Distributed and Parallel Computing

Lecture 08

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Parallel Sorting

There are many serial algorithms that do not parallelize well. We want algorithms with:

- All many threads to work together on the problem
  - Serial algorithms are often inherently sequential
- Minimize branch divergence
  - Serial algorithms tend to do a lot of branching
- Coalesce memory access
  - Serial algorithms tend to access memory very randomly

Odd-Even Sort

The algorithm proceeds in a sequence of steps:

- In every even step, compare the elements in the even locations (0,2,4,...) with their neighbours to the right (1,3,5,...) and swap if out of order
- In every odd step, compare the elements in the odd locations with their neighbours to the right and swap if out of order

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- n inputs, steps: $O(n)$, work: $O(n^2)$

Parallel Merge Sort

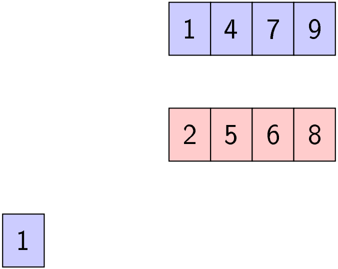
In the simplest form, Parallel Merge Sort works as follows:

- Start with a set of (trivially sorted) sequences of length 1
    - i.e. single elements
- In each step, merge independent pairs of sequences from the set of sorted sequences together to make a set of half the number of longer sorted sequences
- Finish when the last pair of sequences is merged into one final sorted sequence

Merging 2 sequences

Sequentially merging 2 sequences:

while neither sequence is empty
    Compare the elements at the head of the 2 sequences
    Pop smaller and append to the output sequence
append the elements of the non-empty sequence to the output



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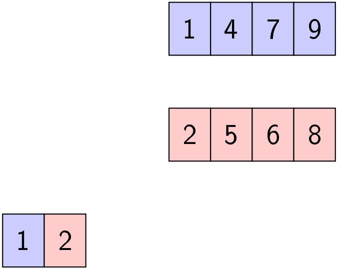

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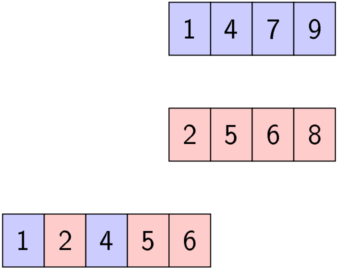

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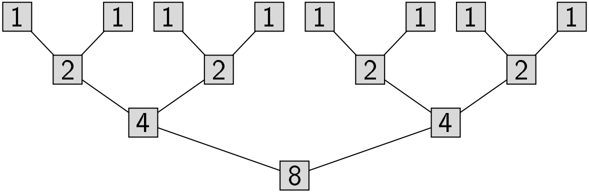
Parallel Merge Sort

Progressive sequence sizes shown:



- n inputs
- steps: $O(\log n)$
- work: $O(n \log n)$
  - In each step we are generating n elements.
  - Each element generated (except the last in each merge) is the result of one comparison
  - $n(1 - \frac{1}{2}) + n(1 - \frac{1}{4}) + n(1 - \frac{1}{8}) + \ldots$ with $\log n$ terms
  - $= n \log n - (n - 1)$
  - $= O(n \log n)$

Merge Sort on NVidia GPUs

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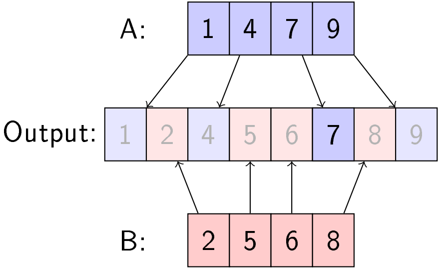

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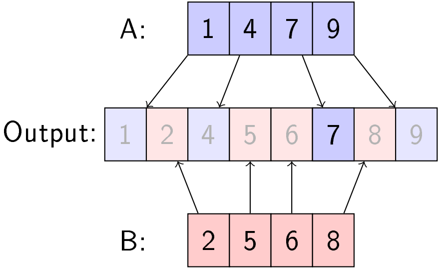

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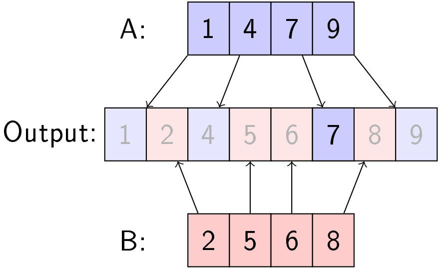

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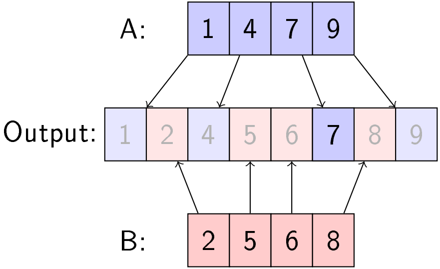

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- Thread does binary search to find insertion location in B is 3
- Hence location in output is $2 + 3 = 5$

Merge: 1 Block of Threads to 1 Merge

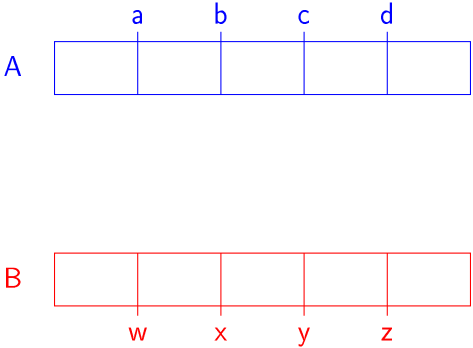
- Merge in a sequence of kernels
- Blocks per Grid is the number of merges to execute
- Threads per block is the number of elements that the merge will produce
- Copy sequences from global to shared memory, merge and copy back
- Thus (on GTX960s) suitable for merges that produce sequences of length 64 to 1024
  - GTX960 allows up to 32 blocks per SM, but can manage 2048 threads per SM. So less than 64 threads per block and the SM will not be fully occupied
- Can handle merges larger than 1024 elements:
  - Read chunks of sequences from global to shared memory, merge chunks and copy back
  - Slightly tricky to handle the *streams* of chunks

Merge: Multiple Blocks of Threads to 1 Merge

Problem is to break up a large merge so that different blocks can work on different parts of the merge independently

- Choose *splitters*, max K elements apart, from both sequences
- Merge the splitters into a single sorted list, remembering their locations in their home sequences
  - our previous medium merge method can do this
- Find the insertion location of each splitter in its *foreign* sequence (binary search)
  - Each splitter now has locations for both sequences
- Each consecutive pair of splitters thus defines a section of both sequences that can be merged independently of any other sections
- None of these sections can merge into more than $2K$ elements
- Choose K to be maximum 512 and each merge section can be handled by 1 block of 1024 threads

Merge: Multiple Blocks of Threads to 1 Merge



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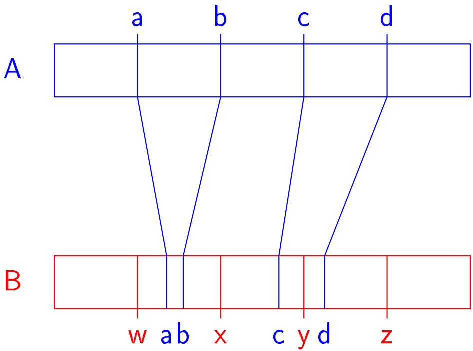

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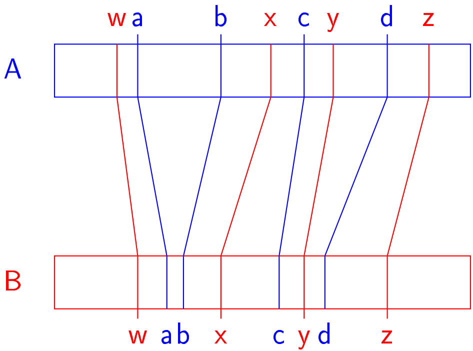

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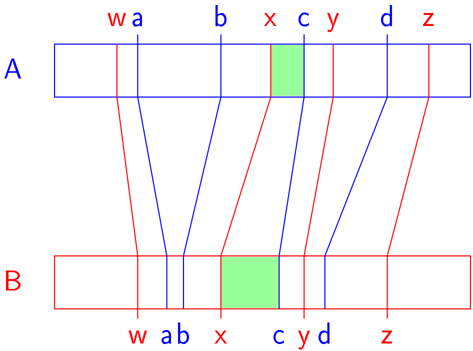
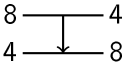


- $|[b, c]|$ in A is $K \Rightarrow |[x, c]| \leq K$ in A
- Similarly $|[x, c]| \leq K$ in B $+ \rightarrow \leq 2K$
- Hence the merge of the $[x, c]$ segments is no more than $2K$
- Similarly for all other segment pairs

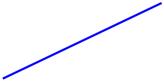
Bitonic Sort

Some definitions:

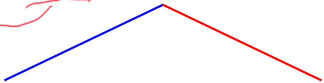
- A comparator is a function that swaps two elements if they are in the wrong order



- A monotonic increasing/decreasing sequence is one where every element is equal to or greater/less than every preceding element in the sequence
  - 1, 4, 8, 16, 16, 18, 19, 22



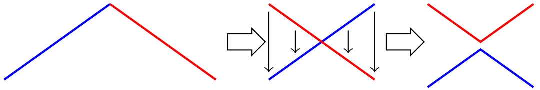
- A bitonic sequence is a sequence which changes order direction at most once, or a circular shift of such a sequence
  - 15, 12, 5, 2, 1, 4, 8, 16
  - 1, 4, 8, 16, 15, 12, 5, 2



Bitonic Split

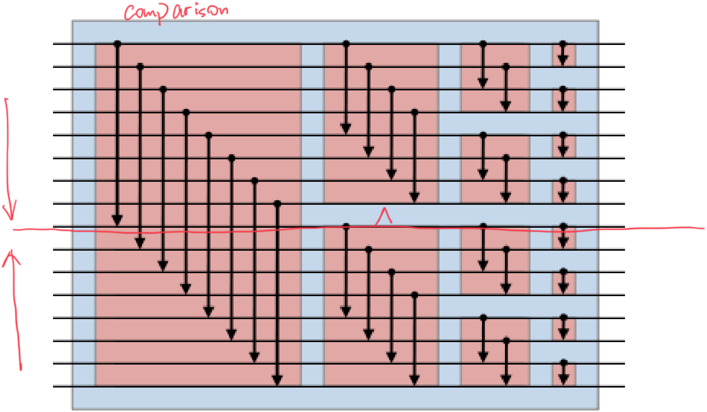
The central idea in Bitonic sort is that:

- A simple parallel arrangement of comparators can split a bitonic sequence into two bitonic sequences, where all elements of the first are less than all elements of the second:



									15	12	5	2		1	4	5	2	smaller
1	4	8	16	15	12	5	2	$\Rightarrow$					$\Rightarrow$					
									1	4	8	16		15	12	8	16	larger

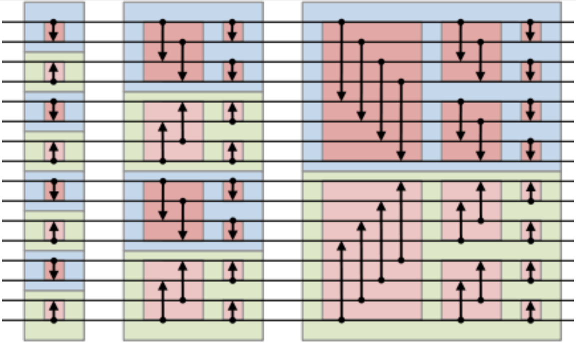
Bitonic Sort: Second Phase



- If the inputs along the left are a bitonic sequence:
  - First red block splits it into two bitonic sequences, where all upper half elements are less than all lower half ones
  - The next two red blocks splits these 2 into 4 similarly, etc.
  - Final output is sorted

image from `https://en.wikipedia.org/wiki/Bitonic_sorter`

Bitonic Sort: First Phase



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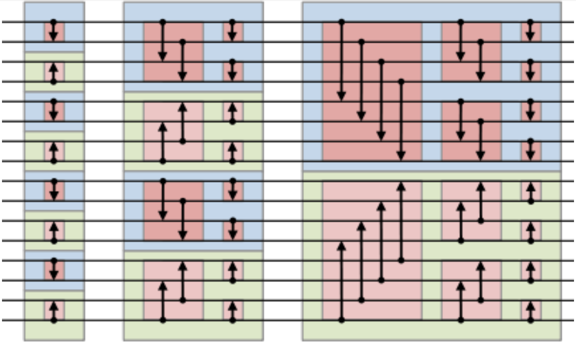

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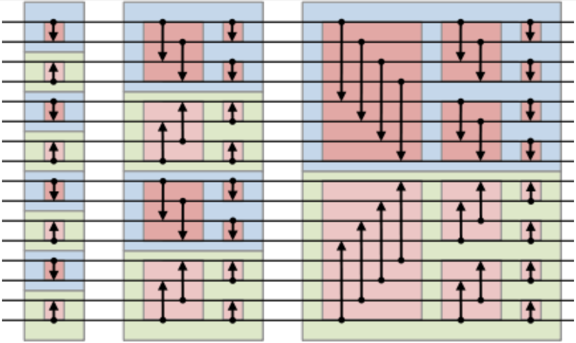

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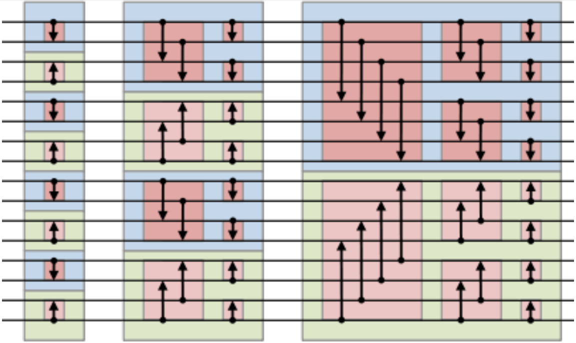

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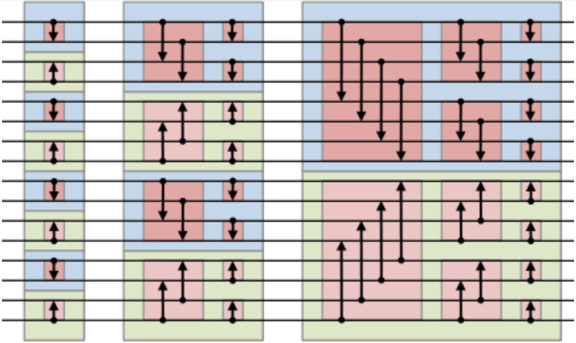

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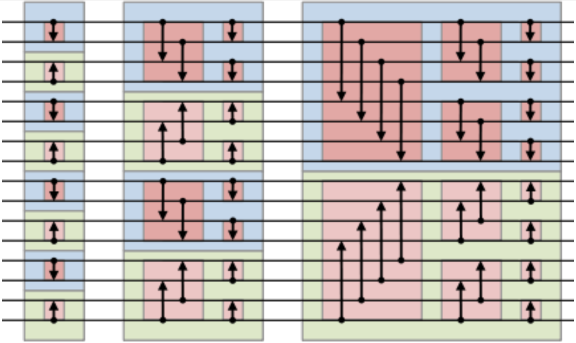

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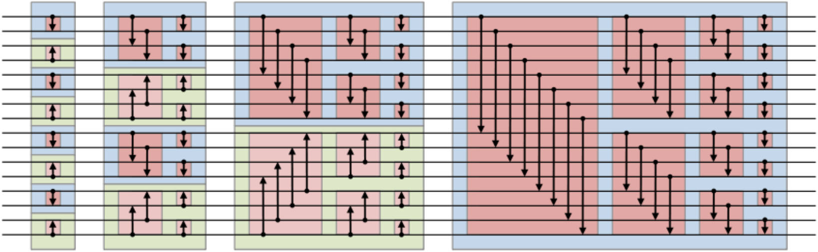
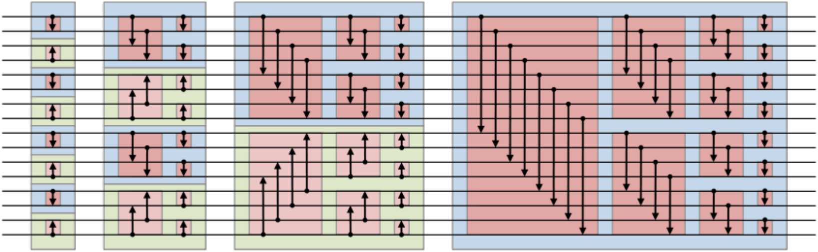


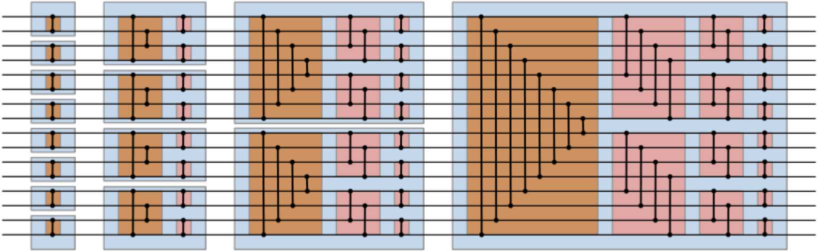
- Each column of red blocks runs in parallel with no races
- Assign each thread to one data element (Some implementations: 1 thread to one comparison)
- Each comparison executed twice:
  - At lower end, thread stores the smaller of the two values
  - At Upper end, thread stores the larger of the two values
- Complexity: $O(n \log^2 n)$ steps: but fastest sort for small sets
- Excellent for first stage of merge sort

image from `https://en.wikipedia.org/wiki/Bitonic_sorter`

Bitonic Sort



Can be rearranged with all arrows down:



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Radix Sort

Radix sort works by doing a series of **stable** splits based on ascending significance bits of the input values.

- A stable split preserves the relative orginal order of the elements in each part of the split

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- Fastest CUDA GPU sort for medium to large inputs

Radix Sort Implementation

- Each split section can be generated with a compact operation:
  - Map on LSB = 0, followed by an exclusive sum scan to calculate the first section scatter addresses

    least significant bit.
  - Use the last scatter address calculated as an offset to the scatter addresses for the second section
  - If using multi-bit radix steps, run a histogram to calculate the number in each section and hence the offsets