# Probabilistic Robotics*

## Partially Observable Markov Decision Process

**Mohan Sridharan**

University of Birmingham, UK

*m.sridharan@bham.ac.uk*

*Revised original slides that accompany the book by Thrun, Burgard and Fox.

# POMDPs

- **State is not observable** – agent has to make decisions based on belief state which is a **posterior distribution over states**.
- Let $b$ be the belief of the agent about the state under consideration.
- POMDPs compute a **value function over belief space**:

$$V_T(b) = \gamma \max_u \left\{ r(b,u) + \int V_{T-1}(b')\, p(b' \mid u,b)\, db' \right\}$$

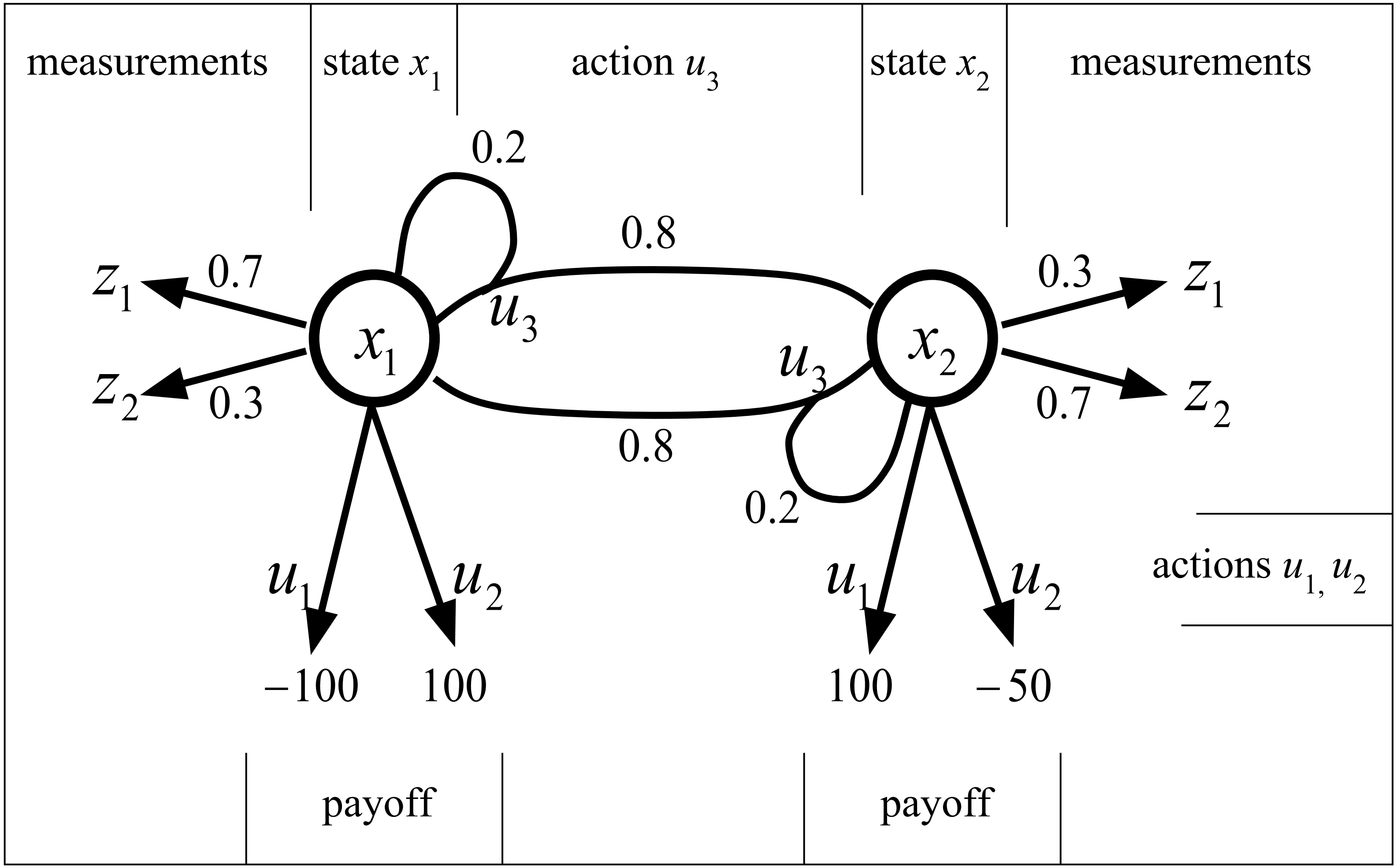
$$\pi_T(b) = \arg\max_u \left\{ r(b,u) + \int V_{T-1}(b')\, p(b' \mid u,b)\, db' \right\}$$

$$V_T(x) = \gamma \max_u \left\{ r(x,u) + \int V_{T-1}(x')\, p(x' \mid u,x)\, dx' \right\}$$

# Problems

- Belief is a probability distribution – each value in a POMDP is a function of an entire probability distribution!
- Probability distributions are continuous.
- Huge complexity of belief spaces.
- For **finite worlds** with finite state, action, and observation spaces and finite horizons, we can **effectively represent the value functions by piecewise linear functions**.

# An Illustrative Example



# The Parameters of the Example

- The actions $u_1$ and $u_2$ are terminal actions.
- The action $u_3$ is a sensing action that potentially leads to a state transition.
- The horizon is finite and $\gamma$=1.

$$r(x_1, u_1) = -100, \quad r(x_2, u_1) = 100$$

$$r(x_1, u_2) = 100, \quad r(x_2, u_2) = -50$$

$$r(x_1, u_3) = -1, \quad r(x_2, u_3) = -1$$

$$p(x_1^{'} \mid x_1, u_3) = 0.2 \quad p(x_2^{'} \mid x_1, u_3) = 0.8$$

$$p(x_1^{'} \mid x_2, u_3) = 0.8 \quad p(x_2^{'} \mid x_2, u_3) = 0.2$$

$$p(z_1 \mid x_1) = 0.7 \quad p(z_2 \mid x_1) = 0.3$$

$$p(z_1 \mid x_2) = 0.3 \quad p(z_2 \mid x_2) = 0.7$$

$$p_1 = b(x_1), \;\; p_2 = b(x_2), \;\; p_2 = 1 - p_1$$

$$\text{Policy } \pi : [0, 1] \rightarrow u$$

# Payoff in POMDPs

- In MDPs, the payoff (or return) depends on the state of the system.
- In POMDPs, the true state is not known.

- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$
\begin{aligned}
r(b,u) &= E_x[r(x,u)] \\
&= \int r(x,u)p(x)\,dx \\
&= p_1\,r(x_1,u) + p_2\,r(x_2,u)
\end{aligned}
$$

# Payoffs in Our Example (1)

- If we are certain that we are in state $x_1$ and execute action $u_1$ we receive reward of -100.
- If we definitely know that we are in $x_2$ and execute $u_1$ the reward is +100.

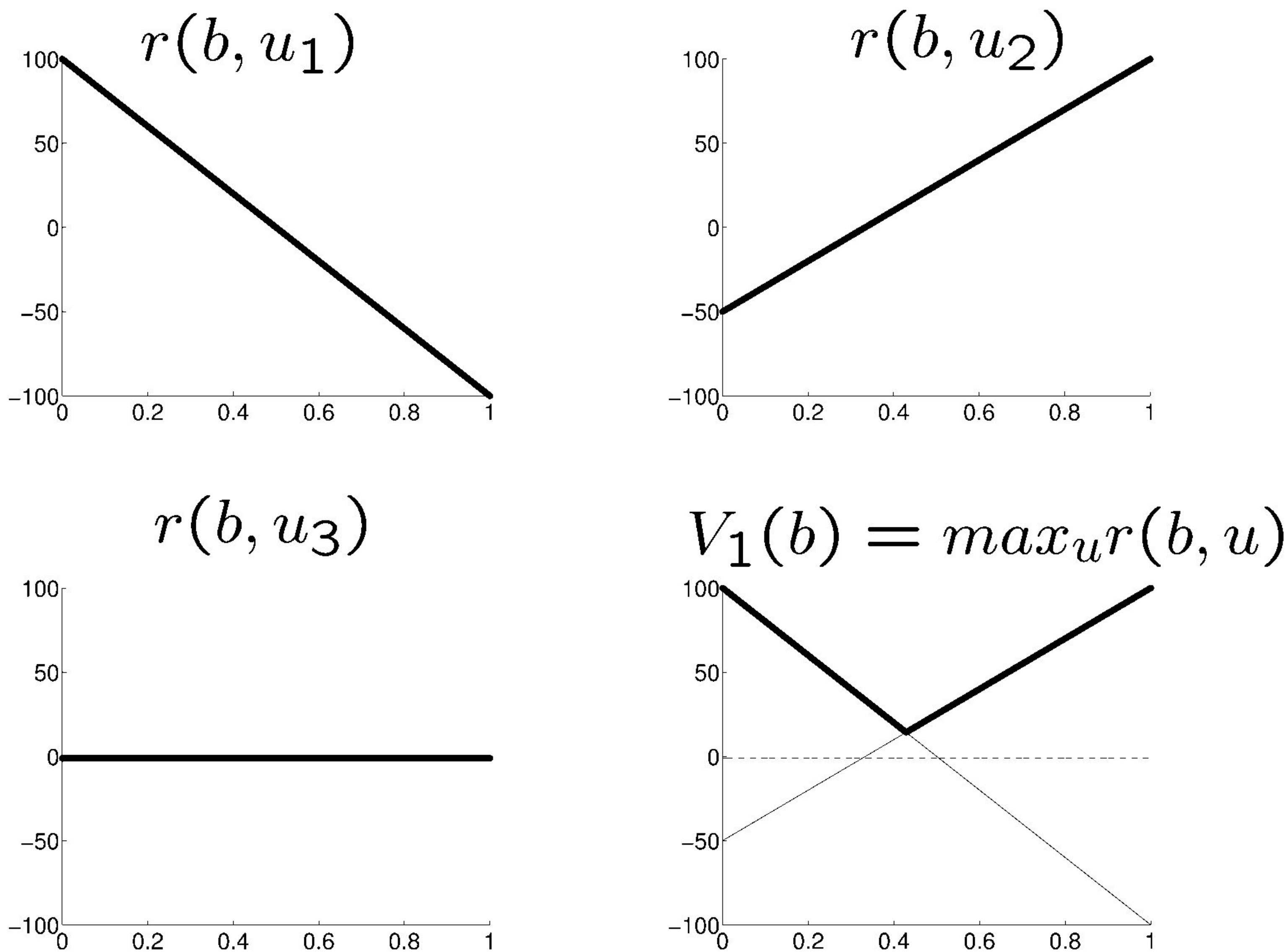
- In between it is the linear combination of the extreme values weighted by the probabilities:

$$\begin{aligned} r(b, u_1) &= -100\, p_1 + 100\, p_2 \\ &= -100\, p_1 + 100\, (1 - p_1) \end{aligned}$$

$$r(b, u_2) = 100\, p_1 - 50\, (1 - p_1)$$

$$r(b, u_3) = -1$$

# Payoffs in Our Example (2)



# The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.

- In our example, the optimal policy for T=1 is:

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \leq \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

# Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point:

$$\begin{aligned} V_1(b) &= \max_u r(b,u) \\ &= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\, (1-p_1) \\ 100\, p_1 & -50\, (1-p_1) \\ -1 & \end{array} \right\} \end{aligned}$$

- It is piecewise linear and convex.

# Pruning

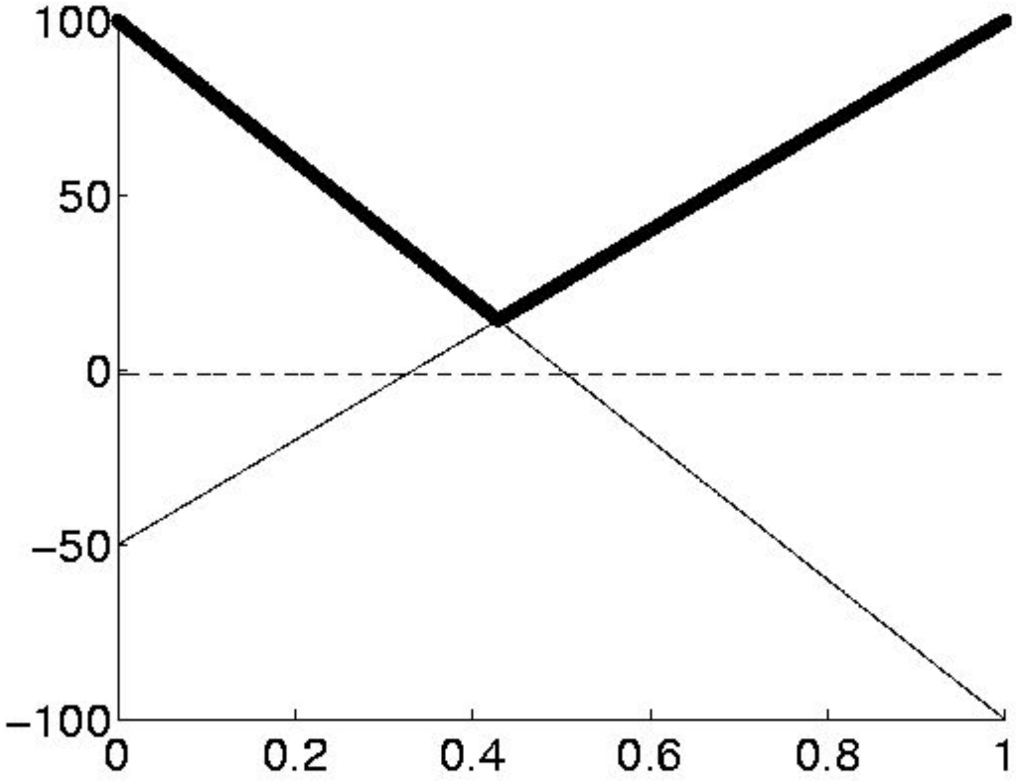
- Carefully consider $V_1(b)$ – only the first two components contribute.

- The third component can be pruned away from $V_1(b)$:

$$V_1(b) \;=\; \max\left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

# Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.




$V_1(b)$

# Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.

- Suppose the robot perceives $z_1$ for which:

  $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.

- Given the observation $z_1$ we update the belief using Bayes rule:

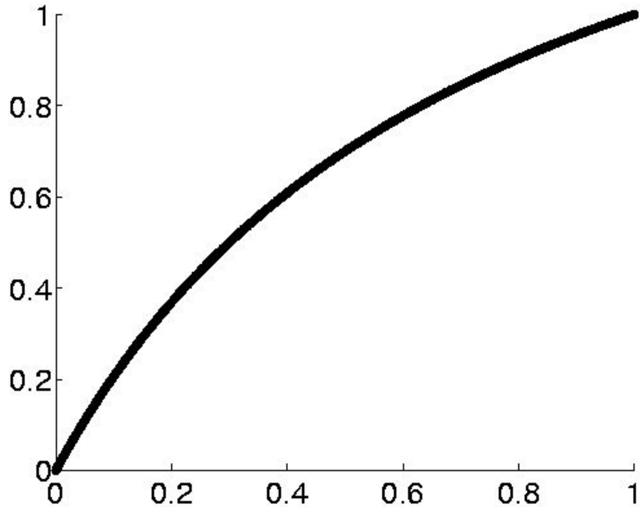
$$p'_1 = p(x_1 \mid z) = \frac{p(z_1 \mid x_1)\,p(x_1)}{p(z_1)} = \frac{0.7\,p_1}{p(z_1)}$$

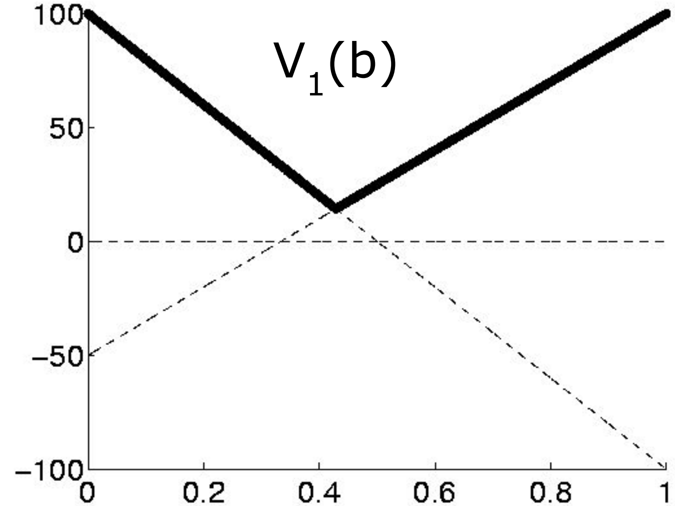
$$p'_2 = \frac{0.3(1-p_1)}{p(z_1)}$$

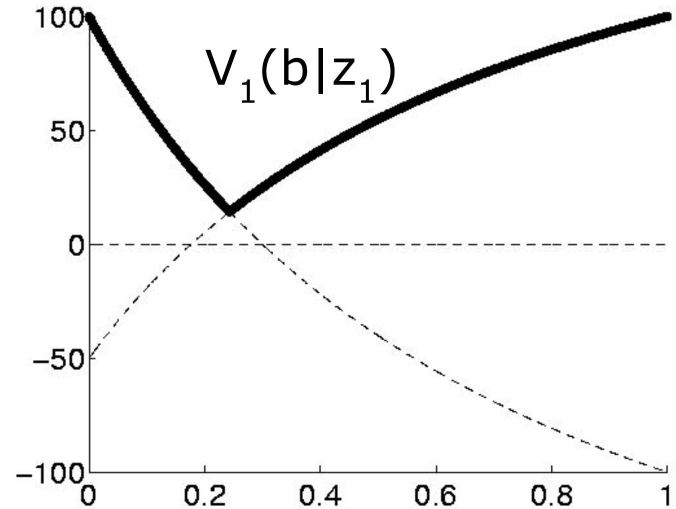
$$p(z_1) = 0.7\,p_1 + 0.3(1-p_1) = 0.4\,p_1 + 0.3$$

# Value Function

$p_1' = p_1$ after sensing $z_1$



# Increasing the Time Horizon

- Coming back to our assumption that robot can make an observation before deciding on an action.
- If the robot perceives $z_1$:
  $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- We update the belief $V_1(b \mid z_1)$ using Bayes rule to obtain:

$$
\begin{aligned}
V_1(b \mid z_1) &= \max \left\{ \begin{array}{rr} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\} \\
&= \frac{1}{p(z_1)} \max \left\{ \begin{array}{rr} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

# Expected Value after Measuring

- Since we do not know what the next measurement will be, we have to compute the expected belief:

$$
\begin{aligned}
\overline{V_1}(b) = E_z[V_1(b \mid z)] &= \sum_{i=1}^{2} p(z_i) V_1(b \mid z_i) \\
&= \sum_{i=1}^{2} p(z_i) V_1\left( \frac{p(z_i \mid x_1) p_1}{p(z_i)} \right) \\
&= \sum_{i=1}^{2} V_1\big( p(z_i \mid x_1) p_1 \big)
\end{aligned}
$$

# Expected Value after Measuring

- Since we do not know what the next measurement will be, we have to compute the expected belief:

$$
\begin{aligned}
\bar{V}_1(b) &= E_z[V_1(b \mid z)] \\
&= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\
&= \max \left\{ \begin{array}{rr} -70\, p_1 & +30\, (1 - p_1) \\ 70\, p_1 & -15\, (1 - p_1) \end{array} \right\} \\
&\quad + \max \left\{ \begin{array}{rr} -30\, p_1 & +70\, (1 - p_1) \\ 30\, p_1 & -35\, (1 - p_1) \end{array} \right\}
\end{aligned}
$$

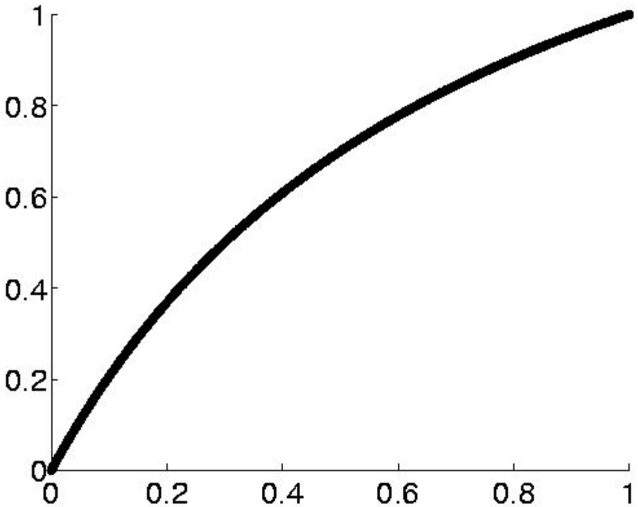
# Resulting Value Function

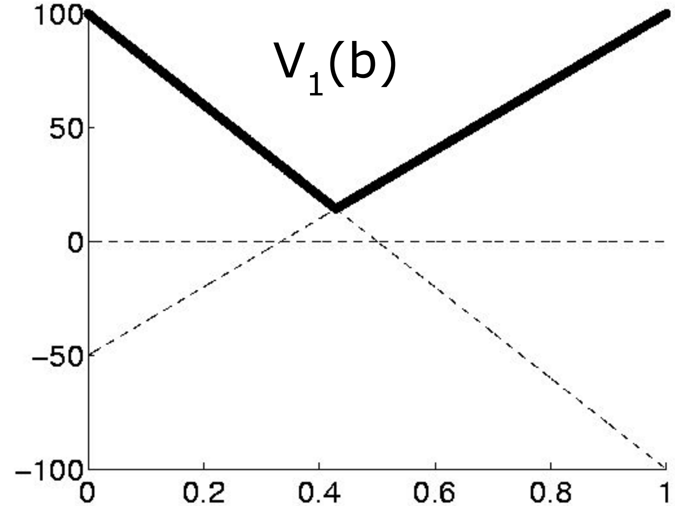
- The four possible combinations yield the following function which then can be simplified and pruned:

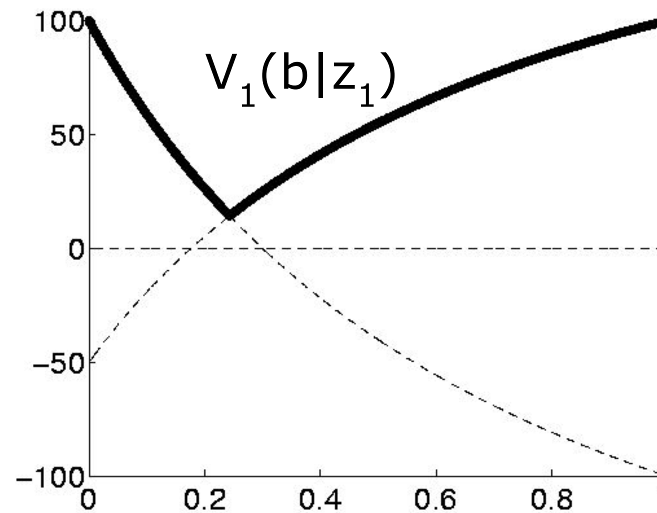
$$
\begin{aligned}
\bar{V}_1(b) &= \max \left\{ \begin{array}{llll} -70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ -70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \end{array} \right\} \\
&= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ +40\, p_1 & +55\,(1-p_1) \\ +100\, p_1 & -50\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

# Value Function

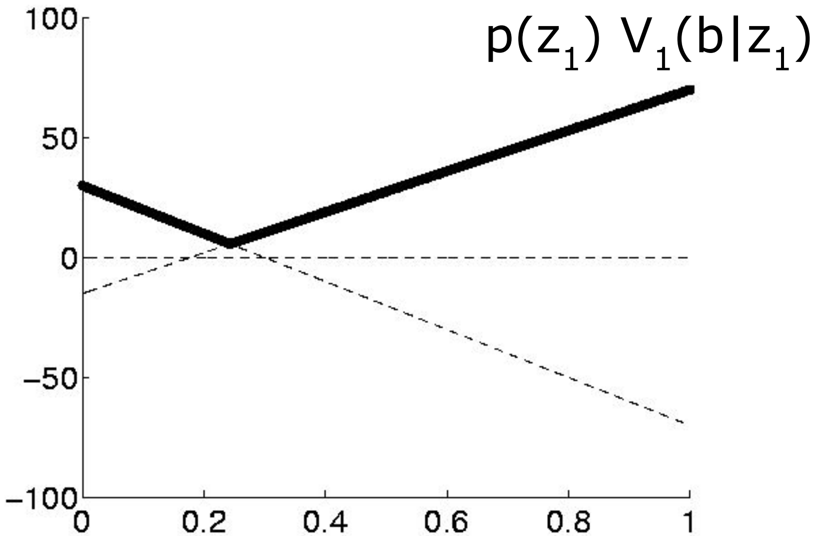
$p_1' = p_1$ after sensing $z_1$

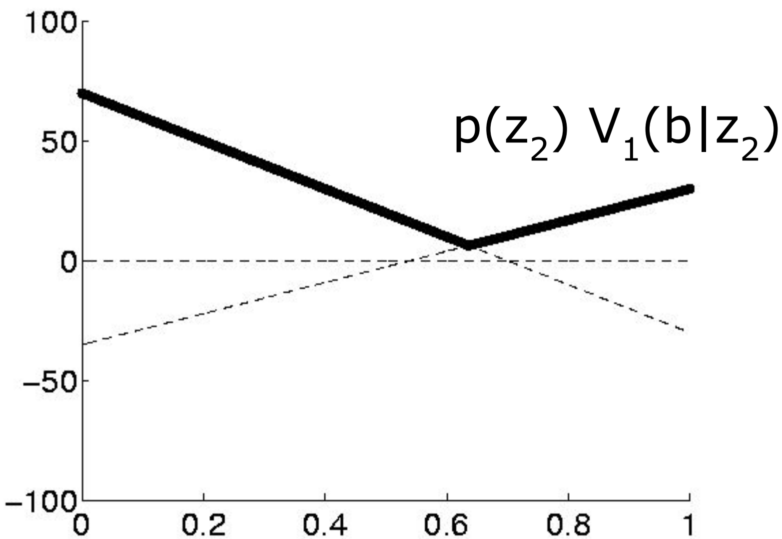


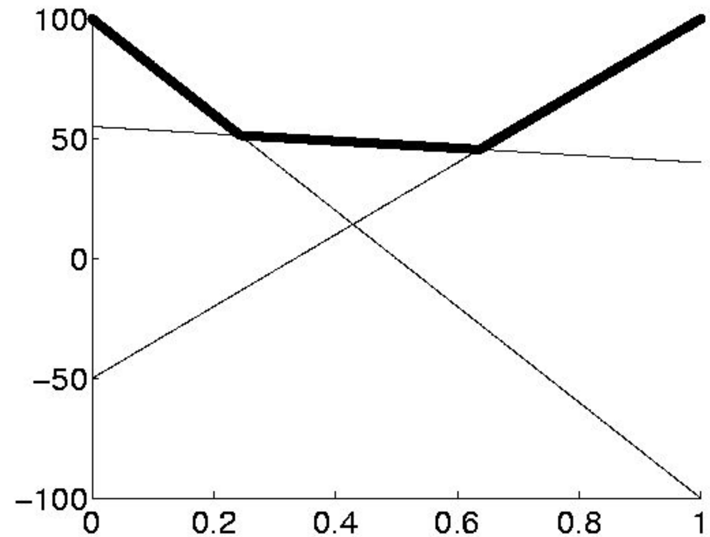


# Value Function




$$\bar{V}_1(b)$$

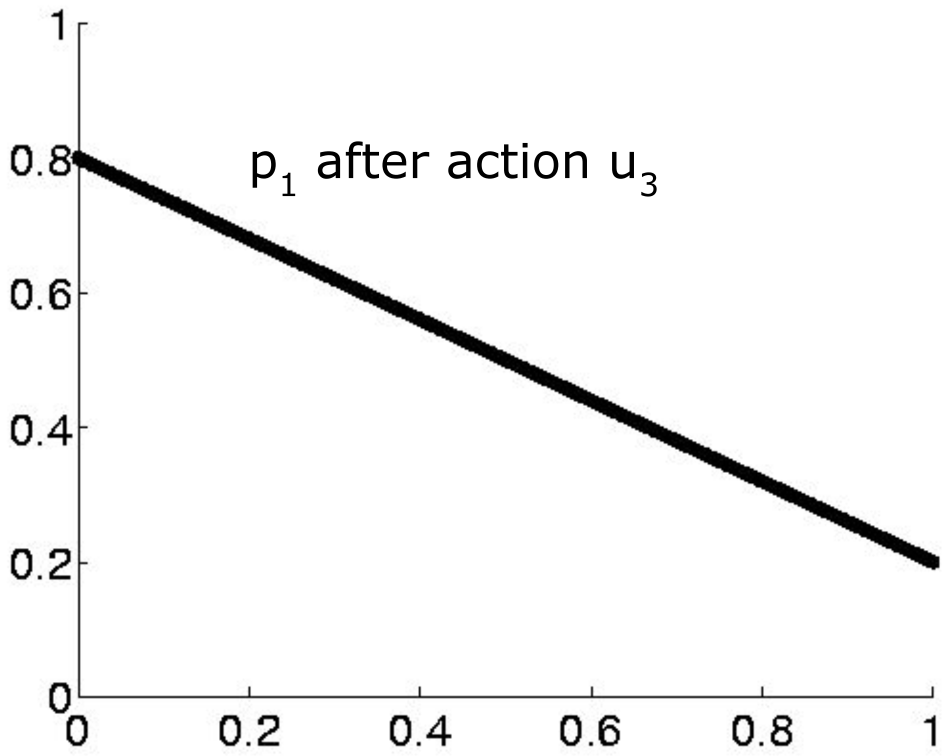
$= p(z_1)\ V_1(b|z_1) + p(z_2)\ V_1(b|z_2)$

# State Transitions (Prediction)

- When the agent selects $u_3$ its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$
\begin{aligned}
p_1' &= E_x\left[p(x_1' \mid x, u_3)\right] \\
&= \sum_{i=1}^{2} p(x_1' \mid x_i, u_3)\; p_i \\
&= 0.2\, p_1 + 0.8(1 - p_1) \\
&= 0.8 - 0.6\, p_1
\end{aligned}
$$

# State Transitions (Prediction)




$$p_1' = 0.8 - 0.6\,p_1$$

# Value Function after executing $u_3$

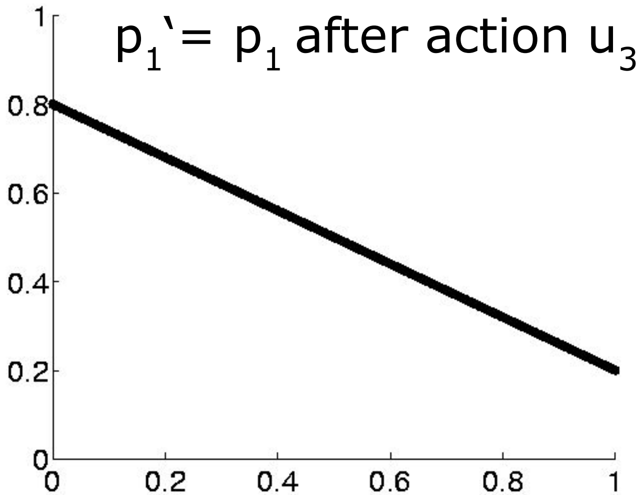
- Taking the state transitions into account:

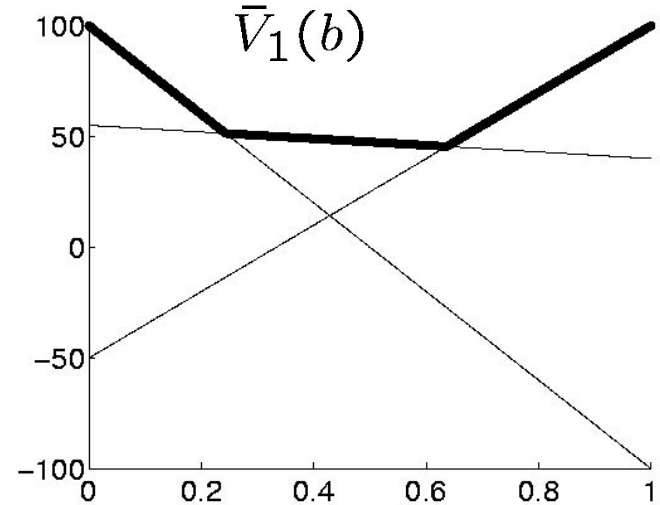
$$\bar{V}_1(b) = \max \left\{ \begin{array}{llll} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

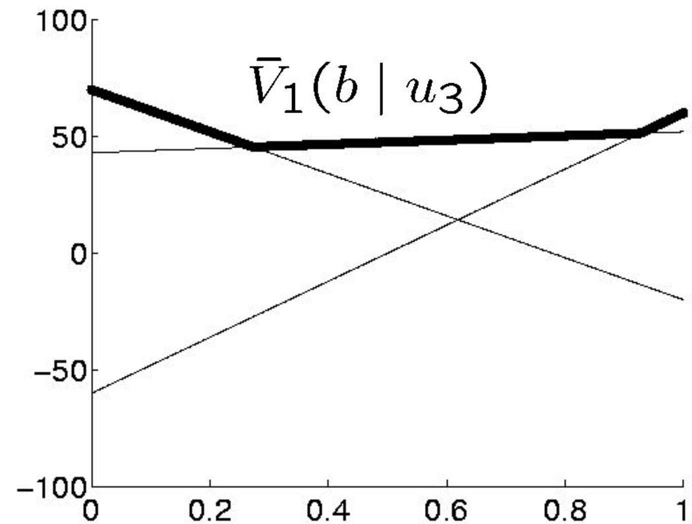
$$= \max \left\{ \begin{array}{ll} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{ll} 60\,p_1 & -60\,(1-p_1) \\ 52\,p_1 & +43\,(1-p_1) \\ -20\,p_1 & +70\,(1-p_1) \end{array} \right\}$$

# Value Function after executing $u_3$



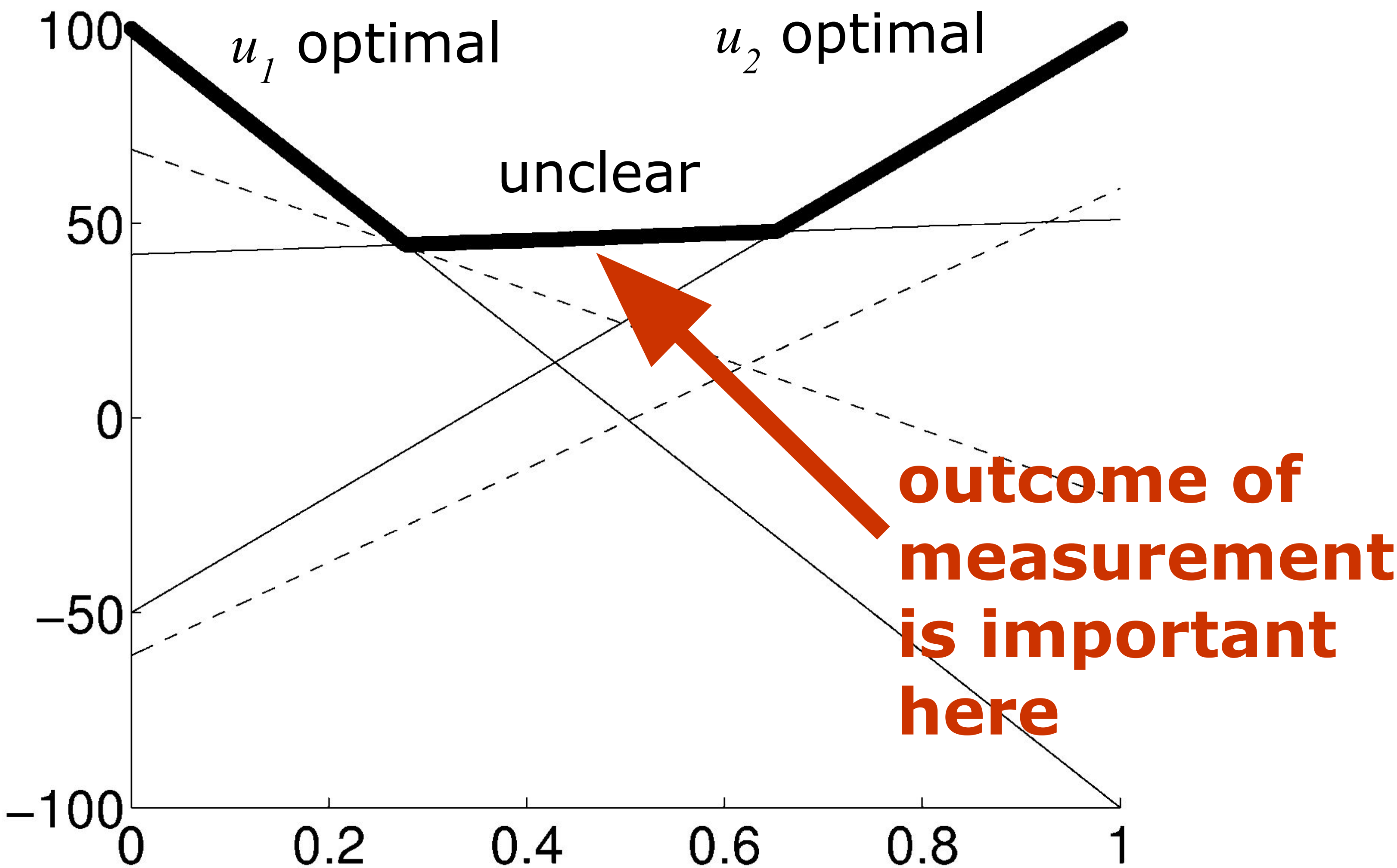



## Value Function for T=2

- Taking into account that the agent can either directly perform $u_1$ or $u_2$ or first $u_3$ and then $u_1$ or $u_2$, we obtain (after pruning).

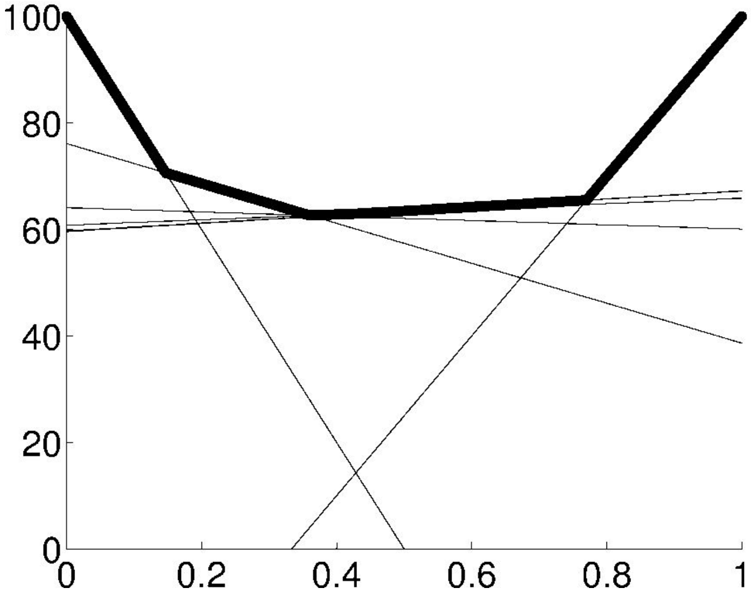
$$\bar{V}_2(b) = \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1 - p_1) \\ 100\, p_1 & -50\,(1 - p_1) \\ 51\, p_1 & +42\,(1 - p_1) \end{array} \right\}$$

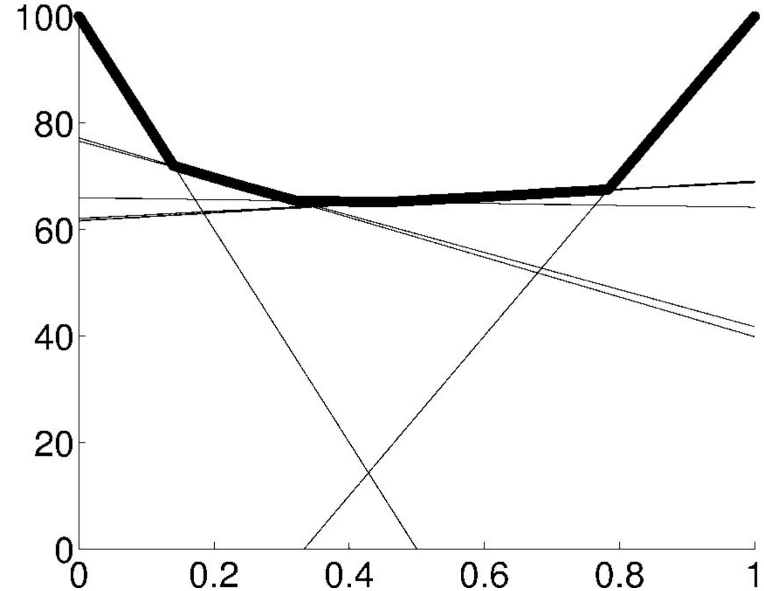
# Graphical Representation of $V_2(b)$



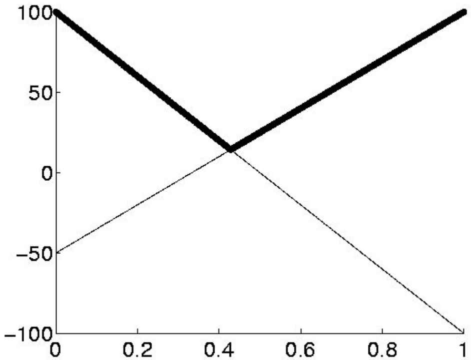
# Deep Horizons and Pruning

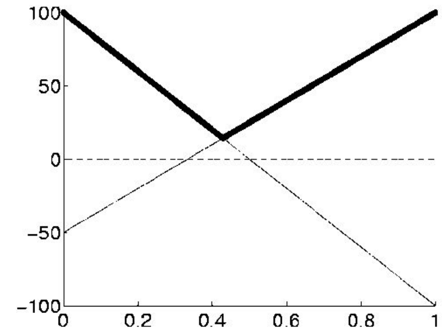
- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20:

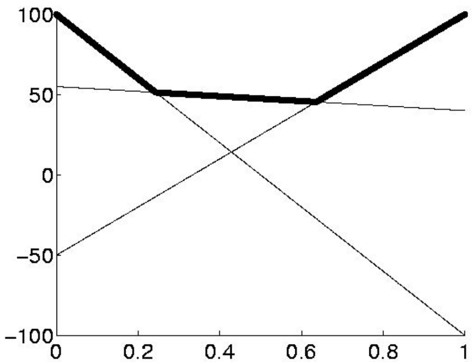


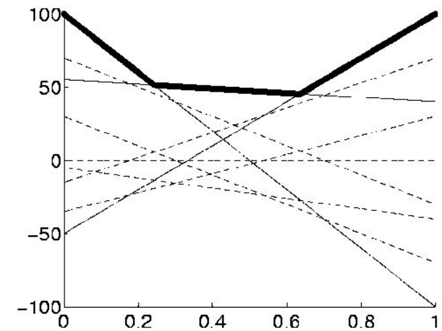



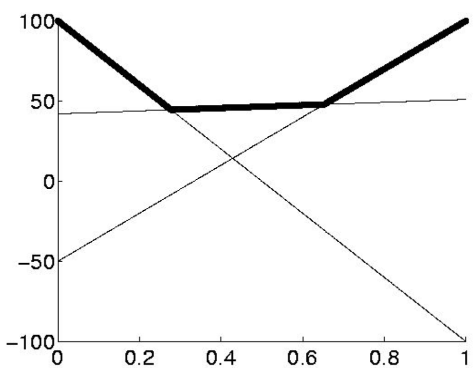
# Deep Horizons and Pruning

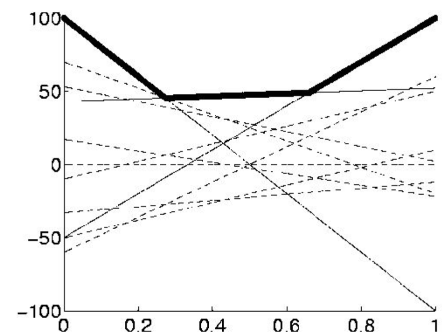


# Why Pruning is Essential

- Each update adds additional linear components to $V$.
- Each measurement squares number of linear components.

- Unpruned value function for T=20 has more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.

- The pruned value functions at T=20, in comparison, contains only 12 linear components.

- The combinatorial explosion of linear components in the value function is the major reason why POMDPs are impractical for most applications.

# POMDP Summary

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.

# POMDP Approximations

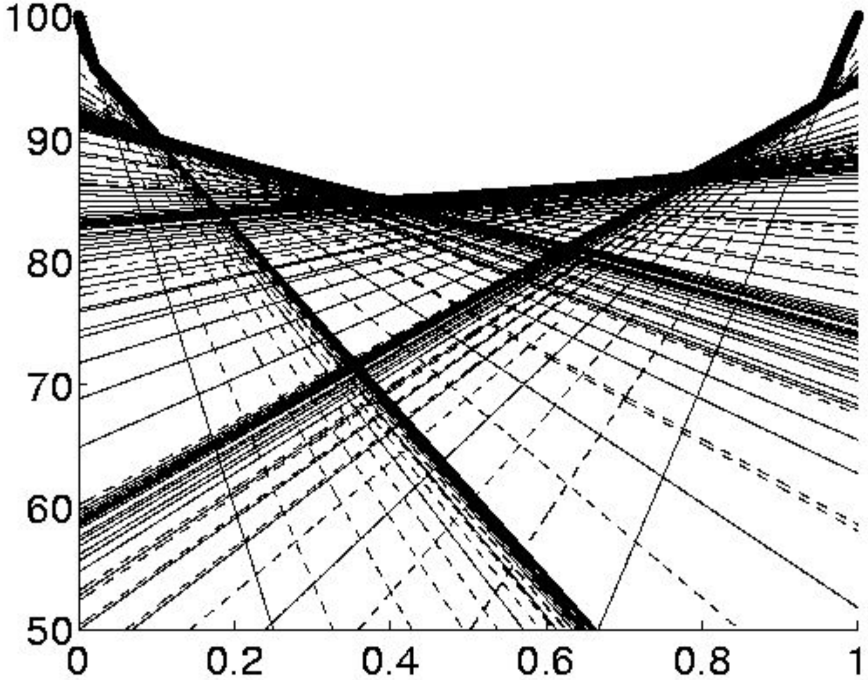
- Point-based value iteration.
- QMDPs.
- AMDPs.
- MC-POMDP.

# Point-based Value Iteration

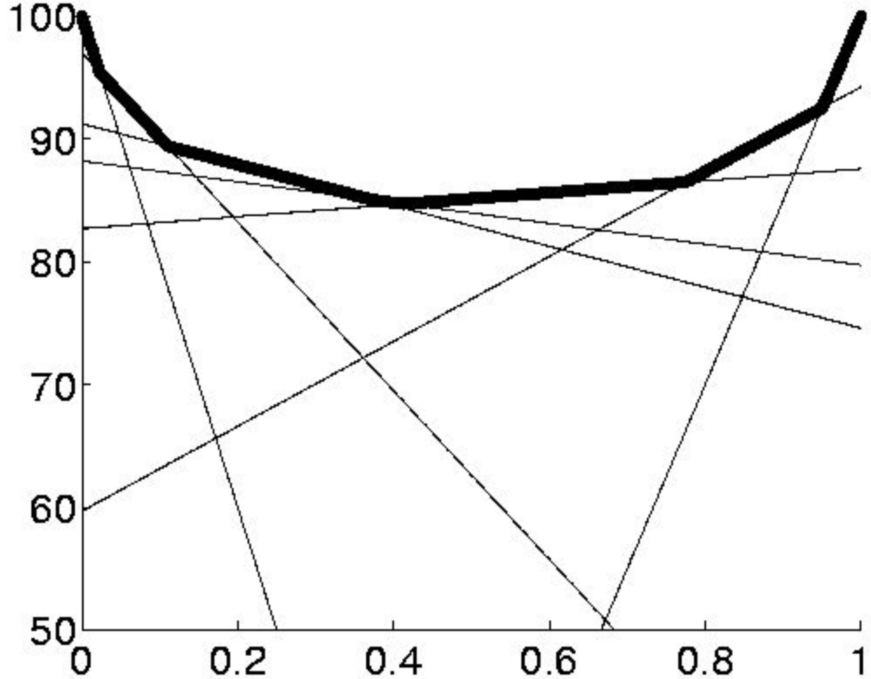
- Maintains a set of example beliefs.
- Only considers constraints that maximize value function for at least one of the examples.

# Point-based Value Iteration

Value functions for T=30




Exact value function




PBVI

# QMDPs

- QMDPs only consider state uncertainty in the first step.
- After that, the world becomes fully observable!
- Planning only marginally less efficient than MDPs, but performance significantly better!

# Monte Carlo POMDPs

- Represent beliefs by samples.
- Estimate value function on sample sets.
- Simulate control and observation transitions between beliefs.

# Summary

- POMDPs ideal for modeling systems with partially observable state and non-deterministic actions.
- Exponential state space explosion is a problem!
- Methods exist to make the problem more tractable – not good enough for real-world robot problems.
- Possible solutions:
  - Impose hierarchy by exploiting inherent structure.
  - Speed up POMDP solution techniques.