Lecture 9: A probabilistic approach to classification

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Learning Outcomes

By the end of this lecture you should be able to

- Develop a simple probabilistic model of a linearly separable binary class dataset
- Use the model to derive classification rules
- Generalise the model to non-linear boundaries and multiple classes
- Generate new samples using the model

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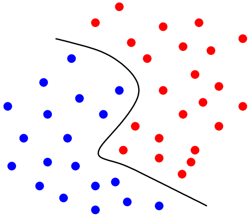

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- Linear combination of variables–a straight line–that best discriminates between two pre-defined groups of points
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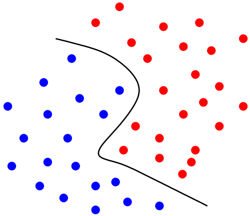

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- Linear combination of variables–a straight line–that best discriminates between two pre-defined groups of points
- Uses the probability distributions of the classes to find the line of equal probability dividing two classes
- *Generative approach* to classification

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- Assign points to most probable classes

Formulating LDA

- Consider a data point $\mathbf{x}$ which we want to assign to one of two predefined classes Π_i $(i = 1, 2)$
- We write that $P(\mathbf{x} \in \Pi_i) = \pi_i$

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- Bayes rule: probability that $\mathbf{x}$ belongs to class Π_i is:

$$\begin{aligned} P(\Pi_i|\mathbf{x}) &= \frac{P(\mathbf{x}|\Pi_i)P(\Pi_i)}{P(\mathbf{x})} \\ &= \frac{f_i(\mathbf{x})\pi_i}{f_1(\mathbf{x})\pi_1 + f_2(\mathbf{x})\pi_2} \end{aligned}$$

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- Equal ratios, randomly assign to either class.

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- Multivariate problem: *covariance matrix* $\mathbf{\Sigma}$ with components

$$\Sigma_{ij} = \frac{1}{N-1}\sum_{n=1}^{N}\left(x_i^{(n)} - \bar{x}_i\right)\left(x_j^{(n)} - \bar{x}_j\right) \tag{1}$$

Class-conditional Likelihood

- Model the classes as normally distributed with different means $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ but identical covariances $\boldsymbol{\Sigma}$.
- Thus, for $n = \{1, 2\}$ the groups distributions are $P(\mathbf{x}|\Pi_i) = f_i(\mathbf{x})$:

$$f_n(\mathbf{x}) = \frac{1}{(2\pi)^{r/2}\,|\boldsymbol{\Sigma}|^{1/2}} \exp\,[-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}}_n)^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \bar{\mathbf{x}}_n)]$$

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$$\begin{aligned} \ln \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} &= (\mathbf{x} - \bar{\mathbf{x}}_1)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \bar{\mathbf{x}}_1) - (\mathbf{x} - \bar{\mathbf{x}}_2)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \bar{\mathbf{x}}_2) \\ &= (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \mathbf{x} - \frac{1}{2} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2) \end{aligned}$$

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- This is a straight line / plane / hyperplane

The Separation Rule

- Writing $\mathbf{M} = \boldsymbol{\Sigma}^{-1}\left(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2\right)$
- And $c = -\left(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2\right)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}\left(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2\right) + \log_e \frac{\pi_1}{\pi_2}$

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- Given the separation rule $\frac{f_1(\mathbf{x})\pi_1}{f_2(\mathbf{x})\pi_2} = 1$ we have:

$$\text{if } L(\mathbf{x}) > 0 \text{ assign } \mathbf{x} \text{ to } \Pi_1 \text{ else } \Pi_2$$

- This is *Gaussian LDA*
- $\mathbf{M}^{\mathrm{T}}\mathbf{x}$ is *Fisher's linear discriminant function*.

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c &= -\frac{1}{2}\left(\log_e \frac{|\boldsymbol{\Sigma}_1|}{|\boldsymbol{\Sigma}_2|} + \bar{\mathbf{x}}_1^{\mathrm{T}}\boldsymbol{\Sigma}_1^{-1}\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2^{\mathrm{T}}\boldsymbol{\Sigma}_2^{-1}\bar{\mathbf{x}}_2\right) - \log_e \frac{\pi_1}{\pi_2}
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The classification rule is:

$$\text{if } Q(\mathbf{x}) > 0 \text{ assign } \mathbf{x} \text{ to } \Pi_1$$

else assign $\mathbf{x}$ to Π_2.

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- $\mathbf{x}$ is assigned to Π_i if $L_{ij} > 0$ for all $j \neq i$.
- The discriminant function between classes i and j is then

$$L_{ij}(\mathbf{x}) = \mathbf{m}_{ij}^{\mathrm{T}}\mathbf{x} + c_{ij}$$

with

$$\begin{aligned} \mathbf{m}_{ij} &= (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j)^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \text{ and} \\ c_{ij} &= -\frac{1}{2} \left(\bar{\mathbf{x}}_i^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}}_j \right) + \log_e \frac{\pi_i}{\pi_j}. \end{aligned}$$

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