Angular Modulation in Classical Systems

A Geometric Proof of Cos(ϕ) and Cos²(ϕ) Decay

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Submitted by Iwl Robran
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Author’s Note

This thesis began with a simple intuition: that the familiar curves of $\cos(\phi)$ and $\cos^2(\phi)$ reflect something deeper than mere coincidence—they inescapably emerge from rotation itself. That core idea guided the structure, examples, and reasoning throughout.

The argument was conceived and developed independently by the author. Computational tools were used solely for editing, formatting, and symbolic refinement. All insights, errors, and conclusions are the responsibility of the author.

Abstract

This work demonstrates that the $\cos(\varphi)$ and $\cos^2(\varphi)$ decay patterns observed across numerous physical systems emerge not from domain-specific dynamics, but from intrinsic geometric properties of Euclidean space. Through both algebraic and geometric derivations, we show how angular alignment between two vectors modulates interaction strength via scalar projection. Linear interactions decay with $\cos(\varphi)$; quadratic interactions—such as energy, intensity, or overlap—decay with $\cos^2(\varphi)$. These patterns are not empirical fits or domain-specific quirks, but natural consequences of projection geometry.

I. Thesis

Whenever an interaction between two classical systems depends on rotational alignment, its strength modulates as:

$\cos(\varphi)$ for linearly projected relationships (e.g., force, shadow length)
$\cos^2(\varphi)$ for interactions depending on the square of that projection (e.g., intensity, energy, probability overlap)

These relationships arise directly from scalar projection and hold in any Euclidean system governed by directional modulation.

II. Definitions and Domain

Let a and b be unit vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ with angle φ between them. Define:

Scalar projection:

$$a \cdot b = \cos(\varphi)$$

Quadratic projection (used in energy/intensity models):

$$(a \cdot b)^2 = \cos^2(\varphi)$$

We restrict $\varphi \in [0°, 90°]$ to match physical observables that are non-negative and symmetric, as both $\cos(\varphi)$ and $\cos^2(\varphi)$ remain within [0, 1].

III. Algebraic Derivation

Let vector F represent a directional input (e.g., force or polarization), and vector X define the rotated axis or receiver.

Linear dependency (e.g., mechanical work):

$W = |F||X|\cos(\varphi) = F \cdot X$ (as derived in classical mechanics texts [4], [3])

Quadratic dependency (e.g., wave intensity, overlapping areas):

$I \propto (F \cdot X)^2 = \cos^2(\varphi)$ (common in optics and electromagnetism [1], [5], [7])

IV. Geometric Derivation

Consider a capsule-shaped object (e.g., a pill or rod) of length L, lying flat on a surface. When rotated by angle φ away from the horizontal axis, its horizontal projection becomes:

$$L\_proj = L \cos(\varphi)$$

This models systems where a fixed shape interacts with a slit, sensor, or boundary by geometric alignment.

Now suppose interaction strength depends on projected area (e.g., uniform field exposure or energy density). Then:

$$Area\_proj \propto L\_proj \times h \propto \cos(\varphi)$$

If interaction scales with energy or density over this area:

$$V(\varphi) \propto [L\_proj \times h]^2 \Rightarrow \propto \cos^2(\varphi)$$

V. Real-World Examples

Domain	Behavior	Dependency
Mechanics	Work = $F\,d\cos(\varphi)$	Linear [3], [4]
Optics (Malus' Law)	$I = I_0 \cos^2(\varphi)$	Quadratic [1], [5]
Overlapping Apertures	Overlap area $\propto \cos^2(\varphi)$	Quadratic [7]
Antenna Radiation	Gain $\propto \cos^2(\varphi)$	Quadratic [2], [6]
Shadow Casting	Shadow length $\propto \cos(\varphi)$	Linear

Note: In shadow casting, the length of the shadow scales with cos(φ), while the blocked intensity (or illuminated area) may scale with $\cos^2(\varphi)$ depending on detector geometry or system configuration.

VI. Context and Limitations

This proof applies specifically to systems where:

- Interaction strength depends on the directional alignment of two components
- Relationships are well-modeled by scalar projection or its square
- The system exists in Euclidean space and allows continuous rotation

This does not apply to:

- Inverse-square laws (e.g., gravity, radiation over distance)
- Exponential decays (e.g., radioactive decay, capacitor discharge)
- Systems with discontinuous alignment, non-Euclidean geometry, or non-projective interactions

Cos(φ) and $\cos^2(\varphi)$ are not universal decay laws — they are the default angular modulation surfaces in rotation-aligned classical systems.

VII. Pedagogical and Cross-Domain Relevance

This principle has high explanatory and instructional value, especially for:

• Linear Algebra: vector projection, basis decomposition
• Classical Mechanics: force alignment, torque, work
• Electromagnetic Theory: polarization, antenna design, radiative energy [2], [6], [7]
• Engineering and Signal Processing: sensor orientation, directional gain

*Though cosine modulation appears in quantum contexts, including interference models, this work is entirely classical in origin and does not invoke quantum formalism [3].

VIII. Conclusion

Cosine is not an arbitrary curve. It is the geometry of directional rotation.
• $\cos(\varphi)$ arises when systems respond to linear projection
• $\cos^2(\varphi)$ emerges when output scales with squared projection — such as in energy, probability, or overlap

This is not a modeling artifact.
It is the decay surface geometry demands.

References

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