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Nature Inspired Search and Optimisation

22 – Runtime Analysis

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Evolutionary Algorithms

(μ+λ) EA

Initialise P_0 with μ individuals chosen uniformly a random from $\{0,1\}^n$
for $t = 0, 1, 2, \ldots$ until stopping condition met **do**
Create λ new individuals by

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choosing the best μ individuals out of $\mu + \lambda$.
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- If $\mu = \lambda = 1$, then we get the (1+1) EA;
- $p = 1/n$ is generally considered a good parameter setting [Bäck, 1993, Droste et al., 1998];
- By introducing stochastic selection and crossover we obtain a Genetic Algorithm (GA)

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If only one bit is flipped per iteration: Random Local Search (RLS).

How does it work?

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$$= \sum_{i=1}^{n} 1 \cdot 1/n = n/n = 1$$

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Is flipping two bits more likely than flipping none?

$$\begin{aligned}\Pr(X = 2) &= \binom{n}{2}\left(\frac{1}{n}\right)^2\left(1 - \frac{1}{n}\right)^{n-2} \\ &= \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2\left(1 - \frac{1}{n}\right)^{n-2} \\ &= \frac{1}{2}\left(1 - \frac{1}{n}\right)^{n-1} \approx 1/(2e)\end{aligned}$$

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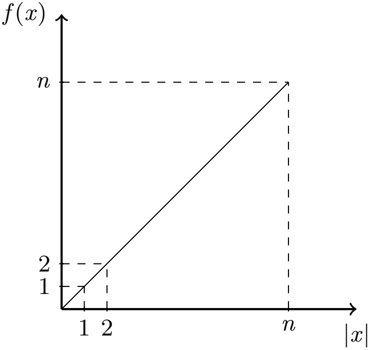
While

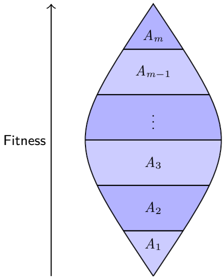
$$\Pr(X = 0) = \binom{n}{0}(1/n)^0 \cdot (1 - 1/n)^n \approx 1/e$$

OneMax

$$\textsc{OneMax}(x) := x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$



Fitness-based Partitions



Definition

A tuple $(A_1, A_2, \ldots, A_m)$ is an f-**based partition** of $f : \mathcal{X} \to \mathbb{R}$ if

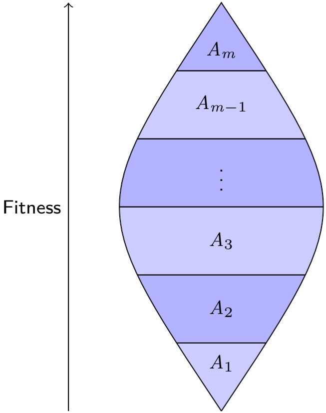
1. $A_1 \cup A_2 \cup \cdots \cup A_m = \mathcal{X}$
2. $A_i \cap A_j = \emptyset$ for $i \neq j$
3. $f(A_1) < f(A_2) < \cdots < f(A_m)$
4. $f(A_m) = \max_x f(x)$

Example

Partition of $\textsc{OneMax}$ into $n + 1$ levels

$$A_j := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = j\}$$

Artificial Fitness Levels - Upper bounds



s_i : prob. of starting in A_i

u_i : prob. of jumping from A_i to any A_j, $i < j$.

T_i : Time to jump from A_i to any A_j, $i < j$.

Expected runtime

$$\begin{aligned}\mathbb{E}[T] &\leq \sum_{i=1}^{m-1} s_i \mathbb{E}\left[\sum_{j=i}^{m-1} T_j\right] \\ &= \sum_{i=1}^{m-1} s_i \sum_{j=i}^{m-1} \mathbb{E}[T_j] \\ &= \sum_{i=1}^{m-1} s_i \sum_{j=i}^{m-1} 1/u_j \leq \sum_{j=1}^{m-1} 1/u_j.\end{aligned}$$

(1+1) EA on ONEMAX

Theorem

The expected runtime of (1+1) EA on ONEMAX *is* $O(n \ln n)$.

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- To reach a higher fitness level it is sufficient to flip a zero into a one and leave the other bits unchanged, which occurs with probability

$$u_j \geq (n-j)\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} \geq \frac{n-j}{en}$$

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- Then by Artificial Fitness Levels

$$\mathbb{E}\left[T\right] \leq \sum_{j=0}^{m-1} 1/u_j \leq \sum_{j=0}^{n-1} \frac{en}{n-j} = en\sum_{i=1}^{n}\frac{1}{i} \leq en(\ln n + 1) = O(n \ln n)$$