GROMOV'S SIMPLICIAL NORM AND BOUNDED COHOMOLOGY

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Abstract. These are lecture notes for the course *Gromov's Simplicial Norm and Bounded Cohomology* in Spring 2022 at the University of Texas at Austin.

Contents

1. Introduction to Gromov's simplicial norm

One interesting topic in geometry and topology is to relate geometric quantities of a manifold to topological invariants. One typical problem asks about which manifolds admit Riemannian metrics with negative (positive, or non-positive) sectional (Ricci, or scalar) curvature.

Here we are interested in volumes of closed manifolds. Usually one needs a Riemannian metric to make sense of it, but is it possible to get a topological invariant out of it? The Mostow rigidity (and Gauss–Bonnet in dimension 2) implies that the volume of a hyperbolic closed manifold is determined by its topology. Gromov's *simplicial volume*, as a special case of the *simplicial norm*, is a way to define this invariant in a purely topological way.

Why should one be interested in such an invariant? The following basic problem is an example where one needs a topological notion of volume/area.

Problem 1.1. *Given two orientable connected closed surfaces S, S', what is the largest possible degree* $\deg(f)$ *of a continuous map $f : S \to S'$?*

As we will see below (Lemma 1.9), the simplicial volumes of S and S', denoted $\|S\|_1$ and $\|S'\|_1$, satisfy

$$\|S\|_1 \geq |\deg(f)| \cdot \|S'\|_1$$

for any continuous map f. Intuitively, S needs to have enough area to cover S' for $|\deg(f)|$ times. This provides an upper bound $\|S\|_1/\|S'\|_1$ when $\|S'\| > 0$, or equivalently when S' has genus at least two as we will prove. Moreover, the upper bound obtained this way is actually sharp, and in Section 1.3 we will exactly determine the set of all possibly degrees

$$\deg(S, S') := \{\deg(f) \mid f : S \to S'\}.$$

Date: January 19, 2022.

1.1. **The simplicial norm.** Fix $n \in \mathbb{Z}_{\geq 0}$. Given a topological space X, Gromov [Gro82] introduced a semi-norm $\|\cdot\|_1$ on the singular homology $H_n(X;\mathbb{R})$ for each n as a real vector space to measure the size of each homology class. Recall that $H_n(X;\mathbb{R})$ is the homology of the singular chain complex

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n_1} \xrightarrow{\partial_{n-1}} \cdots,$$

where $C_n(X;\mathbb{R})$ is the space of singular n-chains, namely the real vector space spanned by the set $S_n(X)$ of all singular n-simplices. As usual, we have the subspaces $B_n \subset Z_n \subset C_n$, where $Z_n := \ker \partial_n$ and $B_n := \operatorname{Im} \partial_{n+1}$ are the spaces of cycles and boundaries respectively. So by definition $H_n(X;\mathbb{R})$ is the quotient Z_n/B_n.

Given the standard basis $S_n(X)$, equip the space $C_n(X;\mathbb{R})$ with the ℓ^1-norm, i.e. $|c|_1 = \sum_{i=1}^k |\lambda_i|$ for any $c = \sum_{i=1}^k \lambda_i c_i$ expressed uniquely as a (finite) linear combination of basis elements $c_i \in S_n(X)$ with coefficients $\lambda_i \in \mathbb{R}$.

Definition 1.2 (Simplicial norm)**.** The restriction of this ℓ^1-norm to Z_n induces a semi-norm on its quotient $H_n(X;\mathbb{R})$, explicitly,

$$\|\sigma\|_1 := \inf_{[c]=\sigma} |c|_1,$$

where the infimum is taken over all cycles $c \in Z_n$ representing the homology class $\sigma \in H_n(X;\mathbb{R})$. This semi-norm is called *Gromov's simplicial norm.*

In words, $\|\sigma\|_1$ is the infimal number of simplices that we need to represent σ.

The following property is immediate from the definition but important.

Proposition 1.3 (Functorial)**.** *For any continuous map $f : X \to Y$, then the induced map $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is non-increasing with respect to the simplicial norm, i.e.*

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1$$

for any $\sigma \in H_n(X;\mathbb{R})$.

Proof. For any cycle $c = \sum_i \lambda_i c_i \in Z_n(X;\mathbb{R})$ representing σ, the cycle $f_*c = \sum \lambda_i f_* c_i = \sum_i \lambda_i (f \circ c_i)$ represents $f_*\sigma$. Hence by definition

$$\|f_*\sigma\|_1 \leq |f_*c|_1 \leq \sum_i |\lambda_i| = |c|_1.$$

Since c is arbitrary, taking infimum implies

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1.$$

□

Corollary 1.4 (Invariance)**.** *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X;\mathbb{R}) \to H_n(Y;\mathbb{R})$ is an isometric isomorphism (i.e. an isomorphism that is norm-preserving) with respect to the simplicial norm.*

More generally, if for a map $f : X \to Y$ there is $g : Y \to X$ such that $g_* f_*$ is the identity on $H_n(X;\mathbb{R})$, then f_* is an isometric embedding (i.e. injective and norm-preserving).

Proof. The first part easily follows from the second part by taking g to be a homotopy inverse of f.

For the second part, by functoriality of g and the fact that $g_* f_* = id$, we have $\|\sigma\|_1 = \|(g_* f_*)\sigma\|_1 \leq \|f_*\sigma\|_1$. Combining with the functoriality of f, we must have $\|\sigma\|_1 = \|f_*\sigma\|_1$ for any $\sigma \in H_n(X;\mathbb{R})$. Hence f_* is norm-preserving. Injectivity easily follows from the fact that $g_* f_* = id$. □

Exercise 1.5. *Recall that $H_0(X;\mathbb{R})$ is isomorphic to the $\mathbb{R}$-vector space with basis corresponding to the path connected components of the space X. For any path component C and a point $c \in C$, thought of as a singular 0-simplex, we have a homology class $\sigma = [c]$. Show that $\|\sigma\|_1 = 1$.*

Remark 1.6. If A is a subspace of X, then we can define a simplicial (semi-)norm similarly on the relative homology group $H_n(X, A; \mathbb{R})$. Here one can treat $H_n(X, A; \mathbb{R})$ as the homology of the chain complex $C_n(X, A) = C_n(X)/C_n(A)$ (with the induced differentials). These vector spaces are equipped with semi-norms induced from $C_n(X)$ and thus we can define an induced semi-norm on $H_n(X, A; \mathbb{R})$ as before. When A is empty, this agrees with our definition above.

More generally, one can analogously define simplicial norm for any *normed* chain complex; see [Fri17].

Exercise 1.7. *Concretely, we can think of $H_n(X, A; \mathbb{R}) = Z_n(X, A)/B_n(X, A)$, where $B_n(X, A) = B_n(X) \cup C_n(A)$ and $Z_n(X, A) = \partial_n^{-1} C_{n-1}(A)$, with $C_i(A)$ treated naturally as a subspace of $C_i(X)$ for both $i = n-1, n$. Show that the semi-norm induced from this quotient agrees with the definition in the remark above.*

1.2. **The simplicial volume.** Now we specialize to measure the size of an oriented connected compact manifold M with (possibly empty) boundary ∂M. Let $n = \dim M$. The orientation picks out a generator $[M] \in H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$, called the *fundamental class*. We think of it as a class in $H_n(M, \partial M; \mathbb{R}) \cong \mathbb{R}$ using the map $H_n(M, \partial M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{R})$ induced by the standard inclusion $\mathbb{Z} \to \mathbb{R}$. Concretely, if M has a triangulation, then the sum of all n-simplices with compatible orientation is a cycle representing the fundamental class.

Definition 1.8 (Simplicial volume)**.** The simplicial volume of M is $\|[M]\|_1$, which we often abbreviate as $\|M\|_1$. Note that the choice of orientation does not affect the simplicial volume.

If M is non-orientable, then M has an orientable double cover N, and we define $\|M\|_1 := \|N\|_1/2$.

Recall that, for any continuous map $f : M^n \to N^n$ between oriented connected closed (occ) manifolds, the degree $\deg(f)$ is the unique integer such that $f_*[M] = \deg(f) \cdot [N]$.

Lemma 1.9. *For any continuous map $f : M^n \to N^n$ between occ manifolds, we have*

$$|\deg(f)| \cdot \|N\|_1 \leq \|M\|_1.$$

Moreover, if f is a (finite) covering map, then equality holds.

Proof. The inequality follows from functoriality (Proposition 1.3) since $\|f_*[M]\|_1 = \|\deg(f)\cdot[N]\|_1 = \deg(f) \cdot \|[N]\|_1$.

For any $\epsilon > 0$, let $c = \sum_i \lambda_i c_i$ be a cycle representing the fundamental class $[N]$. Each map $c_i : \Delta^n \to N$ has $d := |\deg(f)|$ lifts $\tilde{c}_i^j$ to M, $j = 1, \cdots, d$. Then $\tilde{c} = \sum_i \sum_{j=1}^d \tilde{c}_i^j$ is a cycle and clearly $f_*[\tilde{c}] = |\deg(f)| \cdot [N] = \pm f_*[M]$. Hence $[\tilde{c}] = \pm[M]$, and $\|M\|_1 \leq |\deg(f)| \sum_i |\lambda_i|$. Since c is arbitrary, minimizing the summation $\sum_i |\lambda_i|$ gives the reversed inequality we desire. □

Corollary 1.10. *If an orientable closed connected manifold M admits a selfmap $f : M \to M$ with $|\deg(f)| > 1$, then $\|M\|_1 = 0$.*

Example 1.11.

(1) For any sphere S^n, $n \geq 1$, we have $\|S^n\|_1 = 0$.
(2) For the n-torus $T^n = (S^1)^n$, $n \geq 1$ we have $\|T^n\|_1 = 0$.
(3) More generally, if $M = S^1 \times N$ for a closed manifold N, then $\|M\|_1 = 0$.

Lemma 1.12. *For $n \geq 1$, if a homology class $\sigma \in H_n(X; \mathbb{R})$ is represented by a sphere, i.e. there is a map $f : S^n \to X$ with $f_*[S^n] = \sigma$, then $\|\sigma\|_1 = 0$.*

Corollary 1.13. *For any X, the simplicial norm $\|\cdot\|_1$ vanishes on $H_1(X; \mathbb{R})$.*

1.3. **Application: degrees of maps between surfaces.**

2. Bounded cohomology

3. Quasimorphisms

4. More on Gromov's simplicial norm

5. Mostow's rigidity

6. Actions on the circle and the bounded Euler class

References

[Fri17] Roberto Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[Gro82] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

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