CASSELMAN–WALLACH PROPERTY FOR HOMOLOGICAL THETA LIFTING

ZHIBIN GENG AND HANG XUE

ABSTRACT. In this paper, we establish the Casselman–Wallach property for homological theta lifting over archimedean local fields. As a consequence, the Euler–Poincaré characteristic is a well-defined element in the Grothendieck group of Casselman–Wallach representations. Our main tool is a corank-one parabolic stable filtration on the Weil representation.

CONTENTS

1. INTRODUCTION

Let F be an archimedean local field. Let $\mathbb{W}$ be a finite-dimensional symplectic space over F. Fix a non-trivial unitary character $\psi : F \to \mathbb{C}^\times$. Let (G', H') be a reductive dual pair in $\mathrm{Sp}(\mathbb{W})$. In the context of theta lifting, one associates to (G', H') a corresponding pair (G, H) (roughly speaking, (G, H) differs from (G', H') only in that certain symplectic factors are replaced by their metaplectic double cover; see Subsection 2.2 for the precise statement). Denote by $\Omega_{G,H,\psi}$ the associated Weil representation of $G \times H$. We may omit the subscript when no confusion arises.

Write Rep_G^{CW} (resp. Rep_H^{CW}) for the category of Casselman–Wallach representations of G (resp. H). For $\pi \in \text{Rep}_G^{\text{CW}}$, let $\pi^\vee$ denote its contragredient. Consider the coinvariant space

$$\Theta_0(\pi) := (\Omega \mathbin{\widehat{\otimes}} \pi^\vee)_G.$$

It should be noted that, a priori, it is not clear whether $\Theta_0(\pi)$ is Hausdorff. The usual full theta lift $\Theta(\pi)$ of π is defined as the maximal Hausdorff quotient of $\Theta_0(\pi)$, namely

$$\Theta(\pi) := (\Omega \mathbin{\widehat{\otimes}} \pi^\vee)_G / \overline{\{0\}}.$$

For a general introduction to the theory of theta lifting, we refer the reader to [MVW87] and [GKT25]. A brief overview is also provided in Subsection 2.2.

A natural question is whether $\Theta_0(\pi)$ is already Hausdorff, so that the quotient by $\overline{\{0\}}$ is unnecessary. An affirmative answer will lead to the homological formulation of the theta lifting (in the sense of Schwartz homology), and naturally raises the further question of whether the higher homology groups are also Casselman–Wallach representations. See Section 3 for the basic notions of Schwartz homology. The main result of this paper provides an affirmative answer to the above question and serves as a preliminary result for the homological theta lifting. The precise statement is as follows.

Theorem 1.1. *Let π be a Casselman–Wallach representation of G. For all $i \geq 0$, the Schwartz homology groups*

$$\text{H}_i^{\mathcal{S}}(G, \ \Omega \mathbin{\widehat{\otimes}} \pi),$$

are Casselman–Wallach representations of H, and equal to zero when $i > \dim(G/K)$, where K is a maximal compact subgroup of G. In particular, they are Hausdorff.

By this theorem, the Euler–Poincaré characteristic

$$\text{EP}_G(\Omega \mathbin{\widehat{\otimes}} \pi) := \sum_{i \geq 0} (-1)^i \text{H}_i^{\mathcal{S}}(G, \ \Omega \mathbin{\widehat{\otimes}} \pi)$$

is a well-defined element in the Grothendieck group of Rep_H^{CW}. As pointed out in [APS17], this Euler–Poincaré characteristic is often more accessible than the full theta lift itself, and is expected to provide insights into the structure of the latter.

The study of homological aspects of theta lifts in the p-adic setting was initiated in [APS17]. Our results can be regarded as the corresponding archimedean analogue. In this setting, we work with topological vector spaces in contrast to merely vector spaces in the p-adic setting, which gives rise to many additional difficulties and technical complications.

Our main tool is a corank-one parabolic stable filtration on the Weil representation. This is the archimedean counterpart of the filtration first studied by Kudla. In the archimedean setting, due to the non-exactness of the Jacquet functors and the presence of the transverse derivatives, the general filtration takes a much more complicated shape. In this paper, we restrict ourselves to the corank-one case, which suffices for our purposes and admits a particularly clean form.

We now briefly describe the proof of Theorem 1.1, which contains some of the major innovations of this paper. First, we may assume (G, H) is an irreducible dual pair. Using Casselman's subrepresentation theorem together with a lemma from [HT90] concerning the topology of long exact sequences (see Lemma 3.13), we reduce Theorem 1.1 to the following.

Theorem 1.2. *Let (G, H) and Ω be as before, and let $P_{\min} = L \ltimes N$ be a minimal parabolic subgroup of G. Let V be an irreducible representation of L. Denote by $\text{Ind}_{P_{\min}}^G(V)$ the*

corresponding principal series representation. Then, for all $i \geq 0$,

$$\mathrm{H}_i^{\mathcal{S}}(G, \Omega \,\widehat{\otimes}\, \mathrm{Ind}_{P_{\min}}^G(V))$$

is a Casselman–Wallach representation of H.

Frobenius reciprocity quickly reduces Theorem 1.2 to

$$\mathrm{H}_i^{\mathcal{S}}(P_{\min}, \Omega \,\widehat{\otimes}\, V \otimes \delta_{P_{\min}}^{-1}).$$

Thus, we need to study the restriction of Ω to parabolic subgroups. This is where we make use of the corank-one filtration on the Weil representation. It allows us to argue by induction on the split rank of G. For Type I dual pairs, the base case is the compact dual pair, for which Hausdorffness is immediate. The finite length property follows from Howe's original work [How89a] and [How89b, Section 3]. In fact, in Howe's original proof of Howe duality [How89b], compact dual pairs already play a fundamental role in determining the general case. For Type II dual pairs, the base case is $(\mathrm{GL}_1, \mathrm{GL}_m)$, which we calculated explicitly in Subsection 4.1. For the induction step, we treat Type I and Type II dual pairs separately, in Sections 5 and 4, respectively.

2. Preliminaries and Background

In this section, we recall some preliminary material and fix the notation to be used in the paper.

2.1. Casselman–Wallach representation.
Let G be a real reductive group with complexified Lie algebra $\mathfrak{g}$ and maximal compact subgroup K. A Casselman-Wallach representation of G is a smooth Fréchet representation of moderate growth and of finite length. The readers may consult [Wal92, Chapter 11] or [BK14] for details. Denote by $\mathrm{Rep}_G^{\mathrm{CW}}$ the category of Casselman–Wallach representations of G.

By the Casselman–Wallach globalization theorem, the functor of taking K-finite vectors induces an equivalence between $\mathrm{Rep}_G^{\mathrm{CW}}$ and the category of Harish–Chandra $(\mathfrak{g}, K)$-modules. Consequently, morphisms in $\mathrm{Rep}_G^{\mathrm{CW}}$ have well-behaved topological properties, as shown in the following lemma.

Lemma 2.1 ([AGS15a, Corollary 2.2.5]). *(1) $\mathrm{Rep}_G^{\mathrm{CW}}$ is an abelian category.*
(2) Any morphism in $\mathrm{Rep}_G^{\mathrm{CW}}$ has closed image.

Every Casselman–Wallach representation is in fact a nuclear Fréchet space, NF-space for short. See [CHM00, Appendix A] for more details about NF-spaces. If W is an NF-space, and $V \subseteq W$ is a closed subspace, then V and W/V are both NF-spaces. Additionally, all surjective morphisms between Fréchet spaces are open.

For two topological vector spaces V and W, write $V \,\widehat{\otimes}\, W$ for their completed projective tensor product. If V and W are both NF-spaces, so is $V \,\widehat{\otimes}\, W$. Furthermore, for a fixed NF-space V, the functor $W \mapsto V \,\widehat{\otimes}\, W$ is exact on the category of NF-spaces.

The following lemma plays an important role in our reduction step.

Lemma 2.2 (Casselman's subrepresentation theorem, [Wal88, Subsection 3.8]). *Let π be an irreducible representation of G. Write $P_{\min} = L \ltimes N$ for a minimal parabolic subgroup of G. Then there exists an irreducible finite-dimensional representation V of L, such that π can be imbedded into $\operatorname{Ind}_{P_{\min}}^{G}(V)$.*

2.2. **Theta correspondence.** In this section, we briefly review the theory of theta lifting and fix notation. For more details, the reader may consult [MVW87; Ada07; GKT25]. Let F be an archimedean local field. Let $\mathbb{W}$ be a finite-dimensional symplectic space over F. Fix a non-trivial unitary character $\psi : F \to \mathbb{C}^{\times}$.

We first recall basic facts about reductive dual pairs in $\operatorname{Sp}(\mathbb{W})$. Our conventions follow [GKT25, Chapter 10.5] and [Ada07]. Let (G', H') be a reductive dual pair in $\operatorname{Sp}(\mathbb{W})$. Then there is an orthogonal decomposition $\mathbb{W} = \oplus_{i=1}^{k} \mathbb{W}_i$, such that

$$(G', H') = (G'_1, H'_1) \times \cdots \times (G'_k, H'_k),$$

where each (G'_i, H'_i) is an irreducible reductive dual pair in $\operatorname{Sp}(\mathbb{W}_i)$. The following tables present a complete list of irreducible reductive dual pairs, together with the associated division algebra and involution (D, κ) in the Type I case. Note that if (G', H') is a reductive dual pair in $\operatorname{Sp}(\mathbb{W})$, then so is (H', G').

	G'	H'	$\operatorname{Sp}(\mathbb{W})$	(D, κ)
Type I	$\operatorname{Sp}(2n, \mathbb{R})$	$\operatorname{O}(p, q, \mathbb{R})$	$\operatorname{Sp}(2n(p+q), \mathbb{R})$	$(\mathbb{R}, 1)$
	$\operatorname{U}(r, s)$	$\operatorname{U}(p, q)$	$\operatorname{Sp}(2(p+q)(r+s), \mathbb{R})$	$(\mathbb{C}, \bar{\cdot})$
	$\operatorname{O}^*(2n)$	$\operatorname{Sp}(p, q)$	$\operatorname{Sp}(4n(p+q), \mathbb{R})$	$(\mathbb{H}, \bar{\cdot})$
Type II	$\operatorname{GL}(n, \mathbb{R})$	$\operatorname{GL}(m, \mathbb{R})$	$\operatorname{Sp}(2nm, \mathbb{R})$	
	$\operatorname{GL}(n, \mathbb{H})$	$\operatorname{GL}(m, \mathbb{H})$	$\operatorname{Sp}(8nm, \mathbb{R})$	

Irreducible reductive dual pairs over $\mathbb{R}$

	G'	H'	$\operatorname{Sp}(W)$	(D, κ)
Type I	$\operatorname{Sp}(2n, \mathbb{C})$	$\operatorname{O}(m, \mathbb{C})$	$\operatorname{Sp}(2nm, \mathbb{C})$	$(\mathbb{C}, 1)$
Type II	$\operatorname{GL}(n, \mathbb{C})$	$\operatorname{GL}(m, \mathbb{C})$	$\operatorname{Sp}(2nm, \mathbb{C})$	

Irreducible reductive dual pairs over $\mathbb{C}$

Let

$$G := G_1 \times \cdots \times G_k, \qquad H := H_1 \times \cdots \times H_k,$$

where for each i the pair (G_i, H_i) is obtained from (G'_i, H'_i) by the following rule: if neither factor is an odd orthogonal group then $G_i := G'_i$ and $H_i := H'_i$, while if one factor is an odd orthogonal group then the other factor is replaced by its metaplectic double cover. Denote by $\operatorname{Mp}(\mathbb{W})^{\mathbb{C}^1}$ the $\mathbb{C}^1$-metaplectic cover of $\operatorname{Sp}(\mathbb{W})$, where $\mathbb{C}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. There exists a splitting homomorphism

$$s : G \times H \longrightarrow \operatorname{Mp}(\mathbb{W})^{\mathbb{C}^1}.$$

Let Ω_ψ be the Weil representation of $\operatorname{Mp}(\mathbb{W})^{\mathbb{C}^1}$ with respect to ψ. Via the above splitting homomorphism s, we regard Ω_ψ as a representation of $G \times H$.

Suppose π is an irreducible representation of G, define the full theta lift of π to be

$$\Theta(\pi) := (\Omega_\psi \,\widehat{\otimes}\, \pi^{\vee})_G / \overline{\{0\}}.$$

By Howe's pioneering work [How79] [How89a] [How89b], $\Theta(\pi)$ is of finite length as H representation, and hence belongs to $\operatorname{Rep}_H^{\mathrm{CW}}$. Moreover, when $\Theta(\pi)$ is non-zero, it admits

a unique irreducible quotient. The analogous statement over p-adic fields, formulated by Howe as the Howe duality conjecture, was subsequently established in full generality by Gan–Takeda [GT16] and Gan–Sun [GS17].

3. Schwartz homology and reduction step

In this section, we review necessary facts on Schwartz analysis and Schwartz homology. In Subsection 3.3, we show that the proof of Theorem 1.1 can be reduced to that of Theorem 1.2.

3.1. **Schwartz induction.** We work with induced representations in the Schwartz setting. Let G be an almost linear Nash group, $H \subseteq G$ a Nash subgroup, and V a smooth Fréchet representation of H of moderate growth. We refer to [CS21, Subsection 6.2] for the definition of the Schwartz induced representation $\mathrm{ind}_H^G V$. When $H \backslash G$ is compact, it coincides with the usual smooth induced representation.

For a tempered bundle $\mathcal{E}$ over a Nash manifold X, we write $\Gamma^{\mathcal{S}}(X, \mathcal{E})$ for the space of Schwartz sections of $\mathcal{E}$ over X in the sense of [CS21, Definition 6.1]. The Schwartz induction $\mathrm{ind}_H^G V$ admits the following geometric interpretation. Let $\mathcal{V} = G \times^H V$ be the associated homogeneous bundle. It is a (right) G-tempered bundle over $H \backslash G$; see [CS21, Subsection 3.3] and [Xue20, Subsection 2.1]. By [CS21, Proposition 6.7], we have

$$(1) \qquad \mathrm{ind}_H^G V \cong \Gamma^{\mathcal{S}}(H \backslash G, \mathcal{V}).$$

To study the filtration for the Type II dual pairs, we need the following generalization of equality (1).

Lemma 3.1. *Let G be an almost linear Nash group and $H \subseteq G$ a closed Nash subgroup. Let X' be a Nash manifold with a right H-action. We define an H-action on $X' \times G$ by*

$$h \cdot (x', g) = (x'h, h^{-1}g).$$

Let

$$X := X' \times_H G$$

be the quotient of $X' \times G$ by this H-action, which admits a right G-action. Let $\mathcal{E}$ be a (right) G-tempered bundle over X. Denote by π the natural action of G on $\Gamma^{\mathcal{S}}(X, \mathcal{E})$, and ρ the action of H on $\Gamma^{\mathcal{S}}(X', \mathcal{E}|_{X'})$, then

$$\pi \cong \mathrm{ind}_H^G \rho.$$

Proof. We construct the isomorphism explicitly as follows:

$$\begin{array}{ccl} \pi & \longrightarrow & \mathrm{ind}_H^G \rho \\ \phi & \longmapsto & (g \mapsto (\pi(g)\phi)|_{X'}) \end{array},$$

$$\begin{array}{ccl} \mathrm{ind}_H^G \rho & \longrightarrow & \pi \\ f & \longmapsto & (x \mapsto r_g(f(g)(x'))) \end{array},$$

where $x = (x', g)$, and r_g denotes the action of $g \in G$ on $\mathcal{E}$. One can check that the second map is well-defined and that these two maps are inverses of each other. $\square$

3.2. **Schwartz homology.** The theory of Schwartz homology is developed in [CS21]. In this subsection, we introduce the necessary tools that will be used in our subsequent proof. We refer the readers to [CS21] for more details.

Let G be an almost linear Nash group. Denote by $\mathcal{S}\mathrm{mod}_G$ the category of smooth Fréchet representations of G of moderate growth. For $V \in \mathcal{S}\mathrm{mod}_G$, consider the space of G-coinvariants

$$V_G := V/(\sum_{g \in G}(g-1) \cdot V),$$

which is given the quotient topology and becomes a locally convex (not necessarily Hausdorff) topological vector space. In [CS21], the Schwartz homology groups $\mathrm{H}_i^{\mathcal{S}}(G, -)$ are introduced so that

$$\mathrm{H}_0^{\mathcal{S}}(G, V) \cong V_G,$$

and each $\mathrm{H}_i^{\mathcal{S}}(G, V)$ is naturally a locally convex (not necessarily Hausdorff) topological vector space. This homology theory possesses the following favorable properties.

Proposition 3.2 (Shapiro's Lemma, [CS21, Theorem 7.5])**.** *Let H be a Nash subgroup of G and $V \in \mathcal{S}\mathrm{mod}_H$. Then for all $i \in \mathbb{Z}_{\geq 0}$, there is an identification of topological vector spaces*

$$\mathrm{H}_i^{\mathcal{S}}(G, (\mathrm{ind}_H^G(V \otimes \delta_H)) \otimes \delta_G^{-1}) \cong \mathrm{H}_i^{\mathcal{S}}(H, V).$$

Here, δ_G (resp. δ_H) denotes the modular character of G (resp. H).

Proposition 3.3 ([CS21, Theorem 7.7])**.** *For every representation V in $\mathcal{S}\mathrm{mod}_G$ and every $i \in \mathbb{Z}_{\geq 0}$, there is an identification of topological vector spaces*

$$\mathrm{H}_i^{\mathcal{S}}(G, V) \cong \mathrm{H}_i(\mathfrak{g}, K; V).$$

Here and henceforth, for $V \in \mathcal{S}\mathrm{mod}_G$, $\mathrm{H}_*(\mathfrak{g}, K; V)$ denotes the homology groups of the Koszul complex

$$\cdots \to (\wedge^{l+1}(\mathfrak{g}/\mathfrak{k}) \otimes V)_K \to (\wedge^l(\mathfrak{g}/\mathfrak{k}) \otimes V)_K \to (\wedge^{l-1}(\mathfrak{g}/\mathfrak{k}) \otimes V)_K \to \cdots$$

Proposition 3.4 ([CS21, Corollary 7.8])**.** *Every short exact sequence*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

in the category $\mathcal{S}\mathrm{mod}_G$ yields a long exact sequence

$$\cdots \to \mathrm{H}_{i+1}^{\mathcal{S}}(G, V_3) \to \mathrm{H}_i^{\mathcal{S}}(G, V_1) \to \mathrm{H}_i^{\mathcal{S}}(G, V_2) \to \mathrm{H}_i^{\mathcal{S}}(G, V_3) \to \cdots$$

of (not necessarily Hausdorff) locally convex topological vector spaces.

It is important in practice to determine the Hausdorffness of the Schwartz homology. We have the following useful propositions.

Proposition 3.5 ([CS21, Theorem 5.9, Proposition 5.7])**.** *Let V be a relatively projective representation in $\mathcal{S}\mathrm{mod}_G$. Then the coinvariant space V_G is a Fréchet space. Moreover, when G is compact, every representation in $\mathcal{S}\mathrm{mod}_G$ is relatively projective.*

Proposition 3.6 ([BW80, Lemma 3.4], [CS21, Proposition 1.9])**.** *Let $V \in \mathcal{S}\mathrm{mod}_G$ and $i \in \mathbb{Z}_{\geq 0}$. If $\mathrm{H}_i^{\mathcal{S}}(G; V)$ is finite-dimensional, then it is Hausdorff.*

Proposition 3.7 (Künneth formula, [Gen25, Proposition 3.8])**.** *Let G_1 and G_2 be two almost linear Nash groups. Assume $V_i \in \mathcal{S}\mathrm{mod}_{G_i}$, $i = 1, 2$, are both NF-spaces. If for all $j \in \mathbb{Z}_{\geq 0}$, $\mathrm{H}_j^{\mathcal{S}}(G_i, V_i)$ are NF-spaces, then there is an isomorphism*

$$\mathrm{H}_m^{\mathcal{S}}(G_1 \times G_2, V_1 \widehat{\otimes} V_2) \cong \bigoplus_{p+q=m} \mathrm{H}_p^{\mathcal{S}}(G_1, V_1) \widehat{\otimes} \mathrm{H}_q^{\mathcal{S}}(G_2, V_2), \quad \forall\, m \in \mathbb{Z}_{\geq 0},$$

as topological vector spaces. In particular, $\mathrm{H}_m^{\mathcal{S}}(G_1 \times G_2, V_1 \widehat{\otimes} V_2)$ is an NF-space.

Proposition 3.8 (Hochschild-Serre spectral sequence for nilpotent normal subgroups)**.** *Let $H = L \ltimes N$ be an almost linear Nash group with N being a nilpotent normal subgroup. Consider $V \in \mathcal{S}\mathrm{mod}_H$, if for all $j \in \mathbb{Z}_{\geq 0}$, $\mathrm{H}_j^{\mathcal{S}}(N, V) \in \mathcal{S}\mathrm{mod}_L$, then there exists convergent first quadrant spectral sequences:*

$$E_{p,q}^2 = \mathrm{H}_p^{\mathcal{S}}(L, \mathrm{H}_q^{\mathcal{S}}(N, V)) \implies \mathrm{H}_{p+q}^{\mathcal{S}}(H, V).$$

Furthermore, assume $V \in \mathcal{S}\mathrm{mod}_{G \times H}$, if for all $p, q \in \mathbb{Z}_{\geq 0}$, $\mathrm{H}_p^{\mathcal{S}}(L, \mathrm{H}_q^{\mathcal{S}}(N, V))$ are Casselman–Wallach representations of G, then so is $\mathrm{H}_i^{\mathcal{S}}(H, V)$ for all $i \in \mathbb{Z}_{\geq 0}$.

Proof. The first assertion is exactly [Gen25, Proposition 3.9]. The second assertion follows from [WZ25, Lemma 2.15] together with Lemma 2.1. $\qquad\square$

Lemma 3.9 ([AGS15b, Lemma 6.2.2])**.** *Let X be a Nash manifold and let V be a real vector space. Let $\phi : X \to V^*$ be a Nash map. Suppose $0 \in V^*$ is a regular value of ϕ. Let $\mathcal{T}(X)$ be the space of tempered functions on X. Then ϕ induces a map*

$$\chi : V \to \mathcal{T}(X), \quad \chi(v)(x) = \exp(\pi\sqrt{-1}\mathrm{Re}(\phi(x)(v))),$$

which gives an action of V on $S(X)$ by $\pi(v)(f) := \chi(v) \cdot f$ (the usual multiplication). We have the following assertions.

(i) $\mathrm{H}_i^{\mathcal{S}}(\mathfrak{v}, S(X)) = 0$ for $i > 0$.

(ii) Let $X_0 := \phi^{-1}(0)$, which is smooth by the regularity assumption. Denote by r the restriction map $r : S(X) \to S(X_0)$. Then r gives an isomorphism $\mathrm{H}_0^{\mathcal{S}}(\mathfrak{v}, S(X)) \xrightarrow{\sim} S(X_0)$. In particular, if X_0 is empty, then $\mathrm{H}_0^{\mathcal{S}}(\mathfrak{v}, S(X)) = 0$.

We also need the following two lemmas when studying the filtration on the Weil representation.

Lemma 3.10. *Let G, H be two almost linear Nash groups, and P (resp. Q) be a Nash subgroup of G (resp. H). Take $\pi \in \mathcal{S}\mathrm{mod}_{P \times Q}$. Assume that $\mathrm{H}_i^{\mathcal{S}}(P, \pi \otimes \delta_{G/P})$ is Hausdorff for all $i \geq 0$. Then for all $i \geq 0$, there is an identification in $\mathcal{S}\mathrm{mod}_H$*

$$\mathrm{H}_i^{\mathcal{S}}(G, \mathrm{ind}_{P \times Q}^{G \times H} \pi) \cong \mathrm{ind}_Q^H \mathrm{H}_i^{\mathcal{S}}(P, \pi \otimes \delta_{G/P}).$$

Here, $\delta_{G/P} := (\delta_G)|_P \cdot \delta_P^{-1}$.

Proof. Let $R_\bullet \twoheadrightarrow \pi$ be a $P \times Q$ strong relative projective resolution of π, with each R_i an NF-space. Then $\mathrm{ind}_{P \times Q}^{G \times H} R_\bullet \twoheadrightarrow \mathrm{ind}_{P \times Q}^{G \times H} \pi$ is a $G \times H$ strong relative projective resolution of $\mathrm{ind}_{P \times Q}^{G \times H} \pi$. Note that

$$(\mathrm{ind}_{P \times Q}^{G \times H} R_i)_G = (S(H) \widehat{\otimes} R_i \otimes \delta_{G/P} \otimes \delta_Q^{-1})_{P \times Q} = \mathrm{ind}_Q^H (R_i \otimes \delta_{G/P})_P.$$

Thus $\mathrm{H}_i^{\mathcal{S}}(G, \mathrm{ind}_{P \times Q}^{G \times H} \pi)$ is the i-th homology group of the chain complex

$$\cdots \to \mathrm{ind}_Q^H (R_i \otimes \delta_{G/P})_P \to \cdots.$$

Denote $R_\bullet' := R_\bullet \otimes \delta_{G/P}$. Consider the chain complex

$$\cdots \to (R_{i+1}')_P \xrightarrow{d_{i+1}} (R_i')_P \xrightarrow{d_i} (R_{i-1}')_P \to \cdots.$$

Set $K_i := \mathrm{Ker}\, d_i$, $B_i := \mathrm{Im}\, d_{i+1}$, both equipped with subspace topology of $(R_i')_P$. Note that K_i is clearly closed in $(R_i')_P$, and B_i is also closed in $(R_i')_P$ since $\mathrm{H}_i^{\mathcal{S}}(P, \pi \otimes \delta_{G/P})$ is Hausdorff. As ind_Q^H is an exact functor, we obtain

$$0 \to \mathrm{ind}_Q^H K_i \to \mathrm{ind}_Q^H (R_i')_P \to \mathrm{ind}_Q^H (R_{i-1}')_P,$$

$$0 \to \mathrm{ind}_Q^H K_{i+1} \to \mathrm{ind}_Q^H (R_{i+1}')_P \to \mathrm{ind}_Q^H B_i \to 0.$$

Hence,
$$\mathrm{H}_i^{\mathcal{S}}(G, \mathrm{ind}_{P \times Q}^{G \times H} \pi) = \frac{\mathrm{ind}_Q^H K_i}{\mathrm{ind}_Q^H B_i}.$$

Moreover, we have
$$0 \to \mathrm{ind}_Q^H B_i \to \mathrm{ind}_Q^H K_i \to \mathrm{ind}_Q^H \mathrm{H}_i^{\mathcal{S}}(P, \pi \otimes \delta_{G/P}) \to 0$$

According to the open mapping theorem, it follows that
$$\mathrm{H}_i^{\mathcal{S}}(G, \mathrm{ind}_{P \times Q}^{G \times H} \pi) \cong \mathrm{ind}_Q^H \mathrm{H}_i^{\mathcal{S}}(P, \pi \otimes \delta_{G/P}).$$

$\square$

The following lemma provides us a sufficient condition under which inverse limits and Schwartz homology can be switched.

Lemma 3.11. *Assume that $V \in \mathcal{S}\mathrm{mod}_{G \times H}$ admits a $G \times H$ decreasing filtration*
$$V =: V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq \cdots$$
such that $V \cong \varprojlim_k V/V_k$. Suppose that:

(1) For all i and k, $\mathrm{H}_i^{\mathcal{S}}(G, V_k/V_{k+1})$ is a Casselman–Wallach representation of H;
(2) There exists an integer K such that for all $k > K$ and all i, $\mathrm{H}_i^{\mathcal{S}}(G, V_k/V_{k+1}) = 0$.
Then, for all i, we have a natural topological isomorphism
$$\mathrm{H}_i^{\mathcal{S}}(G, \varprojlim_k V/V_k) \cong \varprojlim_k \mathrm{H}_i^{\mathcal{S}}(G, V/V_k),$$
and the space on either side is a Casselman–Wallach representation of H.

Proof. This follows from [Gro61, Proposition 13.2.3], with cochain complexes K_α^* in [Gro61, Proposition 13.2.3] replaced by chain complexes $P_{k,\bullet}$ where
$$P_{k,\bullet} \to V/V_k \to 0$$
is a strong relative projective resolution. Recall that an inverse system indexed by $\mathbb{Z}_{\geq 0}$
$$\cdots \to U_{\alpha+1} \to U_\alpha \to \cdots$$
satisfies the Mittag-Leffler condition if for any $\alpha \in \mathbb{Z}_{\geq 0}$, there is some $\beta \geq \alpha$ so that the image of
$$U_\gamma \to U_\alpha$$
are the same for all $\gamma \geq \beta$. We note that the system
$$\cdots \to V/V_{k+1} \to V/V_k \to \cdots$$
satisfies the Mittag-Leffler condition since the maps are all surjective. The system
$$\cdots \to \mathrm{H}_{i+1}^{\mathcal{S}}(G, V/V_{k+1}) \to \mathrm{H}_{i+1}^{\mathcal{S}}(G, V/V_k) \to \cdots$$
also satisfies the Mittag-Leffler condition by assumption (2). The proof of [Gro61, Proposition 13.2.3] can now be copied word by word. $\square$

We conclude this subsection by recalling a lemma concerning the action on coinvariant spaces.

Lemma 3.12 ([FSX18, Lemma 2.5]). *Let $\mathrm{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and set $G := \mathrm{GL}_n(\mathrm{D})$. Then $S(G)$ carries the natural left and right translation actions of $G \times G$. Write G_l for the subgroup $G \times \{1\}$ of $G \times G$. For $\pi \in \mathcal{S}\mathrm{mod}_G$, we have*
$$(S(G) \widehat{\otimes} \pi)_{G_l} \cong \pi,$$
as representations of G.

3.3. **Reduction step.** In this subsection, we reduce the proof of Theorem 1.1 to the case of the principal series. The following lemma is crucial.

Lemma 3.13 ([HT90, Corallory A.11]). *Let $0 \to E \to F \to G \to 0$ be a short exact sequence of complexes of nuclear Fréchet spaces. Let $A \to B \to C$ be any three consecutive terms in the associated long exact sequence. If C is Hausdorff, then $A \to B$ is a topological homomorphism, i.e., it is continuous and is an open map onto its image. If, in addition, $A \to B$ has closed kernel, then B is Hausdorff.*

We are now ready to prove the following proposition, which reduces Theorem 1.1 to Theorem 1.2.

Proposition 3.14. *For (G, H) as in Subsection 2.2, Theorem 1.2 implies Theorem 1.1.*

Proof. Let (G, H) be as in Subsection 2.2, and denote by Ω the Weil representation of $G \times H$.

Let π be a Casselman–Wallach representation of G. We aim to show that, for all $i \geq 0$, $\mathrm{H}_i^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi)$ is a Casselman–Wallach representation of H. We proceed by downward induction on i. Let $n = \dim(G/K)$, where K is a maximal compact subgroup of G. The base case is $i = n + 1$, for which all the homology groups vanish.

Now we do the induction step. Assume that for every $\pi \in \mathrm{Rep}_G^{\mathrm{CW}}$, $\mathrm{H}_{k+1}^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi)$ is a Casselman–Wallach representation of H, we claim that, for every $\pi \in \mathrm{Rep}_G^{\mathrm{CW}}$, $\mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi)$ is also Casselman–Wallach. We prove this claim by induction on the length of π. First, assume that π is irreducible. In this case, by Casselman's subrepresentation theorem, we have a short exact sequence

$$0 \to \pi \to I \to \sigma \to 0,$$

where I is a principal series representation. Then we have

$$\cdots \to \mathrm{H}_{k+1}^{\mathcal{S}}(G, \Omega \widehat{\otimes} I) \to \mathrm{H}_{k+1}^{\mathcal{S}}(G, \Omega \widehat{\otimes} \sigma) \to \mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi) \to \mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} I) \to \cdots.$$

By Theorem 1.2, the induction hypothesis, Lemma 2.1 and Lemma 3.13, we deduce that $\mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi)$ is Casselman–Wallach.

Assume the claim is proved for every π with length at most l. Let π be a Casselman–Wallach representation of length $l + 1$, and we take a short exact sequence

$$0 \to \rho \to \pi \to \tau \to 0,$$

such that τ is irreducible and ρ is of length l (such an exact sequence clearly exists). It gives rise to the long exact sequence

$$\cdots \to \mathrm{H}_{k+1}^{\mathcal{S}}(G, \Omega \widehat{\otimes} \tau) \to \mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \rho) \to \mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi) \to \mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \tau) \to \cdots.$$

Lemma 3.13 thus implies that $\mathrm{H}_k^{\mathcal{S}}(G, \Omega \widehat{\otimes} \pi)$ is Casselman–Wallach. This concludes the induction step. $\qquad\square$

4. TYPE II DUAL PAIRS

In this section, we prove Theorem 1.2 for the Type II dual pairs. Our main tool is the corank-one parabolic stable filtration on the Weil representation. Using this filtration, we reduce the problem for $(\mathrm{GL}_{n+1}, \mathrm{GL}_m)$ to the case $(\mathrm{GL}_n, \mathrm{GL}_m)$, and the basic case $(\mathrm{GL}_1, \mathrm{GL}_m)$ is handled by direct calculation in Subsection 4.1.

We fix the notation for this section. Let F be an archimedean local field and let D be a central division algebra over F of dimension r^2. Let $\psi : F \to \mathbb{C}^{\times}$ be a non-trivial unitary character of F. For positive integers n, m, denote by $M_{n,m}$ the space of $n \times m$ matrices with entries in D, and set $M_{0,m} = M_{n,0} := \{0\}$. Let $\mathrm{Nrd} : M_{n,n} \to F$ be the

reduced norm and denote $\nu := |\operatorname{Nrd}|_F^r$, where $|\cdot|_F$ is the normalized absolute value of F. Let $\operatorname{Trd} : M_{n,n} \to F$ be the reduced trace. For $x \in M_{n,m}$, we write ${}^t x$ for the transpose of x when $\mathrm{D} = \mathbb{R}$, and conjugate transpose of x when $\mathrm{D} = \mathbb{C}$ or $\mathbb{H}$.

To distinguish the two factors in the Type II dual pair, we set

$$G_n := \operatorname{GL}_n(\mathrm{D}), \qquad H_m := \operatorname{GL}_m(\mathrm{D}),$$

and also put $G_0 = H_0 := \{1\}$. For $k \geq 1$, denote by $S(G_k)$ the space of Schwartz functions on G_k and let ρ_k be the natural action of $G_k \times H_k$ on $S(G_k)$:

$$(\rho_k(g,h)\phi)(x) := \phi(g^{-1}xh), \qquad g, x \in G_k, \ h \in H_k, \ \phi \in S(G_n).$$

Let $S(M_{n,m})$ denote the space of Schwartz functions on $M_{n,m}$. Denote by $\sigma_{n,m}$ the natural action of $G_n \times H_m$ on $S(M_{n,m})$, which is given by

$$(\sigma_{n,m}(g,h)\phi)(x) := \phi(g^{-1}xh), \qquad g \in G_n, \ h \in H_m, \ x \in M_{n,m}, \ \phi \in S(M_{n,m}).$$

The Weil representation $\Omega_{n,m}$ of the dual pair (G_n, H_m) is given by

$$\Omega_{n,m}(g,h) := \nu(g)^{-\frac{m}{2}} \nu(h)^{\frac{n}{2}} \sigma_{n,m}(g,h).$$

For $0 \leq t \leq n$, let $P_{t,n-t}$ (or P_t when no confusion arises) denote the standard parabolic subgroup of G_n attached to the partition $(t, n-t)$. We write the Levi decomposition as $P_t = L_t \ltimes N_t$ with $L_t = G_t \times G_{n-t}$. Similarly, we define the standard parabolic subgroup Q_k of H_m for $0 \leq k \leq m$.

4.1. **The case of** $(\operatorname{GL}_1, \operatorname{GL}_m)$. The case of $(\operatorname{GL}_1, \operatorname{GL}_m)$ is our basic case in the induction step. In this subsection, we explicitly treat the full theta lift for this case. To simplify notation, we use the action $\sigma_{1,m}$. Since $\Omega_{1,m}$ differs from $\sigma_{1,m}$ only by a character, this causes no loss of generality.

Consider $\mathrm{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Note that

$$(2) \qquad \mathbb{R}^\times = \{\pm 1\} \times \mathbb{R}_{>0}^\times, \ \mathbb{C}^\times = \mathrm{U}(1) \times \mathbb{R}_{>0}^\times, \ \mathbb{H}^\times = \mathrm{SU}(2) \times \mathbb{R}_{>0}^\times,$$

where the first factor is always compact.

Let $V := M_{1,m}$, $U := M_{1,m} \setminus \{0\}$. Set $r = \dim_{\mathbb{R}}(V)$. Then U is a principal $\mathbb{R}_{>0}^\times$-bundle over the sphere $\mathbb{S}^{r-1}$. Hence, by [CS21, Proposition 5.2], $S(U)$ is a relative projective representation of $\mathbb{R}_{>0}^\times$.

Let χ be a character of $\mathbb{R}_{>0}^\times$. Consider the short exact sequence

$$0 \to S(U) \to S(V) \to \mathbb{C}[\![V]\!] \to 0,$$

then we have the associated long exact sequence

$$(3) \quad 0 \to \mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(V) \otimes \chi) \to \mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi) \xrightarrow{\delta} \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(U) \otimes \chi)$$
$$\to \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(V) \otimes \chi) \to \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi) \to 0.$$

For $k \in \mathbb{Z}_{\geq 0}$, define a character χ_k of $\mathbb{R}_{>0}^\times$ by $\chi_k(x) = x^k$. When $\chi \neq \chi_k$ for any $k \in \mathbb{Z}_{\geq 0}$, using Borel's Lemma [CS21, Proposition 8.2, Proposition 8.3, and Theorem 8.5], one can easily show that

$$\mathrm{H}_i^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi) = 0, \quad \forall i \in \mathbb{Z}_{\geq 0}.$$

For the character χ_k, we have the following lemma.

Lemma 4.1. *Both spaces* $\mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi_k)$ *and* $\mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi_k)$ *are finite dimensional. Indeed, they coincide and are equal to the space of homogeneous polynomials of total degree k:*

$$\mathrm{Span}_{\mathbb{C}}\{\prod_{i=1}^r x_i^{a_i} \mid \sum_{i=1}^r a_i = k, a_i \in \mathbb{Z}_{\geq 0}\},$$

where $x_1, \ldots, x_r$ *denote the* $\mathbb{R}$*-coordinates on* V.

Proof. The lemma follows by direct calculation. $\qquad\square$

Using Lemma 3.13 together with the long exact sequence (3), we obtain the following corollary.

Corollary 4.2. $\mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(V) \otimes \chi)$ *is Hausdorff.*

Proof. According to [CS21, Proposition 5.5], $S(U) \otimes \chi$ is a relative projective representation of $\mathbb{R}_{>0}^\times$. Hence, by Proposition 3.5, $\mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(U) \otimes \chi)$ is Hausdorff.

Consider the exact sequence (3)

$$\mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi) \xrightarrow{\delta} \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(U) \otimes \chi) \to \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(V) \otimes \chi) \to \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi).$$

According to Lemma 4.1 and the discussion preceding it, the first and the last terms are finite-dimensional, and hence Hausdorff by Proposition 3.6. Since $\mathrm{Im}\,\delta$ is finite-dimensional, it is a closed subspace of $\mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(U) \otimes \chi)$. Therefore, the corollary follows from Lemma 3.13. $\qquad\square$

We now turn to the $\mathrm{H}_1^{\mathcal{S}}$ term.

Lemma 4.3.
$$\mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(V) \otimes \chi) = 0$$

Proof. Consider the exact sequence (3). When $\chi \neq \chi_k$, the lemma follows immediately from the discussion preceding Lemma 4.1. Hence, it remains to treat the case $\chi = \chi_k$. It suffices to verify that the connecting map δ in (3) is injective, which will be done by a diagram chase:

$$
\begin{array}{ccc}
\mathfrak{t} \otimes S(V) \otimes \chi_k & \longrightarrow & \mathfrak{t} \otimes \mathbb{C}[\![V]\!] \otimes \chi_k \\
\downarrow & & \\
& & \\
S(U) \otimes \chi_k & \longrightarrow & S(V) \otimes \chi_k
\end{array}
\qquad .
$$

The map $\mathrm{H}_1^{\mathcal{S}}(\mathbb{R}_{>0}^\times, \mathbb{C}[\![V]\!] \otimes \chi_k) \xrightarrow{\delta} \mathrm{H}_0^{\mathcal{S}}(\mathbb{R}_{>0}^\times, S(U) \otimes \chi) = S(\mathbb{S}^{r-1})$ is then given by

$$\prod_{i=1}^r x_i^{d_i} \mapsto F(a_1, \cdots, a_r) = -\frac{1}{k!} \prod_{i=i}^r a_i^{d_i}.$$

Since the image of every element is a homogeneous polynomial, it cannot vanish identically. Thus, the connecting map δ in (3) is injective. $\qquad\square$

Corollary 4.4. *Let* $\pi \in \mathrm{Rep}_{G_1}^{\mathrm{CW}}$. *Then for every* $i \geq 1$, *one has*

$$\mathrm{H}_i^{\mathcal{S}}(\mathrm{D}^\times, S(V) \otimes \pi) = 0.$$

Proof. Using the standard spectral sequence argument with respect to the group decomposition (2), this follows from Lemma 4.3 and Proposition 3.5. $\qquad\square$

Hence, for $\pi \in \mathrm{Rep}_{G_1}^{\mathrm{CW}}$, there is a long exact sequence

$$(4) \quad 0 \to \mathrm{H}_1^{\mathcal{S}}(\mathrm{D}^\times, \mathbb{C}[\![V]\!] \otimes \pi) \to \mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, S(U) \otimes \pi) \to \mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, S(V) \otimes \pi)$$
$$\to \mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, \mathbb{C}[\![V]\!] \otimes \pi) \to 0.$$

Based on Lemma 4.1 and the preceding discussion, the spectral sequence arising from the decomposition (2) shows that the first and last terms are finite-dimensional.

Proposition 4.5. *Let $\pi \in \mathrm{Rep}_{G_1}^{\mathrm{CW}}$. Then for every $m \geq 1$, one has*

$$\mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, S(V) \mathbin{\widehat{\otimes}} \pi) \in \mathrm{Rep}_{H_m}^{\mathrm{CW}}.$$

Proof. Denote by $\overline{Q_1}$ the opposite parabolic subgroup of Q_1 in H_m. Note that

$$\mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, S(U) \otimes \pi) = \mathrm{H}_0^{\mathcal{S}}(\mathrm{D}^\times, \mathrm{ind}_{G_1 \times \overline{Q_1}}^{G_1 \times H_m}(\rho_1 \boxtimes \mathbb{C}) \otimes \pi) = \mathrm{ind}_{\overline{Q_1}}^{H_m}(\pi \boxtimes \mathbb{C}),$$

where the first equality is obtained by direct calculation, and the second is a consequence of Lemmas 3.10 and 3.12. The proposition now follows from Lemma 3.13, the long exact sequence (4), and the discussion following (4). $\qquad\square$

Combining Corollary 4.4 with Proposition 4.5, we complete the proof of Theorem 1.1 for dual pairs $(\mathrm{GL}_1, \mathrm{GL}_m)$.

4.2. **Corank-one filtration.** In this subsection, we introduce the corank-one filtration for Type II dual pairs, namely a $P_1 \times H_m$-stable filtration on the Weil representation.

We write elements of $M_{n,m}$ in column-block form as $\begin{bmatrix} a \\ b \end{bmatrix}$, where $a \in M_{1,m}, b \in M_{n-1,m}$. For $f \in S(M_{n,m})$, we define the partial Fourier transform in the $M_{1,m}$-variable by

$$\widehat{f}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) := \int_{M_{1,m}} f\left(\begin{bmatrix} x \\ b \end{bmatrix}\right) \psi(\mathrm{Trd}(^t a x)) \, dx.$$

Conjugating the natural action $\sigma_{n,m}$ by this partial Fourier transform yields an equivalent model $\sigma_{n,m}^*$ of the $G_n \times H_m$-action on $S(M_{n,m})$. This realization is more convenient for our purposes. Below, we record several explicit formulas for $\sigma_{n,m}^*$.

$$(5) \qquad \sigma_{n,m}^*\left(\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix}, h\right) f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \nu(g_1)^m \nu(h)^{-1} f\left(\begin{bmatrix} ^t g_1 \cdot a \cdot {}^t h^{-1} \\ g_2^{-1} \cdot b \cdot h \end{bmatrix}\right),$$

where $g_1 \in G_1$, $g_2 \in G_{n-1}$, and $h \in H_m$.

$$(6) \qquad \sigma_{n,m}^*\left(\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}, 1\right) f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \psi(\mathrm{Trd}(b \cdot {}^t a \cdot u)) f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right),$$

where $u \in M_{1,n-1}$.

Let $M := M_{n,m}$, $Z := \left\{\begin{bmatrix} 0 \\ b \end{bmatrix} \in M_{n,m} | b \in M_{n-1,m}\right\}$, $U := M \setminus Z$. The space $S(M)$ contains a $P_1 \times H_m$ stable subspace $S(U)$. Set $S_Z(M) := S(M)/S(U)$. Then we have

$$(\mathrm{F\text{-}II}) \qquad\qquad 0 \to S(U) \to S(M) \to S_Z(M) \to 0.$$

Denote $V := M_{1,m}$ and view it as a real manifold. Note that $S_Z(M) \cong \mathbb{C}[\![V]\!] \mathbin{\widehat{\otimes}} S(M_{n-1,m})$.

Denote by $\mathfrak{n}_1$ the complexified Lie algebra of N_1. We first study the $\mathfrak{n}_1$-homology of the open piece. The following lemma holds.

Lemma 4.6.

$$\mathrm{H}_i(\mathfrak{n}_1, S(U)) = \begin{cases} \mathrm{ind}_{L_1 \times Q_1}^{L_1 \times H_m}(\lambda_{1,1} \otimes \rho_1 \,\widehat{\otimes}\, \sigma_{n-1,m-1}) & \text{if } i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

Here:

- $G_1 \times H_1$ *acts on* ρ_1;
- $G_{n-1} \times H_{m-1}$ *acts on* $\sigma_{n-1,m-1}$;
- $\lambda_{1,1}$ *is the character defined by*

$$\lambda_{1,1} = \begin{cases} \nu^m & \text{on } G_1 \\ \nu^{-1} & \text{on } Q_1 \end{cases}.$$

Proof. Define

$$\mathcal{N} := \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in M_{n,m} \,|\, a \in M_{1,m}, b \in M_{n-1,m}, b \cdot {}^t a = 0 \right\}.$$

Consider the map

$$\phi: \begin{array}{ccc} U & \to & N_1^* \\ \begin{bmatrix} a \\ b \end{bmatrix} & \mapsto & (u \mapsto \mathrm{Trd}(b \cdot {}^t a \cdot u)) \end{array}.$$

One can verify directly that 0 is a regular value of ϕ, and $\phi^{-1}(0) = \mathcal{N} \setminus Z$. Following Lemma 3.9, we have

$$\mathrm{H}_i(\mathfrak{n}_1, S(U)) = \begin{cases} S(\mathcal{N} \setminus Z) & \text{if } i = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then, together with Lemma 3.1, the proposition follows from the same formal calculation as in [Mín08, Section 3]. $\qquad\square$

Now we turn to the closed piece. Recall that $S_Z(M) \cong \mathbb{C}[\![V]\!] \,\widehat{\otimes}\, S(M_{n-1,m})$. It admits a decreasing filtration by degree $S_Z(M)_k := \mathbb{C}[\![V]\!]_{\geq k} \,\widehat{\otimes}\, S(M_{n-1,m})$. The associated graded pieces are given by

$$S_Z(M)_k / S_Z(M)_{k+1} = F_k \otimes S(M_{n-1,m}),$$

where F_k is the space of homogeneous polynomials on V of total degree k.

Lemma 4.7. *As a representation of $P_1 \times H_m$,*

$$S_Z(M)_k / S_Z(M)_{k+1} \cong \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m}.$$

Here:

- N_1 *acts trivially;*
- $G_{n-1} \times H_m$ *acts on* $\sigma_{n-1,m}$;
- F_k *carries a $G_1 \times H_m$-action induced from the following action on V:*

$$(g_1, h) \cdot v = {}^t g_1^{-1} \cdot v \cdot {}^t h, \quad v \in V;$$

- $\lambda_{1,0}$ *is the character defined by*

$$\lambda_{1,0} = \begin{cases} \nu^m & \text{on } G_1 \\ \nu^{-1} & \text{on } H_m \end{cases}.$$

Proof. Using the formulas (5) and (6), the lemma follows from a direct calculation. $\qquad\square$

13

4.3. **Proof of Theorem 1.2: Type II case.** In this subsection, we prove the following proposition, and use it to prove Theorem 1.2 at the end.

Proposition 4.8. *Let $n \geq 2$, and let χ be a Casselman–Wallach representation of G_1. Assume that for all i, m, and every $\tau \in \mathrm{Rep}^{\mathrm{CW}}_{G_{n-1}}$,*

$$\text{(C-II)} \qquad \mathrm{H}^{\mathcal{S}}_i(G_{n-1}, \Omega_{n-1,m} \widehat{\otimes} \tau) \text{ is a Casselman–Wallach representation of } H_m.$$

Then for all i, m, and every $\pi \in \mathrm{Rep}^{\mathrm{CW}}_{G_{n-1}}$,

$$\mathrm{H}^{\mathcal{S}}_i(G_n, \Omega_{n,m} \widehat{\otimes} \mathrm{Ind}^{G_n}_{P_1}(\chi \boxtimes \pi))$$

is also a Casselman–Wallach representation of H_m.

Following Shapiro's lemma, we know that

$$\mathrm{H}^{\mathcal{S}}_i(G_n, \Omega_{n,m} \widehat{\otimes} \mathrm{Ind}^{G_{n+1}}_{P_1}(\chi \boxtimes \pi)) \cong \mathrm{H}^{\mathcal{S}}_i(P_1, \Omega_{n,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1}).$$

Consider the short exact sequence (F-II) in Subsection 4.2,

$$0 \to S(U) \to S(M) \to S_Z(M) \to 0.$$

According to Lemma 3.13, Proposition 4.8 follows once we show that, under the condition (C-II),

$$\mathrm{H}^{\mathcal{S}}_i(P_1, S(U) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1}) \quad \text{and} \quad \mathrm{H}^{\mathcal{S}}_i(P_1, S_Z(M) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1})$$

are Casselman–Wallach representations for all $i \geq 0$. These will be proved in Lemmas 4.9–4.11. Throughout these lemmas, we assume the condition (C-II).

We begin with the open piece.

Lemma 4.9. *For all $i \geq 0$, $\mathrm{H}^{\mathcal{S}}_i(P_1, S(U) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1})$ is a Casselman–Wallach representation of H_m.*

Proof. Using Lemma 4.6 and the spectral sequence argument, we obtain

$$\mathrm{H}^{\mathcal{S}}_i(P_1, S(U) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1}) \cong \mathrm{H}^{\mathcal{S}}_i(L_1, \mathrm{ind}^{L_1 \times H_m}_{L_1 \times Q_1}(\lambda_{1,1} \otimes \rho_1 \widehat{\otimes} \sigma_{n-1,m-1}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1}),$$

provided that the right-hand side is a Casselman–Wallach representation of H_m. According to Lemma 3.10, we have

$$\mathrm{H}^{\mathcal{S}}_i(L_1, \mathrm{ind}^{L_1 \times H_m}_{L_1 \times Q_1}(\lambda_{1,1} \otimes \rho_1 \widehat{\otimes} \sigma_{n-1,m-1}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta^{-1}_{P_1}) \cong \mathrm{ind}^{H_m}_{Q_1} \mathrm{H}^{\mathcal{S}}_i(L_1, \rho_1 \widehat{\otimes} \Omega_{n-1,m-1} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \eta),$$

where η is a certain product of modular character and normalization character. Since it has no influence on our result, we just denote it by η to simplify the notation. We shall denote its restriction to G_1 (resp. G_{n-1}) by η_1 (resp. η_{n-1}). According to the Künneth formula, we have

$$\mathrm{H}^{\mathcal{S}}_i(L_1, \rho_1 \widehat{\otimes} \Omega_{n-1,m-1} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \eta) \cong \bigoplus_{p+q=i} \mathrm{H}^{\mathcal{S}}_p(G_1, \rho_1 \otimes \chi \otimes \eta_1) \widehat{\otimes} \mathrm{H}^{\mathcal{S}}_q(G_{n-1}, \Omega_{n-1,m-1} \widehat{\otimes} \pi \otimes \eta_{n-1}).$$

By the condition (C-II), the right-hand side of this equation is a Casselman–Wallach representation of $H_1 \times H_{m-1}$. Therefore, all the representations appearing above are Casselman–Wallach representations of H_m. $\qquad \square$

Now we turn to the closed piece. As in Lemma 4.7, $S_Z(M)$ admits a decreasing filtration $S_Z(M)_k$ with graded pieces

$$S_Z(M)_k/S_Z(M)_{k+1} \cong \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m},$$

where F_k is the space of homogeneous polynomials on V of total degree k.

Lemma 4.10. *For all i and k,*

$$\mathrm{H}_i^{\mathcal{S}}(P_1, (\chi \boxtimes \pi) \widehat{\otimes}(S_Z(M)_k/S_Z(M)_{k+1}) \otimes \delta_{P_1}^{-1})$$

is a Casselman–Wallach representation of H_m. Moreover, there exists an integer K such that for all $k > K$ and all i,

$$\mathrm{H}_i^{\mathcal{S}}(P_1, (\chi \boxtimes \pi) \widehat{\otimes}(S_Z(M)_k/S_Z(M)_{k+1}) \otimes \delta_{P_1}^{-1}) = 0.$$

Proof. By the spectral sequence, it suffices to prove the statement for

$$\mathrm{H}_i^{\mathcal{S}}(L_1, \mathrm{H}_j(\mathfrak{n}_1, \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}).$$

Let $\xi_j \boxtimes \beta_j$ be an irreducible $G_1 \times G_{n-1}$ constituent of $\mathrm{H}_j(\mathfrak{n}_1, \mathbb{C})$. By Lemma 4.7 and the long exact sequence argument, we further reduce to

$$\mathrm{H}_i^{\mathcal{S}}(L_1, (\chi \boxtimes \pi) \widehat{\otimes}(\xi_j \boxtimes \beta_j) \otimes \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m} \otimes \delta_{P_1}^{-1}) \cong$$
$$\bigoplus_{p+q=i} \mathrm{H}_p^{\mathcal{S}}(G_1, \xi_j' \otimes \chi \otimes F_k) \otimes \mathrm{H}_q^{\mathcal{S}}(G_{n-1}, \beta_j' \otimes \pi \widehat{\otimes} \Omega_{n-1,m}),$$

where β_j' (resp. ξ_j') denotes β_j (resp. ξ_j) twisted by certain inessential character. By the condition (C-II), $\mathrm{H}_q^{\mathcal{S}}(G_{n-1}, \beta_j' \otimes \pi \widehat{\otimes} \Omega_{n-1,m})$ is a Casselman–Wallach representation of H_m. Meanwhile, $\mathrm{H}_p^{\mathcal{S}}(G_1, \xi_j' \otimes \chi \otimes F_k)$ is finite-dimensional. Consequently,

$$\mathrm{H}_i^{\mathcal{S}}(L_1, (\chi \boxtimes \pi) \widehat{\otimes}(\xi_j \boxtimes \beta_j) \otimes \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m} \otimes \delta_{P_1}^{-1})$$

is a Casselman–Wallach representation of H_m.

Moreover, since $\xi_j' \otimes \chi$ is fixed, there must exist a constant $K_{i,j}$, such that for all $k > K_{i,j}$, the $\mathbb{R}_{>0}^\times$-component of G_1 acts on $\xi_j' \otimes \chi \otimes F_k$ via a nontrivial character. Hence, for all $k > K_{i,j}$,

$$\mathrm{H}_i^{\mathcal{S}}(L_1, \mathrm{H}_j(\mathfrak{n}_1, \lambda_{1,0} \otimes F_k \otimes \sigma_{n-1,m}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) = 0.$$

Let $K := \max\{K_{i,j}\}$. The lemma follows. $\qquad\square$

Lemma 4.11. *For all $i \geq 0$, $\mathrm{H}_i^{\mathcal{S}}(P_1, S_Z(M) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$ is a Casselman–Wallach representation of H_m.*

Proof. It follows directly from Lemmas 3.11 and 4.10. $\qquad\square$

Proof of Proposition 4.8. Using Lemma 3.13, the proposition follows from Lemmas 4.9 and 4.11. $\qquad\square$

We are now ready to prove Theorem 1.2 for the Type II case.

Proof of Theorem 1.2: Type II case. We prove that Theorem 1.2 holds for all dual pairs (G_n, H_m) by induction on n. The base case $(\mathrm{GL}_1, \mathrm{GL}_m)$ has been treated in Subsection 4.1.

For the induction step, fix $n \geq 2$ and assume that for all m, Theorem 1.2 holds for dual pairs (G_{n-1}, H_m). By Proposition 3.14, it follows that Theorem 1.1 holds for all pairs (G_{n-1}, H_m), so the hypothesis of Proposition 4.8 is satisfied. Therefore, via induction by stages, the induction step follows from Proposition 4.8. The theorem follows. $\qquad\square$

5. Type I dual pairs

In this section, we prove Theorem 1.2 for the Type I dual pairs. Our main tool remains the corank-one parabolic stable filtration on the Weil representation.

Although the same idea also works, we do not treat irreducible dual pairs containing an odd orthogonal factor here. The issue of the associated double cover does not affect our final result but complicates the notation; we leave the details of this case to the interested reader.

We fix the notation for this section. Our conventions are mainly taken from [GKT25, Section 16.2]. Although [GKT25, Section 16.2] treats only the non-archimedean case, the formula for the mixed model remains valid in the archimedean setting.

Let F be an archimedean local field and let D be a division algebra over F with center E, together with an involution κ such that $E^\kappa = F$. For $d \in D$, write $\bar{d} := \kappa(d)$. Let $\psi : F \to \mathbb{C}^\times$ be a non-trivial unitary character of F. Let $\mathrm{Tr}_{D/F} : D \to F$ be the trace map, defined by $\mathrm{Tr}_{D/F}(x) = x$ if $D = F$, and $\mathrm{Tr}_{D/F}(x) = x + \bar{x}$ if $D \neq F$. For a D-module X, let $\mathrm{Nrd} : \mathrm{GL}(X) \to E^\times$ be the reduced norm and denote $\nu := |\mathrm{Nrd}|_E^r$, where $| \cdot |_E$ is the normalized absolute value of E, and $r = \sqrt{\dim_E D}$.

For a right D-module X, there is a canonical way to define a left D-module $^\kappa X$. Namely, we set $^\kappa X := X$ as an abelian group, and let D act on it from the left as

$$d \cdot x := x\bar{d}, \quad d \in D, \quad x \in X.$$

Note that we have $\mathrm{GL}(^\kappa X) = \mathrm{GL}(X)$ even as sets of maps on X.

Fix $\epsilon \in \{\pm 1\}$. Let

$$(G_n, H_m) = (G(W_n), H(V_m))$$

be an irreducible Type I reductive dual pair, where (W_n, q_W) is a $-\epsilon$-Hermitian right D-module of $\dim_D W_n = n$ and (V_m, q_V) an ϵ-Hermitian left D-module of $\dim_D V_m = m$. We assume that neither is an odd orthogonal group.

We fix a splitting data (χ, ψ) for G_n (relative to V_m) and one (μ, ψ) for H_m (relative to W_n), where χ, μ are unitary characters of $E^\times$. Using this splitting data, we associate the Weil representation $\Omega_{n,m}$ to $G_n \times H_m$.

5.1. Corank-one filtration.
In this subsection, we introduce the corank-one filtration for the Type I dual pairs, namely a $P_1 \times H_m$-stable filtration on the Weil representation. Here, P_1 is the parabolic subgroup of G_n defined in (8) below.

We introduce notation for the case where both W_n and V_m are isotropic. Let x_1, x_1^* be isotropic vectors in W_n such that $q_W(x_1, x_1^*) = 1$, and let y_1, y_1^* be isotropic vectors in V_m such that $q_V(y_1, y_1^*) = 1$. Set

$$X_1 := \mathrm{Span}_D\{x_1\}, \quad X_1^* := \mathrm{Span}_D\{x_1^*\}, \quad Y_1 := \mathrm{Span}_D\{y_1\}, \quad Y_1^* := \mathrm{Span}_D\{y_1^*\}.$$

Then we have the standard decompositions

$$(7) \qquad W_n = X_1 \oplus W_{n-2} \oplus X_1^* \quad \text{and} \quad V_m = Y_1 \oplus V_{m-2} \oplus Y_1^*,$$

where W_{n-2} (resp. V_{m-2}) is the orthogonal complement of $X_1 \oplus X_1^*$ (resp. $Y_1 \oplus Y_1^*$).

Let

$$(8) \qquad\qquad\qquad P_1 = L_1 N_1 \subseteq G_n$$

be the maximal parabolic subgroup of G_n stabilizing X_1. The elements in L_1 and N_1 are, respectively, of the form

$$\ell(a, g) = \begin{pmatrix} a & & \\ & g & \\ & & (a^*)^{-1} \end{pmatrix} \quad \text{and} \quad n(b, d) = \begin{pmatrix} 1 & b & d - \frac{1}{2}bb^* \\ & 1 & -b^* \\ & & 1 \end{pmatrix},$$

where $g \in G(W_{n-2})$, $a \in \mathrm{GL}(X_1)$, $b \in \mathrm{Hom}(W_{n-2}, X_1)$, and $d \in \mathrm{Herm}(X_1^*, X_1)$, with
$$\mathrm{Herm}(X_1^*, X_1) := \{d \in \mathrm{Hom}(X_1^*, X_1) | d^* = -d\}.$$

Here, the elements $a^* \in \mathrm{GL}(X_1^*)$, $b^* \in \mathrm{Hom}(X_1^*, W_{n-2})$, and $d^* \in \mathrm{Hom}(X_1^*, X_1)$ are defined by requiring that, for $w \in W_{n-2}$,
$$q_W(ax_1, x_1^*) = q_W(x_1, a^*x_1^*), \quad q_W(bw, x_1^*) = q_W(w, b^*x_1^*), \quad q_W(dx_1^*, x_1^*) = q_W(x_1^*, d^*x_1^*).$$

Denote
$$N_{1,0} := \left\{ n(0, d) = \begin{pmatrix} 1 & & d \\ & 1 & \\ & & 1 \end{pmatrix} : d \in \mathrm{Herm}(X_1^*, X_1) \right\}.$$

Thus we have
$$L_1 = \mathrm{GL}(X_1) \times G(W_{n-2}), \quad N_1 = \mathrm{Hom}(W_{n-2}, X_1) \times N_{1,0}.$$

Similarly, we let
$$Q_1 = M_1 U_1 \subseteq H_m$$
be the maximal parabolic subgroup of H_m stabilizing Y_1, where $M_1 = \mathrm{GL}(Y_1) \times H(V_{m-2})$.

We work in the mixed model. The Weil representation admits the realization
$$\Omega_{n,m} = S(X_1^* \otimes V_m) \widehat{\otimes} \Omega_{n-2,m} = S(X_1^* \otimes V_m, \ \Omega_{n-2,m}),$$
where $\Omega_{n-2,m}$ is the Weil representation of $G(W_{n-2}) \times H(V_m)$ with respect to the splitting data (χ, μ, ψ). We now describe the action explicitly. Let $\varphi \in S(X_1^* \otimes V_m, \ \Omega_{n-2,m})$, and $x = x_1^* \otimes v \in X_1^* \otimes V_m$.

- $(\Omega_{n,m}(h)\varphi)(x) = \Omega_{n-2,m}(h)\varphi(h^{-1}x), \quad h \in H(V_m),$
- $(\Omega_{n,m}(l(1, g_0))\varphi)(x) = \Omega_{n-2,m}(g_0)\varphi(x), \quad g_0 \in G(W_{n-2}),$
- $(\Omega_{n,m}(l(a, 1))\varphi)(x) = \chi_V(\mathrm{Nrd}\, a)\nu(a)^{\frac{m}{2}}\varphi(a^*x), \quad a \in \mathrm{GL}(X_1),$
- $(\Omega_{n,m}(n(b, 0))\varphi)(x) = \rho_0((b^*x, 0))\varphi(x), \quad b \in \mathrm{Hom}(W_{n-2}, X_1),$
- $(\Omega_{n,m}(n(0, d))\varphi)(x) = \psi\left(\frac{1}{2}\mathrm{Tr}_{D/F}(q_W(x_1^*, -dx_1^*)\overline{q_V(v, v)})\right)\varphi(x), \ d \in \mathrm{Herm}(X_1^*, X_1).$

Here, ρ_0 denotes the Heisenberg representation of the Heisenberg group $\mathrm{H}(W_{n-2} \otimes V_m)$ realized on the space of $\Omega_{n-2,m}$, and χ_V is the character associated with the splitting data, as defined in [GKT25, Formula (12.5)].

When V_m is isotropic, we further realize $\Omega_{n-2,m}$ using the mixed model,
$$\Omega_{n-2,m} = S(W_{n-2} \otimes Y_1^*) \widehat{\otimes} \Omega_{n-2,m-2} = S(W_{n-2} \otimes Y_1^*, \ \Omega_{n-2,m-2}),$$
where $\Omega_{n-2,m-2}$ is the Weil representation of $G(W_{n-2}) \times H(V_{m-2})$ with respect to the splitting data (χ, μ, ψ). We describe the action explicitly. Let $\varphi_0 \in S(W_{n-2} \otimes Y_1^*, \ \Omega_{n-2,m-2})$, and $x = w \otimes y_1^* \in W_{n-2} \otimes Y_1^*$.

- $(\Omega_{n-2,m}(g_0)\varphi_0)(x) = \Omega_{n-2,m-2}(g_0)\varphi_0(g_0^{-1}x), \quad g_0 \in G(W_{n-2}),$
- $(\Omega_{n-2,m}(m(1, h_0))\varphi_0)(x) = \Omega_{n-2,m-2}(h_0)\varphi_0(x), \quad h_0 \in H(V_{m-2}),$
- $(\Omega_{n-2,m}(m(a, 1))\varphi_0)(x) = \chi_W(\mathrm{Nrd}\, a)\nu(a)^{\frac{n-2}{2}}\varphi_0(a^*x), \quad a \in \mathrm{GL}(Y_1),$
- $(\Omega_{n-2,m}(u(b, 0))\varphi_0)(x) = \rho_{00}((b^*x, 0))\varphi_0(x), \quad b \in \mathrm{Hom}(V_{m-2}, Y_1),$
- $(\Omega_{n-2,m}(u(0, d))\varphi_0)(x) = \psi\left(\frac{1}{2}\mathrm{Tr}_{D/F}(q_W(w, w)\overline{q_V(y_1^*, -dy_1^*)})\right)\varphi_0(x), \ d \in \mathrm{Herm}(Y_1^*, Y_1),$

where ρ_{00} denotes the Heisenberg representation of the Heisenberg group $\mathrm{H}(W_{n-2} \otimes V_{m-2})$ realized on the space of $\Omega_{n-2,m-2}$, χ_W is the character associated with the splitting data, as defined in [GKT25, Formula (12.5)].

Let $M := X_1^* \otimes V_m$, $Z := \{0\} \subseteq M$, $U := M \setminus Z$. The space $S(M) \widehat{\otimes} \Omega_{n-2,m}$ contains a $P_1 \times H_m$ stable subspace $S(U) \widehat{\otimes} \Omega_{n-2,m}$. Set $S_Z(M) := S(M)/S(U)$. Then we have

(F-I) $\qquad 0 \to S(U) \widehat{\otimes} \Omega_{n-2,m} \to S(M) \widehat{\otimes} \Omega_{n-2,m} \to S_Z(M) \widehat{\otimes} \Omega_{n-2,m} \to 0.$

Note that $S_Z(M) \cong \mathbb{C}[\![M]\!]$, where we view M as a real manifold.

We first study the $\mathfrak{n}_1$-homology of the open orbit piece. Let $\mathrm{Isom}(^\kappa X_1, Y_1)$ be the space of invertible D-linear map from $^\kappa X_1$ to Y_1. The following lemma holds.

Lemma 5.1. *If V_m is anisotropic, then*
$$\mathrm{H}_i(\mathfrak{n}_1, S(U) \widehat{\otimes} \Omega_{n-2,m}) = 0 \quad \forall\, i \in \mathbb{Z}_{\geq 0}.$$

If V_m is isotropic, we have
$$\mathrm{H}_i(\mathfrak{n}_1, S(U) \widehat{\otimes} \Omega_{n-2,m}) = \begin{cases} \mathrm{ind}_{L_1 \times Q_1}^{L_1 \times H_m}(\lambda_{1,1} \otimes \rho_1 \widehat{\otimes} \Omega_{n-2,m-2}) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here:

- *ρ_1 is the natural action of $\mathrm{GL}(X_1) \times \mathrm{GL}(Y_1)$ on $S(\mathrm{Isom}(^\kappa X_1, Y_1))$ as*
$$((g,h) \cdot f)(x) := f(g^{-1}xh),$$
for $(g,h) \in \mathrm{GL}(X_1) \times \mathrm{GL}(Y_1)$, $f \in S(\mathrm{Isom}(^\kappa X_1, Y_1))$ and $x \in \mathrm{Isom}(^\kappa X_1, Y_1)$;
- *$\Omega_{n-2,m-2}$ is the Weil representation of $G(W_{n-2}) \times H(V_{m-2})$ with respect to the splitting data (χ, μ, ψ);*
- *$\lambda_{1,1}$ is the character defined by*
$$\lambda_{1,1} = \begin{cases} \chi_V \cdot \nu^{\frac{m}{2}} & \text{on } \mathrm{GL}(X_1) \\ \chi_W \cdot \nu^{\frac{n-2}{2}} & \text{on } \mathrm{GL}(Y_1) \end{cases},$$
where χ_V (resp. χ_W) is regarded as a character of $\mathrm{GL}(X_1)$ (resp. $\mathrm{GL}(Y_1)$) via the reduced norm Nrd.

Proof. Consider the map
$$\begin{aligned} \phi: \quad U &\to N_{1,0}^* \\ x_1^* \otimes v &\mapsto \left(d \mapsto \mathrm{Tr}_{D/F}(q_W(x_1^*, -dx_1^*)\overline{q_V(v,v)})\right). \end{aligned}$$

When V_m is anisotropic, ϕ has no zeros. By Lemma 3.9 and the $N_{1,0}-$action on the mixed model, we obtain
$$\mathrm{H}_i(\mathfrak{n}_{1,0}, S(U) \widehat{\otimes} \Omega_{n-2,m}) = 0, \quad \forall\, i \in \mathbb{Z}_{\geq 0}.$$

Now assume that V_m is isotropic, and define
$$\mathcal{N} := \left\{ x_1^* \otimes v \in X_1^* \otimes V_m : q_V(v,v) = 0 \right\}.$$

One can verify directly that 0 is a regular value of ϕ, and $\phi^{-1}(0) = \mathcal{N} \setminus \{0\}$. Applying Lemma 3.9 again, we obtain
$$\mathrm{H}_i(\mathfrak{n}_{1,0}, S(U) \widehat{\otimes} \Omega_{n-2,m}) = \begin{cases} S(\mathcal{N} \setminus \{0\}) \widehat{\otimes} \Omega_{n-2,m} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x_1^* \otimes y_1 \in \mathcal{N} \setminus \{0\}$. Consider the action of $L_1 \times H_m$ on $\mathcal{N} \setminus \{0\}$, we have an isomorphism
$$S(\mathcal{N} \setminus \{0\}) \widehat{\otimes} \Omega_{n-2,m} \cong \mathrm{ind}_{(L_1 \times Q_1) \ltimes (N_1/N_{1,0})}^{(L_1 \times H_m) \ltimes (N_1/N_{1,0})}(\lambda' \otimes \rho_1 \widehat{\otimes} S(W_{n-2} \otimes Y_1^*) \widehat{\otimes} \Omega_{n-2,m-2}).$$

The actions on the right hand side are described as follows.

- *$N_1/N_{1,0}$ acts only on $S(W_{n-2} \otimes Y_1^*)$ by*
$$n(b,0)\phi(w \otimes y_1^*) = \psi(\mathrm{Tr}_{D/F}(q_W(w, w_b)))\phi(w \otimes y_1^*), \quad \text{where } w_b := b^* x_1^*;$$

18

- ρ_1 is the natural action of $\mathrm{GL}(X_1) \times \mathrm{GL}(Y_1)$ on $S(\mathrm{Isom}({}^\kappa X_1, Y_1))$ as

$$((g, h) \cdot f)(x) := f(g^{-1} x h),$$

for $(g, h) \in \mathrm{GL}(X_1) \times \mathrm{GL}(Y_1)$, $f \in S(\mathrm{Isom}({}^\kappa X_1, Y_1))$ and $x \in \mathrm{Isom}({}^\kappa X_1, Y_1)$;
- $G(W_{n-2}) \times Q_1$ acts on $\Omega_{n-2,m} = S(W_{n-2} \otimes Y_1^*) \widehat{\otimes} \Omega_{n-2,m-2}$ via the restriction of the Weil representation;
- λ' is a character on $\mathrm{GL}(X_1)$ defined by $\lambda' = \chi_V \cdot \nu^{\frac{m}{2}}$.

Consider the map

$$\psi : \begin{array}{ccc} W_{n-2t} \otimes Y_1^* & \to & \mathrm{Hom}(W_{n-2}, X_1)^* \\ w \otimes y_1^* & \mapsto & (b \mapsto \mathrm{Tr}_{\mathrm{D}/F}(q_W(w, w_b))) \end{array}, \quad \text{where } w_b := b^* x_1^*,$$

which is an isomorphism. Using Lemma 3.9 once more, together with Lemma 3.10, we obtain

$$\mathrm{H}_i^{\mathcal{S}}(N_1/N_{1,0}, S(\mathcal{N} \setminus \{0\}) \widehat{\otimes} \Omega_{n-2,m}) = \begin{cases} \mathrm{ind}_{L_1 \times Q_1}^{L_1 \times H_m}(\lambda_{1,1} \otimes \rho_1 \otimes \Omega_{n-2,m-2}) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases},$$

Here the actions are as described in the lemma. $\qquad \square$

Now we turn to the closed orbit. Recall that $S_Z(M) \cong \mathbb{C}[\![M]\!]$. It admits a decreasing filtration by degree $S_Z(M)_k = \mathbb{C}[\![V_m]\!]_{\geq k}$. The associated graded piece

$$F_k := S_Z(M)_k / S_Z(M)_{k+1}$$

is the space of homogeneous polynomials on M of total degree k.

Lemma 5.2. *As a representation of $P_1 \times H_m$,*

$$(S_Z(M)_k / S_Z(M)_{k+1}) \otimes \Omega_{n-2,m} \cong \lambda_{1,0} \otimes F_k \otimes \Omega_{n-2,m}.$$

Here:

- *N_1 acts trivially;*
- *$\Omega_{n-2,m}$ is the Weil representation of $G(W_{n-2}) \times H_m$ with respect to the splitting data (χ, μ, ψ);*
- *F_k carries a $\mathrm{GL}(X_1) \times H_m$-action induced from the natural $\mathrm{GL}(X_1) \times H_m$-action on M;*
- *$\lambda_{1,0} := \chi_V \cdot \nu^{\frac{m}{2}}$ is a character of $\mathrm{GL}(X_1)$.*

Proof. Using the explicit formulas in the mixed model, we observe that N_1 acts trivially on M, hence it acts trivially on F_k. The same formulas also show that, when restricted to the point $0 \in M$, N_1 acts trivially on $\Omega_{n-2,m}$. The remaining assertions follow directly from the mixed model formulas. $\qquad \square$

5.2. **Proof of Theorem 1.2: Type I case.** In this subsection, we prove the following proposition and use it to prove Theorem 1.2 for the Type I case at the end.

Let (W_n, V_m) be as before, and denote by $\mathfrak{t}_W$ (resp. $\mathfrak{t}_V$) the Witt tower containing W_n (resp. V_m).

Proposition 5.3. *Assume that W_n is isotropic. Let $W_n = X_1 \oplus W_{n-2} \oplus X_1^*$ be the decomposition as in (7). Let χ be a Casselman–Wallach representation of $\mathrm{GL}(X_1)$. Assume that for all i, for all $V_m \in \mathfrak{t}_V$, and every $\tau \in \mathrm{Rep}_{G(W_{n-2})}^{\mathrm{CW}}$,*

(C-I) $\quad \mathrm{H}_i^{\mathcal{S}}(G(W_{n-2}), \Omega_{n-2,m} \widehat{\otimes} \tau)$ *is a Casselman–Wallach representation of $H(V_m)$.*

Then for all i, for all $V_m \in \mathfrak{t}_V$, and every $\pi \in \mathrm{Rep}_{G(W_{n-2})}^{\mathrm{CW}}$,

$$\mathrm{H}_i^{\mathcal{S}}(G(W_n), \Omega_{n,m} \widehat{\otimes} \mathrm{Ind}_{P_1}^{G(W_n)}(\chi \boxtimes \pi)).$$

is also a Casselman–Wallach representation of $H(V_m)$.

Following Shapiro's lemma, we know that

$$\mathrm{H}_i^{\mathcal{S}}(G_n, \Omega_{n,m} \widehat{\otimes} \mathrm{Ind}_{P_1}^{G_{n+1}}(\chi \boxtimes \pi)) \cong \mathrm{H}_i^{\mathcal{S}}(P_1, \Omega_{n,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$$

Consider the short exact sequence (F-I) in Subsection 5.1,

$$0 \to S(U) \widehat{\otimes} \Omega_{n-2,m} \to S(M) \widehat{\otimes} \Omega_{n-2,m} \to S_Z(M) \widehat{\otimes} \Omega_{n-2,m} \to 0.$$

According to Lemma 3.13, Propsition 5.3 follows once we show that, under the condition (C-I),

$$\mathrm{H}_i^{\mathcal{S}}(P_1, S(U) \widehat{\otimes} \Omega_{n-2,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) \quad \text{and} \quad \mathrm{H}_i^{\mathcal{S}}(P_1, S_Z(M) \widehat{\otimes} \Omega_{n-2,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$$

are Casselman–Wallach representations for all $i \geq 0$. These will be proved in Lemmas 5.4–5.6. Throughout these lemmas, we assume the condition (C-I).

We begin with the open orbit.

Lemma 5.4. *For all $i \geq 0$, $\mathrm{H}_i^{\mathcal{S}}(P_1, S(U) \widehat{\otimes} \Omega_{n-2,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$ is a Casselman–Wallach representation of H_m.*

Proof. When V_m is anisotropic, this lemma follows trivially from Lemma 5.1.

Now we assume V_m is isotropic. Using Lemma 5.1 again and the spectral sequence argument, we obtain

$$\mathrm{H}_i^{\mathcal{S}}(P_1, S(U) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) \cong \mathrm{H}_i^{\mathcal{S}}(L_1, \mathrm{ind}_{L_1 \times Q_1}^{L_1 \times H_m}(\lambda_{1,1} \otimes \rho_1 \widehat{\otimes} \Omega_{n-2,m-2}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}),$$

provided that the right-hand side is a Casselman–Wallach representation of H_m. According to Lemma 3.10, we have

$$\mathrm{H}_i^{\mathcal{S}}(L_1, \mathrm{ind}_{L_1 \times Q_1}^{L_1 \times H_m}(\lambda_{1,1} \otimes \rho_1 \widehat{\otimes} \Omega_{n-2,m-2}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) \cong$$
$$\mathrm{ind}_{Q_1}^{H_m} \mathrm{H}_i^{\mathcal{S}}(L_1, (\rho_1 \widehat{\otimes} \Omega_{n-2,m-2}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \eta)$$

where η is a certain product of modular character and normalization character. Since it has no influence on our result, we just denote it by η to simplify the notation. We denote its restriction to $\mathrm{GL}(X_1)$ (resp. $G(W_{n-2})$) by η_1 (resp. η_{n-2}). According to the Künneth formula, we have

$$\mathrm{H}_i^{\mathcal{S}}(L_1, (\rho_1 \widehat{\otimes} \Omega_{n-2,m-2}) \widehat{\otimes}(\chi \boxtimes \pi) \otimes \eta) \cong$$
$$\bigoplus_{p+q=i} \mathrm{H}_p^{\mathcal{S}}(\mathrm{GL}(X_1), \rho_1 \otimes \chi \otimes \eta_1) \widehat{\otimes} \mathrm{H}_q^{\mathcal{S}}(G(W_{n-2}), \Omega_{n-2,m-2} \widehat{\otimes} \pi \otimes \eta_{n-2})$$

By the condition (C-I), the right-hand side of this equation is a Casselman–Wallach representation of $\mathrm{GL}(Y_1) \times H(V_{m-2})$. Therefore, all the representations appearing above are Casselman–Wallach representations of H_m. $\qquad \square$

Now, we turn to the closed orbit. As in Lemma 5.2, $S_Z(M) \widehat{\otimes} \Omega_{n-2,m}$ admits a decreasing filtration $S_Z(M)_k \widehat{\otimes} \Omega_{n-2,m}$ with graded pieces

$$(S_Z(M)_k / S_Z(M)_{k+1}) \otimes \Omega_{n-2,m} \cong \lambda_{1,0} \otimes F_k \otimes \Omega_{n-2,m\cdot},$$

where F_k is the space of homogeneous polynomials on V of total degree k.

Lemma 5.5. *For all i and k,*

$$\mathrm{H}_i^{\mathcal{S}}(P_1, (S_Z(M)_k / S_Z(M)_{k+1}) \otimes \Omega_{n-2,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$$

is a Casselman–Wallach representation of H_m. Moreover, there exists an integer K such that for all $k > K$ and all i,

$$\mathrm{H}_i^{\mathcal{S}}(P_1, (S_Z(M)_k / S_Z(M)_{k+1}) \otimes \Omega_{n-2,m} \widehat{\otimes}(\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) = 0.$$

Proof. The proof is similar to that of Lemma 4.10. It suffices to prove the statement for
$$\mathrm{H}_i^{\mathcal{S}}(L_1, \mathrm{H}_j(\mathfrak{n}_1, \mathbb{C}) \otimes \lambda_{1,0} \otimes F_k \otimes \Omega_{n-2,m} \,\widehat{\otimes}\, (\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}).$$
Let $\xi_j \boxtimes \beta_j$ be an irreducible $\mathrm{GL}(X_1) \times G(W_{n-2})$ constituent of $\mathrm{H}_j(\mathfrak{n}_1, \mathbb{C})$. We can further reduce to

$$\mathrm{H}_i^{\mathcal{S}}(L_1, (\xi_j \boxtimes \beta_j) \otimes \lambda_{1,0} \otimes F_k \otimes \Omega_{n-2,m} \,\widehat{\otimes}\, (\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1}) \cong$$
$$\bigoplus_{p+q=i} \mathrm{H}_p^{\mathcal{S}}(\mathrm{GL}(X_1), \xi_j' \otimes \chi \otimes F_k) \,\widehat{\otimes}\, \mathrm{H}_q^{\mathcal{S}}(G(W_{n-2}), \beta_j' \otimes \Omega_{n-2,m} \,\widehat{\otimes}\, \pi).$$

where β_j' (resp. ξ_j') denotes β_j (resp. ξ_j) twisted by a certain inessential character. Since $\xi_j' \otimes \chi$ is fixed, there must exist a constant $K_{i,j}$, such that for all $k > K_{i,j}$, the $\mathbb{R}_{>0}^\times$-component of $\mathrm{GL}(X_1)$ acts on $\xi_j' \otimes \chi \otimes F_k$ via a nontrivial character. Thus, the lemma follows from a similar argument as in the proof of Lemma 4.10. $\square$

Lemma 5.6. *For all $i \geq 0$, $\mathrm{H}_i^{\mathcal{S}}(P_1, S_Z(M) \,\widehat{\otimes}\, \Omega_{n-2,m} \,\widehat{\otimes}\, (\chi \boxtimes \pi) \otimes \delta_{P_1}^{-1})$ is a Casselman–Wallach representation of H_m.*

Proof. It follows from Lemmas 3.11 and 5.5. $\square$

Proof of Proposition 5.3. Using Lemma 3.13, the proposition follows from Lemmas 5.4 and 5.6. $\square$

We are now ready to prove Theorem 1.2 for the Type I case.

Proof of Theorem 1.2: Type I case. Let (W_n, V_m) be as before, and denote by $\mathfrak{t}_W$ (resp. $\mathfrak{t}_V$) the Witt tower containing W_n (resp. V_m). We prove that Theorem 1.2 holds for all dual pairs $(G(W), H(V))$ with $W \in \mathfrak{t}_W$ and $V \in \mathfrak{t}_V$. The proof proceeds by induction on the split rank r_W of W.

The base case is the compact dual pair, namely when $G(W)$ is compact. In this case, all higher-degree homology groups vanish. For the $\mathrm{H}_0^{\mathcal{S}}$ term, Hausdorffness is automatic, and the finite length property follows from Howe's original work [How89a] and [How89b, Section 3].

For the induction step, fix $W \in \mathfrak{t}_W$ with $r_W \geq 1$. Assume that for all $V_m \in \mathfrak{t}_V$, Theorem 1.2 holds for dual pairs $(G(W'), H(V_m))$, where $W' \in \mathfrak{t}_W$ has split rank $r_{W'} = r_W - 1$. By Proposition 3.14, it follows that Theorem 1.1 holds for all pairs $(G(W'), H(V_m))$, so the hypothesis of Proposition 4.8 is satisfied. Therefore, via induction by stages, the induction step follows from Proposition 4.8. The theorem follows. $\square$

References

[Ada07]   Jeffrey Adams. "The theta correspondence over $\mathbb{R}$". In: *Harmonic analysis, group representations, automorphic forms and invariant theory*. Vol. 12. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. World Sci. Publ., Hackensack, NJ, 2007, pp. 1–39. ISBN: 978-981-277-078-3. DOI: 10.1142/9789812770790_0001. URL: https://doi.org/10.1142/9789812770790_0001.

[APS17]   Jeffrey D. Adams, Dipendra Prasad, and Gordan Savin. "Euler-Poincaré characteristic for the oscillator representation". In: *Representation theory, number theory, and invariant theory*. Vol. 323. Progr. Math. Birkhäuser/Springer, Cham, 2017, pp. 1–22. ISBN: 978-3-319-59728-7. DOI: 10.1007/978-3-319-59728-7_1. URL: https://doi.org/10.1007/978-3-319-59728-7_1.

[AGS15a]   Avraham Aizenbud, Dmitry Gourevitch, and Siddhartha Sahi. "Derivatives for smooth representations of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$". In: *Israel J. Math.* 206.1 (2015), pp. 1–38. ISSN: 0021-2172,1565-8511. DOI: 10.1007/s11856-015-1149-9. URL: https://doi.org/10.1007/s11856-015-1149-9.

[AGS15b]   Avraham Aizenbud, Dmitry Gourevitch, and Siddhartha Sahi. "Twisted homology for the mirabolic nilradical". In: *Israel Journal of Mathematics* 206 (2015), pp. 39–88.

[BK14]   Joseph Bernstein and Bernhard Krötz. "Smooth Fréchet globalizations of Harish-Chandra modules". In: *Israel J. Math.* 199.1 (2014), pp. 45–111. ISSN: 0021-2172,1565-8511. DOI: 10.1007/s11856-013-0056-1.

[BW80]   Armand Borel and Nolan R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Vol. No. 94. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1980, pp. xvii+388. ISBN: 0-691-08248-0.

[CHM00]   William Casselman, Henryk Hecht, and Dragan Milicic. "Bruhat filtrations and Whittaker vectors for real groups". In: *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998).* Vol. 68. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2000, pp. 151–190. ISBN: 0-8218-1197-5. DOI: 10.1090/pspum/068/1767896.

[CS21]   Yangyang Chen and Binyong Sun. "Schwartz homologies of representations of almost linear Nash groups". In: *Journal of Functional Analysis* 280.7 (2021), p. 108817. ISSN: 0022-1236. DOI: https://doi.org/10.1016/j.jfa.2020.108817.

[FSX18]   Yingjue Fang, Binyong Sun, and Huajian Xue. "Godement–Jacquet L-functions and full theta lifts". In: *Mathematische Zeitschrift* 289 (2018), pp. 593–604.

[GKT25]   Wee Teck Gan, Stephen S. Kudla, and Shuichiro Takeda. "The Local Theta Correspondence". Theta book (July 26, 2025 version). 2025. URL: https://sites.google.com/view/stakeda/theta-book.

[GS17]   Wee Teck Gan and Binyong Sun. "The Howe duality conjecture: quaternionic case". In: *Representation theory, number theory, and invariant theory.* Vol. 323. Progr. Math. Birkhäuser/Springer, Cham, 2017, pp. 175–192. ISBN: 978-3-319-59728-7. DOI: 10.1007/978-3-319-59728-7_6. URL: https://doi.org/10.1007/978-3-319-59728-7_6.

[GT16]   Wee Teck Gan and Shuichiro Takeda. "A proof of the Howe duality conjecture". In: *J. Amer. Math. Soc.* 29.2 (2016), pp. 473–493. ISSN: 0894-0347,1088-6834. DOI: 10.1090/jams/839. URL: https://doi.org/10.1090/jams/839.

[Gen25]   Zhibin Geng. *On the existence of twisted Shalika periods: the Archimedean case.* 2025. arXiv: 2501.11917 [math.RT]. URL: https://arxiv.org/abs/2501.11917.

[Gro61]   A. Grothendieck. "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I." In: *Inst. Hautes Études Sci. Publ. Math.* 11 (1961), p. 167. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1961__11__167_0.

[HT90]   Henryk Hecht and Joseph L. Taylor. "Analytic localization of group representations". In: *Adv. Math.* 79.2 (1990), pp. 139–212. ISSN: 0001-8708,1090-2082. DOI: 10.1016/0001-8708(90)90062-R. URL: https://doi.org/10.1016/0001-8708(90)90062-R.

[How79]   R. Howe. "θ-series and invariant theory". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Vol. XXXIII. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1979, pp. 275–285. ISBN: 0-8218-1435-4.

[How89a]   Roger Howe. "Remarks on classical invariant theory". In: *Trans. Amer. Math. Soc.* 313.2 (1989), pp. 539–570. ISSN: 0002-9947,1088-6850. DOI: 10.2307/2001418. URL: https://doi.org/10.2307/2001418.

[How89b]   Roger Howe. "Transcending classical invariant theory". In: *J. Amer. Math. Soc.* 2.3 (1989), pp. 535–552. ISSN: 0894-0347,1088-6834. DOI: 10.2307/1990942. URL: https://doi.org/10.2307/1990942.

[Mín08]   Alberto Mínguez. "Correspondance de Howe explicite: paires duales de type II". In: *Ann. Sci. Éc. Norm. Supér. (4)* 41.5 (2008), pp. 717–741. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2080. URL: https://doi.org/10.24033/asens.2080.

[MVW87]   Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*. Vol. 1291. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1987, pp. viii+163. ISBN: 3-540-18699-9. DOI: 10.1007/BFb0082712. URL: https://doi.org/10.1007/BFb0082712.

[Wal88]   Nolan R. Wallach. *Real reductive groups. I*. Vol. 132. Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1988, pp. xx+412. ISBN: 0-12-732960-9.

[Wal92]   Nolan R. Wallach. *Real reductive groups. II*. Vol. 132-II. Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1992, pp. xiv+454. ISBN: 0-12-732961-7.

[WZ25]   Kaidi Wu and Hongfeng Zhang. *Archimedean Bernstein-Zelevinsky Theory and Homological Branching Laws*. 2025. arXiv: 2509.08719 [math.RT]. URL: https://arxiv.org/abs/2509.08719.

[Xue20]   Hang Xue. "Bessel models for unitary groups and Schwartz homology". preprint. 2020.

DEPARTMENT OF MATHEMATICS AND NEW CORNERSTONE SCIENCE LABORATORY, THE UNIVERSITY OF HONG KONG
   Email address: gengzb@hku.hk

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ARIZONA
   Email address: xuehang@arizona.edu