Balancing a Ball with a Robotic Arm

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***Abstract*— We propose to use a 2 DOF robot arm to balance a ball on its secondary link. We will derive the equations of motion using the Euler-Lagrange method. We propose to control the robot arm using its joint torques in order to stabilize the position and velocity of the ball resting on the secondary link. The system is inhearantly non-linear given the trigonometric functions and offset in the dynamics. We further propose to model the speed torque relationship of the electric motors.**

I. Introduction

The goal of this course project is to employ a robotic arm to balance a ball on a level surface and compensate for ball movement induced by human interaction or environmental disturbances [1]. Considering that the surface of the ball is spherical, we know that it forms a higher kinematic couple with other objects, and since it has three degrees of freedom, the process of balancing it is also quite a complex task, which we will attempt to do this throughout the course project. Though the ball balancing problem may appear to be limited in application, one must remember that Marc Raibert has made a career out of making robots dance (something my grandmother does not approve of), and he got to give the keynote speech at IROS in 2022.

II. Simulation (Modeling)

We model the system deriving our dynamical model using the Euler-Lagrange method of determining the Equations of Motion. Since we have three degrees of freedom (DOF). We end up with three coupled dynamic equations given by.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_{nc} \tag{1}$$

Where L is the Langrangian or the difference between the kinetic and potential energies for the system of rigid bodies. Since we have three DOF we have $q = [\theta_1\, \theta_2\, s]^\intercal$ and $\dot{q} = [\dot{\theta}_1\, \dot{\theta}_2\, \dot{s}]^\intercal$ and the index variable $k = 1, 2, 3$. Expanding Equation 1 we have these three coupled dynamic equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \tau_1 \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = \tau_2 \tag{3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0 \tag{4}$$

Equations 2-4 are given in the attached mathematical appendix in section IV

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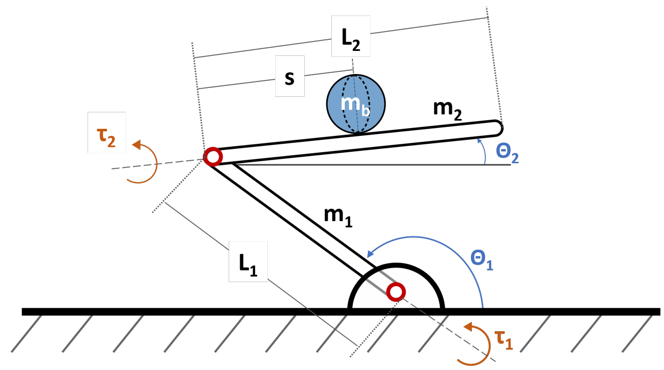

Fig. 1: The two linkage ball balancing robot arm.

These equations have been recast into the more compact form given below:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q})\,\mathbf{q} + \mathbf{U} \tag{5}$$

Where the matrices $\mathbf{M}\,\mathbf{C}$ and $\mathbf{G}$ are also provided in the appendix.

In order to validate the generated equations of motion we numerically integrate (simulate) the system using Matlab's built-in multi-step integration tool ODE45. Position states are given in Fig. 2(a) and (b) We verify that if the inputs are zero then the system's sum of kinetic and potential energy remains constant as the system evolves in time. This is required by the first law of thermodynamics and essentially functions as a check that the Lagrangian derivation of the equations of motion is correct. Moreover, we animate for various initial conditions and show that the outputs appear correct intuitively. Note that the ball always falls, and its state is unbounded. It bears mentioning that this is of course the case because the system needs to be controlled to be stable. Animation of the simulation was accomplished using the fixed time stepping method taught in class.

III. Control System Description

We propose to linearize equation 5, giving us a system that can be written in state space form, such as

$$\dot{\mathbf{X}} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$$
$$\mathbf{Y} = \mathbf{C}\,\mathbf{X} + \mathbf{D}\,\mathbf{U}$$

We propose to control (specifically regulate) as **outputs** the ball position s and velocity $\dot{s}$ of the ball in Fig. 1, thus balancing the ball. The **control inputs** to the arm will be the joint torques τ_1 and τ_2 We will test the stability of this by simulating the system response under various

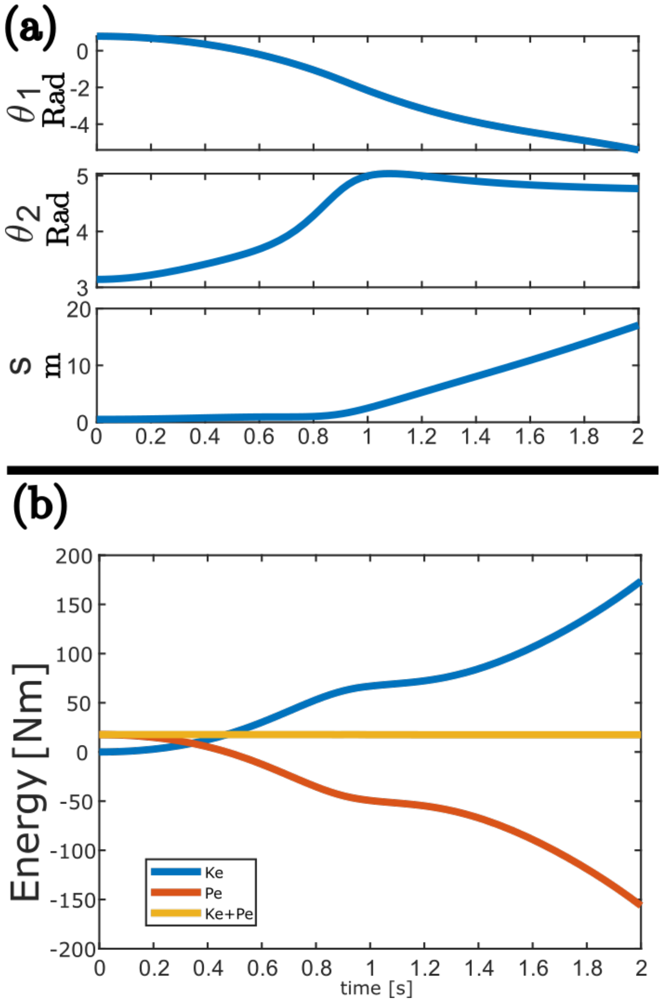

Fig. 2: (a) Position state variables as a function of time for initial conditions that are the mirror image of Fig.1. (b) Plots of kinetic and potential energy as a function of time for the same uncontrolled run. Note the sum of kinetic and potential energy remains constant since there are no inputs. Animation of these outputs are included online (*ReportFig2.mp4*) along with various other initial conditions.

initial conditions for the system, essentially discovering the system's basin of attraction.

We propose to add a **first non-linearity** namely, **motor saturation** to equation 5 by limiting the commanded torques to some maximum we call τ_{max}. Further, we know that **a second source of nonlinearity** will come in the form of **trig functions** in equation 5. In terms of **sensing**, we will assume that the full system state is available for control this will work since it is a simulation project.

References

[1] A. Kassem, H. Haddad, and C. Albitar, "Commparison between different methods of control of ball and plate system with 6dof stewart platform," *IFAC-PapersOnLine*, vol. 48, no. 11, pp. 47–52, 2015.

IV. Mathematical Appendix

$$I_1\,\ddot{\theta}_1+\frac{{L_1}^2\,m_1\,\ddot{\theta}_1}{4}+{L_1}^2\,m_2\,\ddot{\theta}_1+{L_1}^2\,\mathrm{mb}\,\ddot{\theta}_1+\frac{L_1\,g\,m_1\,\cos\left(\theta_1\right)}{2}+L_1\,g\,m_2\,\cos\left(\theta_1\right)+L_1\,g\,\mathrm{mb}\,\cos\left(\theta_1\right)-L_1\,\mathrm{mb}\,\ddot{s}\,\sin\left(\theta_1-\theta_2\right)+L_1\,\mathrm{mb}\,s\,\ddot{\theta}_2\,\cos\left(\theta_1-\theta_2\right)+2\,L_1\,\mathrm{mb}\,\dot{s}\,\dot{\theta}_2\,\cos\left(\theta_1-\theta_2\right)+\frac{L_1\,L_2\,m_2\,\dot{\theta}_2^2\,\sin\left(\theta_1-\theta_2\right)}{2}+L_1\,\mathrm{mb}\,s\,\dot{\theta}_2^2\,\sin\left(\theta_1-\theta_2\right)+\frac{L_1\,L_2\,m_2\,\ddot{\theta}_2\,\cos\left(\theta_1-\theta_2\right)}{2} \tag{6}$$

$$I_2\,\ddot{\theta}_2+\mathrm{mb}\,s^2\,\ddot{\theta}_2+\frac{{L_2}^2\,m_2\,\ddot{\theta}_2}{4}+2\,\mathrm{mb}\,s\,\dot{s}\,\dot{\theta}_2+\frac{L_2\,g\,m_2\,\cos\left(\theta_2\right)}{2}+g\,\mathrm{mb}\,s\,\cos\left(\theta_2\right)+L_1\,\mathrm{mb}\,s\,\ddot{\theta}_1\,\cos\left(\theta_1-\theta_2\right)-\frac{L_1\,L_2\,m_2\,\dot{\theta}_1^2\,\sin\left(\theta_1-\theta_2\right)}{2}-L_1\,\mathrm{mb}\,s\,\dot{\theta}_1^2\,\sin\left(\theta_1-\theta_2\right)+\frac{L_1\,L_2\,m_2\,\ddot{\theta}_1\,\cos\left(\theta_1-\theta_2\right)}{2} \tag{7}$$

$$-\mathrm{mb}\,\left(L_1\,\cos\left(\theta_1-\theta_2\right)\,\dot{\theta}_1^2+s\,\dot{\theta}_2^2-\ddot{s}-g\,\sin\left(\theta_2\right)+L_1\,\ddot{\theta}_1\,\sin\left(\theta_1-\theta_2\right)\right) \tag{8}$$

extracting the matrices...

$$M=\left(\begin{array}{ccc} I_1+\frac{{L_1}^2\,m_1}{4}+{L_1}^2\,m_2+{L_1}^2\,\mathrm{mb} & \frac{L_1\,\cos\left(\theta_1-\theta_2\right)\,\left(L_2\,m_2+2\,\mathrm{mb}\,s\right)}{2} & -L_1\,\mathrm{mb}\,\sin\left(\theta_1-\theta_2\right)\\ \frac{L_1\,\cos\left(\theta_1-\theta_2\right)\,\left(L_2\,m_2+2\,\mathrm{mb}\,s\right)}{2} & \frac{m_2\,{L_2}^2}{4}+\mathrm{mb}\,s^2+I_2 & 0\\ -L_1\,\mathrm{mb}\,\sin\left(\theta_1-\theta_2\right) & 0 & \mathrm{mb} \end{array}\right) \tag{9}$$

$$C=\left(\begin{array}{ccc} 0 & \frac{L_1\,\sin\left(\theta_1-\theta_2\right)\,\left(L_2\,m_2+2\,\mathrm{mb}\,s\right)}{2} & 2\,L_2\,m_b\,\frac{\dot{\theta}_2}{\dot{s}}\,\cos\left(\theta_1-\theta_2\right)\\ -\frac{L_1\,\sin\left(\theta_1-\theta_2\right)\,\left(L_2\,m_2+2\,\mathrm{mb}\,s\right)}{2} & 0 & 2\,m_b\,\frac{\dot{\theta}_2}{\dot{s}}\\ -L_1\,\mathrm{mb}\,\cos\left(\theta_1-\theta_2\right) & -\mathrm{mb}\,s & 0 \end{array}\right) \tag{10}$$

$$G=\left(\begin{array}{c} \frac{L_1\,g\,\cos\left(\theta_1\right)\,\left(m_1+2\,m_2+2\,\mathrm{mb}\right)}{2}\\ \frac{g\,\cos\left(\theta_2\right)\,\left(L_2\,m_2+2\,\mathrm{mb}\,s\right)}{2}\\ g\,\mathrm{mb}\,\sin\left(\theta_2\right) \end{array}\right) \tag{11}$$