Introduction to linear algebra 2
Linear independence, bases and coordinate systems

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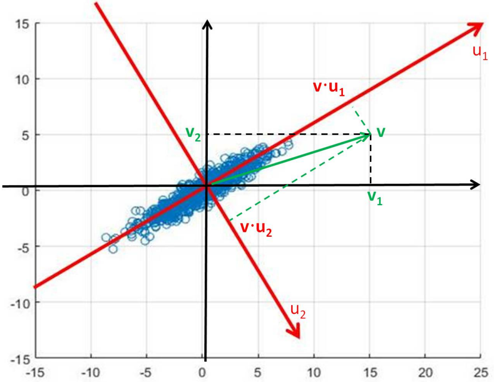
Overview

Alternative coordinate systems



Linear dependence

- Let $\vec{v_1}, \vec{v_2}, ..., \vec{v_M} \in \mathbb{R}^N$
- The set $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_M}\}$ is **linearly dependent** if there are scalars $\lambda_1, \lambda_2, ..., \lambda_M$, **not all of which are zero**, such that

$$\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + ... + \lambda_M \vec{v_M} = 0 \tag{1}$$

- The set $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_M}\}$ is **linearly independent** if it is not linearly dependent.

Examples

- Example 1:

$$\vec{v_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2}$$

  is linearly independent
- Example 2:

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \tag{3}$$

  is linearly independent
- Example 3:

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \tag{4}$$

  is linearly dependent

Demonstrating linear dependence/independence

- Equation (1) is a set of linear simultaneous equations

$$\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} + ... + \lambda_M \vec{v_M} = 0 \tag{5}$$

- If there is a *non-trivial* solution (i.e. not all of the λ_is are zero) then the vectors $\vec{v_1}, ..., \vec{v_M}$ are linearly dependent, otherwise they are linearly independent
- So, to determine linear (in)dependence, solve equation (5)
- This can be done, for example, by Gaussian elimination
  - Form the matrix V whose columns are $\vec{v_1}, ..., \vec{v_M}$
  - If V can be reduced to a diagonal matrix with non-zero diagonal elements by Gaussian elimination, then $\vec{v_1}, ..., \vec{v_M}$ are linearly independent, otherwise they are linearly dependent

Example 1

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \tag{6}$$

$$V = \begin{bmatrix} -1 & 3 & 4 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \sim_{-r_1, r_2 - r_1, r_3 - 2r_1} \begin{bmatrix} 1 & -3 & -4 \\ 0 & 5 & 6 \\ 0 & 7 & 9 \end{bmatrix} \tag{7}$$

$$\sim_{5r_1+3r_2, 5r_3-7r_2, \frac{r_3}{3}} \begin{bmatrix} 5 & 0 & -2 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix} \sim_{r_1+2r_3, r_2-6r_3} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$\tag{9}$$

Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\vec{v_1}, \vec{v_2}$ and $\vec{v_3}$ are linearly independent.

Example 2

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix}, \tag{10}$$

$$V = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 5 \\ 2 & 1 & 7 \end{bmatrix} \sim_{-r_1, r_2 - r_1, r_3 - 2r_1} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 5 & 5 \\ 0 & 7 & 7 \end{bmatrix} \tag{11}$$

$$\sim_{5r_1+3r_2, 5r_3-7r_2} \begin{bmatrix} 5 & 0 & 15 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \sim_{5r_1+3r_2, 5r_3-7r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$\tag{13}$$

Example 2 (continued)

$$V \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\tag{15}$$

Hence $\lambda_2 + \lambda_3 = 0$ and $\lambda_1 + 3\lambda_3 = 0$.
So, for example, if we set $\lambda_3 = 1$, then $\lambda_2 = -1$ and $\lambda_1 = -3$, and

$$-3 \times \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

$$-3\vec{v_1} - \vec{v_2} + \vec{v_3} = 0 \tag{17}$$

Hence $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ are linearly dependent.

Linear dependence - interpretation

- The set $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_M}\}$ is **linearly dependent** if at least one member of the set can be written as a linear combination of the others.
- In $\mathbb{R}^2$ two vectors $\vec{v_1}, \vec{v_2}$ are linearly dependent if and only if $\vec{v_1}$ and $\vec{v_2}$ point in the same direction
- Any set of three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ in $\mathbb{R}^2$ is linearly dependent
- What about $\mathbb{R}^3$?

What is a basis for a vector space?

- A **basis** for a vector space is just a coordinate system.
- In $\mathbb{R}^3$ we have the standard "x, y, z" coordinate system
- Any vector

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \in \mathbb{R}^3 \tag{18}$$

can be written

$$\vec{v} = v_x \times \vec{u_x} + v_y \times \vec{u_y} + v_z \times \vec{u_z} \tag{19}$$

where $\vec{u_x}, \vec{u_y}$ and $\vec{u_z}$ are the **unit vectors**

$$\vec{u_x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u_y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u_z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{20}$$

What is a basis? (continued)

- $\vec{u_x}, \vec{u_y}$ and $\vec{u_z}$ are unit vectors because

$$\|\vec{u_x}\| = \|\vec{u_y}\| = \|\vec{u_z}\| = 1 \tag{21}$$

- $\vec{u_x}, \vec{u_y}$ and $\vec{u_z}$ are also **orthogonal** because

$$\vec{u_x} \cdot \vec{u_y} = \vec{u_x} \cdot \vec{u_z} = \vec{u_y} \cdot \vec{u_z} = 0 \tag{22}$$

- In fact, any set of 3 orthogonal unit vectors can be used as a coordinate system for $\mathbb{R}^3$. Such a set is called a **basis** for $\mathbb{R}^3$
- Strictly speaking, what is describe here is an **orthonormal basis**, but we will call it a basis.

Definition of a basis of $\mathbb{R}^N$

A **basis** for the vector space $\mathbb{R}^N$ is a set of vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_N}$ such that:

1. $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_N}$ are unit vectors:
$$\|\vec{u_1}\| = \|\vec{u_2}\| = \cdots = \|\vec{u_N}\| = 1 \tag{23}$$
and
2. $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_N}$ are mutually orthogonal:
$$\vec{u_n} \cdot \vec{u_m} = 0, n \neq m \tag{24}$$

Notice that equations (23) and (24) can be combined into a single condition: $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_N}$ is a basis for $\mathbb{R}^N$ if and only if

$$\vec{u_n} \cdot \vec{u_m} = \begin{cases} 0, n \neq m \\ 1, n = m \end{cases} \tag{25}$$

Example 1

The set

$$\vec{u_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \tag{26}$$

is a basis for $\mathbb{R}^2$. To show this you need to show that $\vec{u_1}, \vec{u_2}$ satisfy conditions (23) and (24).

$$\|\vec{u_1}\| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \tag{27}$$

$$\|\vec{u_2}\| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \tag{28}$$

Hence $\vec{u_1}$ and $\vec{u_2}$ are unit vectors. Next show that they are orthogonal

Example 1 (continued)

To see that $\vec{u_1}$ and $\vec{u_2}$ are orthogonal

$$\vec{u_1} \cdot \vec{u_2} = \left(\frac{\sqrt{3}}{2} \times \frac{-1}{2}\right) + \left(\frac{1}{2} \times \frac{\sqrt{3}}{2}\right) \tag{29}$$

$$= \frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0 \tag{30}$$

Since $\vec{u_2} \cdot \vec{u_1} = \vec{u_1} \cdot \vec{u_2}$ this completes the proof.

Example 2

The set

$$\vec{u_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{u_2} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad (31)$$

is **not** a basis for $\mathbb{R}^2$. To see this we need to show that $\vec{u_1}, \vec{u_2}$ fail to satisfy at least one of the properties in equations (23) and (24). First see if $\vec{u_1}$ and $\vec{u_2}$ are unit vectors:

$$\|\vec{u_1}\| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \quad (32)$$

$$\|\vec{u_2}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \quad (33)$$

So, $\vec{u_1}$ and $\vec{u_2}$ are unit vectors. Next test for orthogonality

Example 2 (continued)

To test if $\vec{u_1}$ and $\vec{u_2}$ are orthogonal

$$\vec{u_1} \cdot \vec{u_2} = \left(\frac{\sqrt{3}}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{\sqrt{3}}{2}\right) \tag{34}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} \neq 0 \tag{35}$$

Hence $\vec{u_1}$ and $\vec{u_2}$ are not orthogonal and hence are not a basis for $\mathbb{R}^2$

Two questions

1. Are $\vec{u_1}$ and $\vec{u_2}$ linearly independent?
2. Can $\vec{u_1}$ and $\vec{u_2}$ be made into a basis?

Properties of bases

Any basis $E = \{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N}\}$ for $\mathbb{R}^N$ is linearly independent.

- To see this, suppose $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N}\}$ is linearly dependent. Then there exist $\lambda_1, \lambda_2, \cdots, \lambda_N$ such that

$$\lambda_1 \vec{e_1} + \lambda_2 \vec{e_2} + \cdots + \lambda_N \vec{e_N} = 0 \tag{36}$$

  and not all of the λ_ns are zero. Assume $\lambda_1 \neq 0$ then

$$\vec{e_1} = \phi_2 \vec{e_2} + \phi_3 \vec{e_3} + \cdots + \phi_N \vec{e_N}, (\phi_n = -\frac{\lambda_n}{\lambda_1})$$

  Hence

$$0 = \vec{e_n} \cdot \vec{e_1} = \phi_2 \vec{e_n} \cdot \vec{e_2} + \cdots + \phi_N \vec{e_n} \cdot \vec{e_N} = \phi_n$$

- Hence $0 = \phi_n$ and so $\lambda_n = 0$. Since this can be repeated for all values of $n \neq 1$, $\lambda_n = 0, n \neq 1$. But then it must be the case that $\lambda_1 = 0$.

Properties of bases (2)

Let $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N}$ be a basis for $\mathbb{R}^N$. Then any vector $\vec{v} \in \mathbb{R}^N$ can be written **uniquely** as a linear sum of basis vectors

$$\vec{v} = (\vec{v} \cdot \vec{e_1})\vec{e_1} + (\vec{v} \cdot \vec{e_2})\vec{e_2} + \cdots + (\vec{v} \cdot \vec{e_N})\vec{e_N} \quad (37)$$

Uniqueness means that if

$$\vec{v} = \lambda_1 \vec{e_1} + \lambda_2 \vec{e_2} + \cdots + \lambda_N \vec{e_N} \quad (38)$$

and

$$\vec{v} = \phi_1 \vec{e_1} + \phi_2 \vec{e_2} + \cdots + \phi_N \vec{e_N} \quad (39)$$

Then

$$\lambda_n = \phi_n = (\vec{v} \cdot \vec{e_n}), n = 1, \ldots, N \quad (40)$$

Proof

Note that

$$0 = \vec{v} - \vec{v} = (\lambda_1 - \phi_1)\vec{e_1} + (\lambda_2 - \phi_2)\vec{e_2} + \cdots + (\lambda_N - \phi_N)\vec{e_N} \quad (41)$$

Hence $\lambda_n = \phi_n, n = 1, \cdots, N$, because $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N}\}$ is linearly independent

Summary

- Motivation
- Linear dependent and independent sets of vectors
- Bases