GROMOV'S SIMPLICIAL NORM AND BOUNDED COHOMOLOGY

LVZHOU CHEN

ABSTRACT. These are lecture notes for the course *Gromov's Simplicial Norm and Bounded Cohomology* in Spring 2022 at the University of Texas at Austin.

CONTENTS

1. INTRODUCTION TO GROMOV'S SIMPLICIAL NORM

One interesting topic in geometry and topology is to relate geometric quantities of a manifold to topological invariants. One typical problem asks about which manifolds admit Riemannian metrics with negative (positive, or non-positive) sectional (Ricci, or scalar) curvature.

Here we are interested in volumes of closed manifolds. Usually one needs a Riemannian metric to make sense of it, but is it possible to get a topological invariant out of it? The Mostow rigidity (and Gauss–Bonnet in dimension 2) implies that the volume of a hyperbolic closed manifold is determined by its topology. Gromov's *simplicial volume*, as a special case of the *simplicial norm*, is a way to define this invariant in a purely topological way.

Why should one be interested in such an invariant? The following basic problem is an example where one needs a topological notion of volume/area.

Date: February 28, 2022.

Problem 1.1. *Given two orientable connected closed surfaces S, S', what is the largest possible degree $\deg(f)$ of a continuous map $f : S \to S'$?*

As we will see below (Lemma 1.11), the simplicial volumes of S and S', denoted $\|S\|_1$ and $\|S'\|_1$, satisfy

$$\|S\|_1 \geq |\deg(f)| \cdot \|S'\|_1$$

for any continuous map f. Intuitively, S needs to have enough area to cover S' for $|\deg(f)|$ times. This provides an upper bound $\|S\|_1 / \|S'\|_1$ when $\|S'\| > 0$, or equivalently when S' has genus at least two as we will prove. Moreover, the upper bound obtained this way is actually sharp, and in Section 1.6 we will exactly determine the set of all possibly degrees

$$\deg(S, S') := \{\deg(f) \mid f : S \to S'\}.$$

1.1. **The simplicial norm.** Fix $n \in \mathbb{Z}_{\geq 0}$. Given a topological space X, Gromov [Gro82] introduced a semi-norm $\|\cdot\|_1$ on the singular homology $H_n(X; \mathbb{R})$ for each n as a real vector space to measure the size of each homology class. Recall that $H_n(X; \mathbb{R})$ is the homology of the singular chain complex

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n_1} \xrightarrow{\partial_{n-1}} \cdots,$$

where $C_n(X; \mathbb{R})$ is the space of singular n-chains, namely the real vector space spanned by the set $S_n(X)$ of all singular n-simplices. As usual, we have the subspaces $B_n \subset Z_n \subset C_n$, where $Z_n := \ker \partial_n$ and $B_n := \operatorname{Im} \partial_{n+1}$ are the spaces of cycles and boundaries respectively. So by definition $H_n(X; \mathbb{R})$ is the quotient Z_n / B_n.

Given the standard basis $S_n(X)$, equip the space $C_n(X; \mathbb{R})$ with the ℓ^1-norm, i.e. $|c|_1 = \sum_{i=1}^k |\lambda_i|$ for any $c = \sum_{i=1}^k \lambda_i c_i$ expressed uniquely as a (finite) linear combination of basis elements $c_i \in S_n(X)$ with coefficients $\lambda_i \in \mathbb{R}$.

Definition 1.2 (Simplicial norm). The restriction of this ℓ^1-norm to Z_n induces a semi-norm on its quotient $H_n(X; \mathbb{R})$, explicitly,

$$\|\sigma\|_1 := \inf_{[c]=\sigma} |c|_1,$$

where the infimum is taken over all cycles $c \in Z_n$ representing the homology class $\sigma \in H_n(X; \mathbb{R})$. This semi-norm is called *Gromov's simplicial norm*.

In words, $\|\sigma\|_1$ is the infimal number of simplices that we need to represent σ.
The following property is immediate from the definition but important.

Proposition 1.3 (Functorial). *For any continuous map $f : X \to Y$, then the induced map $f_* : H_n(X; \mathbb{R}) \to H_n(Y; \mathbb{R})$ is non-increasing with respect to the simplicial norm, i.e.*

$$\|f_* \sigma\|_1 \leq \|\sigma\|_1$$

for any $\sigma \in H_n(X; \mathbb{R})$.

Proof. For any cycle $c = \sum_i \lambda_i c_i \in Z_n(X; \mathbb{R})$ representing σ, the cycle $f_* c = \sum \lambda_i f_* c_i = \sum_i \lambda_i (f \circ c_i)$ represents $f_* \sigma$. Hence by definition

$$\|f_* \sigma\|_1 \leq |f_* c|_1 \leq \sum_i |\lambda_i| = |c|_1.$$

Since c is arbitrary, taking infimum implies

$$\|f_* \sigma\|_1 \leq \|\sigma\|_1.$$

$\square$

Corollary 1.4 (Invariance). *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X; \mathbb{R}) \to H_n(Y; \mathbb{R})$ is an isometric isomorphism (i.e. an isomorphism that is norm-preserving) with respect to the simplicial norm.*

More generally, if for a map $f : X \to Y$ there is $g : Y \to X$ such that $g_* f_*$ is the identity on $H_n(X; \mathbb{R})$, then f_* is an isometric embedding (i.e. injective and norm-preserving).

Proof. The first part easily follows from the second part by taking g to be a homotopy inverse of f.

For the second part, by functoriality of g and the fact that $g_* f_* = id$, we have $\|\sigma\|_1 = \|(g_* f_*)\sigma\|_1 \leq \|f_*\sigma\|_1$. Combining with the functoriality of f, we must have $\|\sigma\|_1 = \|f_*\sigma\|_1$ for any $\sigma \in H_n(X; \mathbb{R})$. Hence f_* is norm-preserving. Injectivity easily follows from the fact that $g_* f_* = id$. □

It is often convenient to consider cycles with rational coefficients since they can be scaled to integral cycles. We can always find a rational homology class arbitrarily close to a given homology class with respect to the simplicial norm; see the lemma below. This follows from the fact that B_n and Z_n are rational subspaces. Here a point $c \in C_n(X; \mathbb{R})$ is rational if $c \in C_n(X; \mathbb{Q})$, and an $\mathbb{R}$-linear subspace is rational if it has a basis consisting of rational points. Any point in a rational subspace V is a limit of rational points in V with respect to the norm $|\cdot|_1$ (think about it). Here B_n and Z_n are rational because the boundary maps $\partial_{k+1} : C_{k+1}(X; \mathbb{R}) \to C_k(X; \mathbb{R})$ are rational linear, i.e. obtained from $C_{k+1}(X; \mathbb{Q}) \to C_k(X; \mathbb{Q})$ by tensoring with $\mathbb{R}$ over $\mathbb{Q}$.

Lemma 1.5. *If $\sigma \in H_n(X; \mathbb{Q})$, then $\|\sigma\|_1 = \inf |c|_1$ where the infimum is taken over all rational cycles $c = \sum \lambda_i c_i$ (i.e $\lambda_i \in \mathbb{Q}$ and $\partial c = 0$).*

For a general $\sigma \in H_n(X; \mathbb{R})$ and any $\epsilon > 0$, there is $\sigma' \in H_n(X; \mathbb{Q})$ with $\|\sigma - \sigma'\|_1 \leq \epsilon$.

Proof. For the first part, note that $B_n(X; \mathbb{Q})$ is dense in $B_n(X; \mathbb{R})$ with respect to the norm $|\cdot|_1$, since $B_n(X; \mathbb{R})$ is a rational subspace. As $\sigma \in H_n(X; \mathbb{Q})$, it can be represented by some rational cycle c. All other (resp. rational) cycles take the form $c + b$ with $b \in B_n(X; \mathbb{R})$ (resp. $b \in B_n(X; \mathbb{Q})$), so the result follows by density.

The second part is due to the density of $Z_n(X; \mathbb{Q})$ in $Z_n(X; \mathbb{R})$, which holds since $Z_n(X; \mathbb{R})$ is a rational subspace. □

Exercise 1.6. *Recall that $H_0(X; \mathbb{R})$ is isomorphic to the $\mathbb{R}$-vector space with basis corresponding to the path connected components of the space X. For any path component C and a point $c \in C$, thought of as a singular 0-simplex, we have a homology class $\sigma = [c]$. Show that $\|\sigma\|_1 = 1$.*

Remark 1.7. *If A is a subspace of X, then we can define a simplicial (semi-)norm similarly on the relative homology group $H_n(X, A; \mathbb{R})$. Here one can treat $H_n(X, A; \mathbb{R})$ as the homology of the chain complex $C_n(X, A) = C_n(X)/C_n(A)$ (with the induced differentials). These vector spaces are equipped with semi-norms induced from $C_n(X)$ and thus we can define an induced semi-norm on $H_n(X, A; \mathbb{R})$ as before. When A is empty, this agrees with our definition above.*

More generally, one can analogously define simplicial norm for any normed chain complex; see [Fri17].

Exercise 1.8. *Concretely, we can think of $H_n(X, A; \mathbb{R}) = Z_n(X, A)/B_n(X, A)$, where $B_n(X, A) = B_n(X) \cup C_n(A)$ and $Z_n(X, A) = \partial_n^{-1} C_{n-1}(A)$, with $C_i(A)$ treated naturally as a subspace of $C_i(X)$ for both $i = n - 1, n$. Show that the semi-norm induced from this quotient agrees with the definition in the remark above.*

1.2. **The simplicial volume.** Now we specialize to measure the size of an oriented connected compact manifold M with (possibly empty) boundary ∂M. Let $n = \dim M$. The orientation picks out a generator $[M] \in H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$, called the *fundamental class*. We think of it as a class in $H_n(M, \partial M; \mathbb{R}) \cong \mathbb{R}$ using the map $H_n(M, \partial M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{R})$ induced by the standard inclusion $\mathbb{Z} \to \mathbb{R}$. Concretely, if M has a triangulation, then the sum of all n-simplices with compatible orientation is a cycle representing the fundamental class.

Definition 1.9 (Simplicial volume)**.** The simplicial volume of M is $\|[M]\|_1$, which we often abbreviate as $\|M\|_1$. Note that the choice of orientation does not affect the simplicial volume.

If M is non-orientable, then M has an orientable double cover N, and we define $\|M\|_1 := \|N\|_1/2$. If M is disconnected, define $\|M\|_1$ as the sum of volumes of its components.

Exercise 1.10. *If M is orientable and closed, with finitely many components N_i. Show that $\sum_i \|N_i\|_1 = \|\sum_i [N_i]\|_1$, which explains the definition above for the disconnected case.*

Recall that, for any continuous map $f : M^n \to N^n$ between oriented connected closed (occ) manifolds, the degree $\deg(f)$ is the unique integer such that $f_*[M] = \deg(f) \cdot [N]$.

Lemma 1.11. *For any continuous map $f : M^n \to N^n$ between occ manifolds, we have*
$$|\deg(f)| \cdot \|N\|_1 \le \|M\|_1.$$
Moreover, if f is a (finite) covering map, then equality holds.

Proof. The inequality follows from functoriality (Proposition 1.3) since $\|f_*[M]\|_1 = \|\deg(f) \cdot [N]\|_1 = |\deg(f)| \cdot \|[N]\|_1$.

Let $c = \sum_i \lambda_i c_i$ be a cycle representing the fundamental class $[N]$. Each map $c_i : \Delta^n \to N$ has $d := |\deg(f)|$ lifts $\tilde{c}_i^j$ to M, $j = 1, \cdots, d$. Then $\tilde{c} = \sum_i \sum_{j=1}^d \tilde{c}_i^j$ is a cycle and clearly $f_*[\tilde{c}] = d[c] = |\deg(f)| \cdot [N] = \pm f_*[M]$. Hence $[\tilde{c}] = \pm[M]$, and $\|M\|_1 \le |\deg(f)| \cdot |c|_1$. Since c is arbitrary, minimizing its norm gives the reversed inequality we desire. $\square$

Corollary 1.12. *If an orientable closed connected manifold M admits a selfmap $f : M \to M$ with $|\deg(f)| > 1$, then $\|M\|_1 = 0$.*

Exercise 1.13. *Extend the lemma and corollary above to the case of manifolds with boundary.*

Example 1.14.

    (1) *For any sphere S^n, $n \ge 1$, we have $\|S^n\|_1 = 0$.*
    (2) *For the n-torus $T^n = (S^1)^n$, $n \ge 1$ we have $\|T^n\|_1 = 0$.*
    (3) *More generally, if $M = S^1 \times N$ for a closed manifold N, then $\|M\|_1 = 0$.*

These properties of the simplicial volume help us understand the simplicial norm of certain homology classes.

Lemma 1.15. *For $n \ge 1$, if a homology class $\sigma \in H_n(X; \mathbb{R})$ is represented by a sphere, i.e. there is a map $f : S^n \to X$ with $f_*[S^n] = \sigma$, then $\|\sigma\|_1 = 0$.*

Proof. By functoriality and the fact that spheres (of dimension at least one) have zero simplicial volume, $\|\sigma\|_1 = \|f_*[S^n]\|_1 \le \|S^n\|_1 = 0$. Thus $\|\sigma\|_1 = 0$. $\square$

Corollary 1.16. *For any X, the simplicial norm $\| \cdot \|_1$ vanishes on $H_1(X; \mathbb{R})$.*

Proof. Basically, every 1-cycle is a bunch of circles and thus this should follow from Lemma 1.15. To make it precise, we use the approximation by rational cycles from Lemma 1.5 to reduce the problem to integral cycles, which is a standard trick in these topics.

By the second part of Lemma 1.5, it suffices to show that $\|\sigma\|_1 = 0$ for all rational homology classes $\sigma \in H_n(X; \mathbb{R})$. Any such σ is represented by some rational cycle c, and up to scaling, it suffices to consider the case where c is integral, i.e. $c = \sum_i n_i c_i$ for some $n_i \in \mathbb{Z} \setminus \{0\}$. Up to changing the orientation on c_i we may assume $n_i > 0$.

Now create n_i disjoint oriented segments for each c_i for all i. The fact that $\partial c = 0$ implies that we can pair the boundary points of these segments so that the endpoint of a segment s is always paired with the starting point of some segment s' so that the corresponding paths glue up in X respecting the orientations. The end result is a closed oriented 1-manifold, i.e. a disjoint union of

finitely many oriented circles S_k^1 indexed by k. In other words, there is a map $\varphi : \sqcup_k S_k^1 \to X$ such that $\varphi_* \sum_k [S_k^1] = \sigma$. Hence by Lemma 1.15 and the triangle inequality,

$$\|\sigma\|_1 \leq \sum_k \|\varphi_*[S_k^1]\|_1 = 0,$$

so $\|\sigma\|_1 = 0$ as desired. $\square$

1.3. **Volumes of surfaces.** In this section we aim to obtain the first nontrivial examples. We have seen that the simplicial norm is boring on H_0 and vanishes on H_1. Interesting examples emerge in H_2. For orientable connected closed surfaces, we have seen in Example 1.14 that the simplicial volume vanishes when the genus is zero or one. For surfaces of higher genus, the simplicial volume is nonzero and is proportional to the Euler characteristic.

Theorem 1.17. *For any orientable connected closed surface S of genus at least two, we have $\|S\|_1 = -2\chi(S)$.*

Remark 1.18. *Note that by Gauss–Bonnet, for any hyperbolic metric, S has area $-2\pi\chi(S) = \pi\|S\|_1$, so the simplicial volume is proportional to the hyperbolic volume. The factor π is the area of the ideal hyperbolic triangle, or equivalently, the supremum of areas of all hyperbolic triangles (ideal or not). We will generalize this to higher dimension, which is referred to as Gromov's proportionality theorem.*

To combine the results for all genera, it is convenient to introduce the following χ^- notation.

Notation 1.19. For an orientable connected compact surface S, let $\chi^-(S) = \chi(S)$ if $\chi(S) \leq 0$ and let $\chi^-(S) = 0$ otherwise, i.e. we adjust $\chi(S)$ to 0 when S is a sphere or a disk. For a general orientable compact surface $S = \sqcup\Sigma_i$ with components, let $\chi^-(S) := \sum \chi^-(\Sigma_i)$. In other words, $\chi^-(S)$ is the Euler characteristic of S after deleting all components homeomorphic to spheres or disks.

Then the following theorem easily follows from Theorem 1.17 and the case of the sphere and torus.

Theorem 1.20. *For any orientable closed surface S, we have $\|S\|_1 = -2\chi^-(S)$.*

We will prove Theorem 1.17 by establishing inequalities in both directions, which involve two different kinds of ideas.

The strategy for proving $\|S\|_1 \leq -2\chi(S)$ is to construct a sequence of cycles representing the fundamental class $[S]$ approaching the optimal value. As we explained earlier, one concrete way to represent the fundamental class is to triangulate S and take the formal sum of triangles with compatible orientations.

Suppose S has a triangulation with v vertices, e edges and f faces, we know $\chi(S) = v - e + f$. The cycle described above has norm f.

Lemma 1.21. *We have $2e = 3f$, so $\chi(S) = v - \frac{f}{2}$ and $f = 2v - 2\chi(S)$. Hence $\|S\|_1 \leq 2v - 2\chi(S)$.*

Proof. Each triangle has 3 edges, each of which is shared by two triangles. $\square$

So this is close to be optimal except for the error $2v$. The best one can do here is to take a triangulation with $v = 1$, which exists.

Exercise 1.22. *For any occ surface S, there is a triangulation with a single vertex.*

The bounded error can be remedied by taking finite covers.

Lemma 1.23. *If S has genus at least one, then $\|S\|_1 \leq -2\chi(S)$.*

Proof. For any such S and any $d \in \mathbb{Z}_+$, there is a degree d cover $f : S' \to S$. Then by taking a triangulation on S' with a single vertex, we have $\|S'\| \leq$ by Lemma 1.21. Note that both χ and $\| \cdot \|_1$ are multiplicative, i.e. $d\chi(S) = \chi(S')$ and $d\|S\|_1 = \|S'\|_1$ (by Lemma 1.11). Thus we obtain

$$\|S\|_1 = \frac{\|S'\|_1}{d} \leq \frac{2 - 2\chi(S')}{d} = \frac{2 - 2d\chi(S)}{d} = \frac{2}{d} - 2\chi(S).$$

Taking $d \to \infty$, we obtain the desired inequality. $\square$

We considered above all possible ways of representing (resp. a multiple of) the fundamental class using triangulations (resp. of a finite cover).

The reversed inequality uses a technique called "straightening", which involves hyperbolic geometry. Roughly speaking, every hyperbolic triangle has area no greater than π, and the hyperbolic area of the surface is $-2\pi\chi(S)$, so one needs at least $-2\chi(S)$ triangles to cover the entire surface once. So we just need an argument to straighten an arbitrary singular cycle representing the fundamental class into one only involving hyperbolic triangles. We will explain this in more detail (Section 1.5) after a crash course on hyperbolic geometry (1.4).

1.4. **Some hyperbolic geometry.** We give a quick introduction/review of some hyperbolic geometry, mainly to describe geodesics and isometries. A more detailed treatment can be found in [BP92], [Thu97], or any standard textbook/notes on hyperbolic geometry.

The n-dimensional hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) is the unique (up to isometry) simply connected complete Riemannian manifolds with constant sectional curvature -1. There are several models for $\mathbb{H}^n$, providing different ways to view the space.

1.4.1. *The hyperboloid model.* Consider the bilinear form $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$ on $\mathbb{R}^{n+1}$ for any $x, y \in \mathbb{R}^{n+1}$. The set $H := \{x \mid \langle x, x \rangle = -1\}$ is a hyperboloid of two sheets. The restriction of $\langle \cdot, \cdot \rangle$ on either component gives a complete Riemannian metric of constant curvature -1. The upper sheet H_+ (i.e. with $x_{n+1} > 0$) is the hyperboloid model of $\mathbb{H}^n$.

With this model, the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is identified with $\mathrm{O}^+(n, 1)$, the group of linear transformations preserving the bilinear form $\langle \cdot, \cdot \rangle$ and stabilizing the upper sheet. The isometry group acts simply transitively on the orthonormal frame bundle, i.e. given any two points $x, y \in \mathbb{H}^n$ and two orthonormal bases (i.e. two orthonormal frames) at the two points, there is a unique isometry taking the frame at x to the frame at y.

An advantage of this model is that, any k-dimensional totally geodesic subspace of $\mathbb{H}^n$ is the intersection with some linear subspace of dimension $k + 1$. In particular, bi-infinite geodesics are intersections with planes through the origin.

The linearity provides a way to take *convex combinations* of points. More precisely, given k points $p_1, \cdots, p_k \in \mathbb{H}^n = H_+ \subset \mathbb{R}^{n+1}$, any coefficients $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$ uniquely determine a point $p(\lambda_1, \cdots, \lambda_k) \in \mathbb{H}^n$ as the intersection of H_+ with the segment connecting the origin with $\sum_{i=1}^k \lambda_i p_i \in \mathbb{R}^{n+1}$. Apparently $p(\lambda_1, \cdots, \lambda_k)$ depends continuously on the coefficients λ_i, so this defines a continuous map $p : \Delta^{k-1} \to \mathbb{H}^n$, where $\Delta^{k-1} = \{(\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$ is the standard $(k-1)$-simplex. This gives a way to straighten singular simplices in $\mathbb{H}^n$: For any $c : \Delta^{k-1} \to \mathbb{H}^n$, let $p_1, \cdots, p_k$ be the image of the k vertices in the natural order, and let $\widetilde{\mathrm{str}}(c)$ be the map p defined above. This operation has the following properties which we record for reference later:

Lemma 1.24.

(1) If c_1 and c_2 agree on some face of Δ^{k-1}, then so do $\widetilde{\mathrm{str}}(c_1)$ and $\widetilde{\mathrm{str}}(c_2)$.
(2) For any isometry $g \in \mathrm{Isom}\mathbb{H}^n$, we have $\widetilde{\mathrm{str}}(g \circ c) = g \circ \widetilde{\mathrm{str}}(c)$.

Proof. The first part follows from the construction. The second part holds since the isometries in this model are linear maps, which commute with both taking convex combinations and scaling. $\square$

The straightening operation that we will introduce in Section 1.5 relies on this construction.

1.4.2. *The Poincaré ball model.* In this model, $\mathbb{H}^n$ is identified with the open unit disk $\mathbb{D}^n \subset \mathbb{R}^n$ with the metric $\frac{4ds^2}{(1-\|x\|_2^2)^2}$ (at any $x \in \mathbb{D}^n$), where ds^2 is the Euclidean metric. So the metric gets more distorted in this model compared to the Euclidean one when x is closer to the boundary.

In this model, geodesics are circular arcs perpendicular to the boundary sphere. The isometry group consists of *Möbius transformations* that preserve the unit disk.

Definition 1.25. A self-diffeomorphism f of $S^n = \mathbb{R}^n \cup \{\infty\}$ is a Möbius transformation if one of the following equivalent descriptions holds:

(1) f is a composition of inversions and reflections;
(2) f is conformal (i.e. angle preserving);
(3) f takes round spheres and hyperplanes to round spheres and hyperplanes;
(4) f is a Euclidean similarity possibly composed with an inversion, i.e. $f(x) = \lambda A i(x) + b$, where i is either the identity or an inversion, $A \in O(n)$, $\lambda > 0$, and $b \in \mathbb{R}^n$.

Here an *inversion* with respect to a round sphere $S(p, r)$ in R^n centered at p of radius r is the map on $S^n = \mathbb{R}^n \cup \{\infty\}$ given by $i(x) = p + \frac{x-p}{\|x-p\|} \cdot \frac{r^2}{\|x-p\|}$, which fixes $S(p, r)$ pointwise, swaps p and ∞, and preserves all rays from p so that $\|x - p\| \cdot \|i(x) - p\| = r^2$. The equivalence in the definition above (when $n \geq 3$) essentially follows from Liouville's theorem:

Theorem 1.26 (Liouville)**.** *A conformal diffeomorphisms f between two open subsets of $\mathbb{R}^n$ with $n \geq 3$ takes the form $f(x) = \lambda A i(x) + b$, where i is either the identity or an inversion, $A \in O(n)$, $\lambda > 0$, and $b \in \mathbb{R}^n$.*

See [BP92, Theorem A.3.7] for a detailed proof. A more geometric argument can be found here: https://lamington.wordpress.com/2013/10/28/liouville-illiouminated/.

This is a good model to talk about the boundary at infinity of $\mathbb{H}^n$, denoted $\partial \mathbb{H}^n$. Although $\partial \mathbb{H}^n$ is not part of the hyperbolic space $\mathbb{H}^n$, it compactifies the space (i.e. $\overline{\mathbb{H}}^n := \mathbb{H}^n \cup \partial \mathbb{H}^n$) and helps us understand geodesics and isometries. In this model, the boundary is exactly the unit sphere S^{n-1} and the compactification $\overline{\mathbb{H}}^n$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}^n$. Each bi-infinite geodesic naturally has two endpoints on the boundary which uniquely determines the geodesic. Each isometry extends to a homeomorphism on $\overline{\mathbb{H}}^n$ and in particular acts on the boundary (and the action is 2-transitive).

1.4.3. *The upper-half space model.* This model identifies $\mathbb{H}^n$ with the open upper-half space $\{x \in \mathbb{R}^n \mid x_n > 0\}$ equipped with the metric $\frac{ds^2}{x_n^2}$, where ds^2 is the Euclidean metric. Geodesics in this model are vertical lines and circular arcs perpendicular to the hyperplane $\{x_n = 0\}$. Isometries are Möbius transformations preserving the upper-half space. The boundary can be seen as the union of the hyperplane $\{x_n = 0\}$ with ∞.

1.4.4. *Isometries.* There is a classification of orientation-preserving isometries. Any isometry extends to a continuous homeomorphism on $\overline{\mathbb{H}}^n$, which is topologically a closed ball. Thus by Brouwer's fixed point theorem, each isometry must fix some point in $\overline{\mathbb{H}}^n$. Given $g \neq id \in \text{Isom}^+(\mathbb{H}^n)$, there are three mutually exclusive cases:

(1) g is *elliptic* if it fixes some point in $\mathbb{H}^n$. Up to conjugation, we may assume that g fixes the origin 0 in the disk model, in which case g is conjugate to an orthogonal transformation in $SO(n)$, determined by its action on the unit tangent space at 0.
(2) g is *parabolic* if it has no fixed point in $\mathbb{H}^n$ and has a unique fixed point in $\partial \mathbb{H}^n$. Up to conjugation, we may assume that g fixes ∞ in the upper-half space model, in which case we can deduce that g is conjugate to a horizontal (i.e. preserving x_n) translation on $\mathbb{R}^n$.

(3) g is *hyperbolic* if it has no fixed point in $\mathbb{H}^n$ and has two fixed points in $\partial\mathbb{H}^n$. Up to conjugation, we may assume that g fixes ∞ and 0 in the upper-half space model. Then g is the composition of a scaling with an orthogonal transformation in $SO(n-1) \subset SO(n)$ centered at the origin fixing the x_n axis. This axis is the unique bi-infinite geodesic preserved by g, called the *axis* of g.

1.5. **Straightening.** Let M^n be a hyperbolic manifold. We introduce a linear map $\mathrm{str} : C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$, called *straightening*. Fix the universal covering map $\pi : \mathbb{H}^n \to M$. For any singular simplex $c : \Delta^k \to M$, pick any lift $\tilde{c} : \Delta^k \to \mathbb{H}^n$, which we straighten to $\widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \to \mathbb{H}^n$ using the construction described in Section 1.4.1. Define the straightening of c as $\mathrm{str}(c) := \pi \widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \to M$, which is independent of the choice of the lift since $\widetilde{\mathrm{str}}$ commutes with isometries. This extends to a linear map $\mathrm{str} : C_k(M;\mathbb{R}) \to C_k(M;\mathbb{R})$.

Lemma 1.27. *We have* $|\mathrm{str}(c)|_1 \leq |c|_1$ *for any chain c and* $\partial\mathrm{str} = \mathrm{str}\partial$. *Moreover,* str *induces the identity map on* $H_*(M;\mathbb{R})$.

Proof. The straightening map does not increase the number of simplices, so $|\mathrm{str}(c)|_1 \leq |c|_1$ for any chain c. It commutes with the boundary map by construction.

For any singular simplex $c : \Delta^k \to \mathbb{H}^n$, there is an obvious linear homotopy in $\mathbb{R}^{n+1}$ from c to $\widetilde{\mathrm{str}}(c)$, which scales to one on $\mathbb{H}^n$ and projects down to a homotopy on M. Based on this, one can build a chain homotopy between str and id, or directly observed that $[\mathrm{str}(c)] = [c]$ for any cycle c. $\square$

It follows that, when computing the simplicial norm of any homology class $\sigma \in H_*(M;\mathbb{R})$ it suffices to look at cycles consisting of (straight) hyperbolic simplices. A key fact about hyperbolic simplices is that their volume has a uniform upper bound only depending on the dimension, in contrast with Euclidean simplices.

Lemma 1.28 ([BP92, Theorem C.2.1 and Lemma C.2.3])**.** *For each $n \geq 2$, let v_n be the supremum of volumes of all hyperbolic n-simplices (possibly with vertices at infinity). Then $v_2 = \pi$ and $v_n \leq \frac{\pi}{(n-1)!}$.*

Remark 1.29. *A theorem of Haagerup–Munkholm[HM81] shows that v_n is achieved uniquely by the regular[1] hyperbolic ideal n-simplex.*

Proof. Any hyperbolic simplex has volume no more than some ideal hyperbolic simplex. In fact, for any hyperbolic simplex with vertices $p_0, \ldots, p_n \in \mathbb{H}^n$, choose a point p in its interior. The geodesic rays from p to p_i determines a point $p_i' \in \partial\mathbb{H}^n$ for each i. The ideal hyperbolic simplex with vertices $p_0', \cdots, p_n'$ contains the starting one.

When $n = 2$, all ideal hyperbolic triangles are conjugate up to an isometry and have the same area π, so $v_2 = \pi$. The area can be computed explicitly in the upper-half space model, say for the ideal triangle with vertices $-1, 1, \infty$, or can be seen by Gauss–Bonnet.

For $n \geq 3$, we show $v_n \leq \frac{v_{n-1}}{n-1}$ by a nice computation in the upper-half space model following [BP92, Lemma C.2.3], which implies the bound $\frac{\pi}{(n-1)!}$ by induction. Let σ be any ideal hyperbolic n-simplex in the upper-half space model and put one of its vertex as ∞. The remaining n vertices form an ideal hyperbolic $(n-1)$-simplex τ sitting on a totally geodesic subspace X of dimension $n-1$ not containing ∞ as a boundary point. So X is a round hemisphere centered at some $p \in \mathbb{R}^{n-1} \subset \partial\mathbb{H}^n$. Up to an isometry we may assume $p = 0$ and the sphere has radius 1. So X is the upper unit hemisphere. The vertical projection to $\mathbb{R}^{n-1}$ restricts to a homeomorphism from X to the unit closed ball in $\mathbb{R}^{n-1}$, and let s be the inverse. Explicitly, $s(x) = (x, h(x))$ where $h(x) = \sqrt{1 - \|x\|^2}$.

---

[1]completely symmetric in the sense that the isometry group is the symmetric group on the $n+1$ vertices

Let τ_0 be the image of τ under this projection, which is a Euclidean simplex with vertices on the unit sphere. Then

$$\mathrm{vol}(\sigma) = \int_{\tau_0} \int_{h(x)}^{\infty} \frac{dy\,dx}{y^n}$$

$$= \frac{1}{n-1} \int_{\tau_0} \frac{dx}{h(x)^{n-1}}.$$

It suffices to show that

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \leq \mathrm{vol}(\tau).$$

Note that s gives a way to parameterize τ using τ_0, so we just need to show that the pullback of the volume form is dominates $\frac{dx}{h(x)^{n-1}}$ for each x in the unit ball, where we think of dx as the standard volume form on $\mathbb{R}^{n-1}$.

For any x, let ω be the hyperbolic volume form restricted to $T_{s(x)}X$. Note that ω evaluates to $\frac{1}{h(x)^{n-1}}$ for any orthonormal basis of $T_{s(x)}X$. To find out its pullback, choose an orthonormal basis of $T_x\mathbb{R}^{n-1}$. If $x = 0$, then $Dh = 0$ so s_* is the identity map and $s^*\omega = \frac{dx}{h(x)^{n-1}}$. If $x \neq 0$, we may choose the orthonormal basis so that one of them is $e_1 = \frac{x}{\|x\|}$. Then $Dh = 0$ in all directions perpendicular to e_1 and $Dh(e_1) = \frac{\|x\|}{h(x)}$, so s_* is the identity map in the subspace perpendicular to e_1, and it takes this orthonormal basis to an orthogonal basis where all elements have length 1 except that $\|s_*(e_1)\|^2 = 1 + \frac{\|x\|^2}{1-\|x\|^2} = \frac{1}{1-\|x\|^2} = \frac{1}{h(x)^2}$. Thus $s^*\omega = \frac{1}{h(x)} \cdot \frac{dx}{h(x)^{n-1}} \geq \frac{dx}{h(x)^{n-1}}$ since $h(x) \leq 1$. This verifies that the pullback of the volume form by s dominates $\frac{dx}{h(x)^{n-1}}$, and hence

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \leq \mathrm{vol}(\tau) \leq v_{n-1}.$$

$\square$

Lemma 1.30. *Let M^n be an oriented closed connected hyperbolic manifold. Then $\|M\|_1 \geq \frac{\mathrm{vol}(M)}{v_n}$.*

Proof. Let $c = \sum \lambda_i c_i$ be a cycle representing $[M]$. By Lemma 1.27, we may assume that c consists of straight hyperbolic simplices without increasing $\|c\|_1$. Let vol be the volume form. Then we have

$$\mathrm{vol}(M) = \langle [M], \mathrm{vol} \rangle = \left\langle \sum \lambda_i c_i, \mathrm{vol} \right\rangle \leq \sum |\lambda_i| \cdot \max \mathrm{vol}(c_i) \leq |c|_1 \cdot v_n.$$

Sine c is arbitrary, we conclude that $\mathrm{vol}(M) \leq v_n \cdot \|M\|_1$. $\square$

Remark 1.31. *Conceptually, we obtained this lower bound by some sort of ℓ^1–ℓ^∞ duality, where we used a cocycle (the volume form here) that is bounded on all straight hyperbolic simplices. This suggests the use of bounded cocycles and bounded cohomology as a dual theory to better understand simplicial norms.*

Restricting to the case $n = 2$, we can now finish the proof of Theorem 1.17.

Proof of Theorem 1.17. The above lemma for $M = S$ a hyperbolic surface, we have

$$\|S\|_1 \geq \frac{\mathrm{area}(S)}{v_2} = \frac{-2\pi\chi(S)}{\pi} = -2\chi(S)$$

by Gauss–Bonnet. Combining with Lemma 1.23, we conclude $\|S\|_1 = -2\chi(S)$. $\square$

The bound in Lemma 1.30 is also sharp in higher dimensions; see Theorem 1.36. This geometric argument also works for closed manifolds with varying negative curvature, so one can deduce that any such manifold has positive simplicial volume. More generally, we have the following conjecture attributed to Gromov [Gro82, p.11]:

Conjecture 1.32. *Any closed manifold of non-positive curvature and negative Ricci curvature has* $\|M\|_1 > 0$.

This is still open. See [CW19] for a recent partial positive answer that uses a slightly different straightening and uniformly bounds the Jacobian of the straightened maps.

Similarly, this argument can be used to prove positivity of simplicial norms of other homology classes.

Proposition 1.33. *The simplicial norm is an honest norm (instead of a semi-norm) on $H_k(M; \mathbb{R})$ for any $k \geq 2$ and hyperbolic closed orientable manifold M. That is, $\|\sigma\|_1 > 0$ for any $\sigma \neq 0 \in H_k(M; \mathbb{R})$.*

Proof. For any $\sigma \neq 0$, there is $\sigma^* \in H^k(M; \mathbb{R})$ that pairs nontrivially with σ. Represent σ^* by a differential k-form ω. Up to scaling we may assume that $|\omega(V)| \leq 1$ for all k orthogonal vectors V of norm 1 at any point on M. As a result, the restriction of ω on any straight hyperbolic k-simplex is bounded by the volume form. Thus for any straightened chain $\sum \lambda_i c_i$ representing σ, we have

$$|\langle \sigma, \omega \rangle| \leq \max \operatorname{vol}(c_i) \cdot \sum \|\lambda_i\| \leq v_k \cdot |c|_1.$$

It follows that

$$\|\sigma\|_1 \geq \frac{|\langle \sigma, \omega \rangle|}{v_k} > 0.$$

$\square$

1.6. Application: degrees of maps between surfaces.
Now we apply the calculation in Theorem 1.20 to solve Problem 1.1 as an application.

Theorem 1.34. *Given closed orientable connected surfaces S and S' that are not spheres, the set $\deg(S, S')$ of all possible degrees of maps $f : S \to S'$ is*

$$\deg(S, S') = \{d : |d \cdot \chi(S')| \leq |\chi(S)|\}.$$

Proof. By the degree inequality Lemma 1.11, we have $\deg(f) \cdot \|S'\|_1 \leq \|S\|_1$ for any map $f : S' \to S$. Since spheres are excluded, we know $\|S\|_1 = -2\chi(S)$ and $\|S'\|_1 = -2\chi(S')$. It follows that $|d \cdot \chi(S')| \leq |\chi(S)|$ for any $d \in \deg(S, S')$.

It suffices to show that any such d can be realized as $\deg(f)$ for some $f : S \to S'$. We restrict our attention to those $d > 0$ since $d = 0$ is realized by a constant map and those negative values can be obtained by composing with a orientation-reversing homeomorphism. Given any $d > 0$ with $|d \cdot \chi(S')| \leq |\chi(S)|$, there is a degree d cover $p : S'_d \to S'$. Then $|\chi(S'_d)| = |d\chi(S')| \leq |\chi(S)|$, i.e. S'_d has genus no more than that of S. So there is a map $g : S \to S'_d$ with $\deg(g) = 1$ by pinching part of S to a point. Thus the composition $f = gp$ has $\deg(f) = \deg(g) \cdot \deg(p) = d$. $\square$

Exercise 1.35.
   (1) If $S = S^2$, show that $\deg(S, S') = \{0\}$ unless $S' = S^2$, in which case $\deg(S, S') = \mathbb{Z}$.
   (2) If $S' = S^2$, show that $\deg(S, S') = \mathbb{Z}$.

1.7. Gromov's proportionality.
The goal of this section is to prove the following theorem, showing that the lower bound of $\|M\|_1$ in Lemma 1.30 is sharp:

Theorem 1.36 (Gromov's Proportionality). *Let M^n be an oriented closed connected hyperbolic manifold. Then $\|M\|_1 = \frac{\operatorname{vol}(M)}{v_n}$.*

Note that if for some $\epsilon > 0$ we have a cycle $c = \sum \lambda_i c_i$ with $\lambda_i > 0$, each c_i straight hyperbolic and positively oriented with $\operatorname{vol}(c_i) \geq v_n - \epsilon$, then c represents $\lambda[M]$ for some $\lambda > 0$ and

$$(1.1) \quad \lambda \operatorname{vol}(M) = \langle \lambda[M], \operatorname{vol} \rangle = \sum \lambda_i \operatorname{vol}(c_i) \geq \min\{\operatorname{vol}(c_i)\} \cdot \sum \lambda_i \geq (v_n - \epsilon)|c|_1 \geq \lambda(v_n - \epsilon)\|M\|_1.$$

Thus it suffices to show the existence of such a cycle for any $\epsilon > 0$.

There is no obvious construction of such a cycle. However, if we were allowed to have *infinitely many* terms in a cycle, there is a natural construction. A formal way to make this work is to define Gromov's norm using *measure homology* instead of singular homology. That is, on the set $S_k(X)$ of singular k-simplices in X, let $\mathcal{C}_k(X)$ be the space of signed measures with compact support and bounded total variation, equipped with the total variation norm. This gives a chain complex with boundary maps defined in the usual way and induces a semi-norm on its homology (called measure homology). Zastrow and Hansen both showed that measure homology coincides with singular homology for CW complexes [Han98, Zas98], and Löh showed that the isomorphism is isometric [Lö6], thus this is an equivalent way to define Gromov's norm as Thurston originally claimed [Thu, Chapter 6].

The advantage is that this gives more room to construct the desired cycle in the measure sense. In fact, any singular chain $\sum \lambda_i c_i$ can be thought of as the signed measure $\sum \lambda_i \delta_{c_i}$, where δ_{c_i} is the Dirac mass at c_i. So in this way $C_k(X)$ is a subspace of $\mathcal{C}_k(X)$.

Instead of giving a rigorous introduction of measure homology and proving the equivalence, we briefly describe this measure cycle and then approximate it using an honest singular cycle to finish the proof as in [Cal, Section 3.1]. For any $\epsilon > 0$, let Δ be a straight positively oriented n-simplex in $\mathbb{H}^n$ so that $\mathrm{vol}(\Delta) > v_n - \epsilon$. Let $D(\Delta)$ be the space of all isometric orientation-preserving embeddings of Δ in $\mathbb{H}^n$, which can be identified as $\mathrm{Isom}^+(\mathbb{H}^n)$ and comes with the Haar measure. Then $D(\Delta)/\pi_1(M)$ can be thought of as all isometric positively oriented copies of Δ in M, with the induced measure. This makes it into a measure chain $\mathrm{smear}(\Delta)$ in $\mathcal{C}_n(M)$. Similarly, let $\bar{\Delta}$ be the reflection of Δ across some face. The same construction provides a measure chain $\mathrm{smear}(\bar{\Delta})$ in $\mathcal{C}_n(M)$. The key fact is that $\mathrm{smear}(\Delta) - \mathrm{smear}(\bar{\Delta})$ is a measure *cycle*, since every face of Δ is the face of its reflection across this face, and this reflection is a translate of $\bar{\Delta}$. Each copy of $\bar{\Delta}$ has the opposite orientation, but the negative sign "corrects" it.

For the approximation, we need Δ to be chosen with a stronger property in the beginning. It is a fact that, for any $\epsilon > 0$ and any $C > 0$, there is Δ such that $\mathrm{vol}(\Delta') > v_n - \epsilon$ for any Δ' obtained from Δ by moving each vertex by a distance no more than C.

Proof of Theorem 1.36. Let $F \subset \mathbb{H}^n$ be a compact fundamental domain of M and fix $p \in F$. For any $\epsilon > 0$ and $C = \mathrm{diam} F$, let Δ be chosen as above. For any $(n+1)$-tuple $\vec{g} = (g_0, \cdots, g_n) \in \pi_1(M)^{n+1}$, let $c_{\vec{g}}$ be the straight hyperbolic n-simplex with vertices $(g_0 p, \cdots, g_n p)$, which is an approximation of those positively-oriented isometric embeddings of Δ or $\bar{\Delta}$ whose vertices lie in $g_0 F, \cdots, g_n F$ respectively, if exist. Let $\lambda_{\vec{g}}$ (resp. $\bar{\lambda}_{\vec{g}}$) be the measure of such embeddings of Δ (resp. $\bar{\Delta}$). Note that both $\lambda_{\vec{g}}$ and $\bar{\lambda}_{\vec{g}}$ are $\pi_1(M)$ invariant under the diagonal action of $\pi_1(M)$ on $\pi_1(M)^{n+1}$. Let $\pi : \mathbb{H}^n \to M$ be the universal covering map. Then set the following chains in $C_n(M; \mathbb{R})$

$$c = \sum_{\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)} \lambda_{\vec{g}} \pi c_{\vec{g}} \quad \text{and} \quad \bar{c} = \sum_{\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)} \bar{\lambda}_{\vec{g}} \pi c_{\vec{g}}$$

to approximate $\mathrm{smear}(\Delta)$ and $\mathrm{smear}(\bar{\Delta})$ respectively.

Claim 1.37.

(1) $\lambda_{\vec{g}} = 0$ for all but finitely many $\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)$, i.e. c is a singular chain, and so is c'.

(2) If $\lambda_{\vec{g}} > 0$ (resp. $\bar{\lambda}_{\vec{g}} > 0$), then $\mathrm{vol}(c_{\vec{g}}) > v_n - \epsilon$.

(3) For ϵ small enough, if $\lambda_{\vec{g}} > 0$ (resp. $\bar{\lambda}_{\vec{g}} > 0$), then $c_{\vec{g}}$ is positively oriented (resp. negatively oriented).

(4) $c - \bar{c}$ is a cycle.

Proof.

(1) Up to the $\pi_1(M)$ action, we may assume $g_0 p = p \in F$. Then the space of isometric embeddings of Δ with the first vertex inside F is compact, so the remaining n vertices can only lie in finitely many possible gF.

(2) This means that $c_{\vec{g}}$ can be obtained from some embedding of Δ (or $\bar{\Delta}$) by moving each vertex by distance no more than $C = \mathrm{diam} F$. So the assertion follows from our choice of Δ.

(3) No Δ can be close to $\bar{\Delta}$ when they have large enough volume.

(4) Any face τ of $c_{\vec{g}}$ is a straight $(n-1)$-simplex with vertices $(h_1 p, \cdots, h_n p)$ for some $h_i \in \pi_1(M)$. If a face of Δ has an orientation-preserving isomeric embedding with vertices in $h_1 F, \cdots, h_n F$, then this contributes to the coefficient of τ in ∂c by the measure of the total measure of all such isometric embeddings of Δ. For each such embedding, its reflection across τ is an orientation-preserving isometric embedding of Δ, so this has the same contribution to the coefficient of τ in $\partial \bar{c}$. Since this holds for all faces τ, we observe that $\partial(c - \bar{c}) = 0$.

$\square$

By the claim above, all nontrivial terms in c' are negatively oriented, which can be made positively oriented by adding a negative sign. Thus the cycle $c - c'$ can be expressed as a positive linear combination of positively oriented straight simplices of volume at lest $v_n - \epsilon$, this implies $\mathrm{vol}(M) \geq (v_n - \epsilon)\|M\|_1$ by equation (1.1). Since ϵ can be arbitrarily small, this proves $\frac{\mathrm{vol}(M)}{v_n} \geq \|M\|_1$ and completes the proof in combination with Lemma 1.30. $\square$

The smearing operation in the measure homology setup does something more [Thu, Chapter 6]. If M and N are oriented closed oriented Riemannian manifolds with isometric universal cover X, given a singular k-simplex $c : \Delta \to M$, we can similarly do the "smearing" operation by first lifting it to the universal cover, taking all translates of it, and projecting to N. Equipped with the natural measure from $\mathrm{Isom}^+(X)$, this gives a chain in $\mathcal{C}_k(N)$. This extends to a linear map $\mathrm{smear}_{M,N} : \mathcal{C}_k(M) \to \mathcal{C}_k(N)$. This has the property that $\mathrm{smear}_{M,N}[M] = \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot [N]$ and implies

$$\frac{\|N\|_1}{\mathrm{vol}(N)} \leq \frac{\|M\|_1}{\mathrm{vol}(M)}.$$

Flipping the roles of M and N implies the following more general version of Gromov's proportionality:

Theorem 1.38 (Gromov's Proportionality, general case). *Suppose M and N are orientable closed Riemannian manifolds with isometric universal covers, then*

$$\frac{\|M\|_1}{\mathrm{vol}(M)} = \frac{\|N\|_1}{\mathrm{vol}(N)}.$$

It is unclear in general though how this constant proportion is related to the geometry of the universal cover.

2. Bounded cohomology

Bounded cohomology is a dual theory of homology with simplicial norm, and we had a glimpse of its power in the estimate of simplicial volume of hyperbolic manifolds. It is also interesting on its own, providing new invariants sometimes quite different from ordinary cohomology. We will first focus on (bounded) cohomology of groups for concreteness. We will see later in Gromov's mapping theorem that the bounded cohomology of a connected space X is canonically isomorphic to the bounded cohomology of $\pi_1(X)$, so we are not losing any information.

2.1. **Group cohomology.** We start with ordinary cohomology of groups as a warm-up. A good detailed reference on this topic is Brown's book [Bro82].

2.1.1. $K(G,1)$ *spaces.* Given a group G, a $K(G,1)$ *space* X_G is a connected *aspherical* CW complex with $\pi_1(X_G) = G$. Here being aspherical means that all higher homotopy groups of X_G vanish. Equivalently (by Whitehead's theorem), the universal cover of X_G is contractible. Such a space is unique up to homotopy equivalence, and is also called the *Eilenberg–Maclane space* or the *classifying space* of the group G (as a discrete group). The uniqueness follows from the universal property below:

Lemma 2.1. *Suppose X is a $K(G,1)$ space. Let Y be a connected CW complex with $\pi_1(Y) = H$. Any homomorphism $\varphi : H \to G$ is induced by a continuous map $f : Y \to X_G$, which is unique up to homotopy.*

Corollary 2.2. *Suppose X and Y are both $K(G,1)$ spaces. Then any map $f : X \to Y$ inducing an isomorphism $f_* : G \to G$ is a homotopy equivalence.*

Proof. Let $g_* : G \to G$ be the inverse of f_*. By the universal property, it is induced by a map $g : Y \to X$. Then $gf : X \to Y$ induces the identity map on $\pi_1(X)$ and hence is homotopic to the identity map on X by uniqueness. Similarly fg is homotopic to id_Y, so g is a homotopy inverse of f. $\qquad\square$

One way to build a $K(G,1)$ space is to start with a 2-complex with fundamental group G and add higher dimensional cells to kill all higher homotopy groups inductively. A similar idea can be used to prove the universal property above. See details in [Hat02] for instance.

One can use the $K(G,1)$ space to give a topological definition of the group (co)homology.

Definition 2.3. Given a ring R, the homology $H_*(G;R)$ (resp. cohomology $H^*(G;R)$) of G with R coefficients is $H_*(X_G;R)$ (resp. $H^*(X_G;R)$).

Example 2.4.
   (1) For $G = \mathbb{Z}$, we can take $X_G = S^1$. We know $H_k(G;\mathbb{Z}) = H_k(X_G;\mathbb{Z}) \cong \mathbb{Z}$ for $k = 0,1$. Since it is one-dimensional, we have $H_k(G;\mathbb{Z}) = H_k(X_G;\mathbb{Z}) = 0$ for $k \geq 2$.
   (2) For a free group $G = F_n$ with $n \geq 1$, we can take X_G as a graph (e.g. a wedge of circles). We have $H_0(G;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(G;\mathbb{Z}) = \mathbb{Z}^n$, and $H_k(G;\mathbb{Z}) = 0$ for $k \geq 2$ by dimension restrictions.
   (3) $H_0(G;R) = R$ since $K(G,1)$ space is connected by definition.
   (4) $H_1(G;R) = \mathrm{Ab}(G) \otimes_{\mathbb{Z}} R$.
   (5) $H^1(G;R) = \mathrm{Hom}(G,R)$.
   (6) For a cyclic group $G = \mathbb{Z}/m$, we can choose an infinite lens space as the $K(G,1)$ space, which gives $H_k(\mathbb{Z}/m;\mathbb{Z}) \cong \mathbb{Z}/m$ for all k odd, $H_k(\mathbb{Z}/m;\mathbb{Z}) = 0$ for $k > 0$ and even. See [Hat02, Example 2.43] for details in a similar computation.

It follows from the last example that no $K(\mathbb{Z}/m,1)$ space can be finite dimensional. As an application of this fact, we have:

Proposition 2.5. *For any nontrivial finite group G, there is no free action on $\mathbb{R}^n$.*

Proof. Suppose G acts freely. Let $H \cong \mathbb{Z}/m$ be a nontrivial cyclic subgroup. Then H also acts freely on $\mathbb{R}^n$. The action is also properly discontinuous since H is finite. Thus $\mathbb{R}^n/H$ is a $K(H,1)$ space as $\mathbb{R}^n$ is contractible. This is a contradiction. $\qquad\square$

2.1.2. *Aside: The co-Hopfian property.* The uniqueness of $K(G,1)$ spaces (up to homotopy) has interesting consequences in the context of closed manifolds.

Lemma 2.6. *If M and N are both connected aspherical n-manifolds (without boundary) with isomorphic fundamental group, then M and N are either both compact or both non-compact.*

Proof. Note that a connected n-manifold M (without boundary) is compact if and only if $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Since M and N are $K(G,1)$ spaces for the same G, they are homotopy equivalent and $H_n(M;\mathbb{Z}/2) \cong H_n(N;\mathbb{Z}/2)$, so they have the same compactness. $\qquad\square$

This is related to the *co-Hopfian* property of fundamental groups

Definition 2.7. A group G is co-Hopfian if every injective homomorphism $h : G \to G$ is an isomorphism.

Exercise 2.8. *Show that the following groups are not co-Hopfian.*

    (1) $G = \mathbb{Z}$.
    (2) G is a free group or free abelian group.
    (3) G is a free product $H \star K$ where K is not co-Hopfian.

Lemma 2.9. *If M is a closed connected aspherical manifold, then any subgroup of $\pi_1(M)$ that is isomorphic to $\pi_1(M)$ must have finite index. In particular, if $\pi_1(M)$ is not co-Hopfian and M is orientable, then M has a self-map f with $|\deg(f)| > 1$.*

Proof. Let H be a subgroup of $G = \pi_1(M)$ that is isomorphic to G. Let $\pi : M' \to M$ be the covering map corresponding to the inclusion $H \to G$. As M is aspherical, so is M'. Since M is closed, M' must be closed by Lemma 2.6, so π is a finite cover.

The isomorphism $\pi_1(M) \to H$ can be realized as a homotopy equivalence $\varphi : M \to M'$, which necessarily has $|\deg \varphi| = 1$. So the composition $f = \pi\varphi$ is a self-map with $|\deg f| = |\deg \pi| > 1$ if H is a proper subgroup. $\qquad\square$

Corollary 2.10. *If M is an orientable closed connected aspherical manifold with $\|M\|_1 > 0$, then $\pi_1(M)$ is co-Hopfian.*

Proof. As $\|M\|_1 > 0$, M cannot have a self-map f with $|\deg f| > 1$ by the degree inequality. $\qquad\square$

Asphericity is often deduced from geometry of the manifold:

Lemma 2.11. *A complete Riemannian manifold M with non-positive sectional curvature is aspherical. Moreover, the universal cover of M is diffeomorphic to the Euclidean space.*

Proof. For any $p \in M$, the exponential map $T_pM \to M$ is a covering map by the Cartan–Hadamard theorem. Hence T_pM is one realization of the universal cover. $\qquad\square$

Corollary 2.12. *If M is occ with negative sectional curvature, then $\pi_1(M)$ is co-Hopfian.*

Proof. Negative curvature implies $\|M\|_1 > 0$ by a straightening argument, so the assumptions of Corollary 2.10 are met. $\qquad\square$

Example 2.13. *Let S be an occ surface of genus $g > 1$. Then $\pi_1(S)$ is co-Hopfian.*

One can also prove this using the fact that infinite-index subgroups of $\pi_1(S)$ are free groups.

Exercise 2.14. *It is crucial to assume M to be aspherical in Corollary 2.10. Let $N = M\#(S^2 \times S^1)$ be a connected sum, where M is an occ hyperbolic 3-manifold. Show that $\pi_1(N)$ is not co-Hopfian although N is occ and has $\|N\|_1 > 0$.*

2.1.3. *The bar complex.* There is also an explicit and purely algebraic definition of group (co)-homology, which we will see to be equivalent. Consider the following *bar complex* with coefficients in R. Let $C_n(G; R)$ be the free R-module with basis consisting of n-tuples $(g_1, \cdots, g_n) \in G^n$, and let $C^n(G, R) = \mathrm{Hom}_R(C_n(G; R), R)$, where each element assigns a value in R to each element of G^n. The differential $\partial : C_n(G; R) \to C_{n-1}(G; R)$ is determined by
(2.1)

$$\partial(g_1, \cdots, g_n) := (g_2, \cdots, g_n) + \sum_{i=1}^{n-1}(-1)^i(g_1, \cdots g_{i-1}, g_ig_{i+1}, g_{i+2}, \cdots, g_n) + (-1)^n(g_1, \cdots, g_{n-1}).$$

It is standard to check that $\partial^2 = 0$ (which also follows from the topological explanation below), and one can define $H_*(G; R)$ as the homology of this chain complex. Similarly, take $\delta : C^{n-1}(G; R) \to C^n(G; R)$ as the dual of ∂ as usual, we can define $H^*(G; R)$ using the cochains $C^n(G; R)$.

This weird-looking differential (2.1) in the bar complex comes from the following topological interpretation. Let EG be the simplicial complex where n-simplices correspond to elements in G^{n+1} for each $n \geq 0$, where the faces of $(g_0, \cdots, g_n)$ are $(g_0, \cdots, \hat{g}_i, \cdots, g_n)$ (i.e. omitting g_i), $i = 0, \cdots, n$. Then EG is contractible since each simplicial map can be coned off, say by adding a fixed element $g \in G$ as the last coordinate. The group G acts by $g(g_0, \cdots, g_n) = (gg_0, \cdots, gg_n)$, which is a free action. Thus the quotient $BG := EG/G$ is a $K(G, 1)$ space.

One can think of an n-simplex as an equivalence class $[(g_0, \cdots, g_n)]$, where $(g_0, \cdots, g_n) \sim (g'_0, \cdots, g'_n)$ if $(g'_0, \cdots, g'_n) = (gg_0, \cdots, gg_n)$ for some $g \in G$. This leads to a way to define group (co)homology using the so-called homogeneous coordinates.

Instead, we represent each equivalence class $[(g_0, \cdots, g_n)]$ by an n-tuple $(g_0^{-1}g_1, g_1^{-1}g_2, \cdots, g_{n-1}^{-1}g_n)$, which is independent of the representative $(g_0, \cdots, g_n)$. This is called the *inhomogeneous coordinate*. Then $(g_1, \cdots, g_n)$ in inhomogeneous coordinate corresponds to the equivalence class $[(id, g_1, g_1g_2, \cdots, g_1g_2 \cdots g_n)]$ in homogeneous coordinate. Geometrically, one can think of the homogeneous coordinate as marking on vertices and the inhomogeneous coordinate as marking on edges. Under this identification, the boundary map in inhomogeneous coordinates is exactly the differential (2.1). This justifies that the (co)homology defined using bar complex is exactly the (simplicial) homology of the $K(G, 1)$ space BG, and thus agrees with our topological definition.

Under this setup, a k-cochain is an assignment $f : G^k \to R$, labeling each k-tuple in inhomogeneous coordinate an element in R and extending to an R-linear map on the space of k-chains.

Example 2.15. *A 0-cochain is a constant in R, and $\delta : C^0(G; R) \to C^1(G; R)$ is the zero map. Hence every 0-chchain is a cocycle and $H^0(G; R) = R$.*

For a 1-cochain $f : G \to R$, its coboundary is determined by $(\delta f)(g_1, g_2) = f\partial(g_1, g_2) = f(g_2 - g_1) = f(g_2) - f(g_1g_2) + f(g_1)$. So it is a cocycle if and only if f is a homomorphism to the abelian group R (forgetting the ring structure). This shows that $H^1(G; R) = \mathrm{Hom}(G, R)$.

2.2. Bounded cohomology of groups.

Roughly speaking, in bounded cohomology, we consider cochains that are bounded functions instead of arbitrary functions on G^k. For this to make sense, we need to measure the size of elements in R. For simplicity, we consider $R = \mathbb{R}$ or $\mathbb{Z}$, equipped with the usual absolute value.

Then a k-cochain f in inhomogeneous coordinates as a map $f : G^k \to R$ is bounded if

$$|f|_\infty := \sup |f(g_1, \cdots, g_k)| < \infty.$$

Equivalently, $|f|_\infty$ is the sup norm as a linear map on the space of k-chains.

For each k, let $C_b^k(G, R)$ be the subspace of $C^k(G; R)$ consisting of bounded cochains. Then the coboundary restricts to map $\delta_k : C_b^k(G; R) \to C_b^{k+1}(G; R)$ since $|\delta_k f|_\infty \leq (k+1)|f|_\infty$. This gives rise to a new chain complex.

Definition 2.16. (Bounded cohomology) Let $Z_b^n(G; R) := \ker \delta_n$ and $B_b^n(G; R) := \mathrm{Im}\, \delta_{n-1}$. The n-th *bounded cohomology* of G is $H_b^n(G; R) := Z_b^n(G; R)/B_b^n(G; R)$.

The norm $|\cdot|_\infty$ on $C_b^n(G; R)$ restricts to $Z_b^n(G; R)$ and induces a semi norm $\|\cdot\|_\infty$ on $H_b^n(G; R)$. Explicitly, for any $\alpha \in H_b^n(G; R)$,

$$\|\alpha\|_\infty = \inf_{[f]=\alpha} |f|_\infty.$$

It is natural to ask how bounded cohomology differs from ordinary cohomology. There is a natural map connecting them, by treating a bounded cocycle as an ordinary cocycle.

Definition 2.17. (Comparison map) The inclusion $C_b^n(G; R) \to C^n(G; R)$ induces a homomorphism $c : H_b^n(G; R) \to H^n(G; R)$ called the *comparison map*.

Example 2.18. *As shown in Example 2.15, for degree $n = 0$, a 0-cochain is a constant map and thus bounded. Thus $H_b^0(G; R) = H^0(G; R) = R$.*

For degree $n = 1$, a 1-cochain $f : G \to R$ is a cocycle if and only if it is a homomorphism. However, a homomorphism to $R = \mathbb{Z}, \mathbb{R}$ is always unbounded except the trivial one. Thus $H_b^1(G; R) = 0$ and the comparison map is not surjective in general.

2.3. **Quasimorphisms.** Nontrivial bounded cohomology classes emerge in degree $n = 2$. Here is the idea: For any $(n-1)$-cochain f, δf is always a cocycle, but it might happen that δf is bounded while f is not, in which case δf is a bounded n-cocycle potentially nontrivial. We will focus on $R = \mathbb{R}$ throughout this section and thus often omit the coefficient.

When $n = 2$, as we calculated in Example 2.15, for any $f : G \to \mathbb{R}$, we have $(\delta f)(g, h) = f(g) + f(h) - f(gh)$. So δf is bounded exactly when f is a quasimorphism, defined as follows.

Definition 2.19. (Quasimorphism) A map $\varphi : G \to \mathbb{R}$ is a *quasimorphism* if

$$D(\varphi) := \sup_{g,h \in G} |\varphi(g) + \varphi(h) - \varphi(gh)| < \infty.$$

The number $D(\varphi)$ is called the *defect* of φ. Quasimorphisms can be thought of as homomorphisms with bounded error, measured by the defect.

A quasimorphism φ is *homogeneous* if $\varphi(g^n) = n\varphi(g)$ for all $g \in G$ and all $n \in \mathbb{Z}$, i.e. φ restricts to a homomorphism on every cyclic subgroup.

Note that quasimorphisms on G form an $\mathbb{R}$-vector space, which we denote as $\widehat{Q}(G)$. Homogeneous quasimorphisms form a linear subspace, denoted as $Q(G)$. Clearly, homomorphisms to $\mathbb{R}$ are homogeneous quasimorphisms, i.e. $H^1(G) \subset Q(G)$. Also note that any bounded function $\varphi : G \to \mathbb{R}$ is trivially a quasimorphism. This gives another subspace $C_b^1(G) = C_b^1(G; \mathbb{R})$ of $\widehat{Q}(G)$.

Every quasimorphism can be made homogeneous by the following process.

Definition 2.20. (Homogenization) For any $\varphi \in \widehat{Q}(G)$, the homogenization $\bar{\varphi}$ is defined as

$$\bar{\varphi}(g) := \lim_{+\infty} \frac{\varphi(g^n)}{n}.$$

Lemma 2.21. *The homogenization $\bar{\varphi}$ is a well-defined homogeneous quasimorphism, and $\bar{\varphi} - \varphi$ is a bounded function on G. Quantitatively, we have $|\varphi - \bar{\varphi}|_\infty \leq D(\varphi)$.*

In the proof of this lemma, we need the following fact about sub-additive sequence.

Lemma 2.22. *Let a_n be a real-valued sequence that is sub-additive, i.e. $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$. Then we have*

$$\lim_{n \to +\infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}.$$

In particular, the limit exists iff $\frac{a_n}{n}$ is bounded below.

Proof. Clearly we have $\liminf \frac{a_n}{n} \geq \inf_{n \geq 1} \frac{a_n}{n}$. So it suffices to show that $\limsup \frac{a_n}{n} \leq \inf_{n \geq 1} \frac{a_n}{n}$.

Fix $m \geq 1$ and express any n as $n = qm + r$ with $0 < r \leq m$. Let $B = \max_{0 < r \leq m} a_r$. By sub-additivity and induction, we have $a_n \leq qa_m + a_r \leq qa_m + B$. Thus

$$\frac{a_n}{n} \leq \frac{qa_m + B}{qm + r} = \frac{a_m + \frac{B}{q}}{m + \frac{r}{q}}.$$

Hence as $n \to \infty$, we have $q \to \infty$ and

$$\limsup \frac{a_n}{n} \leq \frac{a_m}{m}.$$

Since m is arbitrary, we obtain

$$\limsup \frac{a_n}{n} \leq \inf_{m \geq 1} \frac{a_m}{m}.$$

$\square$

There is an analogous result for sup-additive sequences, replacing inf by sup, which can be deduced by considering the sequence $-a_n$.

Now we prove Lemma 2.21.

Proof of Lemma 2.21. Fixing $g \in G$, we show that the limit defining $\bar{\varphi}(g)$ exists. Although the sequence $\varphi(g^n)$ is not sup- or sub-additive, a small modification does the job.

Let $\varphi_+(g) := \varphi(g) + D(\varphi)$ and let $\varphi_-(g) := \varphi(g) - D(\varphi)$. By definition, we have $|\varphi(g) + \varphi(h) - \varphi(gh)| \le D(\varphi)$, which implies

$$\varphi(gh) - D(\varphi) \le \varphi(g) + \varphi(h) \le \varphi(gh) + D(\varphi),$$

and hence

$$\varphi_+(gh) = \varphi(gh) - D(\varphi) + 2D(\varphi) \le \varphi(g) + \varphi(h) - 2D(\varphi) = \varphi_+(g) + \varphi_+(h).$$

So φ_+ is sub-additive and similarly φ_- is sup-additive.

It follows that

$$\varphi_-(g) \le \frac{\varphi_-(g^n)}{n} \le \frac{\varphi_+(g^n)}{n} \le \varphi_+(g)$$

for all $n \ge 1$. So $\frac{\varphi_+(g^n)}{n}$ is bounded below and $\frac{\varphi_-(g^n)}{n}$ is bounded above. Thus both have finite limit by Lemma 2.22 and its sup-additive analogue. The limits agree since $\varphi_+(g^n) - \varphi_-(g^n) = 2D(\varphi)$. As $\varphi_-(g^n)\varphi(g^n) \le \varphi_+(g^n)$, we have

$$\bar{\varphi}(g) = \lim_{n \to +\infty} \frac{\varphi_-(g^n)}{n} = \lim_{n \to +\infty} \frac{\varphi(g^n)}{n} = \lim_{n \to +\infty} \frac{\varphi_+(g^n)}{n}.$$

This shows that $\bar{\varphi}$ is a well-defined function.

This also gives a way to bound the difference $\varphi - \bar{\varphi}$. As

$$\lim_{n \to +\infty} \frac{\varphi_+(g^n)}{n} = \inf_{n \ge 1} \frac{\varphi_+(g^n)}{n} \le \varphi_+(g) = \varphi(g) + D(\varphi),$$

we get

$$\bar{\varphi}(g) \le \varphi(g) + D(\varphi).$$

Similarly using φ_- we have $\bar{\varphi}(g) \ge \varphi(g) - D(\varphi)$. As g is arbitrary, we conclude that $|\varphi - \bar{\varphi}|_\infty \le D(\varphi)$. In particular, $\bar{\varphi}$ is a quasimorphism as the sum of φ and the bounded function $\bar{\varphi} - \varphi$.

Finally, it remains to check that $\bar{\varphi}$ is homogeneous. For every $k \ge 1$, we have

$$\bar{\varphi}(g^k) = \lim_{n \to +\infty} \frac{\varphi(g^{kn})}{n} = k \cdot \lim_{n \to +\infty} \frac{\varphi(g^{kn})}{kn} = k\bar{\varphi}(g).$$

For $k = 0$, $\bar{\varphi}(id) = \lim \frac{\varphi(id)}{n} = 0$. So it suffices to show that $\bar{\varphi}(g) + \bar{\varphi}(g^{-1}) = 0$ to deal with $k < 0$. Indeed, this easily follows from $|\varphi(g^n) + \varphi(g^{-n})| \le |\varphi(id)| + D(\varphi)$. $\qquad\square$

Remark 2.23. *The triangle inequality and the bound $|\varphi - \bar{\varphi}|_\infty \le D(\varphi)$ implies that $D(\bar{\varphi}) \le 4\,D(\varphi)$. A more involved argument shows that $D(\bar{\varphi}) \le 2\,D(\varphi)$.*

Every quasimorphism uniquely decomposes as the sum of a homogeneous quasimorphism (namely, its homogenization) and a bounded function.

Lemma 2.24. *We have $\widehat{Q}(G) = C_b^1(G) \oplus Q(G)$. That is, $C_b^1(G) \cap Q(G) = 0$ and $\widehat{Q}(G) = C_b^1(G) + Q(G)$.*

Proof. If $\varphi \in C_b^1(G) \cap Q(G)$, then $|\varphi(g)| = |\frac{\varphi(g^n)}{n}| \le \frac{|\varphi|_\infty}{|n|}$. As n is arbitrary, we must have $\varphi(g) = 0$ for all g.

For any $\varphi \in \widehat{Q}(G)$, we have $\varphi = \bar{\varphi} + (\varphi - \bar{\varphi})$, where the homogenization $\varphi \in Q(G)$ and the difference $\varphi - \bar{\varphi}$ is a bounded function by Lemma 2.21. $\qquad\square$

Proposition 2.25. *We have the following exact sequence*

$$0 \to H^1(G; \mathbb{R}) \to Q(G) \xrightarrow{\delta} H_b^2(G; \mathbb{R}) \xrightarrow{c} H^2(G; \mathbb{R}).$$

In particular, the kernel of the comparison map $c : H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ is isomorphic to the quotient $Q(G)/H^1(G)$, which can be thought of as the space of "interesting" homogeneous quasimorphisms.

Proof. We just check that $\ker \delta = H^1(G)$ (as homomorphisms to $\mathbb{R}$) and $\ker c = \operatorname{Im} \delta$.

As we calculated in Example 2.15, $(\delta\varphi)(g,h) = \varphi(g) + \varphi(h) - \varphi(gh)$, so clearly $H^1(G) \subset \ker \delta$. Conversely, if $[\delta\varphi] = 0 \in H_b^2(G)$, then $\delta\varphi = \delta f$ for some bounded function $f : G \to \mathbb{R}$. This implies that $\varphi - f$ is a homomorphism. Then we get two homogeneous quasimorphisms $\varphi - f$ and φ that differ by a bounded function f. By Lemma 2.24, we must have $f = 0$ and thus φ is a homomorphism. Hence $\ker \delta = H^1(G)$.

$\operatorname{Im} \delta \subset \ker c$ holds by definition. Suppose $\alpha \in \ker c$, i.e. $\alpha = [\delta f]$ for some function $f : G \to \mathbb{R}$. Then δf must be bounded, so f is a quasimorphism. Since f and $\bar{f}$ differ by a bounded function, we have $\alpha = [\delta f] = [\delta \bar{f}]$. As $\bar{f} \in Q(G)$ we conclude that $\alpha \in \operatorname{Im} \delta$. Thus $\operatorname{Im} \delta = \ker c$. $\square$

This gives a way to characterize the kernel of the comparison map in degree two.

Corollary 2.26. *The kernel of the comparison map $c : H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ is identified with $Q(G)/H^1(G; \mathbb{R})$.*

Exercise 2.27. *Prove the following variant of the exact sequence in Proposition 2.25*

$$0 \to C_b^1(G) + H^1(G) \to \widehat{Q}(G) \xrightarrow{\delta} H_b^2(G; \mathbb{R}) \xrightarrow{c} H^2(G; \mathbb{R}).$$

We will often use the following basic estimate, which is immediate from the definition and induction.

Lemma 2.28. *Let φ be a quasimorphism. For $g = g_1 \cdots g_n$, we have*

$$\left| \varphi(g) - \sum_i \varphi(g_i) \right| \le (n-1)D(\varphi).$$

In particular,

$$|\varphi(g)| \le \sum_i |\varphi(g_i)| + (n-1)D(\varphi).$$

Homogeneous quasimorphisms have the following nice properties:

Lemma 2.29. *Let φ be a homogeneous quasimorphism.*

   (1) *If g and h commute, then $\varphi(gh) = \varphi(g) + \varphi(h)$. So φ restricts to homomorphisms on abelian subgroups.*
   (2) *φ is conjugation-invariant, i.e. $\varphi(g) = \varphi(hgh^{-1})$ for all $g, h \in G$.*

Proof.
   (1) For any $n \in \mathbb{Z}_+$, we have $(gh)^n = g^n h^n$, so

$$n|\varphi(g) + \varphi(h) - \varphi(gh)| = |\varphi(g^n) + \varphi(h^n) - \varphi((gh)^n)| = |\varphi(g^n) + \varphi(h^n) - \varphi((g^n h^n))| \le D(\varphi).$$

   Letting $n \to \infty$ we see $\varphi(g) + \varphi(h) - \varphi(gh) = 0$.
   (2) For any $n \in \mathbb{Z}_+$, we have

$$n|\varphi(g) - \varphi(hgh^{-1})| \le |\varphi(g^n) - \varphi(hg^n h^{-1})| = |[\varphi(h) + \varphi(g^n) + \varphi(h^{-1})] - \varphi(hg^n h^{-1})| \le 2D(\varphi),$$

   where we used the fact that $\varphi(h) + \varphi(h^{-1}) = 0$. Letting $n \to \infty$ we get $\varphi(g) = \varphi(hgh^{-1})$. $\square$

Corollary 2.30. *If G is abelian, then $Q(G) = H^1(G; \mathbb{R})$.*

Exercise 2.31. *Let φ be a homogeneous quasimorphism. Prove that $|\varphi([g,h])| \leq D(\varphi)$ for all g, h, where $[g, h] = ghg^{-1}h^{-1}$ is the commutator.*

Remark 2.32. *Actually Bavard [Bav91, Lemma 3.6] showed that $D(\varphi) = \sup_{g,h} |\varphi([g,h])|$. See also [Cal09, Lemma 2.24].*

Definition 2.33. A group G is *perfect* if it has trivial abelianization, or equivalently, G agrees with its commutator subgroup, i.e. each $g \in G$ is a product of commutators.

We say G is *uniformly perfect*, if there is a uniform $n \in \mathbb{Z}_+$ such that each $g \in G$ is a product of at most n commutators.

Corollary 2.34. *If G is uniformly perfect, then $Q(G) = H^1(G) = 0$.*

Proof. Suppose for $n \in \mathbb{Z}_+$, each $g \in G$ is a product of at most n commutators. Then for any homogeneous quasimorphism $\varphi \in Q(G)$, and $g = [a_1, b_1] \cdots [a_k, b_k]$ with $k \leq n$ and $a_i, b_i \in G$, we have

$$|\varphi(g)| \leq |\sum_i \varphi([a_i, b_i])| + (k-1)D(\varphi) \leq (2k-1)D(\varphi) \leq (2n-1)D(\varphi)$$

by Exercise 2.31. This shows that φ is a bounded function. Hence $\varphi = 0$ by Lemma 2.24. $\square$

Lemma 2.35. *For any surjective homomorphism $f : G \to H$, the pullback map $f^* : Q(H) \to Q(G)$ is injective and defect-preserving.*

Proof. The pullback is given by $(f^*\varphi)(g) = \varphi(f(g))$, so injectivity is immediate. As for the defect, we have

$$D(f^*\varphi) = \sup_{g_1,g_2 \in G} |\varphi(f(g_1)f(g_2)) - \varphi(f(g_1)) - \varphi(f(g_2))| = \sup_{h_1,h_2 \in H} |\varphi(h_1 h_2) - \varphi(h_1) - \varphi(h_2)| = D(\varphi).$$

$\square$

This gives a way to obstruct homomorphisms from groups with few quasimorphisms to those with lots of quasimorphisms.

2.4. de Rham quasimorphisms.

Let ω be a 1-form on a connected closed hyperbolic manifold M^n. Fix a based point $p \in M$ and let $G = \pi_1(M, p)$. Then any $g \in G$ is uniquely represented by an oriented geodesic loop ℓ_g based at p. The de Rham quasimorphism (due to Barge–Ghys [BG88]) associated to ω is

$$\varphi_\omega(g) := \int_{\ell_g} \omega.$$

Lemma 2.36. *φ_ω defined above is indeed a quasimorphism.*

Proof. We need to bound $\varphi_\omega(g) + \varphi_\omega(h) - \varphi_\omega(gh)$ for any $g, h \in G$.

Let $\widetilde{M} \cong \mathbb{H}^n$ be the universal cover and $\tilde{p}$ be a lift of p. Denote by $\tilde{\omega}$ the pullback of ω on $\widetilde{M}$. Then for any $g \in G$, the unique lift of ℓ_g starting at $\tilde{p}$ is the geodesic $\tilde{\ell}_g$ from $\tilde{p}$ to $g\tilde{p}$. So $\varphi_\omega(g) = \int_{\tilde{\ell}_g} \tilde{\omega}$.

Given $g, h \in G$, we have an oriented geodesic triangle Δ with sides $\tilde{\ell}_g$, $g\tilde{\ell}_h$ and $\tilde{\ell}_{gh}$, where the induced orientation of Δ is opposite to the orientation on $\tilde{\ell}_{gh}$. Thus

$$|\varphi_\omega(g) + \varphi_\omega(h) - \varphi_\omega(gh)| = \left|\int_{\partial\Delta} \tilde{\omega}\right| = \left|\int_\Delta d\tilde{\omega}\right| \leq \|d\tilde{\omega}\|_{\widetilde{M}} \cdot \mathrm{area}(\Delta) \leq \pi\|d\omega\|_M,$$

where $\|d\omega\|$ is the supremum of $|d\omega(v)|$ over all orthonormal 2-frames v on M, which is finite by compactness. Note that we also used the fact that the area of a hyperbolic triangle is uniformly bounded by π. $\square$

Also note that φ_ω is a homomorphism if ω is closed, as the integral over Δ. When ω is exact, then $\varphi_\omega \equiv 0$.

There is also a nice description of the homogenization $\bar{\varphi}_\omega$. For any $g \in G$, there is a unique closed geodesic loop L_g (which is length minimizing in the free homotopy class) representing the conjugacy of g, and $\bar{\varphi}_\omega(g) = \int_{L_g} \omega$. The reason is that, changing the base point p only varies φ_ω by a bounded amount and does not affect the homogenization. So one can move p so that ℓ_g agrees with L_g.

2.5. **Quasimorphisms on free groups.** Various kinds of quasimorphisms were constructed on free groups. Brooks constructed lots quasimorphisms that imply $Q(F_n)$ is infinite-dimensional. A recent construction by Rolli [Rol09] gives a simpler way to prove this, so we will start with his construction.

2.5.1. *Rolli's construction.* Let $\ell^\infty(\mathbb{Z}_+)$ be the space of $\mathbb{R}$-valued bounded functions on the set $\mathbb{Z}_+$, which is an infinite dimensional space. For any $f \in \ell^\infty(\mathbb{Z}_+)$, extend it uniquely to an odd function $f : \mathbb{Z} \to \mathbb{R}$, i.e. $f(-n) = -f(n)$ for all n.

Given $f, g \in \ell^\infty(\mathbb{Z}_+)$, extended as above, define a quasimorphism $\varphi_{f,g}$ on $F_2 = \langle a, b \rangle$ as follows. For any element $w \in F_2$, express it as a reduced word in the generators $w = a_1^{m_1} b_1^{n_1} \cdots a_k^{m_k} b_k^{n_k}$ with $k \geq 1$ and each $m_i, n_i \in \mathbb{Z} \setminus \{0\}$ except that m_1 or n_k could be 0. Define $\varphi_{f,g}(w) = \sum_{i=1}^k f(m_i) + \sum_{i=1}^k g(n_i)$.

It is straightforward to check that

$$|\varphi(u) + \varphi(v) - \varphi(uv)| \leq \max\{3|f|_\infty, 3|g|_\infty\},$$

so φ is a quasimorphism.

The homogenization $\bar{\varphi}_{f,g}$ can be described as follows. For any $w \in F_2$, it has a reduced expression $w = uvu^{-1}$ where v is the unique cyclically reduced word in the conjugacy class of w. Then $\bar{\varphi}_{f,g}(w) = \bar{\varphi}_{f,g}(v) = \varphi_{f,g}(v)$ if v is not a power of the generator a or b, and clearly $\bar{\varphi}_{f,g}(a) = \bar{\varphi}_{f,g}(b) = 0$.

Lemma 2.37. *We have an embedding $\ell^\infty(\mathbb{Z}_+) \to Q(F_2)$ sending f to $\bar{\varphi}_{f,f}$. Moreover, the image intersects $H^1(F_2)$ trivially.*

Proof. By the description above, for any $n \in \mathbb{Z}_+$, we have $\bar{\varphi}_{f,f}(a^n b^n) = 2f(n)$. Thus $\bar{\varphi}_{f,f} = 0$ if and only if $f = 0$. The intersection with $H^1(F_2)$ is trivial since $\bar{\varphi}_{f,f}$ vanishes on the two generators. $\square$

Theorem 2.38. *Both $Q(F_2)$ and $H_b^2(F_2; \mathbb{R})$ are infinite-dimensional.*

Proof. Since $\ell^\infty(\mathbb{Z}_+)$ is infinite-dimensional, the embedding above shows that $Q(F_2)$ is infinite-dimensional. The same space embeds in $Q(F_2)/H^1(F_2) \cong H_b^2(F_2; \mathbb{R})$, so it is also infinite-dimensional. $\square$

Corollary 2.39. *If G surjects F_2, then $Q(G)$ is infinite-dimensional.*

Proof. $Q(F_2)$ embeds $Q(G)$ by Lemma 2.35. $\square$

Note that this applies to all non-abelian free groups and closed hyperbolic surface groups.

Exercise 2.40. *Generalize Rolli's construction to a free product $G = A \star B$. What kind of functions on A, B do you need for the construction to work out?*

2.5.2. *Brooks quasimorphisms.* Now we turn to the quasimorphisms constructed by Brooks, which can be generalized to groups acting on δ-hyperbolic groups. There are two versions, big counting quasimorphisms and little counting quasimorphisms. We follow the exposition in [Cal09, Section 2.3.2].

Let G be a free group generated by S. Fix a reduced word σ. For any $g \in G$, define the big counting function $C_\sigma(g)$ as the number of copies of σ that appear as subwords in the reduced word

representing g. Similarly, define the little counting function $c_\sigma(g)$ as the maximal number n such that there are n *disjoint* copies of σ in the reduced expression of g.

Example 2.41.

    (1) For $S = \{a, b\}$, $\sigma = ab$, and $g = aba^{-1}baba$, we have $C_\sigma(g) = c_\sigma(g) = 2$.

    (2) For $S = \{a, b\}$, $\sigma = aba$, and $g = ababa$, we have $C_\sigma(g) = 2$ but $c_\sigma(g) = 1$. Similarly for $h = abababa$, we have $C_\sigma(h) = 3$ while $c_\sigma(h) = 2$.

For the discussion below, we will use the following observations.

Lemma 2.42.   (1) $C_\sigma(g^{-1}) = C_{\sigma^{-1}}(g)$ and similarly for c_σ.

    (2) σ and σ^{-1} cannot overlap as subwords.

Definition 2.43 (Counting quasimorphisms). The big counting quasimorphism associated to σ is $H_\sigma(g) := C_\sigma(g) - C_\sigma(g^{-1})$. The little counting quasimorphism associated to σ is $h_\sigma(g) := c_\sigma(g) - c_\sigma(g^{-1})$.

Lemma 2.44. *Clearly from the definition, $H_\sigma(g^{-1}) = -H_\sigma(g)$ and similarly for h_σ.*

We show below that these are indeed quasimorphisms and we estimate their defect. For a reduced expression $u = u_1 u_2$, let $s = 1$ (resp. $s = -1$) if a copy of σ (resp. σ^{-1}) appears in the juncture, and let $s = 0$ if no w or w^{-1} appears.

Lemma 2.45. *With the notation above , we have*

$$0 \le s \cdot (H_\sigma(u) - H_\sigma(u_1) - H_\sigma(u_2)) \le |\sigma| - 1,$$

and $h_\sigma(u) - h_\sigma(u_1) - h_\sigma(u_2) = 0$ or s.

Proof. If σ does not appear in the juncture, then $c_\sigma(u) = c_\sigma(u_1) + c_\sigma(u_2)$ and $C_\sigma(u) = C_\sigma(u_1) + C_\sigma(u_2)$.

If σ appears in the juncture, there are at most $|\sigma| - 1$ subwords of length $|\sigma|$ in the juncture, so $C_\sigma(u) - C_{\sigma^{-1}}(u_1) - C_{\sigma^{-1}}(u_2) \le |\sigma| - 1$.

As for c_σ, clearly $c_\sigma(u_1) + c_\sigma(u_2) \le c_\sigma(u)$. Consider a collection of $c_\sigma(u)$ disjoint copies of σ in u. There is at most one copy that lies in the juncture. The remaining copies are disjoint and either in u_1 or u_2. Thus $c_\sigma(u_1) + c_\sigma(u_2) \ge c_\sigma(u) - 1$. Hence $c_\sigma(u) - c_\sigma(u_1) + c_\sigma(u_2) = 0, 1$.

Combining these observations proves the lemma. $\qquad\square$

Lemma 2.46. *We have $D(H_\sigma) \le 3(|\sigma| - 1)$ and $D(h_\sigma) \le 3$. Thus H_σ and h_σ are indeed quasimorphisms.*

Proof. For any $g, h \in G$, there are unique reduced expressions $g = uv^{-1}$, $h = vw^{-1}$ and $gh = uw^{-1}$. Then

$$
\begin{aligned}
H_\sigma(g) + H_\sigma(h) - H_\sigma(gh) &= H_\sigma(uv^{-1}) + H_\sigma(vw^{-1}) + H_\sigma(wu^{-1}) \\
&= (H_\sigma(uv^{-1}) - H_\sigma(u) - H_\sigma(v^{-1})) \\
&\quad + (H_\sigma(vw^{-1}) - H_\sigma(v) - H_\sigma(w^{-1})) \\
&\quad + (H_\sigma(wu^{-1}) - H_\sigma(w) - H_\sigma(u^{-1})),
\end{aligned}
$$

where each parenthesis has absolute value is at most $|\sigma| - 1$ by Lemma 2.45. Hence $D(H_\sigma) \le 3(|\sigma| - 1)$. The same method shows that $D(h_\sigma) \le 3$. $\qquad\square$

Intuitively, we can think of $g, h, (gh)^{-1}$ as a tripod with legs u, v, w. Whenever there is a copy of σ that appears in one of the three legs, there is a copy of σ^{-1} that appears on the opposite side to cancel out the contribution to the defect. So the contribution only comes from copies of σ and σ^{-1} that lie in the juncture, but there are limited spaces at the junctures, giving the defect bound.

It is straightforward to check that

Lemma 2.47. *The homogenization of H_σ applied to g is the number of copies of σ (minus the number of copies of σ^{-1}) as cyclic subwords in the cyclically reduced word representing the conjugacy class of g.*

Exercise 2.48. *Prove the following.*

    (1) *If g is a word shorter than σ, then $H_\sigma(g) = 0$.*

    (2) *If g is cyclically reduced word and $\sigma = g^2$, then $\bar{H}_\sigma(g) \geq 1 > 0$, although g is shorter than σ.*

    (3) *If $\bar{H}_\sigma(g) > 0$ for some g shorter than σ, show that $\sigma = uvu$ as a reduced expression for some nontrivial subword u.*

    (4) *In every conjugacy class, there is some (cyclically reduced) σ such that $\bar{H}_\sigma(g) = 0$ for all g shorter than σ.*

        Hint: Use a lexicographical order on reduced words and take a minimal one in the cyclical reduced words in the given conjugacy class. This is from [Tao16, Lemma 3.1].

    (5) *Given the above result, find an infinite sequence of distinct σ_n so that $\{\bar{H}_{\sigma_n}\}_n$ are linearly independent homogeneous quasimorphisms.*

2.6. **The rotation quasimorphism.** Now we construct a quasimorphism in relation to circle dynamics. Throughout this section, let $T = \mathrm{Homeo}^+(S^1)$, the group of orientation-preserving homeomorphisms on the circle. Every such homeomorphism f lifts to an orientation-preserving homeomorphism $\tilde{f}$ on $\mathbb{R}$, where we think of $\mathbb{R}$ as the universal cover of S^1 via the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$. Then $\tilde{f}$ commutes with the deck transformation, i.e. $\tilde{f}(x+1) = \tilde{f}(x) + 1$, and conversely any orientation-preserving homeomorphism on $\mathbb{R}$ descends to one on S^1. This leads to a new group

$$\widehat{T} = \{g \in \mathrm{Homeo}^+(\mathbb{R}) \mid g(x+1) = g(x) + 1\},$$

which has a natural map $\pi : \widehat{T} \to T$. The kernel $\ker \pi = \mathbb{Z}$ is the set of translations by integers as the lifts of a given map are unique up to a deck transformation. Thus we have a central extension

$$1 \to \mathbb{Z} \to \widehat{T} \to T \to 1.$$

We aim to build a quasimorphism $\widetilde{\mathrm{rot}}$ on $\widehat{T}$, which descends to a map $\mathrm{rot} : T \to \mathbb{R}/\mathbb{Z}$ so that $\mathrm{rot}(f)$ captures the dynamical properties of f for any $f \in T = \mathrm{Homeo}^+(S^1)$.

For any $p \in \mathbb{R}$, let $\tau_p(f) = f(p) - p$ for any $f \in \widehat{T}$.

Lemma 2.49. *τ_p is a quasimorphism and $D(\tau_p) \leq 1$.*

Proof. For any $f, g \in \widehat{T}$, we need to estimate $\tau_p(fg) - \tau_p(f) - \tau_p(g) = f(g(p)) - g(p) - (f(p) - p)$. Note that changing g to $g+n$ does not affect this quantity for any $n \in \mathbb{Z}$ and the same for f. Hence we may assume $p \leq g(p) < p + 1$, and $p \leq f(p) < f(p) + 1$. As f is orientation-preserving, the bound for $g(p)$ implies

$$f(p) \leq f(g(p)) < f(p+1) = f(p) + 1.$$

Combining with the bound for $f(p)$ we obtain

$$(f(p) - p) - 1 = f(p) - (p+1) \leq f(g(p)) - g(p) \leq (f(p) + 1) - p = (f(p) - p) + 1.$$

Thus

$$|\tau_p(fg) - \tau_p(f) - \tau_p(g)| = |f(g(p)) - g(p) - (f(p) - p)| \leq 1.$$

Hence $D(\tau_p) \leq 1$. □

Lemma 2.50. *For any $p, q \in \mathbb{R}$, the difference $\tau_p - \tau_q$ is a bounded function on $\widehat{T}$.*

Proof. Note that $\tau_{p+n}(f) = f(p+n) - (p+n) = f(p) - p = \tau_p$, so we may assume $q \leq p < q + 1$. Then $f(q) \leq f(p) < f(q) + 1$. Thus

$$-1 \leq f(p) - f(q) - (p - q) \leq 1,$$

i.e. $|\tau_p(f) - \tau_q(f)| \leq 1$ for all f. □

It follows that the homogenization $\overline{\tau_p}$ does not depend on p.

Definition 2.51 (Rotation quasimorphism)**.** The rotation quasimorphism $\tilde{\mathrm{rot}} \in Q(\widehat{T})$ is the homogenization of τ_p for any $p \in \mathbb{R}$. As an explicit formula, taking $p = 0$, we have $\tilde{\mathrm{rot}}(f) = \lim_{n \to +\infty} \frac{f^n(0)}{n}$.

$\tilde{\mathrm{rot}}(f)$ is sometimes called the translation number of f. It measures how fast f moves forward on average.

2.7. **Amenable groups.**

3. More on Gromov's simplicial norm

4. Mostow's rigidity

5. Actions on the circle and the bounded Euler class

References

[Bav91] Christophe Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[BG88] Jean Barge and Étienne Ghys. Surfaces et cohomologie bornée. *Invent. Math.*, 92(3):509–526, 1988.

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[Bro82] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Cal] Danny Calegari. *Kleinian groups*. Lecture notes.

[Cal09] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[CW19] Chris Connell and Shi Wang. Positivity of simplicial volume for nonpositively curved manifolds with a Ricci-type curvature condition. *Groups Geom. Dyn.*, 13(3):1007–1034, 2019.

[Fri17] Roberto Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[Gro82] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[Han98] Søren Kold Hansen. Measure homology. *Math. Scand.*, 83(2):205–219, 1998.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HM81] Uffe Haagerup and Hans J. Munkholm. Simplices of maximal volume in hyperbolic n-space. *Acta Math.*, 147(1-2):1–11, 1981.

[Lö6] Clara Löh. Measure homology and singular homology are isometrically isomorphic. *Math. Z.*, 253(1):197–218, 2006.

[Rol09] Pascal Rolli. Quasi-morphisms on free groups, 2009.

[Tao16] Jing Tao. Effective quasimorphisms on free chains, 2016.

[Thu] William P. Thurston. *Geometry and topology of three-manifolds*. Princeton Lecture Notes.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[Zas98] Andreas Zastrow. On the (non)-coincidence of Milnor-Thurston homology theory with singular homology theory. *Pacific J. Math.*, 186(2):369–396, 1998.

Department of Mathematics, The University of Texas at Austin, Austin, TX, USA

Email address, L. Chen: `lvzhou.chen@math.utexas.edu`