Arbitrary-order conservative and consistent remapping and a theory of linear maps, part 1

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3. (Monotone)

   $\mathbf{R}$ monotone $\iff \forall\ \boldsymbol{\psi}^s, \forall\ i \in [1, \ldots, f^t], \min \boldsymbol{\psi}^s \leq \psi_i^t \leq \max \boldsymbol{\psi}^s.$

Geometric locality: let Ω^t and Ω^s denote the geometric regions associated with degrees of freedom $\mathscr{F}^t$ and $\mathscr{F}^s$, the *overlap region* associated with Ω^t and Ω^s is denoted by $\Omega_{ij}^{ov} = \Omega^t \bigcap \Omega^s$. If $\Omega_{ij}^{ov} \neq \varnothing$, then $\mathscr{F}_i^t$ and $\mathscr{F}_j^s$ are said to be local. Regions on the overlap mesh are associated with corresponding local weights J_{ij}^{ov}.

Calculation of the local weights J_{ij}^{ov} is performed via integration over the overlap region,

$$J_{ij}^{ov} = \int_{\Omega_{ij}^{ov}} C_j^s(\mathbf{x}) C_i^t(\mathbf{x})$$

where $C_j^s(\mathbf{x})$ and $C_i^t(\mathbf{x})$ are functions associated with the degrees of freedom $\mathscr{F}_j^s$ and $\mathscr{F}_i^t$ that satisfy

$$\int_{\Omega} C_j^s(\mathbf{x}) = J_j^s, \quad \int_{\Omega} C_i^t(\mathbf{x}) = J_i^t$$

Local Sub-map

Theorem (

Let $\hat{\mathbf{R}}^{(1)}, \ldots, \hat{\mathbf{R}}^{(N)}$ *be a complete set of linear sub-maps which are conservative in* $A^{(1)}, \ldots, A^{(N)}$. *Then the global linear map constructed via*

$$R_{ij} = \frac{1}{J_i^t} \sum_{k=1}^{N} \hat{R}_{ij}^{(k)} \left(\sum_{\ell \in A^{(k)}} J_{i\ell}^{ov} \right)$$

is conservative and consistent.

Remapping Finite Elements to Finite Volumes

This paper focuses specifically on nodal finite element methods over the set of Gauss-Lobatto-Legendre (GLL) nodes. The set of degrees of freedom on the sourcemesh are defined at the N_p^2 GLL nodes within the reference element with discontinuous basis functions.

(a) Geometric distribution of degrees of freedom in a fourth-order Gauss-Lobatto-Legendre finite element. (b) Degrees of freedom in the fourth-order Gauss-Lobatto-Legendre reference element, with coordinate axes $a \in [-1, 1]$ and $b \in [-1, 1]$.

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- $C_m(x)$

$$\psi_j^s(\mathbf{x}) = \sum_{m=0}^{N_p-1} \sum_{n=0}^{N_p-1} (\psi_j^s)_{(m,n)} C_m(\alpha(\mathbf{x})) C_n(\beta(\mathbf{x}))$$

N_p

$P_{N_p-1}(x)$ P_{N_p-1} x_j $\frac{dP_{N_p-1}(x)}{dx}$ $N_p - 2$ ± 1:

$$C_m(x) \equiv \frac{(x^2-1)}{N_p(N_p-1)P_{N_p-1}(x_j)(x-x_j)} \frac{dP_{N_p-1}(x)}{dx},$$

with corresponding weights

$$w_m \equiv \int_{-1}^{1} C_m(x)dx = \frac{2}{N_p(N_p-1)\left[P_{N_p-1}(x_j)\right]^2}.$$

Third- and fourth-order GLL basis functions used for the continuous reconstruction.

Building a "first guess" sub-map

"First guess" sub-map

Subdivision of a quadrilateral and pentagon into triangles.

First-, fourth- and eighth-order triangular quadrature nodes.

$$\int_{\mathscr{F}_{ij}^{ov}} \psi_j^s(\mathbf{x}) dA \approx \sum_{k=1}^{N_q} \psi_j^s(\mathbf{x}_k) \hat{w}_k J_{ij}^{ov}$$

Consistency and Conservation Enforcement

Minimize $\displaystyle\sum_{i=1}^{f^t}\sum_{j=1}^{f^s}\frac{1}{2}(\hat{R}_{ij}-\hat{R}^*_{ij})^2$ subject to conservation and consistency conditions.

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The least squares problem is solved directly via the Lagrangian. Vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ are defined with elements λ_i, $i \in [1,\ldots,f^t]$, and κ_j, $j \in [1,\ldots,\, f^s-1]$ respectively. The Lagrangian takes the form

$$\mathscr{L}(\mathbf{R},\boldsymbol{\lambda},\boldsymbol{\kappa}) = \sum_{i=1}^{f^t}\sum_{j=1}^{f^s}\frac{1}{2}(\hat{R}_{ij}-\hat{R}^*_{ij})^2 - \underbrace{\sum_{i=1}^{f^t}\lambda_i\left[\left(\sum_{j=1}^{f^s}\hat{R}_{ij}\right)-1\right]}_{\text{consistency}} - \underbrace{\sum_{j=1}^{f^s-1}\kappa_j\left[\left(\sum_{i=1}^{f^t}\hat{R}_{ij}J^t_i\right)-J^s_j\right]}_{\text{conservation}}.$$

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$$\Rrightarrow \begin{pmatrix} \mathbf{I} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{R}}_{ij} \\ \boldsymbol{\lambda} \\ \boldsymbol{\kappa} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{R}}^*_{ij} \\ -\mathbf{1} \\ -\mathbf{J}^s \end{pmatrix}$$

where $\mathbf{C}$ is the $(f^t+f^s-1)\times f^tf^s$ matrix defined by the derivatives $\partial\mathscr{L}/\partial\boldsymbol{\lambda}$ and $\partial\mathscr{L}/\partial\boldsymbol{\kappa}$.

Monotonicity Preservation

In order to impose monotonicity, the least squares problem can be augmented with an additional boundedness condition given by monotone property.

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Theorem (The linear combination of R_{ij})

If $\hat{R}_{ij}^{(1)}$ and $\hat{R}_{ij}^{(2)}$ are conservative and consistent linear sub-maps over $A^{(1)} = A^{(2)}$ and $B^{(1)} = B^{(2)}$ respectively, then for $\omega \in [0, 1]$, $\hat{R}_{ij} = \omega \hat{R}_{ij}^{(1)} + (1 - \omega)\hat{R}_{ij}^{(2)}$ is a consistent and conservative linear sub-map.

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$$\hat{R}_{ij} = \omega \hat{R}_{ij}^{0} + (1 - \omega)\hat{R}_{ij}^{s}$$

Finding a value of ω sufficiently large that $\hat{R}_{ij}$ has no negative entries.

$$\omega = \max_{i,j}\left[\max\left(\frac{-\hat{R}_{ij}^{s}}{|\hat{R}_{ij}^{0} - \hat{R}_{ij}^{s}|}, 0\right)\right].$$

Fig. Contour plots of the three test fields used in this study. Fields (a) and (b) take on values in the range [1,3]. Field (c) takes on values in the approximate range [0.46, 1.54].

$$\psi = 2 + \cos^2\theta\cos(2\lambda) \quad (Y_2^2)$$

$$\psi = 2 + \sin^{16}(2\theta)\cos(16\lambda) \quad (Y_{32}^{16})$$

$$\psi = 1 - \tanh\left[\frac{\rho'}{d}\sin(\lambda' - \omega' t)\right] \ (V_X)$$

The first two fields are used to test performance for a smooth well-resolved field and a high-frequency poorly resolved field with rapidly changing gradients. The third field is a dual stationary vortex.

A depiction of the four meshes studied in this manuscript: (a) Cubed-sphere(

a1. $L_1, L_2 L_\infty$

Cubed-sphere mesh to great circle latitude-longitude mesh (non-monotonic):

Standard L_1, L_2 and L_∞ error norms reported for conservative and consistent remapping of the three idealized fields from the cubed-sphere mesh to the 1° great circle latitude-longitude mesh for cubed-sphere resolutions $n_e = 15, 30, 60$ and for three orders of accuracy $N_p = 2, 3, 4$.

a2. $L_{min} L_{max}$

Standard L_{min} and L_{max} error norms reported for conservative and consistent remapping of the three idealized fields from the cubed-sphere mesh to the 1° great circle latitude-longitude mesh for cubed-sphere resolutions $n_e = 15, 30, 60$ and for three orders of accuracy $N_p = 2, 3, 4$. Circled data points indicate that the global minimum / maximum has been enhanced (i.e. that monotonicity was not maintained).

b1. $L_1, L_2 L_\infty$

Cubed-sphere mesh to great circle latitude-longitude mesh (monotonic):

Standard L_1, L_2 and L_∞ error norms reported for conservative, consistent and strictly monotonic remapping of the three idealized fields from the cubed-sphere mesh to the 1° great circle latitude-longitude mesh for cubed-sphere resolutions $n_e = 15, 30, 60$ and for three orders of accuracy $N_p = 2, 3, 4$.

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c1. $L_1, L_2 L_\infty$

Cubed-sphere mesh to geodesic mesh (non-monotonic):

Standard L_1, L_2 and L_∞ error norms reported for conservative and consistent remapping of the three idealized fields from the cubed-sphere mesh to the $N_i = 72$ geodesic mesh for cubed-sphere resolutions $n_e = 15, 30, 60$ and for three orders of accuracy $N_p = 2, 3, 4$.

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d. Real data test

Refined cubed-sphere mesh to great circle latitude-longitude mesh (real data)

Two 8220 8220 plusminustests for remapping from the variable resolution cubed-sphere mesh with $N_p = 4$ to 0.25° great circle latitude-longitude grid. (a) Surface pressure from a variable resolution simulation using conservative and consistent remapping. Observe that the detail of the result is much finer between 135W and 90W in the Northern hemisphere, corresponding to the region of highest mesh refinement. (b) Percentage plant functional type (barren land) using conservative, consistent and monotone remapping. This field is highly discontinuous and requires that the data be constrained to the interval [0, 100] to be considered meaningful.

The End

Thank you!