i(A)$ treated naturally as a subspace of $C_i(X)$ for both $i = n-1, n$. Show that the semi-norm induced from this quotient agrees with the definition in the remark above.*

1.2. **The simplicial volume.** Now we specialize to measure the size of an oriented connected compact manifold M with (possibly empty) boundary ∂M. Let $n = \dim M$. The orientation picks out a generator $[M] \in H_n(M, \partial M;\mathbb{Z}) \cong \mathbb{Z}$, called the *fundamental class*. We think of it as a class in $H_n(M, \partial M;\mathbb{R}) \cong \mathbb{R}$ using the map $H_n(M, \partial M;\mathbb{Z}) \to H_n(M, \partial M;\mathbb{R})$ induced by the standard inclusion $\mathbb{Z} \to \mathbb{R}$. Concretely, if M has a triangulation, then the sum of all n-simplices with compatible orientation is a cycle representing the fundamental class.

Definition 1.9 (Simplicial volume)**.** The simplicial volume of M is $\|[M]\|_1$, which we often abbreviate as $\|M\|_1$. Note that the choice of orientation does not affect the simplicial volume.

If M is non-orientable, then M has an orientable double cover N, and we define $\|M\|_1 := \|N\|_1/2$. If M is disconnected, define $\|M\|_1$ as the sum of volumes of its components.

Exercise 1.10. *If M is orientable and closed, with finitely many components N_i. Show that $\sum_i \|N_i\|_1 = \|\sum_i [N_i]\|_1$, which explains the definition above for the disconnected case.*

Recall that, for any continuous map $f : M^n \to N^n$ between oriented connected closed (occ) manifolds, the degree $\deg(f)$ is the unique integer such that $f_*[M] = \deg(f) \cdot [N]$.

Lemma 1.11. *For any continuous map $f : M^n \to N^n$ between occ manifolds, we have*

$$|\deg(f)| \cdot \|N\|_1 \leq \|M\|_1.$$

Moreover, if f is a (finite) covering map, then equality holds.

Proof. The inequality follows from functoriality (Proposition 1.3) since $\|f_*[M]\|_1 = \|\deg(f)\cdot[N]\|_1 = |\deg(f)| \cdot \|[N]\|_1$.

Let $c = \sum_i \lambda_i c_i$ be a cycle representing the fundamental class $[N]$. Each map $c_i : \Delta^n \to N$ has $d := |\deg(f)|$ lifts $\tilde{c}_i^j$ to M, $j = 1, \cdots, d$. Then $\tilde{c} = \sum_i \sum_{j=1}^d \tilde{c}_i^j$ is a cycle and clearly $f_*[\tilde{c}] = d[c] = |\deg(f)| \cdot [N] = \pm f_*[M]$. Hence $[\tilde{c}] = \pm[M]$, and $\|M\|_1 \leq |\deg(f)| \cdot |c|_1$. Since c is arbitrary, minimizing its norm gives the reversed inequality we desire. ☐

Corollary 1.12. *If an orientable closed connected manifold M admits a selfmap $f : M \to M$ with $|\deg(f)| > 1$, then $\|M\|_1 = 0$.*

Exercise 1.13. *Extend the lemma and corollary above to the case of manifolds with boundary.*

Example 1.14.

(1) For any sphere S^n, $n \geq 1$, we have $\|S^n\|_1 = 0$.
(2) For the n-torus $T^n = (S^1)^n$, $n \geq 1$ we have $\|T^n\|_1 = 0$.
(3) More generally, if $M = S^1 \times N$ for a closed manifold N, then $\|M\|_1 = 0$.

These properties of the simplicial volume help us understand the simplicial norm of certain homology classes.

Lemma 1.15. *For $n \geq 1$, if a homology class $\sigma \in H_n(X; \mathbb{R})$ is represented by a sphere, i.e. there is a map $f : S^n \to X$ with $f_*[S^n] = \sigma$, then $\|\sigma\|_1 = 0$.*

Proof. By functoriality and the fact that spheres (of dimension at least one) have zero simplicial volume, $\|\sigma\|_1 = \|f_*[S^n]\|_1 \leq \|S^n\|_1 = 0$. Thus $\|\sigma\|_1 = 0$. ☐

Corollary 1.16. *For any X, the simplicial norm $\|\cdot\|_1$ vanishes on $H_1(X; \mathbb{R})$.*

Proof. Basically, every 1-cycle is a bunch of circles and thus this should follow from Lemma 1.15. To make it precise, we use the approximation by rational cycles from Lemma 1.5 to reduce the problem to integral cycles, which is a standard trick in these topics.

By the second part of Lemma 1.5, it suffices to show that $\|\sigma\|_1 = 0$ for all rational homology classes $\sigma \in H_n(X; \mathbb{R})$. Any such σ is represented by some rational cycle c, and up to scaling, it suffices to consider the case where c is integral, i.e. $c = \sum_i n_i c_i$ for some $n_i \in \mathbb{Z} \setminus \{0\}$. Up to changing the orientation on c_i we may assume $n_i > 0$.

Now create n_i disjoint oriented segments for each c_i for all i. The fact that $\partial c = 0$ implies that we can pair the boundary points of these segments so that the endpoint of a segment s is always paired with the starting point of some segment s' so that the corresponding paths glue up in X respecting the orientations. The end result is a closed oriented 1-manifold, i.e. a disjoint union of finitely many oriented circles S_k^1 indexed by k. In other words, there is a map $\varphi : \sqcup_k S_k^1 \to X$ such that $\varphi_* \sum_k [S_k^1] = \sigma$. Hence by Lemma 1.15 and the triangle inequality,

$$\|\sigma\|_1 \leq \sum_k \|\varphi_*[S_k^1]\|_1 = 0,$$

so $\|\sigma\|_1 = 0$ as desired. ☐

1.3. **Volumes of surfaces.** In this section we aim to obtain the first nontrivial examples. We have seen that the simplicial norm is boring on H_0 and vanishes on H_1. Interesting examples emerge in H_2. For orientable connected closed surfaces, we have seen in Example 1.14 that the simplicial volume vanishes when the genus is zero or one. For surfaces of higher genus, the simplicial volume is nonzero and is proportional to the Euler characteristic.

Theorem 1.17. *For any orientable connected closed surface S of genus at least two, we have $\|S\|_1 = -2\chi(S)$.*

Remark 1.18. *Note that by Gauss–Bonnet, for any hyperbolic metric, S has area $-2\pi\chi(S) = \pi\|S\|_1$, so the simplicial volume is proportional to the hyperbolic volume. The factor π is the area of the ideal hyperbolic triangle, or equivalently, the supremum of areas of all hyperbolic triangles (ideal or not). We will generalize this to higher dimension, which is referred to as Gromov's proportionality theorem.*

To combine the results for all genera, it is convenient to introduce the following χ^- notation.

Notation 1.19. For an orientable connected compact surface S, let $\chi^-(S) = \chi(S)$ if $\chi(S) \leq 0$ and let $\chi^-(S) = 0$ otherwise, i.e. we adjust $\chi(S)$ to 0 when S is a sphere or a disk. For a general orientable compact surface $S = \sqcup \Sigma_i$ with components, let $\chi^-(S) := \sum \chi^-(\Sigma_i)$. In other words, $\chi^-(S)$ is the Euler characteristic of S after deleting all components homeomorphic to spheres or disks.

Then the following theorem easily follows from Theorem 1.17 and the case of the sphere and torus.

Theorem 1.20. *For any orientable closed surface S, we have $\|S\|_1 = -2\chi^-(S)$.*

We will prove Theorem 1.17 by establishing inequalities in both directions, which involve two different kinds of ideas.

The strategy for proving $\|S\|_1 \leq -2\chi(S)$ is to construct a sequence of cycles representing the fundamental class $[S]$ approaching the optimal value. As we explained earlier, one concrete way to represent the fundamental class is to triangulate S and take the formal sum of triangles with compatible orientations.

Suppose S has a triangulation with v vertices, e edges and f faces, we know $\chi(S) = v - e + f$. The cycle described above has norm f.

Lemma 1.21. *We have $2e = 3f$, so $\chi(S) = v - \frac{f}{2}$ and $f = 2v - 2\chi(S)$. Hence $\|S\|_1 \leq 2v - 2\chi(S)$.*

Proof. Each triangle has 3 edges, each of which is shared by two triangles. $\square$

So this is close to be optimal except for the error $2v$. The best one can do here is to take a triangulation with $v = 1$, which exists.

Exercise 1.22. *For any occ surface S, there is a triangulation with a single vertex.*

The bounded error can be remedied by taking finite covers.

Lemma 1.23. *If S has genus at least one, then $\|S\|_1 \leq -2\chi(S)$.*

Proof. For any such S and any $d \in \mathbb{Z}_+$, there is a degree d cover $f : S' \to S$. Then by taking a triangulation on S' with a single vertex, we have $\|S'\| \leq$ by Lemma 1.21. Note that both χ and $\|\cdot\|_1$ are multiplicative, i.e. $d\chi(S) = \chi(S')$ and $d\|S\|_1 = \|S'\|_1$ (by Lemma 1.11). Thus we obtain

$$\|S\|_1 = \frac{\|S'\|_1}{d} \leq \frac{2 - 2\chi(S')}{d} = \frac{2 - 2d\chi(S)}{d} = \frac{2}{d} - 2\chi(S).$$

Taking $d \to \infty$, we obtain the desired inequality. $\square$

We considered above all possible ways of representing (resp. a multiple of) the fundamental class using triangulations (resp. of a finite cover).

The reversed inequality uses a technique called "straightening", which involves hyperbolic geometry. Roughly speaking, every hyperbolic triangle has area no greater than π, and the hyperbolic area of the surface is $-2\pi\chi(S)$, so one needs at least $-2\chi(S)$ triangles to cover the entire surface once. So we just need an argument to straighten an arbitrary singular cycle representing the fundamental class into one only involving hyperbolic triangles. We will explain this in more detail (Section 1.5) after a crash course on hyperbolic geometry (1.4).

1.4. **Some hyperbolic geometry.** We give a quick introduction/review of some hyperbolic geometry, mainly to describe geodesics and isometries. A more detailed treatment can be found in [BP92], [Thu97], or any standard textbook/notes on hyperbolic geometry.

The n-dimensional hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) is the unique (up to isometry) simply connected complete Riemannian manifolds with constant sectional curvature -1. There are several models for $\mathbb{H}^n$, providing different ways to view the space.

1.4.1. *The hyperboloid model.* Consider the bilinear form $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$ on $\mathbb{R}^{n+1}$ for any $x, y \in \mathbb{R}^{n+1}$. The set $H := \{x \mid \langle x, x\rangle = -1\}$ is a hyperboloid of two sheets. The restriction of $\langle \cdot, \cdot \rangle$ on either component gives a complete Riemannian metric of constant curvature -1. The upper sheet H_+ (i.e. with $x_{n+1} > 0$) is the hyperboloid model of $\mathbb{H}^n$.

With this model, the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is identified with $\mathrm{O}^+(n, 1)$, the group of linear transformations preserving the bilinear form $\langle \cdot, \cdot \rangle$ and stabilizing the upper sheet. The isometry group acts simply transitively on the orthonormal frame bundle, i.e. given any two points $x, y \in \mathbb{H}^n$ and two orthonormal bases (i.e. two orthonormal frames) at the two points, there is a unique isometry taking the frame at x to the frame at y.

An advantage of this model is that, any k-dimensional totally geodesic subspace of $\mathbb{H}^n$ is the intersection with some linear subspace of dimension $k + 1$. In particular, bi-infinite geodesics are intersections with planes through the origin.

The linearity provides a way to take *convex combinations* of points. More precisely, given k points $p_1, \cdots , p_k \in \mathbb{H}^n = H_+ \subset \mathbb{R}^{n+1}$, any coefficients $\lambda_1, \cdots , \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$ uniquely determine a point $p(\lambda_1, \cdots , \lambda_k) \in \mathbb{H}^n$ as the intersection of H_+ with the segment connecting the origin with $\sum_{i=1}^k \lambda_i p_i \in \mathbb{R}^{n+1}$. Apparently $p(\lambda_1, \cdots , \lambda_k)$ depends continuously on the coefficients λ_i, so this defines a continuous map $p : \Delta^{k-1} \to \mathbb{H}^n$, where $\Delta^{k-1} = \{(\lambda_1, \cdots , \lambda_k) \in \mathbb{R}^k \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$ is the standard $(k - 1)$-simplex. This gives a way to straighten singular simplices in $\mathbb{H}^n$: For any $c : \Delta^{k-1} \to \mathbb{H}^n$, let $p_1, \cdots , p_k$ be the image of the k vertices in the natural order, and let $\widetilde{\mathrm{str}}(c)$ be the map p defined above. This operation has the following properties which we record for reference later:

Lemma 1.24.

1. *If c_1 and c_2 agree on some face of Δ^{k-1}, then so do $\widetilde{\mathrm{str}}(c_1)$ and $\widetilde{\mathrm{str}}(c_2)$.*
2. *For any isometry $g \in \mathrm{Isom}\mathbb{H}^n$, we have $\widetilde{\mathrm{str}}(g \circ c) = g \circ \widetilde{\mathrm{str}}(c)$.*

Proof. The first part follows from the construction. The second part holds since the isometries in this model are linear maps, which commute with both taking convex combinations and scaling. □

The straightening operation that we will introduce in Section 1.5 relies on this construction.

1.4.2. *The Poincaré ball model.* In this model, $\mathbb{H}^n$ is identified with the open unit disk $\mathbb{D}^n \subset \mathbb{R}^n$ with the metric $\frac{4ds^2}{(1-\|x\|_2^2)^2}$ (at any $x \in \mathbb{D}^n$), where ds^2 is the Euclidean metric. So the metric gets more distorted in this model compared to the Euclidean one when x is closer to the boundary.

In this model, geodesics are circular arcs perpendicular to the boundary sphere. The isometry group consists of *Möbius transformations* that preserve the unit disk.

Definition 1.25. A self-diffeomorphism f of $S^n = \mathbb{R}^n \cup \{\infty\}$ is a Möbius transformation if one of the following equivalent descriptions holds:

1. f is a composition of inversions and reflections;
2. f is conformal (i.e. angle preserving);
3. f takes round spheres and hyperplanes to round spheres and hyperplanes;
4. f is a Euclidean similarity possibly composed with an inversion, i.e. $f(x) = \lambda Ai(x) + b$, where i is either the identity or an inversion, $A \in O(n)$, $\lambda > 0$, and $b \in \mathbb{R}^n$.

Here an *inversion* with respect to a round sphere $S(p,r)$ in R^n centered at p of radius r is the map on $S^n = \mathbb{R}^n \cup \{\infty\}$ given by $i(x) = p + \frac{x-p}{\|x-p\|} \cdot \frac{r^2}{\|x-p\|}$, which fixes $S(p,r)$ pointwise, swaps p and ∞, and preserves all rays from p so that $\|x-p\| \cdot \|i(x)-p\| = r^2$. The equivalence in the definition above (when $n \geq 3$) essentially follows from Liouville's theorem:

Theorem 1.26 (Liouville)**.** *A conformal diffeomorphisms f between two open subsets of $\mathbb{R}^n$ with $n \geq 3$ takes the form $f(x) = \lambda A i(x) + b$, where i is either the identity or an inversion, $A \in O(n)$, $\lambda > 0$, and $b \in \mathbb{R}^n$.*

See [BP92, Theorem A.3.7] for a detailed proof. A more geometric argument can be found here: `https://lamington.wordpress.com/2013/10/28/liouville-illiouminated/`.

This is a good model to talk about the boundary at infinity of $\mathbb{H}^n$, denoted $\partial\mathbb{H}^n$. Although $\partial\mathbb{H}^n$ is not part of the hyperbolic space $\mathbb{H}^n$, it compactifies the space (i.e. $\overline{\mathbb{H}}^n := \mathbb{H}^n \cup \partial\mathbb{H}^n$) and helps us understand geodesics and isometries. In this model, the boundary is exactly the unit sphere S^{n-1} and the compactification $\overline{\mathbb{H}}^n$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}^n$. Each bi-infinite geodesic naturally has two endpoints on the boundary which uniquely determines the geodesic. Each isometry extends to a homeomorphism on $\overline{\mathbb{H}}^n$ and in particular acts on the boundary (and the action is 2-transitive).

1.4.3. *The upper-half space model.* This model identifies $\mathbb{H}^n$ with the open upper-half space $\{x \in \mathbb{R}^n \mid x_n > 0\}$ equipped with the metric $\frac{ds^2}{x_n^2}$, where ds^2 is the Euclidean metric. Geodesics in this model are vertical lines and circular arcs perpendicular to the hyperplane $\{x_n = 0\}$. Isometries are Möbius transformations preserving the upper-half space. The boundary can be seen as the union of the hyperplane $\{x_n = 0\}$ with ∞.

1.4.4. *Isometries.* There is a classification of orientation-preserving isometries. Any isometry extends to a continuous homeomorphism on $\overline{\mathbb{H}}^n$, which is topologically a closed ball. Thus by Brouwer's fixed point theorem, each isometry must fix some point in $\overline{\mathbb{H}}^n$. Given $g \neq id \in \mathrm{Isom}^+(\mathbb{H}^n)$, there are three mutually exclusive cases:

(1) g is *elliptic* if it fixes some point in $\mathbb{H}^n$. Up to conjugation, we may assume that g fixes the origin 0 in the disk model, in which case g is conjugate to an orthogonal transformation in $\mathrm{SO}(n)$, determined by its action on the unit tangent space at 0.
(2) g is *parabolic* if it has no fixed point in $\mathbb{H}^n$ and has a unique fixed point in $\partial\mathbb{H}^n$. Up to conjugation, we may assume that g fixes ∞ in the upper-half space model, in which case we can deduce that g is conjugate to a horizontal (i.e. preserving x_n) translation on $\mathbb{R}^n$.
(3) g is *hyperbolic* if it has no fixed point in $\mathbb{H}^n$ and has two fixed points in $\partial\mathbb{H}^n$. Up to conjugation, we may assume that g fixes ∞ and 0 in the upper-half space model. Then g is the composition of a scaling with an orthogonal transformation in $\mathrm{SO}(n-1) \subset \mathrm{SO}(n)$ centered at the origin fixing the x_n axis. This axis is the unique bi-infinite geodesic preserved by g, called the *axis* of g.

1.5. **Straightening.** Let M^n be a hyperbolic manifold. We introduce a linear map $\mathrm{str} : C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$, called *straightening*. Fix the universal covering map $\pi : \mathbb{H}^n \to M$. For any singular simplex $c : \Delta^k \to M$, pick any lift $\tilde{c} : \Delta^k \to \mathbb{H}^n$, which we straighten to $\widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \to \mathbb{H}^n$ using the construction described in Section 1.4.1. Define the straightening of c as $\mathrm{str}(c) := \pi\, \widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \to M$, which is independent of the choice of the lift since $\widetilde{\mathrm{str}}$ commutes with isometries. This extends to a linear map $\mathrm{str} : C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$.

Lemma 1.27. *We have $|\mathrm{str}(c)|_1 \leq |c|_1$ for any chain c and $\partial \mathrm{str} = \mathrm{str} \partial$. Moreover,* str *induces the identity map on $H_*(M;\mathbb{R})$.*

Proof. The straightening map does not increase the number of simplices, so $|\mathrm{str}(c)|_1 \le |c|_1$ for any chain c. It commutes with the boundary map by construction.

For any singular simplex $c : \Delta^k \to \mathbb{H}^n$, there is an obvious linear homotopy in $\mathbb{R}^{n+1}$ from c to $\widetilde{\mathrm{str}}(c)$, which scales to one on $\mathbb{H}^n$ and projects down to a homotopy on M. Based on this, one can build a chain homotopy between str and id, or directly observed that $[\mathrm{str}(c)] = [c]$ for any cycle c. □

It follows that, when computing the simplicial norm of any homology class $\sigma \in H_*(M;\mathbb{R})$ it suffices to look at cycles consisting of (straight) hyperbolic simplices. A key fact about hyperbolic simplices is that their volume has a uniform upper bound only depending on the dimension, in contrast with Euclidean simplices.

Lemma 1.28 ([BP92, Theorem C.2.1 and Lemma C.2.3])**.** *For each $n \ge 2$, let v_n be the supremum of volumes of all hyperbolic n-simplices (possibly with vertices at infinity). Then $v_2 = \pi$ and $v_n \le \frac{\pi}{(n-1)!}$.*

Remark 1.29. *A theorem of Haagerup–Munkholm*[HM81] *shows that v_n is achieved uniquely by the regular*[1] *hyperbolic ideal n-simplex.*

Proof. Any hyperbolic simplex has volume no more than some ideal hyperbolic simplex. In fact, for any hyperbolic simplex with vertices $p_0, \dots, p_n \in \mathbb{H}^n$, choose a point p in its interior. The geodesic rays from p to p_i determines a point $p_i' \in \partial\mathbb{H}^n$ for each i. The ideal hyperbolic simplex with vertices $p_0', \cdots, p_n'$ contains the starting one.

When $n = 2$, all ideal hyperbolic triangles are conjugate up to an isometry and have the same area π, so $v_2 = \pi$. The area can be computed explicitly in the upper-half space model, say for the ideal triangle with vertices $-1, 1, \infty$, or can be seen by Gauss–Bonnet.

For $n \ge 3$, we show $v_n \le \frac{v_{n-1}}{n-1}$ by a nice computation in the upper-half space model following [BP92, Lemma C.2.3], which implies the bound $\frac{\pi}{(n-1)!}$ by induction. Let σ be any ideal hyperbolic n-simplex in the upper-half space model and put one of its vertex as ∞. The remaining n vertices form an ideal hyperbolic $(n-1)$-simplex τ sitting on a totally geodesic subspace X of dimension $n-1$ not containing ∞ as a boundary point. So X is a round hemisphere centered at some $p \in \mathbb{R}^{n-1} \subset \partial\mathbb{H}^n$. Up to an isometry we may assume $p = 0$ and the sphere has radius 1. So X is the upper unit hemisphere. The vertical projection to $\mathbb{R}^{n-1}$ restricts to a homeomorphism from X to the unit closed ball in $\mathbb{R}^{n-1}$, and let s be the inverse. Explicitly, $s(x) = (x, h(x))$ where $h(x) = \sqrt{1 - \|x\|^2}$.

Let τ_0 be the image of τ under this projection, which is a Euclidean simplex with vertices on the unit sphere. Then

$$\begin{aligned}\mathrm{vol}(\sigma) &= \int_{\tau_0}\int_{h(x)}^{\infty} \frac{dydx}{y^n} \\ &= \frac{1}{n-1}\int_{\tau_0} \frac{dx}{h(x)^{n-1}}.\end{aligned}$$

It suffices to show that

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \le \mathrm{vol}(\tau).$$

Note that s gives a way to parameterize τ using τ_0, so we just need to show that the pullback of the volume form dominates $\frac{dx}{h(x)^{n-1}}$ for each x in the unit ball, where we think of dx as the standard volume form on $\mathbb{R}^{n-1}$.

For any x, let ω be the hyperbolic volume form restricted to $T_{s(x)}X$. Note that ω evaluates to $\frac{1}{h(x)^{n-1}}$ for any orthonormal basis of $T_{s(x)}X$. To find out its pullback, choose an orthonormal basis

[1] completely symmetric in the sense that the isometry group is the symmetric group on the $n+1$ vertices

of $T_x\mathbb{R}^{n-1}$. If $x = 0$, then $Dh = 0$ so s_* is the identity map and $s^*\omega = \frac{dx}{h(x)^{n-1}}$. If $x \neq 0$, we may choose the orthonormal basis so that one of them is $e_1 = \frac{x}{\|x\|}$. Then $Dh = 0$ in all directions perpendicular to e_1 and $Dh(e_1) = \frac{\|x\|}{h(x)}$, so s_* is the identity map in the subspace perpendicular to e_1, and it takes this orthonormal basis to an orthogonal basis where all elements have length 1 except that $\|s_*(e_1)\|^2 = 1 + \frac{\|x\|^2}{1-\|x\|^2} = \frac{1}{1-\|x\|^2} = \frac{1}{h(x)^2}$. Thus $s^*\omega = \frac{1}{h(x)} \cdot \frac{dx}{h(x)^{n-1}} \leq \frac{dx}{h(x)^{n-1}}$ since $h(x) \leq 1$. This verifies that the pullback of the volume form by s dominates $\frac{dx}{h(x)^{n-1}}$, and hence

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \leq \mathrm{vol}(\tau) \leq v_{n-1}.$$

$\square$

Lemma 1.30. *Let M^n be an oriented closed connected hyperbolic manifold. Then $\|M\|_1 \geq \frac{\mathrm{vol}(M)}{v_n}$.*

Proof. Let $c = \sum \lambda_i c_i$ be a cycle representing $[M]$. By Lemma 1.27, we may assume that c consists of straight hyperbolic simplices without increasing $\|c\|_1$. Let vol be the volume form. Then we have

$$\mathrm{vol}(M) = \langle [M], \mathrm{vol}\rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol} \right\rangle \leq \sum |\lambda_i| \cdot \max \mathrm{vol}(c_i) \leq |c|_1 \cdot v_n.$$

Sine c is arbitrary, we conclude that $\mathrm{vol}(M) \leq v_n \cdot \|M\|_1$. $\square$

Remark 1.31. *Conceptually, we obtained this lower bound by some sort of ℓ^1–ℓ^∞ duality, where we used a cocycle (the volume form here) that is bounded on all straight hyperbolic simplices. This suggests the use of bounded cocycles and bounded cohomology as a dual theory to better understand simplicial norms.*

Restricting to the case $n = 2$, we can now finish the proof of Theorem 1.17.

Proof of Theorem 1.17. The above lemma for $M = S$ a hyperbolic surface, we have

$$\|S\|_1 \geq \frac{\mathrm{area}(S)}{v_2} = \frac{-2\pi\chi(S)}{\pi} = -2\chi(S)$$

by Gauss–Bonnet. Combining with Lemma 1.23, we conclude $\|S\|_1 = -2\chi(S)$. $\square$

The bound in Lemma 1.30 is also sharp in higher dimensions; see Theorem 1.33. This geometric argument also works for closed manifolds with varying negative curvature, so one can deduce that any such manifold has positive simplicial volume. More generally, we have the following conjecture attributed to Gromov [Gro82, p.11]:

Conjecture 1.32. *Any closed manifold of non-positive curvature and negative Ricci curvature has $\|M\|_1 > 0$.*

This is still open. See [CW19] for a recent partial positive answer that better exploits the straightening argument.

1.6. **Application: degrees of maps between surfaces.** Now we apply the calculation in Theorem 1.20 to solve Problem 1.1 as an application.

1.7. **Gromov's proportionality.**

Theorem 1.33 (Gromov's Proportionality)**.** *Let M^n be an oriented closed connected hyperbolic manifold. Then $\|M\|_1 = \frac{\mathrm{vol}(M)}{v_n}$.*

2. Bounded cohomology

3. Quasimorphisms

4. More on Gromov's simplicial norm

5. Mostow's rigidity

6. Actions on the circle and the bounded Euler class

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