Cálculo de Programas

Aula T01

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From a mobile phone manufacturer

(...) For each **list of calls** *stored in the mobile phone (eg. numbers dialed, SMS messages, lost calls), the* **store** *operation should work in a way such that* **(a)** *the more recently a call is made the more accessible it is;* **(b)** *no number appears twice in a list;* **(c)** *only the last 10 entries in each list are stored.*

From a mobile phone manufacturer

```
store ::  Call -> [Call] -> [Call]
store c l = take 10 (store' c l)

store' ::  Call -> [Call] -> [Call]
store' c l = c :  filter (/=c) l
```

From a mobile phone manufacturer

```
store :: Call -> [Call] -> [Call]
store c l = take 10 (c : filter (/=c) l)
```

Compare with ...

```
public void store10(string phoneNumber)
{
  System.Collections.ArrayList auxList =
      new System.Collections.ArrayList();
  auxList.Add(phoneNumber);
  auxList.AddRange(
      this.filteratmost9(phoneNumber) );
  this.callList = auxList;
}
```

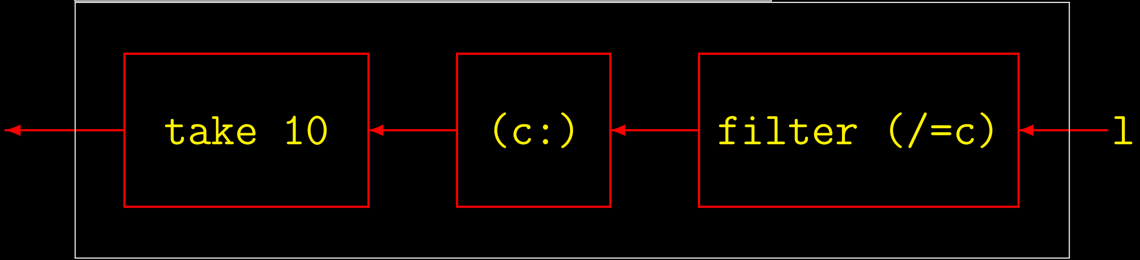
+ `filteratmost9` (next slide)

Compare with ...

```
public System.Collections.ArrayList filteratmost9(string n)
{
  System.Collections.ArrayList retList =
      new System.Collections.ArrayList();
      int i=0, m=0;
  while((i < this.callList.Count) && (m < 9))
  {
      if ((string)this.callList[i] != n)
      {
          retList.Add(this.callList[i]);
          m++;
      }
      i++;
  }
  return retList;
}
```

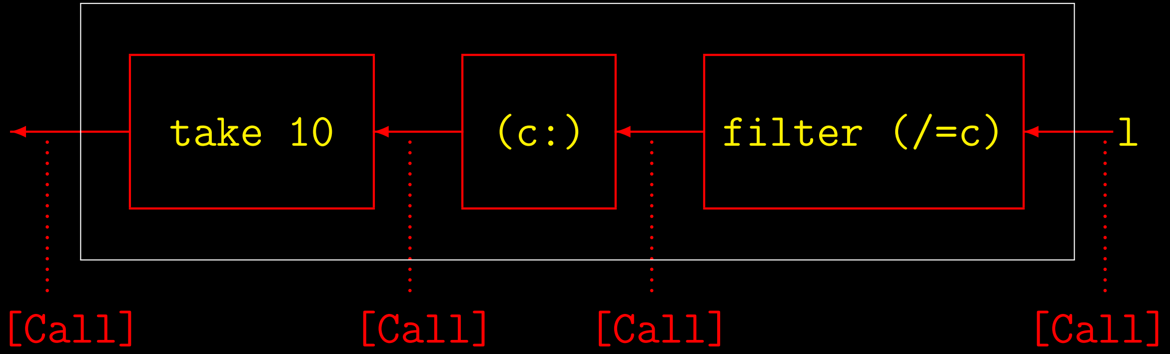
From a mobile phone manufacturer

```
store c :: [Call] -> [Call]
```

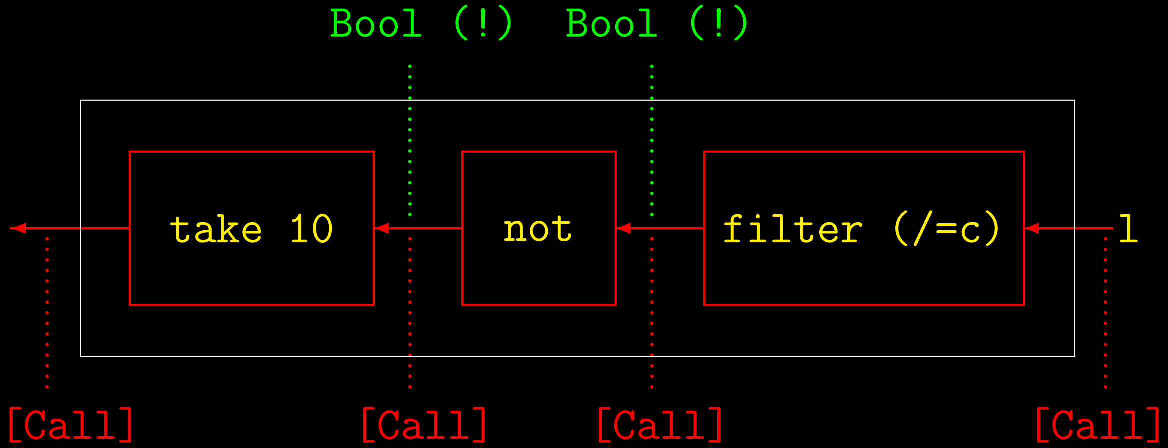


From a mobile phone manufacturer

`store c :: [Call] -> [Call]`

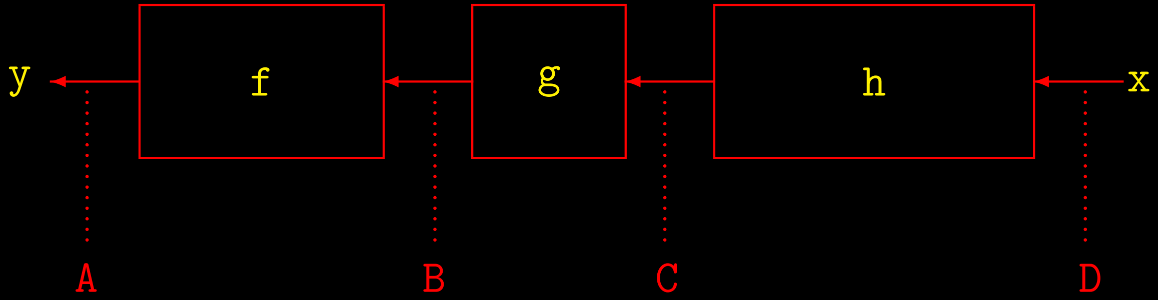


Uups!



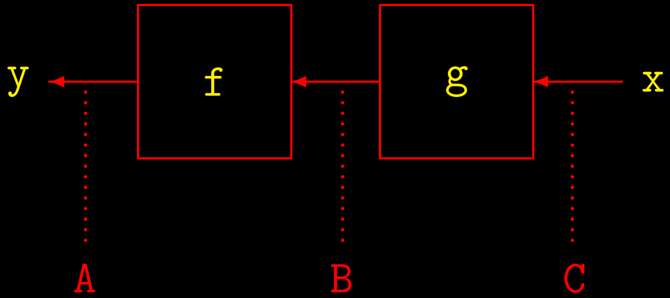
Em geral

```
y = f(g(h x))
```



Em geral

$$y = f(g\ x)$$



Simplificação

$$y = f(g\ x)$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

Composição

$$y = f(g\ x)$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

$$A \xleftarrow{f\ .\ g} C$$

$$y = (f\ .\ g)\ x$$

Composição

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$
$$(a + b) + c \;=\; a + (b + c)$$

Composição

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$
$$(a + b) + c = a + (b + c)$$

$$f \cdot g \cdot h$$
$$a + b + c$$

Composição

$$\mathit{store}\ c = \mathit{take}\ 10 \cdot \underbrace{(c:) \cdot \mathit{filter}\ (\neq c)}_{\mathit{store}'\ c}$$

Composição

$$\mathit{store}\ c = \mathit{take}\ 10 \cdot \underbrace{(c:) \cdot \mathit{filter}\ (\neq c)}_{\mathit{store}'\ c}$$

isto é

$$\mathit{take}\ 10 \cdot ((c:) \cdot \mathit{filter}\ (\neq c))$$

Composição

$$store\ c = take\ 10 \cdot \underbrace{(c:) \cdot filter\ (\neq c)}_{store'\ c}$$

isto é

$$take\ 10 \cdot ((c:) \cdot filter\ (\neq c))$$

igual a

$$(take\ 10 \cdot (c:)) \cdot filter\ (\neq c)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$
$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

Composição

$$(f \cdot g) \cdot h \;=\; f \cdot (g \cdot h)$$

$$(a + b) + c \;=\; a + (b + c)$$

$$a + 0 = 0 + a = a$$

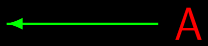
$$f \cdot ? = ? \cdot f = f$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

$$C \xrightarrow{g} B \xrightarrow{f} A$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

A


A

$$A \longleftarrow A$$

$$A \xleftarrow{id} A$$

$$A \xleftarrow{id} A$$

$$id\ a = a$$

Identidade

$$\begin{tikzcd} A \arrow[d, "f"'] & A \arrow[l, "id"'] \\ B & \end{tikzcd}$$

Identidade

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B
\end{tikzcd}
$$

Identidade

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}
$$

Identidade

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[d, "f"] \arrow[dl, "f"'] \\
B & B \arrow[l, "id"]
\end{tikzcd}
$$

Identidade

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[dl, "f"] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}$$

$$f \cdot id = f = id \cdot f$$

Composição e identidade

Associatividade:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Composição e identidade

Associatividade:

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

" Natural-*id*" :

$$f \cdot id = f = id \cdot f$$

$$f \cdot g$$

$$f \cdot g$$

$$f \times g \, ?$$

$$f \cdot g$$

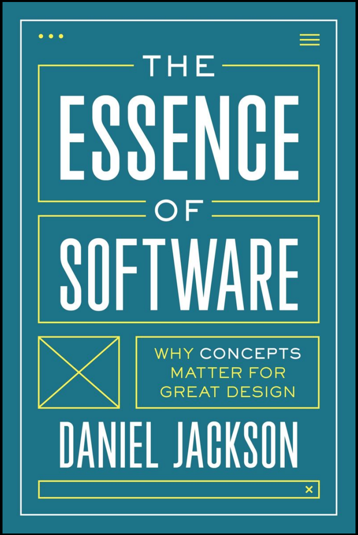
$$f \times g \; ?$$

$$f + g \; ?$$

Cálculo de Programas

Aula T02

"(...) The best services revolve around **a small number of concepts** that are **well designed and easy** (...) **to understand and use**, and their innovations often involve simple but compelling new concepts."



In
The Essence of Software
by
Prof. Daniel Jackson, MIT
(2021)

... small number

... small number
... well defined

… small number
… well defined
… easy to understand

$C \xrightarrow{f} B$ e $A \xrightarrow{g} C$

$$C \xrightarrow{f} B \quad \text{e} \quad A \xrightarrow{g} C$$

Composição: $A \xrightarrow{f \cdot g} B$

$$C \xrightarrow{f} B \quad \text{e} \quad A \xrightarrow{g} C$$

Composição: $A \xrightarrow{f \cdot g} B$

$$B \xleftarrow{f} C \quad \text{e} \quad C \xleftarrow{g} A$$

$$D \xrightarrow{f} B$$

$$A \xrightarrow{g} C$$

?

$D = A$?

$$\begin{tikzcd} B & & C \\ & A \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

$D = A$?

$$\begin{tikzcd} B & & C \\ & A \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

$f \; a \; \ldots \; g \; a$

$$\begin{tikzcd} B & & C \\ & A \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

$$(f\ a, g\ a)$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$f \; a \in B$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$f\ a \in B$$
$$g\ a \in C$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$\frac{f\ a \in B \qquad g\ a \in C}{(f\ a, g\ a) \in B \times C}$$

“Split”

$$\begin{tikzcd}
B & B \times C & C \\
 & A \arrow[ul, "f"'] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"] &
\end{tikzcd}$$

$$\langle f, g \rangle\ a = (f\ a, g\ a)$$

Produto

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Produto

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$\pi_1 : A \times B \to A$$

$$\pi_1 (a, b) = a$$

Produto

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$\pi_1 : A \times B \to A$$
$$\pi_1 (a, b) = a$$

$$\pi_2 : A \times B \to B$$
$$\pi_2 (a, b) = b$$

Produto

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f{,}g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Produto

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\pi_1 \cdot \langle f, g \rangle = f$$

Produto

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$\pi_1 \cdot \langle f, g \rangle = f \qquad \pi_2 \cdot \langle f, g \rangle = g$$

Produto

$$\langle f, g \rangle$$

Produto

$$\langle f, g \rangle$$

f e g em paralelo

f "split" g

Produto

$$\langle f, g \rangle$$

f e g em paralelo

f "split" g

$$\langle f, g \rangle\ a = (f\ a, g\ a)$$

Produto

$$\begin{tikzcd} D \\ B \arrow[u, "f"] \end{tikzcd}$$

Produto

$$
\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & & C \arrow[u, "g"']
\end{tikzcd}
$$

Produto

$$
\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \arrow[u, "g"']
\end{tikzcd}
$$

Produto

$$
\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"'] \arrow[ul] \arrow[r, "\pi_2"] & C \arrow[u, "g"']
\end{tikzcd}
$$

Produto

$$
\begin{tikzcd}
D & & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[ul] \arrow[ur] \arrow[r, "\pi_2"'] & C \arrow[u, "g"']
\end{tikzcd}
$$

Produto

$$
\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}
$$

Produto

$$
\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[ul] \arrow[ur] \arrow[u, "f \times g"] & C \arrow[u, "g"']
\end{tikzcd}
$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

Produto

$$\begin{tikzcd} D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\ B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "f \times g"] \arrow[ul] \arrow[ur] & C \arrow[u, "g"'] \end{tikzcd}$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

Recapitulando

$f \cdot g$

Recapitulando

$f \cdot g$ Composição sequencial

Recapitulando

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$

Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela

Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)

Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	

Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela

Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

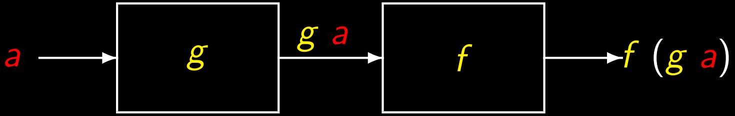
Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

Programação composicional

Recapitulando

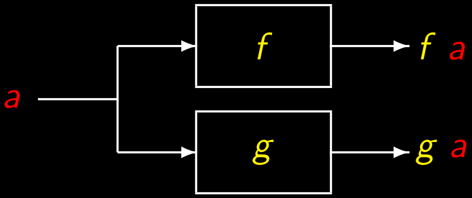
$$(f \cdot g)\ a = f\ (g\ a) \tag{2.6}$$



Composição de funções

Recapitulando

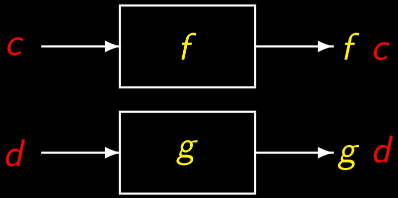
$$\langle f, g \rangle \; a = (f \; a, g \; a) \tag{2.20}$$



"Splits" de funções

Recapitulando

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \tag{2.24}$$



Produtos de funções

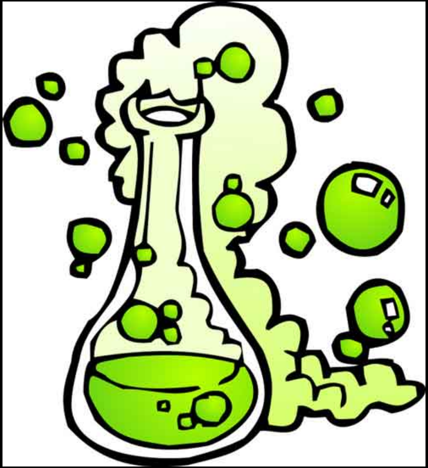
$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela (síncrona)

$f \times g$ Composição paralela (assíncrona)

Programação composicional

Cálculo?



$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle"] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & D \arrow[u, "\langle f , g \rangle \cdot h" description] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle" description] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] \arrow[uul, "f \cdot h"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] & \\
 & D \arrow[u, "h"] \arrow[uul, "f \cdot h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & D \arrow[ul, "f \cdot h"] \arrow[u, "\langle f \cdot h, g \cdot h \rangle"] \arrow[ur, "g \cdot h"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[u, "\langle f, g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle" description] &
\end{tikzcd}
$$

Fusão-$\times$

$$\langle f, g\rangle \cdot h = \langle f \cdot h, g \cdot h\rangle \tag{2.26}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f,g\rangle"] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] \arrow[uul, "f \cdot h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & & \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & & D \times E \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & E \\
B \arrow[u, "h"] & & & & C \arrow[u, "k"'] \\
& & A \arrow[llu, "f"] \arrow[rru, "g"'] \arrow[lluu, "h \cdot f"'] & &
\end{tikzcd}
$$

$$\begin{tikzcd}
D & & D \times E \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & E \\
B \arrow[u, "h"] & & & & C \arrow[u, "k"'] \\
& & A \arrow[ull, "f"] \arrow[urr, "g"'] \arrow[uull, "h \cdot f"'] \arrow[uurr, "k \cdot g"] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\langle h \cdot f, k \cdot g \rangle" description] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}
$$

Absorção-$\times$

$$(h \times k) \cdot \langle f, g \rangle = \langle h \cdot f, k \cdot g \rangle \tag{2.27}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
D \arrow[u, "h"] & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"'] & E \arrow[u, "k"'] \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}
$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}
$$

$$\pi_1 \cdot (h \times k) = h \cdot \pi_1 \qquad\qquad \pi_2 \cdot (h \times k) = k \cdot \pi_2$$

Natural-π_1, natural-π_2

$$\pi_1 \cdot (h \times k) \;=\; h \cdot \pi_1 \tag{2.28}$$

$$\pi_2 \cdot (h \times k) = k \cdot \pi_2 \tag{2.29}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"']
\end{tikzcd}$$

Functor-id-$\times$

$$id \times id = id \tag{2.31}$$

$$\begin{tikzcd}
c \arrow[r] & \boxed{id} \arrow[r] & c \\
d \arrow[r] & \boxed{id} \arrow[r] & d
\end{tikzcd}$$

Produto de identidades é a identidade.

Functor-$\times$

$$(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k) \tag{2.30}$$

Composição de produtos é o **produto** das composições.

Duas leis que faltam

Reflexão-$\times$

$$\langle \pi_1, \pi_2 \rangle = id \qquad (2.32)$$

Duas leis que faltam

Reflexão-$\times$
$$\langle \pi_1, \pi_2 \rangle = id \tag{2.32}$$

Eq-$\times$
$$\langle i, j \rangle = \langle f, g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} i = f \\ j = g \end{array} \right. \tag{2.64}$$

$$\langle \pi_1, \pi_2 \rangle$$

$$
\begin{tikzcd}
A & {} \\
 & A \times B \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, \pi_2 \rangle"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A & {} & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, \pi_2 \rangle"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A & A \times B & B \\
& A \times B \arrow[ul, "\pi_1"'] \arrow[u, "\langle \pi_1, \pi_2 \rangle"] \arrow[ur, "\pi_2"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, \pi_2 \rangle"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "id" description] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

E agora o mais importante...

Recordar o **cancelamento**-$\times$:

$$\begin{tikzcd} B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\ & A \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle"'] \arrow[ur, "g"'] & \end{tikzcd}$$

$\pi_1 \cdot \langle f, g \rangle = f$ $\qquad\qquad$ $\pi_2 \cdot \langle f, g \rangle = g$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Rightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

Existência

$$k = \langle f, g \rangle \;\Rightarrow\; \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

“Existe uma solução — $k = \langle f, g \rangle$ — para as equações da direita”

Unicidade

$$k = \langle f, g \rangle \quad \Leftarrow \quad \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

"As equações da direita só têm uma solução: $k = \langle f, g \rangle$"

Equações!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases}$$

Equações!

$$\begin{cases} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{cases} \Leftrightarrow \begin{cases} x = 6 \\ z = 1 \\ y = 3 \end{cases}$$

Equações!

Problema

Resolver a equação

$$\langle f, g \rangle = id$$

em ordem a f *e a* g.

Equações!

Resolução

Em $\quad k = \langle f, g \rangle \Leftrightarrow \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$

Equações!

Resolução

Em $\quad k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases} \quad$ fazer $k = id$

Equações!

Resolução

Em $\quad k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot id = f \\ \pi_2 \cdot id = g \end{cases}$$

Equações!

Resolução

Em $\quad k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equações!

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equações!

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituindo:

$$id = \langle \pi_1, \pi_2 \rangle$$

Equações!

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituindo:

$$id = \langle \pi_1, \pi_2 \rangle$$

Reflexão-$\times$

Problema

Resolver a equação

$$\langle h, k \rangle = \langle f, g \rangle$$

Problema

Resolver a equação

$$\langle h, k \rangle = \langle f, g \rangle$$

(1 equação, 4 incógnitas)

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \qquad \{ \text{ universal-}\times \ \}$$

$$\left\{ \begin{array}{l} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{array} \right.$$

Resolução

$$
\begin{array}{ll}
 & \langle h, k \rangle = \langle f, g \rangle \\
\Leftrightarrow & \quad \{ \text{ universal-}\times \} \\
 & \left\{ \begin{array}{l} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{array} \right. \\
\Leftrightarrow & \quad \{ \text{ cancelamento-}\times \} \\
 & \left\{ \begin{array}{l} h = f \\ k = g \end{array} \right.
\end{array}
$$

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \qquad \{ \text{ universal-}\times \text{ } \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases}$$

$$\Leftrightarrow \qquad \{ \text{ cancelamento-}\times \text{ } \}$$

$$\begin{cases} h = f \\ k = g \end{cases}$$

Eq-$\times$!

$$\langle \pi_1, \pi_2 \rangle = \mathit{id}$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$\langle \pi_2, \pi_1 \rangle$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$
\begin{tikzcd}
B & \\
& \cdot \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"']
\end{tikzcd}
$$

$\langle \pi_1, \pi_2 \rangle = id$ $\langle \pi_2, \pi_1 \rangle$?

$$
\begin{tikzcd}
B & {} & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[ur, "\pi_1"'] \arrow[u, "{\langle \pi_2, \pi_1 \rangle}"] &
\end{tikzcd}
$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$
\begin{tikzcd}
B & B \times A & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}
$$

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$
\begin{tikzcd}
B & B \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}
$$

Problema

Resolver

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

em ordem a k

Problema

Resolver

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

em ordem a k

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "\langle \pi_2,\pi_1 \rangle"] & & B \times A \arrow[ll, bend left, "k"] \end{tikzcd}$$

Resolução

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

Resolução

$$
\begin{aligned}
& \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow \quad & \{ \ \text{fusão-}\times \ \} \\
& \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id
\end{aligned}
$$

Resolução

$$
\begin{aligned}
& \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow \quad & \{ \text{ fusão-}\times \text{ } \} \\
& \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id \\
\Leftrightarrow \quad & \{ \text{ universal-}\times \text{ } \} \\
& \begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}
\end{aligned}
$$

Resolução

$$
\begin{aligned}
& \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow \quad & \{ \text{ fusão-}\times \} \\
& \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id \\
\Leftrightarrow \quad & \{ \text{ universal-}\times \} \\
& \begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & \{ \text{ trivial } \} \\
& \begin{cases} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{cases}
\end{aligned}
$$

Resolução

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

$\Leftrightarrow$ { fusão-$\times$ }

$$\langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id$$

$\Leftrightarrow$ { universal-$\times$ }

$$\begin{cases} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{cases}$$

$\Leftrightarrow$ { trivial }

$$\begin{cases} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{cases}$$

$\Leftrightarrow$ { universal-$\times$ }

$$k = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "\langle \pi_2, \pi_1 \rangle"] & & B \times A \arrow[ll, bend left, "\langle \pi_2, \pi_1 \rangle"] \end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

Swap

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "\langle \pi_2, \pi_1 \rangle"] & & B \times A \arrow[ll, bend left, "\langle \pi_2, \pi_1 \rangle"] \end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

$$swap \cdot swap = id$$

Até agora

$f \cdot g$ Composição sequencial

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associatividade?

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle ?$$

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associatividade?

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle ?$$

Não! mas...

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd} A & \\ & D \arrow[ul, "f"'] \end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd} A & & B \\ & D \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& D \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$
\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "{\langle f, g\rangle}"] \arrow[ur, "h"'] &
\end{tikzcd}
$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & & C \\
& D \arrow[ul, "{\langle f, g\rangle}"'] \arrow[ur, "h"'] \arrow[u, dashed, "{\langle\langle f, g\rangle, h\rangle}"] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd} B & & C \\ & D \arrow[ul, "g"] \arrow[ur, "h"'] & \end{tikzcd}$$

$\langle f, \langle g, h \rangle\rangle$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "g"] \arrow[u, "\langle g, h \rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd} A & & B \times C \\ & D \arrow[ul, "f"] \arrow[ur, "{\langle g, h \rangle}"'] & \end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "{\langle f, \langle g, h \rangle \rangle}"] \arrow[ur, "{\langle g, h \rangle}"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$
\begin{tikzcd}
A & A \times (B \times C) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "{\langle f, \langle g, h \rangle \rangle}"] \arrow[ur, "{\langle g, h \rangle}"'] &
\end{tikzcd}
$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd} A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\ & D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle"] \arrow[ur, "h"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f,g\rangle,h\rangle"] \arrow[ur, "\langle f,\langle g,h\rangle\rangle"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f,g\rangle,h\rangle"] \arrow[ur, "\langle f,\langle g,h\rangle\rangle"'] &
\end{tikzcd}
$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle$$

$$k \cdot \langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle$$

$$k \cdot \underbrace{\langle \langle f, g \rangle, h \rangle}_{id?} = \langle f, \langle g, h \rangle \rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id?} = \langle f, \langle g, h\rangle\rangle$$

Resolver $\langle\langle f, g\rangle, h\rangle = id$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\langle\langle f, g\rangle, h\rangle = id$$

$$\Leftrightarrow \qquad \{\ \text{universal-}\times\ \}$$

$$\begin{cases} \pi_1 = \langle f, g\rangle \\ \pi_2 = h \end{cases}$$

$$
\begin{aligned}
& \langle \langle f, g \rangle, h \rangle = id \\
\Leftrightarrow \quad & \{ \text{ universal-}\times \ \} \\
& \begin{cases} \pi_1 = \langle f, g \rangle \\ \pi_2 = h \end{cases} \\
\Leftrightarrow \quad & \{ \text{ universal-}\times \ \} \\
& \begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}
\end{aligned}
$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k \cdot \underbrace{\langle \langle f, g \rangle, h \rangle}_{id} = \langle f, \langle g, h \rangle \rangle$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Podemos melhorar...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Podemos melhorar...

$$
\begin{aligned}
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle \\
\Leftrightarrow \quad & \quad \{ \ \pi_2 = id \cdot \pi_2 \ \} \\
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle
\end{aligned}
$$

Podemos melhorar...

$$
\begin{aligned}
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle \\
\Leftrightarrow \quad & \{ \ \pi_2 = id \cdot \pi_2 \ \} \\
& k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle \\
\Leftrightarrow \quad & \{ \ f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \ \} \\
& k = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle
\end{aligned}
$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle \langle f , g \rangle , h \rangle"] \arrow[ur, "\langle f , \langle g , h \rangle \rangle"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \arrow[ll, bend left, "j"] \\
& D \arrow[ul, "\langle \langle f, g \rangle, h \rangle"] \arrow[ur, "\langle f, \langle g, h \rangle \rangle"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \arrow[ll, bend left, "\langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle"]
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "assocr"] & & A \times (B \times C) \arrow[ll, bend left, "assocl"]
\end{tikzcd}$$

$$assocr = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$$
$$assocl = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}
$$

$$assocr \cdot assocl = id$$

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorfismo

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "assocr"] \arrow[r, phantom, "\cong"] & & A \times (B \times C) \arrow[ll, bend left, "assocl"]
\end{tikzcd}
$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorfismo

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

$$swap \cdot swap = id$$

Isomorfismo

$$
\begin{tikzcd}
A \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\cong"] & & B \arrow[ll, bend left, "g"]
\end{tikzcd}
$$

$$f \cdot g = id$$

$$g \cdot f = id$$

Isomorfismo

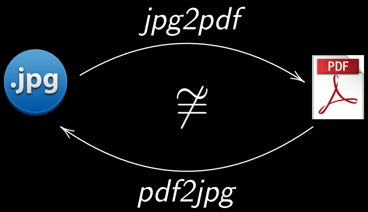
$$\underbrace{iso}_{a\ \ mesma}\ (\iota\sigma o)$$

Isomorfismo

$$\underbrace{\textit{iso}}_{\textit{a mesma}}\ (\iota\sigma o) + \underbrace{\textit{morfismo}}_{\textit{forma}}\ \ (\mu o\rho\phi\iota\sigma\mu o\zeta)$$

"Forma semelhante"

Problema prático!



Problema prático!

$$\begin{tikzcd}[column sep=huge] \text{.jpg} \arrow[r, bend left, "jpg2pdf"] \arrow[r, phantom, "\ncong"] & \text{PDF} \arrow[l, bend left, "pdf2jpg"] \end{tikzcd}$$

$$jpg2pdf \cdot pdf2jpg \neq id$$

$$pdf2jpg \cdot jpg2pdf \neq id$$

Problema prático!

$$\begin{tikzcd} \text{.jpg} \arrow[r, bend left, "jpg2pdf"] \arrow[r, phantom, "\ncong"] & \text{PDF} \arrow[l, bend left, "pdf2jpg"] \end{tikzcd}$$

$$jpg2pdf \cdot pdf2jpg \neq id$$

$$pdf2jpg \cdot jpg2pdf \neq id$$

Conversão de formatos

Necessidade

$$\begin{array}{c} A \\ \Big\downarrow f \\ B \end{array}$$

Conversão de formatos

Necessidade

$$A \xrightarrow{f} B$$

Reutilizável

$$C \xrightarrow{r} D$$

Conversão de formatos

Necessidade Reutilizável

$$
\begin{tikzcd}
A \arrow[rrr, "exporta"] \arrow[d, "f"'] & & & C \arrow[d, "r"] \\
B & & & D
\end{tikzcd}
$$

Conversão de formatos

Necessidade | Reutilizável

$$\begin{tikzcd}[column sep=huge, row sep=large]
A \arrow[r, "exporta"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}$$

Conversão de formatos

Necessidade Reutilizável

$$
\begin{tikzcd}
A \arrow[r, "exporta"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}
$$

$$f = importa \cdot r \cdot exporta$$

Conversão de formatos

Necessidade Reutilizável

$$\begin{tikzcd}
A \arrow[r, "exporta"] \arrow[d, "f"'] \arrow[dr, phantom, "(\cong)"] & C \arrow[d, "r"] \\
B & D \arrow[l, "importa"]
\end{tikzcd}$$

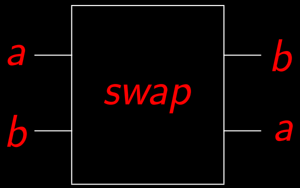
$$f = importa \cdot r \cdot exporta$$

A propósito de *swap*

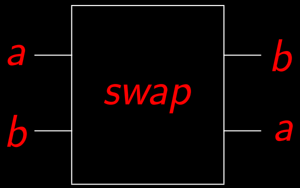
$$
\begin{tikzcd}
A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"]
\end{tikzcd}
$$

Isomorfismos são computações **reversíveis**

A propósito de *swap*



A propósito de *swap*



swap é uma das unidades básicas da **programação quântica**

A propósito de *swap*