

**Problem 1** For the singlet state  $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ , prove that Alice and Bob's outcomes are always anti-correlated when they measure two particles respectively along the same direction.

**Solution:** Suppose that Alice and Bob both choose to measure spin in  $\vec{v}$  direction, then the operator for them are

$$A = \vec{v} \cdot \vec{\sigma}, \quad B = \vec{v} \cdot \vec{\sigma}. \quad (1)$$

The joint operation is thus  $A \otimes B$ .

Suppose  $|v+\rangle$  and  $|v-\rangle$  are the  $\pm 1$ -eigenstates of  $\vec{v} \cdot \vec{\sigma}$ . Then there exist complex numbers  $\alpha, \beta, \gamma, \delta$  such that

$$|0\rangle = \alpha|v+\rangle + \beta|v-\rangle, \quad |1\rangle = \gamma|v+\rangle + \delta|v-\rangle. \quad (2)$$

Substituting them into the expression of singlet state, we obtain

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma) \frac{|v+\rangle|v-\rangle - |v-\rangle|v+\rangle}{\sqrt{2}} \quad (3)$$

But  $\alpha\delta - \beta\gamma$  is the determinant of the unitary matrix  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ , and thus is equal to a phase factor  $e^{i\theta}$  for some real  $\theta$ . (this is because that the eigenvalue of a unitary matrix is of the form  $\lambda_j = e^{i\theta_j}$ , then  $\det U = \prod_j e^{i\theta_j} = e^{i\sum_j \theta_j}$ ). This means that we can write the singlet state as

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|v+\rangle|v-\rangle - |v-\rangle|v+\rangle}{\sqrt{2}} \quad (4)$$

up to an unobservable global phase factor.

As a result, if a measurement of  $\vec{v} \cdot \vec{\sigma}$  is performed on both qubits, i.e., Alice and Bob perform

$$A \otimes B = \vec{v} \cdot \vec{\sigma} \otimes \vec{v} \cdot \vec{\sigma} \quad (5)$$

on singlet state, they always obtain the respective outcomes  $a, b$  and  $a \times b = -1$ , they are anti-correlated. This completes the proof. ■

**Problem 2** PPT (Positive Partial Transposition) criterion is a strong separability criterion for quantum state, which is very convenient and practical for entanglement detection.

(1) Describe the PPT (Positive Partial Transposition) criterion and the realignment criterion.

(2) For the 2-qubit state  $\rho = p|\phi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$ , where,  $0 \leq p \leq 1$ ,  $|\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$ , calculate the  $p$ 's lower bound when  $\rho$  is entangled state using PPT criterion and realignment criterion respectively.

**Solution:**

(1) PPT criterion states that, if  $\rho$  is separable, then the partial transpose  $\rho^{TA}$  has no negative eigenvalues. Namely, if the partial transpose of a density operator  $\rho$  has negative eigenvalue, then the density operator is entangled.

Realignment criterion states that, for any bipartite separable state  $\rho$ , the sum of all singular values of the realignment of  $\rho$ , denoted as  $\tilde{\rho}$  must satisfy that  $\|\tilde{\rho}\| = \sum_i \lambda_i \leq 1$ . Namely, if  $\sum_i \lambda_i > 1$ , then  $\rho$  is entangled.

(2) For the state  $\rho = p|\phi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$ , if we choose the ordered basis as  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ , the density matrix is of the form

$$\rho = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ -\frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \quad (6)$$

- For PPT criterion, notice that  $(\rho^{TA})_{ij;kl} = \rho_{kj;il}$ , thus the partial transpose is as

$$\rho^{TA} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \quad (7)$$

The eigenvalues of  $\rho^{TA}$  are  $\left\{\frac{1}{4}(1-3p), \frac{p+1}{4}, \frac{p+1}{4}, \frac{p+1}{4}\right\}$ . If  $\rho$  is entangled,  $\rho^{TA}$  has negative eigenvalue, we thus have  $\frac{1}{4}(1-3p) < 0$  this implies that  $1 \geq p > \frac{1}{3}$ . Notice that PPT criterion sufficient and necessary for two-qubit case, the entangled range for  $\rho_p$  is thus sufficient and necessary.

- For realignment criterion, notice that the realigned density matrix is defined as  $\tilde{\rho}_{ij;kl} = \rho_{ik;jl}$ , thus we have

$$\tilde{\rho} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{1-p}{4} \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ \frac{1-p}{4} & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \quad (8)$$

The singular values of  $\tilde{\rho}$  are  $\left\{\frac{1}{2}, \frac{p}{2}, \frac{p}{2}, \frac{p}{2}\right\}$ , then  $\|\tilde{\rho}\| = \sum_i \lambda_i = 1/2 + 3p/2 = \frac{3p+1}{2}$ . If  $\rho$  is entangled, we must have  $\|\tilde{\rho}\| > 1$ , which implies that  $1 \geq p > \frac{1}{3}$ .

We see that the result of two criteria match well. ■

**Problem 3** (1) Calculate the amount of entanglement of the state  $\rho = \lambda |\phi^+\rangle\langle\phi^+| + (1-\lambda) |\psi^+\rangle\langle\psi^+|$ , ( $0 \leq \lambda \leq 1$ ) with negativity measure, where  $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ ,  $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

(2) Derive the value scope for  $\lambda$  when the state  $\rho$  is entangled using negativity measure.

**Solution:**

(1) The corresponding density matrix is of the form

$$\rho = \frac{1}{2} \begin{pmatrix} \lambda & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 0 \\ -\lambda & 0 & 0 & \lambda \end{pmatrix} \quad (9)$$

The partial transpose is of the form

$$\rho^{TA} = \frac{1}{2} \begin{pmatrix} \lambda & 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & -\lambda & 0 \\ 0 & -\lambda & 1-\lambda & 0 \\ 1-\lambda & 0 & 0 & \lambda \end{pmatrix} \quad (10)$$

Recall that the negativity of  $\rho$  is defined

$$N(\rho) = \frac{\|\rho^{TA}\|_1 - 1}{2} \quad (11)$$

where  $\|\rho^{TA}\|_1$  is the 1-norm, namely the sum of all singular values of  $\rho$ .

Since  $\rho$  is Herimition square matrix, the singular value is just the absolute value of the eigenvalues. The eigenvalues of  $\rho^{TA}$  are  $\left\{\frac{1}{2}, \frac{1}{2}, \frac{2\lambda-1}{2}, \frac{1-2\lambda}{2}\right\}$ , thus the singular values are  $\left\{\frac{1}{2}, \frac{1}{2}, \left|\frac{2\lambda-1}{2}\right|, \left|\frac{1-2\lambda}{2}\right|\right\}$ , this further

implies that  $\|\rho^{TA}\|_1 = 1 + \left| \frac{2\lambda-1}{2} \right| + \left| \frac{1-2\lambda}{2} \right|$ . Thus the amount of entanglement of  $\rho$  is:

$$\begin{aligned} N(\rho) &= \frac{\|\rho^{TA}\|_1 - 1}{2} = \left( \left| \frac{2\lambda-1}{2} \right| + \left| \frac{1-2\lambda}{2} \right| \right) / 2 \\ &= \begin{cases} \lambda - \frac{1}{2}, & \lambda > \frac{1}{2} \\ 0, & \lambda = \frac{1}{2} \\ \frac{1}{2} - \lambda, & \lambda < \frac{1}{2} \end{cases} \end{aligned} \quad (12)$$

(2) The state is entangled if  $N(\rho) > 0$ , from the derivation of (1) we see that if  $\lambda \neq 1/2$ , the state is entangled, thus the entanglement range is  $[0, 1/2) \cup (1/2, 1]$ . ■

**Problem 4** (1) Describe the definition of the Entanglement Witness (EW).

(2) For the three-qubit GHZ state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

prove that the entanglement witness  $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$  detects three-qubit entanglement around it.

(3) A mixed state  $\rho = p\frac{\mathbf{I}}{8} + (1-p)|GHZ\rangle\langle GHZ|$  ( $0 \leq p \leq 1$ ), calculate the  $p$ 's upper bound when  $\rho$  is entangled state using the EW given above.

**Solution:**

(1) The entanglement witness is the observables which can be used to distinguish the entangled state from separable states. By definition, the operator  $W$  is called an entanglement witness, if it satisfies

- For any separable state  $\rho_{AB}$ ,  $\text{Tr}(W\rho_{AB}) \geq 0$ .
- For at least one entangled state  $\rho_{AB}^e$ ,  $\text{Tr}(W\rho_{AB}^e) < 0$ . This condition is equivalent to say that  $W$  has at least one negative eigenvalue.

(2) To prove that  $\mathcal{W}$  is an entanglement witness, we need to show that  $\mathcal{W}$  satisfy the conditions for an entanglement witness.

- We need to show that for  $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$ ,  $\text{Tr}(\rho_{\text{sep}}\mathcal{W}) \geq 0$  for all separable states  $\rho_{\text{sep}}$ . This is equivalent to that, for all separable states,  $\text{Tr}(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|) \leq \frac{1}{2}$ . Let's try to calculate the maximum value of  $\text{Tr}(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|) = \langle GHZ|\rho_{\text{sep}}|GHZ\rangle$ . By definition, three particle separable states is probabilistic mixture of three particle product states  $|\psi_i\rangle_A|\phi_i\rangle_B|\chi_i\rangle_C$ , i.e.,

$$\rho_{\text{sep}} = \sum_i p_i (|\psi_i\rangle\langle\psi_i|)_A \otimes (|\phi_i\rangle\langle\phi_i|)_B \otimes (|\chi_i\rangle\langle\chi_i|)_C. \quad (13)$$

The means that the maximum value can be taken over all product state

$$\max_{\rho_{\text{sep}}} \left\{ \langle GHZ|\rho_{\text{sep}}|GHZ\rangle \right\} = \max_{|\Psi_{\text{prod}}\rangle=|\psi\rangle|\phi\rangle|\chi\rangle} \left\{ |\langle GHZ|\Psi_{\text{prod}}\rangle|^2 \right\} \quad (14)$$

This implies that

$$\max_{\rho_{\text{sep}}} \text{Tr}(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|) = 1/2. \quad (15)$$

Thus  $\min_{\rho_{\text{sep}}} \text{Tr}(\rho_{\text{sep}}\mathcal{W}) = 0$ ,  $\text{Tr}(\rho_{\text{sep}}\mathcal{W}) \geq 0$  for all separable states  $\rho_{\text{sep}}$ , we complete the proof.

- We can choose the entangled state as  $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$ , it's easily checked that  $\text{Tr}(\mathcal{W}\rho_{GHZ}) = -1/2 < 0$ , thus it detects entangled states around GHZ state. The second condition is satisfied.

(3)  $\rho$  is an entangled state, then

$$\text{Tr}(\rho\mathcal{W}) = \frac{p}{8} \text{Tr}(\mathcal{W}) + (1-p)\langle GHZ|\mathcal{W}|GHZ\rangle = \frac{3p}{8} - \frac{1-p}{2} < 0 \quad (16)$$

this implies that  $p < \frac{4}{7}$ . ■

**Problem 5** (1) What conditions should a good entanglement measures meet?

(2) Describe the definition of distillable entanglement and entanglement cost and their relationship.

(3) Write down the monogamy of entanglement and describe its physical meanings.

**Solution:**

(1) A good entanglement measure  $E(\cdot)$  should satisfy that,

- For any separable state  $\rho$ , there is no entanglement, thus we must have  $E(\rho) = 0$ ;
- Monotonicity under local operation and classical communication (LOCC) operation: no increase under LOCC operations, namely  $E(\Lambda_{LOCC}(\rho)) \leq E(\rho)$ ;
- Continuity: mathematically,  $E$  is continuous, i.e.  $E(\rho) - E(\sigma) \rightarrow 0$ , when  $\|\rho - \sigma\| \rightarrow 0$ ;
- Convexity: mathematically,  $E$  is convex function, i.e.  $E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma)$ ;
- Normalization, i.e.  $E(P_+^d) = \log d$ .

(2) Their definitions are as follows,

- Distillable entanglement  $E_D(\rho)$  of  $\rho$  is the supremum of achievable conversion rates for converting the state  $\rho$  to the maximally entangled states asymptotically. Mathematically, it is defined as

$$E_D(\rho) := \sup \left\{ r : \lim_{n \rightarrow \infty} \left[ \inf_{\Psi} D(\Psi(\rho^{\otimes n}), \Phi(2^{rn})) \right] = 0 \right\} \quad (17)$$

where  $D(\cdot, \cdot)$  is trace distance,  $\Psi$  is a general trace preserving LOCC operation, and  $\Phi(K)$  is the density operator corresponding to the maximally entangled state in  $K$  dimensions,

$$\Phi(K) = |\phi_K^+\rangle \langle \phi_K^+|. \quad (18)$$

- Entanglement cost  $E_C(\rho)$  of  $\rho$  is the infimum of achievable conversion rates for converting maximally entangled states into state  $\rho$  asymptotically. Mathematically, it's defined as

$$E_C(\rho) = \inf \left\{ r : \lim_{n \rightarrow \infty} \left[ \inf_{\Psi} D(\rho^{\otimes n}, \Psi(\Phi(2^{rn}))) \right] = 0 \right\} \quad (19)$$

where  $D(\cdot, \cdot)$  is trace distance,  $\Psi$  is a general trace preserving LOCC operation, and  $\Phi(K)$  is the density operator corresponding to the maximally entangled state in  $K$  dimensions,

$$\Phi(K) = |\phi_K^+\rangle \langle \phi_K^+|. \quad (20)$$

For pure state  $\rho = |\psi\rangle\langle\psi|$ , we know that entanglement cost and distillable entanglement are the same  $E_C(|\psi\rangle) = E_D(|\psi\rangle)$ .

(3) Monogamy relation of entanglement states that: for any tripartite state of systems  $A, B_1, B_2$  and entanglement measure  $E(\cdot)$ , we have

$$E(A : B_1) + E(A : B_2) \leq E(A : B_1 B_2) \quad (21)$$

This means that if A and B are maximally entangled, then A and C must not be entangled.

If the above tripartite monogamy relation holds in general, i.e. not only for qubits, then it can be immediately generalized by induction to the multipartite case:

$$E(A : B_1) + E(A : B_2) + \cdots + E(A : B_N) \leq E(A : B_1 B_2 \cdots B_N). \quad (22)$$

Squashed-entanglement satisfy the entanglement monogamy relation.

$$E_{\text{sq}}(A : B) + E_{\text{sq}}(A : C) \leq E_{\text{sq}}(A : BC). \quad (23)$$

With the same spirit, there are also some other kind of expressions of the the monogamy relation of entanglement. For instance, consider concurrence (we know that  $\mathcal{C}(\rho) = 1$  means the  $\rho$  is maximally entangled and  $\mathcal{C}(\rho) = 0$  means the  $\rho$  is not entangled), consider Coffman-Kundu-Wootters monogamy inequality is as

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq \mathcal{C}_{A:(BC)}^2 \quad (24)$$

This means that when A and B are maximally entangled, then  $\mathcal{C}_{AB}^2 = 0$ , the relation implies that  $\mathcal{C}_{AC}^2 = 0$ , i.e., A and C must not be entangled. For negativity, similar result holds.

**Problem 6** The four Bell states have the following mathematical expressions on the basis  $\{0,1\}$  (the eigenstates of  $\sigma_z$ ),

$$\begin{aligned} |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\ |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \end{aligned}$$

(1) Prove that the four Bell states can be transformed to each other using single qubit rotations  $\{I, \sigma_x, \sigma_y, \sigma_z\}$ .

(2) Give the representation of the four Bell states on the basis  $\{+, -\}$  (the eigenstates of  $\sigma_x$ ).

**Solution:**

(1) Recall that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (25)$$

Then it's easily checked that

$$\Phi^+ \begin{cases} \xrightarrow{\sigma_x \otimes I} |\Psi^+\rangle \\ \xrightarrow{\sigma_y \otimes I} -i |\Psi^-\rangle \\ \xrightarrow{\sigma_z \otimes I} |\Phi^-\rangle \end{cases} \quad \Phi^- \begin{cases} \xrightarrow{\sigma_x \otimes I} -|\Psi^-\rangle \\ \xrightarrow{\sigma_y \otimes I} i |\Psi^+\rangle \\ \xrightarrow{\sigma_z \otimes I} |\Phi^+\rangle \end{cases} \quad (26)$$

$$\Psi^+ \begin{cases} \xrightarrow{\sigma_x \otimes I} |\Phi^+\rangle \\ \xrightarrow{\sigma_y \otimes I} -i |\Phi^-\rangle \\ \xrightarrow{\sigma_z \otimes I} |\Psi^-\rangle \end{cases} \quad \Psi^- \begin{cases} \xrightarrow{\sigma_x \otimes I} -|\Phi^-\rangle \\ \xrightarrow{\sigma_y \otimes I} i |\Phi^+\rangle \\ \xrightarrow{\sigma_z \otimes I} |\Psi^+\rangle \end{cases} \quad (27)$$

(2) Notice that we can directly replace

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \quad (28)$$

Then we obtain

$$\begin{cases} |\Phi^+\rangle \rightarrow |\check{\Phi}^+\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \\ |\Phi^-\rangle \rightarrow |\check{\Psi}^+\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |\Psi^+\rangle \rightarrow |\check{\Phi}^-\rangle = \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle) \\ |\Psi^-\rangle \rightarrow -|\check{\Psi}^-\rangle = -\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \end{cases} \quad (29)$$

This completes the problem. ■

- Problem 7** (1) Describe the physical meanings of von Neumann entropy.  
 (2) Prove that  $S(\rho) \leq \log D$ , where  $D$  is the number of the non-zero eigenvalues of  $\rho$   
 (3) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B)$$

- (4) Prove the concavity of the von Neumann entropy

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i)$$

- (5) Prove that the two body pure state  $|\psi_{AB}\rangle$  is a entangled state if and only if  $S(B|A) < 0$ , in which  $S(B|A) = S(B, A) - S(A)$ ,  $S(\cdot)$  is the von Neumann entropy.

**Solution:**

(1) Like its classical counterpart Shannon entropy, the von Neumann entropy  $S(\rho)$  quantizes the quantum information of each character of the quantum source characterized by a quantum state  $\rho$ .

(2) The proof is completely the same as the proof for Shannon entropy by diagonalizing the density matrix.

$$S(\rho) = -\text{tr}(\rho \log \rho) = -\sum_i \lambda_i \log \lambda_i = \sum_{i=1}^D p_i \log \frac{1}{p_i} \leq \log \left( \sum_{i=1}^D p_i \frac{1}{p_i} \right). \quad (30)$$

Where in the last step, we need to use the concavity of logarithmic function  $\log(x)$ , we regard  $x_i = 1/p_i$ , then

$$\log \left( \sum_i p x_i \right) \geq \sum_i p_i \log x_i. \quad (31)$$

Thus we complete the proof.

(3) We first prove that  $S(A, B) \leq S(A) + S(B)$ . Recall that relative entropy between arbitrary two quantum states are nonnegative. Consider the relative entropy of  $\rho_{AB}$  and  $\rho_A \otimes \rho_B$

$$\begin{aligned} S(\rho_{AB} \| \rho_A \otimes \rho_B) &= \text{tr}(\rho_{AB} \log \rho_{AB}) - \text{tr}(\rho_{AB} \log (\rho_A \otimes \rho_B)) \\ &= -S(\rho_{AB}) - \text{tr}(\rho_{AB} \log \rho_A) - \text{tr}(\rho_{AB} \log \rho_B) \\ &= -S(\rho_{AB}) + S(\rho_A) + S(\rho_B) \\ &\geq 0 \end{aligned} \quad (32)$$

This directly implies that  $S(A, B) \leq S(A) + S(B)$

For  $|S(A) - S(B)| \leq S(A, B)$ , consider a purification of  $\rho_{AB} = \text{tr}_C |\phi\rangle_{ABC} \langle \phi|$ , apply subadditivity to  $\rho_{BC}$ , we can get that

$$S(B, C) \leq S(B) + S(C) \quad (33)$$

since  $S(B, C) = S(A)$ ,  $S(C) = S(A, B)$ , so we get that

$$S(A, B) \geq S(A) - S(B) \quad (34)$$

Similarly,  $S(A, B) \geq S(B) - S(A)$ . To summarize, we have

$$|S(A) - S(B)| \leq S(A, B). \quad (35)$$

This completes the proof.

- (4) By Applying subadditivity of von Neumann entropy to the state

$$\rho_{AB} = \sum_i p_i \rho_i \otimes |i\rangle \langle i|_B \quad (36)$$

where  $|i\rangle$  is a set of orthonormal states for B, we obtain that

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) = S\left(\sum_i p_i \rho_i\right) + H(p_i) \quad (37)$$

By invoking joint entropy theorem which states that  $S(\sum_i p_i \rho_i \otimes |i\rangle\langle i|) = H(p_i) + \sum_i p_i S(\rho_i)$ , we obtain that

$$S(\rho_{AB}) = S\left(\sum_i \rho_i \otimes p_i |i\rangle\langle i|_B\right) = \sum_i p_i S(\rho_i) + H(p_i). \quad (38)$$

Using this and equation (37) we obtain

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i). \quad (39)$$

(5) Since  $|\psi_{AB}\rangle$  is a pure state, we known that  $S(A, B) = 0$ . If  $|\psi_{AB}\rangle$  is an entangled state, taking its Schmidt decomposition, we have

$$|\psi_{AB}\rangle = \sum_i \sqrt{p_i} |i_A\rangle |i_B\rangle, \quad (40)$$

the number of nozero  $p_i$  is equal or great than 2. From the Schmidt decomposition we have

$$\rho_A = \sum_i p_i |i_A\rangle\langle i_A|, \quad S(A) = -\sum_i p_i \log p_i > 0 \quad (41)$$

Thus we have

$$S(B | A) = S(A, B) - S(A) = -S(A) < 0. \quad (42)$$

The completes the problem. ■

**Problem 8** Prove that  $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$  is invariant under transformation  $U(\theta, \vec{n}) \otimes U(\theta, \vec{n})$  where  $U(\theta, \vec{n}) = e^{-\frac{i}{2}\theta\vec{n}\cdot\vec{\sigma}}$

**Solution:** Recall that

$$U(\theta, \vec{n}) = e^{-\frac{i}{2}\theta\vec{n}\cdot\vec{\sigma}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma} \quad (43)$$

from which we have

$$U(\theta, \vec{n}) \otimes U(\theta, \vec{n}) = \cos^2\frac{\theta}{2}I \otimes I - i\sin\frac{\theta}{2}\cos\frac{\theta}{2}(n\cdot\vec{\sigma}_B + n\cdot\vec{\sigma}_A) - \sin^2\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})_A \otimes (\vec{n}\cdot\vec{\sigma})_B. \quad (44)$$

Since

$$\sigma_i \otimes \sigma_i |\psi^-\rangle = -|\psi^-\rangle, \quad i = x, y, z, \quad (45)$$

$$\sigma_i \otimes \sigma_j |\psi^-\rangle = 0, \quad i \neq j, \quad (46)$$

we have

$$(\vec{n}\cdot\vec{\sigma})_A \otimes (\vec{n}\cdot\vec{\sigma})_B |\psi^-\rangle = -|\psi^-\rangle. \quad (47)$$

And notice that

$$\cos^2\frac{\theta}{2}I \otimes I |\psi^-\rangle = \cos^2\frac{\theta}{2} |\psi^-\rangle, \quad (48)$$

$$(\vec{n}\cdot\vec{\sigma}_A + \vec{n}\cdot\vec{\sigma}_B) |\psi^-\rangle = 0 \quad (49)$$

We have

$$U(\theta, \vec{n}) \otimes U(\theta, \vec{n}) |\psi^-\rangle = |\psi^-\rangle. \quad (50)$$

We thus completes the proof. ■

**Problem 9** The entropy of quantum state, expressed as a density matrix  $\rho$ , is  $S(\rho) = -\text{tr}(\rho \log_2 \rho)$ ; in terms of its eigenvalues  $\lambda_k$ , this is  $S(\rho) = -\sum_k \lambda_k \log_2 \lambda_k$ . A state  $\rho$  is a pure state if and only if  $\text{tr}(\rho^2) = 1$ . Prove that this is equivalent to  $S(\rho) = 0$ . You may use the fact  $\rho$  is a valid density matrix if and only if  $\text{tr}(\rho) = 1$  and  $\rho$  is a positive operator (i.e. its eigenvalues are  $\geq 0$ ).

**Solution:** Let's first prove that:  $\rho$  pure  $\Leftrightarrow \text{Tr}(\rho^2) = 1 \Rightarrow S(\rho) = 0$ . If  $\text{Tr}(\rho^2) = 1$

$$\sum_k \lambda_k^2 = \sum_k \lambda_k = 1 \quad (51)$$

But since  $0 \leq \lambda_k \leq 1, \forall k, \lambda_k^2 \leq \lambda_k$ , the only way the saturate the equality is that there exist one  $k$  such that  $\lambda_k = 1$ . This implies that  $\rho$  only has one nonzero eigenvalue, i.e.,  $\rho = |\psi\rangle\langle\psi|$  for some  $|\psi\rangle$ , it's a pure state. Thus  $S(\rho) = 0$ .

For the other direction,  $S(\rho) = 0 \Rightarrow \text{Tr}(\rho^2) = 1 \Leftrightarrow \rho$  pure. Consider

$$S(\rho) = -\sum_k \lambda_k \log_2 \lambda_k = 0 \quad (52)$$

since  $0 \leq \lambda_k \leq 1, \forall k$ , we know that  $\lambda_k \log_2 \lambda_k \leq 0$ , for all  $k$ , Therefore, the only way for the above condition to be satisfied is for  $\lambda_k = 0, 1$ , for all  $k$ , this implies that  $\text{tr}(\rho^2) = 1$ , i.e.,  $\rho$  is a pure state.

To summarize, the following three statements are equivalent: (i)  $\rho$  pure; (ii)  $\text{Tr}(\rho^2) = 1$ ; (iii)  $S(\rho) = 0$ . ■

**Problem 10** For the 2-qubit state

$$\rho = p |\Psi^-\rangle\langle\Psi^-| + (1-p) \frac{I}{4},$$

where  $0 \leq p \leq 1, |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$  calculate the EOF (Entanglement of Formation) of  $\rho$ .

**Solution:**

This is the 2-qubit Werner state, to calculate the entanglement of formation  $E_F(\rho)$ , we can first calculate the concurrence  $\mathcal{C}(\rho)$  and then use the famous relation between concurrence and entanglement of formation

$$E_F(\rho) = H\left(\frac{1 + \sqrt{1 - \mathcal{C}^2(\rho)}}{2}\right), \quad (53)$$

here  $H(x) = -x \log x - (1-x) \log x$  is the Shannon entropy function.

The density matrix of  $\rho$  in the ordered basis  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$  is

$$\rho = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & \frac{-p}{2} & 0 \\ 0 & \frac{-p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \quad (54)$$

And since

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (55)$$

then from definition that  $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ , we see that  $\tilde{\rho} = \rho$ . The square roots of eigenvalues for  $\rho\tilde{\rho} = \rho^2$  are nothing but the eigenvalue of  $\rho$ . Four eigenvalues are (in decreasing order)

$$\lambda_4 = \frac{1+3p}{4}, \lambda_3 = \lambda_2 = \lambda_1 = \frac{1-p}{4}. \quad (56)$$



Then by definition, the concurrence is

$$\mathcal{C}(\rho) = \max\{0, \lambda_4 - \lambda_3 - \lambda_2 - \lambda_1\}. \quad (57)$$

we see that

$$\begin{cases} \mathcal{C}(\rho) = 0, & \text{for } 0 \leq p \leq 1/3 \\ \mathcal{C}(\rho) = \frac{3p-1}{2}, & \text{for } 1/3 < p \leq 1 \end{cases} \quad (58)$$

From this, we see that

$$\begin{cases} H(\rho) = 0, & \text{for } 0 \leq p \leq 1/3 \\ H(\rho) = H\left(\frac{1+\sqrt{1-(\frac{3p-1}{2})^2}}{2}\right) & \text{for } 1/3 < p \leq 1 \end{cases} \quad (59)$$

Thus we complete the calculation. ■

**Problem 11** Consider the state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$ ,  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$ . Calculate the Von Neumann entropy of  $\rho_A$ .

**Solution:** By direct calculation

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (60)$$

this implies that

$$S(\rho_A) = -\left(\frac{1}{2} \log\left(\frac{1}{2}\right) + \frac{1}{2} \log\left(\frac{1}{2}\right)\right) = 1. \quad (61)$$

The maximally mixed state has the maximal entropy. ■

**Problem 12** Give a noisy entanglement state with purity  $F$  for the singlet state  $|\Psi^-\rangle$ ,

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|$$

Supposing  $F = \frac{3}{5}$ , please design a two-way LOCC purification protocol that can obtain the singlet state  $|\Psi^-\rangle$  with as high fidelity as possible from the above mixed state in five steps.

**Solution:**

The solution here I give is referred to the paper "*Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels*" by C. Bennett *et al.*

Notice that Bell diagonal state (also called symmetric Werner state by some author)

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|. \quad (62)$$

has the fidelity with  $|\Psi^-\rangle$  as  $F(|\Psi^-\rangle, W_F) = F$ . Alice and Bob share two pairs of  $W_F$  states respectively, i.e.  $W_F^{12}$  and  $W_F^{34}$ , with 1 and 3 in Alice's side, 2 and 4 in Bob's side. Then the two-way purification protocol works as follows:

- Alice and Bob make unilateral transformation  $\sigma_y$ , namely  $\sigma_y \otimes I$  for Alice and  $I \otimes \sigma_y$  for Bob, on their two pairs of  $W_F$  states. Notice that  $\sigma_y|0\rangle = i|1\rangle$  and  $\sigma_y|1\rangle = -i|0\rangle$ . Thus under  $\sigma_y \otimes I$  we have

$$|\Psi^-\rangle \rightarrow -i|\Phi^+\rangle, \quad |\Psi^+\rangle \rightarrow -i|\Phi^-\rangle, \quad |\Phi^-\rangle \rightarrow i|\Psi^+\rangle, \quad |\Phi^+\rangle \rightarrow -i|\Psi^-\rangle. \quad (63)$$

Similarly for  $I \otimes \sigma_y$  (with only minus sign difference). This means that after the unilateral transformation, the state becomes

$$W_F \xrightarrow{\sigma_y \otimes I} W'_F = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{1-F}{3}|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+|. \quad (64)$$

- Alice and Bob perform the bilateral controlled-NOT operations CNOT on their two pairs of  $W'_F$  states with 1 and 2 as 'source' particles and 3 and 4 as 'target' particles. For clarity, we explain here what we mean by bilateral controlled-NOT operations: CNOT operation is done for 3 conditioned to 1 and CNOT operation is done for 4 conditioned to 2. More precisely, we have

$$\text{CNOT}_{13} \otimes \text{CNOT}_{24}(W'_F \otimes W'_F). \quad (65)$$

Recall that

$$\text{CNOT}|00\rangle = |00\rangle, \quad \text{CNOT}|01\rangle = |01\rangle, \quad \text{CNOT}|10\rangle = |11\rangle, \quad \text{CNOT}|11\rangle = |10\rangle. \quad (66)$$

For Bell states  $|\psi\rangle_{12} \otimes |\varphi\rangle_{34}$  with  $|\psi\rangle_{12}$  source state and  $|\varphi\rangle_{34}$  the target state. Then for the operation

$$\text{CNOT}_{13} \otimes \text{CNOT}_{24}(|\text{source}_{\text{in}}\rangle_{12} \otimes |\text{target}_{\text{in}}\rangle_{34}) = |\text{source}_{\text{out}}\rangle_{12} \otimes |\text{target}_{\text{out}}\rangle_{34} \quad (67)$$

we have the following result:

source <sub>in</sub>	target <sub>in</sub>	source <sub>out</sub>	target <sub>out</sub>
$\Phi^\pm$	$\Phi^+$	$\Phi^\pm$	$\Phi^+$
$\Psi^\pm$	$\Phi^+$	$\Psi^\pm$	$\Psi^+$
$\Psi^\pm$	$\Psi^+$	$\Psi^\pm$	$\Phi^+$
$\Phi^\pm$	$\Psi^+$	$\Phi^\pm$	$\Psi^+$
$\Phi^\pm$	$\Phi^-$	$\Phi^\mp$	$\Phi^-$
$\Psi^\pm$	$\Phi^-$	$\Psi^\mp$	$\Psi^-$
$\Psi^\pm$	$\Psi^-$	$\Psi^\mp$	$\Phi^-$
$\Phi^\pm$	$\Psi^-$	$\Phi^\mp$	$\Psi^-$

(68)

Based on this, they choose to measure the output state  $|\text{target}_{\text{out}}\rangle_{34}$  of two target particles along the z-axis. If their z-spins are parallel, keep the correspond out source state; otherwise, discard the out source state.

Notice the measurements along the Z axis can only distinguish  $\Phi$  from  $\Psi$  but can't distinguish  $-$  from  $+$ , we can only kept all  $\Phi$  target state or  $\Psi$  target state, here we choose keep the  $\Phi$  target state, namely, we keep the 1,3,5,7 rows' source states. Notice that this implies that resulted source state is a mixed state of Bell states with their respective probabilities.

If we set  $G = \frac{1-F}{3}$ , we obtain that

$$\rho = \frac{1}{F^2 + 5G^2 + 2FG} \left( (F^2 + G^2)|\Phi^+\rangle\langle\Phi^+| + 2FG|\Phi^-\rangle\langle\Phi^-| + 2G^2|\Psi^-\rangle\langle\Psi^-| + 2G^2|\Psi^+\rangle\langle\Psi^+| \right) \quad (69)$$

Notice that, now the fidelity between the resulting state and  $|\Phi^+\rangle$  is

$$F' = \frac{F^2 + G^2}{F^2 + 5G^2 + 2FG} = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}. \quad (70)$$

This is a recurrence relation when we repeat this step.

Notice that because  $F'(F)$  is continuous and exceeds  $F$  over the entire range  $\frac{1}{2} < F < 1$ , iteration of the above step can make the fidelity between  $\Phi^+$  and the resulted state arbitrarily high.

- The last step is the implement unilateral  $\sigma_y$  operation to rotate  $\Phi^+$  into  $\Psi^-$ , i.e.,

$$\rho = F'|\Phi^+\rangle\langle\Phi^+| + \dots \mapsto F'|\Psi^-\rangle\langle\Psi^-| + \dots \quad (71)$$

Thus we can distill Bell diagonal states of arbitrarily high purity  $F_{\text{out}} < 1$  from a supply of input mixed states  $W_F$  of any purity  $F_{\text{in}} > \frac{1}{2}$ , (here  $F = 3/5 > 1/2$  satisfy this condition). ■