Linear algebra 4
Eigenvectors and eigenvalues

Martin Russell

School of Computer Science, University of Birmingham

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Overview

1. Eigenvectors
   - Reminder - eigenvectors and eigenvalues
   - Eigenvectors, orthogonal matrices and change of bases
2. The Spectral Theorem
   - Every symmetric real matrix is diagonalizable
   - The MatLab *eig* function

Eigenvectors and eigenvalues in $\mathbb{R}^N$

- Let A be an $N \times N$ matrix
- $\vec{v} \in \mathbb{R}^N$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$ if $\vec{v} \neq 0$ and

$$A\vec{v} = \lambda\vec{v} \tag{1}$$

- Note that:
  - $\vec{v}$ and $A\vec{v} = \lambda\vec{v}$ point in the same direction
  - A scalar multiple of $\vec{v}$ is also an eigenvector of A. Hence we talk about an eigen**space** and choose $\vec{v}$ to be the **unit** vector that defines the space - i.e. assume $\|\vec{v}\| = 1$
- Not all matrices have real eigenvectors. If R_θ is a rotation matrix ($\theta \neq 0$) there is no vector $\vec{v}$ such that $\vec{v}$ and $R_\theta\vec{v}$ point in the same direction, so R_θ has no real eigenvectors.
- Note: **Complex** eigenvalues and eigenvectors are outside the scope of this discussion

Eigenvectors and eigenvalues in $\mathbb{R}^N$

The simplest case - a diagonal real-valued matrix

- Let $D = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$ then the eigenvectors of D are

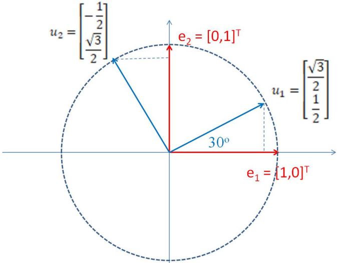
$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ with eigenvalue } \lambda_1 = 7, \text{and,} \tag{2}$$

$$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ with eigenvalue } \lambda_2 = 4 \tag{3}$$

Eigenvectors and eigenvalues in $\mathbb{R}^N$

- Let R be the rotation matrix $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
- From the previous lecture R is orthogonal and R transforms the standard basis $\vec{e_1}, \vec{e_2}$ to a new basis

$$\vec{u_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \tag{4}$$



Eigenvectors and eigenvalues in $\mathbb{R}^N$

- Now consider a new matrix B defined by

$$B = RDR^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 6.25 & 1.299 \\ 1.299 & 4.75 \end{bmatrix} \tag{5}$$

- Solving the characteristic equation (see Tutorial sheet) gives eigenvalues and eigenvectors:

$$\lambda_1 = 7, \vec{e_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix} = \vec{u_1} \tag{6}$$

$$\lambda_2 = 4, \vec{e_2} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.866 \end{bmatrix} = \vec{u_2} \tag{7}$$

Eigenvectors and eigenvalues in $\mathbb{R}^N$: Summary

In summary:

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \text{ has eigenvalues and eigenvectors} \tag{8}$$

$$\lambda_1 = 7, \lambda_2 = 4, \vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

If U is an orthogonal (change of basis) matrix then $A = UDU^T$ is symmetric and has the same eigenvalues as D

$$\lambda_1 = 7, \lambda_2 = 4 \tag{10}$$

The eigenvectors of A are $U\vec{e_1}$ (the 1st column of U, 1st new basis vector), and $U\vec{e_2}$ (the 2nd column of U, 2nd new basis vector)

The Spectral Theorem

- Let A be a **real symmetric** $N \times N$ matrix
- Then there is an $N \times N$ orthogonal matrix U and an $N \times N$ diagonal matrix D such that

$$A = UDU^T \tag{11}$$

- The diagonal elements of D are the eigenvalues of A
- The columns of U are the corresponding eigenvectors
- Eigenvectors for different eigenvalues are orthogonal
- UDU^T is the **eigenvalue decomposition** of A
- If $A = UDU^T$, mathematicians say the A is **diagonalizable**

The MatLab *eig* function

- In MatLab the function `eig` calculates the eigenvalue decomposition of a matrix
- If A is a real $N \times N$ symmetric matrix, then

$$\texttt{[U,D] = eig(A)} \tag{12}$$

  gives a real $N \times N$ orthogonal matrix U and a real $N \times N$ diagonal matrix D such that $UDU^T = A$
- The diagonal elements of D are the eigenvalues of A
- The columns of U are the eigenvectors of A
- Both D and U are real-valued
- U is a "change of basis" transformation. Relative to the new basis A is a diagonal matrix

Summary

- Eigenvalues and eigenvectors revisited
- The effect of a change-of-basis transformation on eigenvectors and eigenvalues
- The spectral theorem for a real, symmetric matrix
- The MatLab `eig` function