Steady State Numerical Solution of Couette-Poiseuille Flows

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ME428 - Summer 2017

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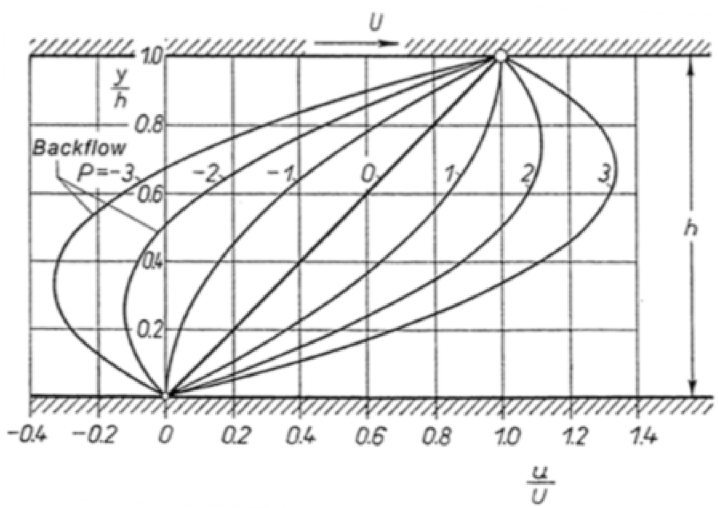


Fig. 1: Flow configuration of a Couette-Pouiseuille parallel plate channel flow for various pressures.

Contents

1 Abstract **2**

2 Introduction **2**

3 Analysis **2**

3.1 Non-Dimensionalization . . . 2

3.2 Analytical Solution . . . 3

3.2.1 Flow Rate, V . . . 3

3.2.2 Wall Shear Stress, τ_w . . . 4

4 Numerical Solution **4**

4.1 Discretization of Governing Equations . . . 4

4.2 Calculation of Flow Rate, V . . . 5

4.3 Wall Shear Stress, τ_w . . . 5

5 Results **5**

5.1 Analytical Solution . . . 5

5.1.1 Velocity Profile . . . 5

5.1.2 Flow Rate and Wall Shear Stress . . . 6

5.2 Numerical Solution . . . 6

5.2.1 Velocity Profile . . . 6

5.2.2 Flow Rate and Wall Shear Stress . . . 7

5.2.3 Error . . . 7

5.3 Effect of Step Size Δy . . . 7

5.4 Effect of U and P . . . 8

5.4.1 $U = 1$ for Variable P . . . 8

5.4.2 $P = 1$ for Variable U . . . 8

6 Conclusion **9**

List of Figures

1 Velocity Profile for Analytic Solution . . . 5

2 Numerical vs. Analytic Velocity Profile . . . 6

3 Effect of Step Size on Solution . . . 7

4 Velocity Profile for $U = 1$ and Variable P . . . 8

5 Velocity Profile for $P = 1$ and Variable U . . . 9

1 Abstract

This project will focus on the fluid dynamics problem of fluid flow between two plates with a given top plate velocity and pressure. In this particular case we assume a no-slip condition, which means that the top plate is fixed. We assumed that the pressure gradient was constant, and obtained both analytic and numerical solutions for the velocity profile, mass flow rate, and wall shear stress. Numerical results were compared with the analytical results to determine the error magnitude in the numerical results. By varying the step size, the numerical approximations can be shown to be very accurate when compared to the analytical results.

2 Introduction

This project is concerned with the problem of steady state fluid flow between two parallel plates separated by a distance h where the top plate moves with constant velocity U and the bottom plate remains stationary. The governing Navier-Stokes equations and boundary conditions for this flow can be written

$$0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$u(0) = 0, \tag{2.2}$$

$$u(h) = U, \tag{2.3}$$

where u is the velocity of the fluid, μ is the viscosity, and p is the pressure. This project will examine the specific case where $U = 0$ and $P = 1$. In other words, the top plate remains stationary and the pressure is constant and not a function of x.

3 Analysis

3.1 Non-Dimensionalization

Before Eq. 2.1 can be solved numerically, it is necessary to find an analytical solution to compare our numerical results against. To do this, Eq. 2.1 must be non-dimensionalized using the following dimensionless scales

$$u^* = \frac{u}{U},$$

$$y^* = \frac{y}{h},$$

$$P^* = \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right),$$

where * indicates a dimensionless variable. These equations can be rearranged to

$$u = u^* U, \tag{3.1}$$

$$y = y^* h, \tag{3.2}$$

$$-\frac{dp}{dx} = \frac{2P^*\mu U}{h^2}, \tag{3.3}$$

and inserted into Eq. 2.1

$$0 = \frac{2P^* \mu U}{h^2} + \mu \frac{\partial^2 (u^* U)}{\partial (y^* h)^2} \tag{3.4}$$

The constants U and h^2 can be pulled out of the partial derivative term, and the entire equation can be reduced to the dimensionless form

$$\frac{\partial^2 u^*}{\partial y^{*2}} + 2P^* = 0 \tag{3.5}$$

Note that the given boundary conditions must also be nondimensionalized as follows

$$u\left(\frac{0}{h}\right) = u(0) = 0$$

$$u\left(\frac{h}{h}\right) = u(1) = U$$

3.2 Analytical Solution

An analytical solution to the differential equation 3.5 may be found for the specific case $U = 0$ and $P = 1$ by integrating twice and using the nondimensionalized boundary conditions to solve for integration constants

$$\iint \frac{\partial^2 u^*}{\partial y^{*2}} dy = \iint -2(1) dy$$

$$u(y) = -y^2 + C_1 y + C_2,$$

$$u(0) = 0, \ C_2 = 0$$

$$u(1) = 0, \ C_1 = 1$$

Therefore, Eq. 3.5 can be solved analytically:

$$u(y) = -y^2 + y \tag{3.6}$$

3.2.1 Flow Rate, V

The flow rate for fluid in a channel flow is defined as

$$V = \int_0^h u(y) dy$$

Substituting Eq. 3.6 and the dimensionless boundary conditions, we obtain

$$V = \int_0^1 (-y^2 + y) dy \tag{3.7}$$

which can be evaluated to show that flow rate $V = \frac{1}{6}$. This value will be compared to the numerical result later in this report.

3.2.2 Wall Shear Stress, τ_w

The wall shear stress is the force per unit area exerted by the wall on the fluid in a direction on the local tangent plane. It is given by the equation

$$\tau_w = \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{3.8}$$

Substituting Eq. 3.6 and evaluating, we obtain

$$\tau_w = \frac{\partial}{\partial y}(-y^2 - y) = -2y - 1\Big|_{y=0},\ \tau_w = -1 \tag{3.9}$$

This value will also be compared with a numerical result later in this report.

4 Numerical Solution

4.1 Discretization of Governing Equations

Equation 3.5 can be discretized into n solution points, each spaced Δy apart, between the upper and lower boundaries of the system. This is accomplished using a second order forward difference approximation. The general form of this approximation can be written

$$f''(y_i) \approx \frac{f(y_{i-1}) - 2f(y_i) + f(y_{i+1})}{(\Delta y)^2} + O[(\Delta y)^2]$$

When applied to Eq. 3.5, this approximation forms a system of n equations of the form

$$u_{i-1} - 2u_i + u_{i+1} = -2P^*(\Delta y)^2,\ i = 1, 2, 3...n$$

which can be placed into matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ -2P^*(\Delta y)^2 \\ -2P^*(\Delta y)^2 \\ -2P^*(\Delta y)^2 \\ \vdots \\ -2P^*(\Delta y)^2 \\ U \end{bmatrix} \tag{4.1}$$

where the first and last rows represent the boundary conditions, and rows 2 through $n-1$ represent the discretized finite difference equations.

Because this matrix is a tridiagonal system, it can be solved using an iterative technique called the Thomas algorithm. The Thomas algorithm assigns the righthand matrix and each diagonal in the coefficient matrix to a vector, and then iterates to eliminate and replace all lower diagonal values with zero, all dominant diagonal values with 1, and all upper diagonal values with a new, modified value. Finally, backwards substitution is performed beginning with the n^{th} row to solve the system and provide a velocity profile for the fluid flow within the channel.

4.2 Calculation of Flow Rate, V

As discussed in section 3.2.1, flow rate V can be evaluated using the integral in Eq. 3.7. This integral can be approximated numerically using the Trapezoidal Rule. The Trapezoidal Rule can be written

$$\int_a^b f(y)dy \approx \sum_{i=0}^{n-1} \left[\left(\frac{f_i + f_{i+1}}{2} \right) \Delta y + O[(\Delta y)^3] \right] \tag{4.2}$$

Values for f_i and f_{i+1} may be obtained from the solution vector provided by the Thomas algorithm and iterated through to approximate flow rate.

4.3 Wall Shear Stress, τ_w

Wall shear stress may be evaluated by taking the first derivative of the velocity profile at the lower wall, as shown in Eq. 3.9. Because the wall is a boundary, a central difference approximation cannot be used. Instead, shear stress can be evaluated using a second order forward difference approximation, which has the form

$$f_0' \approx \frac{-\frac{3}{2}f_0 + 2f_1 - \frac{1}{2}f_2}{\Delta y} \tag{4.3}$$

Once again, values for f_0, f_1, and f_2 may be obtained from the Thomas algorithm solution in order to evaluate this equation.

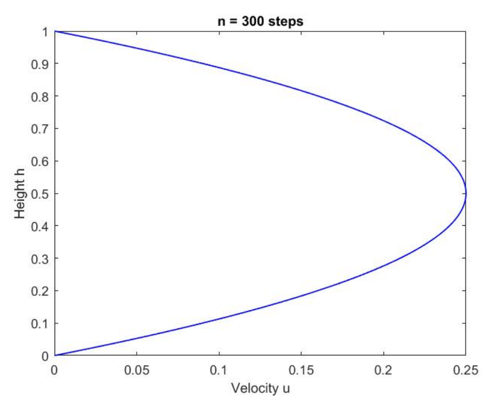
5 Results

5.1 Analytical Solution

5.1.1 Velocity Profile

A plot of the analytical solution for the velocity profile is shown below. As expected, velocity is zero at the top and bottom plates, and reaches a maximum at the midpoint between the two plates. The plot below is for a high resolution case, meaning the number of solution points n is high, making the step size Δy between points very small.

Figure 1: Velocity Profile for Analytic Solution



5.1.2 Flow Rate and Wall Shear Stress

Values for flow rate and wall shear stress were evaluated analytically in Eqs. 3.7 and 3.9 respectively. The results were

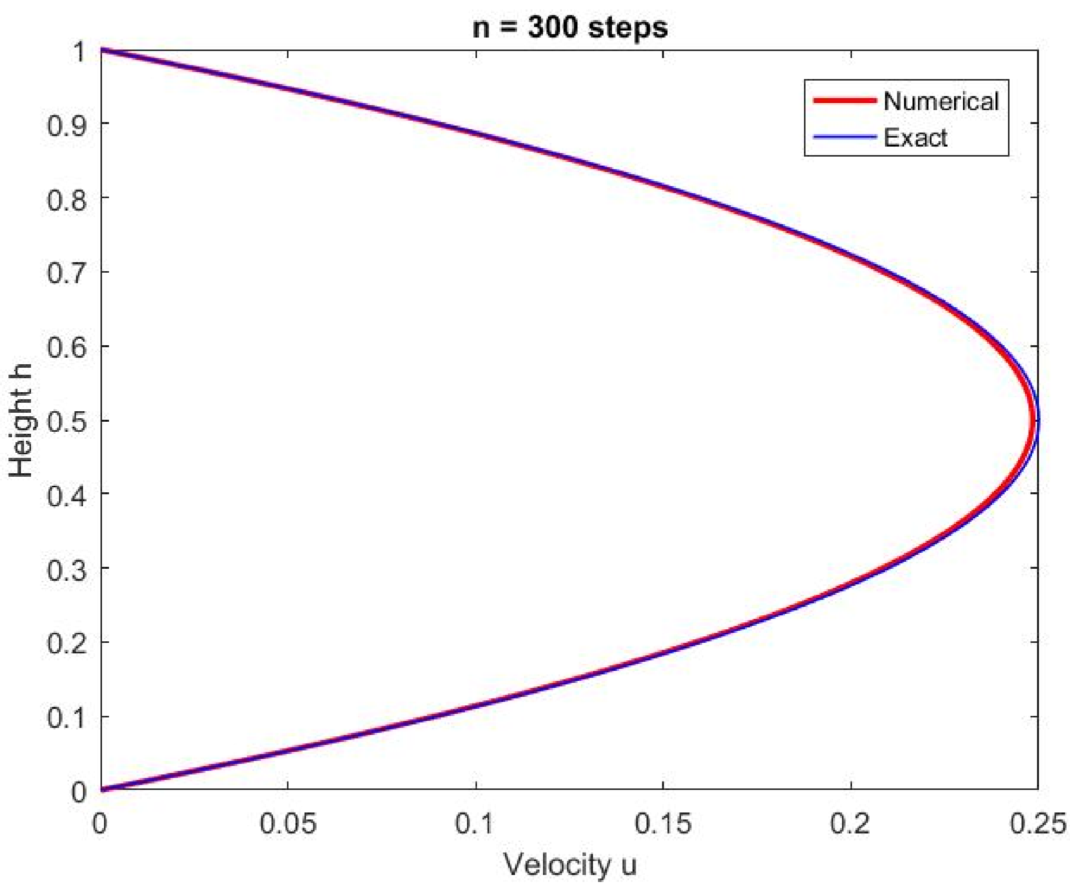
$$V = \frac{1}{6} \approx 0.1667$$
$$\tau_w = -1$$

5.2 Numerical Solution

5.2.1 Velocity Profile

Accuracy for the numerical solution increased as step size decreased. As seen in the plot below, for the same high resolution case there is small degree of inaccuracy, which is greatest at the midpoint of the channel.

Figure 2: Numerical vs. Analytic Velocity Profile



5.2.2 Flow Rate and Wall Shear Stress

For the high resolution $n = 300$ case, flow rate and wall shear stress can be evaluated using Eqs. 4.2 and 4.3 to obtain

$$V = 0.1650 \tag{5.1}$$

$$\tau_w = 0.9967 \tag{5.2}$$

5.2.3 Error

For the velocity profiles shown in Figure 2, error can be calculated by comparing the analytic and numerical solutions at the midpoint or maximum value as follows

$$\text{Velocity } (u)\text{: } \%Error = \left|\frac{0.2500 - 0.2483}{0.2500}\right| \text{ x } 100\% = 0.68\%$$

Similarly, the error percentages for flow rate and wall shear stress can be obtained by comparing the analytic values obtained in sections 3.2.1 and 3.2.2 respectively with the numerical values shown in section 5.2.2. These equations are shown below

$$\text{Flow Rate } V\text{: } \%Error = \left|\frac{0.1667 - 0.1650}{0.1667}\right| \text{ x } 100\% = 1.02\%$$

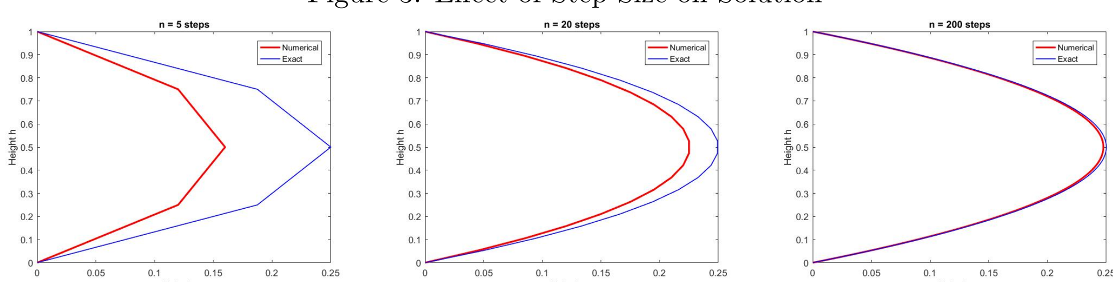
$$\text{Wall Shear Stress } \tau_w\text{: } \%Error = \left|\frac{1 - 0.9967}{1}\right| \text{ x } 100\% = 0.33\%$$

These error calculations make it clear that for high resolution cases with many solution points, the numerical approximation is very accurate when compared to the analytical solution.

5.3 Effect of Step Size Δy

The step size can be varied by altering the number of solution points between the upper and lower bounds. As n is increased, Δy decreases, and both the analytical and numerical solutions become more accurate. The error between the two solutions also decreases as step size is increased. This can be seen with the follow series of plots, where n is increased with each plot: Figure 2 in section 5.2.1

Figure 3: Effect of Step Size on Solution



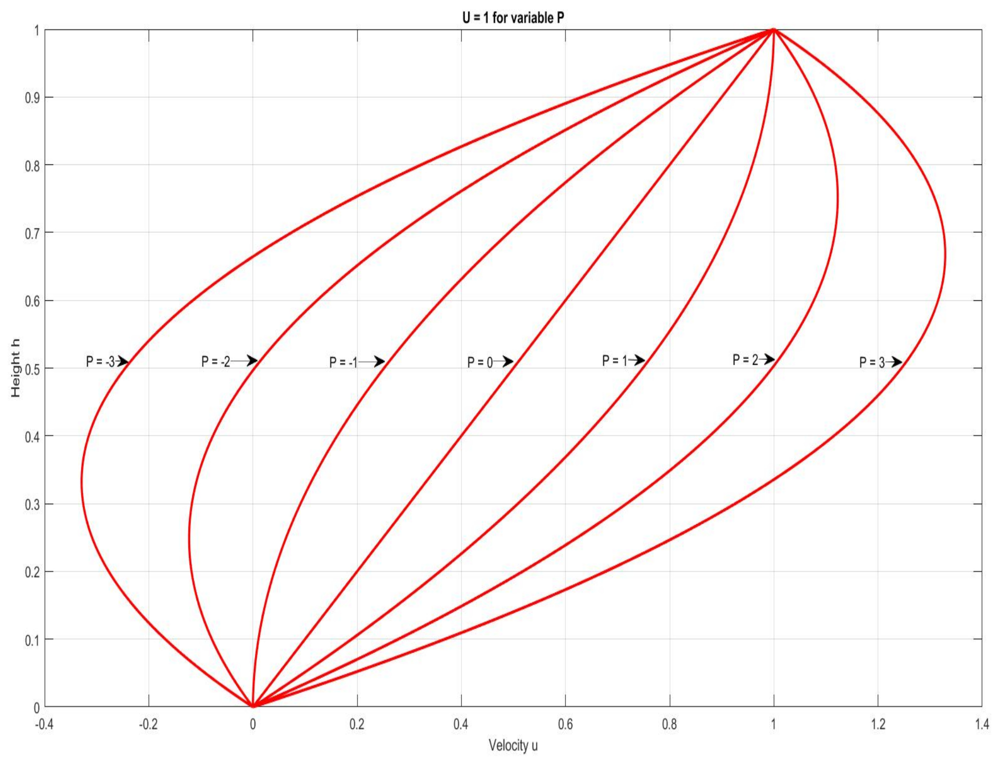
shows the highest resolution case, where $n = 300$ steps. Beyond this point, flow rate and wall shear stress did not change significantly and overall error is less than 1%, meaning grid independence has been achieved.

5.4 Effect of U and P

5.4.1 $U = 1$ for Variable P

The bulk of this report is concerned with the case where both walls of the fluid channel remain stationary. However, we can easily model how the fluid flow changes when the top wall is given a velocity. Figure 4 shows how the velocity profile changes when $U = 1$ and pressure is varied across

Figure 4: Velocity Profile for $U = 1$ and Variable P

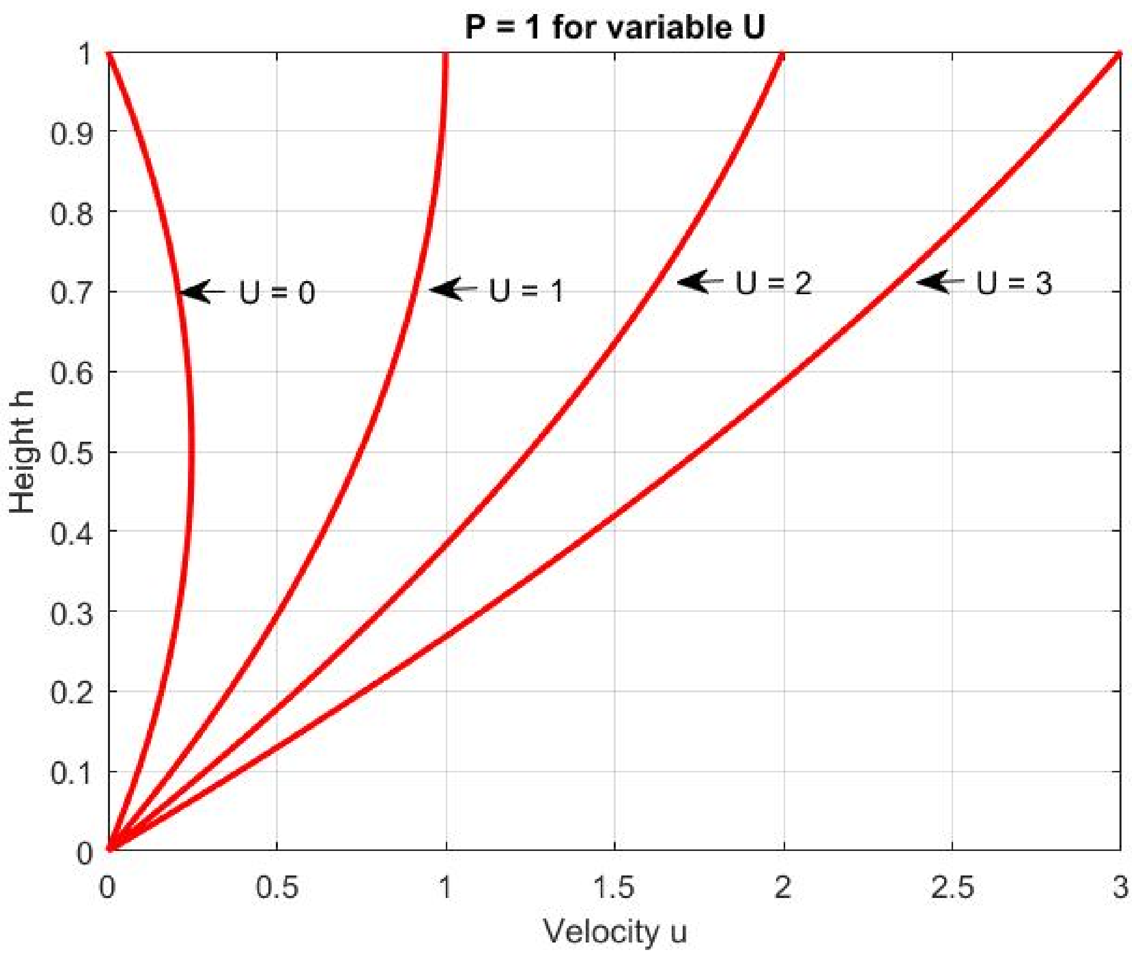


several values. This figure is a recreation of the figure on the title page of this report. As pressure changes, the maximum velocity of the fluid changes as well, with a pressures of greater magnitude resulting in velocities of greater magnitude. Similarly, as pressure increases, both the wall shear stress and mass flow rate increase at a similar rate.

5.4.2 $P = 1$ for Variable U

We can also examine how the velocity profile changes when P is kept constant and top wall velocity U is varied. Figure 5 illustrates the change in velocity profile when top plate velocity is varied but pressure kept constant. As U increases with respect to P, the velocity profile begins to flatten out, with maximum velocity occurring closer to the top plate. Both mass flow rate and wall shear

Figure 5: Velocity Profile for $P = 1$ and Variable U



stress increase as U is increased, however the lower wall shear increases at a significantly greater rate than the mass flow rate in the channel.

6 Conclusion

This project tasked us with using numerical methods to approximate the velocity profile, wall shear stress, and flow rate for steady state fluid flow between two fixed plates. We made several assumptions which allowed the governing equation to be easily solved analytically. This was useful, as it gave us something to compare our numerical results against in order to test their accuracy. After discretizing the governing equation, the resulting system of equations was solved using the Thomas algorithm, which is an iterative method for solving tridiagonal systems. These results were compared against the analytical results and we found that for high resolution cases (cases where the range was divided into many solution points), accuracy was within 1% of the analytical values. These results illustrate how useful numerical methods can be, as many real-world problems will not have a convenient analytical solution.

References

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