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# Cryptography III

Public-key Cryptosystems,

Digital Signatures and Hash functions

# Weaknesses of symmetric cryptosystems

- Managing and distributing shared secret keys is so difficult in a model environment with too many parties and relationships
  - N parties ➔ n(n-1)/2 relationships ➔ each manages (n-1) keys
- No way for digital signatures
  - No non-repudiation service

# Diffie-Hellman new ideas for PKC

- In principle, a PK cryptosystem is designed for a single user, not for a pair of communicating users
  - More uses other than just encryption
- Proposed in Diffie and Hellman (1976) “New Directions in Cryptography”
  - public-key encryption schemes
  - public key distribution systems
    - Diffie-Hellman key agreement protocol
  - digital signature

# Diffie-Hellman's proposal

- Each user creates 2 keys: a secret (private) key and a public key → published for everyone to know

  - The PK is for encryption and the SK for decryption

    X = D(z, E(Z, X))

  - The SK is for creating signatures and the PK for verifying these signatures

    X = E(Z, D(z, X)) → D() for creating signatures, E → verifying

- Also, called asymmetric key cryptosystems
  - Knowing the public-key and the cipher, it is computationally infeasible to compute the private key

# Principles of designing a PK system (trapdoor)

- Using one-way function:
  - Given X, it is easy to compute Y = f(X)
  - Given Y it is hard to compute X = $f^{-1}(Y)$

Example:

  - Given $p_1, p_2, \ldots p_n$ it is easy to compute $N = p_1 * p_2 * \ldots * p_n$ but given N it is hard to find $p_1, p_2, \ldots p_n$

- Such an one-way function can be used as a trapdoor to create a PKC
  - Encryption is easy
  - Decryption is difficult (if not knowing the secret key)

# Merkle – Hellman's encryption scheme using ***Trapdoor Knapsack***

- 1978, Merkle & Hellman proposed an encryption scheme using this Knapsack problem:
  - Given a set of positive numbers $a_i$, $1 \le i \le n$ and $0 < T < \sum_{i=1,n} a_i$; Find a set of indexes $S \subset \{1,2,...,n\}$ such that: $\sum_{i \in S} a_i = T$
  - <u>Example:</u>

    $(a_1, a_2, a_3, a_4) = (2, 3, 5, 7)$ $T = 7$.

    There are 2 solutions: $S = (1, 3)$ as $T = a_1 + a_3$

    and $S = (4)$ as $T = a_4$
- This is a hard problem (NP-hard):
  - No P-time algorithm has been found
  - Exhaustive search: exponential time .

# Merkle – Hellman's encryption scheme

- Consider attempts to create a PK scheme using Knapsack trapdoor; here is a first attempt
  - Select a cargo vector a = $(a_1, a_2, \ldots, a_n)$
  - Encryption: for a binary plaintext block X = $(X_1, X_2, X_3 \ldots, X_n)$ compute: $T = \sum a_i X_i$ (*)
  - Decryption: Given cipher block T, knowing vector a, find $X_i$ that satisfy (*)
- Trapdoor: One way is definitely easy, the other is HARD
- BUT not yet a PK system, we need to make it easy for the owner who knows a secret key

# Merkle – Hellman's encryption scheme

- Merkle added a trick
  - using a super-increasing vector wherein the (i+1)th element is > the sum of all preceding elements (1÷i)
- Using a super-increasing cargo vector, the decryption is so easy

<u>Example</u>

Super-increasing vector: a=(1,2,4,8)

For T=11, we easily compute $X=(X_1,X_2,X_3,X_4)$ such that $T= \sum a_i X_i$ :

Let $T=T_0$

| | | | |
|---|---|---|---|
| $X_4=1$ | $T_0=T_0 - X_4=3$ | ➔ | $(X_1\ X_2\ X_3\ 1)$ |
| $X_3=0$ | $T_2=T_1=3$ | ➔ | $(X_1\ X_2\ 0\ 1)$ |
| $X_2=1$ | $T_3=T_2 - 2=1$ | ➔ | $(X_1\ 1\ 0\ 1)$ |
| $X_1= 1$ | | ➔ | $(1\ 1\ 0\ 1)$ |

# Merkle – Hellman's encryption scheme

- Exercise

  Draw a diagram/pseudo-code to describe an algorithm for the decryption using a super-increasing cargo vector

- To complete the PK scheme however the owner need to disguise his secret key, the super-increasing vector

# Merkle – Hellman: the disguise mechanism

- *Creating keys*:

  Alice creates a super-increasing vector:

  $$a' = (a_1', a_2', \ldots, a_n')$$

  *a'* will be kept as a part of the secret key

  - Then choose $m > \sum a_i'$ to be used as the modulus and choose $\omega$ that is relatively prime to $m$.
  - Now Alice's public key is the vector *a* as the product of *a'* with $\omega$

    $a = (a_1, a_2, \ldots, a_n)$

    $a_i = \omega \times a_i' \pmod{m};\ i = 1, 2, 3 \ldots n$

  - Alice's secret key is the triple $(a', m, \omega)$

# Merkle-Hellman scheme

- ***Encryption:***
  - When Bob wants to send a message X to Alice, he computes:

$$T=\sum a_i X_i$$

- ***Decryption***:
  - When Alice receives T, she will transform the equation $T = a \times X$ into $T' = a' \times X$ as follows

  She first computes $\omega^{-1}$ i.e. $\omega \times \omega^{-1} = 1$ mod $m$, then compute $T' = T \times \omega^{-1}$ $(mod\ m)$

  - Alice *then solve* $T' = a' \times X$ using the super-increasing vector a'.

- Why?

$$T' = T \times \omega^{-1} = \sum a_i X_i \omega^{-1} = \sum a_i' \omega X_i \omega^{-1}$$
$$= \sum (a_i' \omega \omega^{-1}) X_i = \sum a_i' X_i = a' \times X$$

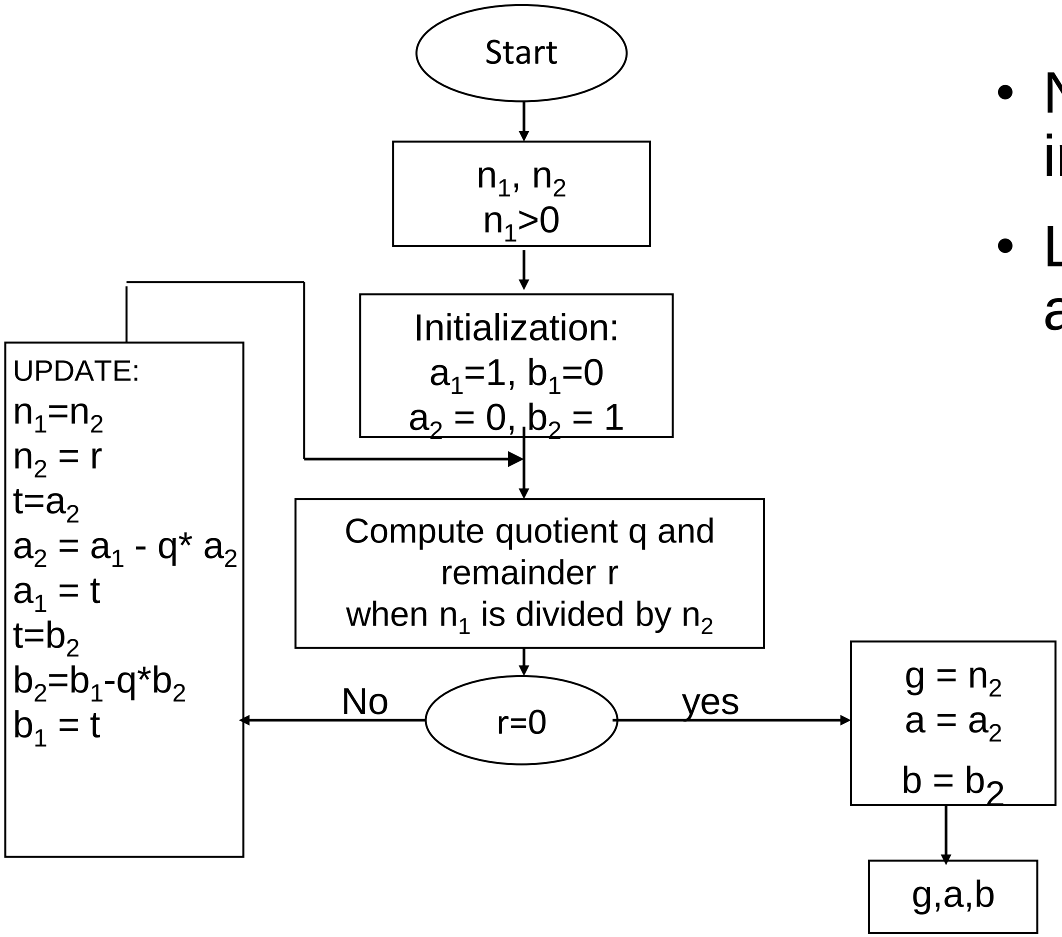
# Failure of Merkle-Hellman PKC

- **Brute Force Attack**
  - For whom not knowing the trapdoor (a', m, $\omega$), decrypting requires the exhaustive search of $2^n$ possible values of X
- **Failure of this Knapsack-based scheme (1982-1984).**
  - Shamir-Adleman showed a weakness by finding a pair ($\omega$',m') to convert a back to a' (finding the private key from the public key)
  - 1984, Brickell announced the collapse of this Knapsack-based system by one hour of computation using Cray -1 for 40 rounds and approx. 100 weights.

# Algorithm for computing modulo inverse

- Computing the inverse of $\omega$ by modulo m
  - Finding $x = \omega^{-1}$ mod m such that $x^*\omega = 1$ (mod m)
  - Many applications such as in the Knapsack trapdoor
- Based on the extended GCD algorithm or the extended Euclidean algorithm (GCD: Greatest common divisor)
  - On finding the GCD of 2 numbers $n_1$ và $n_2$, one will also compute a & b such that $GCD(n_1, n_2) = a \times n_1 + b \times n_2$.
  - If $gcd(n_1,n_2)=1$ then this e-GCD algorithm will find a, b to meet $a \times n_1 + b \times n_2 = 1$, i.e. $n_1$ is the inverse of a by modulo $n_2$

# Homework: prove the correctness of this algorithm




- Numeric example: find the inverse of 11 by modulo 39
- Let $n_1$=39, $n_2$=11 then run the algo as in the following table:

| $n_1$ | $n_2$ | r | q | $a_1$ | $b_1$ | $a_2$ | $b_2$ |
|---|---|---|---|---|---|---|---|
| 39 | 11 | 6 | 3 | 1 | 0 | 0 | 1 |
| 11 | 6 | 5 | 1 | 0 | 1 | 1 | -3 |
| 6 | 5 | 1 | 1 | 1 | -3 | -1 | 4 |

# General remarks on PKC

- Since 1976,many PKC schemes had been proposed many was broken
- A PKC have two main applications
  - Hiding information (including secrete communication)
  - Authentication with digital signatures
- The two algorithms that are most successful are RSA và El-Gamal.
- In general PKC is very slow, not appropriate for on-line encryption
  - Not used for encrypting large volume of date but for special purposes.
  - PKC and SKC are used in combined:
    - Alice and Bob use a PKC system to create a shared secret key between them and then use a SKC system to encrypt the communicated data by using this secret key

# RSA Algorithm

- Invented in **1978** by Ron **R**ivest, Adi **S**hamir and Leonard **A**dleman
  - Published as R L Rivest, A Shamir, L Adleman, "*On Digital Signatures and Public Key Cryptosystems*",Communications of the ACM, vol 21 no 2, pp120-126, Feb 1978
  - Security relies on the difficulty of factoring large composite numbers
- Essentially the same algorithm was discovered in 1973 by Clifford Cocks, who works for the British intelligence

# Main idea

- Encryption and decryption functions are modulo exponential in the field $Z_n = \{0,1,2,..n-1\}$
  - Encryption: $Y=X^e$ mod n (or $\pm$ n)
    - a = b $\pm$n ➔ a=b+k*n, $a \in Z_n$, k = 1,2,3,... e.g. 7 = 37 $\pm$10
  - Decryption: $X= Y^d \pm n$
  - The clue is that e & d must be selected such that

$X^{ed} = X \pmod n$

# Main idea

- The way to create such e&d is by using this Euler theorem: $X^{\varphi(n)}=1 \pmod n$

  - $\varphi(n)$: the size of $Z^*_n = \{k:0<k<n| (k,n)=1\}$

  - $\varphi(n)$ can be computed easily if knowing n factoralization
    - n= p*q, where p, q are primes ➔ $\varphi(n) = (p-1)(q-1)$

  - First choose *e* then compute *d* s.t. $e*d= 1 \pm \varphi(n)$
    or $d \equiv e^{-1} \bmod \varphi(n)$, which will assure that
    $X^{ed}= X^{k.\varphi(n)+1} \equiv (X^{\varphi(n)})^k *X \equiv 1^k * X = X \pmod n$

- Note this works because we know n's factorization
  - From e we compute $d \equiv e^{-1} \bmod \varphi(n)$ since we know $\varphi(n)$, otherwise it is computational infeasible to compute d s.t. $X^{ed} \equiv 1 \bmod n$

# RSA PKC

- **Key generation:**
  - Select 2 large prime numbers of about the same size, p and q
  - Compute $n = pq$, and $\Phi(n) = (q-1)(p-1)$
  - Select a random integer e, $1 < e < \Phi(n)$, s.t. $\gcd(e, \Phi(n)) = 1$
  - Compute d, $1< d< \Phi(n)$ s.t. $ed \equiv 1 \bmod \Phi(n)$
  - **Public key: (e, n) and Private key: d**
    - **Note: p and q must remain secret**

# RSA PKC (cont)

- **Encryption**
  - Given a message M, $0 < M < n$: $M \in Z_n - \{0\}$
  - use public key (e, n) compute

    $C = M^e \bmod n$, i.e. $C \in Z_n - \{0\}$

- **Decryption**
  - Given a ciphertext C, use private key (d) compute $M = C^d \bmod n$

- Why work?
  - $(M^e \bmod n)^d \bmod n = M^{ed} \bmod n = M$

# Example

- Parameters:
  - Select p = 11 vàq = 13
  - n=11*13=143; m= (p-1)(q-1) =10 *12=120
  - Choose e=37 ➔ gcd(37,120=1
  - Using  the algo gcd: e*d =1 ±120 ➔ d= 13 (e*d=481)

- To encrypt a binary string
  - Split it into segments of u bit s.t. $2^u \leq 142$ ➔ u = 7. That is each segment present a number from 0 to 127
  - Compute Y= $X^e$±143

  E.g. For X = (0000010) = 2, we have

  Y= $E_Z(X) = X^{37}$= 12±143 ➔ Y=(00001100)

- Decryption:  X= $D_Z(Y)=12^{13}$=2±143

# RSA implementation

- Execution of RSA is about thousand times slower than DES
  - Even using the fast exponential algorithm and specifically designed hardwares
- n, p, q
  - The security of RSA depends on how large n is, which is often measured in the number of bits for n. Current recommendation is 1024 bits for n.
  - p and q should have the same bit length, so for 1024 bits RSA, p and q should be about 512 bits.
  - p-q should not be small
  - Way to select p and q
    - In general, select large numbers (some special forms), then test for primality
    - Many implementations use the Rabin-Mille test, (probabilistic test)

# *Factorization Prolem*

- Estimated time using the sieve algorithm

$$L(n) \approx 10^{9.7+\frac{1}{50}\log_2 n}$$

- $\log_2 n$: the number of bits in representing n

- By 1996, for n=200, L(n) ≈ 55,000 years.

- Using parallel computing, one can factorize a 129–digit number in 3 months by distributing the workload to the computers throught out the Internet at 1996-7

- Today, for applications requiring high security levels one should values of in 1024-bit or even 2048-bit.

# *Modulo Exponential*

- Fast algorithm to compute exponential in $Z_n$ (modulo n):
  Computing $X^{\alpha}$ (modul n)

- Determine coefficients $\alpha_i$ in the binary representation of $\alpha$ :

$$\alpha = \alpha_0 2^0 + \alpha_1 2^1 + \alpha_2 2^2 + ... + \alpha_k 2^k$$

- Loop in k rounds to compute these *k* modulo exponential, với i=1,k :

$$X^2 = X \times X$$

$$X^4 = X^2 \times X^2$$

$$...$$

$$X^{2^k} = X^{2^{k-1}} \times X^{2^{k-1}}$$

- Now compute $X^{\alpha}$ mod n by multiplying theses $X^{2i}$ computed in the previous steps but only with corresponding coefficients $\alpha_i$ =1:

$$(X^{2^i})^{\alpha_i} = \begin{cases} 1, \alpha_i = 0 \\ X^{2^i}, \alpha_i = 1 \end{cases}$$

# Suggested topics for Reports

- The implementation and correctness of the extended GCD algorithm
- The probabilistic primality test
- Exponential algorithms and implementation
- The correctness of RSA algorithms
- Common Attacks to RSA

# Digital Signatures

- Motivation
  - Diffie-Hellman proposed the idea (1976)
  - Simulation of the real-world into digital worlds
    - Paper contracts need signed to be valid so do electronic versions

- The proofs conveyed in signatures
  - Data integrity: information is original, not modified
  - Authentication: The source of the info is correct, not impersonated

# DS: how they work

- Digital Signature: a data string which associates a message with some originating entity.

- Digital Signature Scheme:
  - a signing algorithm: takes a message and a (private) signing key, outputs a signature
  - a verification algorithm: takes a (public) key verification key, a message, and a signature

- A DS is created based on a PK system
  - Alice signs message X by creating $Y=D_{z_A}(X)$, so the signed document now is $(X, Y=D_{z_A}(X))$.
  - Bob who receives (X,Y), computes $X'=E_{Z_A}(Y)$ then compare if X=X' to confirm the document's validity

# Non-repudiation

- We mention more on applications of DS

- Non-repudiation

    - The signer can't deny that his/her created the document
        - Only Alice knows $z_A$ to create $(X, Y=D_{z_A}(X))$ but everyone else can verify

    - So we say the DS scheme provides non-repudiation

# Public notary

- Motivation
  - Alice may lost her secret key or someone stole it ➔ that bad guy can impersonate Alice to create documents with Alice signatures out of Alice's control
  - Alice can also deny a document truly signed by her in the past: Alice claims the document was impersonated by someone stealing her SK

- Solution: Public notary service
  - A third party – a public notary – can be hired for important documents
  - The trusted notary also signs on the same document, that is to create his signature on the concatenation of the document and Alice's signature

# Proof of delivery (receipts)

- Motivation
  - The sender need proof that the receiver has already got his message
  - The receiver can't deny that once the sender got a receipt

- Solution: An adjudicated protocol
  - A→B: $Y=E_{z_B}(D_{z_A}(X))$
  - B computes: $X'=E_{z_A}(D_{z_B}(Y))$
    - When receiving Y, B computes and checks if X'=X then signs on X' and pass to A as a receipt .
  - B→A: $Y=E_{z_A}(D_{z_B}(X'))$
    - By computing $D_{z_A}(Y)$, A now gets $D_{z_B}(X')$, a B's signature on X
  - Only when A has Y she can consider that B has receive her doc
  - Later, B can not deny receiving X since A can prove otherwise by showing $D_{z_B}(Y)$)

# Weakness of the signature scheme mentioned so far

- When using a PKC to sign X, X can be long ➔ splitting into blocks and signs

  $X = (X_1, X_2, X_3, \ldots X_t) \Rightarrow (SA(X_1), SA(X_2), SA(X_3), \ldots SA(X_t))$

- This creates vulnerability to attack on manipulating blocks
  - The attacker can change order of blocks, remove/ add in a few

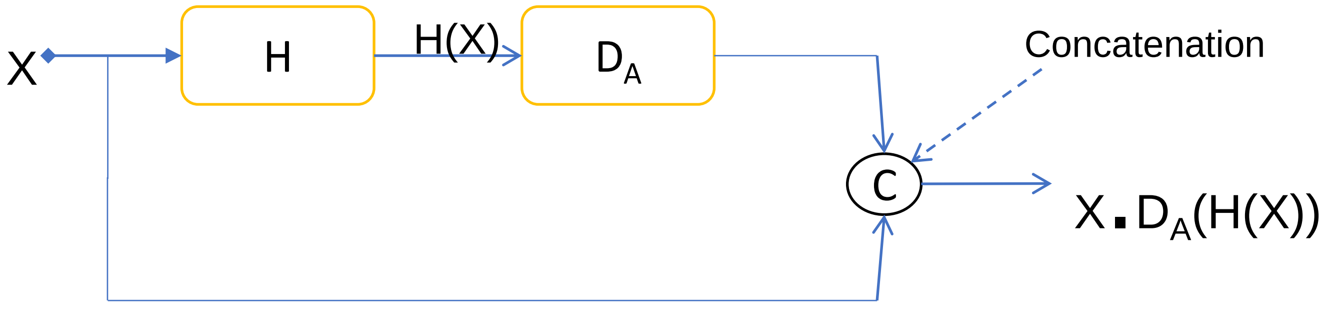
- Slow: PKC is already slow, now is run multiple times

- Signature is long, as long as the message itself.

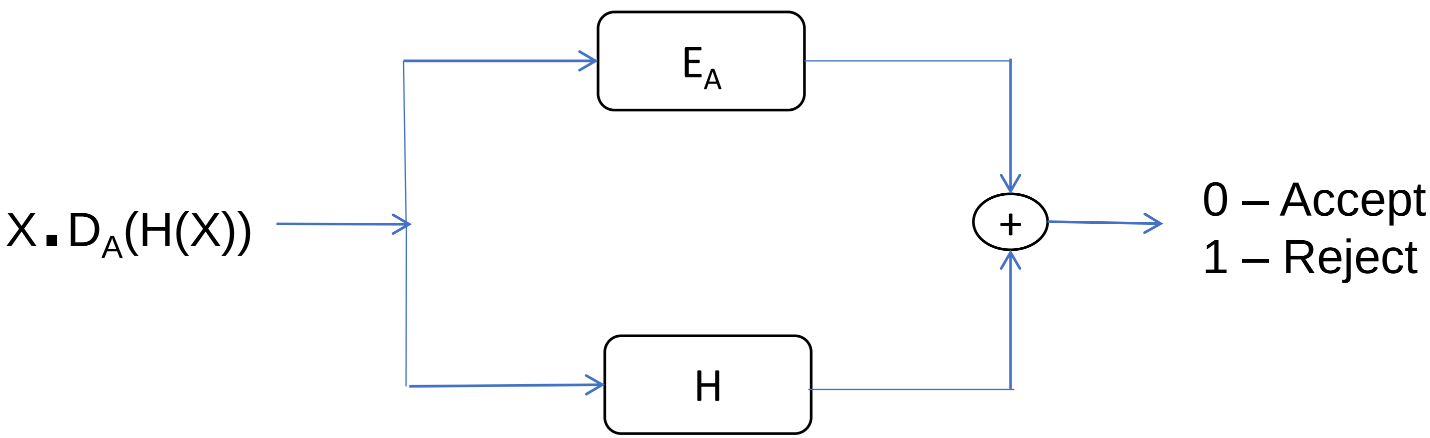
# Hash Functions

- A hash function H maps a message of variable length n bits to a fingerprint of fixed length m bits, with m < n.
  - This hash value is also called a digest (of the original message).
  - Since n>m, there exist many X which are map to the same digest ➔ collision.
- Applications
  - Digital signatures
  - Message authentication

# DS schemes with hash functions




Signature Generator




Signature Verifier

# Main properties

Given a hash function H: $X \rightarrow Y$

- Long message → short, fixed-length hash
- One-way property: given $y \in Y$

  it is computationally infeasible to find a value $x \in X$ s.t. $H(x) = y$
- Collision resistance (collision-free)

  it is computationally infeasible to find any two distinct values $x', x \in X$ s.t. $H(x') = H(x)$
  - This property prevent against signature forgery

# Collisions

- Avoiding collisions is theoretically impossible
  - Dirichlet principle: n+1 rabbits into n cages → at least 2 rabbits go to the same cage
  - This suggest exhaustive search: try |Y|+1 messages then must find a collision (H:X→Y)
- In practice
  - Choose |Y| large enough so exhaustive search is computational infeasible.
    - |Y| not too large or long signature and slow process
  - However, collision-freeness is still hard

# Birthday attack

- Can hash values be of 64 bits?
  - Look good, initially, since a space of size $2^{64}$ is too large to do exhaustive search or compute that many hash values
  - However a birthday attack can easily break a DS with a 64-bit hash function
    - In fact, the attacker only need to create a bunch of $2^{32}$ messages and then launch the attack with reasonably high probability for success.

# How is the attack

- Goal: given H, find x, x' such that H(x)=H(x')
- Algorithm:
  - pick a random set S of q values in X
  - for each $x \in S$, computes $h_x = H(x)$
  - if $h_x = h_{x'}$ for some $x' \neq x$ then collision found: (x,x'), else fail
- The average success probability is

  $\varepsilon = 1\text{-}\exp({}^{q(q-1)}/_{2|Y|})$
  - Suppose Y has size $2^m$, choose $\mathbf{q \approx 2^{m/2}}$ then $\varepsilon$ is almost 0.5!

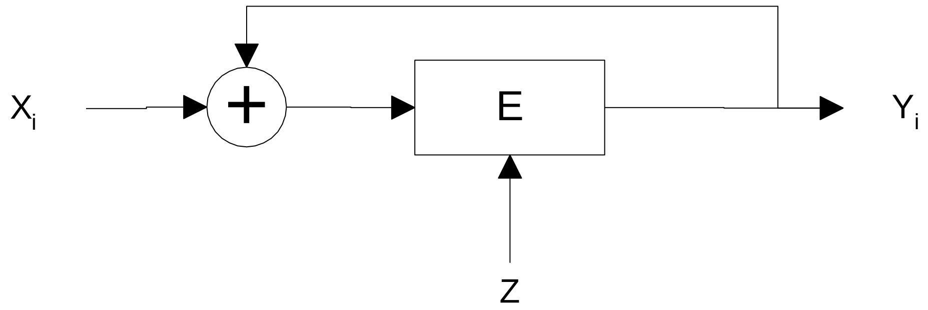
# Birthday paradox

- Given a group of people, the minimum number of people
  - such that two will share the same birthday with probability at least 50%

  is only 23 ➔ why "paradox"
  - Computing the chance
    - 1 - (1- 1/365)(1-2/365)…(1-22/365) = 1-0.493 = 0.507

# Common techniques to build hash functions

- Using SKC
  - E.g. using SKC in CBC mode
- Using modulo arithmetic operations
- Specific designs
  - MD4, MD5, SHA




$X = X_1\ X_2\ X_3\ \ldots\ X_n$

$Y_i = E_Z(X_i \oplus Y_{i-1})$

$H(X) = Y_n$

# MAC: message authentication code

- Hash function is public and the key shared between the sender and the receiver is secret
  - Sender computes mac1 = MAC(M, H, K) and sends it along with the message M
  - Receiver computes mac2 = MAC(M, H, K) and checks if mac1 = mac2 ? Yes ➔ the message is authentic; no => reject it
- The output of MAC can not be produced without knowing the secret key
  - So, this mechanism provides data integrity and ***source authentication***

# More on the Birthday Paradox

What is the probability that two persons in a room of 23 have the same birthday?

# Birthday Paradox

- Ways to assign $k$ different birthdays without duplicates:

$N = 365 * 364 * \ldots * (365 - k + 1)$

$= 365! / (365 - k)!$

- Ways to assign $k$ different birthdays with possible duplicates:

$D = 365 * 365 * \ldots * 365 = 365^k$

# Birthday “Paradox”

Assuming real birthdays assigned randomly:

N/D = probability there are no duplicates

1 - N/D = probability there is a duplicate

$$= 1 - 365! / ((365 - k)!(365)^k)$$

# Generalizing Birthdays



$$P(n, k) = 1 - n!/(n-k)!n^k$$

Given $k$ random selections from $n$ possible values, $P(n, k)$ gives the probability that there is at least 1 duplicate.

# Birthday Probabilities

P(∃ two match) = 1 – *P*(all are different)

*P*(2 chosen from *N* are different)

$= 1 - 1/N$

*P*(3 are all different)

$= (1 - 1/N)(1 - 2/N)$

*P*(*k* trials are all different)

$= (1 - 1/N)(1 - 2/N) \dots (1 - (k - 1)/N)$

ln (*P*)

$= \ln(1 - 1/N) + \ln(1 - 2/N) + \dots \ln(1 - (k - 1)/N)$

# Happy Birthday Bob!

$\ln(P) = \ln(1 - 1/N) + \ldots + \ln(1 - (k-1)/N)$

For $0 < x < 1$: $\ln(1 - x) \leq x$

$$\ln(P) \leq -(1/N + 2/N + \ldots + (k-1)/N)$$

Gauss says:

$$1 + 2 + 3 + 4 + \ldots + (k-1) + \mathrm{k} = ½\,\mathrm{k}\,(k+1)$$

So,

$$\ln(P) \leq -½\,(k\text{-}1)\,k/N$$

$$P \leq e^{½\,(k\text{-}1)k\,/\,N}$$

Probability of match $\geq 1 - e^{-½\,(k\text{-}1)k\,/\,N}$

# Applying Birthdays

$$P(n, k) > 1 - e^{-k*(k-1)/2n}$$

For $n = 365$, $k = 20$:

$$P(365, 20) > 1 - e^{-20*(19)/2*365}$$

$$P(365, 20) > .4058$$

For $n = 2^{64}$, $k = 2^{32}$: $P\,(2^{64}, 2^{32}) > .39$

For $n = 2^{64}$, $k = 2^{33}$: $P\,(2^{64}, 2^{33}) > .86$

For $n = 2^{64}$, $k = 2^{34}$: $P\,(2^{64}, 2^{34}) > .9996$

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# Thank you for your attentions!

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