Lecture 14: Unsupervised Learning: Clustering

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29 November 2019

Introduction

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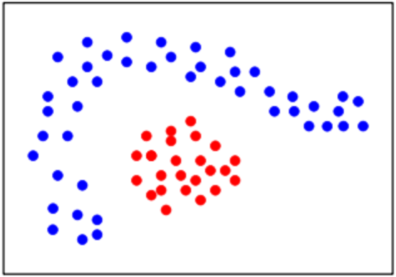
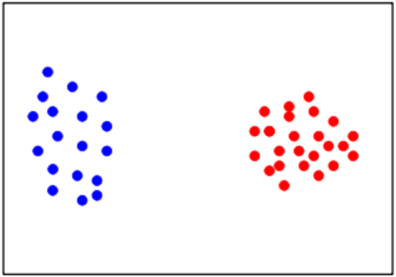
- Can we "discover" concepts/structure in data?
- *Unsupervised learning*
- Main approach is *clustering*





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- Common approaches include:
  - Vector quantisation
  - Agglomerative approaches
  - Mixture modelling

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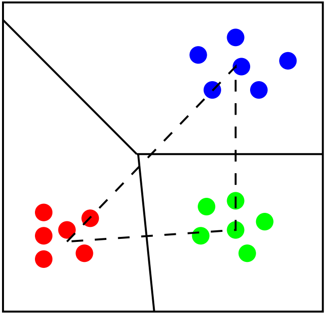
- *Voronoi Tesselation* of the space.
- Prototype vectors are often known as the cluster *centroids*.



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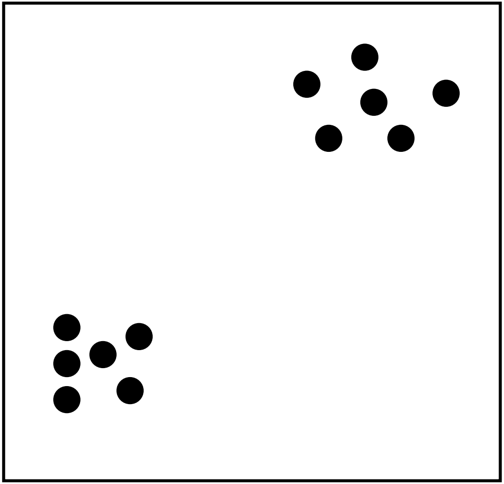

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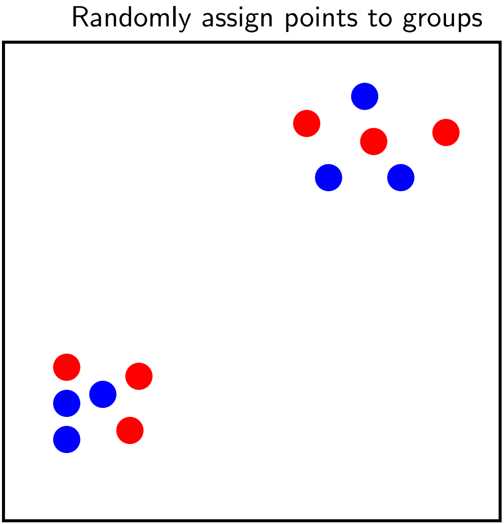
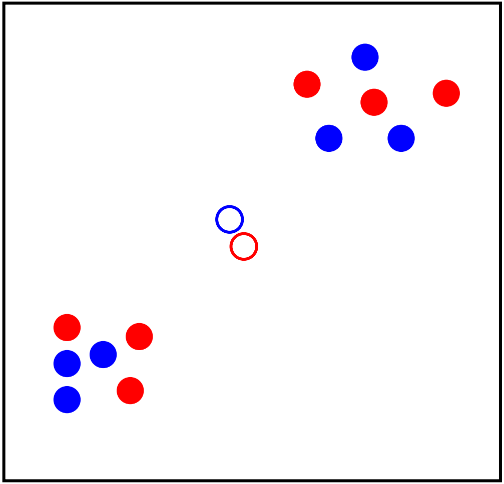


The k-means Algorithm

Compute group average



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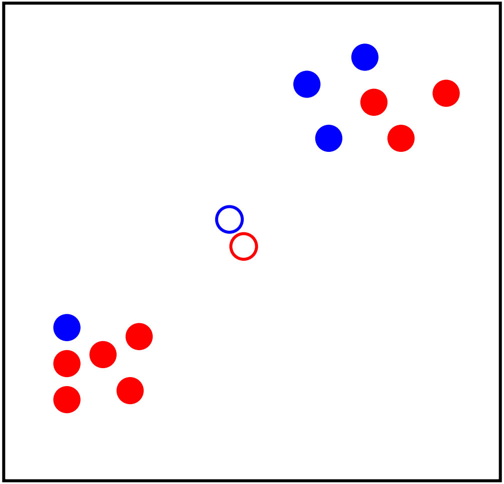

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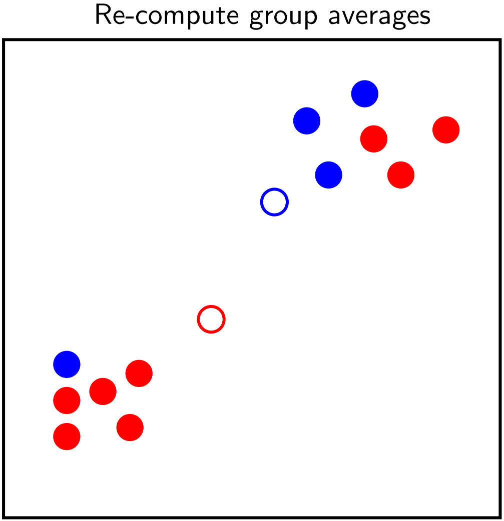

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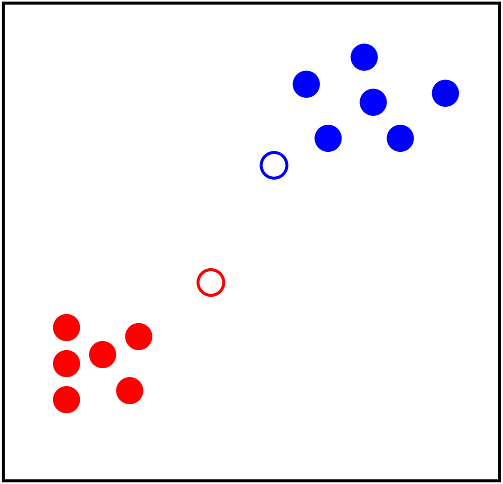

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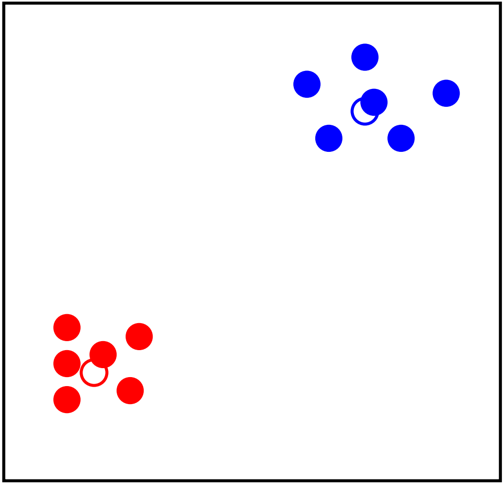
Re-compute group averages



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- All degrees of cluster can be extracted from this

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5) Go to 2) and continue grouping until all points are grouped

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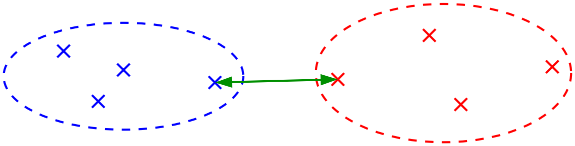

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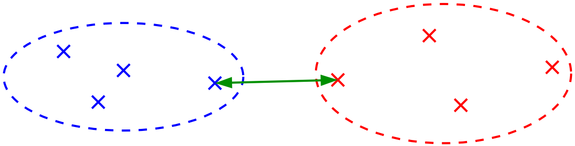

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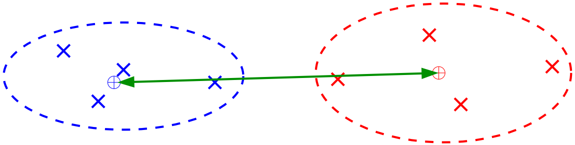

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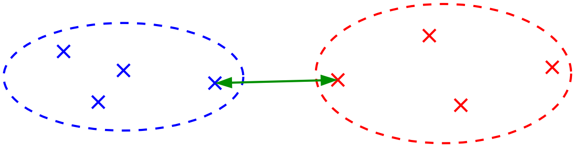

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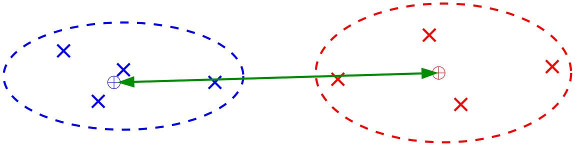

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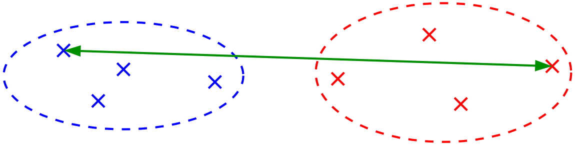

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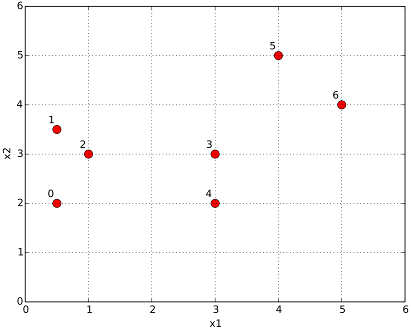

List of points 0,1,2,3,4,5,6

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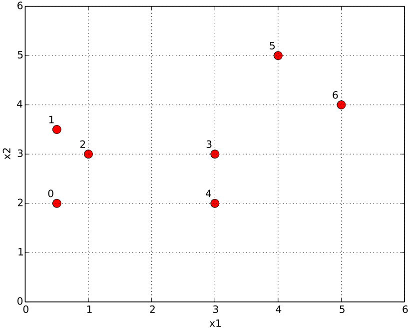

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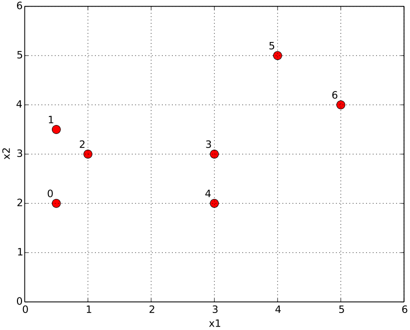

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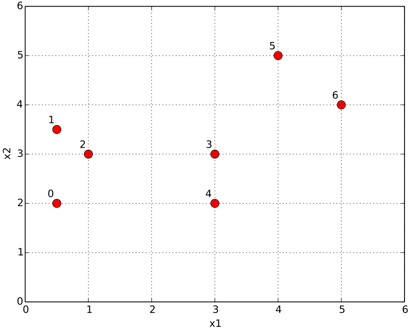

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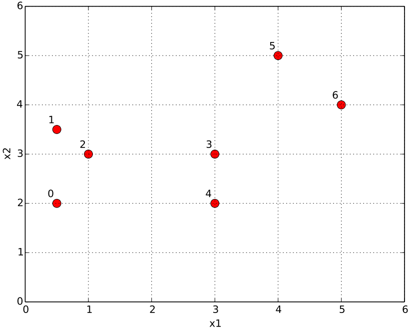

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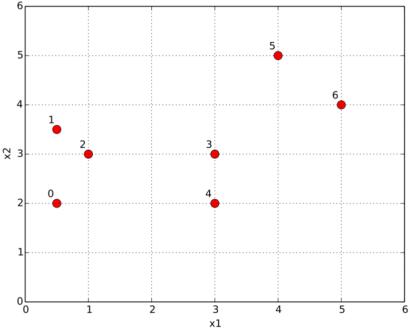

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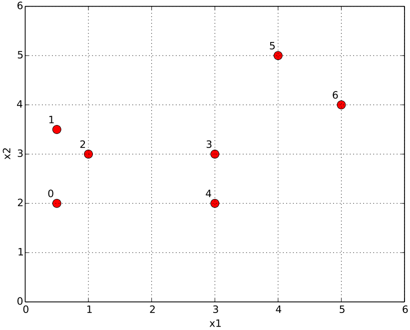

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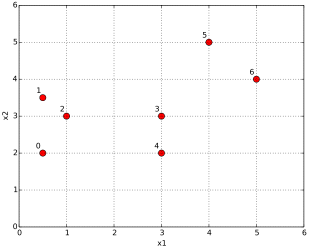

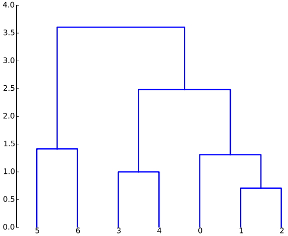

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- Clustering: which component generated the data point

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- Clustering is then formulated as
  1. Learn the parameters of the GMM which best describe the data.
  2. Determine from which component a data point is most likely to have been generated.

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we interpret

- $A_k = p(k)$ as the prior probability of choosing a point from component k
- $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) = p(\mathbf{x}|k)$ as the class-conditional likelihoods.

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- Finally, using Bayes' theorem we compute the posterior *responsibilities*

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- Probability that component k explains $\mathbf{x}$
- GMM gives *soft* cluster assignments

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