INVESTIGATION OF THE EFFECTS OF MASS AND LENGTH ON THE PERIOD OF A PENDULUM

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ABSTRACT

In this investigation, the effects of the dimensions of the bob and the length of the pendulum on the period of the pendulum were analyzed. The angular position data was regressed to find the coefficients for the differential equation describing the motion of the bob. The error incurred by small angle approximation was evaluated in order determine the largest acceptable angle to be 14°. The period was found through regression to be $T(\theta, L) = 2\sqrt{L} + (0.04 \pm .01)\theta$. Finally, the damping coefficient of the pendulum was determined to be (0.0007±.0001) kgs^{-1} for the 99.7g bob, (0.00076±.00001) kgs^{-1} for the 279.1g bob, and (0.00021±.00002) kgs^{-1} for the 20.0g bob.

I INTRODUCTION

Pendulums are used in several devices ranging from old clocks to metronomes to earthquake seismographs. As such, it is important to investigate the effects of the dimensions and mass of the bob on the motion of the pendulum. The effect of air resistance is often overlooked but may also have a non-trivial effect on the pendulum. Therefore, this effect was also examined in this experiment. In this paper, the collection and analysis of pendulum data presents findings on the correlation between mass, length, and initial angle of the pendulum and its effect on the period, as well as the effect of the damping coefficient on the pendulum's motion.

II THEORY

The theory for this experiment consists mainly of three equations.

The period of a simple pendulum is given by:

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad [1]$$

(HRW, 2010)

Where:

T = The period of the pendulum (s)
L = The length of the pendulum (m)

The second-order differential equation describing the motion of the pendulum is in the following form:

$$\ddot{\theta} + \frac{\zeta}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad [2]$$

(HRW, 2010)

Where:

θ = Angular displacement (rad)
ζ = Damping coefficient (kgs^{-1})
m = Mass of bob (kg)

By using a small angle approximation, as well as noting the fact that ζ^2 is insignificant compared to $\frac{g}{L}$, [2] can be solved to yield the following (non-trivial drag) solution:

$$\theta = \theta_{max} e^{\frac{-\zeta t}{2m}} \sin\left(\sqrt{\frac{g}{L}}t + \phi\right) \qquad [3]$$

(HRW, 2010)

Where:

θ_{max} = Maximum angular displacement (rad)
t = Time (s)
ϕ = Phase shift (rad)

III METHOD

Three bobs were examined with masses 99.7 g, 279.1 g, and 20.0 g. They were labelled Bob 1, Bob 2, and Bob 3 respectively. For each bob, three different lengths of the pendulum were considered: 42.75 cm, 40 cm, and 32.7 cm for Bob 1, 44.9 cm, 41.5 cm, and 33.9 cm for Bob 2, 37.74 cm, 34.72 cm, and 27.62 cm for Bob 3. These lengths were chosen by wrapping the string around the support some number of times on top of the previous trial. Five runs, each with a duration of five seconds, were effected for each length with initial angles of 4°, 8°, 12°, 16°, and 30°. The final angle of 30° was chosen to determine the effect of large angles on the period. A sixth run for each length was effected with an angle of 12° to determine the damping effects on the pendulum over the duration of one minute.

A phone was used to record video footage of each run. The bobs were later tracked using the Tracker Video Analysis and Modeling Tool to obtain the position of the pendulum as a function of time. To improve the precision of the tracker, the bobs were marked with a solid green line.

The horizontal displacement data was subtracted by half the range in order to set the equilibrium point of the pendulum as the fiducial. The horizontal displacement was then converted to angular displacement by using the property that $x = L sin\theta$, where x is the horizontal displacement. After substituting θ with equation [3], the data was regressed to observe the period of the oscillation as well as the behaviour of the pendulum over time.

IV DATA

Data concerning each bob as numbered from 1-3 is displayed in the following table.

Bob	Diameter (±.01 cm)	Height (±.01 cm)	Mass (±.1 g)
1	2.511±.007	2.453±.009	99.7±.1
2	2.512±.009	4.94±.01	279.1±.1
3	1.45±.01	1.540±.008	20.0±.1

Fig 1. Table of Measurements of the physical properties of the bobs used in the experiment. Bobs with varying masses and dimensions were assessed.

Data concerning the pendulum motion of each trial was recorded directly from either the Tracker Video Analysis Modeling Tool or Vernier Motion Detector to a Microsoft Excel file. A few trials conducted are shown in the graphs below.

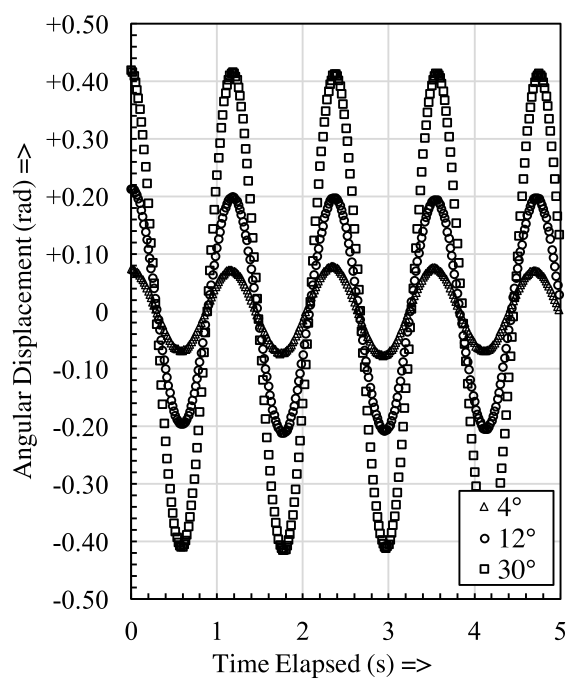

Fig 2. Angular Displacement vs. Time Graph of Pendulum with Mass 279.1g and Length 41.5cm. Each plot has a different initial angle. It can be observed that the periods are relatively the same.

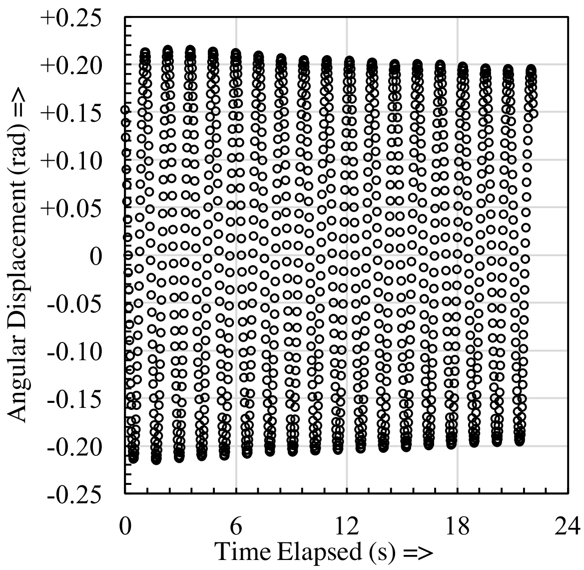

Fig 3. Angular Displacement vs. Time Graph of Pendulum with Mass 20.0g and Length 37.74cm and Initial Angle of 12°. The damping of the oscillation of the pendulum is visible after 20 seconds.

V Analysis

A small angle approximation was done to simplify [2] into a solvable differential equation to model the motion of the bob. This can be done because $\sin\theta \approx \theta$ when θ is small. In formal terms, the error of $\sin\theta$ is given by:

$$\varepsilon = \left|\frac{\theta}{\sin\theta} - 1\right| \qquad [4]$$

For the purpose of this lab, it was decided that an error of less than 1% in the period of the pendulum would be acceptable. Using a CAS on [4], it was determined that the error would be acceptable as long as θ is less than around 14°.

To determine the effects of small angle approximation on a pendulum with large initial angle, the data from all runs were regressed in the form of [3]. The period can be determined from the coefficient and compared to the ideal period.

Angle	Period (s)	Ideal Period (s)	Residue (%)
4°	1.3259±.0005	1.3265±.0003	0.0452
8°	1.3285±.0005	1.3265±.0003	0.151
12°	1.3297±.0003	1.3265±.0003	0.241
16°	1.3339±.0004	1.3265±.0003	0.558
30°	1.3478±.0004	1.3265±.0003	1.61

Fig 4. Initial Angular Displacements and Residuals of Pendulum of Mass 99.7g and Length 42.75cm. Period was obtained from the coefficients by regression. Ideal Period was calculated with real constants. It can be observed that as initial angle increases, residue increases.

It can be observed from the table that residue increases as the initial angle increases. The large angle of 30° produces a residue greater than 1%, while the angle of 16° produces a residue of greater than 0.5%. Thus, 16° still produces a residue within the acceptable range and can be considered acceptable for small angle approximations.

To obtain an expression for the period, the ideal periods were subtracted from the measured periods to obtain the perturbation, and then compared to the length and initial angle of the run.

When the perturbation of the period was compared with the initial angle, the regression chosen was linear. Taking the average of all regressions yields a coefficient of (0.04±.01) s rad^{-1}, which describes a linear increase in perturbation.

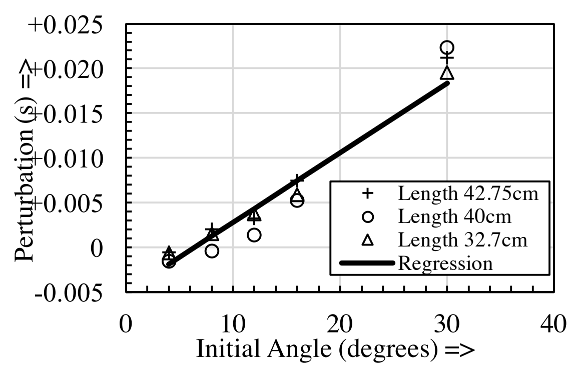

Fig 5. Perturbation vs. Initial Angle for Pendulum with Mass 99.7g. Each plot indicates a different pendulum length. The points were regressed linearly.

The length of the pendulum was compared with the perturbation. It can be observed that length is independent of perturbation.

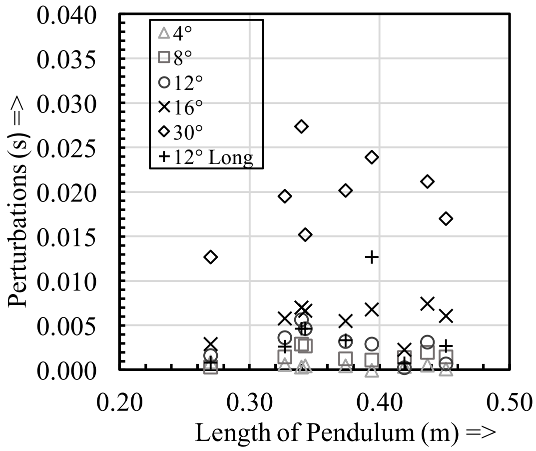

Fig 6. Perturbation vs. Length of Pendulum. Each plot indicates a different initial angle. There is no correlation.

By combining all variables and approximating the square root of g as π for pendulums on Earth, the following expression describing the period of a pendulum was obtained:

$$T(\theta, L) = 2\sqrt{L} + (0.04 \pm .01)\theta \qquad [5]$$

As the damping coefficient is the result of air resistance, by substituting the drag equation into [2] and solving, it can be found that:

$$\zeta = c_D \rho A \qquad [6]$$

By substituting in values obtained from the work by *Halliday and Resnick (2010)* the damping coefficient can be predicted.

Taking the extended runs, the damping coefficient can be derived for each mass.

Bob	Damping Coefficient (kg s^{-1})	Prediction (kg s^{-1})
1	0.0007±.0001	0.00091±.00002
2	0.00076±.00001	0.00184±.00002
3	0.00021±.00002	0.00033±.00002

Fig 7. Damping Coefficient and Prediction for each Bob mass. By comparing the regressed coefficients to [3], the damping coefficient for each long run was found. For each bob, the damping coefficients for each run were averaged and their standard deviation found.

VI Sources of Error

An aspect of the experiment that could be improved is utilizing a more precise way of releasing the mass at the start of the run. The bob was released from one lab member's hand, which caused undesired wobbling and rotation. These factors were especially noticeable in the runs with smaller initial angular displacement and made tracking the bob much more difficult.

It was noticed that due to slack in the string, the third bob did not hang directly downward from the support stand. This resulted in a slightly asymmetric oscillation. This issue could be mitigated by using a slightly heavier bob instead of the third bob used originally.

VII Conclusion

The analysis of the pendulum's motion yielded results on the residuals of large angle pendulums, the period of a pendulum, and the coefficient of drag. The residuals of the period exceed 1% after an initial angle of around 16°. The period of a pendulum can be described with the equation $T(\theta, L) = 2\sqrt{L} + (0.04 \pm .01)\theta$. Finally, the damping coefficient is (0.0007±.0001) kgs^{-1} for the 99.7g bob, (0.04±.02) kgs^{-1} for the 279.1g bob, and (0.00021±.00002) kgs^{-1} for the 20.0g bob.

VIII Sources

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