Minimal time of null controllability of two parabolic equations

Lydia Ouaili*

Abstract

We consider a one-dimensional 2×2 parabolic equations, simultaneously controllable by a localized function in their source term. We also consider a simultaneous boundary control. In each case, we prove the existence of minimal time $T_0(q)$ of null controllability, that is to say, the corresponding problem is null controllable at any time $T > T_0(q)$ and not null controllable for $T < T_0(q)$. We also prove that one can expect any minimal time associated to the boundary control problem.

1 Introduction

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, 1)$, and consider the parabolic systems:

$$\begin{cases} \partial_t y - \Delta y + q(\cdot)Hy = 1_\omega \widetilde{B}\, u & \text{in } Q_T := (0,T)\times(0,1), \\ y(\cdot,0) = 0, \;\; y(\cdot,1) = 0 & \text{on } \;(0,T), \\ y(0,\cdot) = y_0 & \text{in } \;(0,1), \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t y - \Delta y + q(\cdot)Hy = 0 & \text{in } Q_T, \\ y(\cdot,0) = \widetilde{B}\, v, \quad y(\cdot,1) = 0 & \text{on } \;(0,T), \\ y(0,\cdot) = y_0 & \text{in } \;(0,1), \end{cases} \tag{2}$$

In systems (1) and (2), $q \in L^2(0,1)$ is a given function, H is a real matrix, $\widetilde{B} \in \mathbb{R}^2$ is a given vector, 1_ω denotes the characteristic function of ω, $y_0 \in L^2(0,1;\mathbb{R}^2)$ is the initial datum for system (1) and $y_0 \in H^{-1}(0,1;\mathbb{R}^2)$ for system (2), $u \in L^2(Q_T)$ and $v \in L^2(0,T)$ are the control forces.

It is well known that the system (1) (resp., the system (2)) is well-posed for any $T > 0$ (resp., admits a unique solution by transposition) (see [18]), i.e., for any $(y_0, u) \in L^2(0,1;\mathbb{R}^2)\times L^2(Q_T)$ (resp., $(y_0, v) \in H^{-1}(0,1;\mathbb{R}^2)\times L^2(0,T)$) there exists a unique solution

$$\begin{aligned} &y \in L^2(0,T;H_0^1(0,1;\mathbb{R}^2)) \cap C^0([0,T];L^2(0,1;\mathbb{R}^2)) \\ &(\text{resp.}, y \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,1;\mathbb{R}^2))). \end{aligned}$$

Note that the rank of the control operator $\widetilde{B}$ is one and we are interested in studying the approximate controllability and null controllability for systems (1) and (2).

*Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France. E-mail: lydia.ouaili@etu.univ-amu.fr

1. We say that system (1) (resp., system (2)) is approximately controllable in $L^2(0,1;\mathbb{R}^2)$ (resp., $H^{-1}(0,1;\mathbb{R}^2)$) at time T if for every $y_0, y_d \in L^2(0,1;\mathbb{R}^2)$ (resp., $y_0, y_d \in H^{-1}(0,1;\mathbb{R}^2)$) and $\varepsilon > 0$, there exists $u \in L^2(Q_T)$ (resp., $v \in L^2(0,T)$) such that the solution y to system (1) (resp., system (2)) satisfies
$$\|y(T,\cdot) - y_d\|_{L^2(0,1;\mathbb{R}^2)} \leq \varepsilon \text{ (resp., } \|y(T,\cdot) - y_d\|_{H^{-1}(0,1;\mathbb{R}^2)} \leq \varepsilon.)$$
2. We say that system (1) (resp., system (2)) is null controllable at time T if for every $y_0 \in L^2(0,1;\mathbb{R}^2))$ (resp., $y_0 \in H^{-1}(0,1;\mathbb{R}^2),$) there exists $u \in L^2(Q_T)$ (resp., $v \in L^2(0,T)$) such that
$$y(T,\cdot) = 0 \text{ in } L^2(0,1;\mathbb{R}^2) \text{ (resp., } y(T,\cdot) = 0 \text{ in } H^{-1}(0,1;\mathbb{R}^2)).$$

Systems (1) and (2) are a particular class of general $n \times n$ parabolic systems
$$\begin{cases} \partial_t y - D\Delta y + A(t,x)y = 1_\omega \widetilde{B}\, u & \text{in } Q_T := (0,T) \times \Omega, \\ y = \widetilde{C} u 1_{\Gamma_0} & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases} \tag{3}$$
where ω and Γ_0 are respectively open subsets of a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and of its boundary $\partial\Omega$, $D = \mathrm{diag}(d_1, ..., d_n) \in \mathcal{L}(\mathbb{R}^n)$, with $n \geq 1$, $A(t,x) = (a_{i,j}(t,x))_{1 \leq i,j \leq n} \in L^\infty(Q_T)^{n^2}$ is the coupling matrix and $\widetilde{B}, \widetilde{C} \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, with $m \leq n$, are the control matrices.

Concerning the scalar case ($n = 1$), H.O. Fattorini and D.L. Russell prove in [9] and [10] the null controllability in the one-dimentional case ($N = 1$), using the moment method. The N-dimentional case has been established simultaneously by G. Lebeau and L. Robbiano in [15] and by A. Fursikov and O. Yu. Imanuvilov in [12] using Carleman inequalities. The case where A is a constant matrix, has been considered in [2], where the authors prove a necessary and sufficient condition for the approximate and null controllability. In [13], a cascade structure of the matrix $A(t,x)$ has been considered and the null controllability for $n \times n$ systems has been established by Carleman inequalities, under the following assumptions on the coupling terms: there exists a nonempty open set $\omega_0 \subseteq \omega$ and a positive constant c_0 such that
$$a_{i,i-1} > c_0 > 0 \text{ in } (0,T) \times \omega_0 \text{ or } -a_{i,i-1} > c_0 > 0 \text{ in } (0,T) \times \omega_0, \ \forall i \in \{2, ..., n\}. \tag{4}$$
This condition implies in particular that $\mathrm{Supp}\, a_{i,i-1} \cap \omega \neq \emptyset$ for all $2 \leq i \leq n$. The null controllability in the general case $\mathrm{Supp}\, a_{i,i-1} \cap \omega = \emptyset$ remains an open problem. One of the recent steps in this direction has been established for two parabolic equations ($n = 2$) in [5], where the authors analyze the distributed and boundary controllability of system (3) under the following assumption on the structure:
$$D := I_d,\ A(x) := \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix},\ \widetilde{C} := \widetilde{B} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \Omega := (0,1).$$
Under the assumptions
$$\mathrm{Supp}\, q \cap \omega = \emptyset,\ \left| \int_0^1 q(x) |\sin(k\pi x)|^2\, dx \right| + \left| \int_0^a q(x) |\sin(k\pi x)|^2\, dx \right| \neq 0,$$

for distributed control ($\widetilde{C} = 0$ in (3)) and assumption

$$\int_0^1 q(x)|\sin(k\pi x)|^2\,dx \neq 0, \quad \forall k \geq 1,$$

for boundary control ($\widetilde{B} = 0$ in (3)), they prove the existence of a minimal time $T_0(q) \in [0,\infty]$ for the distributed controllability problem (resp., $T_1(q) \in [0,+\infty]$ for the boundary controllability problem), i.e., system (3) with $\widetilde{C} = 0$ (resp., with $\widetilde{B} = 0$) is null controllable if $T > T_0(q)$ (resp., if $T > T_1(q)$), and not null controllable if $T < T_0(q)$ (resp., $T < T_1(q)$). Moreover, for every $\tau_0 \in [0,\infty]$, the authors prove that there exists a function $q \in L^\infty(0,1)$ and $\omega \subset (0,1)$ satisfying $\mathrm{Supp}\ q \cap \omega = \emptyset$ in the case of distributed controls, such that $T_0(q) = \tau_0$ or $T_1(q) = \tau_0$.

To our knowledge, there are few results on boundary control systems ($\widetilde{B} = 0$ in (3)). Most of them concern the one-dimensional case ($N = 1$). In [11], the authors consider the case where $D = Id$ and A is a constant matrix and prove a necessary and sufficient condition for the boundary controllability using the moment method. The case $D \neq I_d$ and $n = 2$ has been considered in [4] with

$$D := \mathrm{diag}(1,d),\ d > 0 \text{ and } d \neq 1,\ A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \widetilde{C} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \Omega := (0,1).$$

The authors prove the existence of a minimal time of null controllability, given by the index of condensation (see Definition 2.1 below) of the sequence of eigenvalues $\Lambda = \{k^2\pi^2,\ d\,k\pi^2\}_{k\geq 0}$ of the system.

Systems (1) and (2) are particular cases of system (3) with $N = 1$, $n = 2$, $m = 1$,

$$D := I_d,\ A(t,x) := H\,q(x), \text{ with } H \in \mathcal{L}(\mathbb{R}^2) \text{ and } \Omega = (0,1).$$

In this paper we consider the case when H is a diagonalizable matrix with two distinct real eigenvalues μ_1, $\mu_2 \in \mathbb{R}$. System (1) and (2) have been studied in [5] when the matrix H has $\lambda = 0$ as a unique eigenvalue. The approach followed in this paper is close to that developed in [5]. However, due to the structure of the matrix H, new and interesting mathematical difficulties arise.

Remark 1.1. In the case when $q(x) = q$ is constant, it is shown in [2] that system (1) is null controllable if and only if the Kalman rank condition

$$\det\left[\widetilde{B}, H\widetilde{B}\right] \neq 0, \tag{5}$$

holds. For system (2), this condition (5) is necessary for both approximate and null controllability (see [6, Remark 25]).
In the case when $q \in L^2(0,1)$ is a given function, observe that conditions $q \not\equiv 0$ and (5) are necessary for both approximate controllability and null controllability for systems (1) and (2).

After an appropriate change of variables, observe that the controllability of systems (1) and (2) is equivalent to the controllability of the following systems:

$$\begin{cases} \partial_t y + Ly = 1_\omega B\,u & \text{in } Q_T := (0,T)\times(0,1), \\ y(\cdot,0) = 0,\ \ y(\cdot,1) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1), \end{cases} \tag{6}$$

and
$$\begin{cases} \partial_t y + Ly = 0 & \text{in } \ Q_T, \\ y(\cdot,0) = B\, v(t), \ \ y(\cdot,1) = 0 & \text{on } \ (0,T), \\ y(0,\cdot) = y_0 & \text{in } \ (0,1), \end{cases} \tag{7}$$
where the operator $(L, D(L))$ and the vector $B \in \mathbb{R}^2$ are respectively given by:
$$L := \begin{pmatrix} -\Delta + \mu_1\, q & 0 \\ 0 & -\Delta + \mu_2\, q \end{pmatrix},\ D(L) = H^2(0,1;\mathbb{R}^2) \cap H_0^1(0,1;\mathbb{R}^2), \tag{8}$$
and $B = (b_1, b_2)^T \neq (0,0)$, with b_1, b_2 a real coefficients.

Remark 1.2. Note that in this case, the Kalman rank condition (5) and condition $q \not\equiv 0$ read as
$$q \not\equiv 0 \quad \text{and} \quad b_1 b_2(\mu_1 - \mu_2) \neq 0, \tag{9}$$
for (6) when L is given by (8). Following Remark 1.1, (9) is necessary for the null controllability of system (1) at time $T > 0$. In fact, the algebraic Kalman condition (5) and condition (4), i.e., there exists a nonempty open subset $\omega_0 \subseteq \omega$ and a positive constant $c_0 > 0$, such that
$$q(x) > c_0 \ \text{ or } \ q(x) < -c_0 \quad \text{in } \ \omega_0,$$
imply the approximate and null controllability of system (1) at time $T > 0$. On the other hand, if Supp $q \cap \omega = \emptyset$, the distributed controllability of system (1) has been studied in [5] when H has a unique eigenvalue, $\lambda = 0$, with geometric multiplicity 1.
We can also deduce that conditions (9) are necessary for the controllability of (7) when L is given by (8). Again, the controllability of (2) has been studied in [5] when H has $\lambda = 0$ as a unique eigenvalue with geometric multiplicity 1.

The main novelty of this paper is to consider H with different eigenvalues and
$$\operatorname{Supp} q \subset [0,a] \ \text{ or } \ \operatorname{Supp} q \subset [b,1], \tag{10}$$
for system (6). We are going to see that, under assumption (10), it appears a minimal time $T_0(q) \in [0, +\infty]$.

Let us denote by $\sigma(L_i) = \{\lambda_{i,k}\}_{k \geq 1}$, with $i = 1,2$, the set of eigenvalues corresponding to the operators $(L_i, D(L_i))$ for $i = 1,2,$ defined by
$$L_i = -\partial_{xx} + \mu_i q, \quad D(L_i) = H^2(0,1) \cap H_0^1(0,1). \tag{11}$$
The main result of this paper is the following one:

Theorem 1.1. *Let us consider $(L, D(L))$ given by* (8)*, $B \in \mathbb{R}^2$ and $q \in L^2(0,1)$, a given function. Let us assume that conditions* (9) *hold. In addition, let us suppose that* (10) *is satisfied for system* (6)*. Then, one has:*

1. *Systems* (6) *and* (7) *are approximately controllable at time T if and only if*
$$\sigma(L_1) \cap \sigma(L_2) = \emptyset. \tag{12}$$
2. *Assume that condition* (12) *holds and define*
$$T_0(q) := \limsup_{n \to +\infty} -\frac{\ln |\lambda_{1,n} - \lambda_{2,n}|}{\lambda_{1,n}}. \tag{13}$$
Then,

(a) If $T > T_0(q)$ systems (6) and (7) are null controllable at time T.

(b) If $T < T_0(q)$ systems (6) and (7) are not null controllable at time T.

Remark 1.3. The controllability result of system (7) has been already proved in [4, Th 2.5, p. 12] by means of condensation grouping. In fact, we will prove that $T_0(q)$ may take any value of $[0, +\infty]$ (see Remark 6). For the sake of completeness we will provide a proof of the controllability result of system (7).

Remark 1.4. To prove the approximate controllability result, we carry out an analysis of the properties of the eigenfunctions of $(L, D(L))$ (see section 2.1) for system (7) and, under the geometrical condition (10) on the function $q \in L^2(0,1)$ for system (6). We establish a necessary and sufficient condition (12) that characterizes the approximate controllability property for systems (6) and (7). Thus, (12) is a necessary condition for the null controllability of these systems at time $T > 0$. Observe that this condition does not depend on the final time T.

Remark 1.5. From the expressions of eigenvalues $\lambda_{1,k}$, $\lambda_{2,k}$ of $(L, D(L))$ (see (18)), we can deduce that, under assumption

$$\int_0^1 q(x)\,dx \neq 0, \tag{14}$$

there exists $k' \in \mathbb{N}^*$ such that

$$\lambda_{1,k} \neq \lambda_{2,l}, \quad \forall k, l \geq k',$$

In particular, condition (12) holds (apart from a finite number of Fourier modes) and we can deduce that $T_0(q) = 0$. As a consequence, under condition (14), we deduce the existence of a finite-dimensional space $X \subset L^2(0,1;\mathbb{R}^2)$ such that one has the null controllability of (6) and (7) at any time $T > 0$ if $y_0 \in X^\perp$.

Remark 1.6. In the case of boundary controllability, we will see in section 6, that there exists a function $q \in L^2(0,1)$, such that $T_0(q) > 0$. In fact, $T_0(q)$ may take any value in $[0, +\infty]$.

This paper is organized as follows: In Section 2, we recall some preliminary results related to the spectrum of the operator $(L, D(L))$, some characterizations of the controllability and a result on the existence of biorthogonal families to real exponentials. Section 3 is devoted to studying the approximate controllability of systems (6) and (7). In Section 4, we prove the existence of a time $T_0(q)$ such that systems (6) and (7) are null controllable for any $T > T_0(q)$. Finally, in Section 5, we prove that systems (6) and (7) are not null controllable when $T < T_0(q)$.

2 Preliminary results

In this paper, we denote by $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ the usual duality pairing between $H^{-1}(0,1;\mathbb{R}^2)$ and $H_0^1(0,1;\mathbb{R}^2)$ and by $\langle \cdot, \cdot \rangle_{L^2}$ the scalar product of either $L^2(0,1;\mathbb{R}^2)$ or $L^2(0,1;\mathbb{R})$, with norm denoted by $\| \cdot \|_{L^2}$.

In this section, we give some spectral properties of the operator $(L, D(L))$ which will be used later. We also recall some controllability properties of systems (6) and (7) and we finish by recalling a known result on the existence and bounds of biorthogonal families to real exponentials.

2.1 Spectral properties

Theorem 2.1. *Let $q \in L^2(0,1)$, $\lambda \in \mathbb{C}$ and consider the following initial value problem:*

$$\left\{ \begin{array}{l} -p'' + q\,p = \lambda\, p \ \textit{in}\ (0,1), \\ p(0) = 0,\ p'(0) = 1. \end{array} \right. \tag{15}$$

Then, problem (15) *admits a unique solution $p \in H^2(0,1)$ which is the solution of the Volterra equation:*

$$p(x) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}}\, q(t)\, p(t)\, dt, \quad \forall x \in (0,1). \tag{16}$$

We denote by $p(\cdot, \lambda, q)$ the solution of (15), corresponding to $\lambda \in \mathbb{C}$ and the function $q \in L^2(0,1)$. We now recall some well-known properties concerning the spectrum of the Sturm-Liouville problem.

Proposition 2.1. *The operators $(L_i, D(L_i))$, given by* (11)*, with $i = 1, 2$, are selfadjoint and admit an increasing sequence of eigenvalues $\sigma(L_i) := \{\lambda_{i,k}\}_{k\geq 1} \subset \mathbb{R}$, $i = 1, 2$, with the following properties:*

$$\lambda_{i,1} < \lambda_{i,2} < ... < \lambda_{i,k} < \lambda_{i,k+1} < ..., \ \textit{with} \lim_{k\to+\infty} \lambda_{i,k} = +\infty, \tag{17}$$

and

$$\lambda_{i,k} = \pi^2 k^2 + \mu_i \int_0^1 q(x)\, dx - \mu_i \int_0^1 \cos(2kx)\, q(x)\, dx + \mathcal{O}\left(\frac{1}{k}\right), \tag{18}$$

for $k \to \infty$. Furthermore, if $\varphi_{i,k}$ is the normalized eigenfunction associated to $\lambda_{i,k}$ with i=1,2, then, the sequence $\{\varphi_{i,k}\}_{k\geq 1}$ is an orthonormal basis of $L^2(0,1)$. Moreover

$$\varphi_{i,k}(x) = \frac{p(x, \lambda_{i,k}, \mu_i q)}{||p(\cdot, \lambda_{i,k}, \mu_i q)||_{L^2}}, \quad \forall k \geq 1,\ x \in (0,1),\ \textit{with}\ i = 1, 2, \tag{19}$$

with the following asymptotic behavior:

$$\left\{ \begin{array}{l} p(x, \lambda_{i,k}, \mu_i q) = \frac{\sin(\sqrt{\lambda_{i,k}}x)}{\sqrt{\lambda_{i,k}}} + \mathcal{O}\left(\frac{1}{k^2}\right), \\ ||p(\cdot, \lambda_{i,k}, \mu_i q)||_{L^2} = \frac{1}{\sqrt{2\lambda_{i,k}}} \sqrt{1 + \mathcal{O}\left(\frac{1}{k}\right)}, \end{array} \right. \tag{20}$$

and

$$\left\{ \begin{array}{ll} \varphi_{i,k}(x) & = \sqrt{2}\sin(k\pi x) + \mathcal{O}\left(\frac{1}{k}\right), \\ \varphi'_{i,k}(x) & = \sqrt{2}\pi k\, \cos(k\pi x) + \mathcal{O}(1), \end{array} \right. \tag{21}$$

as $k \to \infty$ uniformly for $x \in [0,1]$.

For a proof of the previous results, we refer to ([14, Th 4.4, p.125, Th 4.10, p.134 and Th 4.11, p.135], [16, Th 4, p. 35]).

We deduce that the spectrum of $(L, D(L))$ is $\{\lambda_{1,k}, \lambda_{2,k}; k \in \mathbb{N}^*\}$, and the corresponding eigenfunctions are given by:

$$\phi_{1,k} = \begin{pmatrix} \varphi_{1,k} \\ 0 \end{pmatrix} \text{ and } \phi_{2,k} = \begin{pmatrix} 0 \\ \varphi_{2,k} \end{pmatrix}, \ \forall k \geq 1. \tag{22}$$

Moreover the sequence $\{\phi_{1,k}, \phi_{2,k}, k \in \mathbb{N}^*\}$ is an orthonormal basis of $L^2(0,1;\mathbb{R}^2)$.

2.2 Controllability properties

Let us introduce the adjoint problem associated to systems (6) and (7):

$$\begin{cases} -\partial_t \psi + L\psi = 0 & \text{in } Q_T, \\ \psi(\cdot,1) = 0,\, \psi(\cdot,0) = 0 & \text{on } (0,T), \\ \psi(T,\cdot) = \psi_0 & \text{in } (0,1), \end{cases} \tag{23}$$

where $\psi_0 \in L^2(0,1;\mathbb{R}^2)$ or $\psi_0 \in H_0^1(0,1,\mathbb{R}^2)$ is a given initial datum. Let us first see that system (23) is well posed, in the following sense:

Proposition 2.2. *For all $\psi_0 \in L^2(0,1;\mathbb{R}^2)$ system (23) admits a unique solution $\psi \in L^2(0,T;H_0^1(0,1,\mathbb{R}^2)) \cap C^0([0,T];L^2(0,1;\mathbb{R}^2))$, given by*

$$\psi(t,\cdot) = \sum_{k\geq 1} e^{-\lambda_{1,k}(T-t)}\langle \psi_0, \phi_{1,k}\rangle_{L^2}\phi_{1,k} + e^{-\lambda_{2,k}(T-t)}\langle \psi_0, \phi_{2,k}\rangle_{L^2}\phi_{2,k}.$$

Moreover, if $\psi_0 \in H_0^1(0,1;\mathbb{R}^2)$, then the solution satisfies

$$\psi \in L^2(0,T;H^2(0,1;\mathbb{R}^2) \cap H_0^1(0,1;\mathbb{R}^2)) \cap C^0([0,T];H_0^1(0,1;\mathbb{R}^2)).$$

The next proposition, provides a general characterizations of the controllability properties related to systems (6) and (7).

Proposition 2.3. *1. System (6) is approximately controllable at time $T > 0$ if and only if, the following unique continuation property holds:*

"If ψ is the solution of the adjoint problem (23) associated to $\psi_0 \in L^2(0,1;\mathbb{R}^2)$ and

$$B^*\psi = 0 \textit{ in } (0,T)\times\omega, \textit{ then, one has } \psi_0 \equiv 0 \textit{ in } (0,1).\text{"} \tag{24}$$

2. System (6) is null controllable at time $T > 0$, if and only if there exists $C > 0$ such that the observability inequality

$$||\psi(0,\cdot)||_{L^2}^2 \leq C \iint_{(0,T)\times\omega} |B^*\psi(t,x)|^2\, dx\, dt \tag{25}$$

holds for every $\psi_0 \in L^2(0,1;\mathbb{R}^2)$, where ψ is the corresponding solution of (23).

3. System (7) is approximately controllable at time $T > 0$, if and only if, the following unique continuation property holds:

"If ψ is the solution of the adjoint problem (23) associated to $\psi_0 \in H_0^1(0,1;\mathbb{R}^2)$ and

$$B^*\, \partial_x\psi(t,0) = 0 \textit{ on } (0,T), \textit{ then, one has } \psi_0 \equiv 0 \textit{ in } (0,1).\text{"} \tag{26}$$

4. System (7) is null controllable at time $T > 0$, if and only if, there exists $C > 0$ C such that the observability inequality

$$||\psi(0,\cdot)||_{H_0^1(0,1;\mathbb{R}^2)}^2 \leq C \int_0^T |B^*\partial_x\psi(t,0)|^2\, dt \tag{27}$$

holds for every $\psi_0 \in H_0^1(0,1;\mathbb{R}^2)$, where ψ is the associated solution of (23).

For a proof of the previous results see for instance [8], [18] or [19].

2.3 Biorthogonal family and condensation index

In this subsection, we study the existence of a biorthogonal family to the real exponentials in $L^2(0,T)$.
Let us consider a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{R}_+^*$ satisying

$$\begin{cases} \lambda_k < \lambda_{k+1}, \quad \forall k \geq 1, \\ \displaystyle\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty. \end{cases} \tag{28}$$

Definition 2.1. The index of condensation of the sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$ is defined by:

$$c(\Lambda) = \limsup_{k\to+\infty} \frac{1}{\lambda_k} \ln \frac{1}{|C'(\lambda_k)|} \in [0,+\infty], \tag{29}$$

where

$$C(\lambda) := \prod_{k\geq 1} \left(1 - \frac{\lambda^2}{\lambda_k^2}\right) \tag{30}$$

is called the interpolation function.

This definition was introduced by V. Bernstein in [7] to study the overconvergence of Dirichlet series (see [7] and [17]).

Proposition 2.4. *Let us consider a sequence $\Lambda = \{\lambda_k\}_{k\geq 1} \in \mathbb{R}_+^*$ satisfying* (28)*. Then, there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$, i.e.,*

$$\int_0^T q_k(t)\, e^{-\lambda_j t}\, dt = \delta_{k,j}, \quad \forall k, j \in \mathbb{N}.$$

Moreover, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|q_k\|_{L^2} \leq C_\varepsilon\ e^{(c(\Lambda)+\varepsilon)\lambda_k}, \quad \forall k \geq 1 \tag{31}$$

where $c(\Lambda)$ is given by (29) *.*

For a proof of this result, we refer to [4].

3 Approximate controllability

This section is devoted to the proof of the first item of Theorem 1.1. To study the approximate controllability of systems (6) and (7), we use the properties of the spectrum of the operator $(L, D(L))$ (see (8)) given by Proposition 2.1.

3.1 Approximate controllability for system (6)

We recall that $\varphi_{i,k}$ is given by (19) with $p(\cdot, \lambda_{i,k}, \mu_i q)$ given by (see (15) and (16))

$$p(x, \lambda_{i,k}, \mu_i q) = \frac{\sin(\sqrt{\lambda_{i,k}}x)}{\sqrt{\lambda_{i,k}}} + \mu_i \int_0^x \frac{\sin(\sqrt{\lambda_{i,k}}(x-s))}{\sqrt{\lambda_{i,k}}}\, q(s)\, p(s)\, ds,$$

for any $x \in (0,1)$, $k \geq 1$ and $i = 1,2$. In particular

$$-p''(\cdot,\lambda_{i,k},\mu_i q) + \mu_i\, q(\cdot)\, p(\cdot,\lambda_{i,k},\mu_i q) = \lambda_{i,k}\, p(\cdot,\lambda_{i,k},\mu_i q) \quad \text{in } (0,1),$$

and $p(1,\lambda_{i,k},\mu_i q) = p(0,\lambda_{i,k},\mu_i q) = 0$. Using an integration by parts in

$$\int_0^1 \frac{\sin\sqrt{\lambda_{i,k}}(x-s)}{\sqrt{\lambda_{i,k}}}(-p''(s)+\mu_i q(s)\, p(s))\, ds = \lambda_{i,k}\int_0^1 \frac{\sin\sqrt{\lambda_{i,k}}(x-s)}{\sqrt{\lambda_{i,k}}} p(s)\, ds,$$

one gets

$$\mu_i \int_0^1 \frac{\sin\sqrt{\lambda_{i,k}}(x-s)}{\sqrt{\lambda_{i,k}}} q(s)\, p(s)\, ds = -\frac{\sin(\sqrt{\lambda_{i,k}}x)}{\sqrt{\lambda_{i,k}}} + p'_{i,k}(1)\frac{\sin(\sqrt{\lambda_{i,k}}(x-1))}{\sqrt{\lambda_{i,k}}},$$

where $p'_{i,k}(1) = p'(1,\lambda_{i,k},\mu_i q)$. Then we can write for all $x \in (0,1)$ and $k \in \mathbb{N}^*$ that

$$p(x,\lambda_{i,k},\mu_i q) = p'_{i,k}(1)\frac{\sin(\sqrt{\lambda_{i,k}}(x-1))}{\sqrt{\lambda_{i,k}}} + \mu_i \int_1^x \frac{\sin(\sqrt{\lambda_{i,k}}(x-s))}{\sqrt{\lambda_{i,k}}}\, q(s)p(s)\, ds.$$

Using (19), we deduce that if $\operatorname{Supp} q \subset (0,a)$ then the eigenfunctions of $(L_i, D(L_i))$, with i=1,2, satisfy

$$\varphi_{i,k}(x) = \frac{p'_{i,k}(1)\sin(\sqrt{\lambda_{i,k}}(x-1))}{\sqrt{\lambda_{i,k}}||p(\cdot,\lambda_{i,k},\mu_i q)||_{L^2}} = \varphi'_{i,k}(1)\frac{\sin(\sqrt{\lambda_{i,k}}(x-1))}{\sqrt{\lambda_{i,k}}},\ \forall x \in \omega, \quad (32)$$

and if $\operatorname{Supp} q \subset (b,1)$ then

$$\varphi_{i,k}(x) = \frac{\sin(\sqrt{\lambda_{i,k}}\, x)}{\sqrt{\lambda_{i,k}}||p(\cdot,\lambda_{i,k},\mu_i q)||_{L^2}},\ \forall x \in \omega. \quad (33)$$

In this subsection, we assume that $\operatorname{Supp} q \subset (b,1)$, the case $\operatorname{Supp} q \subset (0,a)$ can be treated in the same way.

Necessary condition Let us fix $T > 0$ and assume that condition (12) does not hold, i.e., there exist $k_0, j_0 \in \mathbb{N}$ such that $\lambda_{1,k_0} = \lambda_{2,j_0} = \lambda$. Thus

$$\varphi_{1,k_0}(x) = \alpha_{j_0,k_0}\,\varphi_{2,j_0}(x), \quad \forall x \in \omega, \text{ with } \alpha_{j_0,k_0} = \frac{||p(\cdot,\lambda_{2,j_0},\mu_1 q)||_{L^2}}{||p(\cdot,\lambda_{1,k_0},\mu_2 q)||_{L^2}}.$$

Let us consider

$$\psi_0 := b_2\phi_{1,k_0} - \alpha_{j_0,k_0}\, b_1\phi_{2,j_0},$$

with $\{\phi_{i,k}\}_{k\geq 1}, i = 1,2$, given by (22). Then, the associated solution of the adjoint problem (23) is given by

$$\psi(t,x) = e^{\lambda(T-t)} b_2\,\phi_{1,k_o}(x) - e^{\lambda(T-t)} b_1\,\alpha_{j_0,k_0}\,\phi_{2,j_0}(x), \quad \forall (t,x) \in (0,T)\times\omega.$$

In particular, if $(t,x) \in (0,T)\times\omega$, this function satisfies

$$B^*\psi(t,x) = e^{-\lambda(T-t)}(b_1\, b_2\,\varphi_{1,k_0}(x) - b_1\, b_2\,\alpha_{j_0,k_0}\varphi_{2,j_0}(x)) = 0,$$

but $\psi_0 \neq 0$. We deduce by Proposition 2.3 that the system (6) is not approximately controllable at time $T > 0$.

Sufficient condition Let us now assume that (12) holds and consider $\psi_0 \in L^2(0,1;\mathbb{R}^2)$, such that the corresponding solution to the adjoint problem (23) satisfies

$$B^*\psi(t,x) = 0, \quad \text{in } (0,T)\times\omega.$$

Using Proposition 2.2, $\psi \in L^2(0,T,H_0^1(0,1,\mathbb{R}^2)) \cap C^0([0,T];L^2(0,1,\mathbb{R}^2))$ and

$$\sum_{k\geq 1}\Big(b_1 e^{-\lambda_{1,k}(T-t)}\langle\psi_0,\phi_{1,k}\rangle_{L^2}\varphi_{1,k} \\ + b_2 e^{-\lambda_{2,k}(T-t)}\langle\psi_0,\phi_{2,k}\rangle_{L^2}\,\varphi_{2,k}\Big) = 0 \quad \text{in } (0,T)\times\omega.$$

Without loss of generality, let us assume that $\lambda_{i,k} > 0$, for all $k \geq 1$ and $i = 1,2$. Under the assumption (12), the sequence $\Lambda = \{\lambda_{1,k},\ \lambda_{2,k}\}_{k\geq 1}$ can be ordered increasingly and then satisfies condition (28). Using Proposition 2.4, there exists a family $\{q_{1,k}(t),\ q_{2,k}(t)\}_{k\geq 1}$ biorthogonal to $\{e^{-\lambda_{1,k}t}, e^{-\lambda_{2,k}t}\}_{k\geq 1}$ in $L^2(0,1)$.

Therefore, for all $x \in \omega = (a,b)$ and $k \geq 1$, one has :

$$0 = \int_0^T B^*\,\psi(t,x)\,q_{1,k}(t)\,dt = b_1\langle\psi_0,\phi_{1,k}\rangle_{L^2}\varphi_{1,k}(x),$$
$$0 = \int_0^T B^*\,\psi(t,x)\,q_{2,k}(t)\,dt = b_2\langle\psi_0,\phi_{2,k}\rangle_{L^2}\varphi_{2,k}(x).$$

Since the eigenfunctions $\{\varphi_{1,k}\}_{k\geq 1}$, $\{\varphi_{2,k}\}_{k\geq 1}$ has exactly $k+1$ roots in $[0,1]$ (see (32) and (33)), one has necessarily

$$\langle\psi_0,\phi_{i,k}\rangle_{L^2} = 0, \quad \forall k \geq 1, \quad i = 1,2.$$

By completeness of the eigenfunctions (see (22)), we deduce that $\psi_0 = 0$ on $(0,1)$. Thus by Proposition 2.3, we deduce that the system (6) is approximately controllable at any time $T > 0$.

Remark 3.1. The operator L may contain negative eigenvalues, but by (17) there exists $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\lambda_{i,k} > 0$, taking m large enough and $\lambda'_{i,k} = \lambda_{i,k} + m > 0$, we obtain a strictly positive increasing sequence.

3.2 Approximate controllability for the system (7)

Let us fix $T > 0$ and consider the system (7) (without assumption (10), on the support of the function q).

Necessary condition Let us assume that condition (12) does not hold, i.e., that there are $k_0, j_0 \in \mathbb{N}$ such that $\lambda_{1,k_0} = \lambda_{2,j_0} = \lambda$. Let us take

$$\psi_0(x) = a\,\phi_{1,k_0}(x) + b\,\phi_{2,j_0}(x), \quad \forall x \in (0,1), (a,b) \in \mathbb{R}^2.$$

Thus the solution of the adjoint problem (23) associated to ψ_0 is given by

$$\psi(t,x) = a\,e^{-\lambda(T-t)}\,\phi_{1,k_0}(x) + b\,e^{-\lambda(T-t)}\,\phi_{2,j_0}(x), \quad \forall (t,x) \in Q_T.$$

Then

$$B^*\partial_x\psi(t,0) = a\, b_1\, e^{-\lambda(T-t)}\, \varphi'_{1,k_0}(0) + b\, b_2\, e^{-\lambda(T-t)}\, \varphi'_{2,j_0}(0), \quad \forall t \in (0,T).$$

Taking

$$a = b_2\, \varphi'_{2,j_0}(0) \text{ and } b = -b_1\, \varphi'_{1,k_0}(0),$$

we obtain $B^*\partial_x\psi(t,0) = 0$ on $(0,T)$. On the other hand, from (9), (15) and (19), we deduce $\varphi'_{i,k}(0) \neq 0$, for all $k \geq 1$, and $\psi \neq 0$. So, system (7) is not approximately controllable at time $T > 0$.

Sufficient condition Let us now suppose that condition (12) holds. Let us take $\psi_0 \in H_0^1(0,1;\mathbb{R}^2)$ and assume that the solution of the adjoint problem (23) associated to ψ_0 satisfies

$$B^*\partial_x\psi(t,0) = 0, \quad \forall t \in (0,T),$$

which implies that

$$\sum_{k\geq 1} \Big(b_1 e^{-\lambda_{1,k}(T-t)} \langle \psi_0, \phi_{1,k}\rangle_{L^2}\, \varphi'_{1,k}(0) + b_2 e^{-\lambda_{2,k}(T-t)} \langle \psi_0, \phi_{2,k}\rangle_{L^2}\, \varphi'_{2,k}(0) \Big) = 0, \quad \text{in } (0,T).$$

The same arguments of the previous subsection lead to

$$0 = \int_0^1 B^*\partial_x\psi(t,0)\, q_{1,k}(t)\, dt = b_1 \varphi'_{1,k}(0) \langle \psi_0, \varphi_{1,k}\rangle_{L^2},\ \forall k \in \mathbb{N},$$
$$0 = \int_0^1 B^*\partial_x\psi(t,0)\, q_{2,k}(t)\, dt = b_2 \varphi'_{2,k}(0) \langle \psi_0, \varphi_{2,k}\rangle_{L^2},\ \forall k \in \mathbb{N},$$

where $\{q_{1,k},\, q_{2,k}\}_{k\geq 1}$ is the biorthogonal family to $\{e^{-\lambda_{1,k}t},\, e^{-\lambda_{2,k}t}\}_{k\geq 1}$ in $L^2(0,T)$. As before, $\varphi'_{i,k}(0) \neq 0$, for all $k \geq 1$. By completeness of eigenfunctions (see (22)), we deduce that $\psi_0 = 0$ on $(0,1)$. Then the continuation property (26) holds. So, the system (7) is approximately controllable. This ends the proof of the first item of Theorem 1.1.

4 Positive null controllability result

Let us now prove the second part of Theorem 1.1. To this end, we divide the proof into several steps. We first prove the existence of $T_0(q)$, such that systems (6) and (7) are null controllable, when $T > T_0(q)$, using the moment method (see [9],[10]).

4.1 The positive null controllability result for the system (6)

Let us take $\psi_{0,k} = \phi_{i,k}$, with i=1,2 (see (22)) as the initial datum of the adjoint system (23), then the associated solutions are given by

$$\psi_{i,k}(t,x) = \phi_{i,k}(x)\, e^{-\lambda_{i,k}(T-t)}, \quad (t,x) \in Q_T,\ k \geq 1,\ i = 1,2. \tag{34}$$

Direct computations give that, for all $k \geq 1$ and $i = 1, 2$

$$\iint_{Q_T} u(t,x)\, 1_\omega\, B^* \phi_{i,k}(x)\, e^{-\lambda_{i,k}(T-t)}\, dt\, dx = \langle y(T,\cdot), \phi_{i,k}\rangle_{L^2} - \langle y_0, \phi_{i,k}(\cdot)\, e^{-\lambda_{i,k}T}\rangle_{L^2},$$

where $y \in L^2(0,T; H_0^1(0,1;\mathbb{R}^2)) \cap C^0([0,T]; L^2(0,1;\mathbb{R}^2))$ is the solution of system (6) associated to y_0.

Observe that by completeness of eigenfunctions, given $y_0 \in L^2(0,T;\mathbb{R}^2)$, the control $u \in L^2(Q_T)$ is such that the solution y of system (6) satisfies $y(T,\cdot) = 0$ if and only if

$$\begin{aligned}
&\iint_{Q_T} u(T-t,x)\, 1_\omega\, b_1\, \varphi_{1,k}(x)\, e^{-\lambda_{1,k}t}\, dt\, dx = \langle y_0, \phi_{1,k}(\cdot)\, e^{-\lambda_{1,k}T}\rangle_{L^2},\ \forall k \geq 1,\\
&\iint_{Q_T} u(T-t,x)\, 1_\omega\, b_2\, \varphi_{2,k}(x)\, e^{-\lambda_{2,k}t}\, dt\, dx = \langle y_0, \phi_{2,k}(\cdot)\, e^{-\lambda_{2,k}T}\rangle_{L^2},\ \forall k \geq 1.
\end{aligned} \tag{35}$$

Assumption (12) and Proposition 2.4 ensure the existence of a biorthogonal family $\{q_{1,k},\, q_{2,k}\}_{k\geq 1}$ to $\{e^{-\lambda_{1,k}t}, e^{-\lambda_{2,k}t}\}_{k\geq 1}$ in $L^2(0,T)$, which satisfies the following estimate

$$\forall \varepsilon > 0,\ \exists\, C_\varepsilon \text{ such that } \|q_{i,k}\|_{L^2(0,T)} \leq C_\varepsilon\ e^{(c(\Lambda)+\varepsilon)\lambda_{i,k}}, \forall k \geq 1,\ i = 1, 2, \tag{36}$$

where $c(\Lambda)$ is the condensation index of the sequence $\Lambda = \{\lambda_{1,k}, \lambda_{2,k}\}_{k\geq 1}$ (see (29)). Following the approach of [1], we restrict the control to the following form:

$$\tilde{u}(t,x) = u(T-t,x) = \sum_{k\geq 1} (q_{1,k}(t)\varphi_{1,k}(x) m_{1,k} + q_{2,k}(t)\varphi_{2,k}(x) m_{2,k}), \tag{37}$$

where $m_{i,k}$ are coefficients to be determined. We replace u by (37) in (35), one gets, formally:

$$m_{i,k}\, b_i \int_\omega \varphi_{i,k}^2(x)\, dx = \langle y_0, \phi_{i,k}(x) e^{-\lambda_{i,k}T}\rangle_{L^2}, \quad \forall k \geq 1 \text{ and } i = 1, 2.$$

Since the eigenfunctions $\{\varphi_{1,k}\}_{k\geq 1}$, $\{\varphi_{2,k}\}_{k\geq 1}$ have exactly $k+1$ roots in $[0,1]$, then

$$\int_\omega \varphi_{i,k}^2\, dx > 0, \quad \forall k \geq 1, i = 1, 2,$$

Moreover

$$\int_\omega \sin(k\pi x)^2\, dx \underset{k\to+\infty}{\longrightarrow} b - a > 0.$$

Using (21), we deduce that there exists $C > 0$ such that

$$\inf_{k\geq 1} |b_i \int_\omega \varphi_{i,k}^2\, dx| \geq C > 0. \tag{38}$$

Let us define:

$$m_{i,k} := \frac{\langle y_0, \phi_{i,k}(x)\rangle_{L^2} e^{-\lambda_{i,k}T}}{b_i \int_\omega \varphi_{i,k}^2\, dx}, \quad \forall k \geq 1,\ i = 1, 2.$$

Let us now prove that $\tilde{u} \in L^2(Q_T)$, that is to say, the convergence of the series (37) in $L^2(Q_T)$. Estimate (36) of $\|q_{i,k}\|_{L^2(0,T)}$ and (38), lead to :

$$\|\tilde{u}\|_{L^2(Q_T)} \leq \frac{C_\varepsilon \|y_0\|_{L^2}}{C} \sum_{k\geq 1} \left(e^{-\lambda_{1,k}(T-c(\Lambda)-\varepsilon)} + e^{-\lambda_{2,k}(T-c(\Lambda)-\varepsilon)} \right).$$

Let $T > c(\Lambda)$. Taking $\varepsilon = \frac{T-c(\Lambda)}{2}$, one has

$$\|\tilde{u}\|_{L^2(Q_T)} \leq \frac{C_\varepsilon \|y_0\|_{L^2}}{C} \sum_{k\geq 1} \sum_{i=1}^{2} e^{-\lambda_{i,k}(\frac{T-c(\Lambda)}{2})} < \infty. \tag{39}$$

This inequality shows that if $T > c(\Lambda)$, then $u \in L^2(Q_T)$. We deduce that system (6) is null controllable at time $T > c(\Lambda)$. To conclude with Item (a) of Theorem 1.1, it remains to show that $c(\Lambda) = T_0(q)$, where $T_0(q)$ is given by (13).

Lemma 4.1. *Assume that condition* (12) *holds. Let* $\Lambda = \{\lambda_{1,k}, \lambda_{2,k}, k \in \mathbb{N}\}$, *then:*

$$c(\Lambda) = T_0(q) = \limsup_{k\to+\infty} -\frac{\ln|\lambda_{1,k} - \lambda_{2,k}|}{\lambda_{1,k}}. \tag{40}$$

Proof. Observe that from the expressions of $\lambda_{1,k}$ and $\lambda_{2,k}$ (see (18)), we deduce:

$$\lim_{k\to\infty} |\lambda_{1,k+1} - \lambda_{2,k}| = \lim_{k\to\infty} |\lambda_{2,k+1} - \lambda_{1,k}| = +\infty.$$

and

$$\lim_{k\to+\infty} |\lambda_{1,k} - \lambda_{2,k}| = \left| (\mu_1 - \mu_2) \int_0^1 q(x)\, dx \right|.$$

We deduce by the previous properties that there exists an integer k_0 such that for all $k \geq k_0$, one has

$$\max_{i=1,2} \lambda_{i,k} < \min_{i=1,2} \lambda_{i,l}, \quad \forall l > k, \quad (l,k) \in \mathbb{N}^2.$$

Therefore the sequence Λ can be rearranged into an increasing sequence $\{\lambda_k\}_{k\geq 1}$ defined by

$$\{\lambda_k\}_{1\leq k\leq 2k_0-2} = \{\lambda_{1,k}\}_{1\leq k\leq k_0-1} \cup \{\lambda_{2,k}\}_{1\leq k\leq k_0-1},$$

such that

$$\lambda_k < \lambda_{k+1}, \quad \forall\ 1 \leq k \leq 2k_0 - 3$$

and, from the $(2k_0 - 1)$-th term, by

$$\begin{cases} \lambda_{2k_0+2k-1} = \min_{i=1,2} \lambda_{i,k_0+k}, & \forall\, k \geq 0, \\ \lambda_{2k_0+2k} = \max_{i=1,2} \lambda_{i,k_0+k}, & \forall\, k \geq 0. \end{cases}$$

Without loss of generality, we can assume that $k_0 = 1$, that is to say
$\Lambda := \{\lambda_k,\ k \geq 1\}$, with

$$\lambda_{2k-1} = \min_{i=1,2} \lambda_{i,k} \text{ and } \lambda_{2k} = \max_{i=1,2} \lambda_{i,k}, \quad \forall k \geq 1.$$

Observe that we can write $\lambda_{2k-1} = k^2\pi^2 + \alpha_k$ and $\lambda_{2k} = k^2\pi^2 + \beta_k$, with

$$\lim_{k\to\infty} \alpha_k = \mu_i \int_0^1 q(x)\,dx, \text{ and } \lim_{k\to\infty} \beta_k = \mu_j \int_0^1 q(x)\,dx,$$

for some $i, j \in \{1,2\}$. Then there exists $M > 0$ such that $\alpha_k, \beta_k \in [-M, M]$.

Let $\lambda_k \in \Lambda$. Using the expression (30), one has

$$|C'(\lambda_k)| = \frac{2}{\lambda_k} \prod_{j\neq k} \left| \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right) \right|.$$

From the previous expression, we deduce:

$$\begin{aligned} \frac{\ln|C'(\lambda_k)|}{\lambda_k} &= \frac{\ln|\lambda_{k-1}-\lambda_k|}{\lambda_k} + \frac{\ln|\lambda_{k+1}-\lambda_k|}{\lambda_k} \\ &\quad + \frac{1}{\lambda_k}\ln\left(\frac{2(\lambda_{k+1}+\lambda_k)(\lambda_{k-1}+\lambda_k)}{\lambda_k\lambda_{k-1}^2\lambda_{k+1}^2}\right) + F_k + G_k, \end{aligned} \tag{41}$$

where

$$F_k = \sum_{j<k-1} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_j^2} - 1\right) \text{ and } G_k = \sum_{j>k+1} \frac{1}{\lambda_k} \ln\left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right).$$

From the definition of λ_k, one gets

$$\lim_{k\to\infty} \frac{1}{\lambda_k} \ln\left(\frac{2(\lambda_{k+1}+\lambda_k)(\lambda_{k-1}+\lambda_k)}{\lambda_k\lambda_{k-1}^2\lambda_{k+1}^2}\right) = 0.$$

On the other hand, assume that one has:

$$\lim_{k\to+\infty} F_k = \lim_{n\to+\infty} G_k = 0. \tag{42}$$

Coming back to (41) we deduce

$$\limsup_{k\to+\infty} -\frac{\ln|C'(\lambda_k)|}{\lambda_k} = \limsup_{k\to+\infty} -(a_k + b_k),$$

where

$$a_k = \frac{\ln|\lambda_{k-1}-\lambda_k|}{\lambda_k} \quad \text{and} \quad b_k = \frac{\ln|\lambda_{k+1}-\lambda_k|}{\lambda_k}.$$

Since $\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} a_{2k-1} = 0$, one has

$$\begin{aligned} \limsup_{k\to+\infty} -(a_{2k}+b_{2k}) &= \limsup_{k\to+\infty} -\frac{\ln|\lambda_{1,k}-\lambda_{2,k}|}{\lambda_{2k}}, \\ \limsup_{k\to+\infty} -(a_{2k-1}+b_{2k-1}) &= \limsup_{k\to+\infty} -\frac{\ln|\lambda_{1,k}-\lambda_{2,k}|}{\lambda_{2k-1}}. \end{aligned}$$

This implies the identity (40) and would finalyze the proof of Lemma 4.1. Therfore, our next task will be to prove (42).

1. Study of F_k: Notice that

$$|F_k| \leq \sum_{\substack{j<k-1 \\ \lambda_k > \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_j^2} - 1\right) + \sum_{\substack{j<k-1 \\ \lambda_k < \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_j^2}{\lambda_k^2 - \lambda_j^2}\right),$$

The first term in the right-hand side of the previous inequality, is estimated by

$$\sum_{\substack{j<k-1 \\ \lambda_k > \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_j^2} - 1\right) \leq \sum_{\substack{j<k-1 \\ \lambda_k > \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_1^2} - 1\right) = \frac{k-1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_1^2} - 1\right) \leq 2(k-1)\frac{\ln \lambda_k/\lambda_1}{\lambda_k},$$

and the second term is estimated by

$$\sum_{\substack{j<k-1 \\ \lambda_k < \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_j^2}{\lambda_k^2 - \lambda_j^2}\right) \leq \sum_{\substack{j<k-1 \\ \lambda_k < \sqrt{2}\lambda_j}} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_j^2}{\lambda_k^2 - \lambda_{k-2}^2}\right) \leq \frac{k-1}{\lambda_k} \ln\left(\frac{\lambda_k^2}{\lambda_k^2 - \lambda_{k-2}^2}\right).$$

Therefore,

$$\lim_{k \to +\infty} |F_k| = 0.$$

2. Study of G_k: Notice that

$$|G_k| = \left| \sum_{j>k+1} \frac{1}{\lambda_k} \ln\left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right) \right| \leq \sum_{j>k+1} \frac{1}{\lambda_k} \ln\left(\frac{\lambda_j^2}{\lambda_j^2 - \lambda_k^2}\right) = \sum_{j>k+1} \frac{1}{\lambda_k} \ln\left(1 + \frac{\lambda_k^2}{\lambda_j^2 - \lambda_k^2}\right).$$

Using the inequality $\ln(1+x) \leq x$, when $x > 0$, one has

$$|G_k| \leq \sum_{j>k+1} \frac{1}{\lambda_k} \ln\left(1 + \frac{\lambda_k^2}{\lambda_j^2 - \lambda_k^2}\right) \leq \sum_{j>k+1} \frac{\lambda_k}{(\lambda_j - \lambda_k)(\lambda_j + \lambda_k)} \leq \sum_{j>k+1} \frac{1}{(\lambda_j - \lambda_k)}. \tag{43}$$

Let us analyse the series in the right-hand side of inequality (43). This series can be written as

$$\sum_{j>k+1} \frac{1}{(\lambda_j - \lambda_k)} = \sum_{j=2n-1 \geq k+2} \frac{1}{(\lambda_{2n-1} - \lambda_k)} + \sum_{j=2n \geq k+2} \frac{1}{(\lambda_{2n} - \lambda_k)}.$$

Thus, from the assumptions on the sequences $\{\lambda_{1,k}\}_{k\geq 1}$ and $\{\lambda_{2,k}\}_{k\geq 1}$, we can write:

$$|G_{2k-1}| \leq \sum_{n \geq k+1} \frac{1}{(n^2\pi^2 + \alpha_n - k^2\pi^2 - \alpha_k)} + \sum_{n \geq k+1} \frac{1}{(n^2\pi^2 + \beta_n - k^2\pi^2 - \alpha_k)}$$

and

$$|G_{2k}| \leq \sum_{n \geq k+2} \frac{1}{(n^2\pi^2 + \alpha_n - k^2\pi^2 - \beta_k)} + \sum_{n \geq k+1} \frac{1}{(n^2\pi^2 + \beta_n - k^2\pi^2 - \beta_k)}.$$

It is not difficult to see that if $k \geq \frac{4M - \pi^2}{2\pi^2}$, then

$$0 \leq \frac{1}{n^2\pi^2 + x - k^2\pi^2 - y} \leq \frac{2}{n^2\pi^2 - k^2\pi^2}, \quad \forall n \geq k+1, \quad \forall x, y \in [-M, M].$$

Taking into account that α_n, $\beta_n \in [-M, M]$ for any $n \geq 1$, from the previous inequality, one has:

$$
\begin{aligned}
|G_{2k-1}| &\leq 4 \sum_{n \geq k+1} \frac{1}{n^2\pi^2 - k^2\pi^2} = 4 \sum_{i \geq 1} \frac{1}{(k+i)^2\pi^2 - k^2\pi^2} = \frac{4}{\pi^2} \sum_{i \geq 1} \frac{1}{i^2 + 2ki} \\
&\leq \frac{4}{\pi^2} \left[\frac{1}{1+2k} + \int_1^{+\infty} \frac{1}{t^2 + 2kt}\, dt \right] = \frac{4}{\pi^2} \left[\frac{1}{1+2k} + \frac{1}{2k} \ln(1+2k) \right].
\end{aligned}
$$

A similar inequality can be obtained for $|G_{2k}|$. In particular, we can infer

$$
\lim_{k \to \infty} |G_{2k-1}| = \lim_{k \to \infty} |G_{2k}| = 0.
$$

This ends the proof. $\square$

4.2 Positive null controllability result for system (7)

Let us now analyze the null controllability of system (7). To this end, assume that $T_0(q) < +\infty$ ($T_0(q)$ is given in (13) and satisfies (40)) and fix $T > T_0(q)$. Our objective is, again, to formulate the null controllability for system (7) as a moment problem for the control $v \in L^2(0,T)$.

Let us take $\psi_0 = \phi_{i,k}, k \in \mathbb{N}^*$, with $i = 1, 2$. Then the corresponding solutions of the adjoint system (23) are given by (34). Thus, given $y_0 \in H^{-1}(0,1;\mathbb{R}^2)$, $v \in L^2(0,T)$ drives the solution y of system (7) to zero at time T if and only if $v \in L^2(0,T)$ satisfies

$$
\int_0^T v(T-t)\, b_i\, \varphi'_{i,k}(0)\, e^{-\lambda_{i,k} t}\, dt = -\langle y_0, \phi_{i,k}(x) e^{-\lambda_{i,k} T} \rangle_{H^{-1}, H_0^1}, \tag{44}
$$

for any $k > 1$, and $i = 1, 2$. As in the previous subsection (under condition (12)), we will solve the moment problem (44) using the biorthogonal family $\{q_{1,k}, q_{2,k}\}_{k \geq 1}$ to $\{e^{-\lambda_{1,k} t}, e^{-\lambda_{2,k} t}\}_{k \geq 1}$ in $L^2(0,T)$ provided by Proposition 2.4. We seek a solution of (44) under the form

$$
\tilde{v}(t) = v(T-t) = \sum_{k \geq 1} q_{1,k}(t)\, d_{1,k} + q_{2,k}(t)\, d_{2,k}, \tag{45}
$$

where $d_{i,k}$ is obtained formally by replacing (45) in (44):

$$
d_{i,k} = \frac{-\langle y_0, \phi_{i,k}(x) \rangle_{H^{-1}, H_0^1} e^{-\lambda_{i,k} T}}{b_i\, \varphi'_{i,k}(0)}, \quad \forall k \geq 1, \quad i = 1, 2.
$$

It remains to prove that $\tilde{v} \in L^2(0,T)$. Using (45) one has

$$
\begin{aligned}
||\tilde{v}||_{L^2(0,T)} \leq ||y_0||_{H^{-1}} \sum_{k \geq 1} \Bigg(& \frac{||\phi_{1,k}||_{H_0^1}}{b_1 |\varphi'_{1,k}(0)|} e^{-\lambda_{1,k} T} ||q_{1,k}||_{L^2(0,T)} \\
& + \frac{||\phi_{2,k}||_{H_0^1}}{b_2 |\varphi'_{2,k}(0)|} e^{-\lambda_{2,k} T} ||q_{2,k}||_{L^2(0,T)} \Bigg),
\end{aligned} \tag{46}
$$

Using the asymptotic behavior of eigenfunctions (see (21)), one has

$$
\begin{aligned}
\varphi'_{i,k}(0) &= \sqrt{2}\pi k + \mathcal{O}(1), \quad i = 1, 2, \\
||\varphi'_{i,k}||_{L^2} &= (\pi^2 k^2 + \mathcal{O}(1))^{\frac{1}{2}}, \quad i = 1, 2,
\end{aligned} \tag{47}
$$

as $k \to \infty$ uniformly for $x \in [0,1]$ and $q \in L^2(0,1)$. Moreover, from the estimate of $\{q_{i,k},\ k \in \mathbb{N}\}$ (see (31)), inequality (46) implies that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$||\tilde{v}||_{L^2(0,T)} \le C_\varepsilon ||y_0||_{H^{-1}} \sum_{k\ge 1} \frac{||\phi_{1,k}||_{H_0^1}}{b_1|\varphi'_{1,k}(0)|} e^{-\lambda_{1,k}(T-c(\Lambda)-\varepsilon)} + \frac{||\phi_{2,k}||_{H_0^1}}{b_2|\varphi'_{2,k}(0)|} e^{-\lambda_{2,k}(T-c(\Lambda)-\varepsilon)}.$$

Since $T > T_0(q) = c(\Lambda)$, taking $\varepsilon = \frac{T-c(\Lambda)}{2}$, we deduce that $\tilde{v} \in L^2(0,T)$ and system (7) is then null controllable at time T. This ends the positive null controllability result for system (7).

5 Negative null controllability result for system (6) and (7)

In this section, we prove Item (b) of Theorem 1.1. Let us assume that $T_0(q) > 0$. Arguing by contradiction, we prove the negative null controllability result of the systems (6) and (7) when $T < T_0(q)$.

5.1 Negative null controllability result for system (6)

By Proposition 2.3, system (6) is null controllable at time T if and only if any solution ψ of the adjoint system (23) satisfies the observability inequality (25). Let us consider

$$\psi_{0,k} := a\,\phi_{1,k} + b\,\phi_{2,k}, \quad \forall k \ge 1,$$

with $a, b \in \mathbb{R}$. So, the corresponding solution ψ_k of the adjoint system (23) is

$$\psi_k(t,x) = a\, e^{-\lambda_{1,k}(T-t)}\, \phi_{1,k}(x) + b\, e^{-\lambda_{2,k}(T-t)}\, \phi_{2,k}(x), \quad \forall (t,x) \in Q_T. \tag{48}$$

From (25), if system (6) is null controllable at time T, there exists $C > 0$ such that for all $k \ge 1$ and $(a,b) \in \mathbb{R}^2$ one has:

$$a^2\, e^{-2\lambda_{1,k}T} + b^2\, e^{-2\lambda_{2,k}T} \le C \int_0^T\!\!\int_\omega \left| b_1\, a\, \varphi_{1,k}\, e^{-\lambda_{1,k}(T-t)} + b_2\, b\, \varphi_{2,k}\, e^{-\lambda_{2,k}(T-t)} \right|^2 dx\, dt.$$

In terms of quadratic forms, the previous inequality is equivalent to

$$e^{-2\,L_k T} \le C\, Q_{k,T}, \quad \forall k \ge 1, \tag{49}$$

where

$$L_k := \begin{pmatrix} \lambda_{1,k} & 0 \\ 0 & \lambda_{2,k} \end{pmatrix} \text{ and } Q_{k,T} := \int_0^T e^{-\,L_k(T-t)} B'_k e^{-\,L_k(T-t)}\, dt, \quad \forall k \ge 1,$$

with

$$B'_k := \begin{pmatrix} b_1^2\, ||\varphi_{1,k}||^2_{L^2(\omega)} & b_1\, b_2 \langle \varphi_{1,k}, \varphi_{2,k} \rangle_{L^2(\omega)} \\ b_1\, b_2 \langle \varphi_{1,k}, \varphi_{2,k} \rangle_{L^2(\omega)} & b_2^2\, ||\varphi_{2,k}||^2_{L^2(\omega)} \end{pmatrix}, \quad \forall k \ge 1.$$

Consider the function η defined by

$$\eta(s) := \frac{e^{sT}-1}{s}, \quad \forall s > 0. \tag{50}$$

Then, inequality (49) can be equivalently written as

$$\frac{1}{C} I \leq C_k, \quad \forall k \geq 1, \tag{51}$$

where

$$C_k := \begin{pmatrix} b_1^2 \|\varphi_{1,k}\|^2_{L^2(\omega)} \eta(2\lambda_{1,k}) & b_1 b_2 \langle \varphi_{1,k}, \varphi_{2,k} \rangle_{L^2(\omega)} \eta(\lambda_{1,k} + \lambda_{2,k}) \\ b_1 b_2 \langle \varphi_{1,k}, \varphi_{2,k} \rangle_{L^2(\omega)} \eta(\lambda_{1,k} + \lambda_{2,k}) & b_2^2 \|\varphi_{2,k}\|^2_{L^2(\omega)} \eta(2\lambda_{2,k}) \end{pmatrix}.$$

The following computations are closely related to [3, Sec. 2.2]. Inequality (51) is equivalent to

$$\inf_{k\geq 1} \sigma_k := \inf_{k \geq 1} \inf_{x \neq 0} \frac{(C_k x, x)}{\|x\|^2} \geq C > 0,$$

where C is a positive constant. Clearly σ_k is the smallest eigenvalue of C_k. As $C_k \in \mathcal{L}(\mathbb{R}^2)$, in particular,

$$\frac{\det C_k}{Tr C_k} \leq \sigma_k \leq \frac{2 \det C_k}{Tr C_k}, \quad \forall k \geq 1,$$

where $Tr(C_k)$ denotes the trace of matrix C_k. The objective is to prove that if $T < T_0(q)$ and for a suitable sequence $\{k(n)\}_{n \in \mathbb{N}}$, one has

$$\lim_{n \to \infty} \frac{\det C_{k(n)}}{Tr C_{k(n)}} = 0, \tag{52}$$

in order to contradict (51) and then deduce that the system (6) is not null controllable at time T. To this end, we will study the asymptotic behavior of $\frac{\det C_k}{Tr C_k}$ which depends on the spectrum of $(L, D(L))$ (see Proposition 2.1). We have

$$\begin{aligned} \det C_k = {b_1}^2 \, {b_2}^2 \Big(& \|\varphi_{1,k}\|^2_{L^2(\omega)} \|\varphi_{2,k}\|^2_{L^2(\omega)} \eta(2\lambda_{1,k}) \eta(2\lambda_{2,k}) \\ & - \langle \varphi_{1,k}, \varphi_{2,k} \rangle^2_{L^2(\omega)} \eta(\lambda_{1,k} + \lambda_{2,k})^2 \Big), \end{aligned} \tag{53}$$

and

$$Tr C_k = {b_1}^2 ||\varphi_{1,k}\|^2_{L^2(\omega)} \eta(2\lambda_{1,k}) + {b_2}^2 \|\varphi_{2,k}\|^2_{L^2(\omega)} \eta(2\lambda_{2,k}). \tag{54}$$

Recall that, under assumption (9), one has $b_1 b_2 \neq 0$ and $\mu_1 \neq \mu_2$. Let us consider the function

$$g(s) := \ln(\eta(s)) = \ln \frac{e^{sT}-1}{s}.$$

Appliying a Taylor-formula for any $s_1, s_2 > 0$, with $s_1 \leq s_2$, one gets

$$g(2s_1) = g(s_1+s_2) + (s_1 - s_2) g'(s_1+s_2) + \frac{(s_1-s_2)^2}{2} g''(\overline{s}_1),$$

$$g(2s_2) = g(s_1+s_2) - (s_1 - s_2) g'(s_1+s_2) + \frac{(s_1-s_2)^2}{2} g''(\overline{s}_2),$$

where $\overline{s}_1$, $\overline{s}_2 \in [2s_1, 2s_2]$. So,

$$g(2s_1) + g(2s_2) = 2g(s_1+s_2) + \frac{(s_1-s_2)^2}{2}(g''(\overline{s}_1) + g''(\overline{s}_2)),$$

but

$$\inf_{s\in[2s_1,2s_2]} g''(s) \le \frac{g''(\overline{s}_1) + g''(\overline{s}_2)}{2} \le \sup_{s\in[2s_1,2s_2]} g''(s).$$

Since g'' is continuous, there exists $\overline{s} \in [2s_1, 2s_2]$, such that $\frac{g''(\overline{s}_1)+g''(\overline{s}_2)}{2} = g''(\overline{s})$. We deduce that

$$\eta(2s_1)\eta(2s_2) = \eta(s_1+s_2)^2 e^{(s_1-s_2)^2(\ln(\eta(\overline{s}))'')}, \quad \forall s_1, s_2 > 0, \quad \overline{s} \in [2s_1, 2s_2].$$

Let us denote by $\zeta(s) = \ln(\eta(s))''$ for all $s > 0$. Then

$$\zeta(s) = \frac{1}{s^2} - \frac{T^2}{(e^{sT/2} - e^{-sT/2})^2} \quad \text{and} \quad \frac{T^2}{T^2s^2+24} \le \zeta(s) \le \frac{1}{s^2}. \tag{55}$$

Indeed, observe that

$$e^{Ts/2} - e^{-Ts/2} = Ts + \frac{T^3s^3}{24} + \frac{T^5s^5}{5!\,32} + ... \ge Ts + \frac{T^3s^3}{24} = \frac{Ts}{24}(24 + T^2s^2),$$

then

$$\begin{aligned}\zeta(s) &\ge \frac{1}{s^2} - \frac{24^2}{s^2(24+T^2s^2)^2} = \frac{T^4s^4 + 48\,T^2s^2}{s^2(24+T2s^2)^2} \ge \frac{T^4s^4 + 24\,T^2s^2}{s^2(24+T^2s^2)^2}\\ &= \frac{T^2}{T^2s^2+24}, \quad \forall s > 0.\end{aligned}$$

In particular,

$$\eta(2\lambda_{1,k})\,\eta(2\lambda_{2,k}) = \eta(\lambda_{1,k}+\lambda_{2,k})^2\, e^{(\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k)}, \quad \forall k \ge 1, \tag{56}$$

with

$$\begin{aligned}&\lambda'_k \in [\lambda_{1,k}, \lambda_{2,k}] \text{ and } \frac{T^2}{T^2\lambda_{2,k}^2 + 24} \le \zeta(\lambda'_k) \le \frac{1}{\lambda_{1,k}^2}, \forall k \ge 1, \text{ if } \lambda_{1,k} < \lambda_{2,k},\\ &\lambda'_k \in [\lambda_{2,k}, \lambda_{1,k}] \text{ and } \frac{T^2}{T^2\lambda_{1,k}^2 + 24} \le \zeta(\lambda'_k) \le \frac{1}{\lambda_{2,k}^2}, \forall k \ge 1, \text{ if } \lambda_{2,k} < \lambda_{1,k}.\end{aligned}$$

Coming back to (53) and using the previous expressions, obtain that

$$\begin{aligned}\det C_k = b_1^2\, b_2^2 \eta(2\lambda_{1,k})\,\eta(2\lambda_{2,k}) \Big(&\|\varphi_{1,k}\|^2_{L^2(\omega)}\|\varphi_{2,k}\|^2_{L^2(\omega)}\\ &-\langle\varphi_{1,k},\varphi_{2,k}\rangle^2_{L^2(\omega)} e^{-(\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k)}\Big).\end{aligned}$$

Since $\lim\limits_{k\to\infty}(\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k) = 0$ (see (18) and (55)), let us write

$$e^{-(\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k)} = 1 - (\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k) - \frac{1}{2}(\lambda_{1,k}-\lambda_{2,k})^2\mathbf{o}\left((\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k)\right).$$

Thus from the expression of $\det C_k$ and TrC_k (see (54)), we deduce

$$\frac{\det C_k}{TrC_k} = N_k\, M_k, \quad \forall k \geq 1, \tag{57}$$

where

$$N_k := \frac{{b_1}^2 {b_2}^2 \eta(2\lambda_{1,k})}{{b_1}^2 ||\varphi_{1,k}\|^2_{L^2(\omega)} \eta(2\lambda_{1,k}) + {b_2}^2 \|\varphi_{2,k}\|^2_{L^2(\omega)} \eta(2\lambda_{2,k})}, \tag{58}$$

and

$$\begin{aligned} M_k := \eta(2\lambda_{2,k}) \Big(&\det G_k + (\lambda_{1,k} - \lambda_{2,k})^2 < \varphi_{1,k}, \varphi_{2,k} >^2_{L^2(\omega)} \zeta(\lambda'_k) \Big[1 \\ &+ \frac{1}{2} \mathbf{o} \left((\lambda_{1,k} - \lambda_{2,k})^2 \zeta(\lambda'_k) \right) \Big] \Big). \end{aligned} \tag{59}$$

In the previous expression G_k is the Gram matrix of $(1_\omega B^* \phi_{1,k}, 1_\omega B^* \phi_{2,k})$ in $L^2(0,1;\mathbb{R}^2)$, i.e.,

$$G_k := \begin{pmatrix} \|\varphi_{1,k}\|^2_{L^2(\omega)} & \langle \varphi_{1,k} \varphi_{2,k} \rangle_{L^2(\omega)} \\ \langle \varphi_{1,k}, \varphi_{2,k} \rangle_{L^2(\omega)} & \|\varphi_{2,k}\|^2_{L^2(\omega)} \end{pmatrix}, \quad \forall k \geq 1.$$

Let us now study the behavior of $\det G_k$ for k large enough. We assume that $\operatorname{Supp} q \subset (b, 1)$, the case $\operatorname{Supp} q \subset (0, a)$ can be treated in the same way. Then (see (20), (32) and (33))

$$\varphi_{i,k}(x) = \alpha_{i,k} \sin \sqrt{\lambda_{i,k}} x, \quad \forall x \in \omega,$$

with

$$\alpha_{i,k} = \frac{\sqrt{2}}{\sqrt{1 + \mathcal{O}_i(\frac{1}{k})}} = \sqrt{2} + \mathcal{O}_i(\frac{1}{k}), \quad i = 1, 2.$$

Let us denote by $R_k := \sqrt{\lambda_{1,k}} - \sqrt{\lambda_{2,k}}$. Then, for any $x \in \omega$, one has

$$\begin{aligned} \varphi_{1,k}(x) &= \alpha_{1,k} \sin((\sqrt{\lambda_{2,k}} + R_k)x) = \alpha_{1,k} \sin(\sqrt{\lambda_{2,k}} x) + R_k f_k(x) \\ &= \frac{\alpha_{1,k}}{\alpha_{2,k}} \varphi_{2,k}(x) + R_k f_k(x), \quad \forall k \geq 1, \quad x \in \omega, \end{aligned}$$

where f_k is given by

$$f_k(x) = \alpha_{1,k} \frac{\sin((\sqrt{\lambda_{2,k}} + R_k)x) - \sin \sqrt{\lambda_{2,k}}\, x}{R_k}, \quad \forall x \in \omega, \quad \forall k \geq 1.$$

Observe that f_k can be written as

$$f_k(x) = \alpha_{1,k}\, x\, \cos(\Theta_{k,x}\, x) \quad \forall x \in \omega,$$

for $\Theta_{k,x} \in \mathbb{R}$. In particular, for a positive constant C, one gets

$$|f_k(x)| \leq C, \quad \forall x \in \omega, \quad \forall k \geq 1.$$

Therefore

$$||\varphi_{2,k}||^2_{L^2(\omega)}||\varphi_{1,k}||^2_{L^2(\omega)} = \frac{\alpha^2_{1,k}}{\alpha^2_{2,k}}||\varphi_{2,k}||^4_{L^2(\omega)} + R^2_k||\varphi_{2,k}||^2_{L^2(\omega)}\|f_k\|^2_{L^2(\omega)} + 2\frac{\alpha_{1,k}}{\alpha_{2,k}}R_k\langle f_k, \varphi_{2,k}\rangle_{L^2(\omega)}||\varphi_{2,k}||^2_{L^2(\omega)}$$

and

$$\langle \varphi_{2,k}, \varphi_{1,k}\rangle^2_{L^2(\omega)} = \frac{\alpha^2_{1,k}}{\alpha^2_{2,k}}||\varphi_{2,k}||^4 + R^2_k\,\langle f_k, \varphi_{2,k}\rangle^2_{L^2(\omega)} + 2\frac{\alpha_{1,k}}{\alpha_{2,k}}R_k\,||\varphi_{2,k}||^2\langle f_k, \varphi_{2,k}\rangle_{L^2(\omega)}.$$

We can conclude

$$\det G_k = R^2_k\, I_k, \tag{60}$$

where

$$I_k = ||\varphi_{2,k}||^2_{L^2(\omega)}||f_k||^2_{L^2(\omega)} - \langle f_k, \varphi_{2,k}\rangle^2_{L^2(\omega)}$$

satisfies $|I_k| \leq C$ for any $k \geq 1$, with C a positive constant. In particular, we deduce

$$\det G_k = \mathcal{O}(R^2_k) = \mathcal{O}\left(\frac{(\lambda_{1,k}-\lambda_{2,k})^2}{(\sqrt{\lambda_{1,k}}+\sqrt{\lambda_{2,k}})^2}\right), \quad \text{as } k \to \infty. \tag{61}$$

Coming back to the expressions of N_k and M_k given by (58) and (59), we observe that

$$\lim_{k\to\infty}\frac{\eta(2\lambda_{1,k})}{\eta(2\lambda_{2,k})} = 1, \tag{62}$$

(η given in (50)). We deduce that

$$\begin{cases} \lim\limits_{k\to\infty} N_k &= \lim\limits_{k\to\infty} \frac{b_1^2\, b_2^2\, \eta(2\lambda_{1,k})}{{b_1}^2||\varphi_{1,k}||^2_{L^2(\omega)}\eta(2\lambda_{1,k}) + {b_2}^2\|\varphi_{2,k}\|^2_{L^2(\omega)}\eta(2\lambda_{2,k})} \\ &= \frac{b_1^2\, b_2^2}{2(b-a)(b_1^2+b_2^2)}. \end{cases} \tag{63}$$

On the other hand, by (61), one has

$$M_k = \eta(2\lambda_{2,k})(\lambda_{1,k}-\lambda_{2,k})^2\left[\mathcal{O}\left(\frac{1}{(\sqrt{\lambda_{1,k}}+\sqrt{\lambda_{2,k}})^2}\right) + \langle\varphi_{1,k}, \varphi_{2,k}\rangle_{L^2(\omega)}\zeta(\lambda'_k)(1 + \mathbf{o}\left((\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda'_k)\right))\right].$$

Observe that

$$\eta(2\lambda_{2,k})(\lambda_{1,k}-\lambda_{2,k})^2 = \frac{1-e^{-2\lambda_{2,k}T}}{2\lambda_{2,k}}\, e^{2\lambda_{2,k}(T+\frac{\ln|\lambda_{1,k}-\lambda_{2,k}|}{\lambda_{2,k}})}.$$

Recall we have assumed that $T \in (0, T_0(q))$. In particular $T_0(q) > 0$. From the definition of $c(\Lambda)$ and Lemma 4.1, there exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ such that

$$T_0(q) = c(\Lambda) = \lim_{n\to+\infty} -\frac{\ln|\lambda_{1,k_n}-\lambda_{2,k_n}|}{\lambda_{2,k_n}} \in [0, +\infty].$$

We are going to assume $T_0(q) < +\infty$. The case $c(\Lambda) = +\infty$ is obvious and we also get a contradiction. Then, for any $\varepsilon > 0$, there exists k_1 such that

$$T + \frac{\ln|\lambda_{1,k_n} - \lambda_{2,k_n}|}{\lambda_{2,k_n}} < \varepsilon + T - T_0(q), \quad \forall n \geq k_1. \tag{64}$$

Choosing $\varepsilon = \frac{T-T_0(q)}{2}$, we deduce that

$$e^{2\lambda_{2,k_n}(T + \frac{\ln|\lambda_{1,k_n} - \lambda_{2,k_n}|}{\lambda_{2,k_n}})} \leq e^{2\lambda_{2,k_n}(\frac{T-T_0(q)}{2})}.$$

Finally, since $T < T_0(q)$, one has

$$\lim_{n\to\infty} M_{k_n} = 0.$$

This limit together with (63) prove (52). Thus, system (6) is not null controllable at time T. This proves the negative result in item 2 of Theorem 1.1 for system (6).

5.2 Negative null controllability result for system (7)

Let us now prove the negative null controllability result for the system (7). Assume that $T_0(q) > 0$ and let $0 < T < T_0(q)$. By contradiction, we will prove that system (7) is not null controllable at time T.

Using Proposition 2.3 again, system (7) is null controllable at time T if and only if there exists $C > 0$ such that the observability inequality (27) holds for any solution ψ of the adjoint system (23). Let us work with intial data $\psi_{0,k} = a\,\phi_{1,k} + b\,\phi_{2,k}$, with $k \in \mathbb{N}^*$ and $(a,b) \in \mathbb{R}^2$. Then, the associated solution of the adjoint system (23) is given by (48) and we deduce that the observability inequality (27) becomes

$$A_{1,k} \leq C A_{2,k}, \quad \forall k \geq 1, \quad \forall (a,b) \in \mathbb{R}^2,$$

with

$$A_{1,k} := ||\psi(0,\cdot)||^2_{H_0^1(0,1,\mathbb{R}^2)} = a^2\, c_{1,k}\, e^{-2\lambda_{1,k}(T)} + b^2\, c_{2,k}\, e^{-2\lambda_{2,k}(T)},$$

where (see (22))

$$c_{i,k} := 1 + ||\varphi'_{i,k}||^2_{L^2}, \quad \forall k \geq 1, \quad i = 1,2$$

and

$$\begin{aligned} A_{2,k} &:= \int_0^T |B^* \partial_x \psi(t,0)|^2\, dt \\ &= \int_0^T \left| b_1\, a\, \varphi'_{1,k}(0)\, e^{-\lambda_{1,k}(T-t)} + b_2\, b\, \varphi'_{2,k}(0)\, e^{-\lambda_{2,k}(T-t)} \right|^2\, dt. \end{aligned}$$

In terms of quadratic forms, the previous inequality is equivalent to

$$e^{-2\,L_k T} \leq C \int_0^T e^{-\,L_k(T-t)}\, \tilde{B}_k e^{-\,L_k(T-t)}\, dt, \ \forall k \geq 1, \quad \forall k \geq 1, \tag{65}$$

where $L_k = diag(\lambda_{1,k}, \lambda_{2,k})$ and

$$\tilde{B}_k := \begin{pmatrix} \frac{b_1^2}{c_{1,k}} \varphi'_{1,k}(0)^2 & \frac{b_1 b_2}{\sqrt{c_{1,k}\, c_{2,k}}} \varphi'_{1,k}(0)\varphi'_{2,k}(0) \\ \frac{b_1 b_2}{\sqrt{c_{1,k}\, c_{2,k}}} \varphi'_{1,k}(0)\varphi'_{2,k}(0) & \frac{b_2^2}{c_{2,k}} \varphi'_{2,k}(0)^2 \end{pmatrix}, \ \forall k \geq 1.$$

Computing the integral at the right-hand side of (65), we deduce that it can be written in the following form:

$$I \leq C\, H_k, \quad \forall k \geq 1, \tag{66}$$

where

$$H_k := \begin{pmatrix} \frac{b_1^2}{c_{1,k}} \varphi'_{1,k}(0)^2 \eta(2\lambda_{1,k}) & \frac{b_1 b_2}{\sqrt{c_{1,k} c_{2,k}}} \varphi'_{1,k}(0)\varphi'_{2,k}(0)\eta(\lambda_{1,k}+\lambda_{2,k}) \\ \frac{b_1 b_2}{\sqrt{c_{1,k} c_{2,k}}} \varphi'_{1,k}(0)\varphi'_{2,k}(0)\eta(\lambda_{1,k}+\lambda_{2,k}) & \frac{b_2^2}{c_{2,k}} \varphi'_{2,k}(0)^2 \eta(2\lambda_{2,k}) \end{pmatrix},$$

with η defined by (50). Let $\tilde{\sigma}_k$ be the smallest eigenvalue of H_k, then

$$\frac{2 \det H_k}{Tr H_k} \geq \tilde{\sigma}_k \geq \frac{\det H_k}{Tr H_k}, \quad \forall k \geq 1.$$

Let us analyse the behavior of $\frac{\det H_k}{Tr H_k}$. One has

$$\begin{aligned} \det H_k &= \frac{b_1^2 b_2^2}{c_{1,k} c_{2,k}} \varphi'_{1,k}(0)^2 \varphi'_{2,k}(0)^2 (\eta(2\lambda_{1,k})\eta(2\lambda_{2,k}) - \eta(\lambda_{1,k}+\lambda_{2,k})^2), \\ Tr H_k &= \frac{b_1^2}{c_{1,k}} \varphi'_{1,k}(0)^2 \eta(2\lambda_{1,k}) + \frac{b_2^2}{c_{2,k}} \varphi'_{2,k}(0)^2 \eta(2\lambda_{2,k}). \end{aligned}$$

Same computations as in the previous subsection (see (56)) give

$$\begin{aligned} \det H_k = \; & \frac{b_1^2 b_2^2}{c_{1,k} c_{2,k}} \varphi'_{1,k}(0)^2 \varphi'_{2,k}(0)^2 \eta(2\lambda_{1,k})\eta(2\lambda_{2,k})(\lambda_{1,k}-\lambda_{2,k})^2 \zeta(\lambda'_k) \\ & \times \left[1 + \frac{1}{2} o\left((\lambda_{1,k}-\lambda_{2,k})^2 \zeta(\lambda'_k) \right) \right], \end{aligned}$$

where $\lambda'_k \in [\lambda_{1,k}, \lambda_{2,k}]$ if $\lambda_{1,k} < \lambda_{2,k}$, or $\lambda'_k \in [\lambda_{2,k}, \lambda_{1,k}]$ if $\lambda_{2,k} < \lambda_{1,k}$. Then,

$$\begin{aligned} \frac{\det H_k}{Tr H_k} = \; & \frac{b_1^2 b_2^2 \varphi'_{1,k}(0)^2 \varphi'_{2,k}(0)^2 \eta(2\lambda_{1,k})\eta(2\lambda_{2,k})}{c_{1,k} c_{2,k} (\frac{b_1^2}{c_{1,k}} \varphi'_{1,k}(0)^2 \eta(2\lambda_{1,k}) + \frac{b_2^2}{c_{2,k}} \varphi'_{2,k}(0)^2 \eta(2\lambda_{2,k}))} \\ & \times (\lambda_{1,k}-\lambda_{2,k})^2 \zeta(\lambda'_k) \left[1 + \frac{1}{2} o\left((\lambda_{1,k}-\lambda_{2,k})^2 \zeta(\lambda'_k) \right) \right]. \end{aligned} \tag{67}$$

By (47), one has

$$\lim_{k \to \infty} \frac{\varphi'_{i,k}(0)^2}{c_{i,k}} = 2, \quad i = 1, 2$$

and this limit together with (62) gives

$$\lim_{k \to \infty} \frac{b_1^2 b_2^2 \varphi'_{1,k}(0)^2 \varphi'_{2,k}(0)^2 \eta(2\lambda_{1,k})}{c_{1,k} c_{2,k} (\frac{b_1^2}{c_{1,k}} \varphi'_{1,k}(0)^2 \eta(2\lambda_{1,k}) + \frac{b_2^2}{c_{2,k}} \varphi'_{2,k}(0)^2 \eta(2\lambda_{2,k}))} = 2 \frac{b_1^2 b_2^2}{b_1^2 + b_2^2}.$$

Observe that

$$\eta(2\lambda_{2,k})(\lambda_{1,k}-\lambda_{2,k})^2\zeta(\lambda_k') = e^{2\lambda_{2,k}(T+\frac{\ln|\lambda_{1,k}-\lambda_{2,k}|}{\lambda_{2,k}})}\zeta(\lambda_k')\frac{1-e^{-2\lambda_{2,k}T}}{2\lambda_{2,k}}.$$

From the definition of $T_0(q)$, there exists a subsequence $\{k_n\}_{n\in\mathbb{N}^*}$ which satisfies (64). If we take $\varepsilon = \frac{T-T_0(q)}{2}$ and $T < T_0(q)$ in (67), we obtain

$$\lim_{n\to\infty}\frac{\det H_{k_n}}{TrH_{k_n}} = 0,$$

which gives a contradiction with (66). This ends the proof of Theorem 1.1.

6 A complementary result

This section is devoted to giving a complementary result on the minimal time $T_0(q) \in [0,+\infty]$ associated to the null controllability of the system (7). Let us fix $\mu_1 = 0$, $\mu_2 = 1$ and consider the application $q \in L^2(0,1) \longmapsto T_0(q) \in [0,+\infty]$. We are going to prove that this application is onto, that is to say that one can expect any minimal time. We first recall a result related to the inverse Sturm-Liouville problem.

It is well known that, for all $q \in L^2(0,1)$ the Dirichlet problem

$$\begin{cases} -u'' + q\,u = \kappa\,u & \text{in } (0,1), \\ u(0) = 0, \;\; u(0) = 0 \end{cases} \tag{68}$$

has a sequence $\kappa_k = \kappa_k(q)$ of simple eigenvalues, with $k \geq 1$, such that

$$\kappa_1 < \kappa_2 < ... < \kappa_k < ..., \quad \text{with } \lim_{k\to\infty}\kappa_k = +\infty.$$

We denote by $g_k(x) = g_k(x,\kappa_k,q)$, the corresponding normalized eigenfunctions in $L^2(0,1)$. Given $q \in L^2(0,1)$, the direct Dirichlet problem is to determine the eigenvalues $\{\kappa_k\}_{k\in\mathbb{N}^*}$ and the corresponding eigenfunction $u \neq 0$ of (68). Observe that the corresponding inverse problem is the following one: Given the sequence $\{\alpha_k\}_{k\geq 1} \subset (0,+\infty)$, we want to determine $q \in L^2(0,1)$ such that the sequence of eigenvalues of (68) is

$$\kappa_k = \alpha_k, \quad \forall k \geq 1.$$

The following result, due to P. Trubowitz (see [16]), provides a positive answer to the previous inverse problem with a particular class of eigenvalues:

Theorem 6.1. *The increasing sequence $\{\kappa_k\}_{k\geq 1}$, is the Dirichlet spectrum of problem* (68) *for some $q \in L^2(0,1)$ if and only if, for a constant C, one has*

$$\kappa_k = \pi^2 k^2 + C + r_k, \quad \textit{with } \sum_{k\geq 1} r_k^2 < +\infty. \tag{69}$$

The following result is related to the minimal time of boundary null controllability $T_0(q)$ of the system (7). One has:

Theorem 6.2. *For any $\tau \in [0,+\infty]$, there exists $\mu_1, \mu_2 \in \mathbb{R}$ and $q \in L^2(0,1)$, such that the minimal time $T_0(q)$ associated to the system (7) is $T_0(q) = \tau$.*

Proof. Let us fix $\tau \in [0,+\infty]$ and take $\mu_1 = 0$ and $\mu_2 = 1$ in (7). Let us consider $\gamma = \{\gamma_k\}_{k\in\mathbb{N}^*} \subset \ell^2$, given by

$$\gamma_k = \begin{cases} e^{-\frac{1}{k}} & \text{if } \tau = 0, \\ e^{-\tau\pi^2k^2} & \text{if } \tau \in (0,+\infty), \\ e^{-\tau k^3} & \text{if } \tau = +\infty, \end{cases}$$

for all $k \in \mathbb{N}^*$. Clearly the sequence $\lambda_{2,k} = \pi^2k^2 + \gamma_k$, $k \in \mathbb{N}$, satisfies (69) for $C = 0$. Applying Theorem 6.1, we deduce the existence of $q(\gamma) \in L^2(0,1)$ associated to the Dirichlet problem (68) with $\kappa_k = \lambda_{2,k}$, for any $k \geq 1$. On the other hand, let us introduce the following boundary control problem

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 & \text{in } Q_T := (0,T)\times(0,1), \\ \partial_t y_2 - \Delta y_2 + q(\gamma)\, y_2 = 0 & \text{in } Q_T, \\ y_1(\cdot,0) = b_1 v(t),\ y_2(\cdot,0) = b_2 v(t), & \text{on } (0,T), \\ y_1(\cdot,1) = y_2(\cdot,1) = 0, & \text{on } (0,T), \\ y_1(T,\cdot) = y_{1,0},\ \ y_2(T,\cdot) = y_{2,0} & \text{in } (0,1), \end{cases}$$

where $v \in L^2(0,T)$ is the control force. In this case, $\lambda_{1,k} = \pi^2k^2$, $\lambda_{2,k} = \pi^2k^2 + \gamma_k$, $k \geq 1$, and

$$T_0(q) = \limsup_{k\to+\infty} -\frac{\ln|\pi^2k^2 - \lambda_{2,k}|}{\pi^2k^2} = \limsup_{k\to+\infty} -\frac{\ln|\gamma_k|}{\pi^2k^2} = \tau.$$

This ends the proof. ∎

7 Comments, further result and open problems

1. In this work, we proved a necessary and sufficient condition of approximate and null controllability for system (6) with distributed controls under the geometrical assumption (10). It would be interesting to prove an analogous result to Theorem 1.1 without this geometrical assumption for the function q, i.e., when
$$Supp\, q \subset [0,a] \cup [b,1].$$

2. The null controllability results obtained here for systems (6) and (7) remain valid for $q \in L^\infty(0,1)$. Assumption $q \in L^2(0,1)$ is used in section 6, when, for a given $\tau \in [0,+\infty]$ we provide a potential $q \in L^2(0,1)$, for which the minimal time of null controllability of system (7) is equal to τ. This result is obtained using the inverse spectral theory (see Theorem 6.1). We can also apply the inverse spectral theory for the distributed system, but Theorem 6.1 does not give information on the localization of the support of $q \in L^2(0,1)$.

3. The methods used here for studying the controllability of the system (7), either for the positive result or negative one, require a careful study of the spectrum of the Strum-Liouville operator (see [14], [16]). Thanks to Proposition 2.1, the null controllability result of system (7) can be generalized if we consider the following problem:

$$\begin{cases} \partial_t y + Ly = 0 & \text{in} \quad Q_T, \\ y(\cdot,0) = B\,v(t), \;\; y(\cdot,1) = 0 & \text{on} \;\; (0,T), \\ y(0,\cdot) = y_0 & \text{in} \;\; (0,1), \end{cases}$$

where the operator $(L, D(L))$ and $B \in \mathbb{R}^2$ are respectively given by:

$$L := \begin{pmatrix} -\Delta + q_1 & 0 \\ 0 & -\Delta + q_2 \end{pmatrix}, \; D(L) = H^2(0,1;\mathbb{R}^2) \cap H_0^1(0,1;\mathbb{R}^2),$$

with $q_1, q_2 \in L^2(0,1)$ and $B = (b_1, b_2)^T$.

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