

Problem 1 Describe and prove the no-cloning theorem.

Solution: The no-cloning theorem states that it is impossible to perfectly clone an unknown quantum state using unitary evolution. The proof is as follows, suppose that we have such a unitary operator U such that for arbitrary state $|\phi\rangle$, we have $U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$. Then consider two different and un-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$, we have

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle, \quad (1)$$

$$U(|\varphi\rangle \otimes |0\rangle) = |\varphi\rangle \otimes |\varphi\rangle. \quad (2)$$

Taking the inner product of left hand sides and right hand sides of the above two equalities, we obtain

$$(\langle\psi| \otimes \langle 0|)U^\dagger U(|\varphi\rangle \otimes |0\rangle) = (\langle\psi| \otimes \langle\psi|)(|\varphi\rangle \otimes |\varphi\rangle), \quad (3)$$

equivalently, we have

$$\langle\psi|\varphi\rangle = (\langle\psi|\varphi\rangle)^2. \quad (4)$$

Notice that in the above we have used the assumption that U is unitary. This implies that $\langle\psi|\varphi\rangle$ equals 0 or 1, i.e., ψ and φ are the same or orthogonal, which is contradict with the assumption of $|\psi\rangle$ and $|\varphi\rangle$, thus such U does not exist. ■

Problem 2 Prove that non-orthogonal states can't be reliably distinguished.

Solution: This can be proved using proof by contradiction. To this end, let's first clarify that here by the word 'reliably' we means that the experimenter choose a measurement M_j to measure and according to the measurement outcome, he guess the index of given state ψ_i using the guess function $f(j) = i$ with success probability 1. Now suppose the experimenter is given two nonorthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. Suppose the measurement $\mathcal{M} = \{M_1, \dots, M_n\}$ which can reliably distinguish these two states is possible. If the state $|\psi_1\rangle$ (resp. $|\psi_2\rangle$) is prepared then the probability of measuring j such that $f(j) = 1$ (resp. $f(j) = 2$) must be 1. Defining $E_i \equiv \sum_{j:f(j)=i} M_j^\dagger M_j$, these observations may be written as:

$$\langle\psi_1|E_1|\psi_1\rangle = 1; \quad \langle\psi_2|E_2|\psi_2\rangle = 1. \quad (5)$$

From the completeness of the measurement $\sum_i E_i = I$ it follows that $\sum_i \langle\psi_1|E_i|\psi_1\rangle = 1$, and since $\langle\psi_1|E_1|\psi_1\rangle = 1$ we must have $\langle\psi_1|E_2|\psi_1\rangle = 0$, and thus $\sqrt{E_2}|\psi_1\rangle = 0$. Suppose we decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\varphi\rangle$, where $|\varphi\rangle$ is orthonormal to $|\psi_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, and $|\beta| < 1$ since $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal. Then $\sqrt{E_2}|\psi_2\rangle = \beta\sqrt{E_2}|\varphi\rangle$, which implies a contradiction with (5) as

$$\langle\psi_2|E_2|\psi_2\rangle = |\beta|^2 \langle\varphi|E_2|\varphi\rangle \leq |\beta|^2 < 1 \quad (6)$$

where the second last inequality follows from the observation that

$$\langle\varphi|E_2|\varphi\rangle \leq \sum_i \langle\varphi|E_i|\varphi\rangle = \langle\varphi|\varphi\rangle = 1. \quad (7)$$

This completes the proof. ■

Problem 3 Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y and Z

Solution: Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (8)$$

- Firstly consider Z , since Z is diagonal, it's obvious that

$$\lambda_Z = \pm 1, \quad (9)$$

and the corresponding eigenvectors are

$$|Z+\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |Z-\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (10)$$

The diagonal representation of Z is

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|. \quad (11)$$

• For X , we have

$$\det(X - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad (12)$$

So the eigenvalues of X are $\{1, -1\}$. The eigenvector corresponding to $\lambda = 1$ is

$$X\psi = \psi \implies \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (13)$$

This implies $\alpha = \beta$. Thus the normalized eigenvector (up to an arbitrary phase factor) is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The eigenvector corresponding to $\lambda = -1$ is

$$X\psi = -\psi \implies \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \iff \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \quad (14)$$

This implies that $\alpha = -\beta$. Thus the normalized eigenvector (up to an arbitrary phase factor) is $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then, the diagonal representation of X is given by

$$X = |+\rangle\langle +| - |-\rangle\langle -| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) - \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|). \quad (15)$$

• Now consider Y ,

$$\det(Y - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad (16)$$

So the eigenvalues of Y are $\{1, -1\}$. The eigenvector corresponding to $\lambda = 1$ is

$$Y\psi = \psi \implies \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (17)$$

This implies that $\alpha = -i\beta$. Thus the normalized eigenvector up to an overall phase is $|Y+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. The eigenvector corresponding to $\lambda = -1$ is

$$Y\psi = -\psi \implies \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \iff \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \quad (18)$$

This implies that $\alpha = i\beta$. Thus the normalized eigenvector is $|Y-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Then, the diagonal representation of Y is given by

$$Y = |Y+\rangle\langle Y+| - |Y-\rangle\langle Y-| = \frac{1}{2}(|0\rangle + i|1\rangle)(\langle 0| + i\langle 1|) - \frac{1}{2}(|0\rangle - i|1\rangle)(\langle 0| - i\langle 1|) \quad (19)$$

This completes the problem. ■

Problem 4 Write down the commutation relations and anti-commutation relations for the Pauli matrices and prove them.

Solution: The commutation relations:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad (20)$$

the anti-commutation relations:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$$

where $i, j, k = 1, 2, 3$

In fact, we have

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k. \quad (21)$$

This can be proved by direction calculation using the expression of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (22)$$

by easy matrix calculation, we have

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = I. \quad (23)$$

And

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3 \quad (24)$$

$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \sigma_1 \quad (25)$$

$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = \sigma_2 \quad (26)$$

This can gives the commutation and anti-commutation relations. Alternatively, we can prove it using expression (21).

$$\begin{aligned} [\sigma_i, \sigma_j] &= \sigma_i\sigma_j - \sigma_j\sigma_i = (\delta_{ij}I + i\epsilon_{ijk}\sigma_k) - (\delta_{ji}I + i\epsilon_{jik}\sigma_k) \\ &= i\epsilon_{ijk}\sigma_k - i\epsilon_{jik}\sigma_k = i\epsilon_{ijk}\sigma_k + i\epsilon_{ijk}\sigma_k = 2i\epsilon_{ijk}\sigma_k \end{aligned} \quad (27)$$

where we have used $\delta_{ij} = \delta_{ji}$ and $\epsilon_{jik} = -\epsilon_{ijk}$. Similarly, using expression (21), we have

$$\begin{aligned} \{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_j\sigma_i = (\delta_{ij}I + i\epsilon_{ijk}\sigma_k) + (\delta_{ji}I + i\epsilon_{jik}\sigma_k) \\ &= 2\delta_{ij}I \end{aligned} \quad (28)$$

where we have used $\delta_{ij} = \delta_{ji}$ and $\epsilon_{jik} = -\epsilon_{ijk}$. ■

Problem 5 Prove the Cauchy-Schwarz inequality that for any two vectors $|v\rangle$ and $|w\rangle$, $|\langle v | w \rangle|^2 \leq \langle v | v \rangle \langle w | w \rangle$

Solution: Consider two vectors $|v\rangle = (v_1, \dots, v_n)^T$ and $|w\rangle = (w_1, \dots, w_n)^T$, we see that

$$\langle w | w \rangle = \|w\|^2 = \sum_i w_i^* w_i, \quad \langle v | v \rangle = \|v\|^2 = \sum_i v_i^* v_i, \quad \langle v | w \rangle = \sum_i v_i^* w_i. \quad (29)$$

If $\|v\| = 0$, the inequality trivially holds, namely $0 \leq 0$. Therefore, we assume that $\|v\| \neq 0$, then we can construct a vector $w - \frac{\langle v | w \rangle}{\|v\|^2} v$, the norm of this vector is nonnegative

$$\begin{aligned} 0 &\leq \left\| w - \frac{\langle v | w \rangle}{\|v\|^2} v \right\|^2 = \langle w | w \rangle - \frac{|\langle v | w \rangle|^2}{\|v\|^2} - \frac{|\langle v | w \rangle|^2}{\|v\|^2} + \frac{|\langle v | w \rangle|^2}{\|v\|^4} \|v\|^2 \\ &= \langle w | w \rangle - \frac{|\langle v | w \rangle|^2}{\|v\|^2}. \end{aligned} \quad (30)$$

This implies that

$$\|v\|^2 \langle w | w \rangle \geq |\langle v | w \rangle|^2. \quad (31)$$

and asserted inequality follows.

Problem 6 Let \vec{v} be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(i\theta\vec{v} \cdot \vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma}$$

where $\vec{v} \cdot \vec{\sigma} = \sum_{i=1}^3 v_i \sigma_i$

Solution: From defining properties of Pauli matrices

$$\sigma_i \sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k. \quad (32)$$

we see that

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b}I + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma} \quad (33)$$

This further implies that $(\vec{v} \cdot \vec{\sigma})^2 = I$. From Taylor series expansion for matrices

$$\exp(iA) = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!} \quad (34)$$

and noticing that for interger $k \geq 0$

$$(i\theta\vec{v} \cdot \vec{\sigma})^{2k+1} = i(-1)^k \theta^{2k+1} (\vec{v} \cdot \vec{\sigma}) \quad (i\theta\vec{v} \cdot \vec{\sigma})^{2k} = (-1)^k \theta^{2k} I \quad (35)$$

we can get

$$\begin{aligned} \exp(i\theta\vec{v} \cdot \vec{\sigma}) &= \sum_{n=0}^{\infty} \frac{(i\theta\vec{v} \cdot \vec{\sigma})^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k} I}{2k!} + \sum_{k=0}^{\infty} \frac{i(-1)^k \theta^{2k+1} (\vec{v} \cdot \vec{\sigma})}{(2k+1)!} \\ &= \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma}. \end{aligned} \quad (36)$$

This completes the proof. ■

Problem 7 Prove that for any 2-dimension linear operator A ,

$$A = \frac{1}{2} \text{Tr}(A)I + \frac{1}{2} \sum_{k=1}^3 \text{Tr}(A\sigma_k) \sigma_k$$

in which $\sigma_k (k = 1, 2, 3)$ are Pauli matrices.

Solution: Recall that σ_μ (where $\sigma_0 = I$ and $\sigma_k (k = 1, 2, 3)$ are Pauli matrices) form the basis of complex vector space $M_2(\mathbb{C})$ of 2×2 matrices. This means that we can write $A = \sum_{\mu=0}^3 a_\mu \sigma_\mu$ with some complex coefficients a_μ . Then

$$A\sigma_k = \sum_{\mu=0}^3 a_\mu \sigma_\mu \sigma_k \quad (37)$$

since

$$\text{Tr}(\sigma_\mu \sigma_k) = 2\delta_{\mu k} \quad (38)$$

then

$$\text{Tr}(A\sigma_k) = \sum_{\mu=0}^3 \text{Tr}(a_\mu \sigma_\mu \sigma_k) = \sum_{\mu=0}^3 2a_\mu \delta_{\mu k} = 2a_k \quad (39)$$

Therefore

$$a_k = \frac{1}{2} \text{Tr}(A\sigma_k) \quad (40)$$

From which we obtain

$$A = \frac{1}{2} \text{Tr}(A)I + \frac{1}{2} \sum_{k=1}^3 \text{Tr}(A\sigma_k) \sigma_k \quad (41)$$

this completes the proof. ■

Problem 8 If A and B are two linear operators, show that

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Solution: This is almost obvious from direct calculation

$$\text{Tr}(AB) = \sum_i \sum_j a_{ij} b_{ji} = \sum_i \sum_j b_{ij} a_{ji} = \text{Tr}(BA)$$

This completes the proof. ■

Problem 9 Let ρ be a density operator.

(1) Show that ρ can be written as

$$\rho = \frac{\mathbf{I} + \mathbf{r} \cdot \boldsymbol{\sigma}}{2}$$

where \mathbf{r} is a real three-dimensional vector such that $\|\mathbf{r}\| \leq 1$

(2) Show that $\text{Tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

(3) Show that a state ρ is a pure state if and only if $\|\mathbf{r}\| = 1$.

Solution:

(1) Since σ_μ with $\mu = 0, 1, 2, 3$ form a basis of complex vector space of 2×2 matrices, we have

$$\rho = x_0 \sigma_0 + x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3, \quad (42)$$

for complex numbers x_μ . Since ρ is Hermitian, x_μ are real numbers. Since $\text{Tr} \sigma_0 = 2$ and $\text{Tr} \sigma_i = 0$ for $i = 1, 2, 3$, we obtain

$$\text{Tr}(\rho \sigma_\mu) = 2\delta_{\mu 0}. \quad (43)$$

From which we have $x_\mu = \frac{1}{2} \text{Tr}(\rho \sigma_\mu)$, this directly implies that

$$x_0 = 1/2, \quad (44)$$

and we defined

$$\mathbf{r} = (2x_1, 2x_2, 2x_3)^T = (\text{Tr}(\rho \sigma_1), \text{Tr}(\rho \sigma_2), \text{Tr}(\rho \sigma_3))^T. \quad (45)$$

Taking trace of $\text{Tr}(\rho \sigma_\mu)$, we arrive at

$$x_0 = \frac{1}{2}, \quad (46)$$

thus we get

$$\rho = \frac{\mathbf{I} + \mathbf{r} \cdot \boldsymbol{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}. \quad (47)$$

To show that $\|\mathbf{r}\| \leq 1$, notice that ρ is positive semidefinite, thus $\det \rho \geq 0$, namely

$$\det \frac{1}{2} \begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix} = \frac{1}{4} (1 - \|\mathbf{r}\|^2) \geq 0. \quad (48)$$

This implies that $\|\mathbf{r}\| \leq 1$.

(2) Suppose that $\rho = \sum_i p_i |\phi_i\rangle \langle \phi_i|$, then

$$\begin{aligned} \rho^2 &= \sum_i p_i |\phi_i\rangle \langle \phi_i| \sum_j p_j |\phi_j\rangle \langle \phi_j| \\ &= \sum_{i,j} p_i p_j |\phi_i\rangle \langle \phi_i | \phi_j\rangle \langle \phi_j| \\ &= \sum_i p_i^2 |\phi_i\rangle \langle \phi_i| \end{aligned} \quad (49)$$

then

$$\begin{aligned}
 \text{Tr}(\rho^2) &= \text{Tr}\left(\sum_i p_i^2 |\phi_i\rangle\langle\phi_i|\right) \\
 &= \text{Tr}\left(\sum_i p_i^2 \langle\phi_i|\phi_i\rangle\right) \\
 &= \sum_i p_i^2
 \end{aligned} \tag{50}$$

since $\sum_i p_i = 1$, then $\sum_i p_i^2 \leq \sum_i p_i = 1$ then

$$\text{Tr}(\rho^2) \leq 1 \tag{51}$$

with equality hold if and only if

$$\begin{cases} p_j = 1, \\ p_{i \neq j} = 0 \end{cases} \tag{52}$$

which means that ρ is a pure state.

(3) This can be proved using the criterion we prove in (2) Notice that

$$\rho = \frac{I + \mathbf{r} \cdot \boldsymbol{\sigma}}{2} \tag{53}$$

then

$$\text{Tr}(\rho^2) = \frac{1}{4} \text{Tr}(I + 2\mathbf{r} \cdot \boldsymbol{\sigma} + 2\|\mathbf{r}\|^2) \tag{54}$$

By involking

$$\text{Tr}(I) = 2, \text{Tr}(\sigma_i) = 0, (i = 1, 2, 3) \tag{55}$$

we get that

$$\text{Tr}(\rho^2) = \frac{1}{2} (1 + \|\mathbf{r}\|^2). \tag{56}$$

Since ρ is a pure state if and only if

$$\text{Tr}(\rho^2) = 1 \tag{57}$$

this implies that

$$\|\mathbf{r}\|^2 = 1. \tag{58}$$

Thus we complete the proof. ■

Problem 10 $\rho_A = \frac{I + \mathbf{n}_A \cdot \boldsymbol{\sigma}}{2}, \rho_B = \frac{I + \mathbf{n}_B \cdot \boldsymbol{\sigma}}{2}$, prove that $\text{Tr}(\rho_A \rho_B) = \frac{1 + \mathbf{n}_A \cdot \mathbf{n}_B}{2}$.

Solution: Firstly, Recall that

$$(\mathbf{n}_A \cdot \boldsymbol{\sigma})(\mathbf{n}_B \cdot \boldsymbol{\sigma}) = \mathbf{n}_A \cdot \mathbf{n}_B + i(\mathbf{n}_A \times \mathbf{n}_B) \cdot \boldsymbol{\sigma}. \tag{59}$$

Thus, we have

$$\rho_A \rho_B = \frac{1}{4} (I + \mathbf{n}_A \cdot \boldsymbol{\sigma} + \mathbf{n}_B \cdot \boldsymbol{\sigma} + \mathbf{n}_A \cdot \mathbf{n}_B I + i(\mathbf{n}_A \times \mathbf{n}_B) \cdot \boldsymbol{\sigma}) \tag{60}$$

Since $\text{Tr}(\sigma_i) = 0, \text{Tr}(I) = 2$, we have

$$\text{Tr}(\rho_A \rho_B) = \frac{1 + \mathbf{n}_A \cdot \mathbf{n}_B}{2} \tag{61}$$

This completes the proof. ■

Problem 11 Consider an experiment, in which we prepare the state $|0\rangle$ with the probability $|C_0|^2$, and the state $|1\rangle$ with the probability $|C_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$

Solution: The first case is that this state is a classical mixture, i.e., mixed state, whose density matrix is

$$\rho_C = |C_0|^2 |0\rangle\langle 0| + |C_1|^2 |1\rangle\langle 1| = \begin{pmatrix} |C_0|^2 & 0 \\ 0 & |C_1|^2 \end{pmatrix} \quad (62)$$

There is no coherence of $|0\rangle$ and $|1\rangle$ in this state, when we measurement an observable A we obtain $\langle A \rangle_C = |C_0|^2 \langle 0|A|0\rangle + |C_1|^2 \langle 1|A|1\rangle$.

The second case is The state $|\psi\rangle = C_0|0\rangle + C_1e^{i\theta}|1\rangle$ is a pure state, whose density matrix is

$$\begin{aligned} \rho_Q = |\psi\rangle\langle\psi| &= |C_0|^2 |0\rangle\langle 0| + |C_1|^2 |1\rangle\langle 1| + C_0C_1e^{-i\theta}|0\rangle\langle 1| + C_0C_1e^{i\theta}|1\rangle\langle 0| \\ &= \begin{pmatrix} |C_0|^2 & C_0C_1e^{-i\theta} \\ C_0C_1e^{i\theta} & |C_1|^2 \end{pmatrix} \end{aligned} \quad (63)$$

The $|0\rangle$ and $|1\rangle$ are in a coherent superposition, there is off-diagonal terms of the density matrix. In this case we measure an observable A we will find that $\langle\psi|A|\psi\rangle \neq |C_0|^2 \langle 0|A|0\rangle + |C_1|^2 \langle 1|A|1\rangle$, the appearance of the cross-term is a result of the coherent superposition. Thus, we see that, when measuring these two states in $\{|0\rangle, |1\rangle\}$ basis, the probabilities we get $|0\rangle$ and $|1\rangle$ are same; if other basis is used, the probabilities are different. ■

Problem 12 Please prepare the polarized optical quantum state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$ from an initial state $|0\rangle$, with half wave plate and quarter-wave plate. To implement arbitrary single qubit unitary transformation, how many wave plates are at least needed, and how to perform them?

Solution:

Before doing this let's first recall some basic concepts from linear optics. From Maxwell equation we known that for electromagnetic wave, $\vec{E} \perp \vec{k}$, if the light go along z -axis, then $\vec{k} = (0, 0, k)$ and $\vec{E} = (E_x, E_y, 0)$. A general plane-wave solution of Maxwell equation is then

$$\vec{E} = \vec{A}e^{ikz-i\omega t} \quad (64)$$

where $\vec{A} = (A_x, A_y, 0)$. Thus we can focus on a two-dimensional vector known as Jones vector

$$\vec{J} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad (65)$$

If we set $|0\rangle = \vec{J}_x = (1, 0)^T$ and $|1\rangle = \vec{J}_y = (0, 1)^T$ we see that the general polarization state of the optic is of the form

$$\vec{E} = \|\vec{A}\| \left(a|0\rangle + be^{i\delta}|1\rangle \right) e^{ikz-i\omega t} \quad (66)$$

where the corresponding Jones vector is

$$\vec{J} = a|0\rangle + be^{i\delta}|1\rangle, \quad |a|^2 + |b|^2 = 1. \quad (67)$$

The operations of optic devices can then be modeled as a 2×2 matrix, known as Jones matrix, namely

$$\vec{J}_{out} = U_{device}\vec{J}_{in}. \quad (68)$$

We see that this matches well with qubit and its unitary operations.

The Jones matrix (up to an overall phase factor) of a retardation plate with the fast axis at θ to the x -axis is given by

$$\begin{aligned} U_\delta(\theta) &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta) + e^{-i\delta} \sin^2(\theta) & \cos(\theta) \sin(\theta) - e^{-i\delta} \cos(\theta) \sin(\theta) \\ \cos(\theta) \sin(\theta) - e^{-i\delta} \cos(\theta) \sin(\theta) & e^{-i\delta} \cos^2(\theta) + \sin^2(\theta) \end{pmatrix}. \end{aligned} \quad (69)$$

Then we see that the operations of half wave plate and quarter-wave plate with the fast axis at θ to the x-axis on a given polarization state are given as the following:

- A half wave plate rotates the state vector by $\delta = \pi$. The operator reads

$$U_\pi(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad (70)$$

- A quarter-wave plate rotates the state vector by $\delta = \frac{\pi}{2}$. The operator reads

$$U_{\frac{\pi}{2}}(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i \cos(2\theta) & i \sin(2\theta) \\ i \sin(2\theta) & 1 - i \cos(2\theta) \end{pmatrix} \quad (71)$$

Claim: To realize a general 2×2 unitary matrix U up to a global phase factor, two quarter-wave plate and one half wave plate are needed.

To prove this, suppose that

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (72)$$

notice that the rows and columns of U are unit vectors, let $a = e^{i\theta_a} \cos \varphi$, then $b = e^{i\theta_b} \sin \varphi$ and $c = e^{i\theta_c} \sin \varphi$, this further implies that $d = e^{i\theta_d} \cos \varphi$. Here $\theta_a, \dots, \theta_d$ are some real parameters.

Two rows and two columns are orthogonal respectively, therefore, $a^*c + b^*d = 0$ and $a^*b + c^*d = 0$. This implies that when $\varphi \neq \frac{k\pi}{2}$ we must have

$$e^{\theta_c - \theta_a} + e^{\theta_d - \theta_b} = 0, \quad e^{\theta_b - \theta_a} + e^{\theta_d - \theta_c} = 0. \quad (73)$$

Now suppose that

$$U = U_{\frac{\pi}{2}}(\gamma)U_\pi(\beta)U_{\frac{\pi}{2}}(\alpha) \quad (74)$$

we obtain a system of equation

$$a = e^{i\theta_a} \cos \varphi = \cos(\alpha - \gamma) \cos(\alpha - 2\beta + \gamma) - i \sin(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) \quad (75)$$

$$b = e^{i\theta_b} \sin \varphi = \cos(\alpha - 2\beta + \gamma) \sin(\alpha - \gamma) + i \cos(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) \quad (76)$$

$$c = e^{i\theta_c} \sin \varphi = i \cos(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) - \cos(\alpha - 2\beta + \gamma) \sin(\alpha - \gamma) \quad (77)$$

$$d = e^{i\theta_d} \cos \varphi = \cos(\alpha - \gamma) \cos(\alpha - 2\beta + \gamma) + i \sin(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) \quad (78)$$

It's easily checked that we can choose α, β, γ such that these equation have a solution and conditions (73) is satisfied. This completes the proof of our claim.

To prepare the state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$, perform the unitary transformation $U = e^{i\delta}U_{\frac{\pi}{2}}(\gamma)U_\pi(\beta)U_{\frac{\pi}{2}}(\alpha)$ on the state $|0\rangle$. By solving the equations

$$\begin{pmatrix} C_0 \\ C_1e^{i\theta} \end{pmatrix} = e^{i\delta}U_{\frac{\pi}{2}}(\gamma)U_\pi(\beta)U_{\frac{\pi}{2}}(\alpha) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (79)$$

namely,

$$\begin{pmatrix} C_0 \\ C_1e^{i\theta} \end{pmatrix} = e^{i\delta} \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\alpha - \gamma) \cos(\alpha - 2\beta + \gamma) - i \sin(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) \\ \cos(\alpha - 2\beta + \gamma) \sin(\alpha - \gamma) + i \cos(\alpha + \gamma) \sin(\alpha - 2\beta + \gamma) \end{pmatrix} \quad (80)$$

we can obtain the corresponding angles between the fast axis to the x-axis of the quarter-wave plates and half wave plate. ■

Problem 13 Suppose a two particle pure state is of the form $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$
 (1) Calculate the reduced density matrices ρ_A and ρ_B .
 (2) Do Schmidt decomposition.

Solution:

(1) This can be done by direct calculation.

$$\begin{aligned}\rho_A &= \frac{1}{8}\{(|0\rangle + \sqrt{3}|1\rangle)(\langle 0| + \sqrt{3}\langle 1|) + (\sqrt{3}|0\rangle + |1\rangle)(\sqrt{3}\langle 0| + \langle 1|)\} \\ \rho_B &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0|\end{aligned}\quad (81)$$

(2) To do Schmidt decomposition, we need calculate the eigenvalue and eigenstates of two reduced density matrices. Notice that the eigenvalues of two reduced density matrices are the same, i.e., $\lambda_1 = \lambda_1^A = \lambda_1^B$ and $\lambda_2 = \lambda_2^A = \lambda_2^B$.

- For $\rho_A, \lambda_{1,2}^A = \frac{1}{4}(2 \pm \sqrt{3})$, then get eigenvectors.

$$\begin{aligned}|\lambda_1\rangle_A &= |\lambda_1^A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\lambda_2\rangle_A &= |\lambda_2^A\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned}$$

- For $\rho_B, \lambda_{1,2}^B = \frac{1}{4}(2 \pm \sqrt{3})$, the corresponding eigenstates are

$$\begin{aligned}|\lambda_1\rangle_B &= |\lambda_1^B\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\lambda_2\rangle_B &= |\lambda_2^B\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned}\quad (82)$$

Then, using the Schmidt theorem we know that

$$\begin{aligned}|\Phi\rangle &= \sqrt{\lambda_1} |\lambda_1\rangle_A |\lambda_1\rangle_B + \sqrt{\lambda_2} |\lambda_2\rangle_A |\lambda_2\rangle_B = \frac{\sqrt{2+\sqrt{3}}}{2} |\lambda_1\rangle_A |\lambda_1\rangle_B + \frac{\sqrt{2-\sqrt{3}}}{2} |\lambda_2\rangle_A |\lambda_2\rangle_B \\ &= \frac{1+\sqrt{3}}{2\sqrt{2}} |\lambda_1\rangle_A |\lambda_1\rangle_B + \frac{1-\sqrt{3}}{2\sqrt{2}} |\lambda_2\rangle_A |\lambda_2\rangle_B.\end{aligned}\quad (83)$$

This completes the solution. ■

Problem 14 Prove that suppose $|\psi\rangle$ is a pure state of a composite system, AB . Then there exist orthonormal states $|i_A\rangle$ for system A , and orthonormal states $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients.

Solution:

Suppose that the orthonormal basis $\{|e_A^i\rangle\}$ of \mathcal{H}_A is chose such that $\rho_A = \text{Tr}_B(\rho_{AB})$ is diagonal: $\rho_A = \sum_{i=1}^{d_A} p_i |e_A^i\rangle\langle e_A^i|$, with the orthonormal basis of \mathcal{H}_B is chose as $\{|e_B^j\rangle\}$. Assume that in this basis we

have $|\psi_{AB}\rangle = \sum_{i,j} T_{ij} |\tilde{e}_A^i\rangle \otimes |e_B^j\rangle$, then

$$\begin{aligned} |\psi_{AB}\rangle &= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} T_{ij} |\tilde{e}_A^i\rangle \otimes |e_B^j\rangle, \\ &= \sum_{i=1}^{d_A} |\tilde{e}_A^i\rangle \otimes \left(\sum_{j=1}^{d_B} T_{ij} |e_B^j\rangle \right), \\ &= \sum_{i=1}^{d_A} |\tilde{e}_A^i\rangle \otimes |e'^i_B\rangle. \end{aligned} \quad (84)$$

Where $|e'^i_B\rangle = \sum_{j=1}^{d_B} T_{ij} |e_B^j\rangle$. Note that we still need to prove states $\{|e'^i_B\rangle\}$ are mutually orthogonal or normalized. Since reduced density matrix

$$\begin{aligned} \sum_{i=1}^{d_A} p_i |e_A^i\rangle \langle e_A^i| &= \rho_A = \text{Tr}_B \rho_{AB}, \\ &= \sum_{i,i'=1}^{d_A} |\tilde{e}_A^i\rangle \langle \tilde{e}_A^{i'}| \text{Tr}_B (|e'^i_B\rangle \langle e'^{i'}_B|). \end{aligned} \quad (85)$$

Hence, $\text{Tr}_B |e'^i_B\rangle \langle e'^{i'}_B| = \langle e'^{i'}_B | e'^i_B \rangle = p_i \delta_{ii'}$, let $|\tilde{e}_B^i\rangle = \frac{1}{\sqrt{p_i}} |e'^i_B\rangle$, it turns out that $\{|\tilde{e}_B^i\rangle\}$ are orthonormal. This implies that

$$|\psi_{AB}\rangle = \sum_{i=1}^{d_A} \sqrt{p_i} |\tilde{e}_A^i\rangle \otimes |\tilde{e}_B^i\rangle. \quad (86)$$

We remark that although we write the sum run over $i = 1$ to $i = d_A$, the sum actually runs over the i for which $p_i \neq 0$.

By relabeling $\lambda_i = \sqrt{p_i}$ (thus $\sum_i \lambda_i^2 = \sum_i p_i = 1$), $|i_A\rangle = |\tilde{e}_A^i\rangle$ and $|i_B\rangle = |\tilde{e}_B^i\rangle$, we see that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \quad (87)$$

This completes the proof. ■

Problem 15 Prove that suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi}_i\rangle\}$ are two sets of normalized and orthogonal states in space H then there exist a unitary transformation U , s.t. $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, and construct this U transformation.

Solution:

The unitary transformation can be constructed as

$$U = \sum_i |\tilde{\psi}_i\rangle \langle \psi_i| \quad (88)$$

It's easily checked that

$$U|\psi_i\rangle = |\tilde{\psi}_i\rangle. \quad (89)$$

For checking the unitarity, we have

$$U^\dagger U = \left(\sum_j |\psi_j\rangle \langle \tilde{\psi}_j| \right) \left(\sum_i |\tilde{\psi}_i\rangle \langle \psi_i| \right) = \sum_i |\psi_i\rangle \langle \psi_i| = I \quad (90)$$

The UU^\dagger can be checked similarly. ■

Problem 16 Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle$$

where λ_i are real numbers, and $|i_A\rangle, |i_B\rangle, |i_C\rangle$ are orthonormal bases of the respective systems.

Solution

A characteristic feature of Schmidt-like form expression of

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle \quad (91)$$

is that the reduced density matrices ρ_A, ρ_B and ρ_C have the same eigenvalues λ_i^2 . For a pure tripartite state $|\Psi\rangle$, we know that the reduce density matrix $\rho_{AB} = \rho_C$, since we can use Schmidt decomposition of bipartite system AB and C. But when ρ_{AB} is a mixed state, ρ_A is not necessarily the same as ρ_B . Such kind of example is easy to given, suppose that $\rho_A = |0\rangle\langle 0|$ and $\rho_B = p|0\rangle\langle 0| + q|1\rangle\langle 1|$, we construct

$$\rho_{AB} = \rho_A \otimes \rho_B = p|00\rangle\langle 00| + q|01\rangle\langle 01|. \quad (92)$$

We can purify this mixed state by adding system C, thus we obtain

$$|\Psi\rangle_{ABC} = \sqrt{p}|000\rangle + \sqrt{q}|011\rangle. \quad (93)$$

If $p, q \neq 0, 1$, the state $|\Psi\rangle$ is what we want. ■