STABLE TORSION LENGTH

CHLOE I. AVERY AND LVZHOU CHEN

Abstract. The *stable torsion length* in a group is the stable word length with respect to the set of all torsion elements. We show that the stable torsion length vanishes in crystallographic groups. We then give a linear programming algorithm to compute a lower bound for stable torsion length in free products of groups. Moreover, we obtain an algorithm that exactly computes stable torsion length in free products of finite abelian groups. The nature of the algorithm shows that stable torsion length is rational in this case. As applications, we give the first exact computations of stable torsion length for nontrivial examples.

Contents

1. Introduction

Given a generating set S of a group G, the word length $|g|_S$ measures the least number of generators needed to express an element $g \in G$. For finite generating sets, this is widely studied in geometric group theory, and different finite generating sets give equivalent word lengths up to scaling. On the other hand, many groups come with interesting and natural *infinite* generating sets, for instance, the set of all commutators in G (generating the commutator subgroup $[G, G]$), the set of torsion elements, and the set of words in a surface group representing simple closed loops. All of these examples are invariant under automorphisms.

Understanding the word length of such infinite generating sets is often difficult, even for the basic question of whether the word length is bounded [Cal08b, MP20, BM19]. The first main result in this paper establishes boundedness for the word length with respect to the set of all torsion elements in crystallographic groups, namely those acting properly discontinuously and cocompactly on Euclidean spaces.

Theorem A (Theorem 3.1)**.** *For any crystallographic group generated by torsion, the word length with respect to the set of all torsion elements is bounded.*

Date: March 25, 2021.

In contrast, in non-elementary word-hyperbolic groups, there is no upper bound on word length with respect to the set of all torsion elements (see Remark 2.7). When the word length $|\cdot|_S$ with respect to a set S is unbounded, it is interesting to investigate the *stable word length* $\|g\|_S := \lim_n \frac{|g^n|_S}{n}$, which measures the growth of the word length in the direction of g. Very little is known about stable word length for an infinite generating set in general, and giving good estimates or computing it is notoriously hard [Cal08b]. Most known results are about the stable commutator length [Cal09b, CF10, Che20].

In this paper, we use topological methods to study the stable word length with respect to conjugate-invariant generating sets. We focus on the special case of *stable torsion length*, namely the stable word length with respect to the set of torsion elements in G, but a large portion of the argument works for other conjugate-invariant generating sets that are closed under taking powers.

We show that stable torsion length in a free product of finite abelian groups is rational and can be computed by an algorithm.

Theorem B (Rationality and Computability; Theorem 5.14)**.** *If G is a free product of finite abelian groups, then for any $g \in G$, the stable torsion length of g is rational and computable.*

For more general free products, we give a linear programming algorithm that computes an effective lower bound; see Section 4.4.

We apply these methods to give the first exact computations of stable torsion length for nontrivial examples. These formulas hold true in arbitrary free products by an isometric embedding theorem (Theorem E) that we prove.

Theorem C (Product formula; Theorem 6.6)**.** *Let $G = A * B$ be a free product, and let $a \in A$ and $b \in B$ be torsion elements of order p and q respectively such that $p \leq q$. Then*

$$\mathrm{stl}_G(ab) = 1 - \frac{q}{p(q-1)}.$$

Theorem D (Commutator formula; Theorem 6.4)**.** *Let $G = A*B$ be a free product, and let $a \in A$ and $b \in B$ be torsion elements of orders p and q respectively, where $p, q \geq 2$. Then we have*

$$\mathrm{stl}_G([a,b]) = 1 - \frac{1}{\min(p,q) - 1}.$$

Our results show that the stable torsion length behaves in a way similar to the *stable commutator length*, the stable word length with respect to the set of all commutators. In recent years, the study of stable commutator length has seen many advances [Cal09b, Che20] and interesting applications to the surface subgroup problem [Cal08a, CW15, Wil18] and the simplicial volume [HL21]. However, the tools established for stable commutator length are special in an essential way, and thus new tools are required to understand the stable torsion length despite the similarity; see Section 1.1.

In particular, we are not aware of an analog of the Bavard's duality ([Bav91], [Cal09a, Theorem 2.70]) in the case of stable torsion length. Due to the lack of this duality, it is not easy to verify whether certain groups have trivial stable torsion length, such as amenable groups, which include crystallographic groups that we consider in Theorem A.

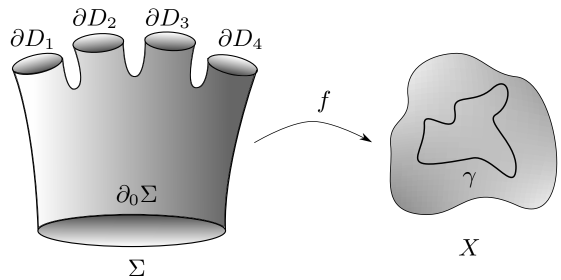

FIGURE 1. A planar surface Σ bounding a loop γ representing the element g

1.1. **Methods.** Given a group G, let X be a topological space with $\pi_1(X) = G$. Given a conjugate-invariant subset S, for a fixed $k \geq 1$ there is an expression $g = s_1 \cdots s_k$ for some $s_i \in S$ if and only if there is a continuous map $f : \Sigma \to X$, where Σ is a disk with k subdisks $D_1, \cdots, D_k$ removed, so that each ∂D_i represents a conjugacy class in S and the remaining boundary component $\partial_0\Sigma$ of Σ represents the conjugacy class of g; see Figure 1 for an illustration. We refer to such a surface as an S-admissible surface.

Thus finding the word length of g with respect to S is to find the least complicated connected *planar* surface in X bounding g in the above way. Similarly, finding the stable word length is to minimize $\frac{-\chi(\Sigma)+1}{n}$ over all connected planar surfaces Σ bounding g^n as above for some $n \in \mathbb{Z}_+$, which turns out to be the same as minimizing $\frac{-\chi(\Sigma)}{n}$ if S is closed under taking powers; see Lemma 2.9.

Such a topological interpretation in terms of surfaces makes the problem of computing stable word length more structured since there are nice operations on surfaces: compression, cut-and-paste, and taking finite covers. However, unlike the case of stable commutator length, where an equation $g = [a_1, b_1] \cdots [a_k, b_k]$ represents a surface of genus k, here we are restricting our attention to *planar* surfaces. This makes the problem harder since the operations above (e.g. taking finite covers) do not necessarily preserve the class of planar surfaces and so we are forced to use certain subclasses of operations.

If S is the set of torsion elements in G, this topological interpretation specializes to the case of stable torsion length of an element g, which we denote as $\mathrm{stl}_G(g)$, and refer to an S-admissible surface bounding g^n as a *torsion-admissible surface* for g of degree n. When G is a free product $G = A * B$, we can take X to be a wedge of spaces X_A and X_B with $\pi_1(X_A) = A$ and $\pi_1(X_B) = B$. Then each element of S is represented by a loop which is supported either in X_A or in X_B. Using this particular structure, we develop a normal form of torsion-admissible surfaces for a given element g which is not conjugate into A or B; see Section 4.1.

To further simplify the problem, we introduce and focus on the family of *simple surfaces*. These are surfaces Σ made of particular pieces such that each is either a disk or an annulus which is supported either in X_A or in X_B. The gluing of pieces are encoded by the *gluing graph* Γ_Σ. Any surface in normal form can be simplified into a simple surface whose gluing graph is a tree.

Therefore, we obtain a lower bound of $\mathrm{stl}_G(g)$ by minimizing the complexity $\frac{-\chi(\Sigma)}{n}$ over all connected simple surfaces Σ for g with $\chi(\Gamma_\Sigma) = 1$; see Lemma 4.11. This can be formulated as a linear programming problem when we further relax to the class of not necessarily connected simple surfaces with $\chi(\Gamma_\Sigma) \geq 0$, which gives a way to compute a nontrivial lower bound of the stable torsion length; see Section 4.4.

When torsion elements in A (resp. B) form a subgroup, any simple surface whose gluing graph is a tree is itself a surface in normal form. Thus $\mathrm{stl}_G(g)$ is exactly the infimal complexity over all connected simple surfaces Σ for g with $\chi(\Gamma_\Sigma)=1$.

This characterization leads to an isometric embedding theorem (Theorem 4.14), that allows us to compute the stable torsion length in simpler free products to obtain more general results. Here we state a special case:

Theorem E (Isometric Embedding, weak version)**.** *Suppose that $i_A : A \to A'$ and $i_B : B \to B'$ are injective homomorphisms from finite groups A and B. Then the induced map $i : A * B \to A' * B'$ preserves the stable torsion length, i.e.*

$$\mathrm{stl}_{A*B}(g) = \mathrm{stl}_{A'*B'}(i(g))$$

for any $g \in A * B$.

To exactly compute the stable torsion length by an algorithm, it is desirable to extend the family of connected simple surfaces Σ with $\chi(\Gamma_\Sigma)=1$ to those with $\chi(\Gamma_\Sigma)\geq 0$. This relaxation does not affect the computation when A and B are finite abelian groups, as we are able to show that connected simple surfaces Σ with $\chi(\Gamma_\Sigma)=0$ can be approximated by those with $\chi(\Gamma_\Sigma)=1$; see Lemma 5.4. This is achieved by considering two operations, splitting and rewiring, that we introduce in Section 5.

Using these two operations, we further show that any connected simple surface Σ with $\chi(\Gamma_\Sigma)\geq 0$ can be simplified into a union of *irreducible* ones; see Section 5.3. Moreover, there are only finitely many different irreducible simple surfaces (Proposition 5.13), which can be enumerated. As a result, the stable torsion length is actually equal to $\frac{-\chi(\Sigma)}{n}$ for some irreducible simple surface Σ, and thus must be a rational number. This yields the Rationality Theorem B.

Finally we carry out explicit computations in free products of cyclic groups and then use Theorem E to generalize the formulas to arbitrary free products and prove Theorems C and D.

1.2. **Organization of the paper.** In Section 2, we provide some basic properties of stable torsion length and formulate the interpretation via torsion-admissible surfaces.

We show crystallographic groups have trivial stable torsion length due to bounded generation in Section 3. In Section 4, we develop a normal form of torsion-admissible surfaces in a free product and introduce simple surfaces, using which we prove the Isometric Embedding Theorem E and give a lower bound estimation via linear programming. Then in Section 5, we specialize to free products of finite abelian groups and introduce the operations of splitting and rewiring to show the Rationality Theorem B. Finally in Section 6 we carry out computations in explicit examples and prove Theorems C and D.

Acknowledgement. The authors would like to thank Benson Farb for suggesting this problem and the study of stl, as well as for comments on the draft and invaluable support from start to finish. We would also like to thank Danny Calegari, Hannah Hoganson, and Kasia Jankiewicz for many helpful conversations.

2. General setup

In Section 2.1, we give the foundational definitions and deduce fundamental properties of stable word length with respect to a conjugate-invariant set S of a group G. Some properties are not used in this paper but could be of independent interest. In Section 2.2, we give a topological formulation when S is closed under taking powers, which is crucial in Sections 4–6.

2.1. The algebraic point of view.

Let G be a group and let S be a conjugate-invariant subset. Let $\langle S\rangle$ be the (normal) subgroup of G generated by S. When S is the set of commutators, we have $\langle S\rangle = [G,G]$. When S is the set of torsion elements, we denote $\langle S\rangle$ as G_{tor}, the subgroup generated by torsion elements.

Definition 2.1. For any element g of $\langle S\rangle$, the word length $|g|_S$ is the minimal k such that $g = s_1\cdots s_k$, where each $s_i \in S$. When S is the set of torsion elements (resp. commutators), we denote $|g|_S$ by $\mathrm{tl}_G(g)$ (resp. $\mathrm{cl}_G(g)$) and refer to it as the *torsion length* (resp. *commutator length*) of g.

The sequence $|g^n|_S$ is subadditive in n, thus

$$\lim_{n\to\infty}\frac{|g^n|_S}{n} = \inf\frac{|g^n|_S}{n},$$

which is called the *stable word length* of g, denoted $\|g\|_S$. When S is the set of torsion elements (resp. commutators), we denote $\|g\|_S$ by $\mathrm{stl}_G(g)$ (resp. $\mathrm{scl}_G(g)$) and refer to it as the *stable torsion length* (resp. *stable commutator length*); When the group G is understood we simply denote it as $\mathrm{stl}(g)$ (resp. $\mathrm{scl}(g)$).

The following properties are standard. They are well known in the case of stable commutator length (see [Cal09a, Chapter 2]).

Lemma 2.2 (Monotonicity)**.** *Let $S\subset G$ and $T\subset H$ be conjugate-invariant subsets. Suppose $\varphi: G\to H$ is a group homomorphism such that $\varphi(S)\subset T$. Then*

$$|\varphi(g)|_T \le |g|_S \quad \text{and} \quad \|\varphi(g)\|_T \le \|g\|_S$$

for all $g\in\langle S\rangle$. In particular, we have

$$\mathrm{tl}_H(\varphi(g)) \le \mathrm{tl}_G(g) \quad \text{and} \quad \mathrm{stl}_H(\varphi(g)) \le \mathrm{stl}_G(g)$$

for all $g\in G_{tor}$.

Proof. If $g = s_1\cdots s_n$ for some $s_i\in S$ and $n\in\mathbb{Z}_+$, then $\varphi(g) = \varphi(s_1)\cdots\varphi(s_n)$ where each $\varphi(s_i)\in T$ by the assumption. Thus the inequality $|\varphi(g)|_T \le |g|_S$ easily follows, which implies the stable version. The assumption clearly holds when S and T are the set of torsion elements in G and H. □

Corollary 2.3 (Retraction)**.** *Suppose that $\varphi: G\to H$ and $\psi: H\to G$ are injective group homomorphisms such that $\psi\circ\varphi: G\to G$ is the identity. Then,*

$$\mathrm{stl}_H(\phi(g)) = \mathrm{stl}_G(g)$$

for all $g\in G_{tor}$.

Proof. This follows immediately from Lemma 2.2. □

Lemma 2.4 (Characteristic)**.** *The functions $|\cdot|_S$ and $\|\cdot\|_S$ are constant on conjugate classes. If S is also invariant under the action of $\mathrm{Aut}(G)$, then $|\cdot|_S$ and $\|\cdot\|_S$ are constant on orbits of $\mathrm{Aut}(G)$.*

Proof. This follows from the definition. $\square$

Thus, both tl_G and stl_G are constant on orbits of $\mathrm{Aut}(G)$.

Lemma 2.5 (Countable subgroup)**.** *Let g be an element in $G_{tor} \leq G$. There exists a countable subgroup $H < G_{tor}$ containing g such that $\mathrm{stl}_H(g) = \mathrm{stl}_G(g)$.*

Proof. For each $n \in \mathbb{N}$, there exists $\mathrm{tl}(g^n)$ torsion elements whose product is g^n. Let H_n be the group generated by those $\mathrm{tl}(g^n)$ torsion elements and let H be the group generated by $\cup_n H_n$. Then H is countable and $\mathrm{tl}_H(g^n) \leq \mathrm{tl}_G(g^n)$ since we have exhibited each g^n as the product of $\mathrm{tl}_G(g^n)$ torsion elements in H. On the other hand, the inequality $\mathrm{tl}_H(g^n) \geq \mathrm{tl}_G(g^n)$ follows from Lemma 2.2. $\square$

A similar statement holds for general stable word length.

Finally, there is an inequality relating the stable commutator length and the stable torsion length. This can be found in [Kot04], but we give a conceptually simper proof using Bavard's duality. Since Bavard's duality and related notions are not used in the rest of the paper, we refer readers to [Cal09a, Chapter 2].

Lemma 2.6 (Kotschick [Kot04])**.** *Let G be a group. For any $g \in G_{tor} \cap [G,G]$, we have*

$$2\mathrm{scl}(g) \leq \mathrm{stl}(g).$$

Proof. Let $\varphi : G \to \mathbb{R}$ be a homogeneous quasimorphism. If the defect $D(\varphi) = 0$ for every φ, then $\mathrm{scl}(g) = 0$, and the inequality holds trivially. Now assume that $D(\varphi) > 0$. If $g = t_1 \cdots t_{\mathrm{tl}(g)}$ for some torsion elements t_i, then we have

$$\left| \varphi(g) - \sum_{i=1}^{\mathrm{tl}(g)} \varphi(t_i) \right| \leq (\mathrm{tl}(g) - 1) D(\varphi).$$

Since each t_i is torsion and φ is homogeneous, we know $\varphi(t_i) = 0$ for all i, and thus

$$\frac{|\varphi(g)|}{D(\varphi)} \leq \mathrm{tl}(g) - 1.$$

Using that φ is homogeneous, applying this to g^n for any $n \in \mathbb{N}$, we have

$$\frac{|\varphi(g)|}{D(\varphi)} \leq \frac{\mathrm{tl}(g^n) - 1}{n}.$$

The Bavard's duality states that the supremum of the left-hand side over all homogeneous quasimorphisms is $2\mathrm{scl}(g)$. As the limit of the right-hand side is $\mathrm{stl}(g)$, this gives the desired inequality. $\square$

Remark 2.7. A theorem of Epstein–Fujiwara [EF97] implies that any non-elementary hyperbolic group G has an infinite dimensional space of homogeneous quasimorphisms, which implies that $\mathrm{scl}_G(g) > 0$ for some g. Therefore, by Lemma 2.6, if G is a non-elementary hyperbolic group generated by torsion, then $\mathrm{stl}_G(g) > 0$ for some g, and torsion length is unbounded in G.

2.2. **A topological point of view.** Fix the group G and conjugate-invariant subset S, which we will assume to be closed under taking powers in Lemma 2.9 below. Let X be a space with fundamental group G and let γ be a loop representing g.

Definition 2.8. Let Σ be a compact, oriented, connected *planar* (i.e. genus zero) surface and let $f : \Sigma \to X$. We say that (Σ, f) is *S-admissible* for g of degree $n(\Sigma, f)$ if Σ has a specified boundary component $\partial_0\Sigma$ such that $f : \Sigma \to X$ takes $\partial_0\Sigma$ to γ, winding around $n(\Sigma, f)$ times, and the image of all other boundary components are loops representing conjugacy classes in S (see Figure 1).

When S is the set of torsion elements in G, we refer to (Σ, f) as a *torsion-admissible surface* instead. We often denote a torsion-admissible surface by Σ instead of (Σ, f) to make f implicit.

We refer to the boundary components of Σ other than $\partial_0\Sigma$ as *holes*. Denote by $H(\Sigma)$ the number of holes on Σ. Note that $-\chi(\Sigma) = H(\Sigma) - 1$. For a connected surface Σ, let $\chi^-(\Sigma)$ be $\chi(\Sigma)$ unless Σ is a sphere or a disk, in which case we define $\chi^-(\Sigma)$ to be 0.

Lemma 2.9. *Suppose $s^n \in S$ for any $s \in S$ and $n \in \mathbb{Z}$. For any $g \in \langle S\rangle$ we have*

$$\|g\|_S = \inf_{\Sigma} \frac{H(\Sigma)}{n(\Sigma)} = \inf_{\Sigma} \frac{-\chi^-(\Sigma)}{n(\Sigma)}, \tag{2.1}$$

where each infimum is taken over all S-admissible surfaces Σ.

Proof. The first equality is simply the topological reformulation of the algebraic definition. For the second, note that if g is torsion then g^n bounds a disk for some n and all three quantities are zero in this case. Suppose g is not a torsion element. Then for any S-admissible surface Σ we have $-\chi^-(\Sigma) = -\chi(\Sigma)$. In particular, $-\chi^-(\Sigma) = H(\Sigma) - 1 \le H(\Sigma)$. Thus it suffices to show that

$$\inf_{\Sigma} \frac{H(\Sigma)}{n(\Sigma)} \le \inf_{\Sigma} \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

Note that given any S-admissible surface Σ and any $N \in \mathbb{Z}_+$, Σ has a degree N cover Σ_N with genus zero such that the preimage of $\partial_0\Sigma$ is a single boundary component; see Figure 2. Note that any other boundary component of Σ_N covers a boundary component of Σ different from $\partial_0\Sigma$. Since S is closed under taking powers, Σ_N is S-admissible of degree $n(\Sigma_N) = N \cdot n(\Sigma)$. Thus

$$\inf_{\Sigma'} \frac{H(\Sigma')}{n(\Sigma')} \le \frac{H(\Sigma_N)}{n(\Sigma_N)} = \frac{-\chi^-(\Sigma_N) + 1}{n(\Sigma_N)} = \frac{-N \cdot \chi^-(\Sigma) + 1}{N \cdot n(\Sigma)}.$$

Taking $N \to \infty$ proves the desired inequality. $\square$

Remark 2.10. We can replace $-\chi^-(\Sigma)$ by $-\chi(\Sigma)$ for any S-admissible surface Σ whenever g is not torsion .

3. Crystallographic groups

Let $E(n)$ be the isometry group of the Euclidean space $\mathbb{R}^n$. Each element $\gamma \in E(n)$ acts on $\mathbb{R}^n$ via $\gamma(x) = Ax + v$ for some uniquely determined orthogonal transformation $A \in O(n)$ and vector $v \in \mathbb{R}^n$. We refer to A as the *rotational part* of γ and v as the *translational part* of γ.

A *crystallographic group* Γ of dimension n is a cocompact discrete subgroup of $E(n)$. By a theorem of Bieberbach [Bie12] (see also [Szc12]) for any such Γ, the

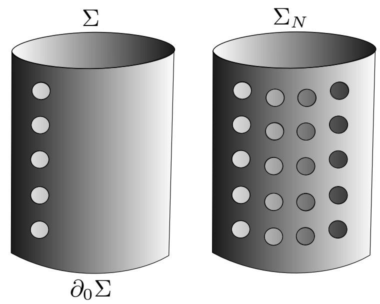

FIGURE 2. On the left is an S-admissible surface Σ and on the right is a planar degree $N = 4$ cover Σ_N of Σ such that the preimage of $\partial_0\Sigma$ is a single boundary component

subgroup H acting by translations is a normal subgroup of Γ of finite index and isomorphic to $\mathbb{Z}^n$. So we have an exact sequence

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where G is a finite subgroup of $O(n)$, and the map $\Gamma \to G$ takes the rotational part $A \in G$ of any $\gamma \in \Gamma$.

Let Γ_{tor} be the subgroup of Γ generated by torsion elements. We show the following more precise version of Theorem A.

Theorem 3.1. *For any crystallographic group Γ, the torsion subgroup Γ_{tor} is boundedly generated by torsion elements, and thus* $\mathrm{stl}_\Gamma \equiv 0$.

Let $H_{tor} := H \cap \Gamma_{tor}$ and let G_{tor} be the image of Γ_{tor} in $G \leq O(n)$. Then H_{tor} is a free abelian subgroup of H and is finite index in Γ_{tor}. We have

$$1 \longrightarrow H_{tor} \longrightarrow \Gamma_{tor} \longrightarrow G_{tor} \longrightarrow 1.$$

Here we think of H both as the translation subgroup of Γ and as a lattice in $\mathbb{R}^n$, where each vector h of the lattice corresponds to the translation $T_h : x \mapsto x + h$. To avoid confusion, we use h to represent an element of H when we regard H as a lattice and use T_h when we regard H as the translation subgroup of Γ. Note that $T_{nh} = T_h^n$ for any $n \in \mathbb{Z}$ and $T_{h+h'} = T_h \cdot T_{h'}$ for all $h, h' \in H$.

We prove Theorem 3.1 by constructing a finite index subgroup H_0 of H_{tor} that is boundedly generated by torsion in Γ_{tor}. We start by finding elements in H_{tor} that can be written as a product of few torsion elements.

Lemma 3.2. *Suppose $\gamma \in \Gamma_{tor}$ is a torsion element with rotational part $A \in G_{tor}$ and let I be the identity element of $O(n)$. Then for any $h \in H$, we have $(A-I)h \in H_{tor}$, and the corresponding translation $T_{(A-I)h} = [\gamma, T_h]$ is a product of two torsion elements.*

Proof. Since γ is a torsion element, it must fix some point $p \in \mathbb{R}^n$, and thus $\gamma(x) = A(x - p) + p$. Hence

$$\gamma T_h \gamma^{-1}(x) = A[A^{-1}(x - p) + p + h - p] + p = x + Ah,$$

so $[\gamma, T_h](x) = x + Ah - h = x + (A-I)h = T_{(A-I)h}(x)$. This shows that $T_{(A-I)h} = [\gamma, T_h]$, which is a product of two torsion elements, as $[\gamma, T_h] = \gamma \cdot (T_h \gamma^{-1} T_h^{-1})$. It follows that $(A - I)h \in H \cap \Gamma_{tor} = H_{tor}$. $\square$

Iterating the previous lemma, we control the torsion length for a larger family of elements in H_{tor}.

Lemma 3.3. *For any $m \geq 1$ and $1 \leq i \leq m$, let $\gamma_i \in \Gamma_{tor}$ be a torsion element with rotational part $A_i \in G_{tor}$. Then for any $h \in H$, we have $(A_1 \cdots A_m - I)h \in H_{tor}$, and the corresponding translation $T_{(A_1 \cdots A_m - I)h}$ is a product of $4m-2$ torsion elements.*

Proof. We prove this by induction on m. The base case $m = 1$ follows from Lemma 3.2. Suppose this holds for $m-1$. Then

$$\begin{aligned}&[\gamma_1, T_{(A_2 \cdots A_m - I)h}] \cdot [\gamma_1, T_h] \cdot T_{(A_2 \cdots A_m - I)h}\\ =&T_{(A_1 - I)(A_2 \cdots A_m - I)h} \cdot T_{(A_1 - I)h} \cdot T_{(A_2 \cdots A_m - I)h}\\ =&T_{(A_1 \cdots A_m - I)h}\end{aligned}$$

by Lemma 3.2. Note that the first row is the product of $2+2+(4m-6) = 4m-2$ torsion elements by Lemma 3.2 and the induction hypothesis. $\square$

As a consequence, we can uniformly bound the torsion length of all elements in H_{tor} of the form $(A-I)h$ for any $h \in H$ and any $A \in G_{tor}$.

Lemma 3.4. *With the notation above, there is some M such that for any $h \in H$ and any $A \in G_{tor}$ we have $(A-I)h \in H_{tor}$, and the corresponding translation $T_{(A-I)h}$ is a product of at most M torsion elements in Γ.*

Proof. For each $A \in G_{tor}$, pick an arbitrary lift $\gamma \in \Gamma_{tor}$. Since G_{tor} is finite, there is some m such that each γ can be written as a product of at most m torsion elements. By Lemma 3.3, the conclusion holds with $M = 4m-2$. $\square$

We will need the following lemma to ensure that we can pick elements of the form $(A-I)h$ considered above to generate a finite index subgroup H_0 in H_{tor}.

Lemma 3.5. *With the notation above, let $X \subset \mathbb{R}^n$ be the subspace spanned by the image of $A - I$ for all $A \in G_{tor}$. Consider $H_{tor} \leq H \leq \mathbb{R}^n$ as a discrete subgroup of $\mathbb{R}^n$. Then X is also the $\mathbb{R}$-linear span of H_{tor}.*

Proof. By Lemma 3.2, we have $(A-I)h \in H_{tor}$ for any $h \in H$ and $A \in G_{tor}$. This shows that the $\mathbb{R}$-linear span of H_{tor} contains X since H spans the entire space $\mathbb{R}^n$. It remains to show that $H_{tor} \subset X$.

Any torsion element $\gamma \in \Gamma$ acts on $\mathbb{R}^n$ by

$$\gamma(x) = A(x-p) + p = Ax + (I-A)p.$$

The translational part $(I-A)p$ lies in X by definition. We show that this holds for all $\gamma \in \Gamma_{tor}$. Since Γ_{tor} is generated by torsion, by induction, it suffices to show that $\eta\gamma$ has translational part in X if both $\eta, \gamma \in \Gamma_{tor}$ do. Indeed, if $\gamma(x) = Ax + u$ and $\eta(x) = Bx + v$ with $A, B \in G_{tor}$ and $u, v \in X$, then

$$\eta\gamma(x) = B(Ax+u) + v = BAx + (B-I)u + u + v$$

has translational part $(B-I)u + u + v \in X$ since all three terms lie in X. Thus any $\gamma \in \Gamma_{tor}$ can be written as $\gamma(x) = Ax + u$ for some $u \in X$. In particular, any translation in Γ_{tor} takes the form T_u for some $u \in X$. This shows $H_{tor} = H \cap \Gamma_{tor} \subset X$. $\square$

Now we are in a place to prove Theorem 3.1.

Proof of Theorem 3.1. We use the notation above. Let d be the dimension of the space X as in Lemma 3.5. Since H spans $\mathbb{R}^n$, by Lemma 3.5 there exists $h_i \in H$ and $A_i \in G_{tor}$ for $1 \le i \le d$ such that $\{(A_i - I)h_i\}_{i=1}^d$ is a basis of X. By Lemma 3.2, the subgroup H_0 generated by $\{(A_i - I)h_i\}_{i=1}^d$ is a subgroup of $H_{tor} \le \mathbb{R}^n$, and by construction its $\mathbb{R}$-linear span is X. As the $\mathbb{R}$-linear span of H_{tor} is also equal to X by Lemma 3.5, we observe that H_0 is finite index in H_{tor}.

Applying Lemma 3.4 to $A_i \in G_{tor}$ and $kh_i \in H_{tor}$ for any $k \in \mathbb{Z}$ and $1 \le i \le d$, we know there is some uniform M such that $(A_i - I)kh_i$ lies in H_{tor} and the corresponding translation $T_{(A_i-I)kh_i}$ is a product of at most M torsion elements. By the definition of H_0, any element $h \in H_0$ can be written as $\sum_{i=1}^d k_i(A_i - I)h_i$ for some $k_i \in \mathbb{Z}$, and thus the corresponding translation

$$T_h = \prod_{i=1}^{d} T_{k_i(A_i-I)h_i}$$

is a product of at most dM torsion elements.

Since H_0 is finite index in H_{tor}, it is also finite index in Γ_{tor}. By fixing coset representatives of $H_0 \le \Gamma_{tor}$ and expressing them as products of torsion elements, the result follows from bounded generation of H_0 that we showed above. □

Example 3.6. Consider the $(3,3,3)$-triangle group

$$\Gamma \coloneqq \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = 1\rangle.$$

Γ acts properly discontinuously and cocompactly on the Euclidean plane with fundamental domain an equilateral triangle T, where the generators a, b, and c act by reflections about the three lines ℓ_1, ℓ_2, and ℓ_3 containing the three sides of T respectively. This realizes Γ as a cocompact discrete subgroup of $E(2)$.

By Theorem 3.1, Γ is boundedly generated by torsion elements. Here, we explicitly show that any $\gamma \in \Gamma$ is a product of at most four torsion elements.

The orbit of T under the Γ actions gives a tiling of $\mathbb{R}^2$ as in Figure 3. As T is a fundamental domain, elements of Γ are in one-to-one correspondence to the image of T. All lines in the tiling are the image of ℓ_1, ℓ_2, or ℓ_3 under Γ. Thus any reflection fixing one of these lines is a conjugate of a, b, or c. and hence an element of Γ. So it suffices to show that one can take T to any other triangle in the tiling using at most four such reflections.

We can take T to any blue triangle in Figure 3 by at most two reflections about horizontal lines in the tiling. Now we can arrive at any red triangle by further applying a reflection about a line in the tiling parallel to ℓ_2. Note that any remaining triangle shares a side with a red triangle, so we can arrive at any remaining triangle by another reflection. Hence we can reach any triangle in the tiling by at most four reflections.

Remark 3.7. For any integers $p, q, r \ge 2$, the group $\Gamma_{p,q,r} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^r = 1\rangle$ is called the (p, q, r)-triangle group. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, then $\Gamma_{p,q,r}$ is finite, so $\mathrm{stl}_{\Gamma_{p,q,r}} \equiv 0$. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then $\Gamma_{p,q,r}$ is crystallographic, so by Theorem 3.1, $\mathrm{stl}_{\Gamma_{p,q,r}} \equiv 0$.

Finally, if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then $\Gamma_{p,q,r}$ is non-elementary hyperbolic. By remark 2.7, $\mathrm{stl}_{\Gamma_{p,q,r}} \not\equiv 0$.

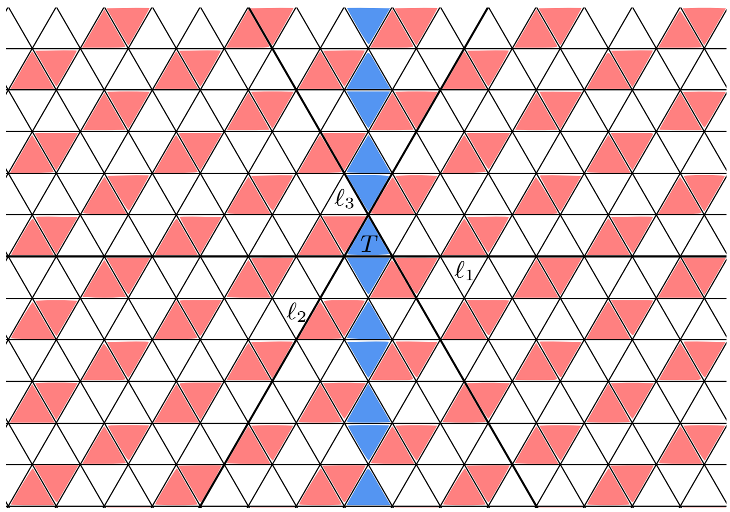

FIGURE 3. The tiling given by the action of the triangle group.

4. Free products

In the rest of this paper, we focus on stable torsion length in free products.

Let $G = A * B$ be a free product of groups A and B. Let X_A be a $K(A, 1)$ space, let X_B be a $K(B, 1)$ space, and let X be the space obtained by connecting X_A, X_B by a line segment with midpoint $*$. In the sequel, we will really think of X_A as including the half segment up to $*$, and similarly for X_B.

We develop a normal form for torsion-admissible surfaces (defined in Section 2.2) in X in Section 4.1. The normal form can be further simplified to *simple surfaces* which we introduce in Section 4.2. Describing the stable torsion length in terms of simple surfaces leads to an isometric embedding theorem (Section 4.3) and a linear programming problem which produces an effective lower bound of $\mathrm{stl}_G(g)$ for any element $g \in G$ (Section 4.4). Specializing to the case where A and B are finite abelian groups, we will further develop an algorithm that computes $\mathrm{stl}_G(g)$ for any g in Section 5.

4.1. **The normal form.** Let $g \in A * B$ be an element which does not conjugate into A or B. Stable word length is constant on conjugacy classes, so it suffices to consider g as a cyclically reduced word $g = a_1 b_1 \cdots a_L b_L$ where $a_i \in A \setminus \{id\}$ and $b_i \in B \setminus \{id\}$.

Let γ be a loop in X representing g such that $*$ decomposes it as a concatenation of $2L$ arcs $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_L, \beta_L$ cyclically, where each α_i (resp. β_i) is supported on the A-side (resp. B-side) and represents $a_i \in A$ (resp. $b_i \in B$) as a loop based at $*$.

Let $f : S \to X$ be a torsion-admissible surface (see Section 2.2). Recall that by definition there is a specified boundary component $\partial_0 S$ of S whose image represents a power of g, and the remaining boundary components are referred to as holes. Since the image of each hole represents a torsion element in $G = A * B$, which must be conjugate to a torsion element in either A or B, up to homotopy we may assume the image of each hole is disjoint from $*$. Perturb f further by a homotopy to make it transverse to $*$ and keep the image of holes disjoint from $*$. Then $F := f^{-1}(*)$ is an embedded proper submanifold of S of codimension one. Thus F is a finite collection of disjoint embedded loops and proper arcs. Moreover, the endpoints of any proper arc in F lie on $\partial_0 S$ since all other boundary components are disjoint from F.

Lemma 4.1. *Up to homotopy and compression of S into another torsion-admissible surface S' of the same degree such that $-\chi^-(S') \leq -\chi^-(S)$ and $H(S') \leq H(S)$, we can assume that $F = f^{-1}(*)$ only consists of proper arcs.*

Proof. If F contains any embedded loop τ that is essential (possibly boundary parallel) in S, then τ must cut S into two components since S is planar. Since $\tau \subset F$ is disjoint from $\partial_0 S$, only one of the components contains $\partial_0 S$ after cutting. Let S' be this component with a disk coning off τ, and extend f by mapping the entire disk to $*$. Further compose f with a homotopy which pushes the image of the disk away from $*$. In this way we obtain a simple torsion-admissible surface and eliminate an embedded essential loop $\tau \subset F$ without changing the map on $\partial_0 S$ (and thus the degree). This deletes an arbitrary embedded essential loop in F.

Now suppose F contains any inessential embedded loop ρ, i.e. ρ bounds a disk in S. Take the inner-most disk D among those bounding such loops. Up to a homotopy, we can modify f on a small neighborhood of D to eliminate an inessential embedded loop $\rho = \partial D \subset F$.

By applying the above operations finitely many times, we obtain a torsion-admissible surface S' with the desired properties. □

From now on, assume $F = f^{-1}(*)$ only consists of proper arcs. Denote $S_A := f^{-1}(X_A)$ and $S_B := f^{-1}(X_B)$. Then F cuts S into S_A and S_B, which are collections of subsurfaces with corners, and map into X_A and X_B respectively. See the example below.

Example 4.2. Suppose $g = a_1 b_1 a_2 b_2$ where a_1 is a product two torsion elements, a_2 is 2-torsion, and $b_2 = b_1^{-1}$. Then a torsion-admissible surface S can be constructed as shown in Figure 4, where the red boundary component represents g^4 and each blue boundary represents a torsion element (in A). Then F is the disjoint union of the green arcs, and the subsurface S_A is the union of those pieces in darker grey.

In general, there are two types of boundary components of S_A:

1. A *polygonal boundary* is one that contains corners, arcs in F, and arcs in $\partial_0 S$. Such a boundary is divided into an even number of sides by corners of S_A, where the sides alternate between arcs on F and arcs on $\partial_0 S$ which are mapped to some α_i; see the "outer" boundary component in Figure 4 of each component in S_A.
2. A *hole* is a boundary component that is disjoint from F. Then by construction, it must come from a hole of S and represent a torsion element in A; see the blue boundary components of S_A in Figure 4.

Lemma 4.3. *With the above setup, each component of S_A has exactly one polygonal boundary.*

Proof. A component of S_A without any polygonal boundary must be itself a component of S with only holes on the boundary, which is absurd since S is connected. If a component of S_A has at least two polygonal boundaries C_1 and C_2, then an

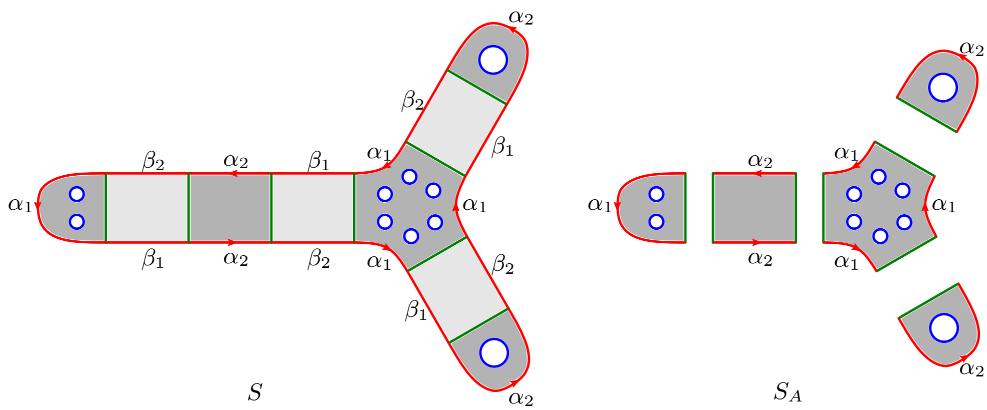

FIGURE 4. The decomposition of a torsion-admissible surface S by cutting along F (the green arcs). The subsurface S_A is pictured on the right.

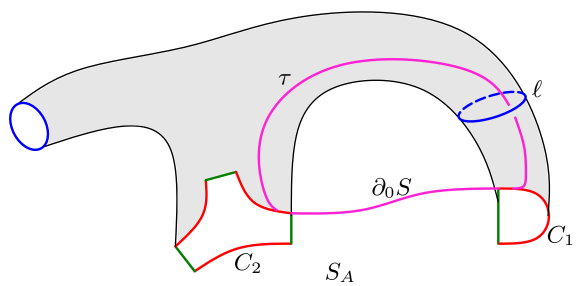

FIGURE 5. An arc τ traveling from one side of ℓ to the other side using part of the boundary $\partial_0 S$.

embedded loop ℓ in S_A homotopic to C_1 is non-separating in S: one can go from one side of ℓ to C_1, follow $\partial_0 S$ to arrive at C_2, and then travel to the other side of ℓ in this component; see the arc τ in Figure 5. This contradicts the fact that S is planar. $\square$

Thus every component of S_A is a polygon with $h \geq 0$ holes and a unique polygonal boundary. The same analysis works for components of S_B.

Definition 4.4. The *normal form* of a torsion-admissible surface S is the decomposition of S into S_A and S_B as above, where each component of S_A (resp. S_B) is a polygon with $h \geq 0$ holes and a unique polygonal boundary.

We summarize the discussion above in the following lemma and corollary.

Lemma 4.5. *Any torsion-admissible surface S can be modified into another torsion-admissible surface S' in normal form of the same degree, such that $-\chi(S) \geq -\chi(S')$ and $H(S) \geq H(S')$.*

Corollary 4.6. *In equation (2.1) of Lemma 2.9 we can take each infimum over torsion-admissible surfaces in normal form instead.*

4.2. **Simple Surfaces.** The collection of torsion-admissible surfaces for an element $g \in G$ can be further simplified to the collection of simple surfaces, which we now introduce. We use simple surfaces to obtain effective estimates of $\mathrm{stl}_G(g)$ in Section 4.4. We push this further in Section 5 when G is a free product of finite abelian groups to compute $\mathrm{stl}_G(g)$.

Roughly speaking, a simple surface S is made of particular pieces either in the A-side, or the B-side. This is similar to a torsion-admissible surface in normal form, however, one main difference is that each piece now only contains at most one hole. Before introducing simple surfaces, we first define the collection of pieces that are allowed in simple surfaces.

Recall that the loop γ representing g decomposes into arcs $\alpha_1, \beta_1, \ldots, \alpha_L, \beta_L$, representing elements $a_1, b_1, \cdots, a_L, b_L$ in A and B.

Definition 4.7. A *polygonal boundary* is an oriented circle together with a map f into X_A, so that the map f naturally divides the loop into an even number of sides alternating between *arcs* and *turns* as follows. The map f collapses each turn to the wedge point $*$ and maps each arc to some α_i, called the the *label* of the arc. See each "outer" boundary components of S_A in Figure 4 or 6 for examples, where arcs are in red and turns are in green.

As a loop in X_A, the polygonal boundary represents a conjugacy class in A, referred to as the *winding class* of the polygonal boundary.

If a polygonal boundary has trivial winding class, the map extends to a disk bounding the polygonal boundary. We call such a disk with polygonal boundary a *disk-piece*.

If a polygonal boundary has nontrivial winding class, we require it to lie in A_{tor}. We represent such a conjugacy class by a loop in X_A which is away from $*$ and homotopic to the polygonal boundary. Then, there is an annulus bounding the polygonal boundary on one side and this homotopic loop (with opposite induced orientation) on the other side. We refer to this annulus as an *annulus-piece*. We refer to the non-polygonal boundary as the *hole* in the annulus-piece.

A *piece on the A-side* is defined to be either a disk-piece or an annulus-piece. We denote the collection of all possible pieces on the A-side as $\widetilde{\mathcal{P}}_A$. Similarly we define *pieces on the B-side* and denote the corresponding collection as $\widetilde{\mathcal{P}}_B$. Let $\widetilde{\mathcal{P}} := \widetilde{\mathcal{P}}_A \cup \widetilde{\mathcal{P}}_B$ be the collection of all pieces.

Each turn on a polygonal boundary in A with the given orientation travels from an arc labeled by some α_i to another labeled by some α_j. We say such a turn is of *type* (α_i, α_j). Similarly each turn on the B-side is of type (β_k, β_ℓ) for some k and ℓ. We say a turn of type (α_i, α_j) is *compatible* with a turn of type (β_{j-1}, β_i), where indices are taken mod L.

Pieces on the A-side can glue to pieces on the B-side along compatible turns and the maps on these pieces can be extended continuously in the obvious way; see the left of Figure 6 where pieces are glued along compatible turns, colored in green.

Definition 4.8 (Simple surfaces)**.** A *simple surface* S is a finite collection of pieces in $\widetilde{\mathcal{P}}$ together with a pairing on the set of turns of the given pieces so that paired turns are compatible. Geometrically, we think of S as a surface obtained by gluing the given finitely many pieces along turns by the given pairing; see the left of Figure 6 for an example where $L = 2$. It follows from the definition of compatible turns that each boundary component of S containing at least one arc must wind around γ by a positive number of times. Define the degree $n(S) > 0$ of S to be the sum of these positive numbers.

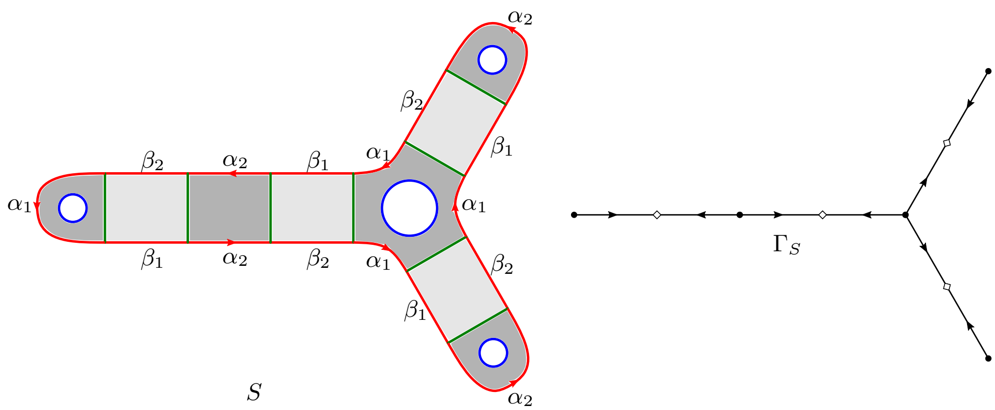

FIGURE 6. A simple surface S and its gluing graph Γ_S. Here S is obtained by simplifying the torsion-admissible surface in Figure 4.

Definition 4.9 (Gluing graph)**.** Associated to each simple surface is a *gluing graph* Γ_S, where each vertex represents a piece and each edge connecting two vertices represents two paired compatible turns that the two pieces glue along; see the right of Figure 6.

Lemma 4.10. *For a simple surface S, let e be the number of edges in Γ_S and let d be the number of disk-pieces in S. Then*

$$-\chi(S) = e - d.$$

Proof. By filling in all holes in annulus-pieces of S (using disks), we obtain a surface S' that deformation retracts to Γ_S. If v is the number of vertices in Γ_S, then,

$$-\chi(S) = -\chi(S') + (v-d) = -\chi(\Gamma_S) + (v-d) = e - v + v - d = e - d,$$

and $v - d$ is the number of annulus-pieces in S. □

To allow some desired flexibility, we do not require simple surfaces to be connected. A simple surface S is connected if and only if the gluing graph Γ_S is connected. Connected simple surfaces S with $\chi(\Gamma_S) = 1$ are closely related to torsion-admissible surfaces in normal form.

Lemma 4.11. *Given an element $g = a_1 b_1 \cdots a_L b_L$ in the free product $A * B$. For a connected simple surface S with $\chi(\Gamma_S) = 1$, if each hole in an annulus-piece on the A-side (B-side) represents a torsion element in A (resp. B), then S is torsion-admissible for g of degree $n(S)$. Conversely, any torsion-admissible surface S for g can be simplified into a connected simple surface S' of the same degree with $\chi(\Gamma_{S'}) = 1$ such that $-\chi(S') \le -\chi(S)$ and $H(S') \le H(S)$. Thus*

$$\mathrm{stl}(g) \ge \inf \frac{H(S)}{n(S)} = \inf \frac{-\chi(S)}{n(S)}, \tag{4.1}$$

where each infimum is taken over all connected simple surfaces S with $\chi(\Gamma_S) = 1$.

Proof. For a connected simple surface S, if we cap off all the holes in its annulus-pieces, then S deformation retracts to Γ_S. Thus if $\chi(\Gamma_S) = 1$, the capped-off surface is a connected surface with boundary of Euler characteristic 1, i.e. a disk.

Therefore a simple surface S with $\chi(\Gamma_S)=1$ has genus zero and a *single* boundary component that represents $g^{n(S)}$, and all other boundary components of S are holes in its annulus-pieces. Hence if each hole in an annulus-piece in $\widetilde{\mathcal{P}}_A$ (resp. $\widetilde{\mathcal{P}}_B$) represents a torsion element in A (resp. B), then S is torsion-admissible for g of degree $n(S)$ by definition.

Conversely, given any torsion-admissible surface S for g of degree n, we can put it in normal form by Lemma 4.5 without changing the degree. Each component of S_A (resp. S_B) gives rise to a piece in $\widetilde{\mathcal{P}}_A$ (resp. $\widetilde{\mathcal{P}}_B$) except that it may contain more than one hole. For each component of S_A containing more than one hole, replace it by a piece with the same polygonal boundary, which is a disk or an annulus depending on whether the winding class of the polygonal boundary is trivial or not. Note that the winding class of such a polygonal boundary on the A-side (resp. B-side) always lies in A_{tor} (resp. B-side) since it is the product of those torsion elements corresponding to the holes (up to conjugacy). The surface S' obtained this way is connected and planar. Thus S' is a connected simple surface with $\chi(\Gamma_{S'})=1$ and degree $n(S')=n(S)$. Moreover, as S' is obtained from the normal form of S by eliminating some holes, we have $-\chi(S')\le-\chi(S)$ and $H(S')\le H(S)$ using Lemma 4.5. See Figure 4 and Figure 6 for an example.

The two infima are equal for a similar reason to that of formula (2.1) by taking suitable covering spaces; see Figure 12 for an illustration of a good covering space of a simple surface. □

It is often convenient to consider a subfamily of simple surfaces, where only a subset of types of pieces are allowed.

Definition 4.12. Given a collection $\mathcal{P}\subset\widetilde{\mathcal{P}}$ of types of pieces, a simple surface S (with respect to g) is called $\mathcal{P}$-*simple* if all pieces used in S have types in $\mathcal{P}$. A collection $\mathcal{P}$ is called *sufficient* if

$$\mathrm{stl}(g)\ge\inf\frac{-\chi(S)}{n(S)},$$

where the infimum is taken over all connected $\mathcal{P}$-simple surfaces S with $\chi(\Gamma_S)=1$.

Lemma 4.11 shows that taking $\mathcal{P}=\widetilde{\mathcal{P}}$ gives a sufficient collection. In many cases large pieces can be simplified into several small pieces (see Section 5.1), which allows a finite small collection $\mathcal{P}$ to be sufficient.

Corollary 4.13. *Suppose*

(1) *either both A_{tor} and B_{tor} are torsion groups,*
(2) *or $g=a_1b_1\cdots a_Lb_L$, where the subgroup generated by $\{a_1,\cdots,a_L\}$ (resp. $\{b_1,\cdots,b_L\}$) is a torsion group.*

Then

$$\mathrm{stl}(g)=\inf\frac{-\chi(S)}{n(S)},$$

where the infimum is taken over all connected $\mathcal{P}$-simple surfaces S with $\chi(\Gamma_S)=1$ for some sufficient.

Proof. The "$\ge$" direction holds by definition. The other direction holds since under both assumptions every connected simple surface S with $\chi(\Gamma_S)=1$ is torsion-admissible by Lemma 4.11. □

4.3. **Isometric embedding.** As an application of Corollary 4.13, injective homomorphisms of factor groups induce an embedding of the free products, which preserves the stable torsion length for generic elements, under suitable assumptions.

Theorem 4.14. *Let $i_A : A \to A'$ and $i_B : B \to B'$ be injective homomorphisms. Let $g \in A * B$ be an element that is not conjugate into A or B. Suppose*

(1) *either both A'_{tor} and B'_{tor} are torsion groups,*
(2) *or g is conjugate to a cyclically reduced word $a_1 b_1 \cdots a_L b_L$, where the subgroup generated by $\{a_1, \cdots, a_L\}$ and the subgroup generated by $\{b_1, \cdots, b_L\}$ are torsion groups.*

Then the induced map $i : A * B \to A' * B'$ preserves the stable torsion length of g, i.e.

$$\mathrm{stl}_{A*B}(g) = \mathrm{stl}_{A'*B'}(i(g)).$$

Proof. It suffices to show that $\mathrm{stl}_{A*B}(g) \le \mathrm{stl}_{A'*B'}(i(g))$ since the other direction follows by monotonicity (Lemma 2.2). Since i_A is injective, torsion elements in A correspond to torsion elements in the image of i_A. Thus A_{tor} is a torsion group if A'_{tor} is, and similarly for B_{tor}. Hence by Corollary 4.13, under either assumption we have $\mathrm{stl}_{A*B}(g) = \inf_S \frac{-\chi(S)}{n(S)}$, where the infimum is taken over connected simple surfaces S with $\chi(\Gamma_S) = 1$.

On the other hand, by Lemma 4.11, we know $\inf_{S'} \frac{-\chi(S')}{n(S')} \le \mathrm{stl}_{A'*B'}(i(g))$, where the infimum is taken over all connected simple surfaces S' for $i(g)$ with $\chi(\Gamma_{S'}) = 1$. Thus it suffices to show that $\inf_S \frac{-\chi(S)}{n(S)} \le \inf_{S'} \frac{-\chi(S')}{n(S')}$. We prove this by showing that every simple surface for $g' := i(g)$ naturally pulls back to a simple surface for g.

Up to conjugation, we may write $g = a_1 b_1 \cdots a_L b_L$, where $a_j \in A \setminus \{id\}$, $b_j \in B \setminus \{id\}$, and $L \ge 1$. Then $g' = a'_1 b'_1 \cdots a'_L b'_L$, where $a'_j = i_A(a_j)$ and $b'_j = i_B(b_j)$.

Let C' be a piece on the A'-side in a simple surface S' for g', and suppose its polygonal boundary consists of arcs corresponding to $a'_{j_1}, \cdots, a'_{j_k}$ in the cyclic order, for some $k \in \mathbb{Z}_+$. Then we can construct a polygonal boundary consisting of arcs corresponding to $a_{j_1}, \cdots, a_{j_k}$ in the cyclic order, say with winding class $w \in A$.

We claim that w is torsion under both assumptions. Since w is a product of a_i's, this is obvious under assumption (2). If A'_{tor} is a torsion group as in assumption (1), then the winding class of the polygonal boundary of C' must be $i_A(w)$, which lies in A'_{tor} and thus must be torsion. Since i_A is injective, we know w must be a torsion element as well in this case.

It follows that the polygonal boundary we construct bounds an A-piece C. Moreover, since $w = id$ if and only if $i_A(w) = id$, the piece C has the same topological type as the piece C'. Similarly we can construct a B-piece corresponding to any B'-piece. Doing this for all pieces of the simple surface S', we obtain pieces that assemble accordingly to a simple surface S for g that has the same degree as S' and $\chi(S) = \chi(S')$. Thus $\inf_S \frac{-\chi(S)}{n(S)} \le \inf_{S'} \frac{-\chi(S')}{n(S')}$ as desired, which completes the proof. $\square$

Now Theorem E follows as a simple corollary.

Proof of Theorem E. If g is conjugate to an element in A, then g is torsion since A is finite. Then $i(g)$ is also torsion and thus $\mathrm{stl}_{A*B}(g) = 0 = \mathrm{stl}_{A'*B'}(i(g))$. Similarly the equality holds if g is conjugate to an element in B.

Now if g is not conjugate into A or B, then the assumption (2) in Theorem 4.14 obviously holds since A and B are finite groups. Hence the equality follows by Theorem 4.14. □

4.4. **Lower bounds via linear programming.** Given a sufficient collection $\mathcal{P}$ of types of pieces (Definition 4.12), we describe a (possibly infinite-dimensional) linear programming problem that produces a lower bound of $\mathrm{stl}_G(g)$. This is based on the following observation.

Lemma 4.15. *Suppose $\mathcal{P}$ is a sufficient collection for a free product $G = A * B$. For any element g not conjugate into factor groups, we have*

$$\mathrm{stl}_G(g) \geq \inf_S \frac{-\chi(S)}{n(S)},$$

where the infimum is taken over all (not necessarily connected) $\mathcal{P}$-simple surfaces S with $\chi(\Gamma_S) \geq 0$.

Proof. Since the family of surfaces we consider here contains all connected $\mathcal{P}$-simple surfaces S with $\chi(\Gamma_S) = 1$, the result follows from the definition of sufficient collections (Definition 4.12). □

The reason to consider $\mathcal{P}$-simple surfaces that are not necessarily connected with the relaxed constraint $\chi(\Gamma_S) \geq 0$ is that this family of surfaces is easier to work with, allowing us to encode such surfaces as vectors in a nice subspace of a vector space as follows.

Given a reduced word g and a sufficient collection $\mathcal{P}$ as above, let $V_{\mathcal{P}} = \mathbb{R}^{\mathcal{P}}$. Let $\{e_P \mid P \in \mathcal{P}\}$ be the standard basis, where e_P is the positive unit vector in the P-direction. For any $\mathcal{P}$-simple surface S and any $P \in \mathcal{P}$, let x_P be the number of pieces of type P in S. Associate to S a vector $v(S) \in V_{\mathcal{P}}$ so that the P-component of $v(S)$ is x_P for any $P \in \mathcal{P}$.

The vector $v(S)$ is a non-negative integer point in $V_{\mathcal{P}}$ satisfying some rational linear constraints that we describe below. For any turn type T (e.g. $T = (\alpha_i, \alpha_j)$ or (β_i, β_j)), there is a linear function $f_T : V_{\mathcal{P}} \to \mathbb{R}$ such that for the standard basis $f_T(e_P)$ counts the number of turns of type T in a piece of type P for each $P \in \mathcal{P}$. For any two compatible turn types T, T', i.e. $T = (\alpha_i, \alpha_j)$ and $T' = (\beta_{j-1}, \beta_i)$, we have

$$f_T(v(S)) = f_{T'}(v(S))$$

for any $\mathcal{P}$-simple surface S since each turn is glued to another turn by definition. We refer to this set of equations as the *gluing conditions.*

Let $\chi_\Gamma : V_{\mathcal{P}} \to \mathbb{R}$ be the linear function determined by $\chi(e_P) = 1 - \frac{e}{2}$ for each $P \in \mathcal{P}$, where e is the number of turns on the polygonal boundary of the piece P. Similarly we have a linear function $\chi_o : V_{\mathcal{P}} \to \mathbb{R}$ with the property that $\chi_o(e_P) = \chi(P) - \frac{e}{2}$ for each $P \in \mathcal{P}$, where $\chi(P)$ is 1 if P is a disk-piece and is 0 if P is an annulus-piece. Then it is straightforward to see that $\chi_\Gamma(v(S)) = \chi(\Gamma_S)$ and $\chi_o(v(S)) = \chi(S)$ for any $\mathcal{P}$-simple surface S with gluing graph Γ_S.

Finally, let $n : V_{\mathcal{P}} \to \mathbb{R}$ be the linear function such that $n(e_P)$ counts the number of copies of α_1 on the polygonal boundary of P for each $P \in \mathcal{P}$. Then for any $\mathcal{P}$-simple surface S, its degree is $n(v(S))$.

Definition 4.16. Given the word g and a sufficient collection $\mathcal{P}$, let $C_{\mathcal{P}}$ be the subspace of $V_{\mathcal{P}}$ consisting of vectors x that satisfy all gluing conditions, $\chi_\Gamma(x) \geq 0$, and $n(x) = 1$.

Summarizing up the discussion above, we have:

Lemma 4.17. *For any $\mathcal{P}$-simple surface of degree n, the vector $v(S)/n$ is a rational point in $C_{\mathcal{P}}$ and $\chi_o(v(S)/n) = \chi(S)/n$.*

Conversely, we have:

Lemma 4.18. *For any rational point $x \in C_{\mathcal{P}}$, there is some $n \in \mathbb{Z}_+$ and a $\mathcal{P}$-simple surface S of degree n, such that $x = v(S)/n$, $\chi(\Gamma_S) \geq 0$ and $\chi(S)/n = \chi_o(x)$.*

Proof. Choose n so that nx is an integer point in $V_{\mathcal{P}}$. Then $nx = \sum_P k_P e_P$ for some non-negative integers k_P. Take k_P pieces of type P for each $P \in \mathcal{P}$. Since x satisfies the gluing conditions, so does nx. Thus by gluing these pieces along compatible pairs of turns, we obtain a $\mathcal{P}$-simple surface S such that $v(S) = nx$. Then we have $\chi(\Gamma_S) = \chi_\Gamma(nx) = n\chi_\Gamma(x) \geq 0$ and $\chi(S)/n = \chi_o(nx)/n = \chi_o(x)$ since both χ_Γ and χ_o are linear on $V_{\mathcal{P}}$. □

It follows that we can compute the infimum in Lemma 4.15 by minimizing the rational linear function $-\chi_o$ on the compact polyhedron $C_{\mathcal{P}}$ (see the lemma below), which is a linear programming problem. This gives a way to compute a nontrivial lower bound of $\mathrm{stl}_G(g)$. We compute two explicit examples in Section 6, where the lower bounds are actually sharp in both cases.

Lemma 4.19. *For a finite sufficient collection $\mathcal{P}$, the set $C_{\mathcal{P}}$ is a rational compact polyhedron. Moreover, it is nonempty if some power of g is a product of torsion elements in G. In this case, the infimum of $-\chi(S)/n(S)$ over all $\mathcal{P}$-simple surfaces S with $\chi(\Gamma_S) \geq 0$ is achieved.*

Proof. By definition, the set $C_{\mathcal{P}}$ is defined by finitely many rational linear inequalities. The normalizing condition $n(x) = 1$ together with gluing conditions implies that each coordinate of x is no more than 1 for any $x \in C_{\mathcal{P}}$. In other words, the normalized number of any piece is at most one since the degree is normalized to be one. Thus $C_{\mathcal{P}}$ is a rational compact polyhedron when $\mathcal{P}$ is finite.

When a power of g is a product of torsion elements in G, torsion-admissible surfaces exist. Thus by Lemma 4.11, we can reduce any torsion-admissible surface to a simple surface S with $\chi(\Gamma_S) = 1$, which yields a rational point in $C_{\mathcal{P}}$ by Lemma 4.17. Hence $C_{\mathcal{P}}$ is nonempty.

Then by Lemmas 4.17 and 4.18, the infimum of $-\chi(S)/n(S)$ over all $\mathcal{P}$-simple surface S with $\chi(\Gamma_S) \geq 0$ can be calculated as the infimum of the rational linear function $-\chi_o$ on $C_{\mathcal{P}}$. Hence by compactness the infimum is achieved by a (rational) vertex x of $C_{\mathcal{P}}$, which by Lemma 4.18 is of the form $v(S)/n$ for a simple surface S of degree n in the above family. □

5. Free products of abelian groups

In this section we focus on the case of a free product $G = A * B$, where A and B are finite abelian groups. We will exhibit an algorithm that computes $\mathrm{stl}_G(g)$ for any given element g. The method works for the free product of arbitrarily many finite abelian groups, but we won't pursue it here.

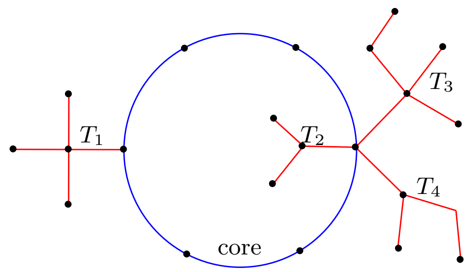

FIGURE 7. A connected graph with Euler characteristic zero that consists of a core (in blue) and four decorative trees (in red).

We adopt the setup and notation in the previous section. Note that by Corollary 4.13, to compute $\mathrm{stl}_G(g)$ for a given element g, it suffices to consider connected simple surfaces S (Definition 4.8) with $\chi(\Gamma_S) = 1$, where Γ_S is the gluing graph (Definition 4.9).

We will introduce two operations that further simplify surfaces: *splitting* and *rewiring*. In terms of the gluing graph, we will use splitting to reduce the valence of vertices, and use rewiring to reduce the diameter of the graph.

However, the family of connected simple surfaces S with $\chi(\Gamma_S) = 1$ is not closed under the two operations above. As a remedy, we consider the larger family of simple surfaces with $\chi(\Gamma_\Sigma) \geq 0$ for each component Σ, which is more convenient to work with for two reasons:

- The two operations (when applied appropriately) preserve this family, and
- The complexity of any connected simple surfaces with $\chi(\Gamma_S) = 0$ can be approximated by a sequence of connected simple surfaces with $\chi(\Gamma_S) = 1$, and thus can still be used to compute stl; see Lemma 5.4.

For a connected simple surface S with $\chi(\Gamma_S) = 0$, the gluing graph Γ_S is connected and has a unique embedded loop, which we refer to as the *core*. The gluing graph can be thought of as obtained from the core by attaching finitely many (rooted) trees to vertices on the core. We refer to each of such trees as *a decorative tree*, and the vertex it attaches to as the *root*. See Figure 7.

We first introduce the two operations, splitting and rewiring, in Sections 5.1 and 5.2 respectively. In particular we show the approximation Lemma 5.4 in Section 5.2 using rewiring. Then in Section 5.3 we define irreducible simple surfaces and show that every connected simple surface S with $\chi(\Gamma_S) \geq 0$ decomposes into a union of irreducible ones after applying splitting and rewiring. This yields an algorithm to compute stl and show that it is rational in free products of finite abelian groups; see Theorem 5.14.

5.1. **Splitting of a piece.** The first operation that we introduce on simple surfaces is *splitting* of a piece.

Let C be a piece on the A-side with polygonal boundary P. Suppose a proper subset of the turns on P can form a polygonal boundary P_1 with trivial winding class, and the remaining turns can form another polygonal boundary P_2. Then P_2 has the same winding class as P if A is abelian.

In this case P_1 bounds a disk-piece C_1 and P_2 bounds a piece C_2 that has the same topological type as the original piece C. When C is a piece on a simple surface

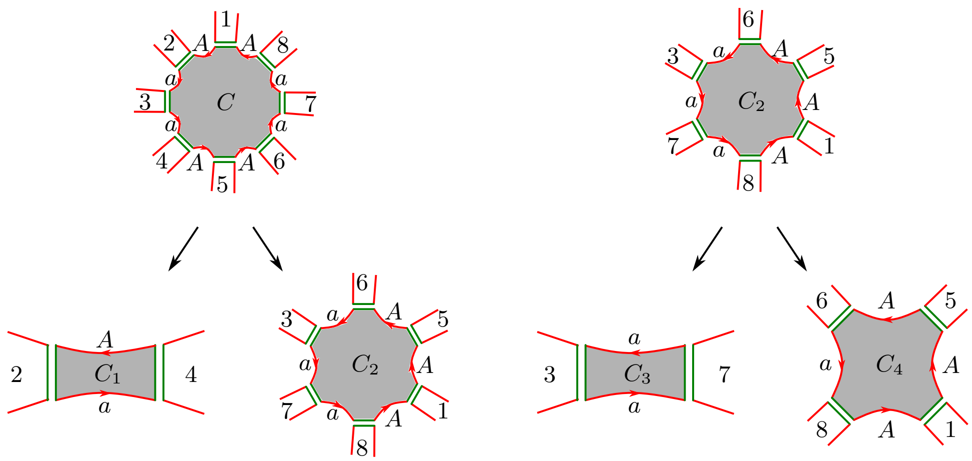

FIGURE 8. Two examples of splitting, where the one on the right assumes $a^2 = 1$.

S, *splitting* is the operation that we replace the piece C above by the two new pieces C_1 and C_2 without changing the gluing of turns. This modifies the simple surface without changing the number of holes while splitting one vertex of the gluing graph Γ_S into two.

Example 5.1. Let $\mathcal{A} = \mathbb{Z}/p$ be a cyclic group generated by a and let $g = abAB \in \mathcal{A} * \mathcal{B}$, where A and B denote a^{-1} and b^{-1} respectively. Let γ be a loop representing g, decomposed into arcs $\alpha_1, \beta_1, \alpha_2, \beta_2$ corresponding to a, b, A, B respectively.

(1) On the left of Figure 8, we have disk-piece C on the $\mathcal{A}$-side with two copies of each of the four turns (α_1, α_1), (α_1, α_2), (α_2, α_2), (α_2, α_1). The two turns (α_1, α_2) and (α_2, α_1) form a disk-piece C_1 and the remaining six turns form another disk-piece C_2. The splitting breaks the piece C into the pieces C_1 and C_2.

(2) There are two consecutive copies of the turn (α_1, α_1) on C_2. If $p = 2$, these two turns form a disk-piece C_3 since $a^2 = 1$, and the remaining turns form another disk-piece C_4, shown on the right of Figure 8. Note that in this case, the turns on the new pieces sit in a cyclic order compatible to the their cyclic order on C_2, which is not the case for the previous splitting.

For the rest of this section, we focus on a special case similar to case (2) in Example 5.1, where splitting works without assuming the factor groups to be abelian. Suppose there is a proper subsequence of sides

$$(T_0, A_1, T_1, \cdots, A_k, T_k)$$

for some $k \geq 1$ on the polygonal boundary P in the positive cyclic order starting and ending at turns T_0, T_k of the *same* type, such that the product of the elements represented by the arcs $A_1, \cdots, A_k$ is the identity in A. Let

$$(A_{k+1}, T_{k+1}, \cdots, A_n)$$

be the complementary sequence of sides in the positive cyclic order, where $n > k$. Then we obtain two polygonal boundaries P_1 and P_2, where the sides are

$(A_1, T_1, \cdots, A_k, T_k)$ and $(A_{k+1}, T_{k+1}, \cdots, A_n, T_n = T_0)$, respectively. Then by the assumption, the winding class of P_1 is trivial and thus P_1 bounds a disk-piece C_1. The winding class of P_2 is the same as that of P and thus P_2 bounds a piece C_2 that has the same topological type as C.

Splitting decomposes such a piece C into two pieces C_1 and C_2 without changing the total number of holes. In addition, it does not affect the gluing of compatible turns. Analogously one can perform this for pieces on the B-side.

It is helpful to think about the effect of this operation conceptually in terms of the gluing graph Γ_S. Orient edges so that they go from vertices representing pieces on the A-side to those on the B-side. Such edges fall into different *types* according to the types of turns. Then splitting of a piece applies to a vertex which necessarily have two adjacent edges e_1, e_2 of the same type. It replaces such a vertex v by two vertices v_1, v_2, where part of the original adjacent edges become edges at v_1 and the others are edges at v_2, so that e_i is an edge at v_i.

Note that Γ_S is actually a *fatgraph* in the sense that there is a cyclic order on the edges at each vertex, which is induced from the orientation on the polygonal boundary of each piece. Hence for this special type of splitting, the cyclic order at the vertex v that we split and the position of e_1, e_2 in the order determine which edges of v become edges of v_1 and v_2.

We are able to apply splitting to any vertex with large valence in the gluing graph if the corresponding factor group is finite.

Lemma 5.2. *Suppose the abelian group A is finite. Then for the element $g = a_1 b_1 \cdots a_L b_L$, we can split any piece on the A-side that has more than $|A| \cdot L^2$ turns on its polygonal boundary.*

Proof. Note that there are L^2 possible types of turns on the A-side. Suppose there are more than $|A| \cdot L^2$ turns, then by the pigeonhole principle there exist $|A|+1$ turns of the same type. These turns cut the polygonal boundary into $|A|+1$ segments. Each segment represents an element in A by taking the product of elements corresponding to the arcs on the segment. Let $x_1, \cdots, x_{|A|+1} \in A$ be the elements corresponding to these segments. Then by the pigeonhole principle, there exist $1 \le m < n \le |A|+1$ such that $x^{(m)} = x^{(n)}$, where $x^{(i)} = x_1 \cdots x_i$. This implies $x_{m+1} \cdots x_n = id$ and hence we can apply splitting to this piece. $\square$

5.2. **Rewiring.** The second operation that we introduce on simple surfaces is *rewiring*. This has been used in a similar setting to understand stable commutator length [Che20]. However, in this setting, it is necessary to apply this operation more carefully here to control the Euler characteristic of each component of the gluing graph. We describe this below.

Suppose that there exist two edges e_1, e_2 in the gluing graph of the same type and that each e_i goes from a vertex u_i to a vertex v_i, where $i = 1, 2$. Geometrically, thinking of vertices as pieces, this means that u_1, v_1 are glued together along compatible turns, which have the same type as the compatible turns along which we glue u_2 and v_2. Then, we can cut along these two pairs of turns and glue u_1 to v_2 and glue u_2 to v_1 instead. In terms of the gluing graph, we remove the edges e_1 and e_2 and construct two new edges connecting u_1 to v_2 and u_2 to v_1 instead; see Figure 9.

Note that rewiring does not change the types of pieces and preserves the number of vertices and edges of the gluing graph. Thus applying the rewiring operation

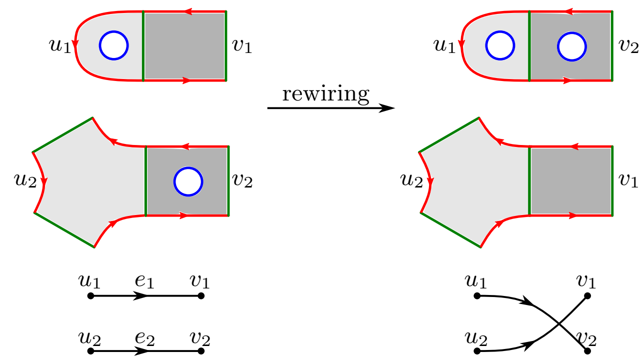

FIGURE 9. The effect of rewiring on the simple surface and gluing graph

to a simple surface S results in another simple surface S' with the same Euler characteristic. However, it might change the number of components and the Euler characteristic of individual connected components. For this reason, we will apply rewiring in a restricted way to preserve the family of simple surfaces with $\chi(\Gamma_\Sigma) \geq 0$ for each connected component Σ.

If e_1, e_2 lie in different components of Γ_S and at least one of them is non-separating, then the rewiring merges the two components into a single component. This shows that the complexity of any connected simple surface with $\chi(\Gamma_S) = 0$ can be approximated by those with $\chi(\Gamma_S) = 1$. Before proving this in Lemma 5.4, we need the following simple observation.

Lemma 5.3. *If A and B are finite, then for the given element $g = a_1 b_1 \cdots a_L b_L$ and for any turn type (α_i, α_j), there is a connected simple surface S with $\chi(\Gamma_S) = 1$ that contains a turn of the given type (α_i, α_j).*

Proof. Since A is finite, there is a piece $P^A_{i,j}$ with exactly two turns (α_i, α_j) and (α_j, α_i). Similarly there is a piece $P^B_{i,j}$ with exactly two turns (β_i, β_j) and (β_j, β_i). So it suffices to put each $P^A_{i,j}$ into a simple surface with the desired properties. Note that there is an arbitrarily long strip of pieces glued together centered at $P^A_{i,j}$, such that on one side we have $P^B_{i,j-1}$, $P^A_{i+1,j-1}$, $P^B_{i+1,j-2}$ and so on, and on the other side we have $P^B_{i-1,j}$, $P^A_{i-1,j+1}$, $P^B_{i-2,j+1}$ and so on; see the top of Figure 10 for an example with $i = 2$ and $j = L = 4$. Here the indices are taken mod L.

For each piece $P^*_{k,\ell}$ in the above sequence, we take the difference $k - \ell$ of the two subindices. Then we observe that the differences between consecutive pieces are consecutive integers and form a monotone sequence. Thus on both sides of $P^A_{i,j}$, we can find pieces of the form $P^*_{m,n}$ with $m \equiv n \mod L$ and $* = A$ or B. We can cut the strip at such a piece and replace this piece by the piece with only one arc α_m (resp. β_m) if $* = A$ (resp. $* = B$); see the bottom of Figure 10. This constructs a connected simple surface S containing $P^A_{i,j}$ such that Γ_S is a tree. $\square$

Lemma 5.4. *Suppose that for the given element g and any turn type (α_i, α_j), there is a connected simple surface S with $\chi(\Gamma_S) = 1$. Then, for any connected simple surface S of degree n with $\chi(\Gamma_S) = 0$, there exists a sequence of connected simple*

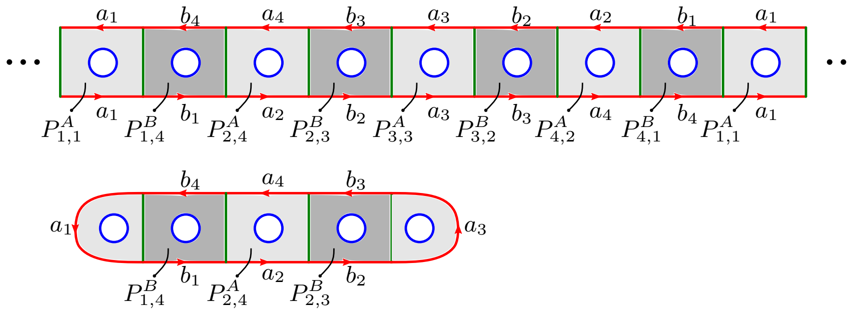

FIGURE 10. An example to include the turn type (α_2, α_4) in a long strip, which can be modified into a simple surface S for $g = a_1 b_1 \cdots a_4 b_4$ with $\chi(\Gamma_S) = 1$.

surfaces S_k of degree n_k with $\chi(\Gamma_{S_k}) = 1$ such that

$$\frac{-\chi(S)}{n} = \lim_k \frac{-\chi(S_k)}{n_k}.$$

Proof. Note that for each $k \in \mathbb{Z}_+$ there is a unique degree k cover Γ_k of Γ_S. This induces a degree k cover Σ_k of S, which is again a connected simple surface (of degree kn) with gluing graph Γ_k. Now pick any edge e on the core of Γ_S corresponding to a turn, say, of type (α_i, α_j). By assumption, we can fix a simple surface S_0 containing a turn of type (α_i, α_j) such that Γ_{S_0} is a tree. Then a lift of e to Γ_k provides a non-separating edge on Σ_k corresponding to a turn of type (α_i, α_j). Thus we can apply rewiring to Σ_k and S_0 to obtain a new simple surface S_k where the gluing graph Γ_{S_k} is connected. Then

$$\chi(\Gamma_{S_k}) = \chi(\Gamma_k) + \chi(S_0) = 0 + 1 = 1.$$

The degree of S_k is $n_k = kn + n_0$, where n_0 is the degree of S_0. Then, we have

$$\lim_{k\to\infty} \frac{-\chi(S_k)}{n_k} = \lim_{k\to\infty} \frac{-\chi(\Sigma_k) - \chi(S_0)}{kn + n_0} = \lim_{k\to\infty} \frac{-k\chi(S) - \chi(S_0)}{kn + n_0} = \frac{-\chi(S)}{n}$$

by construction. $\square$

Corollary 5.5. *If A and B are finite, then*

$$\mathrm{stl}_G(g) = \inf_S \frac{-\chi(S)}{n},$$

where the infimum is taken over all connected simple surfaces with $\chi(\Gamma_S) = 0$ or 1.

Proof. By Lemmas 5.3 and 5.4, the complexity of a connected surface with $\chi(\Gamma_S) = 0$ can be approximated by those with gluing graph being a tree. Thus the infimum remains the same if we restrict the class of surfaces to connected simple surfaces with $\chi(\Gamma_S) = 1$. Then the result follows from Corollary 4.13. $\square$

Remark 5.6. The equality still holds if we consider simple surfaces where each component Σ has $\chi(\Gamma_\Sigma) \geq 0$. Note that this is different from Lemma 4.15, which restricts the Euler characteristic of the gluing graph overall instead of component-wise.

For what follows, we will only apply rewiring to two edges in the same component of the gluing graph, particularly in the following three scenarios.

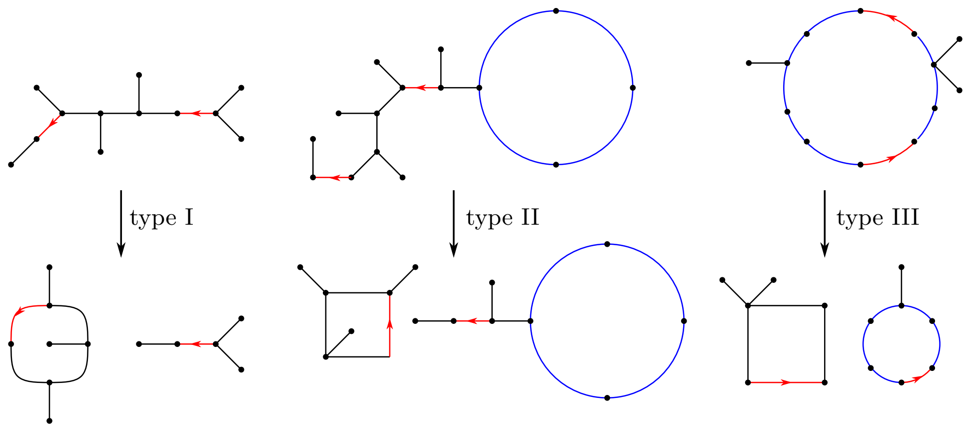

FIGURE 11. In terms of gluing graphs, this shows the effect of the three types of rewiring applied to the two red edges in each graph on top that have the same type and orientation.

The first scenario is when we have a simple surface S whose gluing graph Γ_S is a tree such that there is an embedded oriented path P that starts and ends with two distinct edges of the same type and orientation. Then applying rewiring to these two edges decomposes the simple surface into the union of two connected simple surfaces S_1 and S_2, where the gluing graph of S_2 is still a tree, the gluing graph of S_1 has $\chi(\Gamma_{S_1}) = 0$ and the path P becomes of the core of Γ_{S_1}. See the left of Figure 11. For later reference, we refer to this as *rewiring of type I*.

The second scenario is when we have a connected simple surface S with $\chi(\Gamma_S) = 0$ such that on a decorative tree T there is an embedded oriented path P that starts and ends with two distinct edges e_1, e_2 of the same type and orientation, and the unique path connecting e_2 to the root of T contains P. Then applying rewiring to these two edges decomposes the simple surface into the union of two connected simple surfaces S_1 and S_2, where $\chi(\Gamma_{S_1}) = \chi(\Gamma_{S_2}) = 0$, the core of Γ_{S_1} is inherited from Γ_S, and the core of Γ_{S_2} comes from the path P. See the middle of Figure 11. We refer to this as *rewiring of type II*.

The last scenario is when we have a connected simple surface S with $\chi(\Gamma_S) = 0$ such that, for a fixed orientation of the core of Γ_S as a circle, there are two oriented edges on the core of the same type and orientation. Then applying rewiring to these two edges decomposes the core into two disjoint circles, and accordingly breaks the simple surface into the union of two connected simple surfaces S_1, S_2 with $\chi(\Gamma_{S_1}) = \chi(\Gamma_{S_2}) = 0$ such that their cores are the two circles above. See the right of Figure 11. We refer to this as *rewiring of type III*.

5.3. **Irreducible simple surfaces.** Now we introduce irreducible simple surfaces and show how each connected simple surface S with $\chi(\Gamma_S) \geq 0$ decomposes into a disjoint union of irreducible ones by a sequence of splitting and rewiring.

Definition 5.7. A connected simple surface S is *irreducible* if $\chi(\Gamma_S) \geq 0$, no splitting can be applied to any piece of S, and no rewiring of type I, II or III can be applied.

Proposition 5.8. *For any simple surface S with finitely many components such that each component Σ has $\chi(\Gamma_\Sigma)\ge 0$, there is a sequence of splittings and rewiring of types I, II, or III that modifies S into a disjoint union S' of irreducible simple surfaces. Moreover, there is a component Σ' of S' satisfying*

$$\frac{-\chi(\Sigma')}{n(\Sigma')}\le\frac{-\chi(S)}{n(S)},$$

where $n(\Sigma')$ and $n(S)$ are the degrees of Σ' and S respectively.

Proof. Let $\kappa(S)=2e-2c+\ell$, where e is the number of edges in Γ_S, c is the number of components of Γ_S, and ℓ is the number of embedded loops in Γ_S. Equivalently, ℓ is the number of components of Γ_S that have Euler characteristic zero. Note that $\kappa(S)$ is a non-negative integer since each component contains at least one edge.

For the first assertion, it suffices to check that whenever we apply splitting or rewiring of type I, II or III to modify S into another simple surface S', we have $\kappa(S')<\kappa(S)$. Note that both splitting and rewiring leave the number of edges invariant. So it comes down to checking how $-2c+\ell$ varies in each situation.

If splitting is applied to a component Σ of S, either it breaks Γ_Σ into two components without changing the number of embedded loops, or $\chi(\Gamma_\Sigma)=0$ and it breaks the core of Σ without creating new components. Thus for the simple surface S' obtained this way, we have either $\kappa(S')=\kappa(S)-2$ or $\kappa(S')=\kappa(S)-1$ corresponding to these two cases.

If we apply rewiring of type I to a component Σ, then the tree Γ_Σ breaks into two components, one of which contains a loop. Thus $\kappa(S')=\kappa(S)-1$. For rewiring of type II, we break the graph Γ_Σ into two components each containing a loop, where one of loop is inherited from the core of Γ_Σ. Hence we get one more component and one more loop, yielding $\kappa(S')=\kappa(S)-1$. As for rewiring of type III, we also get one more component and one more loop. Thus for all the three types of rewiring we have $\kappa(S')=\kappa(S)-1$.

For the second assertion, by Lemma 4.10, the (total) Euler characteristic of the simple surface does not change when we apply rewiring, and it increases by 1 every time we apply splitting since we obtain one more vertex representing a disk piece. In addition, both operations do not change the total degree. Suppose we start with S which has degree $n(S)$, and the irreducible simple surfaces we obtain in the end are $\Sigma'_1,\cdots,\Sigma'_k$ with degrees $n(\Sigma'_1),\cdots,n(\Sigma'_k)$ respectively. Then we have

$$\frac{-\chi(S)}{n(S)}\ge\frac{\sum_{i=1}^k-\chi(\Sigma'_i)}{\sum_{i=1}^k n(\Sigma'_i)}\ge\min_{1\le i\le k}\frac{-\chi(\Sigma'_i)}{n(\Sigma'_i)},$$

where the second inequality holds since the term in the middle is a weighted average. Thus the second assertion holds by taking $\Sigma'=\Sigma'_i$ where Σ'_i achieves the minimum above. ☐

Now we bound the size of the gluing graph of any irreducible simple surface to show that there are only finitely many such surfaces for the given element $g=a_1b_1\cdots a_Lb_L$. Note that there are L^2 possible types of turns on each side, giving rise to L^2 types of edges in gluing graphs.

Lemma 5.9. *If the factor groups A and B are finite and S is an irreducible simple surface for g, then the valence of each vertex of Γ_S is at most $L^2\max\{|A|,|B|\}$.*

Proof. Suppose there is a vertex with valence greater than $L^2 \max\{|A|, |B|\}$. Then we can apply splitting to the corresponding piece by Lemma 5.2, which contradicts the assumption that S is irreducible. $\square$

Lemma 5.10. *If S is an irreducible simple surface for g with $\chi(\Gamma_S) = 1$, then the diameter of Γ_S is at most $2L^2$.*

Proof. Suppose the diameter of Γ_S is greater than $2L^2$. Then there is an embedded path P of length at least $2L^2$, which we orient. Since there are at most L^2 types of edges in Γ_S, each with two possible orientations, there are two edges on P that have the same type and orientation by the pigeonhole principle. Hence rewiring of type I is applicable, contradicting that S is irreducible. $\square$

Lemma 5.11. *If S is an irreducible simple surface for g with $\chi(\Gamma_S) = 0$, then for any decorative tree T of Γ_S, the distance from any vertex of T to its root is at most $2L^2$. In particular, the diameter of T is at most $4L^2$.*

Proof. If some vertex has distance more than $2L^2$ to the root, the geodesic connecting them contains more than $2L^2$ edges. So by the same argument as in the proof of Lemma 5.10, rewiring of type II is applicable, contradicting that S is irreducible. $\square$

Lemma 5.12. *If S is an irreducible simple surface with $\chi(\Gamma_S) = 0$, then the core of Γ_S has length at most $2L^2$.*

Proof. If the core has length greater than $2L^2$, the same pigeonhole principle shows that rewiring of type III is applicable, contradicting that S is irreducible. $\square$

Proposition 5.13. *If $G = A * B$, where A and B are finite abelian groups, then for any g not conjugate into the factor groups, there are only finitely many irreducible simple surfaces.*

Proof. By Lemmas 5.10–5.12, the diameter of the gluing graph Γ_S of any irreducible simple surface is bounded above (by $5L^2$). Moreover, the valence of each vertex is bounded above by $L^2 \max\{|A|, |B|\}$. Thus there are only finitely many possible gluing graphs. Since there are finitely many types of edges, and the types of edges around a vertex with a chosen cyclic order determines the type of the corresponding piece, we conclude that there are only finitely many possible irreducible simple surfaces. $\square$

Theorem 5.14. *If $G = A * B$, where A and B are finite abelian groups, then for any g not conjugate into the factor groups, there is an irreducible simple surface S of some degree $n(S)$ with $\chi(\Gamma_S) = 0$ such that*

$$\mathrm{stl}_G(g) = -\frac{\chi(S)}{n(S)}.$$

As a consequence, $\mathrm{stl}_G(g)$ is rational and computable.

Proof. By Corollary 5.5 and Proposition 5.8, we know

$$\mathrm{stl}_G(g) = \inf_S \frac{-\chi(S)}{n(S)},$$

where the infimum is taken over all irreducible simple surfaces S and $n(S)$ is the degree of S. By Proposition 5.13, there are only finitely many irreducible simple

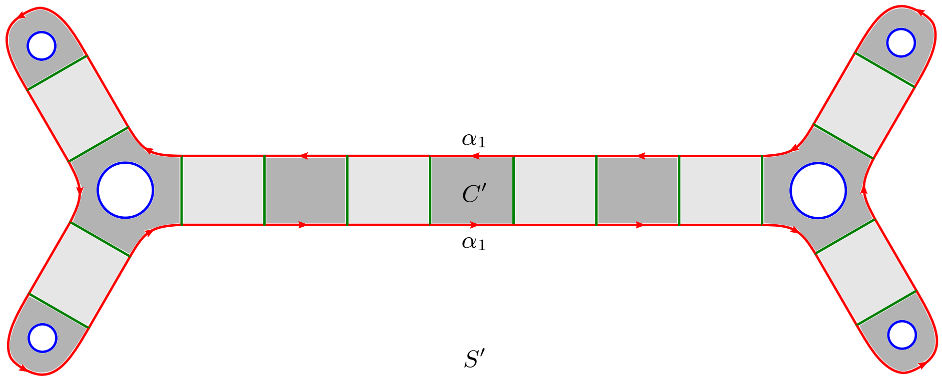

Figure 12. If α_1 represents a 2-torsion in the simple surface in Figure 6, then this is the "branched" degree $k = 2$ cover that we construct with lower complexity, where we pick the piece C as the leftmost piece in Figure 6.

surfaces. Hence one of them achieves the infimum above. Thus $\mathrm{stl}_G(g)$ is rational and can be computed by enumerating the finitely many irreducible simple surfaces.

It remains to observe that the infimum cannot be achieved by a simple surface where Γ_S is a tree. For any such simple surface S of degree $n(S)$, we have at least one annulus-piece C since g is not a torsion element. Suppose the polygonal boundary of C represents a k-torsion element, where $k \geq 2$. Then there is a simple surface S' of degree $n(S') = k \cdot n(S)$ such that $\Gamma_{S'} \setminus \{v'\}$ is formed by k disjoint copies of $\Gamma_S \setminus \{v\}$, where v represents the piece C and v' represents a disk-piece C' whose polygonal boundary is a degree k cover of the polygonal boundary of C; see Figure 12. By Lemma 4.10, we see that

$$\frac{-\chi(S')}{n(S')} = \frac{ke - (kd+1)}{k \cdot n(S)} < \frac{e-d}{n(S)} = \frac{-\chi(S)}{n(S)},$$

where e is the number of edges in Γ_S and d the number of disk-pieces in S. This shows that no connected simple surface with $\chi(\Gamma_S) = 1$ can achieve the minimal complexity. $\square$

Corollary 5.15. *Let $G = A * B$ be a free product, where A and B are finite abelian groups. For any g not conjugate into the factor groups, and for any subfamily $\mathcal{S}$ of simple surfaces for g such that each component has $\chi(\Gamma_S) = 0, 1$, if $\mathcal{S}$ contains all irreducible simple surfaces S with $\chi(\Gamma_S) = 0$, then*

$$\mathrm{stl}_G(g) = \inf_{S \in \mathcal{S}} \frac{-\chi(S)}{n}.$$

Proof. This is a combination of Corollary 5.5 and Theorem 5.14. $\square$

Remark 5.16. In Corollary 5.15, one can take $\mathcal{S}$ to be the subfamily of simple surfaces for g where each component satisfies the valence and diameter bounds (of decorative trees) in Lemmas 5.10–5.12. One can parameterize such surfaces as integer vectors in a rational polyhedral cone, where each variable represents the number of some type of small building blocks used in the surface. Projectively, such surfaces are represented by rational points in a compact rational polyhedron.

This gives a way to use linear programming to compute $\mathrm{stl}_G(g)$ when $G = A * B$ is a free products of finite abelian groups. When A and B are finite cyclic groups, using a setup similar to the one in [Wal13], the numbers of variables and constraints in the linear programming problem are polynomial in $|A|, |B|, |g|$. Thus for a free product G of finite cyclic groups, $\mathrm{stl}_G(g)$ can be computed in polynomial time by [Hač79].

6. Examples

In this section we explicitly compute the stable torsion length in two different examples, for $g = aba^{-1}b^{-1}$ and $g = ab$ in a free product $G = A * B$ with torsion elements $a \in A$ and $b \in B$. In both examples, we first work out a sufficient collection (Definition 4.12) of types of pieces, then use the linear programming problem as in Section 4.4 to compute a lower bound of $\mathrm{stl}_G(g)$, and finally use the Approximation Lemma 5.4 to show that $\mathrm{stl}_G(g)$ actually equals the lower bound.

6.1. **The word** $[a, b]$**.** In this section we consider the word $g = [a, b] = aba^{-1}b^{-1}$ in a free product $G = A * B$, where $a \in A$ and $b \in B$ are torsion elements of orders $p, q \geq 2$. We will focus on the special case where $A = \mathbb{Z}/p$ is generated by a and $B = \mathbb{Z}/q$ is generated by b. The general case will follow from the isometric embedding Theorem 4.14.

Using the setup in Section 4, the word $g = [a, b]$ is represented by a loop γ consisting of arcs $\alpha_1, \beta_1, \alpha_2, \beta_2$, where α_1, α_2 represent a, a^{-1} and β_1, β_2 represent b, b^{-1} respectively. Then there are four types of turns on the A-side: (α_1, α_1), (α_1, α_2), (α_2, α_1), and (α_2, α_2).

Note that the two turns (α_1, α_2) and (α_2, α_1) form a polygonal boundary that bounds a disk-piece, which we denote by R_1. Moreover, on the polygonal boundary of any piece, the number of copies of (α_1, α_2) is equal to that of (α_2, α_1) since the polygonal boundary closes up. Denote this number in a piece C by $N(C) \geq 0$.

If $N(C) \geq 2$ for a piece C, then we can remove a copy of (α_1, α_2) and (α_2, α_1) so that the remaining turns still form a polygonal boundary. Thus we can apply splitting (the general form that works for abelian factor groups) to this piece C to obtain a copy of R_1 and some piece C' with $N(C') = N(C) - 1$; see case (1) of Example 5.1.

If a piece C has $N(C) = 1$, then the arcs on the boundary of C in the cyclic order must be m copies of α_1 followed by n copies of α_2 for some $m, n \geq 1$.

(1) If $m = n$ then we have a disk-piece, which we denote by R_n. For $n > p$, we have p consecutive copies of the turn (α_1, α_1) (resp. (α_2, α_2)) on the boundary of R_n. Thus we can apply splitting twice to reduce R_n to R_{n-p} together with two pieces P_p^+ and P_p^-, where P_p^+ (resp. P_p^-) is the disk-piece with exactly p copies of α_1 (resp. α_2) on the boundary. See case (2) of Example 5.1 for one of the splitting when $p = 2$.

(2) If $m > n$ then we can apply splitting to reduce the piece C into a copy of R_n and a piece with $m - n$ copies of α_2 on the boundary. Similarly for the case $m < n$.

Thus for any simple surface S, after applying splittings as above, we may assume that any piece C other than R_n with $1 \leq n \leq p$ has $N(C) = 0$. Thus any piece different from R_n only contains one type of turns, either (α_1, α_1) or (α_2, α_2).

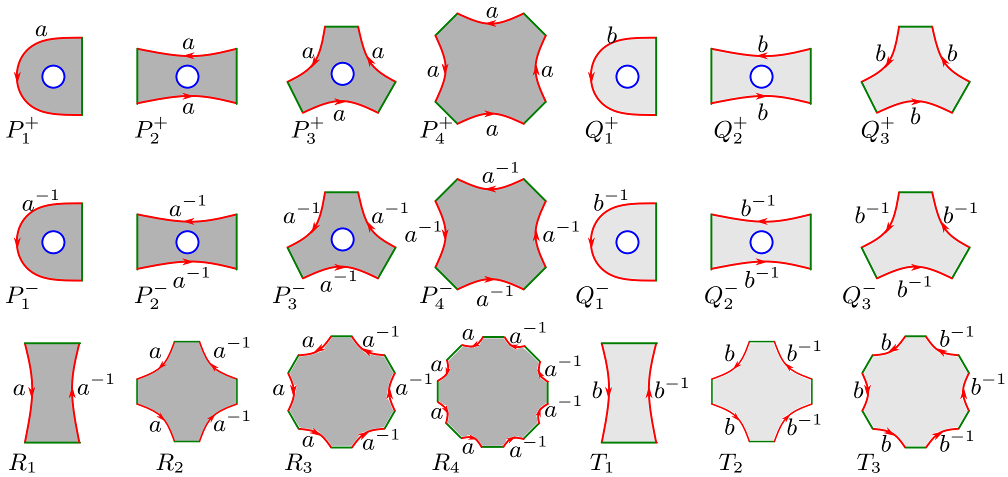

FIGURE 13. The pieces in the collection $\mathcal{P}$ when $p = 4$ and $q = 3$.

Moreover, since a has order p, splitting applies to any piece with more than p copies of the turn (α_1, α_1) (resp. (α_2, α_2)) on the boundary.

Thus there are $3p$ types of remaining pieces on the A-side, which fall into three classes:

(1) A piece with $1 \le n \le p$ arcs α_1 on the boundary, which is a disk only when $n = p$. Denote such pieces as P_n^+; see the left on the first row of Figure 13.
(2) A piece with $1 \le n \le p$ arcs α_2 on the boundary, which is a disk only when $n = p$. Denote such pieces by P_n^-; see the left on the second row of Figure 13.
(3) A disk-piece R_n for $1 \le n \le p$ that has n copies of α_1 followed by n copies of α_2 on the boundary; see the left on the third row of Figure 13.

Similarly, we can reduce simple surfaces by splitting on the B-side so that there are $3q$ types of remaining pieces on the B-side, denoted as Q_n^+, Q_n^-, and T_n for $1 \le n \le q$, where q is the order of b.

Let $\mathcal{P}$ be the collection consisting of the above $3p$ pieces on the A-side and $3q$ pieces on the B-side, depicted in Figure 13 for the case $p = 4$ and $q = 3$.

Lemma 6.1. *The collection $\mathcal{P}$ is sufficient.*

Proof. The discussion above shows that any simple surface can be reduced to a $\mathcal{P}$-simple surface by a sequence of splittings. Note that splitting preserves the number of edges in the gluing graph and adds a vertex representing a disk piece, thus it decreases $-\chi(S)$ by Lemma 4.10. Thus if we start with a connected simple surface S with $\chi(\Gamma_S) = 1$, applying splitting once (if applicable) modifies it into a $\mathcal{P}$-simple surface S' of the same degree with $-\chi(S') < -\chi(S)$. Moreover, the resulting gluing graph $\Gamma_{S'}$ has two components, each of which is a tree. Hence one of the two components has lower complexity than the original one. Thus the infimum of $-\chi(S)/n(S)$ over all connected simple surfaces with $\chi(\Gamma_S) = 1$ does not change as we restrict to connected $\mathcal{P}$-simple surfaces with $\chi(\Gamma_S) = 1$. Hence $\mathcal{P}$ is sufficient by definition. $\square$

Now we can apply the formalism in Section 4.4 to compute a lower bound of $\mathrm{stl}_G([a,b])$ by linear programming. The lower bound turns out to be sharp in this case. The following elementary observation is helpful to simplify our computation.

Lemma 6.2. *For any $n \geq 2$ and any set of numbers $x_k \geq 0$, $1 \leq k \leq n$, there is another set of numbers $x'_k \geq 0$ such that*

(1) $\sum_{k=1}^n kx'_k = \sum_{k=1}^n kx_k$;
(2) $\sum_{k=1}^n x'_k = \sum_{k=1}^n x_k$;
(3) $x'_n \geq x_n$; *and*
(4) $x'_k = 0$ *for all* $1 < k < n$.

Proof. If $x_i > 0$ for some $1 < i < n$, we construct non-negative numbers $\{x'_k\}$ with $x'_i = 0$ and satisfying bullets (1), (2) and (3). Let $\lambda = \frac{n-i}{n-1} \in (0,1)$ and $\mu = \frac{i-1}{n-1} \in (0,1)$. Then $\lambda + n\mu = i$ and $\lambda + \mu = 1$. So the following set of numbers

$$x'_1 = x_1 + \lambda x_i, \quad x'_n = x_n + \mu x_i, \quad x'_i = 0, \quad \text{and } x'_k = x_k \text{ for all } k \neq 1, i, n,$$

satisfies (1), (2) and (3). Hence, the conclusion follows by a sequence of such changes by making one x_i zero at a time. $\square$

Lemma 6.3. *If $A = \mathbb{Z}/p$ and $B = \mathbb{Z}/q$ are generated by a, b, then for $G = A * B$ we have*

$$\mathrm{stl}_G([a,b]) = 1 - \frac{1}{\min(p,q) - 1}.$$

Proof. Let $\mathcal{P}$ be the sufficient collection above. Consider the polyhedron $C_{\mathcal{P}}$ defined in Section 4.4. Let x_n^+ (resp. x_n^-) be the coordinate corresponding to the piece P_n^+ (resp. P_n^-) for each $1 \leq n \leq p$. Let y_n^+ (resp. y_n^-) be the coordinate corresponding to the piece Q_n^+ (resp. Q_n^-). Let z_n and w_n be the coordinates corresponding to the pieces R_n and T_n respectively.

Then the gluing conditions in the definition of the polyhedron $C_{\mathcal{P}}$ as in Section 4.4 become:

$$\sum_{k=1}^p kx_k^+ + \sum_{k=1}^p (k-1)z_k = \sum_{k=1}^p kx_k^- + \sum_{k=1}^p (k-1)z_k = \sum_{k=1}^q w_k, \tag{6.1}$$

$$\sum_{k=1}^q ky_k^+ + \sum_{k=1}^q (k-1)w_k = \sum_{k=1}^q ky_k^- + \sum_{k=1}^q (k-1)w_k = \sum_{k=1}^p z_k. \tag{6.2}$$

By counting the number of copies of the arc α_1, the normalizing condition is

$$\sum_{k=1}^p kx_k^+ + \sum_{k=1}^p kz_k = 1.$$

The left-hand side can be rewritten as $\sum_{k=1}^p kx_k^+ + \sum_{k=1}^p (k-1)z_k + \sum_{k=1}^p z_k$. Thus by the gluing condition (6.1) we can express the normalizing condition equivalently as

$$\sum_{k=1}^p z_k + \sum_{k=1}^q w_k = 1. \tag{6.3}$$

The Euler characteristic constraint $\chi_\Gamma \geq 0$ can be written as

$$\sum_{k=1}^p (1-k)z_k + \sum_{k=1}^q (1-k)w_k + \sum_{k=1}^p (1 - \frac{k}{2})(x_k^+ + x_k^-) + \sum_{k=1}^q (1 - \frac{k}{2})(y_k^+ + y_k^-) \geq 0.$$

Note that by the gluing condition (6.1), we have

$$\begin{aligned}
&\sum_{k=1}^{p}(1-k)z_k+\sum_{k=1}^{p}(1-\frac{k}{2})(x_k^+ + x_k^-)\\
=&\sum_{k=1}^{p}(x_k^+ + x_k^-)-\frac{1}{2}\left[\sum_{k=1}^{p}kx_k^+ +\sum_{k=1}^{p}(k-1)z_k\right]-\frac{1}{2}\left[\sum_{k=1}^{p}kx_k^- +\sum_{k=1}^{p}(k-1)z_k\right]\\
=&\sum_{k=1}^{p}(x_k^+ + x_k^-)-\sum_{k=1}^{q}w_k.
\end{aligned}$$

Similarly, we have

$$\sum_{k=1}^{q}(1-k)w_k+\sum_{k=1}^{q}(1-\frac{k}{2})(y_k^+ + y_k^-)=\sum_{k=1}^{q}(y_k^+ + y_k^-)-\sum_{k=1}^{p}z_k.$$

Using the normalizing condition (6.3), the constraint $\chi_\Gamma \geq 0$ is equivalent to

$$\sum_{k=1}^{p}(x_k^+ + x_k^-)+\sum_{k=1}^{q}(y_k^+ + y_k^-)\geq 1. \tag{6.4}$$

The objective is to minimize $-\chi_o$, which is expressed as

$$\begin{aligned}
&\sum_{k=1}^{p}(k-1)z_k+\sum_{k=1}^{q}(k-1)w_k+\left(\frac{p}{2}-1\right)(x_p^+ + x_p^-)+\left(\frac{q}{2}-1\right)(y_q^+ + y_q^-)\\
&\quad+\sum_{k=1}^{p-1}\frac{k}{2}(x_k^+ + x_k^-)+\sum_{k=1}^{q-1}\frac{k}{2}(y_k^+ + y_k^-)\\
&=\frac{1}{2}\left[\sum_{k=1}^{p}kx_k^+ +\sum_{k=1}^{p}(k-1)z_k\right]+\frac{1}{2}\left[\sum_{k=1}^{p}kx_k^- +\sum_{k=1}^{p}(k-1)z_k\right]\\
&\quad+\frac{1}{2}\left[\sum_{k=1}^{q}ky_k^+ +\sum_{k=1}^{q}(k-1)w_k\right]+\frac{1}{2}\left[\sum_{k=1}^{q}ky_k^- +\sum_{k=1}^{q}(k-1)w_k\right]\\
&\quad-(x_p^+ + x_p^- + y_q^+ + y_q^-)\\
&=\sum_{k=1}^{q}w_k+\sum_{k=1}^{p}z_k-(x_p^+ + x_p^- + y_q^+ + y_q^-)\\
&=1-(x_p^+ + x_p^- + y_q^+ + y_q^-),
\end{aligned}$$

where we used the gluing conditions (6.1) and (6.2), and normalizing condition (6.3) at the last two steps, respectively.

By Lemma 6.2, we may assume $x_k^\pm = 0$ for all $1<k<p$ and $y_k^\pm = 0$ for all $1<k<q$. In addition, if we let $x_k^{+\prime} = x_k^{-\prime} = \frac{1}{2}(x_k^+ + x_k^-)$ for all k without changing z_k and w_k, the constraints (6.1)–(6.4) and the objective function are all unaffected. Thus we can assume $x_k^+ = x_k^-$ for all k and similarly for $y_k^\pm$. Hence the linear

programming problem reduces to

$$\begin{aligned}
\text{minimize:}\quad & 1-2x_p-2y_q\\
\text{subject to:}\quad & x_1+px_p+\sum_{k=1}^{p}(k-1)z_k=\sum_{k=1}^{q}w_k\\
& y_1+qy_q+\sum_{k=1}^{q}(k-1)w_k=\sum_{k=1}^{p}z_k\\
& \sum_{k=1}^{q}w_k+\sum_{k=1}^{p}z_k=1\\
& x_1+x_p+y_1+y_q\ge\frac{1}{2}\\
& x_i,y_i,z_k,w_k\ge 0,
\end{aligned}$$

where $x_i=x_i^\pm$ ($i=1$ or p), $y_i=y_i^\pm$ ($i=1$ or q), and the first four constraints correspond to (6.1)–(6.4).

By symmetry, assume $p\le q$. On the one hand, note that by the first two constraints and the fact that $w_k,z_k\ge 0$, we have

$$x_1+x_p\le\sum_{k=1}^{q}w_k-(p-1)x_p,\quad\text{and}\quad y_1+y_q\le\sum_{k=1}^{p}z_k-(q-1)y_q.$$

Thus using the third and fourth constraints, we get

$$\frac{1}{2}\le x_1+x_p+y_1+y_q\le\sum_{k=1}^{q}w_k-(p-1)x_p+\sum_{k=1}^{p}z_k-(q-1)y_q\le 1-(p-1)(x_p+y_q),$$

which implies $2(x_p+y_q)\le\frac{1}{p-1}$ and thus the objective $1-2(x_p+y_q)\ge 1-\frac{1}{p-1}$. On the other hand, this lower bound $1-\frac{1}{p-1}$ is achieved by the feasible solution $x_1=\frac{1}{2}(1-\frac{1}{p-1})$, $x_p=\frac{1}{2(p-1)}$, $y_1=y_q=0$, $w_1=1$, $w_k=0$ for $k>1$, and $z_k=0$ for all k. Hence we conclude that the minimal value of the linear programming is $1-\frac{1}{\min(p,q)-1}$. Thus

$$\mathrm{stl}_G([a,b])\ge 1-\frac{1}{\min(p,q)-1}$$

by Lemma 4.15.

Moreover, for the feasible solution above, let $n=2(p-1)$. Take $nx_1^+=nx_1^-=nx_1=p-2$ copies of P_1^+ and P_1^-, take $nx_p^+=nx_p^-=nx_p=1$ copy of P_p^+ and P_p^-, and take $nw_1=2(p-1)$ copies of T_1. Such pieces can be glued into a $\mathcal{P}$-simple surface S of degree $n=2(p-1)$ that is connected and has $\chi(\Gamma_S)=0$; see Figure 14 for an example where $q\ge p=5$. Thus by Lemma 5.4, $1-\frac{1}{\min(p,q)-1}=-\chi(S)/n$ is the limit of complexities of a sequence of connected simple surfaces with $\chi_\Gamma=1$. This implies

$$\mathrm{stl}_G([a,b])\le 1-\frac{1}{\min(p,q)-1}$$

by Corollary 4.13. Combining the two parts we obtain the desired equality. □

Now we generalize this formula using Theorem 4.14.

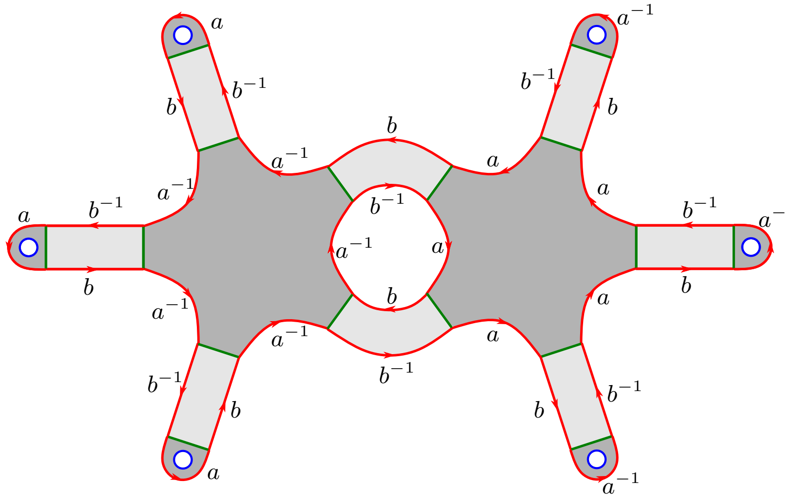

FIGURE 14. A connected $\mathcal{P}$-simple surface S with the given number of pieces satisfying $\chi(\Gamma_S) = 0$ in the case $q \geq p = 5$.

Theorem 6.4. *Let $G = A * B$ be a free product, and let $a \in A$ and $b \in B$ be torsion elements of orders p and q respectively, where $p, q \geq 2$. Then we have*

$$\operatorname{stl}_G([a, b]) = 1 - \frac{1}{\min(p, q) - 1}.$$

Proof. Let $A' = \mathbb{Z}/p$ and $B' = \mathbb{Z}/q$ be the subgroups generated by a and b respectively. Then the inclusions $A' \to A$ and $B' \to B$ satisfy the assumption (2) of Theorem 4.14 since A' and B' are finite. Thus the result directly follows from Lemma 6.3. $\square$

6.2. **The word ab.** In this section we consider the word $g = ab$ in a free product $G = A * B$, where $a \in A$ and $b \in B$ are torsion elements of orders $p, q \geq 2$. We will focus on the special case where $A = \mathbb{Z}/p$ is generated by a and $B = \mathbb{Z}/q$ is generated by b. The general case will follow from the isometric embedding Theorem 4.14.

Using the setup in Section 4, the word $g = ab$ is represented by a loop γ consisting of arcs α and β, where α represents a and β represents b. There is exactly one type of turn on the A-side: (α, α).

Since a has order p, there is a disk-piece with p copies of the arc α on its polygonal boundary. Therefore, we can apply splitting to any A-piece with more than p copies of the turn (α, α) on the boundary. So, on the A-side, after splitting we are left with pieces with $1 \leq n \leq p$ arcs α on the boundary. Furthermore, these pieces are disks only when $n = p$. We denote such pieces by P_n.

Similarly, we can reduce simple surfaces by splitting on the B-side to q possible pieces, each with $1 \leq n \leq q$ arcs β. These pieces are disks only when $n = q$. We denote such pieces by Q_n.

Let $\mathcal{P}$ be the collection above consisting of these p types of pieces on the A-side and these q types of pieces on the B-side. The first row of Figure 13 depicts such pieces when $p = 4$ and $q = 3$.

Lemma 6.5. *The collection $\mathcal{P}$ is sufficient.*

Proof. Since we reduced the collection of pieces to $\mathcal{P}$ by a series of splittings, the argument in Lemma 6.1 shows that $\mathcal{P}$ is sufficient. $\square$

Theorem 6.6 (Product formula)**.** *Let $a \in A$ and $b \in B$ be torsion elements of order p and q respectively such that $p \leq q$, then*

$$\mathrm{stl}_G(ab) = \mathrm{stl}_{\mathbb{Z}/p*\mathbb{Z}/q}(ab) = 1 - \frac{q}{p(q-1)},$$

where $\mathbb{Z}/p$ and $\mathbb{Z}/q$ are the subgroups generated by a and b respectively.

Proof. The first equality follows from Theorem 4.14. So it suffices to compute $\mathrm{stl}_{\mathbb{Z}/p*\mathbb{Z}/q}(ab)$.

Let $\mathcal{P}$ be the sufficient collection above. Consider the polyhedron $C_{\mathcal{P}}$ defined in Section 4.4. Let x_n be the coordinate corresponding to the piece P_n for each $1 \leq n \leq p$. Let y_n be the coordinate corresponding to the piece Q_n for each $1 \leq n \leq q$.

The gluing condition in the definition of $C_{\mathcal{P}}$ becomes:

$$\sum_{k=1}^{p} kx_k = \sum_{k=1}^{q} ky_k.$$

The normalizing condition in the definition of $C_{\mathcal{P}}$ is:

$$\sum_{k=1}^{p} kx_k = 1.$$

The Euler characteristic constraint $\chi_\Gamma \geq 0$ can be written as:

$$\sum_{k=1}^{p} x_k \left(1 - \frac{k}{2}\right) + \sum_{k=1}^{q} y_k \left(1 - \frac{k}{2}\right) \geq 0.$$

Using the gluing and normalizing conditions, this is equivalent to:

$$\sum_{k=1}^{p} x_k + \sum_{k=1}^{q} y_k \geq 1.$$

The objective is to minimize $-\chi_0$, which is expressed as:

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{k}{2} \cdot x_k + \left(\frac{p}{2} - 1\right) x_p + \sum_{k=1}^{q-1} \frac{k}{2} \cdot y_k + \left(\frac{q}{2} - 1\right) y_q \\
&= \frac{1}{2}\left(\sum_{k=1}^{p} k \cdot x_k + \sum_{k=1}^{q} k \cdot y_k\right) - x_p - y_q \\
&= 1 - x_p - y_q.
\end{aligned}$$

Then, by Lemma 6.2, it is sufficient to assume that $x_k = 0$ for $1 < k < p$ and $y_j = 0$ for $1 < j < q$. This reduces the linear programming problem to:

$$\begin{aligned}
\text{minimize:} \quad & 1 - x_p - y_q \\
\text{subject to:} \quad & x_1 + px_p = 1 \\
& y_1 + qy_q = 1 \\
& x_1 + x_p + y_1 + y_q \geq 1 \\
& x_1, x_p, y_1, y_q \geq 0
\end{aligned}$$

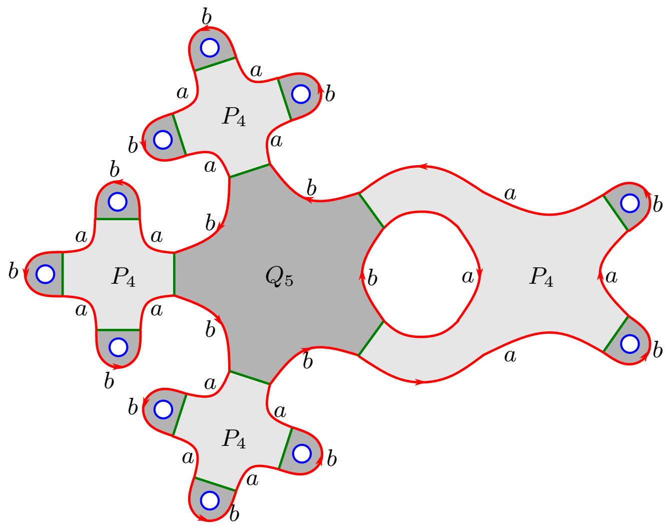

FIGURE 15. A connected $\mathcal{P}$-simple surface S with the given number of pieces satisfying $\chi(\Gamma_S) = 0$ in the case $p = 4$ and $q = 5$.

Then the constraints imply that

$$2 = x_1 + y_1 + px_p + qy_q \geq 1 + (p-1)x_p + (q-1)y_q = 1 + (q-1)(x_p + y_q) - (q-p)x_p.$$

Thus

$$(q-1)(x_p + y_q) \leq 1 + (q-p)x_p \leq 1 + \frac{q-p}{p} = \frac{q}{p},$$

where we used the assumption that $q \geq p$ and the fact that $x_p \leq 1/p$, which is a consequence of the first constraint since $x_1 \geq 0$. Hence it follows that the objective function satisfies

$$1 - (x_p + y_q) \geq 1 - \frac{q}{p(q-1)}.$$

This lower bound is achieved by the feasible solution $x_1 = 0$, $x_p = \frac{1}{p}$, $y_1 = \frac{pq-p-q}{p(q-1)}$, and $y_q = \frac{1}{p(q-1)}$. Therefore, by Lemma 4.15, $\mathrm{stl}_{\mathbb{Z}/p*\mathbb{Z}/q}(ab) \geq 1 - \dfrac{q}{p(q-1)}$.

Moreover, the solution above is (projectively) represented by a connected $\mathcal{P}$-simple surface S of degree $p(q-1)$ with $\chi(\Gamma_S) = 0$ in the following way, depicted in Figure 15 in the case where $p = 4$ and $q = 5$. There is a single piece of type Q_q in S, where 2 out of the q turns are glued with 2 turns in a piece of type P_p, forming the unique embedded loop in Γ_S. The remaining $p-2$ turns of this piece of type P_p are glued to $p-2$ pieces of type Q_1. As for the remaining $q-2$ turns of the unique piece of type Q_q, each of them is glued to a new piece of type P_p. For each of these $q-2$ new pieces of type P_p, the other $p-1$ turns are glued to a piece of type Q_1.

Therefore, we have $\mathrm{stl}_{\mathbb{Z}/p*\mathbb{Z}/q}(ab) \leq 1 - \dfrac{q}{p(q-1)}$ by Corollary 4.13. Thus

$$\mathrm{stl}_G(ab) = \mathrm{stl}_{\mathbb{Z}/p*\mathbb{Z}/q}(ab) = 1 - \frac{q}{p(q-1)}.$$

□

Remark 6.7. There is a product formula [Cal09a, Theorem 2.93] that computes scl of ab in a free product $A * B$ for $a \in A$ and $b \in B$. The result only involves the orders of a and b and $\mathrm{scl}_A(a)$ and $\mathrm{scl}_B(b)$.

It seems unlikely to have such a generalization of Theorem 6.6 for stl when a and b are not necessarily torsion elements. For instance, if a has finite order and b has infinite order, such a formula would express $\mathrm{stl}_G(ab)$ as a function of the order of a and $\mathrm{stl}_B(b)$. This does not seem natural in the following example.

Let $p \le q \le r$, and let $\mathbb{Z}/p$, $\mathbb{Z}/q$, and $\mathbb{Z}/r$ be generated by x, y, and z, respectively. The methods in Sections 5 and 6 generalize to free products of more than two groups. For $G = \mathbb{Z}/p * \mathbb{Z}/q * \mathbb{Z}/r$, a similar calculation as in Theorem 6.6 gives

$$\mathrm{stl}_G(xyz) = 2 - \frac{q}{p(q-1)}.$$

Consider G as the free product of $A = \mathbb{Z}/p$ and $B = \mathbb{Z}/q * \mathbb{Z}/r$, and let $a = x \in A$ and $b = yz \in B$. It seems unnatural to express $2 - \dfrac{q}{p(q-1)}$ as a simple function of p and $\mathrm{stl}_B(b) = 1 - \frac{r}{q(r-1)}$ since the result depends on q but not on r.

References

[Bav91] Christophe Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[Bie12] Ludwig Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume (Zweite Abhandlung.) Die Gruppen mit einem endlichen Fundamentalbereich. *Math. Ann.*, 72(3):400–412, 1912.

[BM19] Michael Brandenbursky and MichałMarcinkowski. Aut-invariant norms and Aut-invariant quasimorphisms on free and surface groups. *Comment. Math. Helv.*, 94(4):661–687, 2019.

[Cal08a] Danny Calegari. Surface subgroups from homology. *Geom. Topol.*, 12(4):1995–2007, 2008.

[Cal08b] Danny Calegari. Word length in surface groups with characteristic generating sets. *Proc. Amer. Math. Soc.*, 136(7):2631–2637, 2008.

[Cal09a] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[Cal09b] Danny Calegari. Stable commutator length is rational in free groups. *J. Amer. Math. Soc.*, 22(4):941–961, 2009.

[CF10] Danny Calegari and Koji Fujiwara. Stable commutator length in word-hyperbolic groups. *Groups Geom. Dyn.*, 4(1):59–90, 2010.

[Che20] Lvzhou Chen. Scl in graphs of groups. *Inventiones mathematicae*, Jan 2020.

[CW15] Danny Calegari and Alden Walker. Random groups contain surface subgroups. *J. Amer. Math. Soc.*, 28(2):383–419, 2015.

[EF97] David B. A. Epstein and Koji Fujiwara. The second bounded cohomology of word-hyperbolic groups. *Topology*, 36(6):1275–1289, 1997.

[Hač79] L. G. Hačijan. A polynomial algorithm in linear programming. *Dokl. Akad. Nauk SSSR*, 244(5):1093–1096, 1979.

[HL21] Nicolaus Heuer and Clara Löh. The spectrum of simplicial volume. *Invent. Math.*, 223(1):103–148, 2021.

[Kot04] D. Kotschick. Quasi-homomorphisms and stable lengths in mapping class groups. *Proc. Amer. Math. Soc.*, 132(11):3167–3175, 2004.

[MP20] Dan Margalit and Andrew Putman. Surface groups, infinite generating sets, and stable commutator length. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(5):2379–2386, 2020.

[Szc12] Andrzej Szczepański. *Geometry of crystallographic groups*, volume 4 of *Algebra and Discrete Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Wal13] Alden Walker. Stable commutator length in free products of cyclic groups. *Exp. Math.*, 22(3):282–298, 2013.

[Wil18] Henry Wilton. Essential surfaces in graph pairs. *J. Amer. Math. Soc.*, 31(4):893–919, 2018.

Department of Mathematics, University of Chicago, Chicago, Illinois, USA
Email address, C. Avery: chloe@math.uchicago.edu

Department of Mathematics, University of Texas at Austin, Austin, Texas, USA
Email address, L. Chen: lvzhou.chen@math.utexas.edu