Lecture 16: Unsupervised Learning: Mixture Models

And a return to dimensionality reduction. . .

Iain Styles

29 November 2019

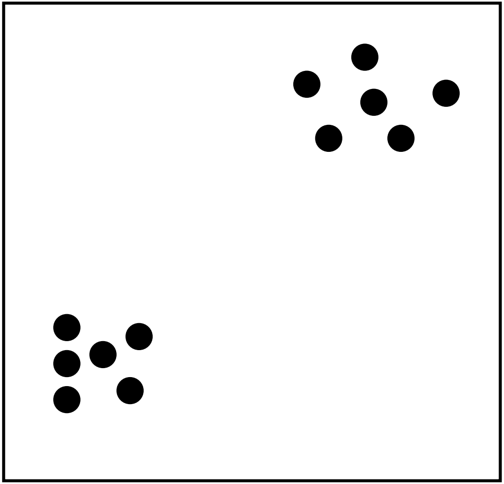
Learning Outcomes

By the end of this lecture you should

- Understand the principles of Gaussian mixture modelling
- Understand how dimensionality reduction can improve classifier performance.

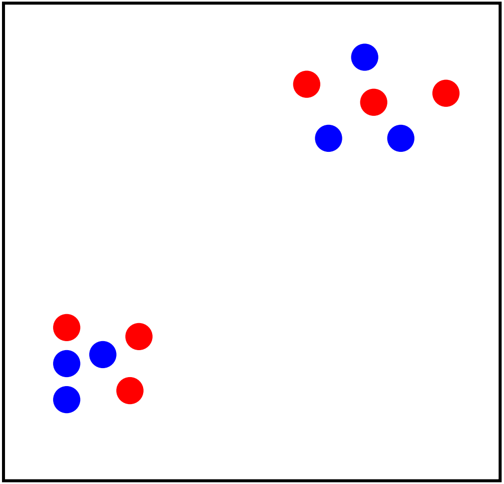
Recap: The k-means Algorithm

Initial data points



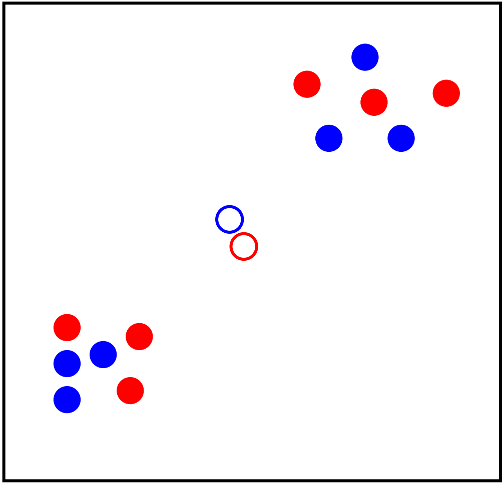
Recap: The k-means Algorithm

Randomly assign points to groups



Recap: The k-means Algorithm

Compute group average



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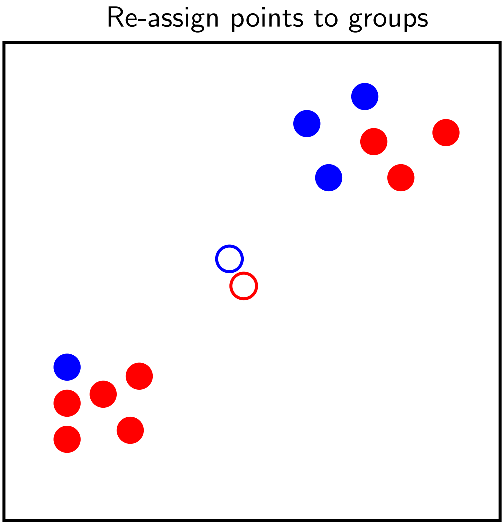
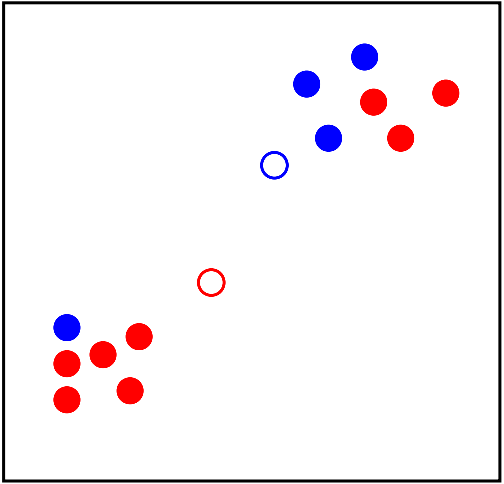


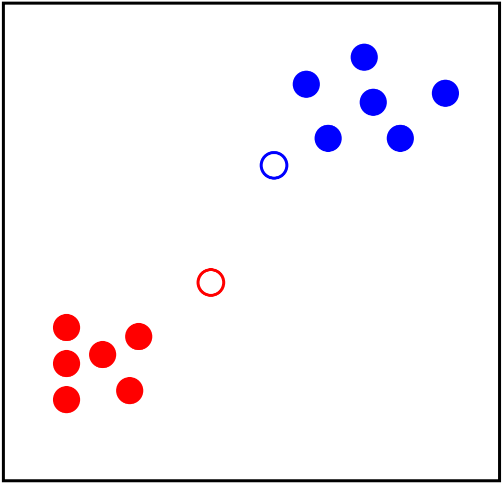
Recap: The k-means Algorithm

Re-compute group averages



Recap: The k-means Algorithm

Re-assign points to groups



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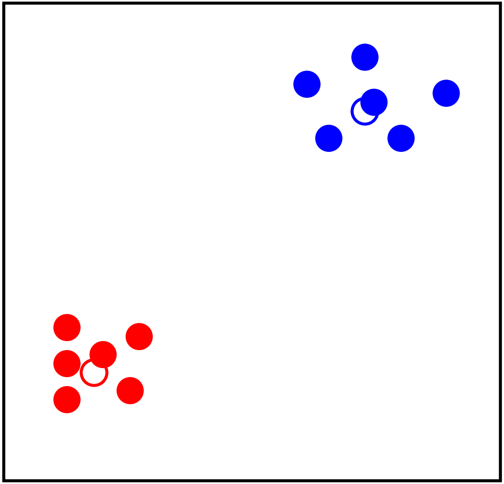
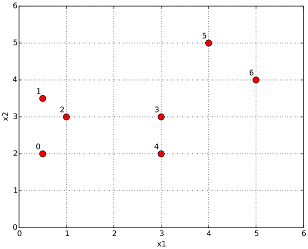
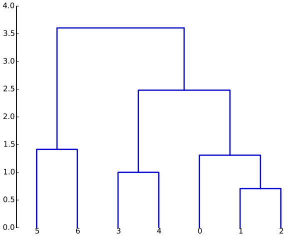


Recap: Agglomerative Clustering





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- Assume data is generated by a statistical process that is a mixture of components
- Clustering: which component generated the data point

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- The sum of K discrete Gaussian clusters
- Clustering is then formulated as
  1. Learn the parameters of the GMM which best describe the data.
  2. Determine from which component a data point is most likely to have been generated.

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we interpret

- $A_k = p(k)$ as the prior probability of choosing a point from component k
- $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) = p(\mathbf{x}|k)$ as the class-conditional likelihoods.

Learning Gaussian Mixture Models

- Finally, using Bayes' theorem we compute the posterior *responsibilities*

$$
\begin{aligned}
r_k(\mathbf{x}) &= p(k|\mathbf{x}) && (5) \\
&= \frac{p(k)p(\mathbf{x}|k)}{\sum_{k'=1}^{K} p(k')p(\mathbf{x}|k')} && (6) \\
&= \frac{A_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{k'=1}^{K} A_{k'} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_{k'}, \boldsymbol{\Sigma}_{k'})} && (7)
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- Probability that component k explains $\mathbf{x}$
- GMM gives *soft* cluster assignments

`https://colab.research.google.com/drive/1Gta0oTu_86UFkAMHqrhpQ7EdKrgmaPXW`

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- **Earlier discussions about this were incorrect due to a bug…**
  - Volume of space is near the "corners"
  - Distance functions converge
  - Computations are expensive
- Is it problematic in practice?
  - Convergence of distances is, but only in data that lacks structure
  - Data is not usually uniformly distributed
  - Computational expense is always a problem – comparing two 10,000-dimensional vectors is expensive

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- f preserves relative distances
- Holds when "f is a random, orthonormal linear transformation"

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- Most data has structure so convergence of distances is not a practical obstacle to learning.
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Why does this matter?

- Most data has structure so convergence of distances is not a practical obstacle to learning.
- But computational expensive is.
- Johnson-Lindenstrass provides a cheap, data-independent way to reduce the dimensionality.
- Distance calculations can be sped up dramatically.

https://colab.research.google.com/drive/1HOoR2sHwKefc5a-AIePoX3u7C9H-GsJL

Summary

- Dimensionality does not necessarily improve classifier accuracy
- But a low-cost classifier such as random projection can dramatically reduce computational cost without affecting accuracy
- Important in practical applications of classifiers
- Next (last) lecture: a few words on fairness and bias in machine learning