The Shortest Path in Line Arrangements

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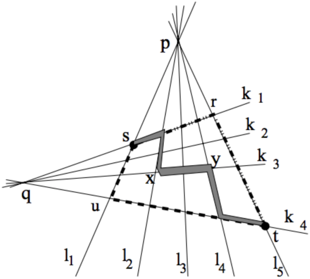
Outline

1. Problem Definition
2. Related Works
3. Faster solution

Problem Definition



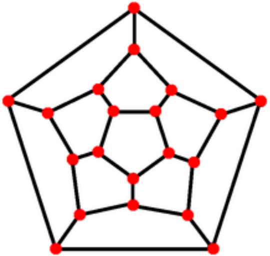
By given lines from two centrals, compute the shortest path between extreme up-left point s and extrem right-bottom point t.

Related Works

Shortest Path on Planar Graph

Theorem

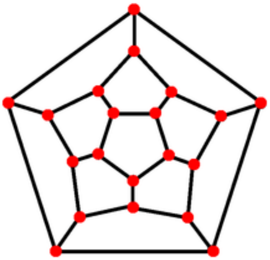
The shortest path in a planar graph is $O(n^2)$. [5]



Planar graph: edges do not intersect.

Related Works

Shortest Path on Planar Graph



Simple proof:

$$dist[s, t] = min(dist[s, m] + dist[m, t])$$

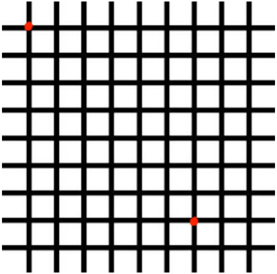
$\forall$m, t is connected

Each edge is considered at most twice.

$$O(|E|) = O(n^2)$$

Related Works

Special case: only vertical and horizontal



Theorem

The shortest path in a grid graph is $O(n^{1.5})$. [3]

Theorem

The special case can be further improved to $O(n \lg n)$. M. van Kreveld

Related Works

Special case: only k-slope

Theorem

The shortest path with k-slope line arrangements is $O(n + k^2)$. [2]

Note: k can be as worse as $2 \times n$.

Theorem

The shortest path in a grid graph has 2 approximation that runs in $O(nlgn)$ time. [4]

Note: Find the shortest path on dual problem.

Theorem

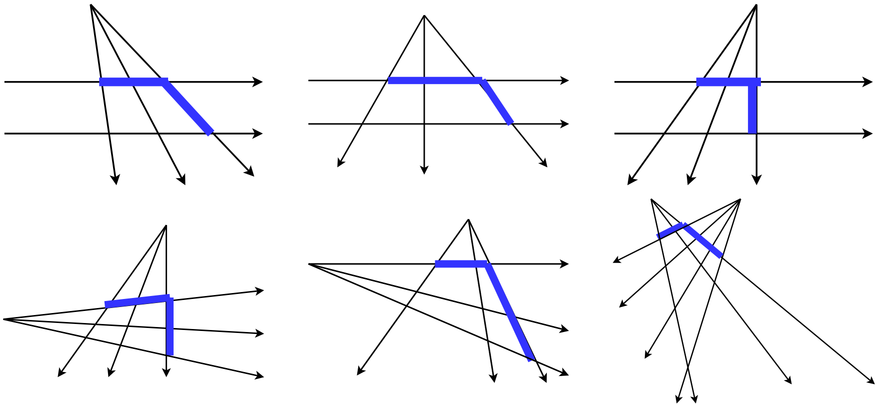
The shortest path in a grid graph has $1+\epsilon$ approximation of SPLA in time $O(nlogn + min(n, \frac{1}{\epsilon^2}) + \frac{1}{\epsilon} log \frac{1}{\epsilon})$

Note: very, very complex.

Constraints	**Time complexity**
No	$O(n^2)$
Verticial and Horizontal	$O(n^{1.5})$, $O(nlgn)$
K slopes	$O(n+k^2)$
2 approximation	$O(nlgn)$
$1+\epsilon$ approximation	$O(nlogn + min(n, \frac{1}{\epsilon^2}) + \frac{1}{\epsilon}log\frac{1}{\epsilon})$

Faster solution

Observation of some cases



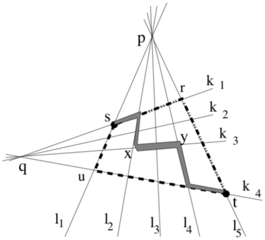
Observation

Mid points are not used in the shortest path.

Assumption

The shortest path will never use the mid points.

True ? Yes. kavitha etc [6] proved it.



Theorem

The shortest s-t path is always one of s-u-t or s-r-t.

Let's start from a simple case.

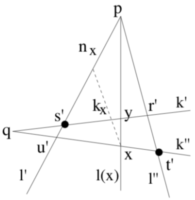


Figure 2: *The line $l(x)$ is any arbitrary line in the open cone defined by the lines l_1 and l_2.*

Path $s - y - x - t$, $s - u - x - t$
Assume $|suxt|$ is the shorter path

Let k be a point that satisfies $|sk| + |kx| = |su| + |ux|$
(k lies between s and y)

Let's start from a simple case.

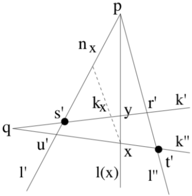


Figure 2: *The line $l(x)$ is any arbitrary line in the open cone defined by the lines l_1 and l_2.*

$|syxt| = |sy| + |yx| + |xt| = |sk| + |ky| + |yx| + |xt|$
$|suxt| = |su| + |ux| + |ut|$

Since $|sk| + |kx| = |su| + |ux|$
We have $|suxt| = |sk| + |kx| + |ut|$

$|suxt| = |sk| + |ky| + |yx| + |xt|$
$|syxt| = |sk| + |kx| + |xt|$

Let's start from a simple case.

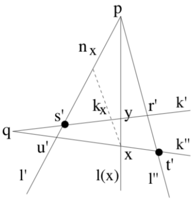


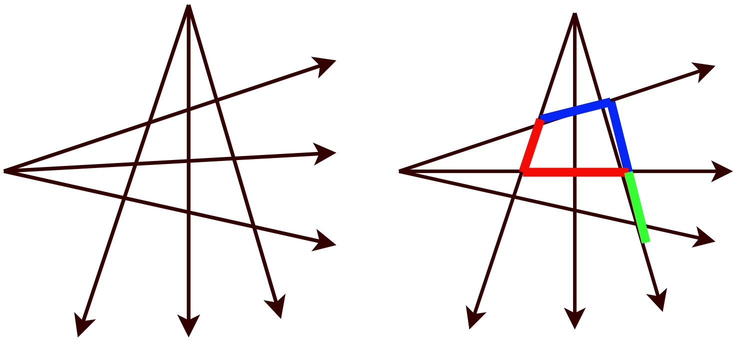
Figure 2: *The line $l(x)$ is any arbitrary line in the open cone defined by the lines l_1 and l_2.*

$$|suxt| = |sk| + |ky| + |yx| + |xt|$$
$$|syxt| = |sk| + |kx| + |xt|$$
$$|suxt| - |syxt| = |ky| + |yx| - |kx| > 0$$

$$|suxt| - |syxt| > 0$$

Similar proof can be done with $|suxt|$ and $|syxt|$.

Generalize to more edges
Suppose the shortest s-t path (call it SP (s, t)) travels along line k1, then bends into li for some $2 \leq i < n$ and then bends into k, j for some $j > 1$.
This contradicts kavitha's theorem applied to the lines l1 , li+1 , k1 , kj , and li.



Faster solution

Kavitha's solution

Given two central points k and j and shoot n and m lines from the central respectively. To find the shortest path between left-up point to right-down point, the algorithm operates as following.

1. Find the lines with smallest and largest slop for k, name as lk_1 and lk_2
2. Find the lines with smallest and largest slop for j, name as jn_1 and jm_2
3. Compute the four intersection points. A: (lk_1, jn_1), B: (lk_1, jm_2), C: (lk_2, jn_1), D: (lk_2, jm_2),
4. The shortest path is either ABC or ACD

Time complexity : $O(n)$

Summary

Method	**Constraints**	**Time complexity**
Naive	No	$O(n^2)$
[3]	Vertical and Horizontal	$O(n^{1.5})$, $O(nlgn)$
[2]	K slopes	$O(n+k^2)$
[1]	2 approximation	$O(nlgn)$
[4]	$1+\epsilon$ approximation	$O(nlogn + min(n, \frac{1}{\epsilon^2}) + \frac{1}{\epsilon} log \frac{1}{\epsilon})$
Kavitha	No	$O(n)$

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