Assignment 0 – Mathematics

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1. A continous probability density function has which of the following properties? Choose all that apply.

   (a) It must be less than 1 everywhere.

   (b) It must be greater than or equal to 0 everywhere.

   (c) The area underneath must be equal to exactly 1.

   (d) Its maximum value must be exactly 1.

   Solution.

   (a) Not true. The requirements for the total area under the PDF to be equal to one does not prevent it from being greater than one.

   (b) True. Negative probability has no meaning in the classical framework of probability.

   (c) True, because only outcomes in the specified sample space can occur.

   (d) Not true, it is only the area that matters.

2. A discrete probability distribution has which of the following properties?

   (a) Individual outcomes can have a probability of greater than one.

   (b) All outcomes must have a probability that is greater than or equal to zero.

   (c) The sum of the probabilities of all of the outcomes must be equal to one.

   (d) All outcomes must have a probability that is less than or equal to one.

   Solution.

   (a) Not true. This would imply that an event was "more than certain".

   (b) True. Negative probability have no meaning in the classical framework of probability.

   (c) True, because only outcomes in the specified sample space can occur.

   (d) True, follows from (b) and (c).

3. What is the length (magnitude) of the vector $\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$?

Give your answer to two decimal places.

Solution.

We answer this using Pythagoras' theorem: $L = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{6^2 + 2^2 + 4^2} = \sqrt{36 + 4 + 16} = \sqrt{56} = 7.48$.

4. The product **XY** of the matrices $\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$ is a 2×2 matrix. What are the values of its elements?

   Solution.

   Using the rules of matrix multiplication, $M_{ij} = \sum_k X_{ik} Y_{kj}$, we have

   $$\mathbf{XY} = \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 \times 3 + 3 \times 2 & 1 \times 1 + 3 \times 5 \\ 4 \times 3 + 4 \times 2 & 4 \times 1 + 4 \times 5 \end{pmatrix} \begin{pmatrix} 9 & 16 \\ 20 & 24 \end{pmatrix}$$

5. Matrix **X** has 13 rows and 6 columns. Matrix **Y** has 6 rows and 5 columns. Their product **XY** has ? rows and ? columns.

   Solution.

   Again, from the rules of matrix multiplication, $M_{ij} = \sum_k X_{ik} Y_{kj}$, $\mathbf{M} = \mathbf{XY}$ must have the same number of rows as **X** (13) and the same number of columns as **Y** (5).

6. A class of students have to choose from a range of options. Two of those options are Machine Learning and Computer Graphics.

   - The proportion of the class that chose Computer Graphics is 0.4
   - The proportion of the class that chose Machine Learning is 0.6
   - Half of those who chose Computer Graphics also chose Machine Learning.

   What is the probability that a student who chose Machine Learning also chose Computer Graphics?

   Give your answer accurate to two decimal places.

   Solution

   Denoting "Computer Graphics" as CG and "Machine Learning" as ML, we have, from the question, that $P(CG = 0.4)$, $P(ML) = 0.6$, and $P(ML|CG) = 0.5$. It then follows from Bayes theorem that $P(CG|ML) = P(ML|CG)P(CG)/P(ML) = 0.5 \times 0.4/0.6 = 0.33$

7. What is the inverse of matrix $\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$?

   (a) $\mathbf{X}^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

   (b) $\mathbf{X}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

(c) $\mathbf{X}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$

Solution

To solve this, we test each answer by computing $\mathbf{X}\mathbf{X}^{-1}$

(a) $\mathbf{X}\mathbf{X}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 20 \end{pmatrix}$

(b) $\mathbf{X}\mathbf{X}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\mathbf{X}\mathbf{X}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & 1 \\ \frac{13}{3} & \frac{5}{2} \end{pmatrix}$

Only case (b) gives the identity matrix and this is therefore the solution.

8. Two discrete variables X and Y have the joint probability distribution shown in the table below.

	$X = x1$	$X = x2$	$X = x3$
$Y = y1$	0.10	0.10	0.20
$Y = y2$	0.30	0.10	0.20

What is $P(X|Y = y2)$?

(a)

$X = x1$	$X = x2$	$X = x3$
0.30	0.10	0.20

(b)

$X = x1$	$X = x2$	$X = x3$
0.25	0.25	0.50

(c)

$X = x1$	$X = x2$	$X = x3$
0.50	0.17	0.33

Note that the values in the tables above are accurate to two decimal places.

Solution

We need the product rule of probability here: $P(X, Y) = P(X|Y)P(Y)$. From this, we have $P(X|Y = y2) = P(X, Y = y2)/P(Y = y2)$. The second row of the table gives us $P(X, Y = y2)P(Y = y2)$, and $P(Y = y2) = \sum_{X=\{x1,x2,x3\}} P(X, Y = y2) = 0.30 + 0.10 + 0.20 = 0.60$. Therefore $P(X = x1|Y = y2) = 0.3/0.6 = 0.5$, $P(X = x2|Y = y2) = 0.1/0.6 = 0.17$, and $P(X = x3|Y = y2) = 0.2/0.6 = 0.33$, answer (c).

9. A function $f(x)$, defined for $-\infty \leq x \leq \infty$, has its minimum value when $x = 0$. Which one of the following statements is true?

(a) The gradient of $f(x)$ at $x = 0$ is equal to the value of $f(x)$ at $x = 0$.

(b) The value of $f(x)$ is zero at $x = 0$.

(c) The gradient of $f(x)$ is zero at $x = 0$.

Solution

The only way that a function can take a minimum value away from the edges of its domain is for there to be a turning point where the gradient is zero (c). The other choices do not imply a minimum.

Although it was not explicitly stated in the question, there are some special cases where this may not be true, and there are stronger conditions needed on the function. For example, $f(x) = 1/x$ has a singularity at $x = 0$ and the minimum values tends to $-\infty$. The question should have stated that the function is required to be continous and differentiable (i.e. smooth).

10. A column vector $\mathbf{v}$ has components v_i. The magnitude, or length of v is given by the formula $L = \sqrt{\sum_i v_i^2)}$. This can also be written as:

(a) $L = \mathbf{v}\mathbf{v}^{\mathrm{T}}$

(b) $L = \sqrt{\mathbf{v}\mathbf{v}^{\mathrm{T}}}$

(c) $L = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$

(d) $L = \mathbf{v}^{\mathrm{T}}\mathbf{v}$

Solution

Noting again the rules of matrix multiplication and treating a column vector as an $N \times 1$ matrix, it follows that $\sum_i v_i^2 = \sum_i v_{1i} v_{i1} = \mathbf{v}^{\mathrm{T}}\mathbf{v}$. Taking the square root leads to the correct answer (c).