Linear algebra 3
Matrices and Linear Transformations

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Overview

1. Matrices
   - Introduction to matrices
   - Matrix multiplication
2. Linear transformations
   - Definition
   - Any matrix implements an linear transform
   - A linear transformation is defined by what it does to a basis
   - Every linear transformation is defined by a matrix
3. Change of basis
   - Change of basis
   - Orthogonal matrices
4. Summary

Matrices

A **matrix** A is just a rectangular grid of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12}, & \cdots & a_{1j}, & \cdots, & a_{1N} \\ a_{21} & a_{22}, & \cdots & a_{2j}, & \cdots, & a_{2N} \\ \vdots & \vdots & \cdots & \vdots, & \vdots, & \vdots \\ a_{i1} & a_{i2}, & \cdots & a_{ij}, & \cdots, & a_{iN} \\ \vdots & \vdots & \cdots & \vdots, & \vdots, & \vdots \\ a_{M1} & a_{M2}, & \cdots & a_{Mj}, & \cdots, & a_{MN} \end{bmatrix} \quad (1)$$

A is an M (number of rows) $\times N$ (number of columns) matrix.
In this course the matrix entries are **real** numbers, $a_{ij} \in \mathbb{R}$

Matrix multiplication

- Given an $M \times N$ matrix A and a $P \times Q$ matrix B, the **matrix product** AB exists if and only if $N = P$. In this case $C = AB$ is the $M \times Q$ matrix whose $(i,j)^{th}$ entry is:

$$c_{ij} = ab_{ij} = \sum_{k=1}^{P} a_{ik} b_{kj} = a_{i*} \cdot b_{*j} \tag{2}$$

where a_{i*} is the i^{th} row of A and b_{*j} is the j^{th} column of B

$$\begin{bmatrix} a_{11} & \cdots & a_{ik}, & \cdots, & a_{1P} \\ \vdots & \cdots & \vdots, & \vdots, & \vdots \\ a_{i1} & \cdots & a_{ik}, & \cdots, & a_{iP} \\ \vdots & \cdots & \vdots, & \vdots, & \vdots \\ a_{M1} & \cdots & a_{ik}, & \cdots, & a_{MP} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b1j, & \vdots, & b1N \\ \vdots & \cdots & \vdots, & \vdots, & \vdots \\ \vdots & \cdots & bkj, & \vdots, & \vdots \\ \vdots & \cdots & \vdots, & \vdots, & \vdots \\ b_{P11} & \cdots & bPj, & \vdots, & b_{PN} \end{bmatrix} \tag{3}$$

Example

Suppose

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}, B = \begin{bmatrix} -7 & 8 \\ 9 & 10 \\ -11 & 12 \end{bmatrix}$$

Then, for example

$$ab_{12} = a_{1*} \cdot b_{*2} = 1 \times 8 + (-2) \times 10 + 3 \times 12 = 24 \tag{4}$$

and

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix} \begin{bmatrix} -7 & 8 \\ 9 & 10 \\ -11 & 12 \end{bmatrix} = \begin{bmatrix} -58 & 24 \\ 83 & 10 \end{bmatrix} \tag{5}$$

Matrix multiplication

- Matrix multiplication is **not commutative**. In other words, even if AB and BA both exist it is not generally true that $AB = BA$
- The $N \times N$ **identity** matrix is the matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- If A is $N \times N$ then $AI = IA = A$

Matrices as transformations

If A is an $N \times N$ matrix and $\vec{v} \in \mathbb{R}^N$ is a real N-dimensional vector, then $A\vec{v}$ exists and

$$\vec{w} = A\vec{v} \in \mathbb{R}^N$$

In other words, $\vec{w}$ is an N-dimensional vector and A defines a **transformation** T_A from $\mathbb{R}^N$ to itself:

$$T_A : \mathbb{R}^N \to \mathbb{R}^N, \, T_A(\vec{v}) = A\vec{v}$$

Definition

- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces
- Let $T : \mathbb{V} \rightarrow \mathbb{W}$ be a transformation.
- So, T is a rule that associates each $\vec{v} \in \mathbb{V}$ with $T(\vec{v}) \in \mathbb{W}$
- T is a *linear* transformation if for $\vec{v_1}, \vec{v_2} \in \mathbb{V}$ and any scalar λ:

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2}) \quad (6)$$

$$T(\lambda \vec{v_1}) = \lambda T(\vec{v_1}) \quad (7)$$

- Alternatively

$$T(\lambda \vec{v_1} + \vec{v_2}) = \lambda T(\vec{v_1}) + T(\vec{v_2}) \quad (8)$$

Example: rotations in $\mathbb{R}^2$

- In $\mathbb{R}^2$, rotation anti-clockwise through an angle θ is realized by multiplication by the rotation matrix

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos\theta \end{bmatrix} \tag{9}$$

- Define a linear transformation $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T_\theta(\vec{v}) = R_\theta \vec{v} \tag{10}$$

  Then T_θ is a linear transformation.
- This follows from properties of matrix multiplication, i.e.

$$T_\theta(\lambda\vec{v_1}+\vec{v_2}) = R_\theta(\lambda\vec{v_1}+\vec{v_2}) = \lambda R_\theta\vec{v_1}+R_\theta\vec{v_2} = \lambda T_\theta(\vec{v_1})+T_\theta(\vec{v_2})$$

Any matrix defines a linear transformation

- In fact, *any* $M \times N$ matrix A defines a linear transformation

$$T_A : \mathbb{R}^N \rightarrow \mathbb{R}^M \tag{11}$$

$$\text{by: } T_A(\vec{v}) = A\vec{v}, (\vec{v} \in \mathbb{R}^N) \tag{12}$$

- So, are there linear transforms $T_A : \mathbb{R}^N \rightarrow \mathbb{R}^M$ that are **not** defined by a matrix in this way?
- The answer is "no". *Every* linear transformation is defined by a matrix. Hence there is a 1-to-1 correspondence between linear transformations $T : \mathbb{R}^N \rightarrow \mathbb{R}^M$ and $M \times N$ matrices

Every linear transformation is defined by a matrix

- Let $T : \mathbb{R}^N \to \mathbb{R}^M$ be a linear transformation
- Let $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}\}$ be any basis for $\mathbb{R}^N$,
- Let $\vec{v} \in \mathbb{R}^N$. Then $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_N\vec{e_N}$, for some scalars $v_1, \cdots, v_N$
- Since T is linear, $T(\vec{v}) = v_1 T(\vec{e_1}) + v_2 T(\vec{e_2}) + \cdots + v_N T(\vec{e_N})$
- T is *completely* defined by what it does to $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}\}$
- This holds for *any* basis $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}\}$

Every linear transformation is defined by a matrix (cont.)

- Let $T : \mathbb{R}^N \to \mathbb{R}^M$ be a linear transformation
- Now let $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}\}$ be the standard basis for $\mathbb{R}^N$, i.e

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e_N} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ and let } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{13}$$

- Let E be the $M \times N$ matrix whose n^{th} column is $T(\vec{e_n})$
- Then,

$$T(\vec{v}) = E\vec{v}, (\vec{v} \in \mathbb{V}) \tag{14}$$

- The linear transformation T is matrix multiplication by E

Proof

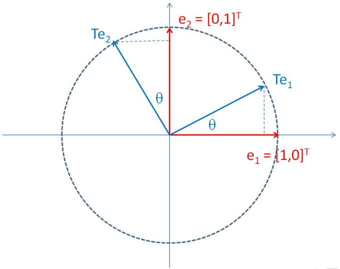
$$E = \begin{bmatrix} T(\vec{e_1})_1 & T(\vec{e_2})_1 & \cdots & T(\vec{e_n})_1 & \cdots & T(\vec{e_N})_1 \\ T(\vec{e_1})_2 & T(\vec{e_2})_2 & \cdots & T(\vec{e_n})_2 & \cdots & T(\vec{e_N})_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T(\vec{e_1})_n & T(\vec{e_2})_n & \cdots & T(\vec{e_n})_n & \cdots & T(\vec{e_N})_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T(\vec{e_1})_M & T(\vec{e_2})_M & \cdots & T(\vec{e_n})_M & \cdots & T(\vec{e_N})_M \end{bmatrix} \quad (15)$$

Proof (continued)

$$
\begin{aligned}
Ev &= \begin{bmatrix} T(\vec{e_1})_1 & T(\vec{e_2})_1 & \cdots & T(\vec{e_n})_1 & \cdots & T(\vec{e_N})_1 \\ T(\vec{e_1})_2 & T(\vec{e_2})_2 & \cdots & T(\vec{e_n})_2 & \cdots & T(\vec{e_N})_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T(\vec{e_1})_n & T(\vec{e_2})_n & \cdots & T(\vec{e_n})_n & \cdots & T(\vec{e_N})_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T(\vec{e_1})_M & T(\vec{e_2})_M & \cdots & T(\vec{e_n})_M & \cdots & T(\vec{e_N})_M \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ \vdots \\ v_N \end{bmatrix} \\
&= \begin{bmatrix} T(\vec{e_1})_1 v_1 + T(\vec{e_2})_1 v_2 + \cdots + T(\vec{e_N})_1 v_N \\ T(\vec{e_1})_2 v_1 + T(\vec{e_2})_2 v_2 + \cdots + T(\vec{e_N})_2 v_N \\ \vdots \\ T(\vec{e_1})_n v_1 + T(\vec{e_2})_n v_2 + \cdots + T(\vec{e_N})_n v_N \\ \vdots \\ T(\vec{e_1})_M v_1 + T(\vec{e_2})_M v_2 + \cdots + T(\vec{e_N})_M v_N \end{bmatrix} = T(\vec{v}) \quad (16)
\end{aligned}
$$

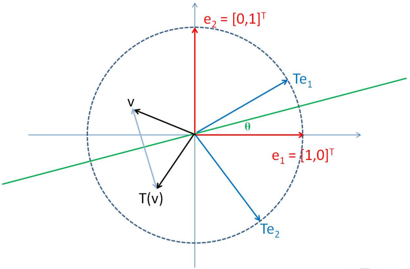
Example 1: Rotations in $\mathbb{R}^2$

- Suppose that the linear transformation T corresponds to rotation anti-clockwise through an angle θ
- $T(\vec{e_1}) = \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix}$, $T(\vec{e_2}) = \begin{bmatrix} -sin(\theta) \\ cos(\theta) \end{bmatrix}$
- Hence T is implemented by $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$



Example 2: Reflections in $\mathbb{R}^2$

- The transformation T corresponds to a reflection through a line through the origin at angle θ to the horizontal axis
- $T(\vec{e_1}) = \begin{bmatrix} cos(2\theta) \\ sin(2\theta) \end{bmatrix}$, $T(\vec{e_2}) = \begin{bmatrix} cos(\frac{\pi}{2} - 2\theta) \\ -sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} sin(2\theta) \\ -cos(2\theta) \end{bmatrix}$
- Hence T is implemented by $\begin{bmatrix} cos(2\theta) & sin(2\theta) \\ sin(2\theta) & -cos(2\theta) \end{bmatrix}$



Summary so far

- There is a 1-to-1 correspondence between:
  - Linear transformations $T : \mathbb{R}^N \to \mathbb{R}^M$, and
  - $N \times M$ real matrices
- If
  - $T : \mathbb{R}^N \to \mathbb{R}^M$,
  - $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}$ is the standard basis for $\mathbb{R}^N$, and
  - E is the $M \times N$ matrix whose n^{th} column is $T(\vec{e_n})$, then

$$T(\vec{v}) = E\vec{v}, (\vec{v} \in \mathbb{R}^N) \tag{17}$$

Change of basis

- Let $e = \vec{e_1}, \vec{e_2}, \cdots, \vec{e_N}$ be the standard basis for $\mathbb{R}^N$
- Let $u = \vec{u_1}, \vec{u_2}, \cdots, \vec{u_N}$ be another basis for $\mathbb{R}^N$
- Suppose that we want to transform $\mathbb{R}^N$ so that vectors are represented with respect to u
- The required transformation would satisfy $T(\vec{e_n}) = \vec{u_n}, (n = 1, \cdots, N)$
- This transformation is implemented by the $N \times N$ matrix U whose n^{th} column is $\vec{u_n}$

Orthogonal matrices

- Let U be the $N \times N$ matrix whose n^{th} column is $\vec{u_n}$, then

$$U^T U = I_N = UU^T. \tag{18}$$

  because the i, j^{th} entry of $U^T U$ is $\vec{u_i} \cdot \vec{u_j}$ and $\vec{u_1}, \cdots \vec{u_N}$ is a basis. where I_N is the $N \times N$ identity matrix.
- An $N \times N$ real matrix that satisfies equation (18) is called an **orthogonal** matrix
- An orthogonal matrix U implements a **change of basis** transformation in $\mathbb{R}^N$:
  - U transforms the standard basis $\vec{e_1}, \cdots, \vec{e_N}$ into the new basis $\vec{u_1}, \cdots, \vec{u_N}$
  - For any $\vec{v} \in \mathbb{R}^N$, $U\vec{v}$ is the vector of coordinates of $\vec{v}$ with respect to $\vec{u_1}, \cdots, \vec{u_N}$

Summary

- Definition of a linear transformation
- A linear transformation is defined by what it does to a basis
- There is a one-to-one correspondence between
  - Linear transformations $T : \mathbb{R}^N \to \mathbb{R}^M$ and
  - $M \times N$ matrices
- Definition of an orthogonal matrix
- Orthogonal matrices correspond to change-of-basis transformations