Lecture 12: A probabilistic approach to classification

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Learning Outcomes

By then end of this lecture you should be able to:

- Understand the solutions to the second assignment and how you could improve
- Understand how **I made a mistake** and how I am trying to correct it
- Explain multiclass classification with LDA and understand how it forms its decision boundaryies

Assignment 2

- Very well done by most people
- Here's how I approached this: https://drive.google.com/open?id=1fQ-8SUYqD8ASeEjABuEXWRi23OFiGIzu

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- Why? Lets' look at the code:

```
def knn(test_set, train_set, train_labels, k):
  ### Returns the most common label in the
      training set of the k-nn for each element in
       the test set.
  predictions = []
  for i in test_set:
    distances = [np.linalg.norm(i-j) for j in
        train_set]
    indices = np.argsort(distances)[0:k]
    predictions.append(mode(train_labels[indices])
        [0][0])
  return predictions
```

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- So distance calculations are wrong...
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- So distance calculations are wrong...
- The perils of dynamically typed languages...
- Demonstrations of dimensionality reduction are not quite as convincing as I had hoped...
- I am working on a revised examples – hopefully next Tuesday.

Recap: Linear Discriminant Analysis

- Build statistical models of classes

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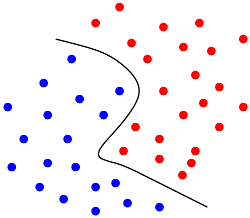
- Construct explicit decision boundary



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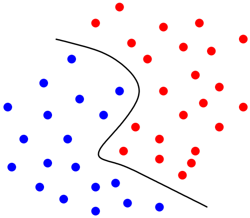

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Formulating LDA

- Bayes rule:

$$P(\Pi_i|\mathbf{x}) = \frac{P(\mathbf{x}|\Pi_i)P(\Pi_i)}{P(\mathbf{x})} \tag{1}$$

$$= \frac{f_i(\mathbf{x})\pi_i}{f_1(\mathbf{x})\pi_1 + f_2(\mathbf{x})\pi_2} \tag{2}$$

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- Equal ratios, randomly assign to either class.

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- $\mathbf{x}$ is assigned to Π_i if $L_{IJ} > 0$ for all $j \neq i$.
- The discriminant function between classes i and j is then

$$L_{ij}(\mathbf{x}) = \mathbf{m}_{ij}^{\mathrm{T}}\mathbf{x} + c_{ij}$$

with

$$\begin{aligned} \mathbf{m}_{ij} &= \left(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j\right)^{\mathrm{T}} \mathbf{\Sigma}^{-1} \text{ and} \\ c_{ij} &= -\frac{1}{2}\left(\bar{\mathbf{x}}_i^{\mathrm{T}}\mathbf{\Sigma}^{-1}\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j^{\mathrm{T}}\mathbf{\Sigma}^{-1}\bar{\mathbf{x}}_j\right) + \log_e\frac{\pi_i}{\pi_j}. \end{aligned}$$

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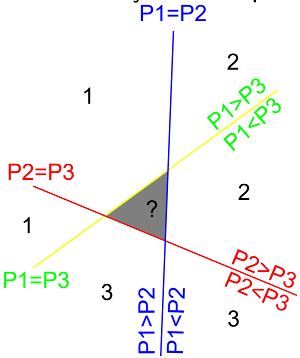

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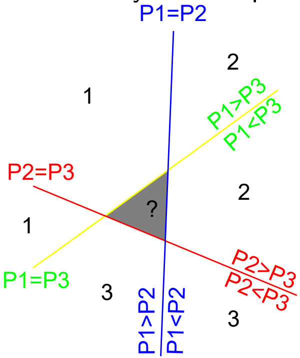

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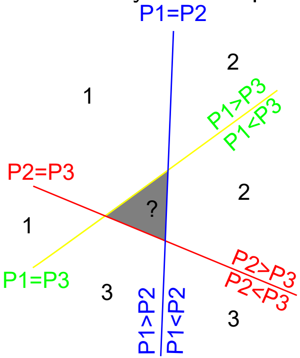

- Pairwise comparisons across all boundaries is the same as taking the class with the max posterior probability
- Let's see how it works. . . `https://colab.research.google.com/drive/1zXjLRI2qhvKoeG_ZAA5xPdWFDc8IENwq`

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- Explicitly construct class-conditional PDFs
- Resample from PDFs for generative modelling
- Strong assumptions – do they always matter?
- Next time: a purely discriminative probabilistic method: logistic regression