Problem 1 Describe and prove the no-cloning theorem.

Solution: The no-cloning theorem states that it is impossible to perfectly clone an unknown quantum state using unitary evolution. The proof is as follows, suppose that we have such a unitary operator U such that for arbitrary state $|\phi\rangle$, we have $U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$. Then consider two different and un-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$, we have

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle, \tag{1}$$
$$U(|\varphi\rangle \otimes |0\rangle) = |\varphi\rangle \otimes |\varphi\rangle. \tag{2}$$

Taking the inner produce of left hand sides and right hand sides of the above two equalities, we obtain

$$(\langle\psi| \otimes \langle 0|)U^{\dagger}U(|\varphi\rangle \otimes |0\rangle) = (\langle\psi| \otimes \langle\psi|)(|\varphi\rangle \otimes |\varphi\rangle), \tag{3}$$

equivalently, we have

$$\langle\psi|\varphi\rangle = (\langle\psi|\varphi\rangle)^2. \tag{4}$$

Notice that in the above we have used the assumption that U is unitary. This implies that $\langle\psi|\varphi\rangle$ equals 0 or 1, i.e., ψ and φ are the same or orthogonal, which is contradict with the assumption of $|\psi\rangle$ and $|\varphi\rangle$, thus such U does not exist. ■

Problem 2 Prove that non-orthogonal states can't be reliably distinguished.

Solution: This can be proved using proof by contradiction. To this end, let's first clarify that here by the word '*reliably*' we means that the experimenter choose a measurement M_j to measure and according to the measurement outcome, he guess the index of given state ψ_i using the guess function $f(j) = i$ with success probability 1. Now suppose the experimenter is given two nonorthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. Suppose the measurement $\mathcal{M} = \{M_1, \cdots, M_n\}$ which can reliably distinguish these two states is possible. If the state $|\psi_1\rangle$ (resp. $|\psi_2\rangle$) is prepared then the probability of measuring j such that $f(j) = 1$(resp. $f(j) = 2$) must be 1. Defining $E_i \equiv \sum_{j:f(j)=i} M_j^{\dagger} M_j$, these observations may be written as:

$$\langle\psi_1 |E_1| \psi_1\rangle = 1; \quad \langle\psi_2 |E_2| \psi_2\rangle = 1. \tag{5}$$

From the completeness of the measurement $\sum_i E_i = I$ it follows that $\sum_i \langle\psi_1 |E_i| \psi_1\rangle = 1$, and since $\langle\psi_1 |E_1| \psi_1\rangle = 1$ we must have $\langle\psi_1 |E_2| \psi_1\rangle = 0$, and thus $\sqrt{E_2}\,|\psi_1\rangle = 0$. Suppose we decompose $|\psi_2\rangle = \alpha\,|\psi_1\rangle + \beta|\varphi\rangle$, where $|\varphi\rangle$ is orthonormal to $|\psi_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, and $|\beta| < 1$ since $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal. Then $\sqrt{E_2}\,|\psi_2\rangle = \beta\sqrt{E_2}|\varphi\rangle$, which implies a contradiction with (5) as

$$\langle\psi_2 |E_2| \psi_2\rangle = |\beta|^2 \langle\varphi |E_2| \varphi\rangle \leq |\beta|^2 < 1 \tag{6}$$

where the second last inequality follows from the observation that

$$\langle\varphi |E_2| \varphi\rangle \leq \sum_i \langle\varphi |E_i| \varphi\rangle = \langle\varphi \mid \varphi\rangle = 1. \tag{7}$$

This completes the proof. ■

Problem 3 Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y and Z

Solution: Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

- Firstly consider Z, since Z is diagonal, it's obvious that

$$\lambda_Z = \pm 1, \tag{9}$$

Problem 3 continued on next page...

and the corresponding eigenvectors are

$$|Z+\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |Z-\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10}$$

The diagonal representation of Z is

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{11}$$

• For X, we have

$$\det(X - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \tag{12}$$

So the eigenvalues of X are $\{1, -1\}$. The eigenvector corresponding to $\lambda = 1$ is

$$X\psi = \psi \Longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{13}$$

This implies $\alpha = \beta$, Thus the normalized eigenvector (up to an arbitrary phase factor) is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The eigenvector corresponding to $\lambda = -1$ is

$$X\psi = -\psi \Longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \tag{14}$$

This implies that $\alpha = -\beta$. Thus the normalized eigenvector (up to an arbitrary phase factor) is $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ Then, the diagonal representation of X is given by

$$X = |+\rangle\langle +| - |-\rangle\langle -| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) - \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|). \tag{15}$$

• Now consider Y,

$$\det(Y - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \tag{16}$$

So the eigenvalues of Y are $\{1, -1\}$. The eigenvector corresponding to $\lambda = 1$ is

$$Y\psi = \psi \Longrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longleftrightarrow \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{17}$$

This implies that $\alpha = -i\beta$. Thus the normalized eigenvector up to an overall phase is $|Y+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. The eigenvector corresponding to $\lambda = -1$ is

$$Y\psi = -\psi \Longrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \Longleftrightarrow \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \tag{18}$$

This implies that $\alpha = i\beta$. Thus the normalized eigenvector is $|Y-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Then, the diagonal representation of Y is given by

$$Y = |Y+\rangle\langle Y+| - |Y-\rangle\langle Y-| = \frac{1}{2}(|0\rangle + i|1\rangle)(\langle 0| + i\langle 1|) - \frac{1}{2}(|0\rangle - i|1\rangle)(\langle 0| - i\langle 1|) \tag{19}$$

This completes the problem. ■

Problem 4 Write down the commutation relations and anti-commutation relations for the Pauli matrices and prove them.

Solution: The commutation relations:

$$\left[\sigma_i, \sigma_j\right] = 2i\epsilon_{ijk}\sigma_k \tag{20}$$

the anti-commutation relations:

$$\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}I$$

where $i, j, k = 1, 2, 3$

In fact, we have

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k. \tag{21}$$

This can be proved by direction calculation using the expression of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

by easy matrix calculation, we have

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = I. \tag{23}$$

And

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3 \tag{24}$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \sigma_1 \tag{25}$$
$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = \sigma_2 \tag{26}$$

This can gives the commutation and anti-commutation relations. Alternatively, we can prove it using expression (21).

$$\begin{aligned}[\sigma_i, \sigma_j] =& \sigma_i\sigma_j - \sigma_j\sigma_i = (\delta_{ij}I + i\epsilon_{ijk}\sigma_k) - (\delta_{ji}I + i\epsilon_{jik}\sigma_k) \\ =& i\epsilon_{ijk}\sigma_k - i\epsilon_{jik}\sigma_k = i\epsilon_{ijk}\sigma_k + i\epsilon_{ijk}\sigma_k = 2i\epsilon_{ijk}\sigma_k\end{aligned} \tag{27}$$

where we have used $\delta_{ij} = \delta_{ji}$ and $\epsilon_{jik} = -\epsilon_{ijk}$. Similarly, using expression (21), we have

$$\begin{aligned}\{\sigma_i, \sigma_j\} =& \sigma_i\sigma_j + \sigma_j\sigma_i = (\delta_{ij}I + i\epsilon_{ijk}\sigma_k) + (\delta_{ji}I + i\epsilon_{jik}\sigma_k) \\ =& 2\delta_{ij}I\end{aligned} \tag{28}$$

where we have used $\delta_{ij} = \delta_{ji}$ and $\epsilon_{jik} = -\epsilon_{ijk}$. ■

Problem 5 Prove the Cauchy-Schwarz inequality that for any two vectors $|v\rangle$ and $|w\rangle$, $|\langle v \mid w\rangle|^2 \leq \langle v \mid v\rangle\langle w \mid w\rangle$

Solution: Consider two vectors $|v\rangle = (v_1, \cdots, v_n)^T$ and $|w\rangle = (w_1, \cdots, w_n)^T$, we see that

$$\langle w|w\rangle = \|w\|^2 = \sum_i w_i^* w_i, \quad \langle v|v\rangle = \|v\|^2 = \sum_i v_i^* v_i, \quad \langle v|w\rangle = \sum_i v_i^* w_i. \tag{29}$$

If $\|v\| = 0$, the inequality trivially holds, namely $0 \leq 0$. Therefore, we assume that $\|v| \neq 0$, then we can construct a vector $w - \frac{\langle v|w\rangle}{\|v\|^2}v$, the norm of this vector is nonnegative

$$\begin{aligned}0 \leq \left\| w - \frac{\langle v|w\rangle}{\|v\|^2}v \right\| =& \langle w|w\rangle - \frac{|\langle v|w\rangle|^2}{\|v\|^2} - \frac{|\langle v|w\rangle|^2}{\|v\|^2} + \frac{|\langle v|w\rangle|^2}{\|v\|^4}\|v\|^2 \\ =& \langle w|w\rangle - \frac{|\langle v|w\rangle|^2}{\|v\|^2}.\end{aligned} \tag{30}$$

This implies that

$$\|v\|^2\langle w|w\rangle \geq |\langle v|w\rangle|^2. \tag{31}$$

and asserted inequality follows.

Problem 6 Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}$$

where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$

Solution: From defining properties of Pauli matrices

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k. \tag{32}$$

we see that

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \tag{33}$$

This further implies that $(\vec{v}\cdot\vec{\sigma})^2 = I$. From Taylor series expansion for matrices

$$\exp(iA) = \sum_{n=0}^{\infty}\frac{(iA)^n}{n!} \tag{34}$$

and noticing that for interger $k \geq 0$

$$(i\theta\vec{v}\cdot\vec{\sigma})^{2k+1} = i(-1)^k\theta^{2k+1}(\vec{v}\cdot\vec{\sigma}) \quad (i\theta\vec{v}\cdot\vec{\sigma})^{2k} = (-1)^k\theta^{2k}I \tag{35}$$

we can get

$$\begin{aligned}\exp(i\theta\vec{v}\cdot\vec{\sigma}) &= \sum_{n=0}^{\infty}\frac{(i\theta\vec{v}\cdot\vec{\sigma})^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}I}{2k!} + \sum_{k=0}^{\infty}\frac{i(-1)^k\theta^{2k+1}(\vec{v}\cdot\vec{\sigma})}{(2k+1)!}\\ &= \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}.\end{aligned} \tag{36}$$

This completes the proof. ■

Problem 7 Prove that for any 2-dimension linear operator A,

$$A = \frac{1}{2}\operatorname{Tr}(A)I + \frac{1}{2}\sum_{k=1}^{3}\operatorname{Tr}(A\sigma_k)\sigma_k$$

in which $\sigma_k(k = 1, 2, 3)$ are Pauli matrices.

Solution: Recall that σ_μ (where $\sigma_0 = I$ and $\sigma_k(k = 1, 2, 3)$ are Pauli matrices) form the basis of complex vector space $M_2(\mathbb{C})$ of 2×2 matrices. This means that we can write $A = \sum_{\mu=0}^{3} a_\mu\sigma_\mu$ with some complex coefficients a_μ. Then

$$A\sigma_k = \sum_{\mu=0}^{3} a_\mu\sigma_\mu\sigma_k \tag{37}$$

since

$$\operatorname{Tr}\left(\sigma_\mu\sigma_k\right) = 2\delta_{\mu k} \tag{38}$$

then

$$\operatorname{Tr}(A\sigma_k) = \sum_{\mu=0}^{3}\operatorname{Tr}\left(a_\mu\sigma_\mu\sigma_k\right) = \sum_{\mu=0}^{3} 2a_\mu\delta_{\mu k} = 2a_k \tag{39}$$

Therefore

$$a_k = \frac{1}{2}\operatorname{Tr}(A\sigma_k) \tag{40}$$

From which we obtain

$$A = \frac{1}{2}\operatorname{Tr}(A)I + \frac{1}{2}\sum_{k=1}^{3}\operatorname{Tr}(A\sigma_k)\sigma_k \tag{41}$$

this completes the proof. ■

Problem 8 If A and B are two linear operators, show that

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$$

Solution: This is almost obvious from direct calculation

$$\operatorname{Tr}(AB) = \sum_i \sum_j a_{ij} b_{ji} = \sum_i \sum_j b_{ij} a_{ji} = \operatorname{Tr}(BA)$$

This completes the proof. ■

Problem 9 Let ρ be a density operator.

(1) Show that ρ can be written as

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2}$$

where r is a real three-dimensional vector such that $\|\boldsymbol{r}\| \leq 1$

(2) Show that $\operatorname{Tr}\left(\rho^2\right) \leq 1$, with equality if and only if ρ is a pure state.

(3) Show that a state ρ is a pure state if and only if $\|r\| = 1$.

Solution:

(1) Since σ_μ with $\mu = 0, 1, 2, 3$ form a basis of complex vector space of 2×2 matrices, we have

$$\rho = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3, \tag{42}$$

for complex numbers x_μ. Since ρ is Hermitian, x_μ are real numbers. Since $\operatorname{Tr}\sigma_0 = 2$ and $\operatorname{Tr}\sigma_i = 0$ for $i = 1, 2, 3$, we obtain

$$\operatorname{Tr}(\sigma_\nu\sigma_\mu) = 2\delta_{\nu\mu}. \tag{43}$$

From which we have $x_\mu = \frac{1}{2}\operatorname{Tr}(\rho\sigma_\mu)$, this directly implies that

$$x_0 = 1/2, \tag{44}$$

and we defined

$$\boldsymbol{r} = (2x_1, 2x_2, 2x_3)^T = (\operatorname{Tr}(\rho\sigma_1), \operatorname{Tr}(\rho\sigma_2), \operatorname{Tr}(\rho\sigma_3))^T. \tag{45}$$

Taking trace of $\operatorname{Tr}(\rho\sigma_\mu)$, we arrive at

$$x_0 = \frac{1}{2}, \tag{46}$$

thus we get

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}. \tag{47}$$

To show that $\|\boldsymbol{r}\| \leq 1$, notice that ρ is positive semidefinite, thus $\det\rho \geq 0$, namely

$$\det\frac{1}{2}\begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix} = \frac{1}{4}(1 - \|\boldsymbol{r}\|^2) \geq 0. \tag{48}$$

This implies that $\|\boldsymbol{r}\| \leq 1$.

(2) Suppose that $\rho = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$, then

$$\begin{aligned}
\rho^2 &= \sum_i p_i|\phi_i\rangle\langle\phi_i| \sum_j p_j|\phi_j\rangle\langle\phi_j| \\
&= \sum_{i,j} p_i p_j|\phi_i\rangle\langle\phi_i \mid \phi_j\rangle\langle\phi_j| \\
&= \sum_i p_i^2 \left|\phi_i\right\rangle \left\langle\phi_i\right|
\end{aligned} \tag{49}$$

then

$$\begin{aligned}\operatorname{Tr}\left(\rho^2\right) &= \operatorname{Tr}\left(\sum_i p_i^2 \left|\phi_i\right\rangle\left\langle\phi_i\right|\right) \\ &= \operatorname{Tr}\left(\sum_i p_i^2 \left\langle\phi_i \mid \phi_i\right\rangle\right) \\ &= \sum_i p_i^2\end{aligned} \tag{50}$$

since $\sum_i p_i = 1$, then $\sum_i p_i^2 \leq \sum_i p_i = 1$ then

$$\operatorname{Tr}\left(\rho^2\right) \leq 1 \tag{51}$$

with equality hold if and only if

$$\begin{cases} p_j = 1, \\ p_{i\neq j} = 0 \end{cases} \tag{52}$$

which means that ρ is a pure state.

(3) This can be proved using the criterion we prove in (2) Notice that

$$\rho = \frac{I + \boldsymbol{r}\cdot\boldsymbol{\sigma}}{2} \tag{53}$$

then

$$\operatorname{Tr}\left(\rho^2\right) = \frac{1}{4}\operatorname{Tr}\left(I + 2\boldsymbol{r}\cdot\boldsymbol{\sigma} + 2\|\boldsymbol{r}\|^2\right) \tag{54}$$

By involking

$$\operatorname{Tr}(I) = 2, \operatorname{Tr}\left(\sigma_i\right) = 0, (i = 1, 2, 3) \tag{55}$$

we get that

$$\operatorname{Tr}\left(\rho^2\right) = \frac{1}{2}\left(1 + \|\boldsymbol{r}\|^2\right). \tag{56}$$

Since ρ is a pure state if and only if

$$\operatorname{Tr}\left(\rho^2\right) = 1 \tag{57}$$

this implies that

$$\|r\|^2 = 1. \tag{58}$$

Thus we complete the proof. ■

Problem 10 $\rho_A = \frac{I+\boldsymbol{n}_A\cdot\boldsymbol{\sigma}}{2}, \rho_B = \frac{I+\boldsymbol{n}_B\cdot\boldsymbol{\sigma}}{2}$, prove that $\operatorname{Tr}\left(\rho_A\rho_B\right) = \frac{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}{2}$.

Solution: Firstly, Recall that

$$(\boldsymbol{n}_A\cdot\cdot\boldsymbol{\sigma})(\boldsymbol{n}_B\cdot\boldsymbol{\sigma}) = \boldsymbol{n}_A\cdot\boldsymbol{n}_B + i(\boldsymbol{n}_A\times\boldsymbol{n}_B)\cdot\boldsymbol{\sigma}. \tag{59}$$

Thus, we have

$$\rho_A\rho_B = \frac{1}{4}\left(I + \boldsymbol{n}_A\cdot\sigma + \boldsymbol{n}_B\cdot\sigma + \boldsymbol{n}_A\cdot\boldsymbol{n}_B I + i(\boldsymbol{n}_A\times\boldsymbol{n}_B)\cdot\boldsymbol{\sigma}\right) \tag{60}$$

Since $\operatorname{Tr}\left(\sigma_i\right) = 0$, $\operatorname{Tr}(I) = 2$, we have

$$\operatorname{Tr}\left(\rho_A\rho_B\right) = \frac{1 + \boldsymbol{n}_A\cdot\boldsymbol{n}_B}{2} \tag{61}$$

This completes the proof. ■

Problem 11 Consider an experiment, in which we prepare the state $|0\rangle$ with the probability $|C_0|^2$, and the state $|1\rangle$ with the probability $|C_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$

Solution: The first case is that this state is a classical mixture, i.e., mixd state, whose density matrix is

$$\rho_C = |C_0|^2 |0\rangle\langle 0| + |C_1|^2 |1\rangle\langle 1| = \begin{pmatrix} |C_0|^2 & 0 \\ 0 & |C_1|^2 \end{pmatrix} \tag{62}$$

There is no coherence of $|0\rangle$ and $|1\rangle$ in this state, when we measurement an observable A we obtain $\langle A\rangle_C = |C_0|^2 \langle 0|A|0\rangle + |C_1|^2 \langle 1|A|1\rangle$.

The second case is The state $|\psi\rangle = C_0|0\rangle + C_1 e^{i\theta}|1\rangle$ is a pure state, whose density matrix is

$$\begin{aligned} \rho_Q = |\psi\rangle\langle\psi| &= |C_0|^2 |0\rangle\langle 0| + |C_1|^2 |1\rangle\langle 1| + C_0 C_1 e^{-i\theta}|0\rangle\langle 1| + C_0 C_1 e^{i\theta}|1\rangle\langle 0| \\ &= \begin{pmatrix} |C_0|^2 & C_0 C_1 e^{-i\theta} \\ C_0 C_1 e^{i\theta} & |C_1|^2 \end{pmatrix} \end{aligned} \tag{63}$$

The $|0\rangle$ and $|1\rangle$ are in a coherent superposition, there is off-diagonal terms of the density matrix. In this case we measure an observable A we will find that $\langle\psi|A|\psi\rangle \neq |C_0|^2 \langle 0|A|0\rangle + |C_1|^2 \langle 1|A|1\rangle$, the appearance of the cross-term is a result of the coherent superposition. Thus, we see that, when measuring these two states in $\{|0\rangle, |1\rangle\}$ basis, the probabilities we get $|0\rangle$ and $|1\rangle$ are same; if other basis is used, the probabilities are different. ■

Problem 12 Please prepare the polarized optical quantum state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$ from an initial state $|0\rangle$, with half wave plate and quarter-wave plate. To implement arbitrary single qubit unitary transformation, how many wave plates are at least needed, and how to perform them?

Solution:

Before doing this let's first recall some basic concepts from linear optics. From Maxwell equation we known that for electromagnetic wave, $\vec{E} \perp \vec{k}$, if the light go along z-axis, then $\vec{k} = (0, 0, k)$ and $\vec{E} = (E_x, E_y, 0)$. A general plane-wave solution of Maxwell equation is then

$$\vec{E} = \vec{A} e^{ikz - i\omega t} \tag{64}$$

where $\vec{A} = (A_x, A_y, 0)$. Thus we can focus on a two-dimensional vector known as Jones vector

$$\vec{J} = \left(A_x, A_y\right) \tag{65}$$

If we set $|0\rangle = \vec{J}_x = (1, 0)^T$ and $|1\rangle = \vec{J}_y = (0, 1)^T$ we see that the general polarization state of the optic is of the form

$$\vec{E} = \|\vec{A}\| \left(a|0\rangle + b e^{i\delta}|1\rangle\right) e^{ikz - i\omega t} \tag{66}$$

where the corresponding Jones vector is

$$\vec{J} = a|0\rangle + b e^{i\delta}|1\rangle, \quad |a|^2 + |b|^2 = 1. \tag{67}$$

The operations of optic devices can then be modeled as a 2×2 matrix, known as Jones matrix, namely

$$\vec{J}_{out} = U_{device} \vec{J}_{in}. \tag{68}$$

We see that this matches well with qubit and its unitary operations.

The Jones matrix (up to an overall phase factor) of a retardation plate with the fast axis at θ to the x-axis is given by

$$\begin{aligned} U_\delta(\theta) &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta) + e^{-i\delta}\sin^2(\theta) & \cos(\theta)\sin(\theta) - e^{-i\delta}\cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) - e^{-i\delta}\cos(\theta)\sin(\theta) & e^{-i\delta}\cos^2(\theta) + \sin^2(\theta) \end{pmatrix}. \end{aligned} \tag{69}$$

Problem 12 continued on next page...

Then we see that the operations of half wave plate and quarter-wave plate with the fast axis at θ to the x-axis on a given polarization state are given as the following:

- A half wave plate rotates the state vector by $\delta = \pi$. The operator reads

$$U_\pi(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \tag{70}$$

- A quarter-wave plate rotates the state vector by $\delta = \frac{\pi}{2}$. The operator reads

$$U_{\frac{\pi}{2}}(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i\cos(2\theta) & i\sin(2\theta) \\ i\sin(2\theta) & 1 - i\cos(2\theta) \end{pmatrix} \tag{71}$$

Claim: To realize a general 2×2 unitary matrix U up to a global phase factor, two quarter-wave plate and one half wave plate are needed.

To prove this, suppose that

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{72}$$

notice that the rows and columns of U are unit vectos, let $a = e^{\theta_a} \cos\varphi$, then $b = e^{i\theta_b} \sin\varphi$ and $c = e^{i\theta_c} \sin\varphi$, this further implies that $d = e^{i\theta_d} \cos\varphi$. Here $\theta_a, \cdots, \theta_d$ are some real parameters.

Two rows and two columns are orthogonal respectively, therefore, $a^*c + b^*d = 0$ and $a^*b + c^*d = 0$. This implies that when $\varphi \neq \frac{k\pi}{2}$ we must have

$$e^{\theta_c - \theta_a} + e^{\theta_d - \theta_b} = 0, \quad e^{\theta_b - \theta_a} + e^{\theta_d - \theta_c} = 0. \tag{73}$$

Now suppose that

$$U = U_{\frac{\pi}{2}}(\gamma) U_\pi(\beta) U_{\frac{\pi}{2}}(\alpha) \tag{74}$$

we obtain a system of equation

$$a = e^{i\theta_a} \cos\varphi = \cos(\alpha - \gamma)\cos(\alpha - 2\beta + \gamma) - i\sin(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) \tag{75}$$
$$b = e^{i\theta_b} \sin\varphi = \cos(\alpha - 2\beta + \gamma)\sin(\alpha - \gamma) + i\cos(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) \tag{76}$$
$$c = e^{i\theta_c} \sin\varphi = i\cos(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) - \cos(\alpha - 2\beta + \gamma)\sin(\alpha - \gamma) \tag{77}$$
$$d = e^{i\theta_d} \cos\varphi = \cos(\alpha - \gamma)\cos(\alpha - 2\beta + \gamma) + i\sin(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) \tag{78}$$

It's easily checked that we can choose α, β, γ such that these equation have a solution and conditions (73) is satisfied. This completes the proof of our claim.

To prepare the state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$, perform the unitary transformation $U = e^{i\delta} U_{\frac{\pi}{2}}(\gamma) U_\pi(\beta) U_{\frac{\pi}{2}}(\alpha)$ on the state $|0\rangle$. By solving the equations

$$\begin{pmatrix} C_0 \\ C_1 e^{i\theta} \end{pmatrix} = e^{i\delta} U_{\frac{\pi}{2}}(\gamma) U_\pi(\beta) U_{\frac{\pi}{2}}(\alpha) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{79}$$

namely,

$$\begin{pmatrix} C_0 \\ C_1 e^{i\theta} \end{pmatrix} = e^{i\delta} \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\alpha - \gamma)\cos(\alpha - 2\beta + \gamma) - i\sin(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) \\ \cos(\alpha - 2\beta + \gamma)\sin(\alpha - \gamma) + i\cos(\alpha + \gamma)\sin(\alpha - 2\beta + \gamma) \end{pmatrix} \tag{80}$$

we can obtain the corresponding angles between the fast axis to the x-axis of the quarter-wave plates and half wave plate. ■

Problem 13 Suppose a two particle pure state is of the form $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$

(1) Calculate the reduced density matrices ρ_A and ρ_B.

(2) Do Schmidt decomposition.

Solution:

(1) This can be done by direct calculation.

$$\begin{aligned}\rho_A &= \frac{1}{8}\{(|0\rangle + \sqrt{3}|1\rangle)(\langle 0| + \sqrt{3}\langle 1|) + (\sqrt{3}|0\rangle + |1\rangle)(\sqrt{3}\langle 0| + \langle 1|)\} \\ \rho_B &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0|\end{aligned} \tag{81}$$

(2)To do Schmidt decomposition, we need calculate the eigenvalue and eigenstates of two reduced density matrices. Notice that the eigenvalues of two reduced density matrices are the same, i.e., $\lambda_1 = \lambda_1^A = \lambda_1^B$ and $\lambda_2 = \lambda_2^A = \lambda_2^B$.

- For ρ_A, $\lambda_{1,2}^A = \frac{1}{4}(2 \pm \sqrt{3})$, then get eigenvectors.

$$\begin{aligned}|\lambda_1\rangle_A &= \left|\lambda_1^A\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\lambda_2\rangle_A &= \left|\lambda_2^A\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned}$$

- For ρ_B, $\lambda_{1,2}^B = \frac{1}{4}(2 \pm \sqrt{3})$, the corresponding eigenstates are

$$\begin{aligned}|\lambda_1\rangle_B &= \left|\lambda_1^B\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\lambda_2\rangle_B &= \left|\lambda_2^B\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned} \tag{82}$$

Then, using the Schmidt theorem we know that

$$\begin{aligned}|\Phi\rangle = \sqrt{\lambda_1}\,|\lambda_1\rangle_A\,|\lambda_1\rangle_B + \sqrt{\lambda_1}\,|\lambda_2\rangle_A\,|\lambda_2\rangle_B =& \frac{\sqrt{2+\sqrt{3}}}{2}\,|\lambda_1\rangle_A\,|\lambda_1\rangle_B + \frac{\sqrt{2-\sqrt{3}}}{2}\,|\lambda_2\rangle_A\,|\lambda_2\rangle_B \\ =& \frac{1+\sqrt{3}}{2\sqrt{2}}\,|\lambda_1\rangle_A\,|\lambda_1\rangle_B + \frac{1-\sqrt{3}}{2\sqrt{2}}\,|\lambda_2\rangle_A\,|\lambda_2\rangle_B\,.\end{aligned} \tag{83}$$

This completes the solution. ■

Problem 14 Prove that suppose $|\psi\rangle$ is a pure state of a composite system, AB. Then there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i\,|i_A\rangle\,|i_B\rangle$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients.

Solution:

Suppose that the orthonormal basis $\{|\tilde{e}_A^i\rangle\}$ of $\mathcal{H}_A$ is chose such that $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ is diagonal: $\rho_A = \sum_{i=1}^{d_A} p_i|e_A^i\rangle\langle e_A^i|$, with the orthonormal basis of $\mathcal{H}_B$ is chose as $\{|e_B^j\rangle\}$. Assume that in this basis we

Problem 14 continued on next page...

have $|\psi_{AB}\rangle = \sum_{i,j} T_{ij}|\tilde{e}^i_A\rangle \otimes |e^j_B\rangle$, then

$$\begin{aligned}|\psi_{AB}\rangle &= \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} T_{ij}|\tilde{e}^i_A\rangle \otimes |e^j_B\rangle, \\ &= \sum_{i=1}^{d_A} |\tilde{e}^i_A\rangle \otimes (\sum_{j=1}^{d_B} T_{ij}|e^j_B\rangle), \\ &= \sum_{i=1}^{d_A} |\tilde{e}^i_A\rangle \otimes |e'^i_B\rangle.\end{aligned} \tag{84}$$

Where $|e'^i_B\rangle = \sum_{j=1}^{d_B} T_{ij}|e^j_B\rangle$. Note that we still need to prove states $\{|e'^i_B\rangle\}$ are mutually orthogonal or normalized. Since reduced density matrix

$$\begin{aligned}\sum_{i=1}^{d_A} p_i|e^i_A\rangle\langle e^i_A| = \rho_A = \mathrm{Tr}_B\rho_{AB}, \\ = \sum_{i,i'=1}^{d_A} |\tilde{e}^i_A\rangle\langle \tilde{e}^{i'}_A|\mathrm{Tr}_B(|e'^i_B\rangle\langle e'^{i'}_B|).\end{aligned} \tag{85}$$

Hence, $\mathrm{Tr}_B|e'^i_B\rangle\langle e'^{i'}_B| = \langle e'^{i'}_B|e'^i_B\rangle = p_i\delta_{ii'}$, let $|\tilde{e}^i_B\rangle = \frac{1}{\sqrt{p_i}}|e'^i_B\rangle$, it turns out that $\{|\tilde{e}^i_B\rangle\}$ are othonormal. This implies that

$$|\psi_{AB}\rangle \sum_{i=1}^{d_A} \sqrt{p_i}|\tilde{e}^i_A\rangle \otimes |\tilde{e}^i_B\rangle. \tag{86}$$

We remark that although we write the sum run over $i = 1$ to $i = d_A$, the sum actually runs over the i for which $p_i \neq 0$.

By relabeling $\lambda_i = \sqrt{p_i}$ (thus $\sum_i \lambda_i^2 = \sum_i p_i = 1$), $|i_A\rangle = |\tilde{e}^i_A\rangle$ and $|i_B\rangle = |\tilde{e}^i_B\rangle$, we see that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \tag{87}$$

This completes the proof. ■

Problem 15 Prove that suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi}_i\rangle\}$ are two sets of normalized and orthogonal states in space H then there exist a unitary transformation U, s.t. $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, and construct this U transformation.

Solution:

The unitary transformation can be constructed as

$$U = \Sigma_i |\tilde{\psi}_i\rangle\langle\psi_i| \tag{88}$$

It's easily checked that

$$U|\psi_i\rangle = |\tilde{\psi}_i\rangle. \tag{89}$$

For checking the unitarity, we have

$$U^\dagger U = \left(\Sigma_j \left|\psi_j\right\rangle \left\langle\tilde{\psi}_j\right|\right)\left(\Sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|\right) = \Sigma_i |\psi_i\rangle\langle\psi_i| = I \tag{90}$$

The $UU^\dagger$ can be checked similarly. ■

Problem 16 Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle$$

where λ_i are real numbers, and $|i_A\rangle, |i_B\rangle, |i_C\rangle$ are orthonormal bases of the respective systems.

Solution

A characteristic feature of Schmidt-like form expression of

$$|\psi\rangle = \sum_i^2 \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle \tag{91}$$

is that the reduced density matrices ρ_A, ρ_B and ρ_C have the same eigenvalues λ_i^2. For a pure tripartite state $|\Psi\rangle$, we know that the reduce density matrix $\rho_{AB} = \rho_C$, since we can use Schmidt decomposition of bipartite system AB and C. But when ρ_{AB} is a mixed state, ρ_A is not necessarily the same as ρ_B. Such kind of example is easy to given, suppose that $\rho_A = |0\rangle\langle 0|$ and $\rho_B = p|0\rangle\langle 0| + q|1\rangle\langle 1|$, we construct

$$\rho_{AB} = \rho_A \otimes \rho_B = p|00\rangle\langle 00| + q|01\rangle\langle 0|. \tag{92}$$

We can purify this mixed state by adding system C, thus we obtain

$$|\Psi\rangle_{ABC} = \sqrt{p}|000\rangle + \sqrt{q}|011\rangle. \tag{93}$$

If $p, q \neq 0, 1$, the state $|\Psi\rangle$ is what we want. ∎

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