

Lecture 14: Unsupervised Learning: Clustering

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29 November 2019

Introduction

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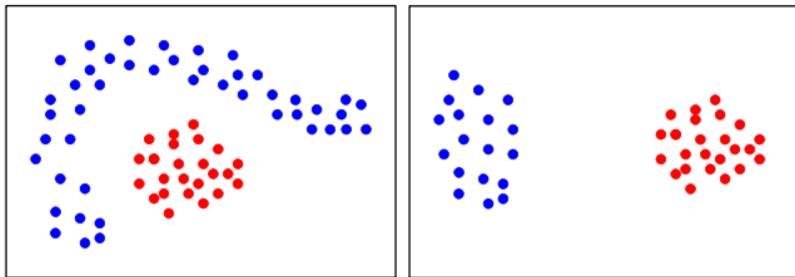
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- ▶ *Unsupervised learning*
- ▶ Main approach is *clustering*



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- ▶ Most are based on grouping points that are “similar”
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- ▶ Common approaches include:
 - ▶ Vector quantisation
 - ▶ Agglomerative approaches
 - ▶ Mixture modelling

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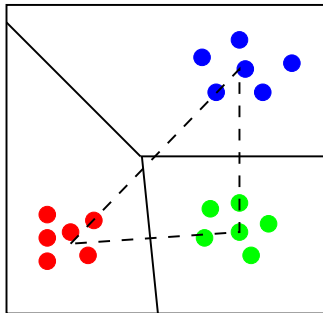
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- ▶ Prototype vectors are often known as the cluster *centroids*.



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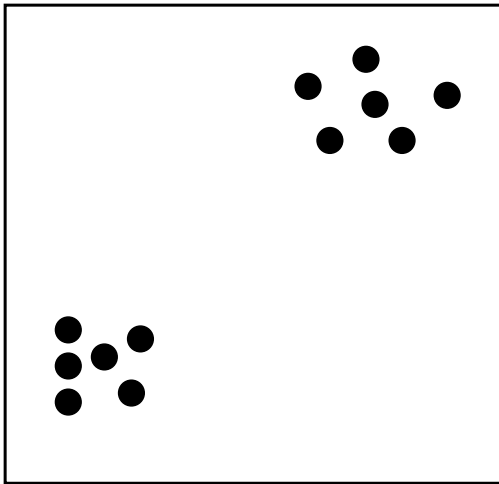
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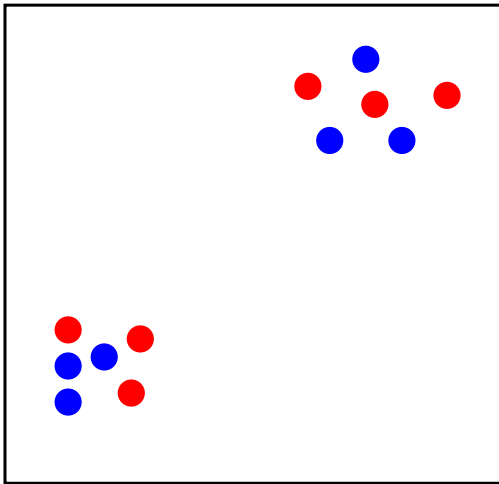
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Initial data points



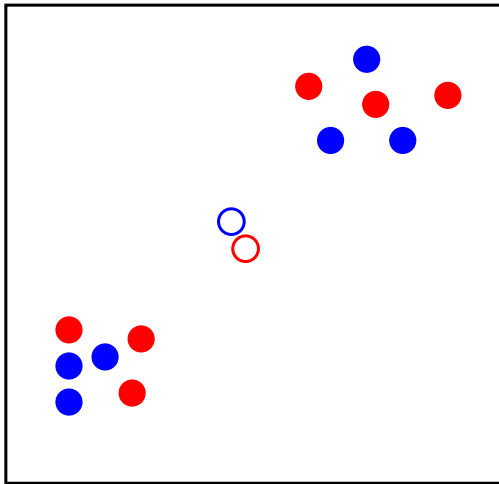
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Randomly assign points to groups



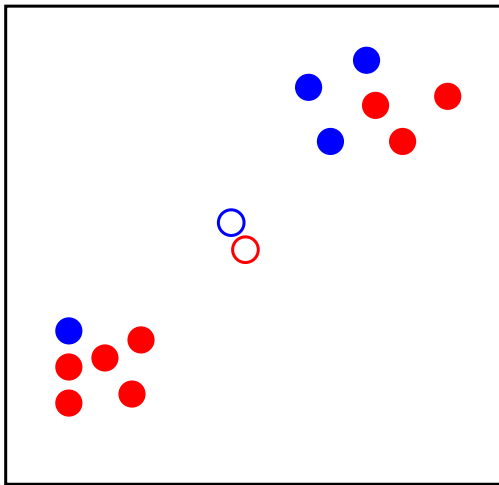
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Compute group average



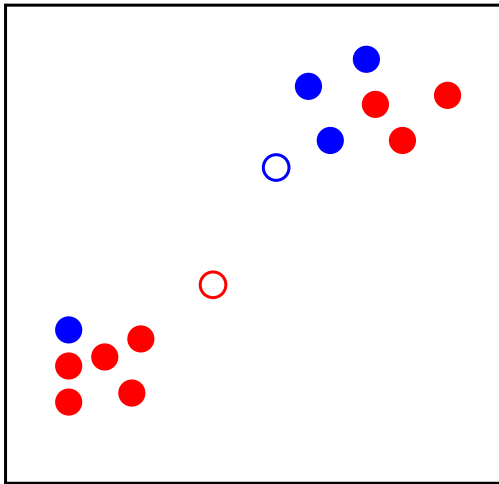
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Re-assign points to groups



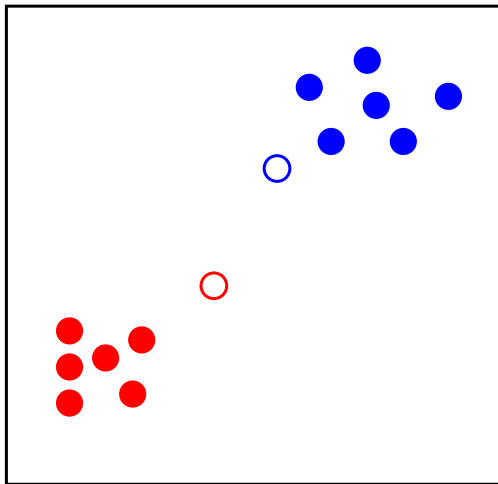
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Re-compute group averages



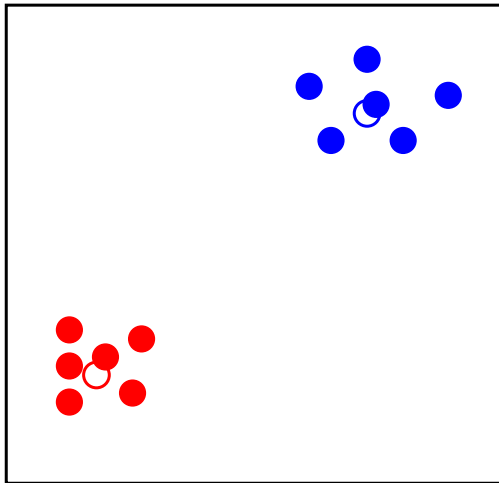
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- ▶ All degrees of cluster can be extracted from this

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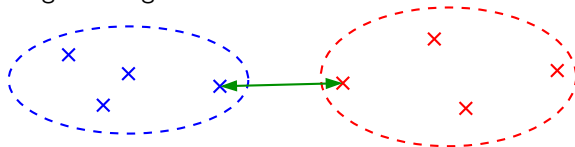
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- 5) Go to 2) and continue grouping until all points are grouped

Linkage

- ▶ How do we measure similarities between groups?

Linkage

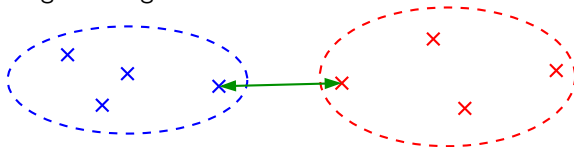
- ▶ How do we measure similarities between groups?
 - ▶ Single Linkage



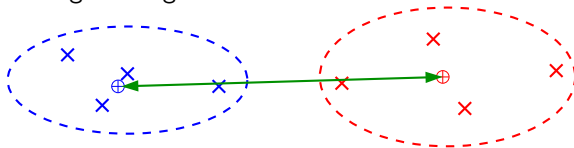
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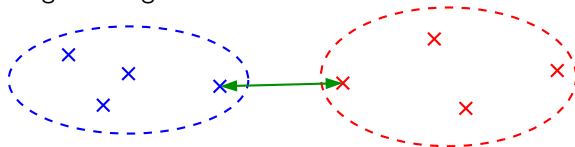
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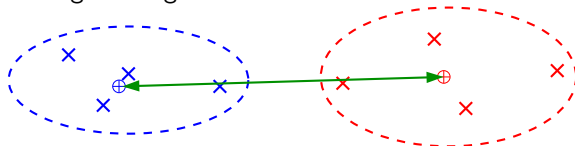
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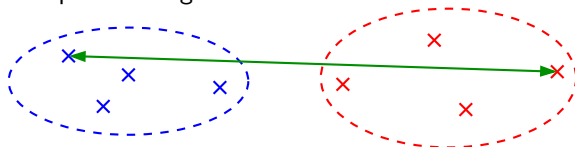
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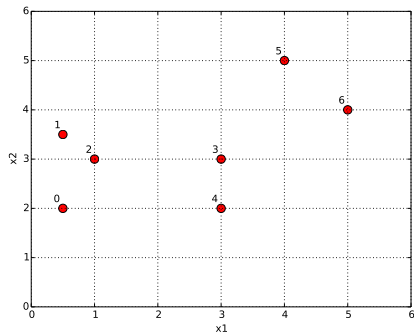
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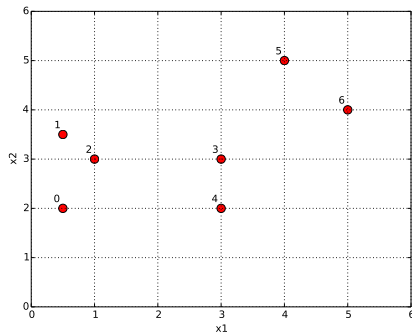
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List of points

0,1,2,3,4,5,6

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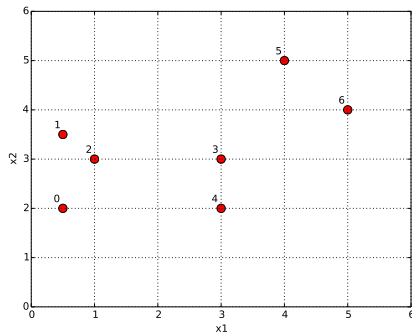
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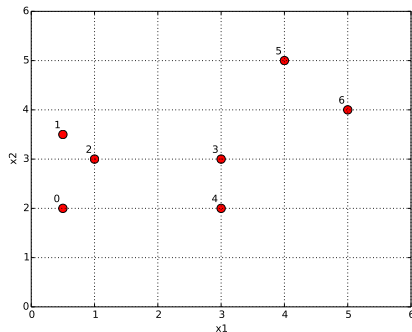
Group 3 with 4

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→ 0,5,6,(1,2),(3,4)

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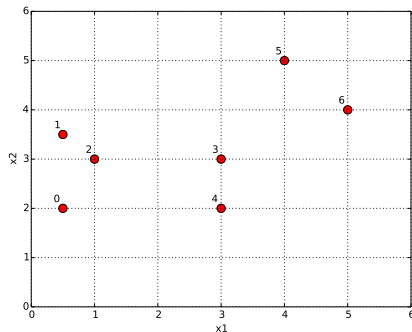
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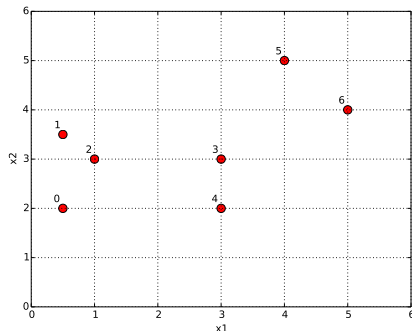
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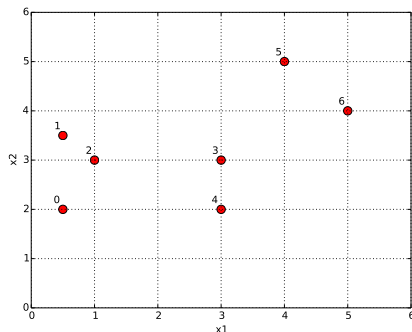
Group 5 with 6

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Group (3,4) with (0,(1,2))

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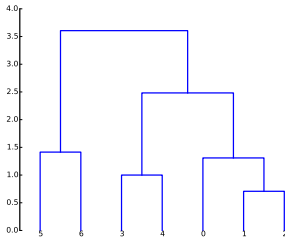
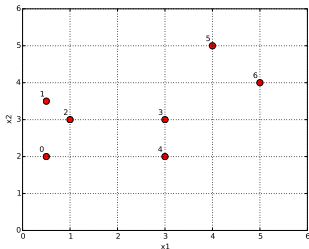
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- ▶ Assume data is generated by a statistical process that is a mixture of components
- ▶ Clustering: which component generated the data point

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- ▶ Clustering is then formulated as
 1. Learn the parameters of the GMM which best describe the data.
 2. Determine from which component a data point is most likely to have been generated.

Learning Gaussian Mixture Models

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we interpret

- ▶ $A_k = p(k)$ as the prior probability of choosing a point from component k
- ▶ $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) = p(\mathbf{x}|k)$ as the class-conditional likelihoods.

Learning Gaussian Mixture Models

- ▶ Finally, using Bayes' theorem we compute the posterior *responsibilities*

$$r_k(\mathbf{x}) = p(k|\mathbf{x}) \quad (5)$$

$$= \frac{p(k)p(\mathbf{x}|k)}{\sum_{k'=1}^K p(k')p(\mathbf{x}|k')} \quad (6)$$

$$= \frac{A_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{k'=1}^K A_{k'} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_{k'}, \boldsymbol{\Sigma}_{k'})} \quad (7)$$

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- ▶ Probability that component k explains \mathbf{x}
- ▶ GMM gives *soft* cluster assignments

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