A Tour of Time Series Analysis with R

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Contents

Preface

This text is designed as an introduction to time series analysis and is used as a support document for the class STAT 429 (Time Series Analysis) given at the University of Illinois at Urbana-Champaign. It preferable to always access the text online rather than a printed to be sure you are using the latest version. The online version so affords additional features over the traditional PDF copy such as a scaling text, variety of font faces, and themed backgrounds. However, if you are in need of a local copy, a **pdf version** is also available.

This document is under active development and as a result is likely to contains many errors. As Montesquieu puts it:

> "*La nature semblait avoir sagement pourvu à ce que les sottises des hommes fussent passagères, et les livres les immortalisent.*"

Contributing

If you notice any errors, we would be grateful if you would let us know. To let us know about the errors, there are two options available to you. The first and subsequently the fastest being if you are familiar with GitHub and know RMarkdown, then make a pull request and fix the issue yourself!. Note, in the online version, there is even an option to automatically start the pull request by clicking the edit button in the top-left corner of the text.

The second option, that will have a slightly slower resolution time is to send an email to `balamut2 AT illinois DOT edu` that includes: the error and a possible revision. Please put in the subject header: `[TTS]`.

Bibliographic Note

This text is heavily inspired by the following three execellent references:

1. "*Time Series Analysis and Its Applications*", Third Edition, Robert H. Shumway & David S. Stoffer.
2. "*Time Series for Macroeconomics and Finance*", John H. Cochrane.
3. "*Cours de Séries Temporelles: Théorie et Applications*", Volume 1, Arthur Charpentier.

Rendering Mathematical Formulae

Throughout the book, there will be mathematical symbols used to express the material. Depending on the version of the book, there are two different render engines.

- For the online version, the text uses MathJax to render mathematical notation for the web. In the event the formulae does not load for a specific chapter, first try to refresh the page. 9 times out of 10 the issue is related to the software library not loading quickly.
- For the pdf version, the text is built using the recommended AMS LaTeX symbolic packages. As a result, there should be no issue displaying equations.

An example of a mathematical rendering capabilities would be given as:

$$a^2 + b^2 = c^2$$

R Code Conventions

The code used throughout the book will predominately be `R` code. To obtain a copy of `R`, go to the Comprehensive R Archive Network (CRAN) and download the appropriate installer for your operating system.

When `R` code is displayed it will be typeset using a `monospace` font with syntax highlighting enabled to ensure the differentiation of functions, variables, and so on. For example, the following adds 1 to 1

```
a = 1L + 1L
a
```

Each code segment may contain actual output from `R`. Such output will appear in grey font prefixed by `##`. For example, the output of the above code segment would look like so:

```
## [1] 2
```

Alongside the PDF download of the book, you should find the R code used within each chapter.

License

Figure 1: This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Chapter 1

Introduction

Prévoir consiste à projeter dans l'avenir ce qu'on a perçu dans le passé. Henri Bergson

After reading this chapter you will be able to:

- Describe what a *time series* is.
- Perform exploratory data analysis on time series data.
- Evaluate different characteristics of a time series.
- Classify basic time series models through equations and plots.
- Manipulate a time series equation using *backsubstitution*.

1.1 Time Series

Generally speaking a *time series* (or stochastic process) corresponds to set of "repeated" observations of the same variable such as price of a financial asset or temperature in a given location. In terms of notation a time series is often written as

$$(X_1, X_2, ..., X_n) \quad \text{or} \quad (X_t)_{t=1,...,n}\,.$$

The time index t is contained within either the set of reals, $\mathbb{R}$, or integers, $\mathbb{Z}$. When $t \in \mathbb{R}$, the time series becomes a *continuous-time* stochastic process such a Brownian motion, a model used to represent the random movement of particles within a suspended liquid or gas, or an ElectroCardioGram (ECG) signal, which corresponds to the palpitations of the heart. However, within this text, we will limit ourselves to the cases where $t \in \mathbb{Z}$, better known as *discrete-time* processes. *Discrete-time* processes are where a variable is measured sequentially at fixed and equally spaced intervals in time akin to 1.1. This implies that we will have two assumptions:

1. t is not random e.g. the time at which each observation is measured is known, and
2. the time between two consecutive observations is constant.

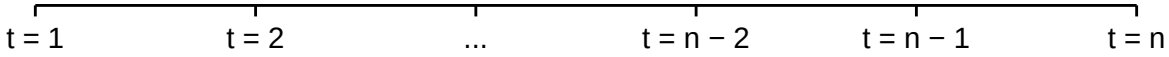


Figure 1.1: Discrete-time can be thought of as viewing a number line with equally spaced points.

Moreover, the term "time series" can also represent a probability model for a set of observations. For example, one of the fundamental probability models used in time series analysis is called a *white noise* process and is defined as

$$W_t \overset{iid}{\sim} N(0, \sigma^2).$$

This statement simply means that (W_t) is normally distributed and independent over time. This model may appear to be dull but as we will see it is a crucial component to constructing more complex models. Unlike the white noise process, time series are typically *not* independent over time. Suppose that the temperature in Champaign is unusually low, then it is reasonable to assume that tomorrow's temperature will also be low. Indeed, such behavior would suggest the existence of a dependency over time. The time series methods we will discuss in this text consists of parametric models used to characterize (or at least approximate) the joint distribution of (X_t). Often, time series models can be decomposed into two components, the first of which is what we call a *signal*, say (Y_t), and the second component is a *noise*, say (W_t), leading to the model

$$X_t = Y_t + W_t.$$

Typically, we have $E[Y_t] \neq 0$ while $E[W_t] = 0$ (although we may have $E[W_t|W_{t-1}, ..., W_1] \neq 0$). Such models impose some parametric structure which represents a convenient and flexible way of studying time series as well as a means to evalute *future* values of the series through forecasting. As we will see, predicting future values is one of the main aspects of time series analysis. However, making predictions is often a daunting task or as famously stated by Nils Bohr:

> "*Prediction is very difficult, especially about the future.*"

There are plenty of examples of predictions that turned out to be completely erroneous. For example, three days before the 1929 crash, Irving Fisher, Professor of Economics at Yale University, famously predicted:

> "*Stock prices have reached what looks like a permanently high plateau*".

Another example is given by Thomas Watson, president of IBM, who said in 1943:

> "*I think there is a world market for maybe five computers.*"

1.2 Exploratory Data Analysis for Time Series

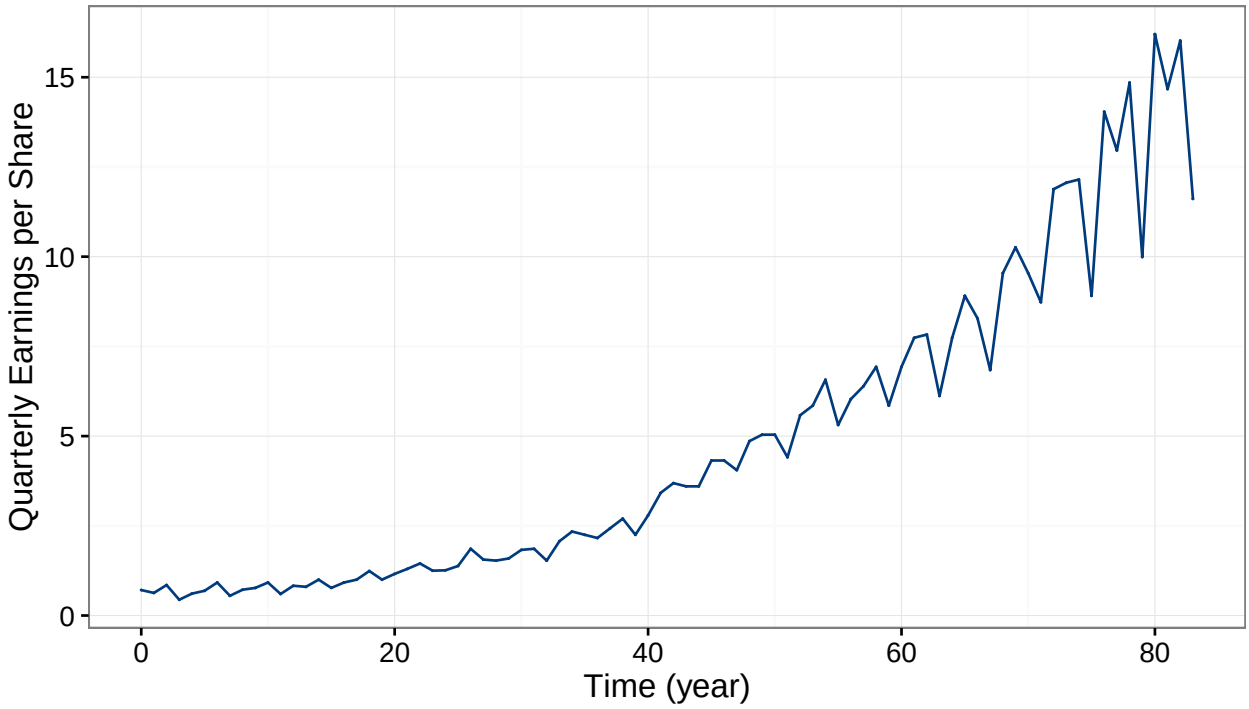
When dealing with relatively small time series (e.g. a few thousands), it is often useful to look at a graph of the original data. These graphs can be informative to "detect" some features of a time series such as trends and the presence of outliers.

Indeed, a trend is typically assumed to be present in a time series when the data exhibit some form of long term increase or decrease or combination of increases or decreases. Such trends could be linear or non-linear and represent an important part of the "signal" of a model. Here are a few examples of non-linear trends:

1. **Seasonal trends** (periodic): These are the cyclical patterns which repeat after a fixed/regular time period. This could be due to business cycles (e.g. bust/recession, recovery).
2. **Non-seasonal trends** (periodic): These patterns cannot be associated to seasonal variation and can for example be due to an external variable such as, for example, the impact of economic indicators on stock returns. Note that such trends are often hard to detect based on a graphical analysis of the data.

3. **"Other" trends**: These trends have typically no regular patterns and are over a segment of time, known as a "window", that change the statistical properties of a time series. A common example of such trends is given by the vibrations observed before, during and after an earthquake.

Example: A traditional example of a time series is the quarterly earnings of the company Johnson and Johson. In the figure below, we present these earnings between 1960 and 1980:

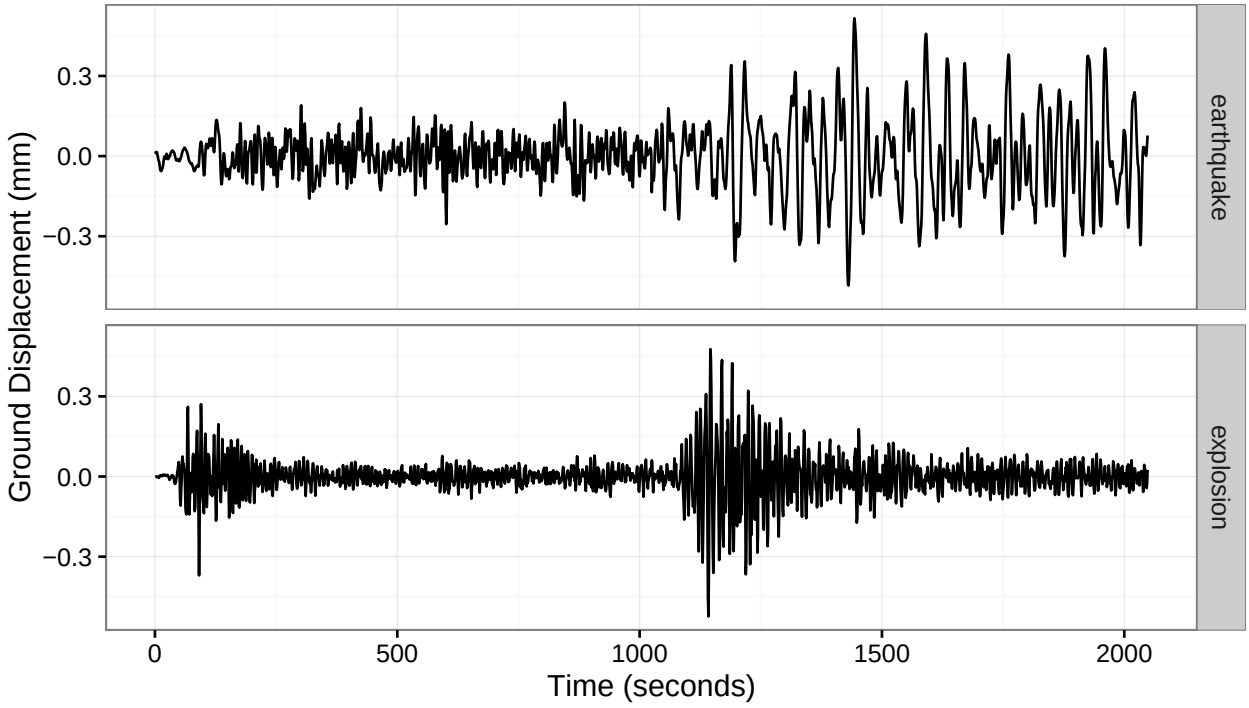


One trait that the graph makes evident is that the data contains a non-linear increasing trend as well as a yearly seasonal component. In addition, one can note that the *variability* of the data seems to increase with time. Being able to make such observations provides important information to select suitable models for the data.

Moreover, when observing "raw" time series data it is also interesting to evaluate if some of the following phenomena occur:

1. **Change in Mean:** Does the mean of the process shift over time?
2. **Change in Variance:** Does the variance of the process evolve with time?
3. **Change in State:** Does the time series appear to change between "states" having distinct statistical properties?
4. **Outliers** Does the time series contain some "extreme" observations? Note that this is typically difficult to assess visually.

Example: In the figure below, we present an example of displacement recorded during an earthquake as well as an explosion.

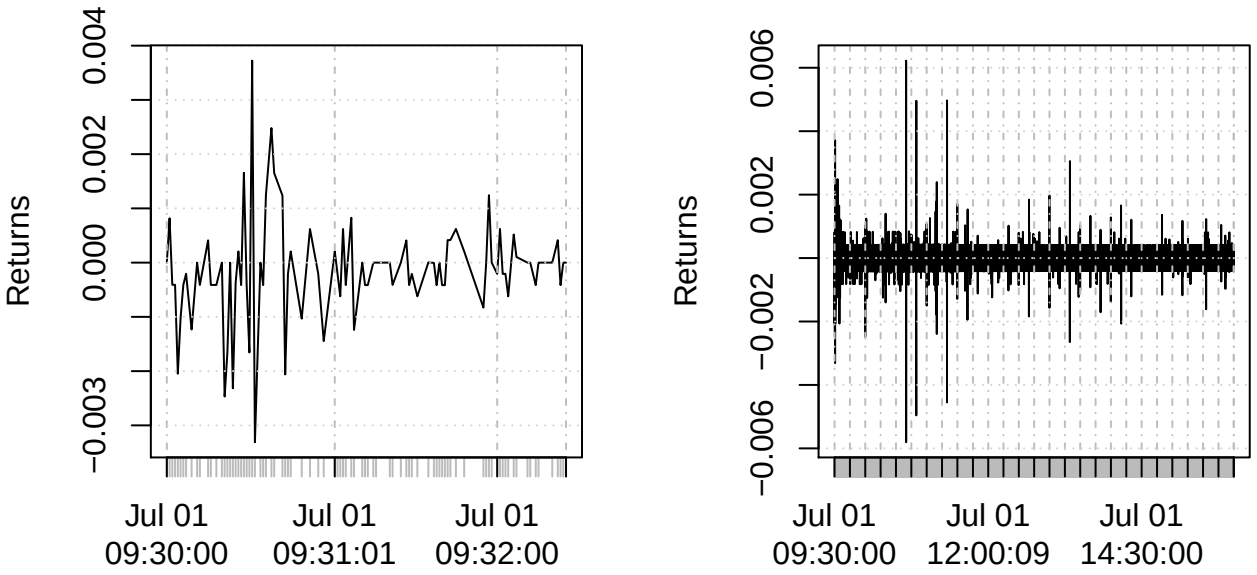


From the graph, it can be observed that the statistical properties of the time series appear to change over time. For instance, the variance of the time series shifts at around $t = 1150$ for both series. The shift in variance also opens "windows" where there appear to be distinct states. In the case of the explosion data, this is particularly relevant around $t = 50, \cdots, 250$ and then again from $t = 1200, \cdots, 1500$. Even within these windows, there are "spikes" that could be considered as outliers most notably around $t = 1200$ for explosion series.

Next, we consider an example coming from high-frequency finance to illustrate the limitations our current framework.

Example: The figure below presents the returns or price innovations (i.e. informally speaking the changes in price from one observation to the other) for the Starbuck's stock on July 1, 2011 for about 150 seconds (left panel) and about 400 minutes (right panel).

```
# Load packages
library(timeDate)

# Load "high-frequency" Starbucks returns for Jul 01 2011
data(sbux.xts, package = "highfrequency")

# Plot returns
par(mfrow = c(1,2))
plot(sbux.xts[1:89], main = " ", ylab = "Returns")
plot(sbux.xts, main = " ", ylab = "Returns")
```



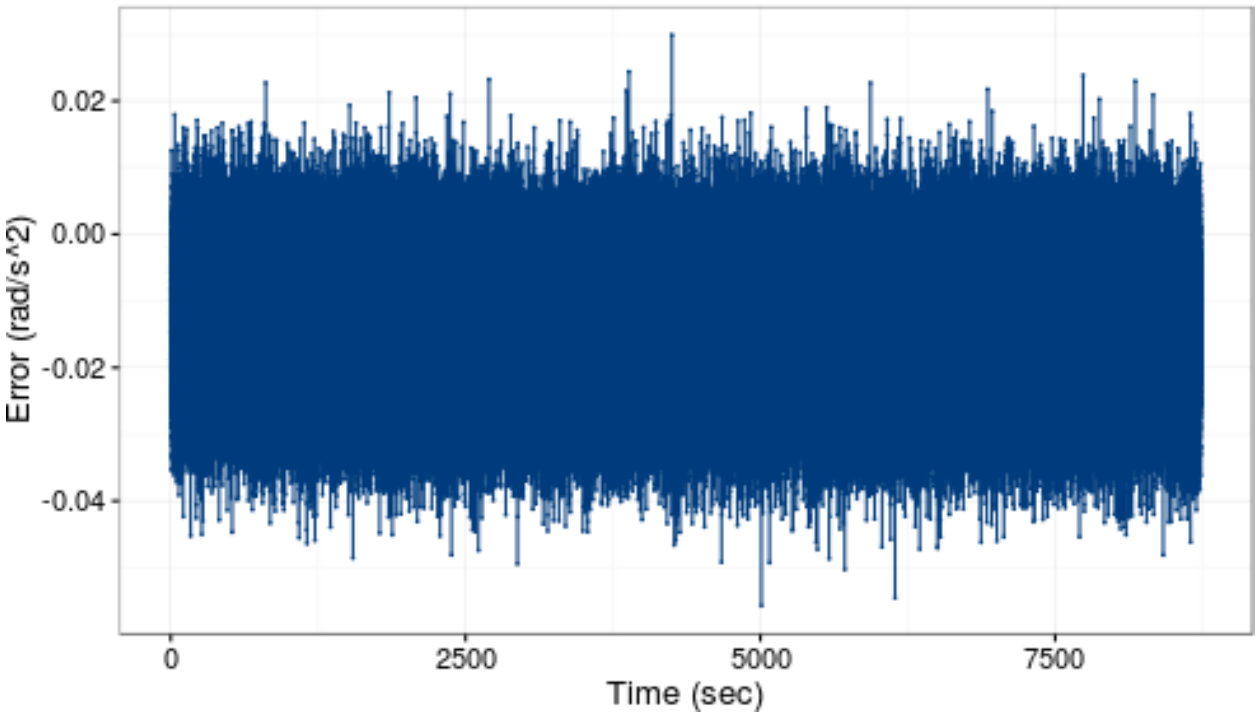
It can be observed on the left panel that observations are not equally spaced. Indeed, in high-frequency data the intervals between two points is typically not constant and, even worse, is a random variable. This implies that the time when a new observation will be available is in general unknown. On the right panel, one can observe that the variability of the data seems to change during the course of the trading day. Such a phenomenon is well known in the finance community since a lot of variation occurs at the start (and the end) of the day while the middle of the day is associated with small changes. Moreover, clear extreme observations can also be noted in this graph at around 11:00

Finally, let us consider the limitations of a direct graphical representation of a time series when the sample size is large. Indeed, due to visual limitations, a direct plotting of the data will probably result in an uninformative aggregation of points between which it is unable to distinguish anything. This is illustrated in the following example.

Example: We consider here the data coming from the calibration procedure of an Inertial Measurement Unit (IMU) which, in general terms, is used to enhance navigation precision or reconstruct three dimensional movements (see e.g. link). These sensors are used in a very wide range of applications such as robotics, virtual reality, vehicle stability control, human and animal motion capture and so forth (see e.g. link). The signals coming from these instruments are measured at high frequencies over a long time and are often characterized by linear trends and numerous underlying stochastic processes. If you have never heard

The code below retrieves some data from an IMU and plots it directly:

```
# Load packages
library(gmwm)
library(imudata)

# Load IMU data
data(imu6, package = "imudata")
Xt = gts(imu6[,1], name = "Gyroscope data", unit = "sec", freq = 100)

# Plot gryoscope data
autoplot(Xt) + ylab("Error (rad/s^2)")
```



Although a linear trend and other processes are present in this signal (time series), it is practically impossible to understand or guess anything from the plot.

1.3 Basic Time Series Models

In this section, we introduce some simple time series models. Before doing so it is useful to define Ω_t as all the information avaiable up to time $t-1$, i.e.

$$\Omega_t = (X_{t-1}, X_{t-2}, ..., X_0)\,.$$

As we will see this compact notation is quite useful.

1.3.1 White noise processes

The building block for most time series models is the Gaussian white noise process, which can be defined as

$$W_t \overset{iid}{\sim} N\left(0, \sigma_w^2\right).$$

This definition implies that:

1. $E[W_t|\Omega_t] = 0$ for all t,
2. $\text{cov}\left(W_t, W_{t-h}\right) = \mathbf{1}_{h=0}\,\sigma^2$ for all t, h.

Therefore, in this process there is an absence of temporal (or serial) dependence and is homoskedastic (i.e it has a constant variance). This definition can be generalized into two sorts of processes, the *weak* and *strong* white noise. The process (W_t) is a weak white noise if

1. $E[W_t] = 0$ for all t,
2. $\text{var}\,(W_t) = \sigma^2$ for all t,
3. $\text{cov}\,(W_t, W_{t-h}) = 0$, for all t, and for all $h \neq 0$.

Note that this definition does not imply that W_t and W_{t-h} are independent (for $h \neq 0$) but simply uncorrelated. However, the notion of independence is used to define a *strong* white noise as

1. $E[W_t] = 0$ and $\text{var}(W_t) = \sigma^2 < \infty$, for all t,
2. $F(W_t) = F(W_{t-h})$, for all t, h (where $F(W_t)$ denotes the distribution of W_t),
3. W_t and W_{t-h} are independent for all t and for all $h \neq 0$.

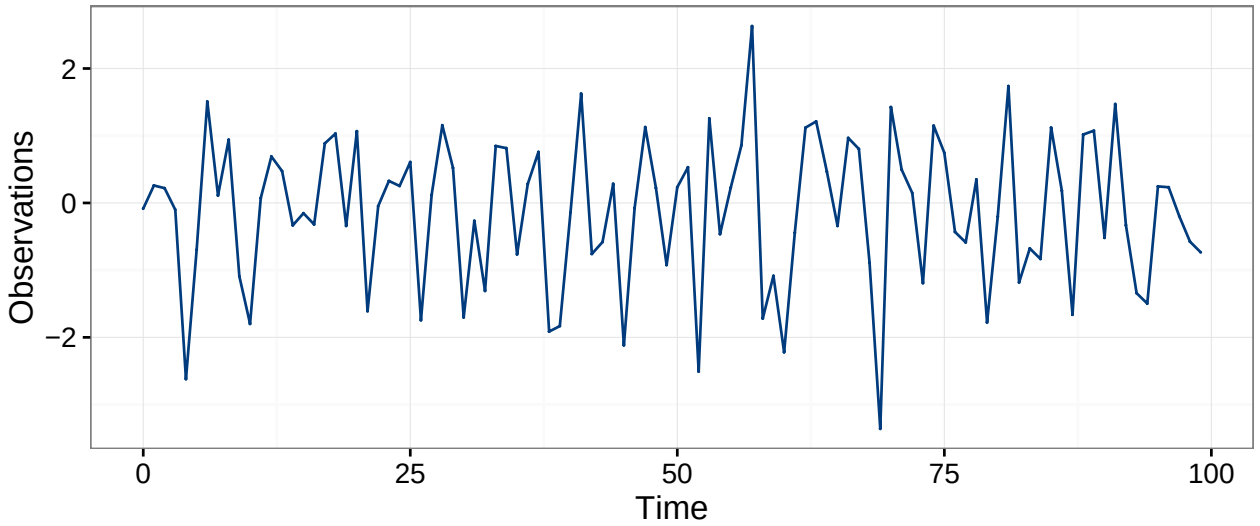
It is clear from these definitions that if a process is a strong white noise it is also a weak white noise. However, the converse is not true as shown in the following example:

Example: Let $Y_t \sim F_{t+2}$, where F_{t+2} denotes a Student distribution with $t+2$ degrees of freedom. Assuming the sequence $(Y_1, \ldots, Y_n)$ to be independent, we let $X_t = \sqrt{\frac{t}{t+2}} Y_t$. Then, the process (X_t) is obviously not a strong white noise as the distribution of X_t changes with t. However, this process is a weak white noise since we have:

- $E[X_t] = \sqrt{\frac{t}{t+2}} E[Y_t] = 0$ for all t.
- $\text{var}(X_t) = \frac{t}{t+2} \text{var}(Y_t) = \frac{t}{t+2} \frac{t+2}{t} = 1$ for all t.
- $\text{cov}(X_t, X_{t+h}) = 0$ (by independence), for all t, and for all $h \neq 0$.

The code below presents an example of how to simulate a Gaussian white noise process

```
# This code simulates a gaussian white noise process
n = 100                                  # process length
sigma2 = 1                               # process variance
Xt = gen.gts(WN(sigma2 = sigma2), N = n)
plot(Xt)
```



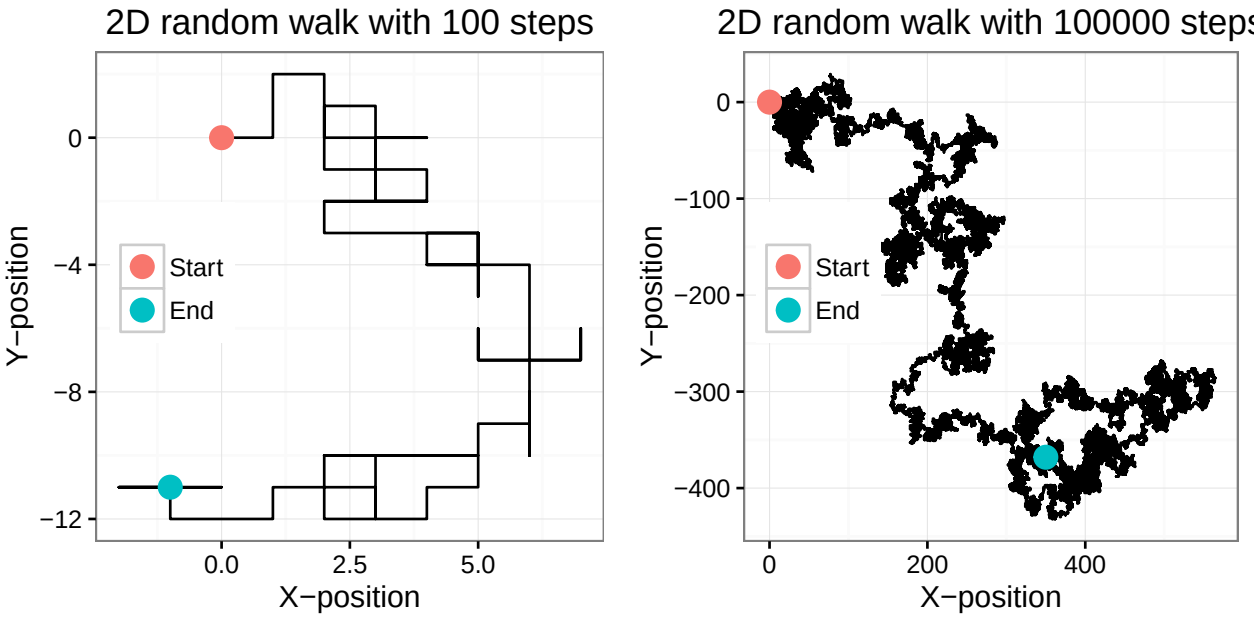
1.3.2 Random Walk Processes

The term *random walk* was first introduce by Karl Pearson in the early 19 hundreds. As for the white noise, there exist a large range of random walk processes. For example, one of the simplest forms of random walk

can be explained as follows: suppose that you are walking on campus and your next step can either be to your left, your right, forward or backward (each with equal probability). Two realizations of such processes are represented below:

```
# Function computes direction random walk moves
RW2dimension = function(steps = 100){
  # Initial matrix
  step_direction = matrix(0, steps+1, 2)

  # Start random walk
  for (i in seq(2, steps+1)){
    # Draw a random number from U(0,1)
    rn = runif(1)

    # Go right if rn \in [0,0.25)
    if (rn < 0.25) {step_direction[i,1] = 1}

    # Go left if rn \in [0.25,0.5)
    if (rn >= 0.25 && rn < 0.5) {step_direction[i,1] = -1}

    # Go forward if rn \in [0.5,0.75)
    if (rn >= 0.5 && rn < 0.75) {step_direction[i,2] = 1}

    # Go backward if rn \in [0.75,1]
    if (rn >= 0.75) {step_direction[i,2] = -1}
  }

  # Cumulative steps
  position = data.frame(x = cumsum(step_direction[, 1]),
                        y = cumsum(step_direction[, 2]))

  # Mark start and stop locations
  start_stop = data.frame(x = c(0, position[steps+1, 1]),
                          y = c(0, position[steps+1, 2]),
                          type = factor(c("Start","End"),
                                        levels = c("Start","End")))

  # Plot results
  ggplot(mapping = aes(x = x, y = y)) +
    geom_path(data = position) +  # Mimics type = 'l'
    geom_point(data = start_stop, aes(color = type), size = 4) +
    theme_bw() +
    labs(
      x = "X-position",
      y = "Y-position",
      title = paste("2D random walk with", steps, "steps"),
      color = ""
    ) + theme(legend.position = c(0.15, 0.55))
}

# Plot 2D random walk with 10^2 and 10^5 steps
set.seed(2)

a = RW2dimension(steps = 10^2)
```

```
b = RW2dimension(steps = 10^5)

library("gridExtra")

grid.arrange(a, b, nrow = 1)
```



Such processes inspired Karl Pearson's famous quote that

> "*the most likely place to find a drunken walker is somewhere near his starting point.*"

Empirical evidence of this phenomenon is not too hard to find on a Friday night in Champaign. In this class, we only consider one very specific form of random walk, namely the Gaussian random walk which can be defined as:

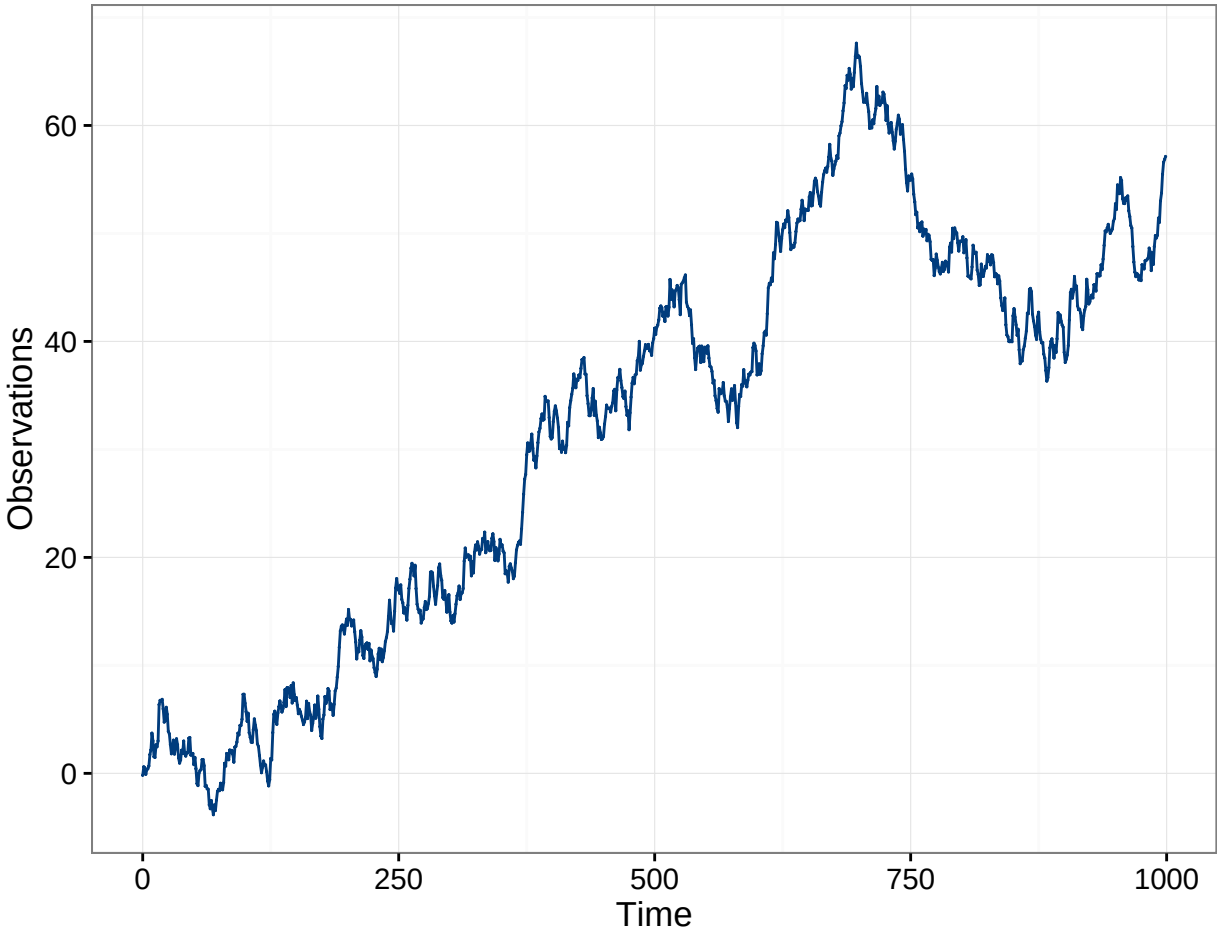
$$X_t = X_{t-1} + W_t,$$

where W_t is a Gaussian white noise and with initial condition $X_0 = c$ (typically $c = 0$). This process can be expressed differently by *backsubstitution* as follows:

$$\begin{aligned} X_t &= X_{t-1} + W_t \\ &= (X_{t-2} + W_{t-1}) + W_t \\ &= \vdots \\ X_t &= \sum_{i=1}^{t} W_i + X_0 = \sum_{i=1}^{t} W_i + c \end{aligned}$$

The code below presents an example of how to simulate a such process

```
# This code simulates a gaussian random walk process
n = 1000                                  # process length
gamma2 = 1                                # innovation variance
Xt = gen.gts(RW(gamma2 = gamma2), N = n)
plot(Xt)
```



1.3.3 Autoregressive Process of Order 1

An autoregressive process of order 1 or AR(1) is a generalization of both the white noise and random walk processes which are both themselves special cases of an AR(1). A (Gaussian) AR(1) process can be defined as

$$X_t = \phi X_{t-1} + W_t,$$

where W_t is a Gaussian white noise. Clearly, an AR(1) with $\phi = 0$ is a Gaussian white noise and when $\phi = 1$ the process becomes a random walk.

Remark: We generally assume that an AR(1), as well as other time series models, have zero mean. The reason for this assumption is only to simplfy the notation but it is easy to consider an AR(1) process around an arbitrary mean μ, i.e.

$$(X_t - \mu) = \phi\,(X_{t-1} - \mu) + W_t,$$

which is of course equivalent to

$$X_t = (1 - \phi)\,\mu + \phi X_{t-1} + W_t.$$

Thus, we will generally only work with zero mean processes since adding means is simple.

Remark: An AR(1) is in fact a linear combination of the past realisations of the white noise W_t. Indeed, we have

$$\begin{aligned} X_t &= \phi_t X_{t-1} + W_t = \phi\left(\phi X_{t-2} + W_{t-1}\right) + W_t \\ &= \phi^2 X_{t-2} + \phi W_{t-1} + W_t = \phi^t X_0 + \sum_{i=0}^{t-1} \phi^i W_{t-i}. \end{aligned}$$

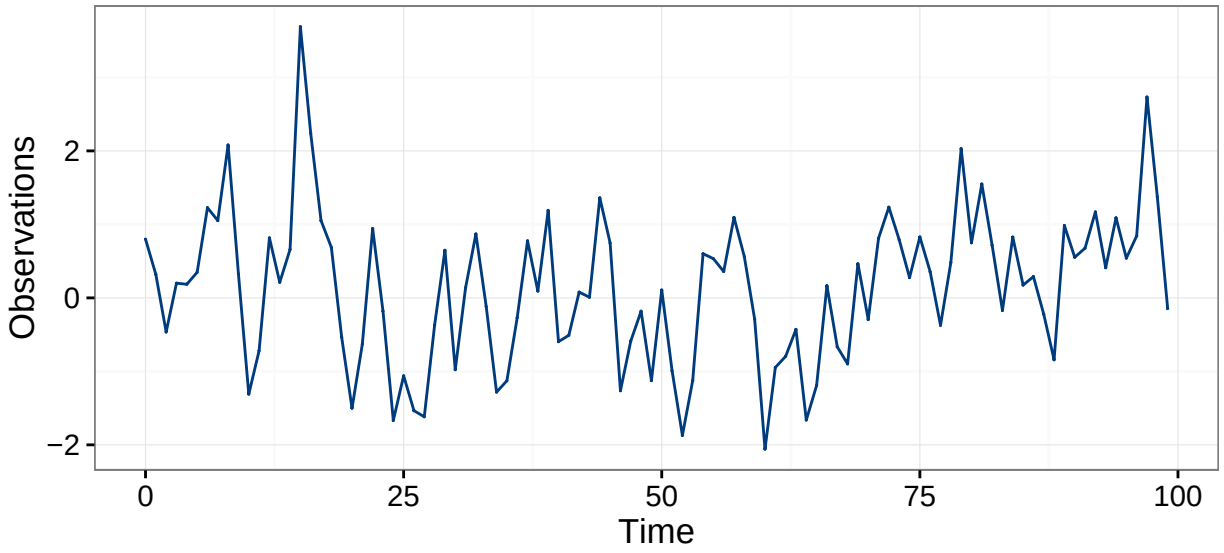
Under the assumption of infinite past (i.e. $t \in \mathbb{Z}$) and $|\phi| < 1$, we obtain

$$X_t = \sum_{i=0}^{\infty} \phi^i W_{t-i},$$

since $\lim_{i\to\infty}\ \phi^i X_{t-i} = 0$.

The code below presents an example of how an AR(1) can be simulated

```
# This code simulate a gaussian random walk process
n = 100                                  # process length
phi = 0.5                                # phi parameter
sigma2 = 1                               # innovation variance
Xt = gen.gts(AR1(phi = phi, sigma2 = sigma2), N = n)
plot(Xt)
```



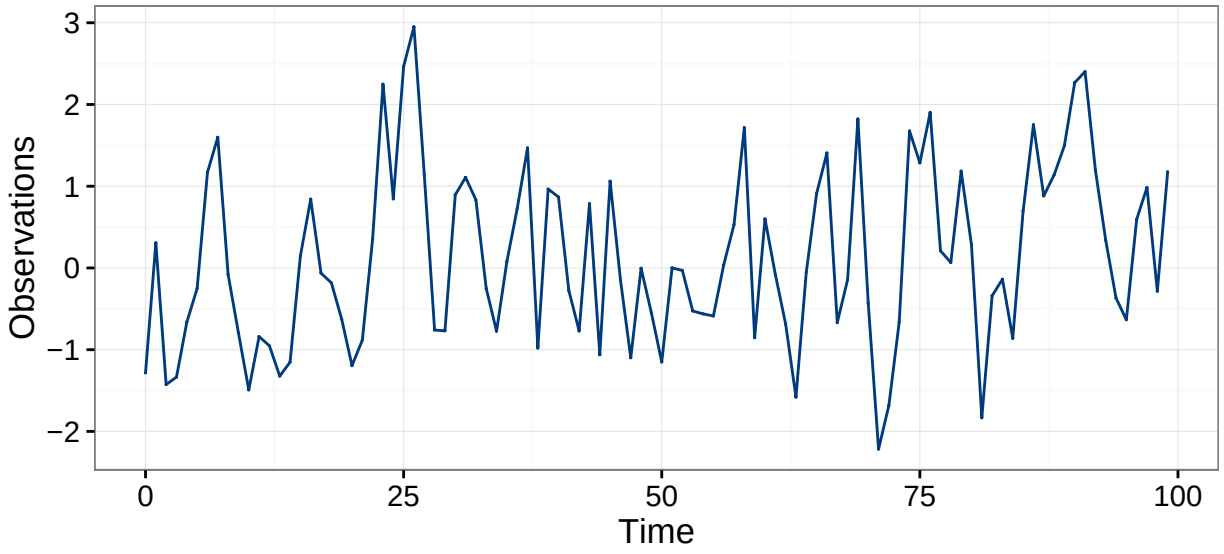
1.3.4 Moving Average Process of Order 1

As we have seen in the previous example, an AR(1) can be expressed as a linear combination of all past observations of (W_t) while the next process, called a moving average process of order 1 or MA(1), is (in some sense) a "truncated" version of an AR(1). It is defined as

$$X_t = \theta W_{t-1} + W_t, \tag{1.1}$$

where (again) W_t denotes a Gaussian white noise process. An example on how to generate an MA(1) is given below:

```
# This code simulates a gaussian white noise process
n = 100                                  # process length
sigma2 = 1                               # innovation variance
theta = 0.5                              # theta parameter
Xt = gen.gts(MA1(theta = theta, sigma2 = sigma2), N = n)
plot(Xt)
```



1.3.5 Linear Drift

A linear drift is a very simple deterministic time series model which can be expressed as

$$X_t = X_{t-1} + \omega,$$

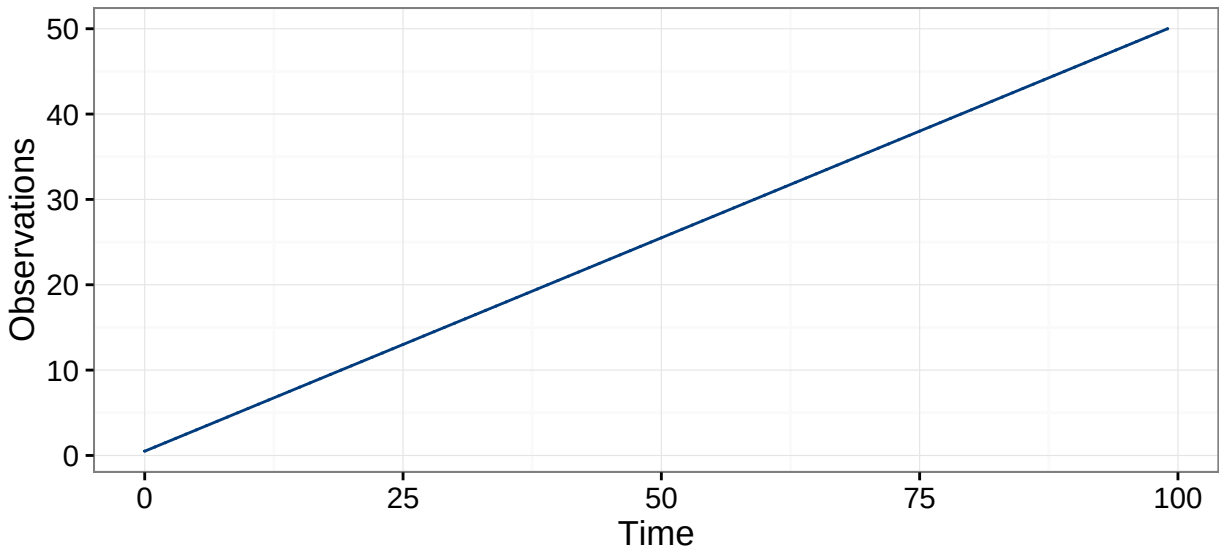
where ω is a constant and with the initial condition $X_0 = c$, an arbitrary constant (typically zero). This process can be expressed in a more familiar form as follows:

$$X_t = X_{t-1} + \omega = (X_{t-2} + \omega) + \omega = t\omega + c$$

Therefore, a (linear) drift corresponds to a simple linear model with slope ω and intercept c.

A drift can simply be generated using the code below:

```
# This code simulate a linear drift with 0 intercept
n = 100                                  # process length
omega = 0.5                              # slope parameter
Xt = gen.gts(DR(omega = omega), N = n)
plot(Xt)
```



1.3.6 Composite Stochastic Processes

A composite stochastic process can be defined as the sum of underlying (or latent) stochastic processes. In this text, we will use the term *latent time series* as a synomym for composite stochastic processes. A simple example of such a process is given by

$$\begin{aligned} Y_t &= Y_{t-1} + W_t + \delta \\ X_t &= Y_t + Z_t, \end{aligned}$$

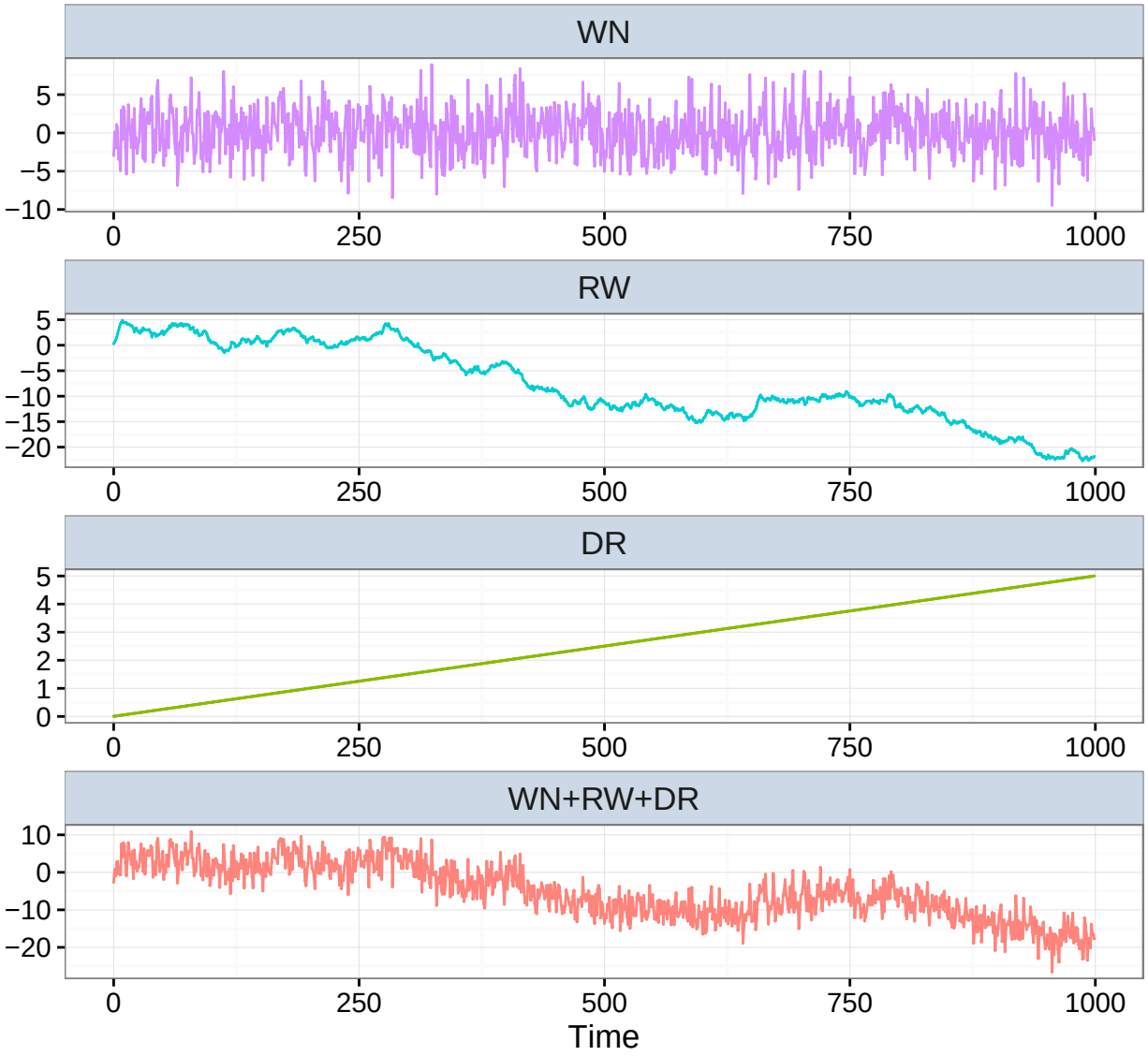
where W_t and Z_t are two independent Gaussian white noise processes. This model is often used as a first tool to approximate the number of individuals in the context ecological population dynamics. For example, suppose we want to study the population of Chamois in the Swiss Alps. Let Y_t denote the "true" number of individuals in this population at time t. It is reasonable that Y_t is (approximately) the population at the previous time $t-1$ (e.g the previous year) plus a random variation and a drift. This random variation is due to the natural randomness in ecological population dynamics and reflects the changes in the number of predators, in the aboundance of food or in the weather conditions. On the other hand, the drift is often of particular interest for ecologists as it can be used to determine the "long" term trends of the population (e.g. is the population increasing, stable or decreasing). Of course, Y_t (the number of individauls) is typically unknown and we observe a noisy version of it, denoted as X_t. This process corresponds to the true population plus a measurement error since some individuals may not be observed while others may have been counted several times. Interestingly, this process can clearly be expressed as a *latent time series model* (or composite stochastic process) as follows:

$$\begin{aligned} R_t &= R_{t-1} + W_t \\ S_t &= \delta t \\ X_t &= R_t + S_t + Z_t, \end{aligned}$$

where R_t, S_t and Z_t denote, respectively, a random walk, a drift and a white noise. The code below can be used to simulate such data:

```
n = 1000                                # process length
delta = 0.005                           # delta parameter (drift)
sigma2 = 10                             # variance parameter (white noise)
gamma2 = 0.1                            # innovation variance (random walk)
```

```
model = WN(sigma2 = sigma2) + RW(gamma2 = gamma2) + DR(omega = delta)
Xt = gen.lts(model, N = n)
plot(Xt)
```



In the above graph, the first three plots represent the latent (unobserved) processes (i.e. white noise, random walk and drift) and the last one represents the sum of the three (i.e. (X_t)).

Bibliography