

# GROMOV'S SIMPLICIAL NORM AND BOUNDED COHOMOLOGY

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ABSTRACT. These are lecture notes for the course *Gromov's Simplicial Norm and Bounded Cohomology* in Spring 2022 at the University of Texas at Austin.

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## 1. INTRODUCTION TO GROMOV'S SIMPLICIAL NORM

One interesting topic in geometry and topology is to relate geometric quantities of a manifold to topological invariants. One typical problem asks about which manifolds admit Riemannian metrics with negative (positive, or non-positive) sectional (Ricci, or scalar) curvature.

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Here we are interested in volumes of closed manifolds. Usually one needs a Riemannian metric to make sense of it, but is it possible to get a topological invariant out of it? The Mostow rigidity (and Gauss–Bonnet in dimension 2) implies that the volume of a hyperbolic closed manifold is determined by its topology. Gromov’s *simplicial volume*, as a special case of the *simplicial norm*, is a way to define this invariant in a purely topological way.

Why should one be interested in such an invariant? The following basic problem is an example where one needs a topological notion of volume/area.

**Problem 1.1.** *Given two orientable connected closed surfaces  $S, S'$ , what is the largest possible degree  $\deg(f)$  of a continuous map  $f : S \rightarrow S'$ ?*

As we will see below (Lemma 1.11), the simplicial volumes of  $S$  and  $S'$ , denoted  $\|S\|_1$  and  $\|S'\|_1$ , satisfy

$$\|S\|_1 \geq |\deg(f)| \cdot \|S'\|_1$$

for any continuous map  $f$ . Intuitively,  $S$  needs to have enough area to cover  $S'$  for  $|\deg(f)|$  times. This provides an upper bound  $\|S\|_1/\|S'\|_1$  when  $\|S'\|_1 > 0$ , or equivalently when  $S'$  has genus at least two as we will prove. Moreover, the upper bound obtained this way is actually sharp, and in Section 1.6 we will exactly determine the set of all possibly degrees

$$\deg(S, S') := \{\deg(f) \mid f : S \rightarrow S'\}.$$

**1.1. The simplicial norm.** Fix  $n \in \mathbb{Z}_{\geq 0}$ . Given a topological space  $X$ , Gromov [Gro82] introduced a semi-norm  $\|\cdot\|_1$  on the singular homology  $H_n(X; \mathbb{R})$  for each  $n$  as a real vector space to measure the size of each homology class. Recall that  $H_n(X; \mathbb{R})$  is the homology of the singular chain complex

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots,$$

where  $C_n(X; \mathbb{R})$  is the space of singular  $n$ -chains, namely the real vector space spanned by the set  $S_n(X)$  of all singular  $n$ -simplices. As usual, we have the subspaces  $B_n \subset Z_n \subset C_n$ , where  $Z_n := \ker \partial_n$  and  $B_n := \text{Im } \partial_{n+1}$  are the spaces of cycles and boundaries respectively. So by definition  $H_n(X; \mathbb{R})$  is the quotient  $Z_n/B_n$ .

Given the standard basis  $S_n(X)$ , equip the space  $C_n(X; \mathbb{R})$  with the  $\ell^1$ -norm, i.e.  $|c|_1 = \sum_{i=1}^k |\lambda_i|$  for any  $c = \sum_{i=1}^k \lambda_i c_i$  expressed uniquely as a (finite) linear combination of basis elements  $c_i \in S_n(X)$  with coefficients  $\lambda_i \in \mathbb{R}$ .

**Definition 1.2** (Simplicial norm). The restriction of this  $\ell^1$ -norm to  $Z_n$  induces a semi-norm on its quotient  $H_n(X; \mathbb{R})$ , explicitly,

$$\|\sigma\|_1 := \inf_{[c]=\sigma} |c|_1,$$

where the infimum is taken over all cycles  $c \in Z_n$  representing the homology class  $\sigma \in H_n(X; \mathbb{R})$ . This semi-norm is called *Gromov’s simplicial norm*.

In words,  $\|\sigma\|_1$  is the infimal number of simplices that we need to represent  $\sigma$ .

The following property is immediate from the definition but important.

**Proposition 1.3** (Functorial). *For any continuous map  $f : X \rightarrow Y$ , then the induced map  $f_* : H_n(X; \mathbb{R}) \rightarrow H_n(Y; \mathbb{R})$  is non-increasing with respect to the simplicial norm, i.e.*

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1$$

for any  $\sigma \in H_n(X; \mathbb{R})$ .

*Proof.* For any cycle  $c = \sum_i \lambda_i c_i \in Z_n(X; \mathbb{R})$  representing  $\sigma$ , the cycle  $f_*c = \sum \lambda_i f_*c_i = \sum_i \lambda_i (f \circ c_i)$  represents  $f_*\sigma$ . Hence by definition

$$\|f_*\sigma\|_1 \leq |f_*c|_1 \leq \sum_i |\lambda_i| = |c|_1.$$

Since  $c$  is arbitrary, taking infimum implies

$$\|f_*\sigma\|_1 \leq \|\sigma\|_1.$$

□

**Corollary 1.4** (Invariance). *If  $f : X \rightarrow Y$  is a homotopy equivalence, then  $f_* : H_n(X; \mathbb{R}) \rightarrow H_n(Y; \mathbb{R})$  is an isometric isomorphism (i.e. an isomorphism that is norm-preserving) with respect to the simplicial norm.*

*More generally, if for a map  $f : X \rightarrow Y$  there is  $g : Y \rightarrow X$  such that  $g_*f_*$  is the identity on  $H_n(X; \mathbb{R})$ , then  $f_*$  is an isometric embedding (i.e. injective and norm-preserving).*

*Proof.* The first part easily follows from the second part by taking  $g$  to be a homotopy inverse of  $f$ .

For the second part, by functoriality of  $g$  and the fact that  $g_*f_* = id$ , we have  $\|\sigma\|_1 = \|(g_*f_*)\sigma\|_1 \leq \|f_*\sigma\|_1$ . Combining with the functoriality of  $f$ , we must have  $\|\sigma\|_1 = \|f_*\sigma\|_1$  for any  $\sigma \in H_n(X; \mathbb{R})$ . Hence  $f_*$  is norm-preserving. Injectivity easily follows from the fact that  $g_*f_* = id$ . □

It is often convenient to consider cycles with rational coefficients since they can be scaled to integral cycles. We can always find a rational homology class arbitrarily close to a given homology class with respect to the simplicial norm; see the lemma below. This follows from the fact that  $B_n$  and  $Z_n$  are rational subspaces. Here a point  $c \in C_n(X; \mathbb{R})$  is rational if  $c \in C_n(X; \mathbb{Q})$ , and an  $\mathbb{R}$ -linear subspace is rational if it has a basis consisting of rational points. Any point in a rational subspace  $V$  is a limit of rational points in  $V$  with respect to the norm  $|\cdot|_1$  (think about it). Here  $B_n$  and  $Z_n$  are rational because the boundary maps  $\partial_{k+1} : C_{k+1}(X; \mathbb{R}) \rightarrow C_k(X; \mathbb{R})$  are rational linear, i.e. obtained from  $C_{k+1}(X; \mathbb{Q}) \rightarrow C_k(X; \mathbb{Q})$  by tensoring with  $\mathbb{R}$  over  $\mathbb{Q}$ .

**Lemma 1.5.** *If  $\sigma \in H_n(X; \mathbb{Q})$ , then  $\|\sigma\|_1 = \inf |c|_1$  where the infimum is taken over all rational cycles  $c = \sum \lambda_i c_i$  (i.e.  $\lambda_i \in \mathbb{Q}$  and  $\partial c = 0$ ).*

*For a general  $\sigma \in H_n(X; \mathbb{R})$  and any  $\epsilon > 0$ , there is  $\sigma' \in H_n(X; \mathbb{Q})$  with  $\|\sigma - \sigma'\|_1 \leq \epsilon$ .*

*Proof.* For the first part, note that  $B_n(X; \mathbb{Q})$  is dense in  $B_n(X; \mathbb{R})$  with respect to the norm  $|\cdot|_1$ , since  $B_n(X; \mathbb{R})$  is a rational subspace. As  $\sigma \in H_n(X; \mathbb{Q})$ , it can be represented by some rational cycle  $c$ . All other (resp. rational) cycles take the form  $c + b$  with  $b \in B_n(X; \mathbb{R})$  (resp.  $b \in B_n(X; \mathbb{Q})$ ), so the result follows by density.

The second part is due to the density of  $Z_n(X; \mathbb{Q})$  in  $Z_n(X; \mathbb{R})$ , which holds since  $Z_n(X; \mathbb{R})$  is a rational subspace. □

**Exercise 1.6.** *Recall that  $H_0(X; \mathbb{R})$  is isomorphic to the  $\mathbb{R}$ -vector space with basis corresponding to the path connected components of the space  $X$ . For any path component  $C$  and a point  $c \in C$ , thought of as a singular 0-simplex, we have a homology class  $\sigma = [c]$ . Show that  $\|\sigma\|_1 = 1$ .*

**Remark 1.7.** *If  $A$  is a subspace of  $X$ , then we can define a simplicial (semi-)norm similarly on the relative homology group  $H_n(X, A; \mathbb{R})$ . Here one can treat  $H_n(X, A; \mathbb{R})$  as the homology of the chain complex  $C_n(X, A) = C_n(X)/C_n(A)$  (with the induced differentials). These vector spaces are equipped with semi-norms induced from  $C_n(X)$  and thus we can define an induced semi-norm on  $H_n(X, A; \mathbb{R})$  as before. When  $A$  is empty, this agrees with our definition above.*

*More generally, one can analogously define simplicial norm for any normed chain complex; see [Fri17].*

**Exercise 1.8.** *Concretely, we can think of  $H_n(X, A; \mathbb{R}) = Z_n(X, A)/B_n(X, A)$ , where  $B_n(X, A) = B_n(X) \cup C_n(A)$  and  $Z_n(X, A) = \partial_n^{-1}C_{n-1}(A)$ , with  $C_i(A)$  treated naturally as a subspace of  $C_i(X)$  for both  $i = n-1, n$ . Show that the semi-norm induced from this quotient agrees with the definition in the remark above.*

**1.2. The simplicial volume.** Now we specialize to measure the size of an oriented connected compact manifold  $M$  with (possibly empty) boundary  $\partial M$ . Let  $n = \dim M$ . The orientation picks out a generator  $[M] \in H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ , called the *fundamental class*. We think of it as a class in  $H_n(M, \partial M; \mathbb{R}) \cong \mathbb{R}$  using the map  $H_n(M, \partial M; \mathbb{Z}) \rightarrow H_n(M, \partial M; \mathbb{R})$  induced by the standard inclusion  $\mathbb{Z} \rightarrow \mathbb{R}$ . Concretely, if  $M$  has a triangulation, then the sum of all  $n$ -simplices with compatible orientation is a cycle representing the fundamental class.

**Definition 1.9** (Simplicial volume). The simplicial volume of  $M$  is  $\|[M]\|_1$ , which we often abbreviate as  $\|M\|_1$ . Note that the choice of orientation does not affect the simplicial volume.

If  $M$  is non-orientable, then  $M$  has an orientable double cover  $N$ , and we define  $\|M\|_1 := \|N\|_1/2$ . If  $M$  is disconnected, define  $\|M\|_1$  as the sum of volumes of its components.

**Exercise 1.10.** If  $M$  is orientable and closed, with finitely many components  $N_i$ . Show that  $\sum_i \|N_i\|_1 = \|\sum_i [N_i]\|_1$ , which explains the definition above for the disconnected case.

Recall that, for any continuous map  $f : M^n \rightarrow N^n$  between oriented connected closed (occ) manifolds, the degree  $\deg(f)$  is the unique integer such that  $f_*[M] = \deg(f) \cdot [N]$ .

**Lemma 1.11.** For any continuous map  $f : M^n \rightarrow N^n$  between occ manifolds, we have

$$|\deg(f)| \cdot \|N\|_1 \leq \|M\|_1.$$

Moreover, if  $f$  is a (finite) covering map, then equality holds.

*Proof.* The inequality follows from functoriality (Proposition 1.3) since  $\|f_*[M]\|_1 = \|\deg(f) \cdot [N]\|_1 = |\deg(f)| \cdot \|N\|_1$ .

Let  $c = \sum_i \lambda_i c_i$  be a cycle representing the fundamental class  $[N]$ . Each map  $c_i : \Delta^n \rightarrow N$  has  $d := |\deg(f)|$  lifts  $\tilde{c}_i^j$  to  $M$ ,  $j = 1, \dots, d$ . Then  $\tilde{c} = \sum_i \sum_{j=1}^d \tilde{c}_i^j$  is a cycle and clearly  $f_*[\tilde{c}] = d[c] = |\deg(f)| \cdot [N] = \pm f_*[M]$ . Hence  $[\tilde{c}] = \pm [M]$ , and  $\|M\|_1 \leq |\deg(f)| \cdot \|c\|_1$ . Since  $c$  is arbitrary, minimizing its norm gives the reversed inequality we desire.  $\square$

**Corollary 1.12.** If an orientable closed connected manifold  $M$  admits a selfmap  $f : M \rightarrow M$  with  $|\deg(f)| > 1$ , then  $\|M\|_1 = 0$ .

**Exercise 1.13.** Extend the lemma and corollary above to the case of manifolds with boundary.

**Example 1.14.**

- (1) For any sphere  $S^n$ ,  $n \geq 1$ , we have  $\|S^n\|_1 = 0$ .
- (2) For the  $n$ -torus  $T^n = (S^1)^n$ ,  $n \geq 1$  we have  $\|T^n\|_1 = 0$ .
- (3) More generally, if  $M = S^1 \times N$  for a closed manifold  $N$ , then  $\|M\|_1 = 0$ .

These properties of the simplicial volume help us understand the simplicial norm of certain homology classes.

**Lemma 1.15.** For  $n \geq 1$ , if a homology class  $\sigma \in H_n(X; \mathbb{R})$  is represented by a sphere, i.e. there is a map  $f : S^n \rightarrow X$  with  $f_*[S^n] = \sigma$ , then  $\|\sigma\|_1 = 0$ .

*Proof.* By functoriality and the fact that spheres (of dimension at least one) have zero simplicial volume,  $\|\sigma\|_1 = \|f_*[S^n]\|_1 \leq \|S^n\|_1 = 0$ . Thus  $\|\sigma\|_1 = 0$ .  $\square$

**Corollary 1.16.** For any  $X$ , the simplicial norm  $\|\cdot\|_1$  vanishes on  $H_1(X; \mathbb{R})$ .

*Proof.* Basically, every 1-cycle is a bunch of circles and thus this should follow from Lemma 1.15. To make it precise, we use the approximation by rational cycles from Lemma 1.5 to reduce the problem to integral cycles, which is a standard trick in these topics.

By the second part of Lemma 1.5, it suffices to show that  $\|\sigma\|_1 = 0$  for all rational homology classes  $\sigma \in H_n(X; \mathbb{R})$ . Any such  $\sigma$  is represented by some rational cycle  $c$ , and up to scaling, it

suffices to consider the case where  $c$  is integral, i.e.  $c = \sum_i n_i c_i$  for some  $n_i \in \mathbb{Z} \setminus \{0\}$ . Up to changing the orientation on  $c_i$  we may assume  $n_i > 0$ .

Now create  $n_i$  disjoint oriented segments for each  $c_i$  for all  $i$ . The fact that  $\partial c = 0$  implies that we can pair the boundary points of these segments so that the endpoint of a segment  $s$  is always paired with the starting point of some segment  $s'$  so that the corresponding paths glue up in  $X$  respecting the orientations. The end result is a closed oriented 1-manifold, i.e. a disjoint union of finitely many oriented circles  $S_k^1$  indexed by  $k$ . In other words, there is a map  $\varphi : \sqcup_k S_k^1 \rightarrow X$  such that  $\varphi_* \sum_k [S_k^1] = \sigma$ . Hence by Lemma 1.15 and the triangle inequality,

$$\|\sigma\|_1 \leq \sum_k \|\varphi_*[S_k^1]\|_1 = 0,$$

so  $\|\sigma\|_1 = 0$  as desired.  $\square$

**1.3. Volumes of surfaces.** In this section we aim to obtain the first nontrivial examples. We have seen that the simplicial norm is boring on  $H_0$  and vanishes on  $H_1$ . Interesting examples emerge in  $H_2$ . For orientable connected closed surfaces, we have seen in Example 1.14 that the simplicial volume vanishes when the genus is zero or one. For surfaces of higher genus, the simplicial volume is nonzero and is proportional to the Euler characteristic.

**Theorem 1.17.** *For any orientable connected closed surface  $S$  of genus at least two, we have  $\|S\|_1 = -2\chi(S)$ .*

**Remark 1.18.** *Note that by Gauss–Bonnet, for any hyperbolic metric,  $S$  has area  $-2\pi\chi(S) = \pi\|S\|_1$ , so the simplicial volume is proportional to the hyperbolic volume. The factor  $\pi$  is the area of the ideal hyperbolic triangle, or equivalently, the supremum of areas of all hyperbolic triangles (ideal or not). We will generalize this to higher dimension, which is referred to as Gromov’s proportionality theorem.*

To combine the results for all genera, it is convenient to introduce the following  $\chi^-$  notation.

**Notation 1.19.** For an orientable connected compact surface  $S$ , let  $\chi^-(S) = \chi(S)$  if  $\chi(S) \leq 0$  and let  $\chi^-(S) = 0$  otherwise, i.e. we adjust  $\chi(S)$  to 0 when  $S$  is a sphere or a disk. For a general orientable compact surface  $S = \sqcup \Sigma_i$  with components, let  $\chi^-(S) := \sum \chi^-(\Sigma_i)$ . In other words,  $\chi^-(S)$  is the Euler characteristic of  $S$  after deleting all components homeomorphic to spheres or disks.

Then the following theorem easily follows from Theorem 1.17 and the case of the sphere and torus.

**Theorem 1.20.** *For any orientable closed surface  $S$ , we have  $\|S\|_1 = -2\chi^-(S)$ .*

We will prove Theorem 1.17 by establishing inequalities in both directions, which involve two different kinds of ideas.

The strategy for proving  $\|S\|_1 \leq -2\chi(S)$  is to construct a sequence of cycles representing the fundamental class  $[S]$  approaching the optimal value. As we explained earlier, one concrete way to represent the fundamental class is to triangulate  $S$  and take the formal sum of triangles with compatible orientations.

Suppose  $S$  has a triangulation with  $v$  vertices,  $e$  edges and  $f$  faces, we know  $\chi(S) = v - e + f$ . The cycle described above has norm  $f$ .

**Lemma 1.21.** *We have  $2e = 3f$ , so  $\chi(S) = v - \frac{f}{2}$  and  $f = 2v - 2\chi(S)$ . Hence  $\|S\|_1 \leq 2v - 2\chi(S)$ .*

*Proof.* Each triangle has 3 edges, each of which is shared by two triangles.  $\square$

So this is close to be optimal except for the error  $2v$ . The best one can do here is to take a triangulation with  $v = 1$ , which exists.

**Exercise 1.22.** For any occ surface  $S$ , there is a triangulation with a single vertex.

The bounded error can be remedied by taking finite covers.

**Lemma 1.23.** If  $S$  has genus at least one, then  $\|S\|_1 \leq -2\chi(S)$ .

*Proof.* For any such  $S$  and any  $d \in \mathbb{Z}_+$ , there is a degree  $d$  cover  $f : S' \rightarrow S$ . Then by taking a triangulation on  $S'$  with a single vertex, we have  $\|S'\|_1 \leq 0$  by Lemma 1.21. Note that both  $\chi$  and  $\|\cdot\|_1$  are multiplicative, i.e.  $d\chi(S) = \chi(S')$  and  $d\|S\|_1 = \|S'\|_1$  (by Lemma 1.11). Thus we obtain

$$\|S\|_1 = \frac{\|S'\|_1}{d} \leq \frac{0 - d\chi(S)}{d} = -\chi(S).$$

Taking  $d \rightarrow \infty$ , we obtain the desired inequality.  $\square$

We considered above all possible ways of representing (resp. a multiple of) the fundamental class using triangulations (resp. of a finite cover).

The reversed inequality uses a technique called “straightening”, which involves hyperbolic geometry. Roughly speaking, every hyperbolic triangle has area no greater than  $\pi$ , and the hyperbolic area of the surface is  $-2\pi\chi(S)$ , so one needs at least  $-\chi(S)$  triangles to cover the entire surface once. So we just need an argument to straighten an arbitrary singular cycle representing the fundamental class into one only involving hyperbolic triangles. We will explain this in more detail (Section 1.5) after a crash course on hyperbolic geometry (1.4).

**1.4. Some hyperbolic geometry.** We give a quick introduction/review of some hyperbolic geometry, mainly to describe geodesics and isometries. A more detailed treatment can be found in [BP92], [Thu97], or any standard textbook/notes on hyperbolic geometry.

The  $n$ -dimensional hyperbolic space  $\mathbb{H}^n$  ( $n \geq 2$ ) is the unique (up to isometry) simply connected complete Riemannian manifolds with constant sectional curvature  $-1$ . There are several models for  $\mathbb{H}^n$ , providing different ways to view the space.

**1.4.1. The hyperboloid model.** Consider the bilinear form  $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$  on  $\mathbb{R}^{n+1}$  for any  $x, y \in \mathbb{R}^{n+1}$ . The set  $H := \{x \mid \langle x, x \rangle = -1\}$  is a hyperboloid of two sheets. The restriction of  $\langle \cdot, \cdot \rangle$  on either component gives a complete Riemannian metric of constant curvature  $-1$ . The upper sheet  $H_+$  (i.e. with  $x_{n+1} > 0$ ) is the hyperboloid model of  $\mathbb{H}^n$ .

With this model, the isometry group  $\text{Isom}(\mathbb{H}^n)$  is identified with  $O^+(n, 1)$ , the group of linear transformations preserving the bilinear form  $\langle \cdot, \cdot \rangle$  and stabilizing the upper sheet. The isometry group acts simply transitively on the orthonormal frame bundle, i.e. given any two points  $x, y \in \mathbb{H}^n$  and two orthonormal bases (i.e. two orthonormal frames) at the two points, there is a unique isometry taking the frame at  $x$  to the frame at  $y$ .

An advantage of this model is that, any  $k$ -dimensional totally geodesic subspace of  $\mathbb{H}^n$  is the intersection with some linear subspace of dimension  $k + 1$ . In particular, bi-infinite geodesics are intersections with planes through the origin.

The linearity provides a way to take *convex combinations* of points. More precisely, given  $k$  points  $p_1, \dots, p_k \in \mathbb{H}^n = H_+ \subset \mathbb{R}^{n+1}$ , any coefficients  $\lambda_1, \dots, \lambda_k \geq 0$  with  $\sum_{i=1}^k \lambda_i = 1$  uniquely determine a point  $p(\lambda_1, \dots, \lambda_k) \in \mathbb{H}^n$  as the intersection of  $H_+$  with the segment connecting the origin with  $\sum_{i=1}^k \lambda_i p_i \in \mathbb{R}^{n+1}$ . Apparently  $p(\lambda_1, \dots, \lambda_k)$  depends continuously on the coefficients  $\lambda_i$ , so this defines a continuous map  $p : \Delta^{k-1} \rightarrow \mathbb{H}^n$ , where  $\Delta^{k-1} = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$  is the standard  $(k-1)$ -simplex. This gives a way to straighten singular simplices in  $\mathbb{H}^n$ : For any  $c : \Delta^{k-1} \rightarrow \mathbb{H}^n$ , let  $p_1, \dots, p_k$  be the image of the  $k$  vertices in the natural order, and let  $\widetilde{\text{str}}(c)$  be the map  $p$  defined above. This operation has the following properties which we record for reference later:

**Lemma 1.24.**

(1) If  $c_1$  and  $c_2$  agree on some face of  $\Delta^{k-1}$ , then so do  $\widetilde{\text{str}}(c_1)$  and  $\widetilde{\text{str}}(c_2)$ .

(2) For any isometry  $g \in \text{Isom} \mathbb{H}^n$ , we have  $\widetilde{\text{str}}(g \circ c) = g \circ \widetilde{\text{str}}(c)$ .

*Proof.* The first part follows from the construction. The second part holds since the isometries in this model are linear maps, which commute with both taking convex combinations and scaling.  $\square$

The straightening operation that we will introduce in Section 1.5 relies on this construction.

**1.4.2. The Poincaré ball model.** In this model,  $\mathbb{H}^n$  is identified with the open unit disk  $\mathbb{D}^n \subset \mathbb{R}^n$  with the metric  $\frac{4ds^2}{(1-\|x\|_2^2)^2}$  (at any  $x \in \mathbb{D}^n$ ), where  $ds^2$  is the Euclidean metric. So the metric gets more distorted in this model compared to the Euclidean one when  $x$  is closer to the boundary.

In this model, geodesics are circular arcs perpendicular to the boundary sphere. The isometry group consists of *Möbius transformations* that preserve the unit disk.

**Definition 1.25.** A self-diffeomorphism  $f$  of  $S^n = \mathbb{R}^n \cup \{\infty\}$  is a Möbius transformation if one of the following equivalent descriptions holds:

- (1)  $f$  is a composition of inversions and reflections;
- (2)  $f$  is conformal (i.e. angle preserving);
- (3)  $f$  takes round spheres and hyperplanes to round spheres and hyperplanes;
- (4)  $f$  is a Euclidean similarity possibly composed with an inversion, i.e.  $f(x) = \lambda A i(x) + b$ , where  $i$  is either the identity or an inversion,  $A \in O(n)$ ,  $\lambda > 0$ , and  $b \in \mathbb{R}^n$ .

Here an *inversion* with respect to a round sphere  $S(p, r)$  in  $\mathbb{R}^n$  centered at  $p$  of radius  $r$  is the map on  $S^n = \mathbb{R}^n \cup \{\infty\}$  given by  $i(x) = p + \frac{x-p}{\|x-p\|} \cdot \frac{r^2}{\|x-p\|}$ , which fixes  $S(p, r)$  pointwise, swaps  $p$  and  $\infty$ , and preserves all rays from  $p$  so that  $\|x-p\| \cdot \|i(x)-p\| = r^2$ . The equivalence in the definition above (when  $n \geq 3$ ) essentially follows from Liouville's theorem:

**Theorem 1.26** (Liouville). *A conformal diffeomorphism  $f$  between two open subsets of  $\mathbb{R}^n$  with  $n \geq 3$  takes the form  $f(x) = \lambda A i(x) + b$ , where  $i$  is either the identity or an inversion,  $A \in O(n)$ ,  $\lambda > 0$ , and  $b \in \mathbb{R}^n$ .*

See [BP92, Theorem A.3.7] for a detailed proof. A more geometric argument can be found here: <https://lamington.wordpress.com/2013/10/28/liouville-illuminated/>.

This is a good model to talk about the boundary at infinity of  $\mathbb{H}^n$ , denoted  $\partial \mathbb{H}^n$ . Although  $\partial \mathbb{H}^n$  is not part of the hyperbolic space  $\mathbb{H}^n$ , it compactifies the space (i.e.  $\overline{\mathbb{H}}^n := \mathbb{H}^n \cup \partial \mathbb{H}^n$ ) and helps us understand geodesics and isometries. In this model, the boundary is exactly the unit sphere  $S^{n-1}$  and the compactification  $\overline{\mathbb{H}}^n$  is homeomorphic to the closed unit disk  $\overline{\mathbb{D}}^n$ . Each bi-infinite geodesic naturally has two endpoints on the boundary which uniquely determines the geodesic. Each isometry extends to a homeomorphism on  $\overline{\mathbb{H}}^n$  and in particular acts on the boundary (and the action is 2-transitive).

**1.4.3. The upper-half space model.** This model identifies  $\mathbb{H}^n$  with the open upper-half space  $\{x \in \mathbb{R}^n \mid x_n > 0\}$  equipped with the metric  $\frac{ds^2}{x_n^2}$ , where  $ds^2$  is the Euclidean metric. Geodesics in this model are vertical lines and circular arcs perpendicular to the hyperplane  $\{x_n = 0\}$ . Isometries are Möbius transformations preserving the upper-half space. The boundary can be seen as the union of the hyperplane  $\{x_n = 0\}$  with  $\infty$ .

**1.4.4. Isometries.** There is a classification of orientation-preserving isometries. Any isometry extends to a continuous homeomorphism on  $\overline{\mathbb{H}}^n$ , which is topologically a closed ball. Thus by Brouwer's fixed point theorem, each isometry must fix some point in  $\overline{\mathbb{H}}^n$ . Given  $g \neq id \in \text{Isom}^+(\mathbb{H}^n)$ , there are three mutually exclusive cases:

- (1)  $g$  is *elliptic* if it fixes some point in  $\mathbb{H}^n$ . Up to conjugation, we may assume that  $g$  fixes the origin 0 in the disk model, in which case  $g$  is conjugate to an orthogonal transformation in  $\text{SO}(n)$ , determined by its action on the unit tangent space at 0.

- (2)  $g$  is *parabolic* if it has no fixed point in  $\mathbb{H}^n$  and has a unique fixed point in  $\partial\mathbb{H}^n$ . Up to conjugation, we may assume that  $g$  fixes  $\infty$  in the upper-half space model, in which case we can deduce that  $g$  is conjugate to a horizontal (i.e. preserving  $x_n$ ) translation on  $\mathbb{R}^n$ .
- (3)  $g$  is *hyperbolic* if it has no fixed point in  $\mathbb{H}^n$  and has two fixed points in  $\partial\mathbb{H}^n$ . Up to conjugation, we may assume that  $g$  fixes  $\infty$  and  $0$  in the upper-half space model. Then  $g$  is the composition of a scaling with an orthogonal transformation in  $\mathrm{SO}(n-1) \subset \mathrm{SO}(n)$  centered at the origin fixing the  $x_n$  axis. This axis is the unique bi-infinite geodesic preserved by  $g$ , called the *axis* of  $g$ .

**1.5. Straightening.** Let  $M^n$  be a hyperbolic manifold. We introduce a linear map  $\mathrm{str} : C_k(M; \mathbb{R}) \rightarrow C_k(M; \mathbb{R})$ , called *straightening*. Fix the universal covering map  $\pi : \mathbb{H}^n \rightarrow M$ . For any singular simplex  $c : \Delta^k \rightarrow M$ , pick any lift  $\tilde{c} : \Delta^k \rightarrow \mathbb{H}^n$ , which we straighten to  $\widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \rightarrow \mathbb{H}^n$  using the construction described in Section 1.4.1. Define the straightening of  $c$  as  $\mathrm{str}(c) := \pi \widetilde{\mathrm{str}}(\tilde{c}) : \Delta^k \rightarrow M$ , which is independent of the choice of the lift since  $\widetilde{\mathrm{str}}$  commutes with isometries. This extends to a linear map  $\mathrm{str} : C_k(M; \mathbb{R}) \rightarrow C_k(M; \mathbb{R})$ .

**Lemma 1.27.** *We have  $|\mathrm{str}(c)|_1 \leq |c|_1$  for any chain  $c$  and  $\partial \mathrm{str} = \mathrm{str} \partial$ . Moreover,  $\mathrm{str}$  induces the identity map on  $H_*(M; \mathbb{R})$ .*

*Proof.* The straightening map does not increase the number of simplices, so  $|\mathrm{str}(c)|_1 \leq |c|_1$  for any chain  $c$ . It commutes with the boundary map by construction.

For any singular simplex  $c : \Delta^k \rightarrow \mathbb{H}^n$ , there is an obvious linear homotopy in  $\mathbb{R}^{n+1}$  from  $c$  to  $\widetilde{\mathrm{str}}(c)$ , which scales to one on  $\mathbb{H}^n$  and projects down to a homotopy on  $M$ . Based on this, one can build a chain homotopy between  $\mathrm{str}$  and  $\mathrm{id}$ , or directly observed that  $[\mathrm{str}(c)] = [c]$  for any cycle  $c$ .  $\square$

It follows that, when computing the simplicial norm of any homology class  $\sigma \in H_*(M; \mathbb{R})$  it suffices to look at cycles consisting of (straight) hyperbolic simplices. A key fact about hyperbolic simplices is that their volume has a uniform upper bound only depending on the dimension, in contrast with Euclidean simplices.

**Lemma 1.28** ([BP92, Theorem C.2.1 and Lemma C.2.3]). *For each  $n \geq 2$ , let  $v_n$  be the supremum of volumes of all hyperbolic  $n$ -simplices (possibly with vertices at infinity). Then  $v_2 = \pi$  and  $v_n \leq \frac{\pi}{(n-1)!}$ .*

**Remark 1.29.** *A theorem of Haagerup–Munkholm [HM81] shows that  $v_n$  is achieved uniquely by the regular<sup>1</sup> hyperbolic ideal  $n$ -simplex.*

*Proof.* Any hyperbolic simplex has volume no more than some ideal hyperbolic simplex. In fact, for any hyperbolic simplex with vertices  $p_0, \dots, p_n \in \mathbb{H}^n$ , choose a point  $p$  in its interior. The geodesic rays from  $p$  to  $p_i$  determines a point  $p'_i \in \partial\mathbb{H}^n$  for each  $i$ . The ideal hyperbolic simplex with vertices  $p'_0, \dots, p'_n$  contains the starting one.

When  $n = 2$ , all ideal hyperbolic triangles are conjugate up to an isometry and have the same area  $\pi$ , so  $v_2 = \pi$ . The area can be computed explicitly in the upper-half space model, say for the ideal triangle with vertices  $-1, 1, \infty$ , or can be seen by Gauss–Bonnet.

For  $n \geq 3$ , we show  $v_n \leq \frac{v_{n-1}}{n-1}$  by a nice computation in the upper-half space model following [BP92, Lemma C.2.3], which implies the bound  $\frac{\pi}{(n-1)!}$  by induction. Let  $\sigma$  be any ideal hyperbolic  $n$ -simplex in the upper-half space model and put one of its vertex as  $\infty$ . The remaining  $n$  vertices form an ideal hyperbolic  $(n-1)$ -simplex  $\tau$  sitting on a totally geodesic subspace  $X$  of dimension  $n-1$  not containing  $\infty$  as a boundary point. So  $X$  is a round hemisphere centered at some  $p \in \mathbb{R}^{n-1} \subset \partial\mathbb{H}^n$ . Up to an isometry we may assume  $p = 0$  and the sphere has radius 1. So  $X$  is the upper unit

<sup>1</sup>completely symmetric in the sense that the isometry group is the symmetric group on the  $n+1$  vertices



hemisphere. The vertical projection to  $\mathbb{R}^{n-1}$  restricts to a homeomorphism from  $X$  to the unit closed ball in  $\mathbb{R}^{n-1}$ , and let  $s$  be the inverse. Explicitly,  $s(x) = (x, h(x))$  where  $h(x) = \sqrt{1 - \|x\|^2}$ .

Let  $\tau_0$  be the image of  $\tau$  under this projection, which is a Euclidean simplex with vertices on the unit sphere. Then

$$\begin{aligned} \text{vol}(\sigma) &= \int_{\tau_0} \int_{h(x)}^{\infty} \frac{dy dx}{y^n} \\ &= \frac{1}{n-1} \int_{\tau_0} \frac{dx}{h(x)^{n-1}}. \end{aligned}$$

It suffices to show that

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \leq \text{vol}(\tau).$$

Note that  $s$  gives a way to parameterize  $\tau$  using  $\tau_0$ , so we just need to show that the pullback of the volume form is dominated by  $\frac{dx}{h(x)^{n-1}}$  for each  $x$  in the unit ball, where we think of  $dx$  as the standard volume form on  $\mathbb{R}^{n-1}$ .

For any  $x$ , let  $\omega$  be the hyperbolic volume form restricted to  $T_{s(x)}X$ . Note that  $\omega$  evaluates to  $\frac{1}{h(x)^{n-1}}$  for any orthonormal basis of  $T_{s(x)}X$ . To find out its pullback, choose an orthonormal basis of  $T_x\mathbb{R}^{n-1}$ . If  $x = 0$ , then  $Dh = 0$  so  $s_*$  is the identity map and  $s^*\omega = \frac{dx}{h(x)^{n-1}}$ . If  $x \neq 0$ , we may choose the orthonormal basis so that one of them is  $e_1 = \frac{x}{\|x\|}$ . Then  $Dh = 0$  in all directions perpendicular to  $e_1$  and  $Dh(e_1) = \frac{\|x\|}{h(x)}$ , so  $s_*$  is the identity map in the subspace perpendicular to  $e_1$ , and it takes this orthonormal basis to an orthogonal basis where all elements have length 1 except that  $\|s_*(e_1)\|^2 = 1 + \frac{\|x\|^2}{1-\|x\|^2} = \frac{1}{1-\|x\|^2} = \frac{1}{h(x)^2}$ . Thus  $s^*\omega = \frac{1}{h(x)} \cdot \frac{dx}{h(x)^{n-1}} \geq \frac{dx}{h(x)^{n-1}}$  since  $h(x) \leq 1$ . This verifies that the pullback of the volume form by  $s$  dominates  $\frac{dx}{h(x)^{n-1}}$ , and hence

$$\int_{\tau_0} \frac{dx}{h(x)^{n-1}} \leq \text{vol}(\tau) \leq v_{n-1}.$$

□

**Lemma 1.30.** *Let  $M^n$  be an oriented closed connected hyperbolic manifold. Then  $\|M\|_1 \geq \frac{\text{vol}(M)}{v_n}$ .*

*Proof.* Let  $c = \sum \lambda_i c_i$  be a cycle representing  $[M]$ . By Lemma 1.27, we may assume that  $c$  consists of straight hyperbolic simplices without increasing  $\|c\|_1$ . Let  $\text{vol}$  be the volume form. Then we have

$$\text{vol}(M) = \langle [M], \text{vol} \rangle = \left\langle \sum \lambda_i c_i, \text{vol} \right\rangle \leq \sum |\lambda_i| \cdot \max \text{vol}(c_i) \leq |c|_1 \cdot v_n.$$

Sine  $c$  is arbitrary, we conclude that  $\text{vol}(M) \leq v_n \cdot \|M\|_1$ . □

**Remark 1.31.** *Conceptually, we obtained this lower bound by some sort of  $\ell^1$ - $\ell^\infty$  duality, where we used a cocycle (the volume form here) that is bounded on all straight hyperbolic simplices. This suggests the use of bounded cocycles and bounded cohomology as a dual theory to better understand simplicial norms.*

Restricting to the case  $n = 2$ , we can now finish the proof of Theorem 1.17.

*Proof of Theorem 1.17.* The above lemma for  $M = S$  a hyperbolic surface, we have

$$\|S\|_1 \geq \frac{\text{area}(S)}{v_2} = \frac{-2\pi\chi(S)}{\pi} = -2\chi(S)$$

by Gauss–Bonnet. Combining with Lemma 1.23, we conclude  $\|S\|_1 = -2\chi(S)$ . □

The bound in Lemma 1.30 is also sharp in higher dimensions; see Theorem 1.36. This geometric argument also works for closed manifolds with varying negative curvature, so one can deduce that any such manifold has positive simplicial volume. More generally, we have the following conjecture attributed to Gromov [Gro82, p.11]:

**Conjecture 1.32.** *Any closed manifold of non-positive curvature and negative Ricci curvature has  $\|M\|_1 > 0$ .*

This is still open. See [CW19] for a recent partial positive answer that uses a slightly different straightening and uniformly bounds the Jacobian of the straightened maps.

Similarly, this argument can be used to prove positivity of simplicial norms of other homology classes.

**Proposition 1.33.** *The simplicial norm is an honest norm (instead of a semi-norm) on  $H_k(M; \mathbb{R})$  for any  $k \geq 2$  and hyperbolic closed orientable manifold  $M$ . That is,  $\|\sigma\|_1 > 0$  for any  $\sigma \neq 0 \in H_k(M; \mathbb{R})$ .*

*Proof.* For any  $\sigma \neq 0$ , there is  $\sigma^* \in H^k(M; \mathbb{R})$  that pairs nontrivially with  $\sigma$ . Represent  $\sigma^*$  by a differential  $k$ -form  $\omega$ . Up to scaling we may assume that  $|\omega(V)| \leq 1$  for all  $k$  orthogonal vectors  $V$  of norm 1 at any point on  $M$ . As a result, the restriction of  $\omega$  on any straight hyperbolic  $k$ -simplex is bounded by the volume form. Thus for any straightened chain  $\sum \lambda_i c_i$  representing  $\sigma$ , we have

$$|\langle \sigma, \omega \rangle| \leq \max \text{vol}(c_i) \cdot \sum \|\lambda_i\| \leq v_k \cdot \|c\|_1.$$

It follows that

$$\|\sigma\|_1 \geq \frac{|\langle \sigma, \omega \rangle|}{v_k} > 0.$$

□

**1.6. Application: degrees of maps between surfaces.** Now we apply the calculation in Theorem 1.20 to solve Problem 1.1 as an application.

**Theorem 1.34.** *Given closed orientable connected surfaces  $S$  and  $S'$  that are not spheres, the set  $\deg(S, S')$  of all possible degrees of maps  $f : S \rightarrow S'$  is*

$$\deg(S, S') = \{d : |d \cdot \chi(S')| \leq |\chi(S)|\}.$$

*Proof.* By the degree inequality Lemma 1.11, we have  $\deg(f) \cdot \|S'\|_1 \leq \|S\|_1$  for any map  $f : S' \rightarrow S$ . Since spheres are excluded, we know  $\|S\|_1 = -2\chi(S)$  and  $\|S'\|_1 = -2\chi(S')$ . It follows that  $|d \cdot \chi(S')| \leq |\chi(S)|$  for any  $d \in \deg(S, S')$ .

It suffices to show that any such  $d$  can be realized as  $\deg(f)$  for some  $f : S \rightarrow S'$ . We restrict our attention to those  $d > 0$  since  $d = 0$  is realized by a constant map and those negative values can be obtained by composing with a orientation-reversing homeomorphism. Given any  $d > 0$  with  $|d \cdot \chi(S')| \leq |\chi(S)|$ , there is a degree  $d$  cover  $p : S'_d \rightarrow S'$ . Then  $|\chi(S'_d)| = |d\chi(S')| \leq |\chi(S)|$ , i.e.  $S'_d$  has genus no more than that of  $S$ . So there is a map  $g : S \rightarrow S'_d$  with  $\deg(g) = 1$  by pinching part of  $S$  to a point. Thus the composition  $f = gp$  has  $\deg(f) = \deg(g) \cdot \deg(p) = d$ . □

**Exercise 1.35.**

- (1) If  $S = S^2$ , show that  $\deg(S, S') = \{0\}$  unless  $S' = S^2$ , in which case  $\deg(S, S') = \mathbb{Z}$ .
- (2) If  $S' = S^2$ , show that  $\deg(S, S') = \mathbb{Z}$ .

**1.7. Gromov's proportionality.** The goal of this section is to prove the following theorem, showing that the lower bound of  $\|M\|_1$  in Lemma 1.30 is sharp:

**Theorem 1.36** (Gromov's Proportionality). *Let  $M^n$  be an oriented closed connected hyperbolic manifold. Then  $\|M\|_1 = \frac{\text{vol}(M)}{v_n}$ .*

Note that if for some  $\epsilon > 0$  we have a cycle  $c = \sum \lambda_i c_i$  with  $\lambda_i > 0$ , each  $c_i$  straight hyperbolic and positively oriented with  $\text{vol}(c_i) \geq v_n - \epsilon$ , then  $c$  represents  $\lambda[M]$  for some  $\lambda > 0$  and

$$(1.1) \quad \lambda \text{vol}(M) = \langle \lambda[M], \text{vol} \rangle = \sum \lambda_i \text{vol}(c_i) \geq \min\{\text{vol}(c_i)\} \cdot \sum \lambda_i \geq (v_n - \epsilon) \|c\|_1 \geq \lambda(v_n - \epsilon) \|M\|_1.$$

Thus it suffices to show the existence of such a cycle for any  $\epsilon > 0$ .

There is no obvious construction of such a cycle. However, if we were allowed to have *infinitely* many terms in a cycle, there is a natural construction. A formal way to make this work is to define Gromov's norm using *measure homology* instead of singular homology. That is, on the set  $S_k(X)$  of singular  $k$ -simplices in  $X$ , let  $\mathcal{C}_k(X)$  be the space of signed measures with compact support and bounded total variation, equipped with the total variation norm. This gives a chain complex with boundary maps defined in the usual way and induces a semi-norm on its homology (called measure homology). Zastrow and Hansen both showed that measure homology coincides with singular homology for CW complexes [Han98, Zas98], and Löh showed that the isomorphism is isometric [LÖ6], thus this is an equivalent way to define Gromov's norm as Thurston originally claimed [Thu, Chapter 6].

The advantage is that this gives more room to construct the desired cycle in the measure sense. In fact, any singular chain  $\sum \lambda_i c_i$  can be thought of as the signed measure  $\sum \lambda_i \delta_{c_i}$ , where  $\delta_{c_i}$  is the Dirac mass at  $c_i$ . So in this way  $C_k(X)$  is a subspace of  $\mathcal{C}_k(X)$ .

Instead of giving a rigorous introduction of measure homology and proving the equivalence, we briefly describe this measure cycle and then approximate it using an honest singular cycle to finish the proof as in [Cal, Section 3.1]. For any  $\epsilon > 0$ , let  $\Delta$  be a straight positively oriented  $n$ -simplex in  $\mathbb{H}^n$  so that  $\text{vol}(\Delta) > v_n - \epsilon$ . Let  $D(\Delta)$  be the space of all isometric orientation-preserving embeddings of  $\Delta$  in  $\mathbb{H}^n$ , which can be identified as  $\text{Isom}^+(\mathbb{H}^n)$  and comes with the Haar measure. Then  $D(\Delta)/\pi_1(M)$  can be thought of as all isometric positively oriented copies of  $\Delta$  in  $M$ , with the induced measure. This makes it into a measure chain  $\text{smear}(\Delta)$  in  $\mathcal{C}_n(M)$ . Similarly, let  $\bar{\Delta}$  be the reflection of  $\Delta$  across some face. The same construction provides a measure chain  $\text{smear}(\bar{\Delta})$  in  $\mathcal{C}_n(M)$ . The key fact is that  $\text{smear}(\Delta) - \text{smear}(\bar{\Delta})$  is a measure *cycle*, since every face of  $\Delta$  is the face of its reflection across this face, and this reflection is a translate of  $\bar{\Delta}$ . Each copy of  $\bar{\Delta}$  has the opposite orientation, but the negative sign “corrects” it.

For the approximation, we need  $\Delta$  to be chosen with a stronger property in the beginning. It is a fact that, for any  $\epsilon > 0$  and any  $C > 0$ , there is  $\Delta$  such that  $\text{vol}(\Delta') > v_n - \epsilon$  for any  $\Delta'$  obtained from  $\Delta$  by moving each vertex by a distance no more than  $C$ .

*Proof of Theorem 1.36.* Let  $F \subset \mathbb{H}^n$  be a compact fundamental domain of  $M$  and fix  $p \in F$ . For any  $\epsilon > 0$  and  $C = \text{diam} F$ , let  $\Delta$  be chosen as above. For any  $(n+1)$ -tuple  $\vec{g} = (g_0, \dots, g_n) \in \pi_1(M)^{n+1}$ , let  $c_{\vec{g}}$  be the straight hyperbolic  $n$ -simplex with vertices  $(g_0 p, \dots, g_n p)$ , which is an approximation of those positively-oriented isometric embeddings of  $\Delta$  or  $\bar{\Delta}$  whose vertices lie in  $g_0 F, \dots, g_n F$  respectively, if exist. Let  $\lambda_{\vec{g}}$  (resp.  $\bar{\lambda}_{\vec{g}}$ ) be the measure of such embeddings of  $\Delta$  (resp.  $\bar{\Delta}$ ). Note that both  $\lambda_{\vec{g}}$  and  $\bar{\lambda}_{\vec{g}}$  are  $\pi_1(M)$  invariant under the diagonal action of  $\pi_1(M)$  on  $\pi_1(M)^{n+1}$ . Let  $\pi : \mathbb{H}^n \rightarrow M$  be the universal covering map. Then set the following chains in  $C_n(M; \mathbb{R})$

$$c = \sum_{\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)} \lambda_{\vec{g}} \pi c_{\vec{g}} \quad \text{and} \quad \bar{c} = \sum_{\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)} \bar{\lambda}_{\vec{g}} \pi c_{\vec{g}}$$

to approximate  $\text{smear}(\Delta)$  and  $\text{smear}(\bar{\Delta})$  respectively.

**Claim 1.37.**

- (1)  $\lambda_{\vec{g}} = 0$  for all but finitely many  $\vec{g} \in \pi_1(M)^{n+1}/\pi_1(M)$ , i.e.  $c$  is a singular chain, and so is  $\bar{c}$ .
- (2) If  $\lambda_{\vec{g}} > 0$  (resp.  $\bar{\lambda}_{\vec{g}} > 0$ ), then  $\text{vol}(c_{\vec{g}}) > v_n - \epsilon$ .
- (3) For  $\epsilon$  small enough, if  $\lambda_{\vec{g}} > 0$  (resp.  $\bar{\lambda}_{\vec{g}} > 0$ ), then  $c_{\vec{g}}$  is positively oriented (resp. negatively oriented).

(4)  $c - \bar{c}$  is a cycle.

*Proof.*

- (1) Up to the  $\pi_1(M)$  action, we may assume  $g_0p = p \in F$ . Then the space of isometric embeddings of  $\Delta$  with the first vertex inside  $F$  is compact, so the remaining  $n$  vertices can only lie in finitely many possible  $gF$ .
- (2) This means that  $c_{\bar{g}}$  can be obtained from some embedding of  $\Delta$  (or  $\bar{\Delta}$ ) by moving each vertex by distance no more than  $C = \text{diam}F$ . So the assertion follows from our choice of  $\Delta$ .
- (3) No  $\Delta$  can be close to  $\bar{\Delta}$  when they have large enough volume.
- (4) Any face  $\tau$  of  $c_{\bar{g}}$  is a straight  $(n-1)$ -simplex with vertices  $(h_1p, \dots, h_np)$  for some  $h_i \in \pi_1(M)$ . If a face of  $\Delta$  has an orientation-preserving isomeric embedding with vertices in  $h_1F, \dots, h_nF$ , then this contributes to the coefficient of  $\tau$  in  $\partial c$  by the measure of the total measure of all such isometric embeddings of  $\Delta$ . For each such embedding, its reflection across  $\tau$  is an orientation-preserving isometric embedding of  $\Delta$ , so this has the same contribution to the coefficient of  $\tau$  in  $\partial \bar{c}$ . Since this holds for all faces  $\tau$ , we observe that  $\partial(c - \bar{c}) = 0$ .

□

By the claim above, all nontrivial terms in  $c'$  are negatively oriented, which can be made positively oriented by adding a negative sign. Thus the cycle  $c - c'$  can be expressed as a positive linear combination of positively oriented straight simplices of volume at least  $v_n - \epsilon$ , this implies  $\text{vol}(M) \geq (v_n - \epsilon)\|M\|_1$  by equation (1.1). Since  $\epsilon$  can be arbitrarily small, this proves  $\frac{\text{vol}(M)}{v_n} \geq \|M\|_1$  and completes the proof in combination with Lemma 1.30. □

The smearing operation in the measure homology setup does something more [Thu, Chapter 6]. If  $M$  and  $N$  are oriented closed oriented Riemannian manifolds with isometric universal cover  $X$ , given a singular  $k$ -simplex  $c : \Delta \rightarrow M$ , we can similarly do the “smearing” operation by first lifting it to the universal cover, taking all translates of it, and projecting to  $N$ . Equipped with the natural measure from  $\text{Isom}^+(X)$ , this gives a chain in  $\mathcal{C}_k(N)$ . This extends to a linear map  $\text{smear}_{M,N} : \mathcal{C}_k(M) \rightarrow \mathcal{C}_k(N)$ . This has the property that  $\text{smear}_{M,N}[M] = \frac{\text{vol}(M)}{\text{vol}(N)} \cdot [N]$  and implies

$$\frac{\|N\|_1}{\text{vol}(N)} \leq \frac{\|M\|_1}{\text{vol}(M)}.$$

Flipping the roles of  $M$  and  $N$  implies the following more general version of Gromov’s proportionality:

**Theorem 1.38** (Gromov’s Proportionality, general case). *Suppose  $M$  and  $N$  are orientable closed Riemannian manifolds with isometric universal covers, then*

$$\frac{\|M\|_1}{\text{vol}(M)} = \frac{\|N\|_1}{\text{vol}(N)}.$$

It is unclear in general though how this constant proportion is related to the geometry of the universal cover.

## 2. BOUNDED COHOMOLOGY

Bounded cohomology is a dual theory of homology with simplicial norm, and we had a glimpse of its power in the estimate of simplicial volume of hyperbolic manifolds. It is also interesting on its own, providing new invariants sometimes quite different from ordinary cohomology. We will first focus on (bounded) cohomology of groups for concreteness. We will see later in Gromov’s mapping theorem that the bounded cohomology of a connected space  $X$  is canonically isomorphic to the bounded cohomology of  $\pi_1(X)$ , so we are not losing any information.

**2.1. Group cohomology.** We start with ordinary cohomology of groups as a warm-up. A good detailed reference on this topic is Brown's book [Bro82].

**2.1.1.  $K(G, 1)$  spaces.** Given a group  $G$ , a  $K(G, 1)$  space  $X_G$  is a connected *aspherical* CW complex with  $\pi_1(X_G) = G$ . Here being aspherical means that all higher homotopy groups of  $X_G$  vanish. Equivalently (by Whitehead's theorem), the universal cover of  $X_G$  is contractible. Such a space is unique up to homotopy equivalence, and is also called the *Eilenberg–MacLane space* or the *classifying space* of the group  $G$  (as a discrete group). The uniqueness follows from the universal property below:

**Lemma 2.1.** *Suppose  $X$  is a  $K(G, 1)$  space. Let  $Y$  be a connected CW complex with  $\pi_1(Y) = H$ . Any homomorphism  $\varphi : H \rightarrow G$  is induced by a continuous map  $f : Y \rightarrow X_G$ , which is unique up to homotopy.*

**Corollary 2.2.** *Suppose  $X$  and  $Y$  are both  $K(G, 1)$  spaces. Then any map  $f : X \rightarrow Y$  inducing an isomorphism  $f_* : G \rightarrow G$  is a homotopy equivalence.*

*Proof.* Let  $g_* : G \rightarrow G$  be the inverse of  $f_*$ . By the universal property, it is induced by a map  $g : Y \rightarrow X$ . Then  $gf : X \rightarrow Y$  induces the identity map on  $\pi_1(X)$  and hence is homotopic to the identity map on  $X$  by uniqueness. Similarly  $fg$  is homotopic to  $id_Y$ , so  $g$  is a homotopy inverse of  $f$ .  $\square$

One way to build a  $K(G, 1)$  space is to start with a 2-complex with fundamental group  $G$  and add higher dimensional cells to kill all higher homotopy groups inductively. A similar idea can be used to prove the universal property above. See details in [Hat02] for instance.

One can use the  $K(G, 1)$  space to give a topological definition of the group (co)homology.

**Definition 2.3.** Given a ring  $R$ , the homology  $H_*(G; R)$  (resp. cohomology  $H^*(G; R)$ ) of  $G$  with  $R$  coefficients is  $H_*(X_G; R)$  (resp.  $H^*(X_G; R)$ ).

**Example 2.4.**

- (1) For  $G = \mathbb{Z}$ , we can take  $X_G = S^1$ . We know  $H_k(G; \mathbb{Z}) = H_k(X_G; \mathbb{Z}) \cong \mathbb{Z}$  for  $k = 0, 1$ . Since it is one-dimensional, we have  $H_k(G; \mathbb{Z}) = H_k(X_G; \mathbb{Z}) = 0$  for  $k \geq 2$ .
- (2) For a free group  $G = F_n$  with  $n \geq 1$ , we can take  $X_G$  as a graph (e.g. a wedge of circles). We have  $H_0(G; \mathbb{Z}) \cong \mathbb{Z}$ ,  $H_1(G; \mathbb{Z}) = \mathbb{Z}^n$ , and  $H_k(G; \mathbb{Z}) = 0$  for  $k \geq 2$  by dimension restrictions.
- (3)  $H_0(G; R) = R$  since  $K(G, 1)$  space is connected by definition.
- (4)  $H_1(G; R) = \text{Ab}(G) \otimes_{\mathbb{Z}} R$ .
- (5)  $H^1(G; R) = \text{Hom}(G, R)$ .
- (6) For a cyclic group  $G = \mathbb{Z}/m$ , we can choose an infinite lens space as the  $K(G, 1)$  space, which gives  $H_k(\mathbb{Z}/m; \mathbb{Z}) \cong \mathbb{Z}/m$  for all  $k$  odd,  $H_k(\mathbb{Z}/m; \mathbb{Z}) = 0$  for  $k > 0$  and even. See [Hat02, Example 2.43] for details in a similar computation.

It follows from the last example that no  $K(\mathbb{Z}/m, 1)$  space can be finite dimensional. As an application of this fact, we have:

**Proposition 2.5.** *For any nontrivial finite group  $G$ , there is no free action on  $\mathbb{R}^n$ .*

*Proof.* Suppose  $G$  acts freely. Let  $H \cong \mathbb{Z}/m$  be a nontrivial cyclic subgroup. Then  $H$  also acts freely on  $\mathbb{R}^n$ . The action is also properly discontinuous since  $H$  is finite. Thus  $\mathbb{R}^n/H$  is a  $K(H, 1)$  space as  $\mathbb{R}^n$  is contractible. This is a contradiction.  $\square$

**2.1.2. Aside: The co-Hopfian property.** The uniqueness of  $K(G, 1)$  spaces (up to homotopy) has interesting consequences in the context of closed manifolds.

**Lemma 2.6.** *If  $M$  and  $N$  are both connected aspherical  $n$ -manifolds (without boundary) with isomorphic fundamental group, then  $M$  and  $N$  are either both compact or both non-compact.*

*Proof.* Note that a connected  $n$ -manifold  $M$  (without boundary) is compact if and only if  $H_n(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$ . Since  $M$  and  $N$  are  $K(G, 1)$  spaces for the same  $G$ , they are homotopy equivalent and  $H_n(M; \mathbb{Z}/2) \cong H_n(N; \mathbb{Z}/2)$ , so they have the same compactness.  $\square$

This is related to the *co-Hopfian* property of fundamental groups

**Definition 2.7.** A group  $G$  is co-Hopfian if every injective homomorphism  $h : G \rightarrow G$  is an isomorphism.

**Exercise 2.8.** Show that the following groups are not co-Hopfian.

- (1)  $G = \mathbb{Z}$ .
- (2)  $G$  is a free group or free abelian group.
- (3)  $G$  is a free product  $H \star K$  where  $K$  is not co-Hopfian.

**Lemma 2.9.** If  $M$  is a closed connected aspherical manifold, then any subgroup of  $\pi_1(M)$  that is isomorphic to  $\pi_1(M)$  must have finite index. In particular, if  $\pi_1(M)$  is not co-Hopfian and  $M$  is orientable, then  $M$  has a self-map  $f$  with  $|\deg(f)| > 1$ .

*Proof.* Let  $H$  be a subgroup of  $G = \pi_1(M)$  that is isomorphic to  $G$ . Let  $\pi : M' \rightarrow M$  be the covering map corresponding to the inclusion  $H \rightarrow G$ . As  $M$  is aspherical, so is  $M'$ . Since  $M$  is closed,  $M'$  must be closed by Lemma 2.6, so  $\pi$  is a finite cover.

The isomorphism  $\pi_1(M) \rightarrow H$  can be realized as a homotopy equivalence  $\varphi : M \rightarrow M'$ , which necessarily has  $|\deg \varphi| = 1$ . So the composition  $f = \pi\varphi$  is a self-map with  $|\deg f| = |\deg \pi| > 1$  if  $H$  is a proper subgroup.  $\square$

**Corollary 2.10.** If  $M$  is an orientable closed connected aspherical manifold with  $\|M\|_1 > 0$ , then  $\pi_1(M)$  is co-Hopfian.

*Proof.* As  $\|M\|_1 > 0$ ,  $M$  cannot have a self-map  $f$  with  $|\deg f| > 1$  by the degree inequality.  $\square$

Asphericity is often deduced from geometry of the manifold:

**Lemma 2.11.** A complete Riemannian manifold  $M$  with non-positive sectional curvature is aspherical. Moreover, the universal cover of  $M$  is diffeomorphic to the Euclidean space.

*Proof.* For any  $p \in M$ , the exponential map  $T_p M \rightarrow M$  is a covering map by the Cartan–Hadamard theorem. Hence  $T_p M$  is one realization of the universal cover.  $\square$

**Corollary 2.12.** If  $M$  is occ with negative sectional curvature, then  $\pi_1(M)$  is co-Hopfian.

*Proof.* Negative curvature implies  $\|M\|_1 > 0$  by a straightening argument, so the assumptions of Corollary 2.10 are met.  $\square$

**Example 2.13.** Let  $S$  be an occ surface of genus  $g > 1$ . Then  $\pi_1(S)$  is co-Hopfian.

One can also prove this using the fact that infinite-index subgroups of  $\pi_1(S)$  are free groups.

**Exercise 2.14.** It is crucial to assume  $M$  to be aspherical in Corollary 2.10. Let  $N = M \# (S^2 \times S^1)$  be a connected sum, where  $M$  is an occ hyperbolic 3-manifold. Show that  $\pi_1(N)$  is not co-Hopfian although  $N$  is occ and has  $\|N\|_1 > 0$ .

**2.1.3. The bar complex.** There is also an explicit and purely algebraic definition of group (co)-homology, which we will see to be equivalent. Consider the following *bar complex* with coefficients in  $R$ . Let  $C_n(G; R)$  be the free  $R$ -module with basis consisting of  $n$ -tuples  $(g_1, \dots, g_n) \in G^n$ , and let  $C^n(G, R) = \text{Hom}_R(C_n(G; R), R)$ , where each element assigns a value in  $R$  to each element of  $G^n$ . The differential  $\partial : C_n(G; R) \rightarrow C_{n-1}(G; R)$  is determined by

$$(2.1) \quad \partial(g_1, \dots, g_n) := (g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) + (-1)^n (g_1, \dots, g_{n-1}).$$

It is standard to check that  $\partial^2 = 0$  (which also follows from the topological explanation below), and one can define  $H_*(G; R)$  as the homology of this chain complex. Similarly, take  $\delta : C^{n-1}(G; R) \rightarrow C^n(G; R)$  as the dual of  $\partial$  as usual, we can define  $H^*(G; R)$  using the cochains  $C^n(G; R)$ .

This weird-looking differential (2.1) in the bar complex comes from the following topological interpretation. Let  $EG$  be the simplicial complex where  $n$ -simplices correspond to elements in  $G^{n+1}$  for each  $n \geq 0$ , where the faces of  $(g_0, \dots, g_n)$  are  $(g_0, \dots, \hat{g}_i, \dots, g_n)$  (i.e. omitting  $g_i$ ),  $i = 0, \dots, n$ . Then  $EG$  is contractible since each simplicial map can be coned off, say by adding a fixed element  $g \in G$  as the last coordinate. The group  $G$  acts by  $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$ , which is a free action. Thus the quotient  $BG := EG/G$  is a  $K(G, 1)$  space.

One can think of an  $n$ -simplex as an equivalence class  $[(g_0, \dots, g_n)]$ , where  $(g_0, \dots, g_n) \sim (g'_0, \dots, g'_n)$  if  $(g'_0, \dots, g'_n) = (gg_0, \dots, gg_n)$  for some  $g \in G$ . This leads to a way to define group (co)homology using the so-called homogeneous coordinates.

Instead, we represent each equivalence class  $[(g_0, \dots, g_n)]$  by an  $n$ -tuple  $(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n)$ , which is independent of the representative  $(g_0, \dots, g_n)$ . This is called the *inhomogeneous coordinate*. Then  $(g_1, \dots, g_n)$  in inhomogeneous coordinate corresponds to the equivalence class  $[(id, g_1, g_1g_2, \dots, g_1g_2 \dots g_n)]$  in homogeneous coordinate. Geometrically, one can think of the homogeneous coordinate as marking on vertices and the inhomogeneous coordinate as marking on edges. Under this identification, the boundary map in inhomogeneous coordinates is exactly the differential (2.1). This justifies that the (co)homology defined using bar complex is exactly the (simplicial) homology of the  $K(G, 1)$  space  $BG$ , and thus agrees with our topological definition.

Under this setup, a  $k$ -cochain is an assignment  $f : G^k \rightarrow R$ , labeling each  $k$ -tuple in inhomogeneous coordinate an element in  $R$  and extending to an  $R$ -linear map on the space of  $k$ -chains.

**Example 2.15.** A 0-cochain is a constant in  $R$ , and  $\delta : C^0(G; R) \rightarrow C^1(G; R)$  is the zero map. Hence every 0-chain is a cocycle and  $H^0(G; R) = R$ .

For a 1-cochain  $f : G \rightarrow R$ , its coboundary is determined by  $(\delta f)(g_1, g_2) = f\partial(g_1, g_2) = f(g_2 - g_1) = f(g_2) - f(g_1g_2) + f(g_1)$ . So it is a cocycle if and only if  $f$  is a homomorphism to the abelian group  $R$  (forgetting the ring structure). This shows that  $H^1(G; R) = \text{Hom}(G, R)$ .

**2.2. Bounded cohomology of groups.** Roughly speaking, in bounded cohomology, we consider cochains that are bounded functions instead of arbitrary functions on  $G^k$ . For this to make sense, we need to measure the size of elements in  $R$ . For simplicity, we consider  $R = \mathbb{R}$  or  $\mathbb{Z}$ , equipped with the usual absolute value.

Then a  $k$ -cochain  $f$  in inhomogeneous coordinates as a map  $f : G^k \rightarrow R$  is bounded if

$$|f|_\infty := \sup |f(g_1, \dots, g_k)| < \infty.$$

Equivalently,  $|f|_\infty$  is the sup norm as a linear map on the space of  $k$ -chains.

For each  $k$ , let  $C_b^k(G, R)$  be the subspace of  $C^k(G; R)$  consisting of bounded cochains. Then the coboundary restricts to map  $\delta_k : C_b^k(G; R) \rightarrow C_b^{k+1}(G; R)$  since  $|\delta_k f|_\infty \leq (k+1)|f|_\infty$ . This gives rise to a new chain complex.

**Definition 2.16.** (Bounded cohomology) Let  $Z_b^n(G; R) := \ker \delta_n$  and  $B_b^n(G; R) := \text{Im } \delta_{n-1}$ . The  $n$ -th bounded cohomology of  $G$  is  $H_b^n(G; R) := Z_b^n(G; R)/B_b^n(G; R)$ .

The norm  $|\cdot|_\infty$  on  $C_b^n(G; R)$  restricts to  $Z_b^n(G; R)$  and induces a semi norm  $\|\cdot\|_\infty$  on  $H_b^n(G; R)$ . Explicitly, for any  $\alpha \in H_b^n(G; R)$ ,

$$\|\alpha\|_\infty = \inf_{[f]=\alpha} |f|_\infty.$$

It is natural to ask how bounded cohomology differs from ordinary cohomology. There is a natural map connecting them, by treating a bounded cocycle as an ordinary cocycle.

**Definition 2.17.** (Comparison map) The inclusion  $C_b^n(G; R) \rightarrow C^n(G; R)$  induces a homomorphism  $c : H_b^n(G; R) \rightarrow H^n(G; R)$  called the *comparison map*.

**Example 2.18.** As shown in Example 2.15, for degree  $n = 0$ , a 0-cochain is a constant map and thus bounded. Thus  $H_b^0(G; R) = H^0(G; R) = R$ .

For degree  $n = 1$ , a 1-cochain  $f : G \rightarrow R$  is a cocycle if and only if it is a homomorphism. However, a homomorphism to  $R = \mathbb{Z}, \mathbb{R}$  is always unbounded except the trivial one. Thus  $H_b^1(G; R) = 0$  and the comparison map is not surjective in general.

**2.3. Quasimorphisms.** Nontrivial bounded cohomology classes emerge in degree  $n = 2$ . Here is the idea: For any  $(n-1)$ -cochain  $f$ ,  $\delta f$  is always a cocycle, but it might happen that  $\delta f$  is bounded while  $f$  is not, in which case  $\delta f$  is a bounded  $n$ -cocycle potentially nontrivial. We will focus on  $R = \mathbb{R}$  throughout this section and thus often omit the coefficient.

When  $n = 2$ , as we calculated in Example 2.15, for any  $f : G \rightarrow \mathbb{R}$ , we have  $(\delta f)(g, h) = f(g) + f(h) - f(gh)$ . So  $\delta f$  is bounded exactly when  $f$  is a quasimorphism, defined as follows.

**Definition 2.19.** (Quasimorphism) A map  $\varphi : G \rightarrow \mathbb{R}$  is a *quasimorphism* if

$$D(\varphi) := \sup_{g, h \in G} |\varphi(g) + \varphi(h) - \varphi(gh)| < \infty.$$

The number  $D(\varphi)$  is called the *defect* of  $\varphi$ . Quasimorphisms can be thought of as homomorphisms with bounded error, measured by the defect.

A quasimorphism  $\varphi$  is *homogeneous* if  $\varphi(g^n) = n\varphi(g)$  for all  $g \in G$  and all  $n \in \mathbb{Z}$ , i.e.  $\varphi$  restricts to a homomorphism on every cyclic subgroup.

Note that quasimorphisms on  $G$  form an  $\mathbb{R}$ -vector space, which we denote as  $\widehat{Q}(G)$ . Homogeneous quasimorphisms form a linear subspace, denoted as  $Q(G)$ . Clearly, homomorphisms to  $\mathbb{R}$  are homogeneous quasimorphisms, i.e.  $H^1(G) \subset Q(G)$ . Also note that any bounded function  $\varphi : G \rightarrow \mathbb{R}$  is trivially a quasimorphism. This gives another subspace  $C_b^1(G) = C_b^1(G; \mathbb{R})$  of  $\widehat{Q}(G)$ .

Every quasimorphism can be made homogeneous by the following process.

**Definition 2.20.** (Homogenization) For any  $\varphi \in \widehat{Q}(G)$ , the homogenization  $\bar{\varphi}$  is defined as

$$\bar{\varphi}(g) := \lim_{n \rightarrow +\infty} \frac{\varphi(g^n)}{n}.$$

**Lemma 2.21.** The homogenization  $\bar{\varphi}$  is a well-defined homogeneous quasimorphism, and  $\bar{\varphi} - \varphi$  is a bounded function on  $G$ . Quantitatively, we have  $|\varphi - \bar{\varphi}|_\infty \leq D(\varphi)$ .

In the proof of this lemma, we need the following fact about sub-additive sequence.

**Lemma 2.22.** Let  $a_n$  be a real-valued sequence that is sub-additive, i.e.  $a_{m+n} \leq a_m + a_n$  for all  $m, n \geq 1$ . Then we have

$$\lim_{n \rightarrow +\infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}.$$

In particular, the limit exists iff  $\frac{a_n}{n}$  is bounded below.

*Proof.* Clearly we have  $\liminf \frac{a_n}{n} \geq \inf_{n \geq 1} \frac{a_n}{n}$ . So it suffices to show that  $\limsup \frac{a_n}{n} \leq \inf_{n \geq 1} \frac{a_n}{n}$ .

Fix  $m \geq 1$  and express any  $n$  as  $n = qm + r$  with  $0 < r \leq m$ . Let  $B = \max_{0 < r \leq m} a_r$ . By sub-additivity and induction, we have  $a_n \leq qa_m + a_r \leq qa_m + B$ . Thus

$$\frac{a_n}{n} \leq \frac{qa_m + B}{qm + r} = \frac{a_m + \frac{B}{q}}{m + \frac{r}{q}}.$$

Hence as  $n \rightarrow \infty$ , we have  $q \rightarrow \infty$  and

$$\limsup \frac{a_n}{n} \leq \frac{a_m}{m}.$$

Since  $m$  is arbitrary, we obtain

$$\limsup \frac{a_n}{n} \leq \inf_{m \geq 1} \frac{a_m}{m}.$$

□



There is an analogous result for sup-additive sequences, replacing  $\inf$  by  $\sup$ , which can be deduced by considering the sequence  $-a_n$ .

Now we prove Lemma 2.21.

*Proof of Lemma 2.21.* Fixing  $g \in G$ , we show that the limit defining  $\bar{\varphi}(g)$  exists. Although the sequence  $\varphi(g^n)$  is not sup- or sub-additive, a small modification does the job.

Let  $\varphi_+(g) := \varphi(g) + D(\varphi)$  and let  $\varphi_-(g) := \varphi(g) - D(\varphi)$ . By definition, we have  $|\varphi(g) + \varphi(h) - \varphi(gh)| \leq D(\varphi)$ , which implies

$$\varphi(gh) - D(\varphi) \leq \varphi(g) + \varphi(h) \leq \varphi(gh) + D(\varphi),$$

and hence

$$\varphi_+(gh) = \varphi(gh) - D(\varphi) + 2D(\varphi) \leq \varphi(g) + \varphi(h) - 2D(\varphi) = \varphi_+(g) + \varphi_+(h).$$

So  $\varphi_+$  is sub-additive and similarly  $\varphi_-$  is sup-additive.

It follows that

$$\varphi_-(g) \leq \frac{\varphi_-(g^n)}{n} \leq \frac{\varphi_+(g^n)}{n} \leq \varphi_+(g)$$

for all  $n \geq 1$ . So  $\frac{\varphi_+(g^n)}{n}$  is bounded below and  $\frac{\varphi_-(g^n)}{n}$  is bounded above. Thus both have finite limit by Lemma 2.22 and its sup-additive analogue. The limits agree since  $\varphi_+(g^n) - \varphi_-(g^n) = 2D(\varphi)$ . As  $\varphi_-(g^n) \leq \varphi_+(g^n)$ , we have

$$\bar{\varphi}(g) = \lim_{n \rightarrow +\infty} \frac{\varphi_-(g^n)}{n} = \lim_{n \rightarrow +\infty} \frac{\varphi(g^n)}{n} = \lim_{n \rightarrow +\infty} \frac{\varphi_+(g^n)}{n}.$$

This shows that  $\bar{\varphi}$  is a well-defined function.

This also gives a way to bound the difference  $\varphi - \bar{\varphi}$ . As

$$\lim_{n \rightarrow +\infty} \frac{\varphi_+(g^n)}{n} = \inf_{n \geq 1} \frac{\varphi_+(g^n)}{n} \leq \varphi_+(g) = \varphi(g) + D(\varphi),$$

we get

$$\bar{\varphi}(g) \leq \varphi(g) + D(\varphi).$$

Similarly using  $\varphi_-$  we have  $\bar{\varphi}(g) \geq \varphi(g) - D(\varphi)$ . As  $g$  is arbitrary, we conclude that  $|\varphi - \bar{\varphi}|_\infty \leq D(\varphi)$ . In particular,  $\bar{\varphi}$  is a quasimorphism as the sum of  $\varphi$  and the bounded function  $\bar{\varphi} - \varphi$ .

Finally, it remains to check that  $\bar{\varphi}$  is homogeneous. For every  $k \geq 1$ , we have

$$\bar{\varphi}(g^k) = \lim_{n \rightarrow +\infty} \frac{\varphi(g^{kn})}{n} = k \cdot \lim_{n \rightarrow +\infty} \frac{\varphi(g^{kn})}{kn} = k\bar{\varphi}(g).$$

For  $k = 0$ ,  $\bar{\varphi}(id) = \lim_{n \rightarrow +\infty} \frac{\varphi(id)}{n} = 0$ . So it suffices to show that  $\bar{\varphi}(g) + \bar{\varphi}(g^{-1}) = 0$  to deal with  $k < 0$ . Indeed, this easily follows from  $|\varphi(g^n) + \varphi(g^{-n})| \leq |\varphi(id)| + D(\varphi)$ .  $\square$

**Remark 2.23.** The triangle inequality and the bound  $|\varphi - \bar{\varphi}|_\infty \leq D(\varphi)$  implies that  $D(\bar{\varphi}) \leq 4D(\varphi)$ . A more involved argument shows that  $D(\bar{\varphi}) \leq 2D(\varphi)$ .

Every quasimorphism uniquely decomposes as the sum of a homogeneous quasimorphism (namely, its homogenization) and a bounded function.

**Lemma 2.24.** We have  $\widehat{Q}(G) = C_b^1(G) \oplus Q(G)$ . That is,  $C_b^1(G) \cap Q(G) = 0$  and  $\widehat{Q}(G) = C_b^1(G) + Q(G)$ .

*Proof.* If  $\varphi \in C_b^1(G) \cap Q(G)$ , then  $|\varphi(g)| = |\frac{\varphi(g^n)}{n}| \leq \frac{|\varphi|_\infty}{n}$ . As  $n$  is arbitrary, we must have  $\varphi(g) = 0$  for all  $g$ .

For any  $\varphi \in \widehat{Q}(G)$ , we have  $\varphi = \bar{\varphi} + (\varphi - \bar{\varphi})$ , where the homogenization  $\bar{\varphi} \in Q(G)$  and the difference  $\varphi - \bar{\varphi}$  is a bounded function by Lemma 2.21.  $\square$

**Proposition 2.25.** *We have the following exact sequence*

$$0 \rightarrow H^1(G; \mathbb{R}) \rightarrow Q(G) \xrightarrow{\delta} H_b^2(G; \mathbb{R}) \xrightarrow{c} H^2(G; \mathbb{R}).$$

*In particular, the kernel of the comparison map  $c : H_b^2(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R})$  is isomorphic to the quotient  $Q(G)/H^1(G)$ , which can be thought of as the space of “interesting” homogeneous quasimorphisms.*

*Proof.* We just check that  $\ker \delta = H^1(G)$  (as homomorphisms to  $\mathbb{R}$ ) and  $\ker c = \text{Im } \delta$ .

As we calculated in Example 2.15,  $(\delta\varphi)(g, h) = \varphi(g) + \varphi(h) - \varphi(gh)$ , so clearly  $H^1(G) \subset \ker \delta$ . Conversely, if  $[\delta\varphi] = 0 \in H_b^2(G)$ , then  $\delta\varphi = \delta f$  for some bounded function  $f : G \rightarrow \mathbb{R}$ . This implies that  $\varphi - f$  is a homomorphism. Then we get two homogeneous quasimorphisms  $\varphi - f$  and  $\varphi$  that differ by a bounded function  $f$ . By Lemma 2.24, we must have  $f = 0$  and thus  $\varphi$  is a homomorphism. Hence  $\ker \delta = H^1(G)$ .

$\text{Im } \delta \subset \ker c$  holds by definition. Suppose  $\alpha \in \ker c$ , i.e.  $\alpha = [\delta f]$  for some function  $f : G \rightarrow \mathbb{R}$ . Then  $\delta f$  must be bounded, so  $f$  is a quasimorphism. Since  $f$  and  $\bar{f}$  differ by a bounded function, we have  $\alpha = [\delta f] = [\delta \bar{f}]$ . As  $\bar{f} \in Q(G)$  we conclude that  $\alpha \in \text{Im } \delta$ . Thus  $\text{Im } \delta = \ker c$ .  $\square$

This gives a way to characterize the kernel of the comparison map in degree two.

**Corollary 2.26.** *The kernel of the comparison map  $c : H_b^2(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R})$  is identified with  $Q(G)/H^1(G; \mathbb{R})$ .*

**Exercise 2.27.** *Prove the following variant of the exact sequence in Proposition 2.25*

$$0 \rightarrow C_b^1(G) + H^1(G) \rightarrow \widehat{Q}(G) \xrightarrow{\delta} H_b^2(G; \mathbb{R}) \xrightarrow{c} H^2(G; \mathbb{R}).$$

We will often use the following basic estimate, which is immediate from the definition and induction.

**Lemma 2.28.** *Let  $\varphi$  be a quasimorphism. For  $g = g_1 \cdots g_n$ , we have*

$$|\varphi(g) - \sum_i \varphi(g_i)| \leq (n-1)D(\varphi).$$

*In particular,*

$$|\varphi(g)| \leq \sum_i |\varphi(g_i)| + (n-1)D(\varphi).$$

Homogeneous quasimorphisms have the following nice properties:

**Lemma 2.29.** *Let  $\varphi$  be a homogeneous quasimorphism.*

- (1) *If  $g$  and  $h$  commute, then  $\varphi(gh) = \varphi(g) + \varphi(h)$ . So  $\varphi$  restricts to homomorphisms on abelian subgroups.*
- (2)  *$\varphi$  is conjugation-invariant, i.e.  $\varphi(g) = \varphi(hgh^{-1})$  for all  $g, h \in G$ .*

*Proof.*

- (1) For any  $n \in \mathbb{Z}_+$ , we have  $(gh)^n = g^n h^n$ , so

$$n|\varphi(g) + \varphi(h) - \varphi(gh)| = |\varphi(g^n) + \varphi(h^n) - \varphi((gh)^n)| = |\varphi(g^n) + \varphi(h^n) - \varphi(g^n h^n)| \leq D(\varphi).$$

Letting  $n \rightarrow \infty$  we see  $\varphi(g) + \varphi(h) - \varphi(gh) = 0$ .

- (2) For any  $n \in \mathbb{Z}_+$ , we have

$$n|\varphi(g) - \varphi(hgh^{-1})| \leq |\varphi(g^n) - \varphi(hg^n h^{-1})| = |[\varphi(h) + \varphi(g^n) + \varphi(h^{-1})] - \varphi(hg^n h^{-1})| \leq 2D(\varphi),$$

where we used the fact that  $\varphi(h) + \varphi(h^{-1}) = 0$ . Letting  $n \rightarrow \infty$  we get  $\varphi(g) = \varphi(hgh^{-1})$ .  $\square$

**Corollary 2.30.** *If  $G$  is abelian, then  $Q(G) = H^1(G; \mathbb{R})$ .*

**Exercise 2.31.** Let  $\varphi$  be a homogeneous quasimorphism. Prove that  $|\varphi([g, h])| \leq D(\varphi)$  for all  $g, h$ , where  $[g, h] = ghg^{-1}h^{-1}$  is the commutator.

**Remark 2.32.** Actually Bavard [Bav91, Lemma 3.6] showed that  $D(\varphi) = \sup_{g, h} |\varphi([g, h])|$ . See also [Cal09, Lemma 2.24].

**Definition 2.33.** A group  $G$  is *perfect* if it has trivial abelianization, or equivalently,  $G$  agrees with its commutator subgroup, i.e. each  $g \in G$  is a product of commutators.

We say  $G$  is *uniformly perfect*, if there is a uniform  $n \in \mathbb{Z}_+$  such that each  $g \in G$  is a product of at most  $n$  commutators.

**Corollary 2.34.** If  $G$  is uniformly perfect, then  $Q(G) = H^1(G) = 0$ .

*Proof.* Suppose for  $n \in \mathbb{Z}_+$ , each  $g \in G$  is a product of at most  $n$  commutators. Then for any homogeneous quasimorphism  $\varphi \in Q(G)$ , and  $g = [a_1, b_1] \cdots [a_k, b_k]$  with  $k \leq n$  and  $a_i, b_i \in G$ , we have

$$|\varphi(g)| \leq \left| \sum_i \varphi([a_i, b_i]) \right| + (k-1)D(\varphi) \leq (2k-1)D(\varphi) \leq (2n-1)D(\varphi)$$

by Exercise 2.31. This shows that  $\varphi$  is a bounded function. Hence  $\varphi = 0$  by Lemma 2.24.  $\square$

**Lemma 2.35.** For any surjective homomorphism  $f : G \rightarrow H$ , the pullback map  $f^* : Q(H) \rightarrow Q(G)$  is injective and defect-preserving.

*Proof.* The pullback is given by  $(f^*\varphi)(g) = \varphi(f(g))$ , so injectivity is immediate. As for the defect, we have

$$D(f^*\varphi) = \sup_{g_1, g_2 \in G} |\varphi(f(g_1)f(g_2)) - \varphi(f(g_1)) - \varphi(f(g_2))| = \sup_{h_1, h_2 \in H} |\varphi(h_1h_2) - \varphi(h_1) - \varphi(h_2)| = D(\varphi).$$

$\square$

This gives a way to obstruct homomorphisms from groups with few quasimorphisms to those with lots of quasimorphisms.

**2.4. de Rham quasimorphisms.** Let  $\omega$  be a 1-form on a connected closed hyperbolic manifold  $M^n$ . Fix a based point  $p \in M$  and let  $G = \pi_1(M, p)$ . Then any  $g \in G$  is uniquely represented by an oriented geodesic loop  $\ell_g$  based at  $p$ . The de Rham quasimorphism (due to Barge–Ghys [BG88]) associated to  $\omega$  is

$$\varphi_\omega(g) := \int_{\ell_g} \omega.$$

**Lemma 2.36.**  $\varphi_\omega$  defined above is indeed a quasimorphism.

*Proof.* We need to bound  $\varphi_\omega(g) + \varphi_\omega(h) - \varphi_\omega(gh)$  for any  $g, h \in G$ .

Let  $\widetilde{M} \cong \mathbb{H}^n$  be the universal cover and  $\tilde{p}$  be a lift of  $p$ . Denote by  $\tilde{\omega}$  the pullback of  $\omega$  on  $\widetilde{M}$ . Then for any  $g \in G$ , the unique lift of  $\ell_g$  starting at  $\tilde{p}$  is the geodesic  $\tilde{\ell}_g$  from  $\tilde{p}$  to  $g\tilde{p}$ . So  $\varphi_\omega(g) = \int_{\tilde{\ell}_g} \tilde{\omega}$ .

Given  $g, h \in G$ , we have an oriented geodesic triangle  $\Delta$  with sides  $\tilde{\ell}_g$ ,  $g\tilde{\ell}_h$  and  $\tilde{\ell}_{gh}$ , where the induced orientation of  $\Delta$  is opposite to the orientation on  $\tilde{\ell}_{gh}$ . Thus

$$|\varphi_\omega(g) + \varphi_\omega(h) - \varphi_\omega(gh)| = \left| \int_{\partial\Delta} \tilde{\omega} \right| = \left| \int_{\Delta} d\tilde{\omega} \right| \leq \|d\tilde{\omega}\|_{\widetilde{M}} \cdot \text{area}(\Delta) \leq \pi \|d\omega\|_M,$$

where  $\|d\omega\|$  is the supremum of  $|d\omega(v)|$  over all orthonormal 2-frames  $v$  on  $M$ , which is finite by compactness. Note that we also used the fact that the area of a hyperbolic triangle is uniformly bounded by  $\pi$ .  $\square$

Also note that  $\varphi_\omega$  is a homomorphism if  $\omega$  is closed, as the integral over  $\Delta$ . When  $\omega$  is exact, then  $\varphi_\omega \equiv 0$ .

There is also a nice description of the homogenization  $\bar{\varphi}_\omega$ . For any  $g \in G$ , there is a unique closed geodesic loop  $L_g$  (which is length minimizing in the free homotopy class) representing the conjugacy of  $g$ , and  $\bar{\varphi}_\omega(g) = \int_{L_g} \omega$ . The reason is that, changing the base point  $p$  only varies  $\varphi_\omega$  by a bounded amount and does not affect the homogenization. So one can move  $p$  so that  $\ell_g$  agrees with  $L_g$ .

**2.5. Quasimorphisms on free groups.** Various kinds of quasimorphisms were constructed on free groups. Brooks constructed lots quasimorphisms that imply  $Q(F_n)$  is infinite-dimensional. A recent construction by Rolli [Rol09] gives a simpler way to prove this, so we will start with his construction.

**2.5.1. Rolli's construction.** Let  $\ell^\infty(\mathbb{Z}_+)$  be the space of  $\mathbb{R}$ -valued bounded functions on the set  $\mathbb{Z}_+$ , which is an infinite dimensional space. For any  $f \in \ell^\infty(\mathbb{Z}_+)$ , extend it uniquely to an odd function  $f : \mathbb{Z} \rightarrow \mathbb{R}$ , i.e.  $f(-n) = -f(n)$  for all  $n$ .

Given  $f, g \in \ell^\infty(\mathbb{Z}_+)$ , extended as above, define a quasimorphism  $\varphi_{f,g}$  on  $F_2 = \langle a, b \rangle$  as follows. For any element  $w \in F_2$ , express it as a reduced word in the generators  $w = a_1^{m_1} b_1^{n_1} \cdots a_k^{m_k} b_k^{n_k}$  with  $k \geq 1$  and each  $m_i, n_i \in \mathbb{Z} \setminus \{0\}$  except that  $m_1$  or  $n_k$  could be 0. Define  $\varphi_{f,g}(w) = \sum_{i=1}^k f(m_i) + \sum_{i=1}^k g(n_i)$ .

It is straightforward to check that

$$|\varphi(u) + \varphi(v) - \varphi(uv)| \leq \max\{3|f|_\infty, 3|g|_\infty\},$$

so  $\varphi$  is a quasimorphism.

The homogenization  $\bar{\varphi}_{f,g}$  can be described as follows. For any  $w \in F_2$ , it has a reduced expression  $w = uvu^{-1}$  where  $v$  is the unique cyclically reduced word in the conjugacy class of  $w$ . Then  $\bar{\varphi}_{f,g}(w) = \bar{\varphi}_{f,g}(v) = \varphi_{f,g}(v)$  if  $v$  is not a power of the generator  $a$  or  $b$ , and clearly  $\bar{\varphi}_{f,g}(a) = \bar{\varphi}_{f,g}(b) = 0$ .

**Lemma 2.37.** *We have an embedding  $\ell^\infty(\mathbb{Z}_+) \rightarrow Q(F_2)$  sending  $f$  to  $\bar{\varphi}_{f,f}$ . Moreover, the image intersects  $H^1(F_2)$  trivially.*

*Proof.* By the description above, for any  $n \in \mathbb{Z}_+$ , we have  $\bar{\varphi}_{f,f}(a^n b^n) = 2f(n)$ . Thus  $\bar{\varphi}_{f,f} = 0$  if and only if  $f = 0$ . The intersection with  $H^1(F_2)$  is trivial since  $\bar{\varphi}_{f,f}$  vanishes on the two generators.  $\square$

**Theorem 2.38.** *Both  $Q(F_2)$  and  $H_b^2(F_2; \mathbb{R})$  are infinite-dimensional.*

*Proof.* Since  $\ell^\infty(\mathbb{Z}_+)$  is infinite-dimensional, the embedding above shows that  $Q(F_2)$  is infinite-dimensional. The same space embeds in  $Q(F_2)/H^1(F_2) \cong H_b^2(F_2; \mathbb{R})$ , so it is also infinite-dimensional.  $\square$

**Corollary 2.39.** *If  $G$  surjects  $F_2$ , then  $Q(G)$  is infinite-dimensional.*

*Proof.*  $Q(F_2)$  embeds  $Q(G)$  by Lemma 2.35.  $\square$

Note that this applies to all non-abelian free groups and closed hyperbolic surface groups.

**Exercise 2.40.** *Generalize Rolli's construction to a free product  $G = A \star B$ . What kind of functions on  $A, B$  do you need for the construction to work out?*

**2.5.2. Brooks quasimorphisms.** Now we turn to the quasimorphisms constructed by Brooks, which can be generalized to groups acting on  $\delta$ -hyperbolic groups. There are two versions, big counting quasimorphisms and little counting quasimorphisms. We follow the exposition in [Cal09, Section 2.3.2].

Let  $G$  be a free group generated by  $S$ . Fix a reduced word  $\sigma$ . For any  $g \in G$ , define the big counting function  $C_\sigma(g)$  as the number of copies of  $\sigma$  that appear as subwords in the reduced word

representing  $g$ . Similarly, define the little counting function  $c_\sigma(g)$  as the maximal number  $n$  such that there are  $n$  disjoint copies of  $\sigma$  in the reduced expression of  $g$ .

**Example 2.41.**

- (1) For  $S = \{a, b\}$ ,  $\sigma = ab$ , and  $g = aba^{-1}baba$ , we have  $C_\sigma(g) = c_\sigma(g) = 2$ .
- (2) For  $S = \{a, b\}$ ,  $\sigma = aba$ , and  $g = ababa$ , we have  $C_\sigma(g) = 2$  but  $c_\sigma(g) = 1$ . Similarly for  $h = abababa$ , we have  $C_\sigma(h) = 3$  while  $c_\sigma(h) = 2$ .

For the discussion below, we will use the following observations.

**Lemma 2.42.** (1)  $C_\sigma(g^{-1}) = C_{\sigma^{-1}}(g)$  and similarly for  $c_\sigma$ .  
 (2)  $\sigma$  and  $\sigma^{-1}$  cannot overlap as subwords.

**Definition 2.43** (Counting quasimorphisms). The *big counting quasimorphism* associated to  $\sigma$  is  $H_\sigma(g) := C_\sigma(g) - C_\sigma(g^{-1})$ . The *little counting quasimorphism* associated to  $\sigma$  is  $h_\sigma(g) := c_\sigma(g) - c_\sigma(g^{-1})$ .

**Lemma 2.44.** Clearly from the definition,  $H_\sigma(g^{-1}) = -H_\sigma(g)$  and similarly for  $h_\sigma$ .

We show below that these are indeed quasimorphisms and we estimate their defect. For a reduced expression  $u = u_1 u_2$ , let  $s = 1$  (resp.  $s = -1$ ) if a copy of  $\sigma$  (resp.  $\sigma^{-1}$ ) appears in the juncture, and let  $s = 0$  if no  $w$  or  $w^{-1}$  appears.

**Lemma 2.45.** With the notation above, we have

$$0 \leq s \cdot (H_\sigma(u) - H_\sigma(u_1) - H_\sigma(u_2)) \leq |\sigma| - 1,$$

and  $h_\sigma(u) - h_\sigma(u_1) - h_\sigma(u_2) = 0$  or  $s$ .

*Proof.* If  $\sigma$  does not appear in the juncture, then  $c_\sigma(u) = c_\sigma(u_1) + c_\sigma(u_2)$  and  $C_\sigma(u) = C_\sigma(u_1) + C_\sigma(u_2)$ .

If  $\sigma$  appears in the juncture, there are at most  $|\sigma| - 1$  subwords of length  $|\sigma|$  in the juncture, so  $C_\sigma(u) - C_{\sigma^{-1}}(u_1) - C_{\sigma^{-1}}(u_2) \leq |\sigma| - 1$ .

As for  $c_\sigma$ , clearly  $c_\sigma(u_1) + c_\sigma(u_2) \leq c_\sigma(u)$ . Consider a collection of  $c_\sigma(u)$  disjoint copies of  $\sigma$  in  $u$ . There is at most one copy that lies in the juncture. The remaining copies are disjoint and either in  $u_1$  or  $u_2$ . Thus  $c_\sigma(u_1) + c_\sigma(u_2) \geq c_\sigma(u) - 1$ . Hence  $c_\sigma(u) - c_\sigma(u_1) + c_\sigma(u_2) = 0, 1$ .

Combining these observations proves the lemma.  $\square$

**Lemma 2.46.** We have  $D(H_\sigma) \leq 3(|\sigma| - 1)$  and  $D(h_\sigma) \leq 3$ . Thus  $H_\sigma$  and  $h_\sigma$  are indeed quasimorphisms.

*Proof.* For any  $g, h \in G$ , there are unique reduced expressions  $g = uv^{-1}$ ,  $h = vw^{-1}$  and  $gh = uw^{-1}$ . Then

$$\begin{aligned} H_\sigma(g) + H_\sigma(h) - H_\sigma(gh) &= H_\sigma(uv^{-1}) + H_\sigma(vw^{-1}) + H_\sigma(wu^{-1}) \\ &= (H_\sigma(uv^{-1}) - H_\sigma(u) - H_\sigma(v^{-1})) \\ &\quad + (H_\sigma(vw^{-1}) - H_\sigma(v) - H_\sigma(w^{-1})) \\ &\quad + (H_\sigma(wu^{-1}) - H_\sigma(w) - H_\sigma(u^{-1})), \end{aligned}$$

where each parenthesis has absolute value is at most  $|\sigma| - 1$  by Lemma 2.45. Hence  $D(H_\sigma) \leq 3(|\sigma| - 1)$ . The same method shows that  $D(h_\sigma) \leq 3$ .  $\square$

Intuitively, we can think of  $g, h, (gh)^{-1}$  as a tripod with legs  $u, v, w$ . Whenever there is a copy of  $\sigma$  that appears in one of the three legs, there is a copy of  $\sigma^{-1}$  that appears on the opposite side to cancel out the contribution to the defect. So the contribution only comes from copies of  $\sigma$  and  $\sigma^{-1}$  that lie in the juncture, but there are limited spaces at the junctures, giving the defect bound.

It is straightforward to check that

**Lemma 2.47.** *The homogenization of  $H_\sigma$  applied to  $g$  is the number of copies of  $\sigma$  (minus the number of copies of  $\sigma^{-1}$ ) as cyclic subwords in the cyclically reduced word representing the conjugacy class of  $g$ .*

**Exercise 2.48.** *Prove the following.*

- (1) *If  $g$  is a word shorter than  $\sigma$ , then  $H_\sigma(g) = 0$ .*
- (2) *If  $g$  is cyclically reduced word and  $\sigma = g^2$ , then  $\bar{H}_\sigma(g) \geq 1 > 0$ , although  $g$  is shorter than  $\sigma$ .*
- (3) *If  $\bar{H}_\sigma(g) > 0$  for some  $g$  shorter than  $\sigma$ , show that  $\sigma = uvu$  as a reduced expression for some nontrivial subword  $u$ .*
- (4) *In every conjugacy class, there is some (cyclically reduced)  $\sigma$  such that  $\bar{H}_\sigma(g) = 0$  for all  $g$  shorter than  $\sigma$ .*

*Hint: Use a lexicographical order on reduced words and take a minimal one in the cyclical reduced words in the given conjugacy class. This is from [Tao16, Lemma 3.1].*

- (5) *Given the above result, find an infinite sequence of distinct  $\sigma_n$  so that  $\{\bar{H}_{\sigma_n}\}_n$  are linearly independent homogeneous quasimorphisms.*

**2.6. The rotation quasimorphism and circle dynamics.** Now we construct a quasimorphism in relation to circle dynamics. Throughout this section, let  $T = \text{Homeo}^+(S^1)$ , the group of orientation-preserving homeomorphisms on the circle. Every such homeomorphism  $f$  lifts to an orientation-preserving homeomorphism  $\tilde{f}$  on  $\mathbb{R}$ , where we think of  $\mathbb{R}$  as the universal cover of  $S^1$  via the map  $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ . Then  $\tilde{f}$  commutes with the deck transformation, i.e.  $\tilde{f}(x+1) = \tilde{f}(x) + 1$ , and conversely any orientation-preserving homeomorphism on  $\mathbb{R}$  descends to one on  $S^1$ . This leads to a new group

$$\widehat{T} = \{g \in \text{Homeo}^+(\mathbb{R}) \mid g(x+1) = g(x) + 1\},$$

which has a natural map  $\pi : \widehat{T} \rightarrow T$ . The kernel  $\ker \pi = \mathbb{Z}$  is the set of translations by integers as the lifts of a given map are unique up to a deck transformation. Thus we have a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \widehat{T} \rightarrow T \rightarrow 1.$$

We aim to build a quasimorphism  $\widetilde{\text{rot}}$  on  $\widehat{T}$ , which descends to a map  $\text{rot} : T \rightarrow \mathbb{R}/\mathbb{Z}$  so that  $\text{rot}(f)$  captures the dynamical properties of  $f$  for any  $f \in T = \text{Homeo}^+(S^1)$ .

For any  $p \in \mathbb{R}$ , let  $\tau_p(f) = f(p) - p$  for any  $f \in \widehat{T}$ .

**Lemma 2.49.**  *$\tau_p$  is a quasimorphism and  $D(\tau_p) \leq 1$ .*

*Proof.* For any  $f, g \in \widehat{T}$ , we need to estimate  $\tau_p(fg) - \tau_p(f) - \tau_p(g) = f(g(p)) - g(p) - (f(p) - p)$ . Note that changing  $g$  to  $g+n$  does not affect this quantity for any  $n \in \mathbb{Z}$  and the same for  $f$ . Hence we may assume  $p \leq g(p) < p+1$ , and  $p \leq f(p) < f(p)+1$ . As  $f$  is orientation-preserving, the bound for  $g(p)$  implies

$$f(p) \leq f(g(p)) < f(p+1) = f(p) + 1.$$

Combining with the bound for  $g(p)$  we obtain

$$(f(p) - p) - 1 = f(p) - (p+1) \leq f(g(p)) - g(p) \leq (f(p) + 1) - p = (f(p) - p) + 1.$$

Thus

$$|\tau_p(fg) - \tau_p(f) - \tau_p(g)| = |f(g(p)) - g(p) - (f(p) - p)| \leq 1.$$

Hence  $D(\tau_p) \leq 1$ . □

**Lemma 2.50.** *For any  $p, q \in \mathbb{R}$ , the difference  $\tau_p - \tau_q$  is a bounded function on  $\widehat{T}$ .*

*Proof.* Note that  $\tau_{p+n}(f) = f(p+n) - (p+n) = f(p) - p = \tau_p$ , so we may assume  $q \leq p < q+1$ . Then  $f(q) \leq f(p) < f(q)+1$ . Thus

$$-1 \leq f(p) - f(q) - (p - q) \leq 1,$$

i.e.  $|\tau_p(f) - \tau_q(f)| \leq 1$  for all  $f$ .  $\square$

It follows that the homogenization  $\overline{\tau}_p$  does not depend on  $p$ .

**Definition 2.51** (Rotation quasimorphism). The *rotation quasimorphism*  $\widetilde{\text{rot}} \in Q(\widehat{T})$  is the homogenization of  $\tau_p$  for any  $p \in \mathbb{R}$ . As an explicit formula, taking  $p = 0$ , we have  $\widetilde{\text{rot}}(f) = \lim_{n \rightarrow +\infty} \frac{f^n(0)}{n}$ .

$\widetilde{\text{rot}}(f)$  is sometimes called the *translation number* of  $f$ . It measures how fast  $f$  moves forward on average.

For any  $f \in T$ , its *rotation number*  $\text{rot}(f) \in \mathbb{R}/\mathbb{Z}$  is  $\widetilde{\text{rot}}(\tilde{f}) \bmod \mathbb{Z}$  for any lift  $\tilde{f} \in \widehat{T}$  of  $f$ .

**Example 2.52.** For any translation  $g(x) = x + \alpha$ ,  $\alpha \in \mathbb{R}$ , we have  $\widetilde{\text{rot}}(g) = \alpha$ .

We will first show that rationality of the rotation number characterizes the existence of periodic orbits. We say  $x \in S^1$  is  $n$ -periodic under  $f \in T$  if  $f^n(x) = x$ . If  $n = 1$ , then  $x$  is a fixed point. We say  $f$  has a periodic orbit if there is some  $x$  is  $n$ -periodic for some  $n$ .

**Lemma 2.53.**  $g \in \widehat{T}$  has a fixed point if and only if  $\widetilde{\text{rot}}(g) = 0$ .

*Proof.* Suppose  $g(p) = p$ , then  $g^n(p) = p$  for all  $n$ , i.e.  $\tau_p(g^n) = 0$ . Hence  $\widetilde{\text{rot}}(g) = 0$ .

Suppose  $g$  has not fixed point, then either  $g(x) < x$  for all  $x \in \mathbb{R}$  or  $g(x) > x$  for all  $x \in \mathbb{R}$ . Without loss of generality, suppose  $g(x) > x$  for all  $x \in \mathbb{R}$ . Note  $t := \inf_{x \in \mathbb{R}} (g(x) - x) = \inf_{x \in [0,1]} (g(x) - x)$  as  $g(x+1) = g(x) + 1$ , it must be achieved by some  $x_0 \in [0,1]$  by compactness. So we have  $g(x) - x \geq g(x_0) - x_0 = t > 0$ . Thus  $g^n(0) > g^{n-1}(0) + t > \dots > nt$ , and  $\widetilde{\text{rot}}(g) = \lim_{n \rightarrow \infty} \frac{g^n(0)}{n} \geq t > 0$ . Thus  $g$  must have fixed point if  $\widetilde{\text{rot}}(g) = 0$ .  $\square$

**Lemma 2.54.**  $g \in \widehat{T}$  has  $\widetilde{\text{rot}}(g) = \frac{m}{n}$  as a reduced fraction with  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{Z} \setminus \{0\}$  if and only if there is  $p \in \mathbb{R}$  such that  $g^n(p) = p + m$ .

*Proof.* If  $g^n(p) = p + m$ , then it is easy to see that  $\widetilde{\text{rot}}(g) = \frac{m}{n}$  as before.

If  $\widetilde{\text{rot}}(g) = \frac{m}{n}$ , then  $\widetilde{\text{rot}}(g^n) = m$  and  $\widetilde{\text{rot}}(g^n - m) = 0$ . Thus by Lemma 2.53 we know  $(g^n - m)(p) = p$  for some  $p \in \mathbb{R}$ , i.e.  $g^n(p) = p + m$ .  $\square$

**Theorem 2.55.** For  $f \in T$ ,  $f$  has a periodic orbit if and only if  $\text{rot}(f) \in \mathbb{Q} \bmod \mathbb{Z}$ . Moreover,  $f$  has an  $n$ -periodic point if and only if  $\text{rot}(f) \in \frac{1}{n}\mathbb{Z} \bmod \mathbb{Z}$ . In particular,  $f$  has fixed points if and only if  $\text{rot}(f) \equiv 0 \bmod \mathbb{Z}$ .

*Proof.* Let  $\tilde{f} \in \widehat{T}$  be an arbitrary lift of  $f$ . Then  $x \in S^1$  is  $n$ -periodic under  $f$  if and only if it has a lift  $p \in \mathbb{R}$  such that  $\tilde{f}^n(p) = p + m$  for some  $m \in \mathbb{Z}$ . So the conclusion follows easily from the previous lemmas.  $\square$

Here is the general structure of an element  $f \in T$  acting with a fixed point. Let  $\text{Fix}(f)$  be the set of fixed points, which is a closed subset of  $S^1$ .

**Lemma 2.56.** The action of  $f$  on each complementary interval  $I$  of  $\text{Fix}(f)$  is conjugate to a nontrivial translation  $T$  on  $\mathbb{R}$ . That is, there is a homeomorphism  $h : I \rightarrow \mathbb{R}$  such that  $hf = Th$ .

*Proof.* Choose any  $x_0 \in I$  and let  $x_n = f^n(x_0)$  for all  $n \in \mathbb{Z}$ . Since  $I \cap \text{Fix}(f) = \emptyset$ ,  $f|_I$  is a monotone. Without loss of generality, assume  $x_{n+1} > x_n$  for all  $n$  (where the order is induced from the orientation on  $I \subset S^1$ ). Let  $I_n = [x_n, x_{n+1}]$ . Then  $\cup_n I_n = I$  since  $\{x_n\}$  has no accumulation point inside  $I$  (which would be a fixed point of  $f$  if otherwise). Note that  $f^n(I_0) = I_n$  for all  $n \in \mathbb{Z}$ .

Choose an arbitrary homeomorphism  $h_0 : I_0 \rightarrow [0,1]$  with  $h_0(x_0) = 0$  and  $h_0(x_1) = 1$ . Define  $h : I \rightarrow \mathbb{R}$  by setting  $h|_{I_n} = T^n h_0 f^{-n}$  for all  $n \in \mathbb{Z}$ , where  $T(x) = x + 1$  is the unit translation on  $\mathbb{R}$ . It follows that  $h(I_n) = [n, n+1]$  and  $h(x_n) = n$  for all  $n \in \mathbb{Z}$ . It is easy to check by construction that  $h$  is a homeomorphism<sup>2</sup> and  $hf = Th$ .  $\square$

<sup>2</sup>If one asks for  $h$  that is orientation preserving where  $\mathbb{R}$  is equipped with the usual orientation and  $I$  has the induced orientation, then we might need to choose  $T$  as  $T(x) = x - 1$

As a byproduct, we have the following observations.

**Lemma 2.57.** *If  $f \in T$  has fixed points, then  $f = [a, b]$  for some  $a, b \in T$ .*

*Proof.* Note that the unit translation  $T(x) = x + 1$  is a commutator  $[u, v]$  of two dilations  $u(x) = 2x$  and  $v(x) = 2(x + 1) - 1$ . Thus by the previous lemma, for each complementary interval  $I$  of  $\text{Fix}(f)$ , there are homeomorphisms  $u_I, v_I$  on  $\bar{I}$  fixing both end points such that  $f|_{\bar{I}} = [u_I, v_I]$ . Define  $u|_{\bar{I}} = u_I$  for each complementary interval  $I$  and  $u|_{\text{Fix}(f)} = \text{id}|_{\text{Fix}(f)}$ , and define  $v$  similarly. Then by construction,  $u, v \in T$  and  $[u, v]|_I = f|_I$ . It follows that  $f = [u, v]$  as desired.  $\square$

**Proposition 2.58.**  *$T = \text{Homeo}^+(S^1)$  is uniformly perfect. More precisely, each  $f \in T$  is a product of at most two commutators.*

*Proof.* By the previous lemma, It suffices to show that for any  $f \in T$  there is a commutator  $[a, b]$  with  $a, b \in T$  such that  $[a, b]f$  has a fixed point. Indeed, for any  $x, y \in S^1$ , there is a commutator that takes  $x$  to  $y$ , which can be done by fixing an arbitrary nontrivial commutator and conjugating it appropriately.  $\square$

By Corollary 2.34, we have

**Corollary 2.59.**  $Q(\text{Homeo}^+(S^1)) = 0$ .

Let us now go back to the relation between rotation numbers and circle dynamics. Ideally, one would like to understand the dynamics up to conjugacy. However, in circle dynamics, it is often natural to consider a weaker equivalence, generated by *semi-conjugacy*.

**Definition 2.60.** A map  $h : S^1 \rightarrow S^1$  is a *semi-conjugacy* between two actions  $\rho_1, \rho_2 : G \rightarrow T$  of a group  $G$  if  $h$  is a surjective continuous map of degree one and  $h\rho_1(g) = \rho_2(g)h$  for all  $g$ . When  $h$  is a homeomorphism, then it is called a conjugacy.

Here we are considering a single map  $f \in T$ , which we can treat as a  $\mathbb{Z}$  action on  $S^1$  via  $n \mapsto f^n$ .

*Denjoy's construction* is a standard way to cook up a new action on the circle from an existing one that is semi-conjugate but often not conjugate.

**Example 2.61** (Denjoy's construction). *Suppose  $\rho : G \rightarrow T$  is an action that admits a countable  $G$ -invariant subset  $O \subset S^1$ . Note that  $O$  always exists if  $G$  is a countable group, say by taking a  $G$ -orbit. Enumerate  $O$  as  $O = \{x_n\}_{n \geq 1}$  and choose a sequence of positive integers  $\{a_n\}_{n \geq 1}$  such that  $\sum_{n \geq 1} a_n < \infty$ .*

*Replace each  $x_n$  by a closed interval  $I_n$  of length  $a_n$ , we obtain a new circle  $Y \cong S^1$  and a natural surjective continuous map  $h : Y \rightarrow S^1$  of degree one, where  $h$  collapses each  $I_n$  to  $x_n$ . We have a new action  $\rho'$  of  $G$  on  $Y$ : for each  $g \in G$ , define  $\rho'(g)(x) = x$  if  $x \notin \cup I_n$  and  $\rho'(g) : I_n \rightarrow I_m$  is the unique orientation preserving linear homeomorphism if  $\rho(g)(x_n) = x_m$ . Then  $h\rho'(g) = \rho(g)h$  for all  $g \in G$ .*

*An action is minimal if the only closed invariant subsets are either empty or the entire space, or equivalently, every orbit is dense. Note that the action on  $Y$  is never minimal since  $Y \setminus \cup_n \text{int}(I_n)$  is a nontrivial closed invariant subset, where  $\text{int}(I_n)$  is the interior of  $I_n$ . Hence if the starting action  $\rho$  is minimal, then the two actions are semi-conjugate but not conjugate.*

Now we consider the case where the rotation number of  $f \in T$  is irrational and show that the action is semi-conjugate to the irrational rigid rotation by  $\text{rot}(f)$ . This is an old theorem due to Poincaré.

**Theorem 2.62** (Poincaré). *If  $f \in T$  has  $\text{rot}(f) = \alpha \notin \mathbb{Q}/\mathbb{Z}$ , then there is a surjective continuous degree one map  $h : S^1 \rightarrow S^1$  such that  $hf = T_\alpha h$ , where  $T_\alpha$  is the rigid rotation on  $S^1 = \mathbb{R}/\mathbb{Z}$  by  $\alpha$ . Moreover, if  $f$  acts minimally, then  $h$  is a homeomorphism.*

To prove this, we need the following lemmas.



**Lemma 2.63.** *For any  $g \in \widehat{T}$ , the set  $\{g^n(x) - x - n\widetilde{\text{rot}}(g)\}_{n \in \mathbb{Z}}$  is bounded for any  $x \in \mathbb{R}$ .*

*Proof.* Recall that  $\tau_x(g) = g(x) - x$  and  $\widetilde{\text{rot}}$  is the homogenization of  $\tau_x$ . Then  $\tau_x - \widetilde{\text{rot}}$  is a bounded function. In particular, its evaluation on  $\{g^n\}_{n \in \mathbb{Z}}$  must be bounded, that is,  $\{g^n(x) - x - n\widetilde{\text{rot}}(g)\}_{n \in \mathbb{Z}}$  is a bounded set.  $\square$

**Remark 2.64.** *One can use this property as the definition of  $\widetilde{\text{rot}}(g)$ .*

**Lemma 2.65.** *For any  $g \in \widehat{T}$  with  $\widetilde{\text{rot}}(g) = \alpha$ , there is a monotone map  $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tilde{h}(x+1) = \tilde{h}(x) + 1$  and  $\tilde{h}g = T_\alpha \tilde{h}$ , where  $T_\alpha$  is the translation by  $\alpha$ .*

*Proof.* Define  $\tilde{h}(x) := \sup_{n \in \mathbb{Z}} \{g^n(x) - n\widetilde{\text{rot}}(g)\}$  for each  $x \in \mathbb{R}$ . Note that this is a finite number as we are taking the supremum of the bounded set in Lemma 2.63 translated by  $x$ .

Clearly  $g^n(x+1) - n\widetilde{\text{rot}}(g) = 1 + [g^n(x) - n\widetilde{\text{rot}}(g)]$ , so  $\tilde{h}(x+1) = \tilde{h}(x) + 1$ . Moreover,  $\tilde{h}(x) \geq \tilde{h}(y)$  for all  $x \geq y$  since  $g^n(\widetilde{x}) \geq g^n(y)$ .

Since  $g^n(g(x)) - n\widetilde{\text{rot}}g = \widetilde{\text{rot}}(g) + [g^{n+1}(x) - (n+1)\widetilde{\text{rot}}(g)]$ , we have  $\tilde{h}(g(x)) = \tilde{h}(x) + \widetilde{\text{rot}}(g) = \tilde{h}(x) + \alpha$ . Thus  $\tilde{h}g = T_\alpha \tilde{h}$ .  $\square$

*Proof of Theorem 2.62.* Fix a lift  $g = \tilde{f} \in \widehat{T}$  of  $f$ , and let  $\tilde{h}$  be the map from Lemma 2.65. It descends to a (degree one) map  $h : S^1 \rightarrow S^1$  since  $\tilde{h}$  commutes with integral translations. Since  $\tilde{h}$  is monotone, it is continuous if and only if it is surjective (i.e. it has no jumps). Let  $\text{Jump}(\tilde{h})$  be the complement of the image of  $\tilde{h}$ , which is a countable union of nontrivial intervals, invariant under integral translations and translation by  $\alpha$ . Thus they descend to a countable union of intervals on  $S^1$  invariant under  $T_\alpha$ .

Since  $\alpha$  is irrational, every orbit of  $T_\alpha$  is dense, so  $\text{Jump}(\tilde{h})$  is empty. Hence  $\tilde{h}$  is continuous and surjective, and so is  $h$ . Then it is easy to check that  $h$  gives the desired semi-conjugacy.

The preimage  $\tilde{h}^{-1}(x)$  of any  $x \in \mathbb{R}$  is an interval. The union  $U$  of the interiors of such intervals is invariant under integral translations and under  $\tilde{f}$ . So it descends to a union of open intervals on  $S^1$  invariant under  $f$ . If every orbit of  $f$  is dense, then  $U$  must be empty. So in this case,  $\tilde{h}$  and  $h$  are injective. Hence  $h$  is a conjugacy when  $f$  acts minimally.  $\square$

**Remark 2.66.** *When  $f$  has rational rotation, it is possible that  $\text{Jump}(\tilde{h})$  is a nontrivial collection of intervals. One can collapse them to get a semi-conjugacy, but the resulting space is not a circle if  $h$  has finite image. However, one can first “blow up” the starting circle by Denjoy’s construction so that  $\text{Fix}(f)$  has interiors, then the semi-conjugacy works for the blow-up circle. Thus  $f$  is equivalent to the rigid rotation by  $\text{rot}(f)$  in all cases under the semi-conjugacy relations.*

**Exercise 2.67.** *Let  $f \in T$  be the boundary map of a parabolic element in  $\text{PSL}_2(\mathbb{R})$ . How does  $\tilde{h}$  behave in the construction above? What if  $f$  is the boundary map of a hyperbolic element?*

For the more general question of understanding  $G$  actions on  $S^1$  for groups  $G$  other than  $\mathbb{Z}$ , we will discuss it later using the bounded Euler class living in  $H_b^2(G; \mathbb{Z})$ , which is closely related to the rotation number.

**2.7. Amenable groups.** We discuss basic properties of amenable groups and their relations to bounded cohomology. There are many equivalent definitions of amenability. Here we start with two most common ones and focus on the case of discrete groups (instead of the more general setting of locally compact groups). Let  $L^\infty(G) = C_b^1(G)$  be the space of bounded functions on  $G$  equipped with the sup norm, which admits a  $G$ -action, where  $(gf)(h) = f(g^{-1}h)$  for all  $h \in G$  and  $f \in L^\infty(G)$ .

**Definition 2.68** (Amenable). A discrete group  $G$  is *amenable* if there is an invariant mean  $m$ , which is a linear functional  $m : L^\infty(G) \rightarrow \mathbb{R}$  such that

- (1)  $m(f) \geq 0$  if  $f \geq 0$ ,
- (2)  $m(1_G) = 1$ , where  $1_G$  is the function taking constant value 1, and

(3)  $m(gf) = m(f)$  for all  $g \in G$  and  $f \in L^\infty(G)$ .

Equivalently, one can think of this as a (left)  $G$ -invariant *finitely additive* (non-negative) measure  $\mu$  on  $G$  with total mass 1, where  $\mu(A) = m(1_A)$  for all  $A \subset G$ . The other direction of the equivalence is given by  $m(f) = \int_G f d\mu$ .

**Example 2.69.**

- (1) Any finite group  $G$  is amenable, where  $m(f) = \frac{1}{|G|} \sum_{g \in G} f(g)$ .
- (2) The free group  $G = F_2$  is not amenable. Suppose there is a  $G$ -invariant finitely additive measure  $\mu$ . Then  $\mu(\{id\}) = 0$  since otherwise  $\mu(G)$  won't be finite. We express  $G \setminus \{id\} = X_a \sqcup X_{a^{-1}} \sqcup X_b \sqcup X_{b^{-1}}$ , where  $X_*$  is the set of elements starting with  $*$ . Note that  $a(X_b \sqcup X_{b^{-1}}) \subset X_a$ , so  $\mu(X_b \sqcup X_{b^{-1}}) \leq \mu(X_a)$ . Similarly  $\mu(X_a \sqcup X_{a^{-1}}) \leq \mu(X_b)$ . Thus  $\mu(X_a) \leq \mu(X_b) \leq \mu(X_a) - \mu(X_{b^{-1}})$ . So we must have  $\mu(X_{b^{-1}}) = 0$ . For the same reason, we must have  $\mu(X_a) = \mu(X_{a^{-1}}) = \mu(X_b) = 0$ , which implies  $\mu(G) = 0$ , contradicting that  $\mu(G) = 1$ .

Amenability interacts nicely with subgroups and quotients.

**Lemma 2.70.**

- (1) If  $G$  is amenable, then every subgroup  $H \leq G$  is amenable.
- (2) If  $G$  is amenable, then every quotient group  $Q = G/N$  is amenable.
- (3) If for a normal subgroup  $N \triangleleft G$ , both  $N$  and  $Q = G/N$  are amenable, then  $G$  is amenable.

*Proof.*

- (1) Let  $\mu$  be a left  $G$ -invariant finitely additive measure with  $\mu(G) = 1$ . Let  $\{Hg_\lambda\}_{\lambda \in \Lambda}$  be the right cosets. For any  $A \subset H$ , define  $\nu(A) := \mu(\cup_{\lambda \in \Lambda} Ag_\lambda)$ . Then clearly  $\nu$  is a finitely additive measure on  $H$  with  $\nu(H) = 1$ , and left  $G$ -invariance of  $\mu$  implies left  $H$ -invariance of  $\nu$ .
- (2) Let  $\mu$  be a  $G$ -invariant finitely additive measure with  $\mu(G) = 1$ . For any  $A \in Q$ , define  $\nu(A) = \mu(\pi^{-1}(A))$ , where  $\pi : G \rightarrow Q$  is the quotient map. Then clearly  $\nu$  is a finitely additive measure on  $Q$  with  $\nu(Q) = 1$ . To see the  $Q$ -invariance, for any  $q \in Q$ , let  $g \in \pi^{-1}(q)$ , and then  $\pi^{-1}(qA) = g\pi^{-1}(A)$ , so  $\nu(qA) = \mu(g\pi^{-1}(A)) = \mu(\pi^{-1}(A)) = \nu(A)$  by  $G$ -invariance of  $\mu$ .
- (3) Let  $\mu_N$  and  $\mu_Q$  be the invariant measures witnessing the amenability of  $N$  and  $Q$  respectively. For any  $A \subset G$ , define a function  $f_A : Q \rightarrow \mathbb{R}$  by  $f_A(gN) = \mu_N(N \cap g^{-1}A)$ . This is a bounded function on  $Q$  so we can define  $\mu(A) := \int_Q f_A d\mu_Q$ . It is easy to check that  $\mu$  is a finitely additive measure with  $\mu(G) = 1$ . To see its invariance, note that  $f_{hA}(hgN) = \mu_N(N \cap g^{-1}A) = f_A(gN)$ , so  $\bar{h}^{-1}f_{hA} = f_A$ , where  $\bar{h}$  is the image of  $h \in G$  in  $Q$ . So  $f_{hA}$  and  $f_A$  have the same integral and hence  $\mu(hA) = \mu(A)$  for all  $h \in G$ .

□

**Proposition 2.71.**  $\mathbb{Z}$  is amenable.

*Proof.* Let  $\mathcal{P}(\mathbb{Z})$  be the power set of  $\mathbb{Z}$ . Finitely additive probability measures on  $\mathbb{Z}$  is a closed subset  $P$  of  $[0, 1]^{\mathcal{P}(\mathbb{Z})}$ , equipped with the product topology. For any  $\epsilon > 0$ , consider the subset  $P_\epsilon \subset P$  of  $\epsilon$ -almost invariant measures, i.e. those  $\mu \in P$  satisfying  $|\mu(zA) - \mu(A)| \leq \epsilon$  for all  $A \subset \mathbb{Z}$  and a chosen generator  $z \in \mathbb{Z}$ . This is a closed subset and thus compact since  $[0, 1]^{\mathcal{P}(\mathbb{Z})}$  is compact by Tychonoff's theorem. The intersection  $\cap_{\epsilon > 0} P_\epsilon$  consists of  $\mathbb{Z}$ -invariant measures, which is nonempty as long as we show  $P_\epsilon \neq \emptyset$  for each  $\epsilon > 0$ .

For each  $N \in \mathbb{Z}_+$ , let  $\mu_N(A) = \frac{|A \cap [-N, N]|}{2N+1}$ . Then  $\mu_N \in P$  and  $|\mu_N(gA) - \mu_N(A)| \leq \frac{2}{2N+1}$ . So  $\mu_N \in P_\epsilon$  if  $N$  is large enough, verifying that  $P_\epsilon$  is nonempty. This completes the proof. □

**Remark 2.72.** The subset  $[-N, N]$  of  $\mathbb{Z}$  has the property that its boundary is small compared to the size of the entire subset, and the ratio goes to zero as  $N$  goes to infinity. This is an example of

a Følner sequence. The argument above generalizes to show that any group with a Følner sequence is amenable, which is actually an equivalent definition.

**Exercise 2.73.** Show that the direct limit of amenable groups is amenable.

**Proposition 2.74.** Solvable groups are amenable.

*Proof.* Note that we have shown that all cyclic groups are amenable. By Lemma 2.70 (3) and induction, all finitely generated abelian groups are amenable. Any abelian group is the direct limit of finitely generated ones, so they are amenable by the exercise above.

For any solvable group, we can see by induction and Lemma 2.70 (3) that its derived subgroups are amenable, based on the fact abelian groups are amenable.  $\square$

**2.8. Vanishing results.** Amenable groups have trivial bounded cohomology and in particular no interesting homogeneous quasimorphisms. We give a direct proof of the latter result.

**Proposition 2.75.** If  $G$  is amenable, then  $Q(G) = H^1(G)$ .

*Proof.* Given any  $\varphi \in Q(G)$ , we aim to show that  $\varphi$  is a homomorphism. Note that  $\varphi(gh) - \varphi(g) - \varphi(h)$  is a bounded function in variables  $g, h \in G$ . So for any fixed  $g$ , the one-variable function  $f_g : G \rightarrow \mathbb{R}$  defined by  $f_g(h) = \varphi(gh) - \varphi(h)$  is bounded. As  $G$  is amenable, let  $f(g) = m(f_g) \in \mathbb{R}$ . This provides a function  $f : G \rightarrow \mathbb{R}$ , which we claim to be a homomorphism and is bounded away from  $\varphi$ . Then  $\varphi = f$  by Lemma 2.24.

Indeed, the function  $f_g - \varphi(g)$  is bounded by  $-D(\varphi)$  and  $D(\varphi)$  from below and above, so its mean  $m(f_g) - \varphi(g) = f(g) - \varphi(g)$  has absolute value bounded by  $D(\varphi)$ , which holds for all  $g \in G$ . So  $|f - \varphi| \leq D(\varphi)$ .

To show that  $f$  is a homomorphism, note that  $f(g_1g_2)$  is the mean of  $f_{g_1g_2}$  and

$$\begin{aligned} f_{g_1g_2}(h) &= \varphi(g_1g_2h) - \varphi(h) \\ &= [\varphi(g_1g_2h) - \varphi(g_2h)] + [\varphi(g_2h) - \varphi(h)] \\ &= f_{g_1}(g_2h) + f_{g_2}(h) \\ &= (g_2^{-1}f_{g_1})(h) + f_{g_2}(h). \end{aligned}$$

So

$$m(f_{g_1g_2}) = m(g_2^{-1}f_{g_1}) + m(f_{g_2}) = m(f_{g_1}) + m(f_{g_2})$$

by the invariance of the mean  $m$ . That is,  $f(g_1g_2) = f(g_1) + f(g_2)$ . So  $f$  is a homomorphism as desired.  $\square$

Instead of directly proving vanishing results for bounded cohomology of amenable groups, we deduce it from the following exact sequence for amenable covers.

**Theorem 2.76** (Johnson, Trauber, Gromov). Suppose there is a short exact sequence of groups

$$1 \rightarrow K \rightarrow G \rightarrow A \rightarrow 1,$$

where  $A$  is amenable. Then there the induced map  $H_b^n(G) \rightarrow H_b^n(K)^A$  is an isometric isomorphism with respect to the standard semi norm, where  $H_b^n(K)^A$  denotes the  $A$  invariant part. In particular (by taking  $K = 1$  and  $G = A$ ),  $H_b^n(A) = 0$  for all  $n$ .

*Proof.* Here we give a topological proof, interpreting  $H_b^n(G)$  as the bounded singular cohomology of a  $K(G, 1)$  space  $X_G$ ; More details about this equivalence will be explained in the next section.

Let  $\pi : X_K \rightarrow X_G$  be the covering space of  $X_G$  corresponding to the normal subgroup  $K$ . Then  $A$  acts on  $X_K$  as deck transformations. Then  $C_b^n(X_G)$  naturally corresponds to  $C_b^n(X_K)^A$  by pullback, the  $A$ -invariant cochains on  $X_K$ . This correspondence commutes with the coboundary and induces a map  $\pi^* : H_b^n(X_G) \rightarrow H_b^n(X_K)^A$ , which is non-increasing with respect to the sup norm, and the image clearly lies in  $H_b^n(X_K)^A$ .

Let  $m$  be an invariant mean on  $A$ . This gives a map going backwards  $m : C_b^n(X_K) \rightarrow C_b^n(X_K)^A$  as follows. For any  $\alpha \in C_b^n(X_K)$  and singular simplex  $\Delta \rightarrow X_K$ , we have a bounded function  $f$  on  $A$  defined as  $f(a) = \alpha(a\Delta)$ . Define  $m(\alpha)(\Delta) = m(f)$ . The  $A$ -invariance implies that  $m(\alpha)$  is  $A$ -invariant. It is straightforward to check that  $m$  is linear, commutes with the coboundary, and  $m \circ \pi = id$ . Thus it induces a map  $m^* : H_b^n(X_K) \rightarrow H_b^n(X_G)$  so that  $m^* \circ \pi^* = id$ . This shows that  $\pi^*$  is injective. Moreover, both  $m^*$  and  $\pi^*$  are non-increasing with respect to the sup norm, so  $m^* \circ \pi^* = id$  also implies that  $\pi^*$  is an isometric embedding.

To see that the image of  $\pi^*$  is  $H_b^n(X_K)^A$ , it suffices to show that any class  $\sigma \in H_b^n(X_K)^A$  is represented by a cocycle in  $C_b^n(X_K)^A$ . Let  $\alpha \in C_b^n(X_K)$  be a cocycle representing  $\sigma$ . Since  $\sigma$  is  $A$ -invariant, we know that, for any  $a \in A$ , we have  $a\alpha - \alpha = \delta f_a$  for some  $f_a \in C_b^{n-1}(X_K)$ . For every singular  $(n-1)$ -simplex  $\Delta \rightarrow X_K$ , define  $f(\Delta)$  as the mean of  $f_a(\Delta)$  as a bounded function over  $a \in A$ . It is bounded because  $\alpha$  is. Then it follows by averaging that  $m(\alpha) - \alpha = \delta f$ , so  $[m(\alpha)] = [\alpha] = \sigma$  and  $m(\alpha) \in C_b^n(X_K)^A$  as desired.  $\square$

There is also a result for surjection with amenable kernel.

**Theorem 2.77** (Gromov). *Suppose there is a short exact sequence of groups*

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1,$$

*where  $A$  is amenable. Then the induced map  $H_b^n(H) \rightarrow H_b^n(G)$  is an isometric isomorphism.*

In degree two, this generalizes to the (left) exactness of  $H_b^2$ .

**Theorem 2.78** (Bouarich). *Suppose there is an exact sequence of groups*

$$K \xrightarrow{i} G \xrightarrow{\pi} H \rightarrow 1.$$

*Then the following induced sequence is exact:*

$$0 \rightarrow H_b^2(H) \rightarrow H_b^2(G) \rightarrow H_b^2(K).$$

*Similarly,*

$$0 \rightarrow Q(H) \rightarrow Q(G) \rightarrow Q(K).$$

*Proof.* We prove this for homogeneous quasimorphisms. This is to show the exactness of  $0 \rightarrow Q(H) \rightarrow Q(G) \rightarrow Q(K)$ . The case of  $H_b^2$  is similar and can be found in [Cal09, Section 2.7.2].

The injectivity  $\pi^* : Q(H) \rightarrow Q(G)$  follows from the fact that this map is defect non-decreasing, which is apparent by the surjectivity of  $G \rightarrow H$ .

The remaining nontrivial part is to show that  $\ker i^* \subset \text{Im } \pi^*$ . Suppose  $\varphi \in Q(G)$  vanishes on the image of  $K$ . We may assume  $K = \ker \pi$ . Then we aim to show that  $\varphi(gk) = \varphi(g)$  for all  $g \in G$  and  $k \in K$ . Note that for any  $n \in \mathbb{Z}$ , we have  $\pi((gk)^n) = \pi(g^n)$ , so  $(gk)^n \cdot g^{-n} \in K$  and  $\varphi((gk)^n \cdot g^{-n}) = 0$ . Hence

$$|\varphi((gk)^n) + \varphi(g^{-n})| \leq D(\varphi),$$

that is  $|\varphi(gk) - \varphi(g)| \leq D(\varphi)/n$ . Taking  $n \rightarrow \infty$  we conclude  $\varphi(gk) = \varphi(g)$  for all  $g \in G$  and  $k \in K$  as desired.  $\square$

**2.9. Bounded cohomology of topological spaces.** Given a topological space  $X$ , its singular cohomology is defined by cochains  $C^n(X; \mathbb{R})$ , which are functions on the space of singular simplices. The *bounded singular cohomology*  $H_b^n(X; \mathbb{R})$  is defined in the same way replacing each  $C^n(X; \mathbb{R})$  by the subspace  $C_b^n(X; \mathbb{R})$  of *bounded* functions, which is equipped with a sup-norm. The norm induces a semi norm  $\|\cdot\|_\infty$  on  $H_b^n(X; \mathbb{R})$  as usual, by minimizing the sup norm of bounded cochains representing the given bounded cohomology class.

Then it is straightforward to see that

**Lemma 2.79.** *Any continuous map  $f : X \rightarrow Y$  induces a map  $f^* : H_b^n(Y) \rightarrow H_b^n(X)$  such that  $\|f^*(\alpha)\|_\infty \leq \|\alpha\|_\infty$  for all  $\alpha \in H_b^n(Y)$ .*

*In particular, if  $f$  is a homotopy equivalence, then  $f^*$  is an isometric isomorphism.*

*Proof.* This is analogous to Proposition 1.3 and Corollary 1.4.  $\square$

Its relation to bounded group cohomology is given by Gromov's mapping theorem.

**Theorem 2.80** (Gromov's mapping theorem). *Let  $X$  be a CW complex with  $\pi_1(X) = G$ . Then the natural map  $K(G, 1) \rightarrow X$  induces an isometric isomorphism  $H_b^n(X) \cong H_b^n(G)$ .*

This is astonishing as the bounded cohomology only depends on  $\pi_1(X)$ . When  $X$  itself is a  $K(G, 1)$ , which up to homotopy we may assume to be the simplicial complex  $BG$  that we built explicitly, the theorem asserts that the bounded cohomology via singular simplices on  $BG$  agrees with the bounded cohomology via simplicial simplices on  $BG$ .

One way to prove this is to establish a theory of strongly injective resolutions along the lines of homological algebra, introduced by Ivanov. We will focus more on applications and skip this. See [Fri17] or [Iva17] for more details.

Combining this with Theorem 2.77 we have

**Theorem 2.81** (Gromov). *For any  $f : X \rightarrow Y$  such that  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is surjective with amenable kernel, the induced map  $f^* : H_b^n(Y) \rightarrow H_b^n(X)$  is an isometric isomorphism.*

As in the bounded group cohomology, we have a comparison map  $c : H_b^n(X; \mathbb{R}) \rightarrow H^n(X; \mathbb{R})$ .

### 3. MORE ON GROMOV'S SIMPLICIAL NORM

Further properties of the simplicial norm can be deduced by a duality principle between homology and bounded cohomology.

**3.1. Duality.** There is duality principle relating the norm on bounded cohomology to the simplicial norm. Note that the usual pairing  $H^n(X; \mathbb{R}) \times H_n(X; \mathbb{R}) \rightarrow \mathbb{R}$  gives a way to pair bounded cohomology classes with homology classes, by composing with the comparison map. The usual  $\ell^1$ - $\ell^\infty$  duality implies that, for any singular chain  $s \in C_n(X)$ , we have

$$|s|_1 = \sup_{f \in C_b^n(X)} \frac{\langle f, s \rangle}{\|f\|_\infty} = \sup_{f \in C_b^n(X), \|f\|_\infty \leq 1} \langle f, s \rangle.$$

Recall that  $\|\sigma\|_1 = \inf_{[s]=\sigma} |s|_1$ , which can be thought of as the induced norm on the quotient by  $\text{Im } \partial$ . The dual space of a quotient consists of the dual functions that vanish on the subspace, in our context those  $f$  with  $\delta f = 0$ , i.e. cocycles. This means, for any  $\sigma \in H_n(X; \mathbb{R})$  we have

$$\|\sigma\|_1 = \sup_{f \in C_b^n(X), \delta f = 0, \|f\|_\infty \leq 1} \langle f, \sigma \rangle.$$

The pairing does not change if we replace  $f$  by another cocycle in the same class, so we get

$$\|\sigma\|_1 = \sup_{\alpha \in H_b^n(X), \|\alpha\|_\infty \leq 1} \langle \alpha, \sigma \rangle.$$

This is called the duality principle.

**Proposition 3.1** (Duality principle). *For any  $\sigma \in H_n(X; \mathbb{R})$ , we have*

$$\|\sigma\|_1 = \sup_{\alpha \in H_b^n(X), \|\alpha\|_\infty \leq 1} \langle \alpha, \sigma \rangle.$$

**Corollary 3.2.** *If  $H_b^n(X) = 0$ , then  $\|\cdot\|_1$  vanishes on  $H_n(X)$ .*

**Example 3.3.** *We showed that  $H_b^1(X; \mathbb{R}) = 0$ , so  $\|\cdot\|_1$  vanishes on  $H_1(X; \mathbb{R})$  by the duality principle, which we proved directly in Corollary 1.16.*

Recall that we showed  $\|M\|_1 = 0$  for  $M = S^n, T^n$  by constructing self maps with nonzero degree. In both examples, the fundamental groups are amenable.

**Proposition 3.4.** *If  $\pi_1(M)$  is amenable, then the simplicial volume  $\|M\|_1 = 0$ .*

*Proof.* We know amenable groups have trivial bounded cohomology by Theorem 2.76, so by Gromov's mapping theorem 2.80 we know  $H_b^n(M) = 0$  for all  $n$ . In particular, taking  $n = \dim M$ , we obtain  $\|M\|_1 = 0$  by Corollary 3.2.  $\square$

In particular, every simply connected closed manifold has vanishing simplicial volume, which is not obvious from the definition.

**Example 3.5.** *For a closed hyperbolic  $n$ -manifold  $M$ , the volume form  $\omega$  represents a cohomology class in  $H^n(M)$ . The volume form itself is not a bounded cocycle, since there are large fat simplices with arbitrarily large volume. However, the composition  $\omega \circ \text{str}$  is bounded. This gives rise to a bounded cohomology class  $\alpha \in H_b^n(M)$  so that its image under the comparison map is  $[\omega]$ . Moreover,  $\|\alpha\|_\infty \leq \|\omega \circ \text{str}\|_\infty = \sup \text{vol}(\Delta^n) = v_n$ . Hence  $\frac{1}{v_n} \|\alpha\|_\infty$  has norm no more than 1. So by the duality principle,*

$$\|M\|_1 \geq \frac{1}{v_n} \langle \alpha, [M] \rangle = \frac{1}{v_n} \langle \omega, [M] \rangle = \frac{\text{vol}(M)}{v_n}.$$

*This essential the same straightening argument we used to show  $\|M\|_1 \geq \frac{\text{vol}(M)}{v_n}$ , written in terms of the duality principle. Since the equality holds by the Proportionality Theorem, we conclude that the bound  $\|\alpha\|_\infty \leq v_n$  is sharp, i.e.  $\|\alpha\|_\infty = v_n$ .*

**3.2. Additivity.** The goal of this section is to show Gromov's additivity theorem

**Theorem 3.6** (Gromov). *For  $n \geq 3$ , let  $M$  be the connected sum of closed orientable  $n$ -manifolds  $M_1$  and  $M_2$ . Then  $\|M\|_1 = \|M_1\|_1 + \|M_2\|_1$ .*

Here the assumption on the dimension  $n \geq 3$  is important. When  $n = 2$ , if  $M_1$  and  $M_2$  have genus  $g_1 \geq 1$  and  $g_2 \geq 1$  respectively, then  $M$  has genus  $g_1 + g_2$ . Then by Theorem 1.20, we know  $\|M\|_1 = -2\chi(M) = 4(g_1 + g_2 - 1)$ , while  $\|M_1\|_1 + \|M_2\|_1 = -2\chi(M_1) - 2\chi(M_2) = 4(g_1 + g_2 - 2)$ . So in this case, we have  $\|M\|_1 > \|M_1\|_1 + \|M_2\|_1$ .

We will deduce Theorem 3.6 from a more general result [Fri17, Theorem 7.6]. Let  $n \geq 2$ , and let  $\{(M_i, \partial M_i)\}_{i \in I}$  be a finite collection of orientable compact connected  $n$ -manifolds with boundary. We say  $(M, \partial M)$  is obtained by gluing  $\{(M_i, \partial M_i)\}_{i \in I}$  together, if there are orientation reversing homeomorphisms  $f_j : S_j^+ \rightarrow S_j^-$  of  $(n-1)$ -manifolds  $\{S_j^\pm\}_{j \in J}$ , such that each  $S_j^\pm$  is a component of some  $\partial M_i$  with induced orientation, and each component of  $M_i$  is identified with at most one  $S_j^\pm$ . Then  $\partial M$  is necessarily the union of those boundary components of  $M_i$ 's that never appear as some  $S_j^\pm$ .

We say the gluing is *compatible* if  $f_{j*} K_j^+ = K_j^-$ ,  $f_{j*}$  is the induced map on  $\pi_1$  and  $K_j^\pm$  is the kernel of the inclusion map  $\pi_1(S_j^\pm) \rightarrow \pi_1(M_i)$  if  $S_j^\pm$  is a component of  $\partial M_i$ . Note that if boundary inclusion  $S_j^\pm \rightarrow M_i$  is  $\pi_1$ -injective, then the gluing is compatible.

**Theorem 3.7.** *Let  $(M, \partial M)$  be obtained by gluing  $\{(M_i, \partial M_i)\}_{i \in I}$  together. If each component of  $\partial M_i$  has amenable fundamental group, then*

$$\|M, \partial M\|_1 \leq \sum_{i \in I} \|M_i, \partial M_i\|.$$

*If in addition the gluing is compatible, then*

$$\|M, \partial M\|_1 = \sum_{i \in I} \|M_i, \partial M_i\|.$$

Here are some consequences and examples.

When the boundary components are simply connected, then the assumptions of the theorem (amenability and compatibility) hold automatically.

**Corollary 3.8.** *For  $n \geq 3$ , if  $(N, \partial N)$  is obtained from a closed orientable  $n$ -manifold  $M$  by removing an open  $n$ -ball. Then  $\|N, \partial N\|_1 = \|M\|_1$ .*

*Proof.* In this case, we have  $\partial N \cong S^{n-1}$ , and  $M$  is obtained by gluing  $(N, \partial N)$  with  $(B^n, S^{n-1})$ . Since  $\pi_1(\partial N) = \pi_1(S^{n-1}) = 1$  as  $n \geq 3$ , the gluing is compatible. So by Theorem 3.7,

$$\|N, \partial N\|_1 = \|M\|_1 - \|B^n, S^{n-1}\|_1 = \|M\|_1.$$

The fact that  $\|B^n, S^{n-1}\|_1 = 0$  can be obtained by a self map with degree greater than one, or by the equality above for the case  $M = S^n$ .  $\square$

This fails when  $n = 2$  as  $\|S\|_1 = -2\chi^-(S)$  for all surfaces possibly with boundary.

Then we can deduce Theorem 3.6.

*Proof of Theorem 3.6.* Let  $(N_i, \partial N_i)$  be  $M_i$  with an open  $n$ -ball removed,  $i = 1, 2$ . Then  $M$  is obtained by gluing  $(N_i, \partial N_i)$  together. As  $N_i \cong S^{n-1}$ , which is simply connected, the gluing is compatible. Thus  $\|M\|_1 = \|N_1\|_1 + \|N_2\|_1 = \|M_1\|_1 + \|M_2\|_1$  by the corollary above.  $\square$

### Example 3.9.

- (1)  $S^2$  can be obtained from gluing two disks along the boundary. The boundary circle has fundamental group  $\mathbb{Z}$ , which is amenable. Although the boundary is not  $\pi_1$ -injective, the kernel for both copies are the entire  $\mathbb{Z} = \pi_1(S^1)$ , so the gluing is compatible. It follows that  $\|S^2\|_1 = 2\|D^2, S^1\|_1$ . Actually we know that both quantities are zero.
- (2) A genus  $g$  surface  $S$  can be obtained by gluing two surfaces  $S_1$  and  $S_2$  both with a single circle boundary, where  $S_i$  has genus  $g_i$ . When  $g_1, g_2 > 0$ , the boundary is  $\pi_1$ -injective, so  $\|S\|_1 = \|S_1\|_1 + \|S_2\|_1$ , which is compatible with the Euler characteristic computation. When  $g_2 = 0$  and  $g_1 = g$ , the gluing is not compatible, and we get a strict inequality  $\|S\|_1 < \|S_1\|_1$  as  $-\chi(S_1) > -\chi(S)$ .

Here is another example, which shows that the equality in Theorem 3.7 fails without compatibility.

**Example 3.10.** Let  $K$  be a knot in  $S^3$ . Let  $(M, \partial M)$  be the knot complement, i.e.  $S^3$  with an open tubular neighborhood of  $K$  removed. A Dehn filling of  $(M, \partial M)$  is a closed manifold  $N$  obtained by gluing  $(M, \partial M)$  with a solid torus  $(S^1 \times D^2, T^2)$ . The topology of  $N$  depends on the identification between  $\partial M$  and  $T^2$ , specifically which element of  $\pi_1(\partial M)$  is identified with  $\partial D^2$ .

This gluing is not compatible when  $K$  is nontrivial, as  $\pi_1(\partial M)$  injects  $\pi_1(M)$  while the kernel of  $\pi_1(T^2) \rightarrow \pi_1(S^1 \times D^2)$  is  $\mathbb{Z}$ . As  $\mathbb{Z}^2 = \pi^1(T^2)$  is amenable, we only get an inequality from Theorem 3.7:

$$\|N\|_1 \leq \|M, \partial M\|_1 + \|S^1 \times D^2, T^2\|_1 = \|M, \partial M\|_1.$$

Here  $\|S^1 \times D^2, T^2\|_1 = 0$  can be seen by doubling (which is a compatible gluing) and  $\|S^1 \times S^2\|_1 = 0$  (for having an  $S^1$  factor). Alternatively, there is a self map of degree greater than one.

Suppose  $K$  is a hyperbolic knot (i.e. the interior of  $M$  has a hyperbolic structure of finite volume), then a theorem of Thurston shows that for all but finitely many Dehn fillings,  $N$  has a hyperbolic structure. Moreover, the hyperbolic volumes of different Dehn fillings  $N$  converge nontrivially to the volume of  $M$ . By the Proportionality Theorem 1.38, we have  $\|N\|_1 = \text{vol}(N)/v_3$ , which is not a constant as we vary the Dehn fillings. So we must have a strict inequality  $\|N\|_1 < \|M, \partial M\|_1$  (for most Dehn fillings  $N$ ). In the limit we observe  $\text{vol}(M)/v_3 \leq \|M, \partial M\|_1$ . Actually this is an equality by smearing in the thick part of  $M$  to produce an efficient representative of the relative fundamental class; see [Thu, Lemma 6.5.4].

A compatible gluing with non-amenable fundamental groups on the boundaries also violates the theorem.

**Example 3.11.** Let  $S$  be a surface of genus at least 2. Let  $M_\varphi$  be the mapping torus associated to a mapping class  $\varphi$ . When  $\varphi$  is pseudo-Anosov,  $M_\varphi$  is hyperbolic, so  $\|M_\varphi\|_1 = \text{vol}(M_\varphi)/v_3$ . It is known that  $\text{vol}(M_\varphi)$  is comparable to the translation length of  $\varphi$  with respect to the Weil-Petersson metric on the Teichmüller space. In particular  $\text{vol}(M_{\varphi^n}) \rightarrow \infty$  as  $n \rightarrow \infty$ .

On the other hand,  $M_\varphi$  is obtained from  $S \times [0, 1]$  by a compatible gluing. However,  $\|M_{\varphi^n}\|_1 > \|S \times [0, 1]\|_1$  for large  $n$ , violating the inequality in Theorem 3.7, and the reason is that  $\pi_1(S)$  is non-amenable as it has free subgroups.

**Example 3.12.** For a closed 3-manifold  $M$ , its simplicial volume can be computed from its prime factors in the prime decomposition. When  $M$  is prime, it can be decomposed along  $\pi_1$ -injective tori so that each component admits one of the eight geometries, by the geometrization theorem. Five out of the eight geometry only allows amenable fundamental groups. Among the remaining three, manifolds with  $\mathbb{H}^2 \times \mathbb{R}$  or  $\widetilde{\text{PSL}}_2$  are Seifert fibered, and the  $S^1$  fiber (which has amenable fundamental group) forces the simplicial volume to vanish. So only the hyperbolic geometry contributes nontrivially to the simplicial volume of  $M$ .

**Proposition 3.13.** If a closed 3-manifold  $M$  admits a complete metric with negative sectional curvature, then it admits a complete hyperbolic metric.

*Proof.* Negative curvature implies that  $M$  is aspherical and the fundamental group  $\pi_1(M)$  is  $\delta$ -hyperbolic. In particular,  $\pi_1(M)$  cannot have any  $\mathbb{Z}^2$  subgroup. It follows that  $M$  is prime and has trivial JSJ decomposition. So itself is geometric. Negative curvature implies that  $\|M\|_1 > 0$ , hence  $M$  can only admit the hyperbolic geometry.  $\square$

From now on, in the setup of Theorem 3.7, we assume all components of  $\partial M_i$  to have amenable  $\pi_1$ . We will first prove the inequality in Theorem 3.7. Naively, we would like to piece together almost optimal cycles representing fundamental classes  $[M_i, \partial M_i]$ . The boundary of the cycles restricted on  $S_j^+$  and  $S_j^-$  represent the fundamental classes respectively, which cancel out after gluing as homology classes, but not necessarily as cycles. The hope is that this can fixed as  $S_j^\pm$  has amenable fundamental group and thus zero simplicial volume. To carry this out, we will use the duality principle in the relative context, between  $H_n(X, Y)$  and  $H_b^n(X, Y)$ , for a subspace  $Y \subset X$ . A key ingredient is the following theorem:

**Theorem 3.14.** Let  $(X, Y)$  be a pair of countable CW complexes. Suppose each component of  $Y$  has amenable fundamental group. Then the natural map  $H_b^n(X, Y) \rightarrow H_b^n(X)$  is an isometric isomorphism for all  $n \geq 2$ .

Note that cochains in  $C_b^n(X, Y)$  are cochains in  $X$  that vanish on  $Y$ . So intuitively, for any  $\alpha \in H_b^n(X)$  one can find a cocycle that vanishes on  $Y$  using amenability of  $\pi_1(Y)$  at an arbitrarily small cost to the norm. See [Fri17, Theorem 5.14] for details.

This allows us to treat any  $\alpha \in H_b^n(M, \partial M)$  as a class in  $H^n(M, \mathcal{S} \cup \partial M)$ , where  $\mathcal{S} = \cup \overline{S}_j$  and  $\overline{S}_j$  is the image of  $S_j^\pm$  in  $M$ . Then we can pull back  $\alpha$  to each  $M_i$  to obtain  $\alpha_i \in H^n(M_i, \partial M_i)$ .

**Lemma 3.15.** For any  $\alpha \in H_b^n(M, \partial M)$ , with the notation above, we have

$$\langle \alpha, [M, \partial M] \rangle = \sum_{i \in I} \langle \alpha_i, [M_i, \partial M_i] \rangle.$$

*Proof.* Let  $c_i$  be a relative cycle in  $C_n(M_i, \partial M_i)$  representing  $[M_i, \partial M_i]$ . Identify it with its image in  $M$ . Then  $\partial c_i$  on each component of  $\partial M_i$  represents the fundamental class. Then for the chain  $c = \sum c_i \in C_n(M)$ , the part of  $\partial c$  on each  $\overline{S}_j$  is the difference of two fundamental cycles since



$f_j : S_j^+ \rightarrow S_j^-$  is orientation reversing, hence it can be written as  $\partial c'_j$  for some chain  $c'_j \in C_n(\bar{S}_j)$ . It follows that  $c - c' \in C_n(M, \partial M)$  represents  $[M, \partial M]$ , where  $c' = \sum c'_j$ .

Representing  $\alpha$  as a cocycle vanishing on all  $\bar{S}_j$ 's, we have  $\langle \alpha, c' \rangle = 0$ , so

$$\langle \alpha, [M, \partial M] \rangle = \langle \alpha, c - c' \rangle = \langle \alpha, c \rangle = \sum_{i \in I} \langle \alpha_i, c_i \rangle = \sum_{i \in I} \langle \alpha_i, [M_i, \partial M_i] \rangle.$$

□

*Proof of the inequality in Theorem 3.7.* By the duality principle, we have

$$\|M, \partial M\|_1 = \sup_{\alpha \in H_b^n(M, \partial M), \|\alpha\|_\infty \leq 1} \langle \alpha, [M, \partial M] \rangle.$$

Given any such  $\alpha$ , in the notation above, each  $\alpha_i$  is obtained by pulling back  $\alpha$ , so  $\|\alpha_i\|_\infty \leq \|\alpha\|_\infty \leq 1$ , so  $\|M_i, \partial M_i\|_1 \geq \langle \alpha_i, [M_i, \partial M_i] \rangle$  by the duality principle on  $M_i$ . Hence the lemma above, we have

$$\sum_{i \in I} \|M_i, \partial M_i\|_1 \geq \sum_{i \in I} \langle \alpha_i, [M_i, \partial M_i] \rangle = \langle \alpha, [M, \partial M] \rangle.$$

As this holds for all  $\alpha \in H_b^n(M, \partial M)$  with  $\|\alpha\|_\infty \leq 1$ , we deduce that

$$\|M, \partial M\|_1 \leq \sum_{i \in I} \|M_i, \partial M_i\|_1$$

as desired. □

The equality in Theorem 3.7 is obtained by a partial converse to Theorem 3.14 for compatible gluing:

**Theorem 3.16.** *Suppose  $M$  is obtained by compatible gluing as in Theorem 3.7. Given  $\varphi_i \in H_b^n(M_i, \partial M_i)$  for all  $i \in I$ , for any  $\epsilon > 0$ , there is  $\alpha \in H_b^n(M, \partial M)$  such that  $\alpha_i = \varphi_i$  for all  $i$  and*

$$\|\alpha\|_\infty \leq \max_{i \in I} \{\|\varphi_i\|_\infty\} + \epsilon.$$

*Proof.* We give a sketch of the proof; Consult [Fri17, Section 9.2] for more details. Let  $G = \pi_1(M)$ . Then cochains in  $H_b^n(M)$  correspond to  $G$ -invariant cochains on the universal cover  $\widetilde{M}$ . The manifold  $M$  has the structure of a graph of spaces, where vertices correspond to interiors of  $M_i$ 's (called vertex spaces) and each edge corresponds to a gluing along some  $S_j^\pm$ . Denote by  $\Gamma$  the corresponding finite graph.

The universal cover  $\widetilde{M}$  has an induced structure of a tree of spaces, where each vertex represents a component of  $p^{-1}(M_v)$  for a vertex space  $M_v$ , where  $p : \widetilde{M} \rightarrow M$  is the covering map. This gives a contraction from  $\widetilde{M}$  to a tree  $T$ , which admits a  $G$  action with quotient being the graph  $\Gamma$ . Compatible gluing ensures that each inclusion of  $M_v$  in  $M$  is  $\pi_1$ -injective by van Kampen. This implies that the restriction of  $p$  to each vertex space  $\widetilde{M}_v$  in  $\widetilde{M}$  is a universal covering map  $\widetilde{M}_v \rightarrow M_v$ , where  $\bar{v}$  is the image of  $v$  under the quotient map  $T \rightarrow \Gamma$ .

Each  $\varphi_i$  can be represented by a cocycle  $f_i$  on  $\widetilde{M}_i$  that is  $\pi_1(M_i)$ -invariant such that  $|f_i|_\infty < \|\varphi_i\| + \epsilon$ . So we have cocycles  $f_v$  defined on each vertex space  $\widetilde{M}_v$  of  $\widetilde{M}$ . The key is to define an extension  $f$  of these cocycles to  $\widetilde{M}$  so that  $|f|_\infty \leq \max_v \{|f_v|_\infty\} = \max_{i \in I} \{|f_i|_\infty\}$ .

For each singular simplex  $\Delta$  in  $\widetilde{M}$ , we define  $f(\Delta)$  as  $f_v(\Delta_v)$  if  $\Delta$  has a ‘‘barycenter’’  $v$  and zero if it has not barycenter, where  $\Delta_v$  is supported on the closure of  $\widetilde{M}_v$  as a ‘‘projection’’ of  $\Delta$ . We say  $v$  is a barycenter of  $\Delta$  if for every pair of vertices  $x_i, x_j$  of  $\Delta$ , all paths connecting them in  $\widetilde{M}$  intersects  $\widetilde{M}_v$ ; One can also view this by first projecting to the tree  $T$ . The barycenter may not exist but is unique if it exists. The projection  $\Delta_v$  is chosen rather arbitrarily, with the property that its vertex  $x'_i$  agrees with the corresponding vertex  $x_i$  of  $\Delta$  if  $x_i \in M_v$ , and lies on the unique boundary of

$\widetilde{M}_v$  separating  $x_i$  from  $M_v$  otherwise. The definition of  $f$  guarantees the bound on its norm. The amenability of the boundary components is used to resolve problems on the boundaries of vertex spaces in this construction.  $\square$

Now we complete the proof of Theorem 3.7.

*Proof of the equality in Theorem 3.7.* For any  $\epsilon > 0$ , by the duality principle, there is  $\varphi_i \in H_b^n(M_i, \partial M_i)$  with  $\|\varphi_i\|_\infty \leq 1$  such that  $\langle \varphi_i, [M_i, \partial M_i] \rangle > \|M_i, \partial M_i\|_1 - \epsilon$ .

Let  $\alpha$  be as in the theorem above for the same  $\epsilon$ . Then  $\|\alpha\|_\infty \leq 1 + \epsilon$

$$(1+\epsilon)\|M, \partial M\|_1 \geq \langle \alpha, [M, \partial M] \rangle = \sum_{i \in I} \langle \alpha_i, [M_i, \partial M_i] \rangle = \sum_{i \in I} \langle \varphi_i, [M_i, \partial M_i] \rangle = \sum_{i \in I} \|M_i, \partial M_i\|_1 - \epsilon|I|.$$

Letting  $\epsilon \rightarrow 0$  gives the inequality in the other direction.  $\square$

**3.3. Volume of a product.** Geometrically, the product  $M \times N$  of two closed Riemannian manifolds  $M^m, N^n$  with the product metric has  $\text{vol}(M \times N) = \text{vol}(M) \cdot \text{vol}(N)$ . For the simplicial volume, in general it is only known that  $\|M \times N\|_1$  is comparable to  $\|M\|_1 \cdot \|N\|_1$  by some universal constants depending on the dimensions  $m, n$ .

**Theorem 3.17** (Gromov). *Let  $M, N$  be closed manifolds of dimension  $m, n$ . Then*

$$\|M\|_1 \cdot \|N\|_1 \leq \|M \times N\|_1 \leq \binom{m+n}{m} \|M\|_1 \cdot \|N\|_1.$$

*Proof.* The product of simplices  $\Delta^m \times \Delta^n$  can be triangulated with  $\binom{m+n}{n}$  simplices of dimension  $m+n$ , where simplices correspond to (geodesic) paths from the lower left corner to the upper right corner of a grid made of  $m \times n$  squares (so the vertices of a path corresponds to vertices of the simplex). This proves the upper bound.

For the lower bound, let  $\alpha_M$  and  $\alpha_N$  be cohomology classes of norm at most 1 in  $H_b^m(M)$  and  $H_b^n(N)$  respectively so that  $\langle \alpha_M, M \rangle > \|M\|_1 - \epsilon_1 \geq -$  and  $\langle \alpha_N, N \rangle > \|N\|_1 - \epsilon_2 \geq 0$ . The cup product  $\alpha = \alpha_M \cup \alpha_N$  satisfies  $\|\alpha\|_\infty \leq \|\alpha_M\|_\infty \cdot \|\alpha_N\|_\infty \leq 1$ , and

$$\langle \alpha, [M \times N] \rangle = \langle \alpha_M, [M] \rangle \cdot \langle \alpha_N, [N] \rangle \geq (\|M\|_1 - \epsilon_1)(\|N\|_1 - \epsilon_2).$$

Hence by the duality principle, we have  $\|M \times N\|_1 \geq (\|M\|_1 - \epsilon_1)(\|N\|_1 - \epsilon_2)$ , where  $\epsilon_1, \epsilon_2 \geq 0$  can be arbitrarily small (and actually can be chosen to be zero). This proves the lower bound.  $\square$

**Remark 3.18.** *The same holds for two arbitrary homology classes instead of fundamental classes. There are various improvements of the bounds. For instance, if  $m = n = 2$  and one of the two classes is the fundamental class of a closed surface, then there is an equality [HL20, Theorem E]:*

$$\|\alpha \times \beta\|_1 = \frac{3}{2} \|\alpha\|_1 \cdot \|\beta\|_1.$$

An immediate consequence generalizes our earlier observation that  $\|S^n \times M\|_1 = 0$  (or with  $S^n$  replaced by any manifold having a self map of degree greater than one).

**Corollary 3.19.**  $\|M \times N\|_1 = 0$  if  $\|M\|_1 = 0$ .

In the special case where  $\pi_1(M)$  is amenable, we can view  $M \times N$  as a trivial  $M$ -bundle over  $N$  where the fibers are amenable. Such genuine bundles in the world of manifolds always have zero volume.

**Theorem 3.20.** *Let  $E$  be a fiber bundle over base  $B$ , where the fiber  $F$  has  $\pi_1(F)$  amenable. Suppose  $F, E, B$  are orientable connected closed manifolds and  $\dim F \geq 1$ . Then  $\|E\|_1 = 0$ .*

*Proof.* From the long exact sequence of homotopy groups for a fibration, we extract the following exact sequence:

$$\pi_1 F \rightarrow \pi_1 E \xrightarrow{p} \pi_1 B \rightarrow 1.$$

As  $\pi_1(F)$  is amenable, Theorem 2.77 and Theorem 2.80 imply that  $H_b^n(B) \cong H_b^n(\pi_1 B) \xrightarrow{p^*} H_b^n(\pi_1 E) \cong H_b^n(E)$  is an isometric isomorphism. Let  $n = \dim E$ . However,

$$\langle E, p^*(\alpha) \rangle = \langle p_*([E]), \alpha \rangle = 0$$

for all  $\alpha \in H_b^n(B)$  since the pairing factors through the comparison map  $H_b^n(B) \rightarrow H^n(B)$ , which is a trivial space as  $\dim B = \dim E - \dim F < \dim E = n$  as  $\dim F \geq 1$ . Hence  $\|E\|_1 = 0$  by the duality principle.  $\square$

This echoes with our earlier comment that Seifert fibered 3-manifolds (with  $\widetilde{\mathrm{PSL}}_2$  or  $\mathbb{H}^2 \times \mathbb{R}$  geometry) have zero simplicial volume so that only hyperbolic pieces contribute nontrivially to the volume of a closed 3-manifold. Although Seifert fibered spaces are not strictly speaking an  $S^1$  fiber bundle over the base surface, but it can be viewed as an  $S^1$  bundle over an orbifold, and the exact sequence above holds with  $\pi_1(B)$  interpreted as the orbifold fundamental group.

#### 4. GROUPS ACTING ON GROMOV-HYPERBOLIC SPACES

The goal of this section is two fold. One is a proof of Mostow's rigidity as an application of Gromov's proportionality, and the other is Epstein–Fujiwara's construction of quasimorphisms for groups acting nicely on negatively curved spaces, which generalizes Brooks' counting quasimorphism and applies to many groups of interest in geometric group theory.

Both require some knowledge of coarse geometry of metric spaces, so we will give a brief introduction, focusing on basics of quasi-isometries, Gromov-hyperbolic spaces, and the Gromov boundary. A good reference for this part is [BH99].

##### 4.1. Quasi-isometries.

**Definition 4.1.** Given constants  $L > 0, C \geq 0$ , for metric spaces  $X$  and  $Y$ , a map  $f : X \rightarrow Y$  is an  $(L, C)$ -quasi-isometric embedding (or  $(L, C)$ -QI embedding for short) if  $f$  is  $L$ -bi-Lipschitz up to a bounded error  $C$ , i.e.

$$\frac{1}{L}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Ld_X(x, y) + C, \forall x, y \in X.$$

We simply say  $f$  is a *quasi-isometric embedding* (QI-embedding) if  $f$  is  $(L, C)$ -QI embedding for some  $L$  and  $C$ . If in addition, there is a QI-embedding  $g : Y \rightarrow X$  such that  $f \circ g$  and  $g \circ f$  are in bounded distance to  $id_X$  and  $id_Y$  respectively, then we say  $f$  is a *quasi-isometry* with *quasi-inverse*  $g$ .

A QI-embedding  $f$  is QI if and only if it is *quasi-onto* in the sense that  $Y$  is contained in the  $r$ -neighborhood of the image of  $f$  for some  $r > 0$ . A quasi-inverse can be chosen by taking each point to a point in the image of  $f$  within distance  $r$ .

**Example 4.2.** Given a group  $G$  with a finite generating set  $S$ , the word length with respect to  $S$  is

$$|g|_S := \inf\{n \mid g = s_1 \cdots s_n, s_i \in S^\pm\}.$$

Then we get an associated word-metric  $d(g, h) := |g^{-1}h|$ . This is indeed a metric on  $G$  as the symmetry follows from the fact that  $|g|_S = |g^{-1}|_S$  and the triangle inequality follows from  $|gh|_S \leq |g|_S + |h|_S$ . The word-metric is left-invariant in the sense that  $d(kg, kh) = d(g, h)$  for all  $k, g, h, \in G$ .

Let  $S'$  be another finite generating set of  $G$ . Distinguish the two word-metrics by denoting them as  $d$  and  $d'$  respectively. The identity map  $id : (G, d) \rightarrow (G, d')$  is a quasi-isometry. Indeed,  $|g|_{S'} \leq L_1|g|_S$  for  $L_1 = \max_{s \in S} |s|_{S'}$ , so

$$d'(g, h) = |g^{-1}h|_{S'} \leq L_1|g^{-1}h|_S \leq L_1d(g, h).$$

Similarly  $|g|_S \leq L_2|g|_{S'}$  for  $L_2 = \max_{s' \in S'} |s'|_S$ , and

$$d(g, h) \leq L_2 d'(g, h).$$

Hence  $\text{id}$  is an  $(L, 0)$ -QI embedding for  $L = \max\{L_1, L_2\}$ . Any bijective QI embedding is a quasi-isometry.

Therefore, a finitely generated group has a well-defined coarse geometry.

**Example 4.3.**  $\mathbb{Z}$  is quasi-isometric to  $\mathbb{R}$  equipped with the Euclidean metric, via the natural inclusion. A quasi-inverse is a contraction  $g$  taking each interval  $[n - 1/2, n + 1/2)$  to  $n$  for all  $n \in \mathbb{Z}$ .

**Example 4.4.** Given a closed Riemannian manifold  $M$ , let  $\widetilde{M}$  be its universal cover with the pullback metric. Then  $G = \pi_1(M)$  acts on  $\widetilde{M}$  by isometries. For any chosen base point  $p \in \widetilde{M}$ , the orbit map  $f : G \rightarrow \widetilde{M}$  given by  $f(g) = gp$  is a quasi-isometry, shown by the Švarc–Milnor lemma below. Note that the fundamental group is purely topological but the Riemannian metric on  $M$  is an additional structure. The coarse geometry of the fundamental group determines what kind of metric a topological manifold  $M$  can carry. For instance, it is known that if  $G = \pi_1(M)$  is virtually nilpotent, then  $M$  cannot have a metric with negative sectional curvature by considering the growth of the size of balls of increasing radius  $r$  in both  $G$  and  $\widetilde{M}$ .

**Lemma 4.5** (Švarc–Milnor Lemma). *If a group  $G$  acts properly discontinuously on a proper geodesic metric space  $X$  by isometries with compact quotient, then  $G$  is finitely generated and any orbit map is a quasi-isometry.*

Here a metric space  $X$  is *proper* if every bounded closed subset (e.g. a closed ball) is compact. It is *geodesic* if any two points  $x, y \in X$  is connected by a geodesic  $\gamma$  so that the length of  $\gamma$  is  $d(x, y)$ . The length of a curve  $\gamma : [0, t] \rightarrow X$  is the supremum of  $\sum_{i=0}^n d(\gamma(t_{i-1}), \gamma(t_i))$  over all increasing sequences  $\{t_i\}_{i=0}^n$  with  $t_0 = 0$  and  $t_n = t$  for all  $n \geq 1$ . It is a (unit speed global) geodesic if  $d(\gamma(s), \gamma(s')) = |s - s'|$  for all  $s, s' \in [0, t]$ .

The finite generation part of the lemma can be deduced from the following more general fact.

**Lemma 4.6.** *Suppose  $G$  acts on a space  $X$  by homeomorphisms such that there is an open subset  $U \subset X$  so that  $GU = X$ . Let  $S = \{g \mid gU \cap U \neq \emptyset\}$ . If  $X$  is connected, then  $S$  generates  $G$ .*

*Proof.* Let  $H$  be the subgroup generated by  $S$ . Let  $V = HU$  and  $V' = (G \setminus H)U$ , both of which are open subsets of  $X$  and  $V \cup V' = GU = X$ . Our goal is to show  $G = H$ , i.e.  $V' = \emptyset$ . Suppose not, since  $X$  is connected, we must have  $V \cap V' \neq \emptyset$ . That is, there is  $h \in H$  and  $g \in G \setminus H$  such that  $hU \cap gU \neq \emptyset$ . Then  $U \cap h^{-1}gU \neq \emptyset$ , so by definition  $h^{-1}g \in S \subset H$ , contradicting  $g \notin H$ .  $\square$

*Proof of Lemma 4.5.* Let  $p \in X$  be any base point. Since  $X/G$  is compact, there is a closed ball  $B = B(p, r)$  of radius  $r > 0$  such that  $GB = X$ . As  $X$  is proper,  $B$  is compact, and  $S = \{g \mid B \cap gB \neq \emptyset\}$  is finite since the action is properly discontinuous. Hence  $G$  is finitely generated by Lemma 4.6. The next step is to compare the word metric  $d_S$  to the metric in  $X$  restricted to the orbit  $Gp$ .

On the one hand, for any  $s \in S$ , there is  $q \in B \cap sB$ , so  $d(p, sp) \leq d(p, q) + d(q, sp) \leq 2r$ . It follows that  $d(p, gp) \leq 2r|g|_S$  for all  $g \in G$ , and  $d(gp, hp) \leq 2rd_S(g, h)$ .

Everything above works for all  $r$  large enough. So in the beginning we may choose  $r$  so that  $GB(p, r/3) = X$ . This helps us obtain the comparison in the other direction. For any  $g \in G$ , let  $\gamma : [0, d] \rightarrow X$  be a geodesic with  $\gamma(0) = p$  and  $\gamma(d) = gp$ , where  $d = d(p, gp)$ . Choose a partition of  $[0, d]$  with break points  $0 = t_0 < t_1 < \dots < t_n = gp$  so that  $|t_i - t_{i-1}| \leq r/3$  for all  $1 \leq i \leq n$ , where  $n \leq 3d/r + 1$ . For each  $0 \leq i \leq n$ , choose  $g_i$  so that  $\gamma(t_i) \in g_i B(p, r/3)$ , where  $g_0 = \text{id}$  and  $g_n = g$ . Then

$$d(g_i p, g_{i-1} p) \leq d(g_i p, \gamma(t_i)) + d(\gamma(t_i), \gamma(t_{i-1})) + d(\gamma(t_{i-1}), g_{i-1} p) \leq r,$$

so  $B(g_i p, r) \cap B(g_{i-1} p, r) \neq \emptyset$ . That is,  $d_S(g_i, g_{i-1}) \leq 1$  for all  $i$ . Hence  $d_S(id, g) \leq n \leq 3d/r + 1 = \frac{3}{r}d(p, gp) + 1$ . As  $G$  acts by isometry, we have

$$d_S(g, h) \leq \frac{3}{r}d(gp, hp) + 1$$

for all  $g, h \in G$ .

Combining the two directions, we see that  $(G, d_S) \rightarrow (X, d)$  is an  $(2r, r/3)$ -QI embedding. As  $X$  is contained in the  $r$  neighborhood of image, we know this is a quasi isometry.  $\square$

**4.2. Gromov-hyperbolic spaces and their boundary.** Gromov-hyperbolic spaces are coarse-geometric generalizations of hyperbolic spaces that are preserved by quasi-isometries. Throughout this section, all metric spaces are assumed to be proper and geodesic.

**Definition 4.7.** Given  $\delta \geq 0$ , a proper geodesic metric space is  $\delta$ -hyperbolic if every geodesic triangle is  $\delta$ -thin: each side is contained in the union of the  $\delta$ -neighborhoods of the other two sides. We simply say  $X$  is *Gromov-hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta$ .

A metric space of finite diameter  $d$  is  $\delta$ -hyperbolic, but it is not of interest.

**Example 4.8.** Every simplicial tree (with the simplicial metric) is 0-hyperbolic since every geodesic triangle is a tripod, so that each side is contained in the union of the other two.

**Example 4.9.** The hyperbolic space  $\mathbb{H}^n$  is  $\delta$ -hyperbolic for a uniform  $\delta$ . It comes down to check an ideal hyperbolic triangle is  $\delta$ -thin, which can be done by an explicit calculation for a nicely chosen one, say in the upper-half plane model.

**Example 4.10.** If a complete Riemannian manifold  $X$  is simply connected and its sectional curvature is no more than  $\kappa < 0$ , then  $X$  is  $\delta$ -hyperbolic for some  $\delta = \delta(\kappa)$ .

**Definition 4.11.** A group  $G$  generated by a finite set  $S$  is  $\delta$ -hyperbolic if its Cayley graph with respect to  $S$  is  $\delta$ -hyperbolic.

**Example 4.12.** Any free group is 0-hyperbolic with respect to a free basis.

If  $n \geq 2$ , then  $\mathbb{Z}^n$  (with respect to the standard basis) is not  $\delta$ -hyperbolic for any  $\delta$ .

Changing the finite generating set  $S$  would change the metric on  $G$  by a quasi-isometry, which always take a  $\delta$ -hyperbolic space to a  $\delta'$ -hyperbolic space for a suitable  $\delta'$ . The key is to understand the image of geodesics under a QI-embedding.

**Definition 4.13.** For an interval  $I$ , a curve  $c : I \rightarrow X$  is a  $(L, C)$ -quasi-geodesic segment (resp. ray or line) if  $c$  is an  $(L, C)$ -QI embedding, where  $I = [a, b]$  (resp.  $I = [0, \infty)$  or  $I = \mathbb{R}$ ) is equipped with the Euclidean metric.

We summarize two key facts about quasi-geodesics in  $\delta$ -hyperbolic spaces.

**Lemma 4.14** (Morse Lemma). For every pair  $(L, C)$  and  $\delta$ , there is a universal constant  $R = R(\delta, L, C) \geq 0$  such that every  $(L, C)$ -quasi-geodesic segment with endpoints  $p, q$  in a  $\delta$ -hyperbolic space  $X$  has Hausdorff distance at most  $R$  to any geodesic segment connecting  $p, q$ .

**Lemma 4.15** (Local-to-global Principle). For every pair  $(L, C)$  and  $\delta$ , there is a universal constant  $K = K(\delta, L, C)$  such that if a curve  $c : \mathbb{R} \rightarrow X$  in a  $\delta$ -hyperbolic space whose restriction to each interval of length  $K$  is an  $(L, C)$ -quasi-geodesic, then  $c$  is a (global)  $(2L, 2C)$ -quasi-geodesic.

We will skip the proofs of these facts but look into a few important consequences.

**Theorem 4.16.** If  $X' \rightarrow X$  is an  $(L, C)$ -quasi-isometric embedding, where  $X$  is  $\delta$ -hyperbolic, then  $X'$  is  $\delta'$ -hyperbolic, where  $\delta'$  only depends on  $L, C, \delta$ . In particular, Gromov-hyperbolicity is QI invariant.

*Proof.* Let  $R$  be as in the Morse lemma, and we show  $\delta' = L(\delta + 2R + C)$  works. Let  $\Delta'$  be a geodesic triangle in  $X'$  with sides  $a', b', c'$ . Without loss of generality, we show that  $c' \subset N_{\delta'}(a') \cup N_{\delta'}(b')$ . Let  $a = fa'$ ,  $b = fb'$ , and  $c = fc'$ , which are  $(L, C)$ -geodesics. By the Morse lemma, there are geodesic segments  $\alpha, \beta, \gamma$  in  $X$  with Hausdorff distance at most  $R$  to  $a, b, c$  respectively. Then

$$c \subset N_R(\gamma) \subset N_{R+\delta}(\alpha) \cup N_{R+\delta}(\beta) \subset N_{2R+\delta}(a) \cup N_{2R+\delta}(b).$$

That is, for each  $c(x)$ , without loss of generality, we may assume that there is  $a(y)$  so that  $d(c(x), a(y)) \leq 2R + \delta$ . Since  $f$  is  $(L, C)$  QI embedding, we have

$$\frac{d(c'(x), a'(y))}{L} - C \leq d(fc'(x), fa'(y)) = d(c(x), a(y)) \leq 2R + \delta.$$

So  $d(c'(x), a'(y)) \leq L(2R + \delta + C) = \delta'$  as desired.  $\square$

**Corollary 4.17.** *Whether a group is Gromov-hyperbolic does not depend on the choice of the finite generating set.*

**Example 4.18.** *If  $G = \pi_1(M)$  for a closed negatively curved manifold  $M$ , then  $G$  is  $\delta$ -hyperbolic for some  $\delta$ .*

**Lemma 4.19.** *If  $\gamma : [0, \infty) \rightarrow X$  is an  $(L, C)$ -quasi-geodesic ray with starting point  $p$  in a  $\delta$ -hyperbolic space  $X$ , then there is a geodesic ray starting at  $p$  within Hausdorff distance  $R$  from the Morse lemma.*

*Proof.* For each  $n \in \mathbb{Z}_+$ , let  $\gamma_n = \gamma|_{[0, n]}$  and  $c_n$  be a geodesic connecting  $\gamma(0)$  and  $\gamma(n)$ . By the Morse lemma,  $c_n$  is contained in the  $R$ -neighborhood of  $\gamma_n$ , where  $R$  does not depend on  $n$ . Fixing each  $m \in \mathbb{Z}_+$ , we claim that Arzelà–Ascoli applies to the sequence of geodesic segments  $\{c_n|_{[0, m]}\}$  for  $n \geq m$ . The sequence is equi-continuous since they are (unit speed) geodesics. They lie in the ball of radius of  $m$  around  $p$ , which is compact.

Hence Arzelà–Ascoli implies that (any subsequence of)  $\{c_n|_{[0, m]}\}$  has a subsequence that converges uniformly to a map  $\ell_m : [0, m] \rightarrow X$ . Here  $\ell_m$  must be a geodesic as the limit of geodesics, and it lies in the  $R$ -neighborhood of  $\gamma$  since each  $c_n$  does. This can be done inductively, so that the subsequence at stage  $m + 1$  is a subsequence of the subsequence at stage  $m$ . It follows that  $\ell_m$  is the restriction of  $\ell_{m'}$  to  $[0, m]$  for all  $m' > m$ , hence this uniquely determines a geodesic ray  $\ell : [0, \infty) \rightarrow X$ . It lies in the  $R$ -neighborhood of  $\gamma$  since each  $\ell_m$  does.

On the other hand, any fixed  $\gamma(t)$  lies in the  $R$ -neighborhood of  $c_n$  for all  $n > t$ . Fix  $m$  much larger than  $t$ , for any  $\epsilon > 0$ , there is some  $n$  such that  $c_n([0, m])$  lies in the  $\epsilon$ -neighborhood of  $\ell([0, m])$ . Then  $\gamma(t)$  lies in the  $(R + \epsilon)$ -neighborhood of  $\ell([0, m])$  if  $m$  is chosen large enough so that  $d(\gamma(t), c_n(s)) > R$  for all  $s > m$ . Letting  $\epsilon \rightarrow 0$ , we see that  $\gamma$  lies in the  $R$ -neighborhood of  $\ell$ .

Hence  $\gamma$  and  $\ell$  has Hausdorff distance at most  $R$ .  $\square$

**Corollary 4.20.** *Any bi-infinite quasi-geodesic line in  $X$  has finite Hausdorff distance to a bi-infinite geodesic.*

*Proof.* Think of the quasi-geodesic line as two quasi-geodesic rays  $\gamma_1, \gamma_2$  starting at the same point  $p$ . By the lemma above, there are geodesic rays  $\ell_1, \ell_2$  starting at  $p$  close to  $\gamma_1$  and  $\gamma_2$  respectively. Take geodesics  $c_n$  connecting  $\ell_1(n)$  to  $\ell_2(n)$ , and take their weak limits using Arzelà–Ascoli as above we find a bi-infinite geodesic close to the union  $\ell_1 \cup \ell_2$  and hence close to the starting quasi-geodesic  $\gamma$ .  $\square$

**Definition 4.21** (Gromov boundary). For a  $\delta$ -hyperbolic space  $X$ , define its (Gromov) boundary (or ideal boundary)  $\partial X$  as the equivalence class of quasi-geodesic rays, where two quasi-geodesic rays are equivalent if and only if they have finite Hausdorff distance to each other.

It does not affect the definition if we restrict our attention to quasi-geodesic rays starting at a chosen base point. This can be seen by an Arzelà–Ascoli argument similar to the proofs above. We can further restrict to genuine geodesic rays starting at a base point by Lemma 4.19.

The topology on  $\partial X$  is given by the following convergence:  $\{y_n\}$  converges  $y$  in  $\partial X$  if there are geodesic rays  $c_n \in y_n$  and  $c \in y$  so that any subsequence of  $c_n$  has a subsequence that converges uniformly on compact sets to  $c$ .

**Example 4.22.** *The Gromov boundary of the hyperbolic space  $\mathbb{H}^n$  is the usual ideal boundary  $\partial \mathbb{H}^n \cong S^{n-1}$ .*

**Theorem 4.23** (Boundary map). *If  $f : X \rightarrow Y$  is a QI-embedding of Gromov-hyperbolic spaces, then  $f$  induces a natural continuous injective map  $\partial f : \partial X \rightarrow \partial Y$ . If  $f$  is QI, then  $\partial f$  is a homeomorphism.*

*Proof.* QI sends quasi-geodesic rays to quasi-geodesic rays, and two sets with Hausdorff distance  $D$  has distance at most  $LD + C$  for their image under  $f$  if  $f$  is  $(L, C)$ -QI. This defines the natural map  $\partial f$ . If the image of two sets have Hausdorff distance  $D$  in  $Y$  then the two sets have Hausdorff distance at most  $D(L + C)$ , which proves injectivity. The map is natural in the sense that, if  $g : Y \rightarrow Z$  is also QI, then  $\partial(gf) = \partial g \circ \partial f$ . Moreover, if two QIs are a bounded distance apart, then they induce the same boundary map. Hence if  $f$  has a quasi-inverse  $g$ , then  $\partial f$  and  $\partial g$  are inverses.

It remains to check the continuity of  $\partial f$ . It suffices to show that if  $c_n : [0, \infty) \rightarrow X$  is a sequence of geodesic rays converging on compact sets to a geodesic  $c$  in  $X$ , then the geodesic rays  $c'_n$  corresponding to the quasi-geodesics  $f \circ c_n$  constructed as in Lemma 4.19 has a subsequence converging to a geodesic ray  $c'$  uniformly on compact sets in  $Y$ , where  $d_H(c', f \circ c) < \infty$  and  $d_H$  is the Hausdorff distance. □

### 4.3. Mostow's rigidity.

## 5. ACTIONS ON THE CIRCLE AND THE BOUNDED EULER CLASS

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