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Fuzzy multiple objective programming and compromise programming with Pareto optimum

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Abstract: A fuzzy multiple objective decision making approach is proposed based on the desirable features of compromise programming and the fuzzy set theory. The proposed two-phase approach guarantees both nondominated and balanced solutions for solving both the crisp and the fuzzy multiple objective decision making problems. Furthermore, it is shown that the original compromise programming does not guarantee nondominated solutions if the distance parameter is assumed to be infinite and the resulting programming problem's solution is not unique.

Keywords: Fuzzy multiobjective programming; compromise programming; Pareto optimum.

1. Introduction

Fuzzy set theory appears to be an ideal approach for obtaining the compromise solution of compromise programming. Leung [3, 4] seems to be the first one to propose this approach. However, he only used fuzzy set in a very limited way in that the constraints in the original compromise programming are replaced by tolerance intervals which are represented by fuzzy membership functions.

Sakawa and coworkers [9, 10] also proposed a fuzzy interactive multiobjective linear programming. The approach proposed by these investigators emphasizes the interactive aspect by the decision maker for problems where the aggregate function of the fuzzy goals cannot be explicitly identified.

In this paper, it is shown that compromise programming [12–14] and fuzzy set approach for multiple objective decision making are essentially equivalent under certain conditions. Furthermore, it is shown that compromise programming does not guarantee nondominated solutions when the distance parameter is assumed the value of infinity and the solution of the resulting programming problem is not unique.

The most important aspect in the fuzzy approach is the compensatory or non-compensatory nature of the aggregate operator. Several investigators [6–8, 16, 17] have discussed this aspect. Zimmerman and coworkers [16, 17] have proposed the use of the operator γ to measure this compensation. However, due to the ease of computation, the most frequently used aggregate operator is the 'min' operator first proposed by Zimmermann. The disadvantages of this operator are that it does not guarantee a nondominated solution and it is completely non-compensatory.

In an earlier paper [5] we proposed a two-phase approach to solve the fuzzy De Novo programming problem, where the noncompensatory operator 'min' was used in the first phase, the same as Zimmermann's approach [15, 17], to obtain the optimal degree of overall satisfaction, then a fully compensatory operator 'averaging' was introduced in the second phase to find a nondominated

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solution. We showed that in this way a nondominated solution can always be obtained in phase two, regardless of the uniqueness of the solution. This two-phase approach is applied to solve the fuzzy multiple objective decision making problems with both fuzzy constraints and fuzzy parameters.

2. Compromise programming

Compromise programming [12–14] seeks the compromise solution among the various objectives of a multi-criteria decision making problem. The idea is based on the minimization of the distance between the ideal and the desired solutions. To introduce the nomenclature, only the essential ideas are summarized in the following [14].

Consider the multiple objective programming problem:

$$\max \quad Z = [Z_1, Z_2, \dots, Z_l]^{\mathrm{T}} = [c_1x, c_2x, \dots, c_lx]^{\mathrm{T}}, \tag{1a}$$

$$\min \quad W = [W_1, W_2, \dots, W_r]^{\mathrm{T}} = [c_1x, x_2x, \dots, c_rx]^{\mathrm{T}}, \tag{1b}$$

$$\text{subject to} \quad Ax * b, \tag{2a}$$

$$x \geqslant 0, \tag{2b}$$

where c_k, $k = 1, 2, \dots, l$, c_s, $s = 1, 2, \dots, l$, and x are n-dimensional vectors, b is an m-dimensional vector, A is an $m \times m$ matrix, and $*$ denotes the operators $\leqslant$, $=$, or $\geqslant$.

The ideal solution of the above problem can be obtained by solving each objective function independently subject to the constraints in (2). This ideal solution can be represented by

$$I^* = (Z_1^*, Z_2^*, \dots, Z_l^*; W_1^*, W_2^*, \dots, W_r^*). \tag{3}$$

The anti-ideal solution of the above problem can be obtained by minimizing the profit objective functions, Z, and by maximizing the cost objective functions, W. All these minimizations and maximizations are carried out with respect to each objective function independently and are subject to the constraints in (2).

This anti-ideal solution can be represented by

$$I^- = (Z_1^-, Z_2^-, \dots, Z_l^-; W_1^-, W_2^-, \dots, W_r^-). \tag{4}$$

To obtain the compromise solution, the following distance expression is minimized:

$$D_p = \sum_{k=1}^{l} \alpha_k^p \left[\frac{Z_k^* - Z_k(x)}{Z_k^* - Z_k^-}\right]^p + \sum_{s=1}^{r} \alpha_s^p \left[\frac{W_s(x) - W_s^*}{W_s^- - W_s^*}\right]^p \tag{5}$$

where p is a distance parameter with values $1 \leqslant p \leqslant \infty$ and α_k and α_s represent the relative importance or weights of each objective function with $\alpha_k > 0$, $\alpha_s > 0$ and $\sum_{k=1}^{l} \alpha_k + \sum_{s=1}^{r} \alpha_s = 1$.

From a practical standpoint, the most important values for p are 1, 2 and ∞. When $p \to \infty$, equation (5) becomes

$$D_\infty = \max_{k,s} \left[\alpha_k \frac{Z_k^* - Z_k(x)}{Z_k^* - Z_k^-},\ \alpha_s \frac{W_s(x) - W_s^*}{W_s^- - W_s^*} \right]. \tag{6}$$

In other words, for p equals infinity, the largest distance completely dominates the results. This is known as the Tchebycheff distance. Geometrically, distances D_1 (the city block distance) and D_2 (the Euclidean distance) are the longest and shortest distances, respectively, between two points in a Euclidean space.

For $p = \infty$, our problem becomes

$$\min \quad D_\infty \tag{7}$$

$$\text{subject to} \quad D_\infty \geq \alpha_k \left[\frac{Z_k^* - Z_k(x)}{Z_k^* - Z_k^-} \right], \quad k = 1, 2, \ldots, l,$$

$$D_\infty \geq \alpha_s \left[\frac{W_s(x) - W_s^*}{W_x^- - W_s^*} \right], \quad s = 1, 2, \ldots, r,$$

$$Ax * b,$$

$$x \geq 0.$$

3. The fuzzy approach

Zimmerman [15] solved the following multiple objective programming problem by using the concept of fuzzy set:

$$\max \quad Z = [c_1x, c_2x, \ldots, c_lx]^{\mathrm{T}}, \tag{8}$$

$$\min \quad Z = [c_1x, c_2x, \ldots, c_rx]^{\mathrm{T}},$$

$$\text{subject to} \quad Ax \leq b,$$

$$x \geq 0.$$

The membership functions for the objective are defined as

$$\mu_k(Z_k) = \frac{Z_k(x) - Z_k^-}{Z_k^* - Z_k^-}, \quad k = 1, 2, \ldots, l, \tag{9a}$$

$$\mu_s(W_s) = \frac{W_s^- - W_s(x)}{W_s^- - W_s^*}, \quad s = 1, 2, \ldots, r, \tag{9b}$$

where Z_k^*, W_s^* and Z_K^-, W_s^- are the ideal and anti-ideal solutions of problem (8).

Zimmermann proposed two aggregation operators, the 'min' operator and the 'product' operator, to aggregate the objective functions. The resulting problem by using the 'product' operator is a nonlinear optimization problem which is difficult to solve. Thus, the 'product' operator is seldom used. If the 'min' operator is used, we obtain the following well-known results.

$$\max \quad \lambda \tag{10}$$

$$\text{subject to} \quad \lambda \leq (Z_k(x) - Z_k^-)/(Z_k^* - Z_k^-), \quad k = 1, 2, \ldots, l,$$

$$\lambda \leq (W_s^- - W_s(x))/(W_s^- - W_s^*), \quad s = 1, 2, \ldots, r,$$

$$x \in X,$$

where λ is defined as

$$\lambda = \min_i \mu_i(x) = \min_{k,s} (\mu_k(Z), \mu_s(W_s)). \tag{11}$$

The biggest disadvantage of (10) which is obtained by using the 'min' aggregation operator is that this 'min' operator is non-compensatory in the sense of Yager [11]. In other words, the results obtained by the 'min' operator represent the worst situation and cannot be compensated by other members which may be very good. Obviously, it is much more desirable if a compensatory operator is used to obtain the compromise solution.

In order to avoid the above mentioned disadvantage, the 'arithmetical average' aggregation operator can be used instead of the 'min' operator. In other words, instead of (10), we have

$$\max \quad \bar{\lambda} = \frac{1}{l+r}\left(\sum_{k=1}^{l} \lambda_k + \sum_{s=1}^{r} \lambda_s\right) \tag{12}$$

$$\begin{aligned} \text{subject to} \quad & \lambda_k \leqslant (Z_k(x) - Z_k^-)/(Z_k^* - Z_k^-), \quad k = 1, 2, \ldots, l, \\ & \lambda_s \leqslant (W_s^- - W_s(x))/(W_s^- - W_s^*), \quad s = 1, 2, \ldots, r, \\ & x \in X. \end{aligned}$$

Notice that $\bar{\lambda}$ is defined as the arithmetic average of individual λ's which correspond to each objective function.

4. Compromise programming versus the fuzzy approach

Consider the compromise programming problem with $p = \infty$ which is represented by (7) and the fuzzy approach problem with the 'min' operator to aggregate the objective functions which is represented by (10). Let us further assume that the objective functions are equally important, or

$$\alpha_k = \alpha_s = \frac{1}{l+r}. \tag{13}$$

It is obvious that

$$D_\infty = \frac{1}{l+r}(l - \lambda) \tag{14}$$

and

$$\min D_\infty \Leftrightarrow \max \lambda, \tag{15}$$

$$D_\infty \geqslant \frac{1}{l+r}\frac{Z_k^* - Z_k(x)}{Z_k^* - Z_k^-} \Leftrightarrow \lambda \leqslant \frac{Z_k(x) - Z_k^-}{Z_k^* - Z_k^-}, \tag{16}$$

$$D_\infty \geqslant \frac{1}{l+r}\frac{W_s(x) - W_s^*}{W_s^- - W_s^*} \Leftrightarrow \lambda \leqslant \frac{W_1^- - W_s(x)}{W_s^- - W_s^*}. \tag{17}$$

Thus, the traditional approach by compromise programming with $p = \infty$ and the fuzzy approach with the 'min' operator are equivalent formulations. They both provide the same solution. The relative distance D_∞ measures the overall regret of using current solution $Z_k(x)$ to replace the ideal solution Z_k^* and the membership function λ measures the overall satisfaction in the same contest. Furthermore, D_∞ considers only the maximum regret and the 'min' operator considers only the minimum satisfaction. Conceptually, the regret and satisfaction functions are complement to each other, or D_∞ is equivalent to $1 - \lambda$. In conclusion, Zimmermann's approach using the 'min' operator is essentially equivalent to the compromise approach with $p = \infty$.

Zeleny [14] has discussed the competitive and compensatory concepts. He pointed out that the fuzzy 'min' operator is noncompensatory and thus is similar to the compromise programming approach with $p = \infty$. However, he did not discuss this similarity in the context of fuzzy optimization or fuzzy linear programming. Furthermore, he failed to point out that compromise programming with $p = \infty$ does not guarantee a nondominated solution. It is obvious from Zimmermann's approach and from its similarity to compromise programming with $p = \infty$ that (7) does not guarantee a nondominated solution.

Now compare the problem represented by (5) with $p = 1$ in compromise programming and the fuzzy approach with the arithmetic average operator which is represented by (12). Again, these two formulas are equivalent,

$$\min D_1 \Leftrightarrow \max \bar{\lambda}, \tag{18}$$

provided that (13) is true. Notice that the arithmetical average operator is completely compensatory while the 'min' operator is completely non-compensatory. Furthermore, from (5) and (6), it is obvious that the problem with $p = \infty$ is completely non-compensatory and with $p = 1$ is completely compensatory.

An example

To illustrate the approach consider the following simple multiple objective example which was considered in [5]:

$$\begin{aligned}
\max\quad & Z_1 = 2x_1 + 5x_2 + 7x_3 + x_4,\\
& Z_2 = 4x_1 + x_2 + 3x_3 + 11x_4,\\
& Z_3 = 9x_1 + 3x_2 + x_3 + 2x_4,\\
\min\quad & W_1 = 1.5x_1 + 2x_2 + 0.3x_3 + 3x_4,\\
& W_2 = 0.5x_1 + x_2 + 0.7x + 2x_4,\\
\text{subject to}\quad & 3x_1 + 4.5x_2 + 1.5x_3 + 7.5x_4 = 150,\\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

The ideal and anti-ideal solutions for this problem can be obtained easily and they are:

$$I^* = (700, 300, 450; 30, 25), \qquad I^- = (20, 33.33, 40; 75, 70)$$

using the fuzzy approach with the 'min' operator, we have

$$\begin{aligned}
\max\quad & \lambda\\
\text{subject to}\quad & \lambda \leq \tfrac{1}{680}(2x_1 + 5x_2 + 7x_3 + x_4 - 20),\\
& \lambda \leq \frac{1}{266.67}(4x_1 + x_2 + 3x_3 + 11x_4 - 33.33),\\
& \lambda \leq \tfrac{1}{410}(9x_1 + 3x_2 + x_3 + 2x_4 - 40),\\
& \lambda \leq \tfrac{1}{45}(75 - 1.5x_1 - 2x_2 - 0.3x_2 - 3x_4),\\
& \lambda \leq \tfrac{1}{45}(70 - 0.5x_1 - x_2 - 0.7x_3 - 2x_4),\\
& 3x_1 + 4.5x_2 + 1.5x_3 + 7.5x_4 = 150,\\
& x_1, x_2, x_3, x_4 \geq 0,
\end{aligned}$$

with $0 \leq \lambda \leq 1$. Notice that the last three constraints are linearly dependent.

This problem was solved depending on the subroutine used, and two solutions were obtained:

Solution 1: $\lambda = 0.5$, $x = (20.71, 3.51, 48.05, 0)$,
$Z = (395.32, 230.52, 245)$, $W = (52.5, 47.5)$,

Solution 2: $\lambda = 0.5$, $x = (21.59, 0, 46.59, 2.05)$,
$z = (371.36, 248.64, 245)$, $W = (52.5, 47.5)$.

Thus, this linear programming problem has a non-unique solution. According to the theory of linear programming, it has an infinite number of solutions. We shall show later that the above two multiple objective programming solutions are dominated solutions and are not desirable.

One possible explanation of the dominated solutions obtained above may be due to the non-compensatory nature of the approach. Since the approach represented by (12) is compensatory, this problem was also solved by the approach represented by (12) with the arithmetic average operator. This linear programming problem can be represented by

$$\max \quad \bar{\lambda} = \tfrac{1}{5}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)$$

$$\begin{aligned}
\text{subject to} \quad & \lambda_1 \leqslant \tfrac{1}{680}(2x_1 + 5x_2 + 7x_3 + x_4 - 20),\\
& \lambda_2 \leqslant \tfrac{1}{266.67}(4x_1 + x_2 + 3x_3 + 11x_4 - 33.33),\\
& \lambda_3 \leqslant \tfrac{1}{410}(9x_1 + 3x_2 + x_3 + 2x_4 - 40),\\
& \lambda_4 \leqslant \tfrac{1}{45}(75 - 1.5x_1 - 2x_2 - 0.3x_3 - 3x_4),\\
& \lambda_2 \leqslant \tfrac{1}{45}(70 - 0.5x_1 - x_2 - 0.7x_3 - 2x_4),\\
& 3x_1 + 4.5x_2 + 1.5x_3 + 7.5x_4 = 150,\\
& x_1, x_2, x_3, x_4 \geqslant 0,\\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in [0, 1].
\end{aligned}$$

The solution for this problem is

$$D_1 = (0.06, 0.02, 0.80, 0.06, 1.00)$$

or

$$\lambda = (0.94, 0.98, 0.20, 0.94, 0.00) \quad \text{and} \quad x = (3.12, 0, 93.75, 0)$$

with system performance results

$$Z = (662.5, 293.8, 121.9), \qquad W = (32.81, 70)$$

where D_1, λ, x and Z are vectors.

This solution is an efficient or nondominated solution. However, it is unbalanced with the last membership function $\lambda_5 = 0$ and λ_3 is also given a very low value. On the other hand, the high performance objectives or goals are given a very high emphasis. Obviously, this is not desirable in compromise programming.

5. Two-phase approach

To overcome the above mentioned difficulty, a two-phase approach is proposed. Zimmermann's approach is used as Phase I in which a solution $(\lambda^{(1)}, x^{(1)})$ is obtained. In the second phase, the compensatory aggregator operator arithmetic average is again used. However, a further constraint by using $\lambda^{(1)}$ which was obtained in Phase I is added. Thus, in Phase II we solve the problem

$$\max \quad \bar{\lambda} = \frac{1}{l + r} \sum_{i=1}^{1+r} \lambda_i \tag{19}$$

$$\begin{aligned}
\text{subject to} \quad & \lambda^{(1)} \leqslant \lambda_k \leqslant (Z_k(x) - Z_k^-)/(Z_k^* - Z_k^-), \quad k = 1, 2, \ldots, l,\\
& \lambda^{(1)} \leqslant \lambda_s \leqslant (Z_s^- - Z_s(x))/(Z_s^- - Z_s^*), \quad s = 1, 2, \ldots, r,\\
& x \in X.
\end{aligned}$$

This solution should yield a nondominated solution. Furthermore, because of the restriction by $\lambda^{(1)}$, the results should be much better balanced. The above numerical example is solved in Phase II by the use of (19) with $\lambda^{(1)} = 0.5$. The solutions obtained are

$$\bar{\lambda} = 0.59, \qquad x = (25, 0, 50, 0),$$

$$Z = (400, 250, 275), \qquad W = (52.5, 47.5).$$

Compared with the Phase I results where two sets of numerical solutions were obtained, it is obvious that both of the solutions obtained in Phase I are dominated solutions.

6. Fuzzy linear programming with multiple objectives

A crisp multiple objective programming can be fuzzified in at least two different ways: (1) fuzzy programming with both fuzzy objectives and fuzzy constraints but with crisp parameters, and (2) fuzzy programming with fuzzy parameters. The fuzzy programming in the first situation can be solved by the approaches discussed in the first part of the paper with minor modifications. In the remaining sections of this paper we shall discuss the second situation which can be expressed as follows:

$$\begin{aligned} &\max \quad \tilde{Z}(x) = (\tilde{c}_1 x, \tilde{c}_2 x, \ldots, \tilde{c}_l x)^{\mathrm{T}}, \\ &\min \quad \tilde{W}(x) = (\tilde{c}'_1 x, \tilde{c}'_2 x, \ldots, \tilde{c}'_r x)^{\mathrm{T}}, \\ &\text{subject to} \quad x \in X = \{x \in R^n \mid \tilde{A}x * \tilde{b}, x \geq 0\}, \end{aligned} \tag{20}$$

where $\tilde{C}_k$ $(k = 1, \ldots, l)$, $\tilde{C}'_s$ $(s = 1, \ldots, r)$ are n-dimensional vectors, $\tilde{b}$ is an m-dimensional vector, $\tilde{A}$ is an $m \times n$ matrix, and the components of $\tilde{C}_k$, $\tilde{C}'_s$, $\tilde{b}$ and $\tilde{A}$ are fuzzy numbers.

The problem represented by (20) has two types of fuzzy variables. The first type are the fuzzy goals resulting from the common multiobjective compromise programming problem. The treatment of these fuzzy goals is similar to that discussed earlier. The second type are the fuzzy coefficients which now have possibilistic distributions instead of deterministic values.

Let $(x)^\alpha_\beta$ be a solution of (20) where $\alpha \in [0, 1]$ denotes the level of possibility at which all fuzzy coefficients are feasible, and $\beta \in [0, 1]$ denotes the grade of compromise to which the solution satisfies all of the fuzzy goals while the coefficients are at a feasible level α. By means of Bellman–Zadeh's [1] rule of conjunction, the fuzzy parameters α can be expressed as

$$\alpha = \min_{k,s,i,j} \{\mu_{\tilde{c}_{kj}}, \mu_{\tilde{c}'_{sj}}, \mu_{\tilde{a}_{ij}}, \mu_{\tilde{b}_i} \mid k = 1, \ldots, l, s = 1, \ldots, r, i = 1, \ldots, m, j = 1, \ldots, n\} \tag{21}$$

which means that the feasibility of the system is equal to the possibility of the most impossible component in the system. Equation (21) means that the higher the possibility of the coefficient is, the stronger the limitations on the coefficient are. Obviously, the optimal solution for a given value of α is reached when

$$\mu_{\tilde{c}_{ij}} = \mu_{\tilde{c}'_{sj}} = \mu_{\tilde{a}_{ij}} = \mu_{\tilde{b}_i} = \alpha \tag{22}$$

for all subscripts.

Let $(\tilde{P})_\alpha$ be the α-cut of a fuzzy number $\tilde{P}$, defined as

$$(\tilde{P})_\alpha = \{p \in S(\tilde{P}) \mid \mu_{\tilde{P}}(p) \geq \alpha\} \tag{23}$$

where $S(\tilde{P})$ is the support of $\tilde{P}$. Let $(\tilde{P})^{\mathrm{L}}_\alpha$ and $(\tilde{P})^{\mathrm{U}}_\alpha$ be the lower bound and the upper bound of the α-cut of $\tilde{P}$ respectively,

$$(\tilde{P})^{\mathrm{L}}_\alpha \leq (\tilde{P})_\alpha \leq (\tilde{P})^{\mathrm{U}}_\alpha. \tag{24}$$

Then, for a given value of α, objectives $\tilde{Z}_k$ to be maximized and $\tilde{W}_s$ to be minimized can be replaced by the upper bound and the lower bound of their respective α-cuts, that is

$$(\tilde{Z}_k)^{\mathrm{U}}_\alpha = \sum_{j=1}^{n} (\tilde{c}_{kj})^{\mathrm{U}}_\alpha \cdot x_j, \quad k = 1, \ldots, l, \tag{25}$$

$$(\tilde{W}_s)^{\mathrm{L}}_\alpha = \sum_{j=1}^{n} (\tilde{c}'_{sj})^{\mathrm{L}}_\alpha \cdot x_j, \quad s = 1, \ldots, r. \tag{26}$$

Furthermore, when $*$ denotes the operators $\leqslant$ for constraints $i = 1, \ldots, m_1$ and $\geqslant$ for constraints $i = m_1 + 1, \ldots, m_2$, these constraints can be similarly replaced by the following constraints:

$$\sum_{j=1}^{n} (\tilde{a}_{ij})_\alpha^{\mathrm{L}} \cdot x_j \leqslant (\tilde{b}_i)_\alpha^{\mathrm{U}}, \quad i = 1, \ldots, m_1, \tag{27}$$

$$\sum_{j=1}^{n} (\tilde{a}_{ij})_\alpha^{\mathrm{U}} \cdot x_j \geqslant (\tilde{b}_i)_\alpha^{\mathrm{L}}, \quad i = m_1 + 1, \ldots, m_2. \tag{28}$$

However, when $*$ denotes the operator $=$ for constraints $i = m_2 + 1, \ldots, m$,

$$\sum_{j=1}^{n} \tilde{a}_{ij} \cdot x_j = \tilde{b}_i, \quad i = m_2 + 1, \ldots, m, \tag{29}$$

we do not know whether to use $(\tilde{a}_{ij})_\alpha^{\mathrm{L}}$ or $(\tilde{a}_{ij})_\alpha^{\mathrm{U}}$ and $(\tilde{b}_i)_\alpha^{\mathrm{L}}$ or $(\tilde{b}_i)_\alpha^{\mathrm{U}}$ because in this case as α is decreased from 1 to 0, the feasible constraint space does not always increase. To deal with equality constraints we give the following proposition.

Proposition. *A fuzzy equality constraint,*

$$\sum_j \tilde{a}_j \cdot x_j = \tilde{b}, \tag{30}$$

is equivalent to two inequality constraints

$$\sum_j (\tilde{a}_j)_\alpha^{\mathrm{L}} \cdot x_j \leqslant (\tilde{b})_\alpha^{\mathrm{U}}, \qquad \sum_j (\tilde{a}_j)_\alpha^{\mathrm{U}} \cdot x_j \geq (\tilde{b})_\alpha^{\mathrm{L}}. \tag{31}$$

Proof. (1) For any $\alpha \in [0, 1]$, if some $\hat{X} \in R^n$ satisfies (30), i.e.,

$$(\tilde{b})_\alpha^{\mathrm{L}} \leqslant \sum_j (\tilde{a}_j)_\alpha \cdot \hat{x}_j \leqslant (\tilde{b})_\alpha^{\mathrm{U}},$$

then $\hat{x}$ satisfies (31) also, because $(\tilde{a}_j)_\alpha^{\mathrm{L}} \leqslant (\tilde{a}_j)_\alpha \leqslant (\tilde{a}_j)_\alpha^{\mathrm{U}}$.

(2) If some $x' \in R^n$ satisfies (31) then if

$$(\tilde{b})_\alpha^{\mathrm{L}} \leqslant \sum_j (\tilde{a}_j)_\alpha^{\mathrm{L}} \cdot x_j' \leqslant (\tilde{b})_\alpha^{\mathrm{U}} \quad \text{or} \quad (\tilde{b})_\alpha^{\mathrm{L}} \leqslant \sum_j (\tilde{a}_j)_\alpha^{\mathrm{U}} \cdot x_j' \leqslant (\tilde{b})_\alpha^{\mathrm{U}},$$

x' already satisfies (30). Otherwise,

$$\sum_j (\tilde{a}_j)_\alpha^{\mathrm{L}} \cdot x_j' \leqslant (\tilde{b})_\alpha^{\mathrm{L}} \quad \text{and} \quad \sum (\tilde{a}_j)_\alpha^{\mathrm{U}} \cdot x_j' \geqslant (\tilde{b})_\alpha^{\mathrm{U}},$$

and obviously, there must exist some $(\tilde{a}_j)_\alpha$, say $(\tilde{a}_{j'})_\alpha$, located between $(\tilde{a}_j)_\alpha^{\mathrm{L}}$ and $(\tilde{a}_j)_\alpha^{\mathrm{U}}$ to make $(\tilde{b})_\alpha^{\mathrm{L}} \leqslant \sum_j (\tilde{a}_j')_\alpha \cdot x_j' \leqslant (\tilde{b})_\alpha^{\mathrm{U}}$. In other words, x' satisfies (30) also.

According to the above proposition, (29) can be replaced by two sets of inequality constraints, that is

$$\sum_{j=1}^{n} (\tilde{a}_{ij})_\alpha^{\mathrm{L}} \cdot x_j \leqslant (\tilde{b}_i)_\alpha^{\mathrm{U}}, \tag{32}$$

and

$$\sum_{j=1}^{n} (\tilde{a}_{ij})_\alpha^{\mathrm{U}} \cdot x_j \geqslant (\tilde{b}_i)_\alpha^{\mathrm{L}}, \tag{33}$$

for $i = m_2 + 1, \ldots, m$.

Thus, for a given value of α, the problem represented by (20) can be transformed to the following problem:

$$\max \quad (\tilde{Z}_k)^{\mathrm{U}}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}_{kj})^{\mathrm{U}}_{\alpha} \cdot x_j, \quad k = 1, \ldots, l, \tag{34a}$$

$$\min \quad (\tilde{W}_s)^{\mathrm{L}}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}'_{sj})^{\mathrm{L}}_{\alpha} \cdot x_j, \quad s = 1, \ldots, r, \tag{34b}$$

$$\text{subject to} \quad \sum_{j=1}^{n} (\tilde{a}_{ij})^{\mathrm{L}}_{\alpha} \cdot x_j \leqslant (\tilde{b}_i)^{\mathrm{U}}_{\alpha}, \quad i = 1, \ldots, m_1, m_2 + 1, \ldots, m, \tag{35a}$$

$$\sum_{j=1}^{n} (\tilde{a}_{ij})^{\mathrm{U}}_{\alpha} \cdot x_j \geqslant (\tilde{b} - i)^{\mathrm{L}}_{\alpha}, \quad i = m_1 + , \ldots, m_2, m_2 + 1, \ldots, m, \tag{35b}$$

$$x_j \geqslant 0, \quad j = 1, \ldots, n. \tag{35c}$$

To simplify the presentation, we shall use X_α to denote the constraints (35).

It is seem that for a given value of α the problem becomes a deterministic linear problem with multiple objectives, which can be solved by using the techniques presented in the previous sections. The resulting equivalent crisp problem is formulated as

$$\max \quad \beta \tag{36}$$

$$\text{subject to} \quad \beta \leqslant \mu^{\alpha}_{k}(Z_k) = \left[\sum_{j=1}^{n} (\tilde{c}_{kj})^{\mathrm{U}}_{\alpha} \cdot x_j - (\tilde{Z}_k)^{-}_{\alpha} \right] \Big/ [(\tilde{Z}_k)^{*}_{\alpha} - (\tilde{Z}_k)^{-}_{\alpha}], \quad k = 1, \ldots, l, \tag{37a}$$

$$\beta \leqslant \mu^{\alpha}_{s}(w_s) = \left[(\tilde{W}_s)^{-}_{\alpha} - \sum_{j=1}^{n} (c'_{sj})^{\mathrm{L}}_{\alpha} \cdot x_j \right] \Big/ [(\tilde{W}_s)^{-}_{\alpha} - (\tilde{W}_s)^{*}_{\alpha}], \quad s = 1, \ldots, r, \tag{37b}$$

$$\beta \in [0, 1], \tag{37c}$$

$$x \in X_\alpha, \tag{37d}$$

$$x_j \geqslant 0, \tag{37e}$$

where $(\tilde{Z}_k)^{*}_{\alpha}$, $(\tilde{W}_s)^{*}_{\alpha}$, and $(\tilde{Z}_k)^{-}_{\alpha}$, $(\tilde{W}_s)^{-}_{\alpha}$ are the ideal and anti-ideal solutions, respectively, which can be obtained by solving each of the following problems independently:

$$\max_{x \in X_\alpha} (\tilde{Z}_k)^{*}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}_{kj})^{\mathrm{U}}_{\alpha} \cdot x_j, \tag{38}$$

$$\min_{x \in X_\alpha} (\tilde{W}_s)^{*}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}'_{ij})^{\mathrm{L}}_{\alpha} \cdot x_j, \tag{39}$$

$$\max_{x \in X_\alpha} (\tilde{W}_s)^{*}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}'_{ij})^{\mathrm{U}}_{\alpha} \cdot x_j, \tag{40}$$

$$\min_{x \in X_\alpha} (\tilde{Z}_k)^{*}_{\alpha} = \sum_{j=1}^{n} (\tilde{c}'_{ij})^{\mathrm{L}}_{\alpha} \cdot x_j, \tag{41}$$

for all the values of k and s.

Assume that all of the fuzzy coefficients are trapezoidal fuzzy numbers, $\tilde{P}$, which can be specified by the foursome $(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})$ with membership function

$$\mu_{\tilde{P}}(p) = \begin{cases} 0, & p \leqslant p^{(1)}, \\ (p - p^{(1)})/(p^{(2)} - p^{(1)}), & p^{(1)} \leqslant p \leqslant p^{(2)}, \\ 1, & p^{(2)} \leqslant p \leqslant p^{(3)}, \\ (p^{(4)} - p)/(p^{(4)} - p^{(3)}), & p^{(3)} \leqslant p \leqslant p^{(4)}, \\ 0, & p \geqslant p^{(4)}. \end{cases} \tag{42}$$

Then the α-cut of $\tilde{P}$ can be expressed by the following interval

$$(\tilde{P})_\alpha = [(\tilde{P})^{\mathrm{L}}_\alpha, (\tilde{P})^{\mathrm{U}}_\alpha] = [p^{(1)} + (p^{(2)} - p^{(1)})\alpha, p^{(4)} - (p^{(4)} - p^{(3)})\alpha]. \tag{43}$$

Note that if $p^{(2)} = p^{(3)}$, then $\tilde{P}$ is reduced to the triangular fuzzy number, specified by $(p^{(1)}, p^{(2),(3)}, p^{(4)})$; if $p^{(1)} = p^{(2)} = p^{(3)} = p^{(4)}$, then $\tilde{P}$ is reduced to a real number.

Using the interval expression, the problem represented by (36) and (37) can be rewritten as

$$\max \quad \beta \tag{44}$$

$$\text{subject to} \quad \beta \leqslant \left\{\sum_{j=1}^{n} [c^{(4)}_{kj} - (c^{(4)}_{kj} - c^{(3)}_{kj})\alpha] \cdot x_j - (\tilde{Z}_k)^-_\alpha\right\} \Big/ [(\tilde{Z}_k)^*_\alpha - (\tilde{Z}_k)^-_\alpha], \quad k = 1, \ldots, l, \tag{45a}$$

$$\beta \leqslant \left\{(\tilde{W}_s)^-_\alpha - \sum_{j=1}^{n} [c^{(1)}_{sj} + (c^{(2)}_{s'j} - c^{(1)}_{s'j})\alpha] \cdot x_j\right\} \Big/ [(\tilde{W}_s)^-_\alpha - (\tilde{W}_s)^*_\alpha], \quad s = 1, \ldots, r, \tag{45b}$$

$$\beta \in [0, 1], \tag{45c}$$

$$[a^{(1)}_{ij} + (a^{(2)}_{ij} - a^{(1)}_{ij})\alpha] \cdot x_j \leqslant b^{(4)}_i - (b^{(4)}_i - b^{(3)}_i)\alpha, \quad i = 1, \ldots, m_1, m_2 + 1, \ldots, m, \tag{45d}$$

$$[a^{(4)}_{ij} - (a^{(4)}_{ij} - a^{(3)}_{ij})\alpha] \cdot x_j \geqslant b^{(1)}_i + (b^{(2)}_i - b^{(1)}_i)\alpha, \quad i = m_1 + 1, \ldots, m_2, m_2 + 1, \ldots, m, \tag{45e}$$

$$x_j \geqslant 0, \quad j = 1, \ldots, n. \tag{45f}$$

For simplicity, the last three sets of constraints will be denoted by X'_α.

Now, let λ be the degree of overall satisfaction to the solution $(x)^\alpha_\beta$ under consideration of both fuzzy goals and fuzzy coefficients. By means of Bellman–Zadeh's rule again, λ is conjuncted as

$$\lambda = \min\{\alpha, \beta\}. \tag{46}$$

In addition to the original unknown x's variables, the problem represented by (44)–(46) has two different types of additional unknown parameters: the α parameter which denotes the level of possibility of the fuzzy coefficients and the β parameter which denotes the compromise between the different objectives. Various numerical solutions schemes can be devised to solve this problem. The following two different approaches are proposed in this work.

The first approach is to solve the problem parametrically by considering α as a variable to be searched. The algorithm is summarized as follows:

Step 1. Define ϵ = step length, τ = accuracy of tolerance, k = iteration counter = 0.

Step 2. Set $\alpha_k = 1 - k\epsilon$, then solve the problem represented by (44) and (45) to obtain β_k and x_k.

Step 3. If $|\beta_k - \alpha_k| \leqslant \tau$ then let $\lambda = \min\{\alpha_k, \beta_k\}$ and $x = x_k$ and go to step 4. If $\alpha_k - \beta_k > \epsilon$, then let $k = k + 1$ and go to step 2. If step size is too large, let $\epsilon = \frac{1}{2}\epsilon$, $k = k$ and go to step 2.

Step 4. Output λ, α_k, β_k and x.

For a given α, this is a linear problem which can be solved easily. After obtaining the values of α and β by the above procedure, we can go to Phase II to solve the following problem to obtain an undominated solution, in case the solution is not unique.

$$\max \quad \bar{\beta} = \frac{1}{l + r}\left(\sum_{k=1}^{l} \beta_k + \sum_{s=1}^{r} \beta_s\right) \tag{47}$$

$$\text{subject to} \quad \beta \leqslant \beta_k = \left\{\sum_{j=1}^{n} [c^{(4)}_{kj} - (x^{(4)}_{kj} - x^{(3)}_{kj})\alpha] - (\tilde{Z}_k)^-_\alpha\right\} \Big/ [(\tilde{Z}_k)^*_\alpha - (\tilde{Z}_k)^-_\alpha,$$

$$\beta \leqslant \beta_s = \left\{(\tilde{W}_s)^-_\alpha - \sum_{j=1}^{n} [c^{(1)}_{s'j} + (c^{(2)}_{s'j} - c^{(1)}_{s'j})\alpha]\right\} \Big/ [(\tilde{w}_s)^-_\alpha - (\tilde{W}_s)^*_\alpha],$$

$$\bar{\beta}, \beta_k, \beta_s \in [0, 1],$$

$$x \in X'_\alpha,$$

where α and β are constants and were obtained in Phase I by the above procedure.

The second approach is to solve α and β simultaneously. The problem to be solved in Phase I is:

$$\max \quad \lambda \tag{48}$$

$$\text{subject to} \quad \lambda \leq \alpha,$$

$$\lambda \leq \beta,$$

$$\beta \leq \left[\sum_{j=1}^{n} (c_{kj}^{(4)} - (c_{kj}^{(4)} - x_{kj}^{(3)})\alpha) \cdot x_j - Z_j^-\right] \Big/ (Z_k^* - Z_k^-), \quad k = 1, \ldots, l,$$

$$\beta \leq \left[W_s^- - \sum_{j=1}^{n} (c_{sj}^{\prime(1)} + (c_{sj}^{\prime(2)} - c_{sj}^{\prime(1)})\alpha) \cdot x_j\right] \Big/ (W_s^- - W_s^*), \quad s = 1, \ldots, r,$$

$$\lambda,\ \alpha,\ \beta \in [0, 1],$$

$$x \in X'_\alpha.$$

This is a nonlinear program. A variety of nonlinear programming algorithms can be used to solve this problem. Some of the available computer codes are SUMT and GREG.

The problem to be solved in the second phase is essentially the same problem as (47) except for replacing $(\tilde{Z}_k)_\alpha^*$, $(\tilde{W}_s)_\alpha^*$ by Z_k^*, W_s^*, and replacing $(\tilde{Z}_k)_\alpha^-$, $(\tilde{W}_s)_\alpha^-$ by Z_k^-, W_s^-, for $k = 1, \ldots, l$ and $s = 1, \ldots, r$.

An example

Consider the following possibilistic linear problem with multiple objective functions:

$$\max \quad \tilde{Z}(x) = 10x_1 + \tilde{6}x_2, \tag{49}$$

$$\min \quad \tilde{W}(x) = \tilde{1}x_1 + 1.5x_2,$$

$$\text{subject to} \quad \tilde{2}x_1 + 2x_2 \leq 1\tilde{4}0,$$

$$x_2 \geq \tilde{8},$$

$$x_1 \geq 0,$$

where all the fuzzy numbers are triangular fuzzy numbers and are given as follows:

$$\tilde{6} = (4, 6, 8), \quad \tilde{1} = (0, 1, 2), \quad \tilde{2} = (1, 2, 3), 1\tilde{4}0 = (100, 140, 180), \quad \tilde{8} = (3, 8, 10),$$

where the maximum possibility of one is achieved at the middle number.

Replacing the fuzzy coefficients by their α-cuts, (49) is transformed to the following problem:

$$\max \quad (\tilde{Z})_\alpha = 10x_1 + (8 - 2\alpha)x_2, \tag{50}$$

$$(\tilde{W})_\alpha = (\alpha)x_1 + 1.5x_2,$$

$$\text{subject to} \quad (1 + \alpha)x_1 + 2x_2 \leq 180 - 40\alpha,$$

$$x_2 \geq 3 + 5\alpha,$$

$$\alpha \in [0, 1],$$

$$x_1, x_2 \geq 0.$$

Problem (50) is first solved parametrically. Table 1 lists some of the results. For this particular problem, when the value of α is decreased gradually, the value of β increases steadily. The best solution is obtained at $\alpha^* = \beta^* = 0.67$, $x^* = (55.7, 6.35)$ with the system performance $Z(x^*) = 599.8$, $W(x^*) = 46.7$.

Assume that the decision maker has given the goals and the tolerances as $Z^* = 883.6$, $Z^- = 33.9$ and $W^* = 9.5$, $W^- = 121.4$ a priori. Then we may aggregate both fuzzy goals and fuzzy coefficients into the

Table 1. The fuzzy results

α	Z^*	Z^-	W^*	W^-	β	x_1	x_2	$Z(x)$	$W(x)$
1.0	668	48	12	105	0.60	37.2	8.0	420	49.2
0.9	725.4	38.2	11.3	108	0.62	41.4	7.5	460.3	48.5
0.8	789.2	39.2	10.5	111	0.63	46.6	7.0	510.9	47.8
0.7	860.5	35.1	9.75	116	0.65	53.0	6.5	572.5	46.8
*0.67	883.6	33.9	9.53	121	0.67	55.7	6.4	599.8	46.7*
0.6	940.8	31.2	9.0	135	0.70	62.8	6.0	668.8	46.6
0.5	1032	27.5	8.25	157	0.75	74.3	5.5	781.4	45.4
0.4	1136	24	7.5	184	0.80	87.8	5.0	914.1	42.6
0.3	1256	20.7	6.75	215	0.85	104	4.5	1071	37.9
0.2	1397	17.6	6.0	252	0.90	123	4.0	1259	30.6
0.1	1564	14.4	5.25	297	0.95	146	3.5	1486	19.8
0.0	1764	12	4.5	352	1.00	174	3.0	1764	4.5

following nonlinear problem:

$$\max \quad \lambda \tag{51}$$

$$\begin{aligned} \text{subject to} \quad & \lambda \leqslant \alpha,\ \beta, \\ & \beta \leqslant \frac{1}{849.7}[(10x_1 + (8-2\alpha)x_2) - 33.9], \\ & \beta \leqslant \frac{1}{111.9}[121.4 - (\alpha x_1 + 1.5x_2)], \\ & \lambda,\ \alpha,\ \beta,\ \in [0,\ 1], \\ & x \in X'_\alpha. \end{aligned}$$

This problem is solved by using a nonlinear SUMT code. The results obtained are $\lambda^* = \alpha^* = \beta^* = 0.67$ and $x^* = (55.7, 6.35)$ which is the same as the best solution shown in Table 1. This is obvious due to the fact that the results obtained in the parametric procedure are used to obtain the current results.

Since (51) has a unique solution, the results obtained in Phase I are nondominated. Thus, Phase II should give the same nondominated results by solving the following problem:

$$\max \quad \bar{\beta} = \tfrac{1}{2}(\beta_1 + \beta_2) \tag{52}$$

$$\begin{aligned} \text{subject to} \quad & 0.67 \leqslant \beta_1 \leqslant \frac{1}{849.7}[(10x_1 + 6.66x_2) - 33.9], \\ & 0.67 \leqslant \beta_2 \leqslant \frac{1}{111.9}[121.4 - (0.67x_1 + 1.5x_2)], \\ & 1.67x_1 + 2x_2 \leqslant 153.2, \\ & x_2 \geqslant 6.35, \\ & \beta_1,\ \beta_2,\ \bar{\beta} \in [0,1], \\ & x_1 \geqslant 0. \end{aligned}$$

7. Discussion

The solution algorithms can also be applied to other types of fuzzy parameters as long as they are convex and normalized. For example, Carlsson and Korhonen [2] assumed that the decision maker can specify two types of intervals, $[p^{\mathrm{L}}, p^{\mathrm{U}})$ or $(p^{\mathrm{L}}, p^{\mathrm{U}}]$, for any fuzzy coefficient $\tilde{P}$, where the square brackets represent risk-free values and the parentheses represent impossible values. Let $\tilde{T}$ be a fuzzy coefficient with interval $[t^{L}, t^{U})$ and $\tilde{Q}$ with $(q^{L}, q^{U}]$. Then $\tilde{T}$ and $\tilde{Q}$ are characterized, respectively, by

$$\mu_{\tilde{T}}(t)=\begin{cases}1, & t \leqslant t^{L},\\ (t^{U}-t)/(t^{U}-t^{L}), & t^{L} \leqslant r \leqslant t^{U},\\ 0, & t \geqslant t^{U},\end{cases} \tag{53}$$

and

$$\mu_{\tilde{T}}(q)=\begin{cases}0, & q \leqslant q^{L},\\ (q-q^{L})/(q^{U}-q^{L}), & q^{L} \leqslant q \leqslant q^{U},\\ 1, & q \geqslant q^{U}.\end{cases} \tag{54}$$

For a given value of α, the α-cuts of $\tilde{T}$ and $\tilde{Q}$ can be expressed as

$$(\tilde{T})_{\alpha}=[t^{L}, t^{U}-(t^{U}-t^{L})\alpha], \tag{55}$$

$$(\tilde{Q})_{\alpha}=[q^{L}+(q^{U}-q^{L})\alpha, q^{U}]. \tag{56}$$

By comparing equations (53)–(56) with equation (42) and (43) we see that the two types of fuzzy parameters $\tilde{T}$ and $\tilde{Q}$ are respectively equal to the right side and the left side of the trapezoidal fuzzy parameter $\tilde{P}$. Furthermore, from earlier discussions we know that only one boundary point, either the left or the right of a fuzzy coefficient, are used in the calculations. Therefore, the formulation of the resulting problem where the fuzzy coefficients are represented as intervals defined by (53) or (54) are essentially the same as that used in formulating the problem represented by trapezoidal fuzzy numbers.

Although only linear problems are discussed, the procedures can be extended easily to nonlinear problems. Obviously, the resulting nonlinear programming problems would be much more difficult to solve.

Another advantage of the two-phase approach is that other types of membership functions, instead of linear, can also be used. Some of the nonlinear membership functions are exponential functions, hyperbolic functions and piecewise-linear functions. It should be emphasized that some nonlinear membership functions are much more desirable for certain applications than linear ones.

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