Introduction to linear algebra 1
Vectors, vector spaces, inner products and norms

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Overview

Vector spaces

- $\mathbb{R}^N$ is the set of N-dimensional real vectors:

$$\mathbb{R}^N = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \mid v_n \in \mathbb{R} \right\} \tag{1}$$

- There are only limited ways to manipulate these vectors:
  - We can add two vectors $\vec{v}$ and $\vec{w}$ together or multiply them by a scalar λ
  - There is no notion of vector multiplication. The dot product $\vec{v} \cdot \vec{w}$ is not a vector and $\vec{v} \times \vec{w}$ is not defined for all N
  - Certainly we can't divide $\vec{v}$ by $\vec{w}$.

Linear spaces

- Vector spaces consist of:
  - A set of vectors,
  - A set of scalars,
  - A rule for multiplying vectors by scalars, and
  - A rule for adding together vectors.
- Operations that involve only scalar multiplication and vector addition are called *linear* operations
- Another word for a vector space is a *linear space* and the branch of mathematics that is concerned with vector spaces in *linear algebra*
- Vector spaces are important because they provide the basic mathematics of linear systems

Definition of a vector space

A vector space consists of:

- A set of *vectors* $\mathbb{V}$ and a set of *scalars* $\mathbb{S}$. We are interested in the case where:
    - $\mathbb{S} = \mathbb{R}, \mathbb{V} = \mathbb{R}^N$, in which case the vector space is **real**
- Two operations, called *scalar multiplication* and *vector addition*, such that if $\lambda \in \mathbb{S}$ and $\vec{v}, \vec{w} \in \mathbb{V}$
    - $\lambda\vec{v} \in \mathbb{V}$ and $\vec{v} + \vec{w} \in \mathbb{V}$
    - $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$
- There is a unique vector $\vec{0} \in \mathbb{V}$ called the *zero vector* such that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- There is a unique vector $-\vec{v} \in \mathbb{V}$ such that $\vec{v} + (-\vec{v}) = \vec{0} = (-\vec{v}) + \vec{v}$

More rules for Vector Spaces...

- For completeness I should also insist that for $\lambda, \mu \in \mathbb{S}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$:
  - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (symmetry)
  - $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)
  - $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$
  - $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
  - $1\vec{v} = \vec{v}$
- These are properties that we normally take for granted
- Formally, a vector space is just a pair $(\mathbb{V}, \mathbb{S})$.

Examples

- N-dimensional real Euclidean space $\mathbb{R}^N$
  - $\mathbb{S} = \mathbb{R}$, $\mathbb{V} = \mathbb{R}^N$ with the normal rules for scalar multiplication and vector addition is a real vector space

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}, v + \lambda w = \begin{bmatrix} v_1 + \lambda w_1 \\ v_2 + \lambda w_2 \\ \vdots \\ v_N + \lambda w_N \end{bmatrix}, \tag{2}$$

- N-dimensional complex space $\mathbb{C}^N$
  - $\mathbb{S} = \mathbb{C}$, $\mathbb{V} = \mathbb{C}^N$ with the corresponding rules for complex scalar multiplication and vector addition is a complex vector space
- For this course we are only concerned with N-dimensional real Euclidean space $\mathbb{R}^N$.

Closure

- The statement "if $\lambda \in \mathbb{S}, \vec{v}, \vec{w} \in \mathbb{V}$ then $\lambda\vec{v} + \vec{w} \in \mathbb{V}$" doesn't just mean that you can multiply vectors by scalars or add vectors.
- It means that the result **must** be in $\mathbb{V}$
- This property is **closure**. We say that $\mathbb{V}$ is **closed for scalar multiplication and vector addition**
- So, for example, the unit disk $S = \{\vec{v} \in \mathbb{R}^2 : |v| \leq 1\}$ with normal scalar multiplication and vector addition is **not** a vector space, because, for example, the sum of two vectors in S is in general not in S.
- For example

$$\vec{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S, \vec{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S, \vec{u_1} + \vec{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S \tag{3}$$

Inner products

- An **inner product** is just a rule that combines two vectors $\vec{v}$ and $\vec{w}$ to give a **scalar** $\langle \vec{v}, \vec{w} \rangle$ such that:

$$\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle \tag{4}$$

$$\langle \lambda\vec{v}, \vec{w} \rangle = \lambda\langle \vec{v}, \vec{w} \rangle, \langle \vec{v}, \lambda\vec{w} \rangle = \lambda\langle \vec{v}, \vec{w} \rangle \tag{5}$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \tag{6}$$

$$\langle \vec{u}, \vec{u} \rangle \geq 0 \tag{7}$$

The "dot product"

- For the real vector space $\mathbb{R}^N$ the "dot product" is an inner product:

$$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w} = \vec{v}^T \vec{w} = \sum_{n=1}^{N} v_n w_n \tag{8}$$

- In this course, the "dot" product is the only inner-product that we are interested in.

Example

- Let

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \tag{9}$$

Then

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \vec{v}^T \vec{w} && (10) \\ &= 2 \times 1 + 1 \times -2 + 4 \times 3 && (11) \\ &= 12 && (12) \end{aligned}$$

The norm of a vector

- The *norm* $\|\vec{v}\|$ of a vector $\vec{v}$ indicates the magnitude of $\vec{v}$
- In general, a norm is a positive valued function defined on elements of a vector space such that for vectors $\vec{v}, \vec{w} \in \mathbb{V}$ and a scalar $\lambda \in \mathbb{S}$:

$$\|\vec{v}\| \geq 0 \tag{13}$$

$$\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\| \tag{14}$$

$$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \tag{15}$$

$$\|\vec{v}\| = 0 \implies \vec{v} = 0 \tag{16}$$

Equation (15) is the *triangle inequality*. The symbol "$\implies$" means "implies that"

Norms and inner-products

- Any inner product defines a norm. For any vector $v \in \mathbb{V}$ define

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \tag{17}$$

In particular, for the real vector space $\mathbb{R}^N$ and the standard dot product

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \tag{18}$$

$$= \sqrt{\vec{v} \cdot \vec{v}} \tag{19}$$

$$= \sqrt{\sum_{n=1}^{n} v_n^2} \tag{20}$$

Equation (20) is just the standard Euclidean norm in $\mathbb{R}^N$

Unit vectors

- A *unit vector* is just a vector $\vec{v} \in \mathbb{V}$ such that

$$\|\vec{v}\| = 1 \tag{21}$$

- Any vector $\vec{v}$ can be converted into a unit vector $\hat{v}$ by multiplying by the inverse of its norm:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} \tag{22}$$

- The most familiar unit vectors in $\mathbb{R}^2$ are

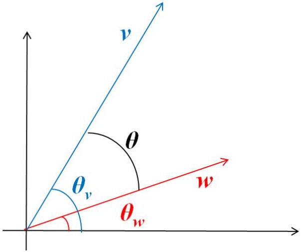
$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

Geometric interpretation of the dot product

- Suppose $\vec{v}, \vec{w} \in \mathbb{R}^N$ and θ is the angle between them. Then

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} \tag{24}$$

Figure: Geometric interpretation of the dot product



Proof (in 2 dimensions)

- In 2 dimensions, let

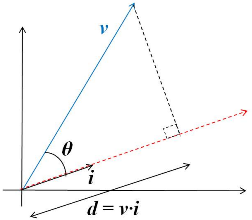
$$
\begin{aligned}
\vec{v} &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} && (25)\\
cos(\theta_v) &= \frac{v_1}{\|\vec{v}\|}, sin(\theta_v) = \frac{v_2}{\|\vec{v}\|} && (26)\\
cos(\theta_w) &= \frac{w_1}{\|\vec{w}\|}, sin(\theta_w) = \frac{w_2}{\|\vec{w}\|} && (27)\\
cos(\theta) &= cos(\theta_v - \theta_w) && (28)\\
&= cos(\theta_v)cos(\theta_w) + sin(\theta_v)sin(\theta_w) && (29)\\
&= \frac{v_1}{\|\vec{v}\|}\frac{w_1}{\|\vec{w}\|} + \frac{v_2}{\|\vec{v}\|}\frac{w_2}{\|\vec{w}\|} && (30)\\
&= \frac{v_1 w_1 + v_2 w_2}{\|\vec{v}\|\|\vec{w}\|} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} && (31)
\end{aligned}
$$

Interpretation of inner-product

- Let $\vec{i}$ be a unit vector and $\vec{v} \in \mathbb{R}^N$. Then $\vec{v} \cdot \vec{i}$ is the magnitude of the component of $\vec{v}$ that points in the direction of $\vec{i}$

$$\frac{\vec{v} \cdot \vec{i}}{\|\vec{v}\|\|\vec{i}\|} = cos\theta = \frac{d}{\|\vec{v}\|}, d = \frac{\|\vec{v}\|(\vec{v} \cdot \vec{i})}{\|\vec{v}\|\|\vec{i}\|} = \vec{v} \cdot \vec{i} \qquad (32)$$

Figure: Geometric interpretation of the dot product



Vector Subspaces

- Suppose $(\mathbb{V}, \mathbb{S})$ is a vector space and $\mathbb{U} \subset \mathbb{V}$.
- Then $\mathbb{U}$ is a *vector subspace* of $\mathbb{V}$ if $(\mathbb{U}, \mathbb{S})$ is a vector space.
- The key condition is that $(\mathbb{U}, \mathbb{S})$ must be **closed**
- In other words in $\vec{u_1}, \vec{u_2} \in \mathbb{U}$ and $\lambda \in \mathbb{S}$, then
  - $\vec{u_1} + \lambda_2 \vec{u_2} \in \mathbb{U}$

Example

- Suppose $\mathbb{V} = \mathbb{R}^2$, $\mathbb{S} = \mathbb{R}$
- Let $\vec{u} \in \mathbb{R}^2$ be a unit vector and let $\mathbb{U} = \{\lambda\vec{u} : \lambda \in \mathbb{R}\}$
- Then $(\mathbb{U}, \mathbb{R})$ is a vector subspace of $(\mathbb{R}^2, \mathbb{R})$
- In fact *all* vector subspaces of $(\mathbb{R}^2, \mathbb{R})$ take this form.
- What are the vector subspaces of $(\mathbb{R}^3, \mathbb{R})$?

Summary

- Formal definition of a vector space and the notion of closure
- Inner products
- Norms
- Interpretations of the inner-product