MATH 2263: Project #1: Comparing Surfaces

Lydia Alem

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1 Part 1: Traces

The cone and paraboloid are two mathematical surfaces with distinct shapes that, at first glance, may appear to be similar. A cursory examination of both Figure 1 and Figure 2 reveals that both shapes exhibit an almost circular-like distribution. However, upon delving deeper into this project, I will demonstrate how significantly different these shapes are.

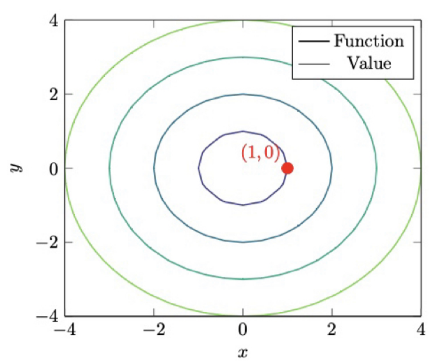


Figure 1: Contour plot of a $z = \sqrt{x^2 + y^2}$.

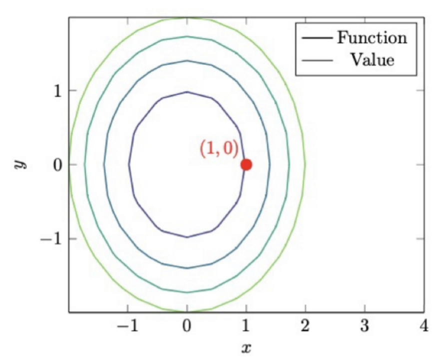


Figure 2: Contour plot of a $z = x^2 + y^2$.

Upon observing the contour maps of the cone and paraboloid, a notable disparity in their growth patterns is immediately discernible. Specifically, the paraboloid exhibits a rate of growth that could be described as approaching an exponential function, evidenced by the discrepancy in growth between $z = 4$ and $z = 3$ when compared to the difference in growth between $z = 2$ and $z = 3$.

In contrast, the growth of the cone appears to be more uniform across its dimensions. This observation could be attributed to the paraboloid's underlying quadratic equation, which imparts a degree of curvature and accentuates the rate of growth in the z direction. Such distinctions are critical to understanding the properties and applications of these shapes in various fields, including physics, engineering, and mathematics.

2 Part 2: Directional Derivatives

I began by calculating the directional derivatives and magnitude of both shapes:

2.1 Step 1

Find the Gradient Vector and Magnitude for $z = x^2 + y^2$:

$$\boxed{\textbf{Recall the Gradient Vector Formula: } \nabla f = \left\langle f_x, f_y, f_z \right\rangle = \tfrac{\partial f}{\partial x}\mathbf{i} + \tfrac{\partial f}{\partial y}\mathbf{j} + \tfrac{\partial f}{\partial z}\mathbf{k}}$$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = -1 \longrightarrow \nabla f = \left\langle 2x, 2y, -1 \right\rangle$$

Plugging in the point $(1, 0, 1)$, we have:

$$\nabla f(1, 0, 1) = \left\langle 2, 0, -1 \right\rangle$$

$$\textbf{Gradient Vector: } \left\langle 2, 0, -1 \right\rangle$$

$$\left\| \left\langle 2, 0, -1 \right\rangle \right\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$$

$$\textbf{Magnitude: } = \sqrt{5}$$

2.2 Step 2

Find the Gradient Vector and Magnitude for $z = \sqrt{x^2 + y^2}$:

$$\boxed{\textbf{Recall the Gradient Vector Formula: } \nabla f = \left\langle f_x, f_y, f_z \right\rangle = \tfrac{\partial f}{\partial x}\mathbf{i} + \tfrac{\partial f}{\partial y}\mathbf{j} + \tfrac{\partial f}{\partial z}\mathbf{k}}$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial z} = -1 \longrightarrow \nabla f = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \rangle$$

Plugging in the point $(1, 0, 1)$, we have:

$$\nabla f(1,0,1) = \langle 1, 0, -1 \rangle$$

$$\textbf{Gradient Vector: } \langle 1, 0, -1 \rangle$$

$$\|\langle 1, 0, -1 \rangle\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\textbf{Magnitude: } = \sqrt{2}$$

2.3 Final Answer

Upon analyzing the rate of change of the surface areas of a paraboloid and a cone, it is evident that the paraboloid has the highest rate of increase, with a value of $\sqrt{5}$, while the cone has a rate of increase of $\sqrt{2}$. The computation of the greatest increase highlights the substantial difference between the two shapes, revealing that the paraboloid increases at a much faster rate than the cone.

Furthermore, observing the contour maps for both shapes provides a graphical representation of their rate of change. From the contour map, it is clear that the paraboloid has a much larger increase as compared to the cone, while the latter seems to increase at a relatively constant rate.

3 Part 3: Tangent Places

Suppose that f has a continuous partial derivative. An equation of the tangent plane to the surface. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by:

$$z - z_0 = \frac{\partial f}{\partial x}\bigg|(x_0, y_0)(x - x_0)\bigg| + \frac{\partial f}{\partial y}\bigg|(x_0, y_0)(y - y_0)\bigg|$$

3.1 Step 1

Find the Tangent Planes on $P(0, 0, 0)$ for $z = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y,$$

Plugging in $x = 0$ and $y = 0$, we get:

$$z - 0 = 2x(0,0)(x - 0) + 2y(0,0)(y - 0)$$

so the equation of the tangent plane is:

$$\mathbf{z = 0}$$

$\square$

3.2 Step 2

Find the Tangent Planes on $P(0,0,0)$ for $Z = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \ \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}},$$

Plugging in $x = 0$ and $y = 0$, we get:

$$z - 0 = \frac{x}{\sqrt{x^2+y^2}}(0,0)(x-0) + \frac{y}{\sqrt{x^2+y^2}}(0,0)(y-0)$$

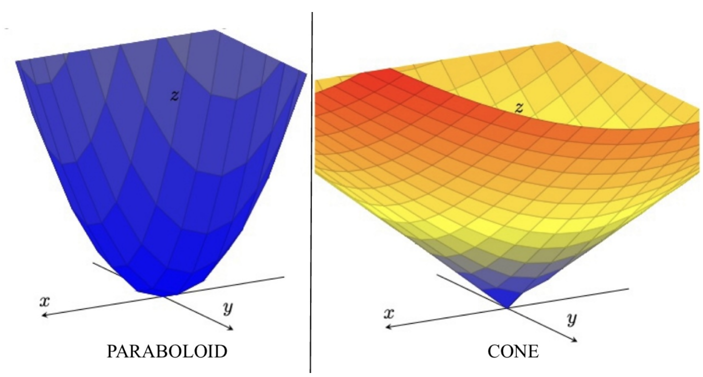
so the equation of the tangent plane is:

$$\mathbf{z = 0}.$$

$\square$

3.3 Step 3

Plotting the surface and the tangent



3D image of both $Z = \sqrt{x^2+y^2}$ and $Z = x^2 + y^2$

To provide a comprehensive understanding of the cone and paraboloid shapes, I decided to utilize the powerful visualization tool, GeoGebra 3D. This tool allows for an interactive 3D representation of the shapes, which provides a better insight into their structure and properties.

3.4 Final Answer

Therefore, in our effort to derive the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(0, 0, 0)$, we have obtained $z = 0$ as the equation of the tangent plane. This result indicates that the tangent line at the origin lies on the x-y plane. This finding can also be clearly observed in the accompanying graphs (figure 1 & 2), where the point $P(0, 0, 0)$ is seen to be untouched by the surface.