Problem 1 solution:

• Recall that Gauss integral formula is

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}. \tag{1}$$

using this the definition of gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, we have

$$c_n = \int_{-\infty}^{\infty} x^n e^{-x^2}dx = \begin{cases} 2\int_0^\infty x^n e^{-x^2}dx = \int_0^\infty (x^2)^{\frac{n+1}{2}-1}e^{-x^2}dx^2 = \Gamma(\frac{n+1}{2}) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}, & n = 2k, k \text{ integer,} \\ 0, \quad n = 2k+1, k \text{ integer} \end{cases} \tag{2}$$

where $(2k-1)!! = (2k-1)(2k-2)\cdots 1$ means double factorial and in the last step we used the property of gamma function $\Gamma(z+1) = z\Gamma(z)$. We have also used the fact that integral of odd function over symmetric interval must vanish for $n = 2k+1$ case. For later use, we list some of the exact values of c_n,

$$c_0 = \sqrt{\pi}, c_1 = 0, c_2 = \frac{1}{2}\sqrt{\pi}, c_3 = 0, c_4 = \frac{3}{4}\sqrt{\pi}, c_5 = 0, c_6 = \frac{15}{8}\sqrt{\pi}, c_7 = 0, c_8 = \frac{105}{16}\sqrt{\pi}. \tag{3}$$

Since $g_0 = f_0 = x^0 = 1$, the norm is

$$\|g_0\| = \sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx} = \pi^{1/4}, \tag{4}$$

therefore

$$e_0 = g_0/\|g_0\| = \pi^{-1/4} = \frac{1}{\sqrt{\sqrt{\pi}2^0 0!}} \times 1. \tag{5}$$

Notice that $f_i = x^i$, we see that

$$\langle e_0, f_1\rangle = \langle e_0, f_3\rangle = \langle e_0, f_5\rangle = 0. \tag{6}$$

Since integral of odd function over symmetric interval must vanish.

By invoking equation (2), we see that

$$\langle e_0, f_2\rangle = \frac{\pi^{1/4}}{2}, \quad \langle e_0, f_4\rangle = \frac{3}{4}\pi^{1/4}. \tag{7}$$

From this we see that

$$g_1 = f_1 = x, \tag{8}$$

using equation (2), the norm is

$$\|g_1\| = \sqrt{\int_{-\infty}^{\infty} x^2 e^{-x^2}} = \frac{\pi^{1/4}}{\sqrt{2}}. \tag{9}$$

Moreover, we have

$$e_1 = \frac{\sqrt{2}}{\pi^{1/4}}x = \frac{1}{\sqrt{\sqrt{\pi}2^1 1!}}(2x). \tag{10}$$

The inner weighted products of e_1 with f_0, f_2, f_4 are zero (integral of odd function over symmetric interval must vanish)

$$\langle e_1, f_2\rangle = \langle e_1, f_4\rangle = 0. \tag{11}$$

By invoking equation (2), we see that

$$\langle e_1, f_1\rangle = \frac{\pi^{1/4}}{\sqrt{2}}, \tag{12}$$

$$\langle e_1, f_3\rangle = \frac{3\pi^{1/4}}{2\sqrt{2}}, \tag{13}$$

$$\langle e_1, f_5\rangle = \frac{15\pi^{1/4}}{4\sqrt{2}}. \tag{14}$$

From the above results, we see that

$$g_2 = f_2 - \langle e_0, f_2 \rangle e_0 - \langle e_1, f_2 \rangle e_1 = x^2 - \frac{1}{2}, \tag{15}$$

and

$$\|g_2\| = \sqrt{c_4 - c_2 + \frac{1}{4}c_0} = \frac{\pi^{1/4}}{\sqrt{2}} \tag{16}$$

which implies that

$$e_2 = g_2/\|g_2\| = \frac{1}{\sqrt{\sqrt{\pi}2^2 2!}}(4x^2 - 2). \tag{17}$$

Notice that e_2 only include x^2 and x^0 terms, then have (integral of odd function over symmetric interval must vanish)

$$\langle e_2, f_3 \rangle = \langle e_2, f_5 \rangle = 0. \tag{18}$$

And

$$\langle e_2, f_4 \rangle = \frac{\sqrt{2}}{\pi^{1/4}}c_6 - \frac{1}{\sqrt{2}\pi^{1/4}}c_2. \tag{19}$$

From the above results, we have

$$\begin{aligned} g_3 =& f_3 - \left\langle e_0, f_3 \right\rangle e_0 - \left\langle e_1, f_3 \right\rangle e_1 - \left\langle e_2, f_3 \right\rangle e_2 \\ =& x^3 - \frac{3}{2}x \end{aligned} \tag{20}$$

it's easily calculated that

$$\|g_3\| = \sqrt{c_6 - 3c_4 + \frac{9}{4}c_2}, \tag{21}$$

this further implies that

$$e_3 = \frac{1}{\sqrt{\sqrt{\pi}2^3 3!}}(8x^3 - 12x). \tag{22}$$

The inner product of e_3 with f_4 is zero, since it only involves x^3 and x^1 terms,

$$\langle e_3, f_4 \rangle = 0. \tag{23}$$

The inner product of e_3 with f_5 is

$$\langle e_3, f_5 \rangle = \frac{1}{\sqrt{\sqrt{\pi}2^3 3!}}(8c_8 - 12c_6). \tag{24}$$

Thus

$$\begin{aligned} g_4 =& f_4 - \left\langle e_0, f_4 \right\rangle e_0 - \left\langle e_1, f_4 \right\rangle e_1 - \left\langle e_2, f_4 \right\rangle e_2 - \left\langle e_3, f_4 \right\rangle e_3 \\ =& x^4 - 3x^2 + \frac{3}{4}. \end{aligned} \tag{25}$$

From this we see that

$$e_4 = \frac{1}{\sqrt{\pi}2^4 4!}(16x^4 - 48x^2 + 12). \tag{26}$$

Since e^4 only contain x^4, x^2, x^0 terms, the inner product of e_4 and f_5 is zero

$$\langle e_4, f_5 \rangle = 0. \tag{27}$$

Therefore

$$\begin{aligned} g_5 =& x^5 - \langle e_0, f_5 \rangle e_0 - \langle e_1, f_5 \rangle e_1 - \langle e_2, f_5 \rangle e_2 - \langle e_3, f_5 \rangle e_3 - \langle e_4, f_5 \rangle e_4 \\ =& x^5 - 5x^3 + \frac{15}{4}x \end{aligned} \tag{28}$$

From the expression, we see that

$$e_5 = \frac{1}{\sqrt{\sqrt{\pi}2^5 5!}}(32x^5 - 160x^3 + 120x). \tag{29}$$

Notice that in the above we have expressed $e_i, i = 0, \cdots, 5$ in such a way that is easy to compare with Hermitian polynomials.

• From the expression

$$H_n(x) = (-1)^n \frac{1}{w(x)} \frac{d^n}{dx^n} \left(B(x)^n w(x)\right) \tag{30}$$

with $w(x) = e^{-x^2}$, we see that

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{31}$$

From this we easily see that

$$H_0(x) = 1, \tag{32}$$
$$H_1(x) = 2x, \tag{33}$$
$$H_2(x) = 4x^2 - 2, \tag{34}$$
$$H_3(x) = 8x^3 - 12x, \tag{35}$$
$$H_4(x) = 16x^4 - 48x^2 + 12, \tag{36}$$
$$H_5(x) = 32x^5 - 160x^3 + 120x. \tag{37}$$

• Using the expressions of e_i and H_i, we obtain

$$H_0(x)/e_0(x) = \sqrt{\sqrt{\pi}2^0 0!}, \tag{38}$$
$$H_1(x)/e_1(x) = \sqrt{\sqrt{\pi}2^1 1!}, \tag{39}$$
$$H_2(x)/e_2(x) = \sqrt{\sqrt{\pi}2^2 2!}, \tag{40}$$
$$H_3(x)/e_3(x) = \sqrt{\sqrt{\pi}2^3 3!}, \tag{41}$$
$$H_4(x)/e_4(x) = \sqrt{\sqrt{\pi}2^4 4!}, \tag{42}$$
$$H_5(x)/e_5(x) = \sqrt{\sqrt{\pi}2^5 5!}. \tag{43}$$

From which we see that $H_n(x)/e_n(x) = \sqrt{\sqrt{\pi}2^n n!} = \|H_n(x)\|$, namely, $\{e_n(x)\}$ is the normalized basis of $L^2_w(-\infty, \infty)$ corresponding to $\{H_n(x)\}$.

• For n-th order derivative $\frac{d^n}{dx^n}$, using Leibniz's rule and proof by induction, we have

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} C_n^k (\frac{d^k}{dx^k} f)(\frac{d^{n-k}}{dx^{n-k}} g) \tag{44}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

Now consider $\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$, we can regard $f = -2x$ and $g = e^{-x^2}$. By invoking equation (58) and noticing the higher order derivatives $f^{(2)} = f^{(3)} = \cdots = f^{(n)} = 0$, we obtain

$$\frac{d^n}{dx^n}\frac{d}{dx}e^{-x^2} = C_n^0 f \frac{d^n}{dx^n} g + C_n^1 (\frac{d}{dx} f)(\frac{d^{n-1}}{dx^{n-1}} g) = -2x \frac{d^n}{dx^n} e^{-x^2} - 2n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}. \tag{45}$$

From the above result, we have

$$\begin{aligned}H_{n+1}(x) &= (-1)^{n+1}e^{x^2}\frac{d^n}{dx^n}\left(\frac{d}{dx}e^{-x^2}\right)\\ &= (-1)^{n+1}e^{x^2}\left(-2x\frac{d^n}{dx^n}e^{-x^2}-2n\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\right)\\ &=2x(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}-2n(-1)^{n-1}e^{x^2}\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\\ &=2xH_n(x)-2nH_{n-1}(x)\end{aligned} \tag{46}$$

This completes the proof.

• By direct calculation

$$\begin{aligned}\frac{d}{dx}H_n(x) &=(-1)^n\frac{d}{dx}(e^{x^2}\frac{d^n}{dx^n}e^{-x^2})\\ &=2x(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}-(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2}\\ &=2xH_n(x)-H_{n+1}(x)=2nH_{n-1}(x).\end{aligned} \tag{47}$$

Notice that in the last step we have used equation (46).

• The differential equation is

$$\frac{d^2}{dx^2}\psi_n(x)+(2n+1-x^2)\psi_n(x)=0 \tag{48}$$

Let's check it.

$$\frac{d}{dx}\psi_n(x)=-xe^{-x^2/2}H_n(x)+e^{-x^2/2}H_n'(x). \tag{49}$$

Then from this and $H_n''-2xH_n'+2nH_n=0$ we have

$$\begin{aligned}\frac{d^2}{dx^2}\psi_n(x) &= -e^{x^2/2}H_n(x)+x^2e^{-x^2/2}H_n(x)+e^{-x^2/2}(-2xH_n'(x)+H_n''(x))\\ &= -\psi_n(x)+x^2\psi_n(x)-2n\psi_n(x)\\ &= [x^2-(2n+1)]\psi_n(x).\end{aligned} \tag{50}$$

This completes the proof. □

Problem 2 solution:

• Since $L^2[-1,1]$ is a Hilbert space, renormalized Legendre polynomials $\{e_i\}_{i=0}^{\infty}$ consist the complete othornamal basis of the space. From the result of the functional analysis we know that the best approximation is given by the one which makes

$$\left\|f-\sum_{i=0}^{4}c_ie_i\right\|^2=\inf\operatorname{dist}(f,\operatorname{span}(e_0,e_1,\cdots,e_4))=\sum_{i=5}^{\infty}\langle e_i,f\rangle^2. \tag{51}$$

This is given by

$$c_i=\langle e_i,f\rangle,\quad i=0,1,\cdots,4. \tag{52}$$

By direct calculation, we have

$$\begin{aligned}c_0&=0\\ c_1&=0\\ c_2&=-\frac{6}{\pi^2}\sqrt{\frac{5}{2}}\\ c_3&=0\\ c_4&=-\frac{30(2\pi^2-21)}{\sqrt{2}\pi^4}.\end{aligned} \tag{53}$$

• By direction calculation, we have

$$\left\| f - \sum_{i=0}^{4} c_i e_i \right\|^2 = \frac{36\pi^4\left(\sqrt{10}+151\right)+\pi^8-113400\pi^2+595350}{\pi^8} \simeq 2.76 \tag{54}$$

$$\|f-g\|^2 = -5+\frac{2\pi^4}{15}-\frac{\pi^6}{84}+\frac{\pi^8}{2592} \simeq 0.203468 \tag{55}$$

Taylor approximation gets more accurate result, this is because that in a Hilbert space with a fixed basis, the distance between projection of a function f into a subspace may be very large, the value is $\|f^{\perp}\|$, this may be very large.

□

Problem 3 solution:
• Since $u(x) = B(x)^n w(x)$, we have

$$B(x)u'(x) = nB'(x)u(x) + A(x)u(x) = G_n(x)u(x) \tag{56}$$

this implies that

$$G_n(x) = nB'(x) + A(x). \tag{57}$$

The degree of G is at most 1.
• To do this, we recall the n-th order Lebniz rule

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} C_n^k \left(\frac{d^k}{dx^k} f\right)\left(\frac{d^{n-k}}{dx^{n-k}} g\right) \tag{58}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. Remember that the order of $B(x)$ is at most 2, the order of $A(x)$ is at most 1 and the order of $G_n(x)$ is at most 1. We obtain

$$\frac{d^{n+1}}{dx^{n+1}}(B(x)u'(x)) = B\frac{d^{n+2}}{dx^{n+2}}u + (n+1)B'\frac{d^{n+1}}{dx^{n+1}}u + \frac{n(n+1)}{2}B''\frac{d^n}{dx^n}u \tag{59}$$

$$\frac{d^{n+1}}{dx^{n+1}}(G_n(x)u(x)) = G_n(x)\frac{d^{n+1}}{dx^{n+1}}u(x) + (n+1)G_n'(x)\frac{d^n}{dx^n}u(x). \tag{60}$$

From the equalities we obtain

$$H(x) = (n+1)B' - G_n = B' - A, \quad J(x) = -(n+1)A' - \frac{n(n+1)}{2}B''. \tag{61}$$

• From the expression

$$\frac{d^n}{dx^n}u(x) = \frac{w(x)f_n(x)}{c_n}, \tag{62}$$

we have

$$\frac{d^{n+1}}{dx^{n+1}}u(x) = \frac{w'f_n + wf_n'}{c_n}, \tag{63}$$

$$\frac{d^{n+2}}{dx^{n+2}}u(x) = \frac{w''f_n + 2w'f_n' + wf_n'}{c_n}. \tag{64}$$

Notice that

$$w' = wA/B, \quad w'' = \frac{wA^2 + wA'B - wAB'}{B^2}. \tag{65}$$

Substituting these expressions into

$$B(x)\frac{d^{n+2}}{dx^{n+2}}u(x) + H(x)\frac{d^{n+1}}{dx^{n+1}}u(x) + J(x)\frac{d^n}{dx^n}u(x) = 0 \tag{66}$$

we have

$$Bwf_n'' + (2Bw' + Hw)f_n' + (Bw'' + Hw' + Jw)f_n = 0. \tag{67}$$

$$\iff Bf_n'' + (2A + H)f_n' + (\frac{A^2 + A'B - AB'}{B} + H\frac{A}{B} + J)f_n = 0 \tag{68}$$

$$\iff Bf_n''(x) + \left(A + B'\right) f_n'(x) - \frac{1}{2}n\left(2A' + (n+1)B''\right) f_n(x) = 0 \tag{69}$$

which completes the proof. □

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