2025 Assignment Dynamics of Structures under Moving Loads – Part A

In this part of the assignment, your task is to solve small problems related to the Canonical Problems 1-4 and interpret the results you obtain. In doing so, you get hands-on experience, and you gain a deeper understanding of the structural responses to moving loads.

<u>Problem 1 (about Canonical Problem 1)</u>
Derive and plot (for a fixed moment in time) the steady-state response of a semi-infinite beam subject to an oscillatory load at the boundary with a relatively small excitation frequency as well as with a relatively large excitation frequency (as compared to the cut-off frequency). Interpret the result.

<u>Problem 2 (about Canonical Problem 1)</u>
Consider an infinite beam with continuous visco-elastic foundation, subject to a constant moving load. The equation of motion reads as follows (η denotes the distributed-dashpot constant; other symbols have the same meaning as in the lecture slides of Canonical Problem 1):

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + \chi w + \eta \frac{\partial w}{\partial t} = Q_0 \delta(x - Vt)$$

Derive and compute the steady-state response in the moving reference system (i.e., $w(\xi)$, where $\xi = x - Vt$) for a sub-critical velocity (note that the critical velocity is approximately the same as that of the undamped system) and plot the result for the following parameter values:

$$\rho A = 268.3 \text{ kg/m}$$
$$EI = 6.42 \cdot 10^6 \text{ Nm}^2$$
$$\chi = 7.3 \cdot 10^6 \text{ N/m}^2$$
$$\eta = 1 \cdot 10^2 \text{ Ns/m}^2$$
$$Q_0 = 80 \cdot 10^3 \text{ N}$$

It is advised to take the integral over frequency numerically, for example using the Trapezium rule (which is most easy). In doing so, make sure you respect the sampling theorem related to the discrete Fourier transform. The use of Maple to conduct the numerical integration is in this case not advised, as the result is very sensitive to the precise settings (accuracy, discretization, etc.).

<u>Problem 3 (about Canonical Problem 1)</u>
For the same problem considered in the previous question, derive and compute the equivalent stiffness at the loading/contact point and plot it versus velocity. Consider in the plot only the sub-critical velocity range (up to the 99% of the critical velocity) and explain what you observe.

<u>Problem 4 (about Canonical Problem 2)</u>
Consider a beam with continuous visco-elastic foundation having finite length and being simply supported at the edges, and excited by a constant moving load. Derive and compute the response of the structure

and plot the response for a number of time moments (include one with $t > L/V$). Explain the observed behaviour (in terms of steady state and transient effects). Use the same parameter values as in Problem 2, take the velocity of the load 0.75 times the critical speed of the infinite beam with distributed elastic foundation and assume the length sufficiently long so that the steady-state profile of the infinite beam under the same load can develop (e.g., 100 m); include as many modes as needed for convergence (if needed, increase the value of η to make sure that the number of modes is not too large; damping decreases the importance of higher modes).

Note that for the specific system (without damping) the following holds:

$$\omega_n = \sqrt{\beta_n^4 \frac{EI}{\rho A} + \frac{\chi}{\rho A}}$$

$$\varphi_n(x) = \sin\left(\beta_n x\right), \quad \beta_n = \frac{n\pi}{L}$$

$$EI\varphi_n''''(x) = \left(\rho A\omega_n^2 - \chi\right)\varphi_n(x)$$

$$\int_0^L \varphi_n(x)\varphi_m(x)dx = \begin{cases} \frac{1}{2}L, & n = m \\ 0, & n \neq m \end{cases}$$

Furthermore, the following modal Green's function (with damping included) can be used when deriving the response (the corresponding equation defining the Green's function is given for completeness; here, the $\bar{\omega}_n$ denote the natural frequencies of the damped system):

$$\ddot{g}_n + 2\zeta\omega_n\dot{g}_n + \omega_n^2 g_n = \delta(t)$$

$$g_n(t) = \frac{1}{\bar{\omega}_n}\mathrm{e}^{-\zeta\omega_n t}\sin(\bar{\omega}_n t)H(t), \quad \bar{\omega}_n = \omega_n\sqrt{1-\zeta^2}$$

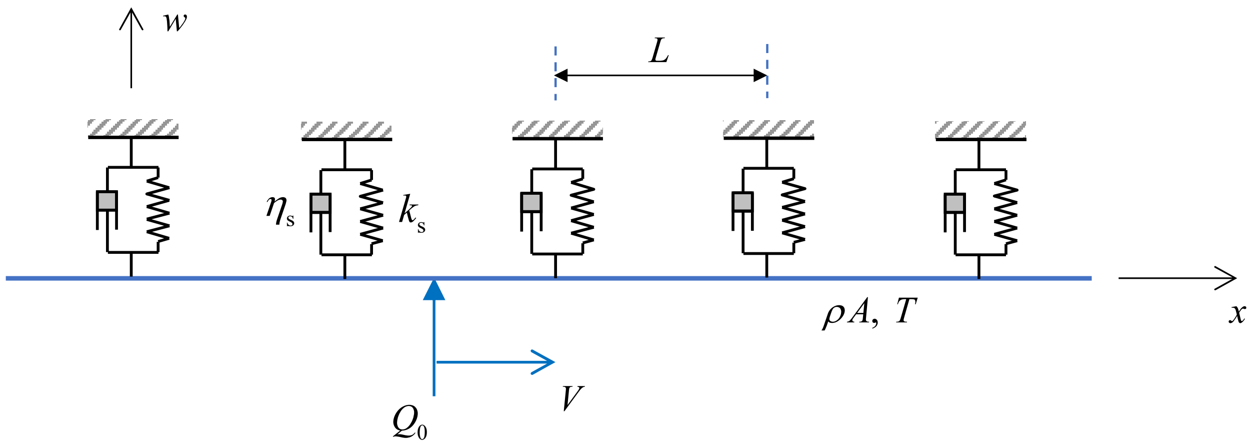
Problem 5 (about Canonical Problem 4)

Consider an overhead powerline structure of a railway track, which is modelled as an infinite tensioned string that is periodically supported by discrete springs and dashpots as shown in the figure below.

The equation of motion reads as follows:

$$\rho\frac{\partial^2 w}{\partial t^2} - T\frac{\partial^2 w}{\partial x^2} + \sum_{n=-\infty}^{\infty}\left(k_s + \eta_s\frac{\partial}{\partial t}\right)w\delta(x - nL) = Q_0\delta(x - Vt)$$

Here, T, η_s denote the (constant) tension force and the dashpot constant of the discrete supports, respectively (other symbols have the same meaning as in the lecture slides of Canonical Problem 4).



a) Proof that the dispersion equation of the system (without damping) reads as follows:

$$\xi^2 - \left(\frac{K_s}{\kappa} \sin(\kappa L) + 2\cos(\kappa L) \right) \xi + 1 = 0, \quad \xi = \exp(-\mathrm{i}kL), \quad K_s = \frac{k_s}{T}, \quad \kappa = \frac{\omega}{c} = \frac{\omega}{\sqrt{T/\rho A}}$$

b) Derive expressions for the wavenumbers, plot the dispersion lines and interpret the result.

c) Derive an expression for the steady-state response in the frequency-space domain (at $x = 0$) and plot the corresponding amplitude spectrum (for the system with damping). Interpret the result. You may wish to use the following relation in the derivation:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{\omega L}{V} - 2\pi n \right)^2 - (\kappa L)^2} = \frac{1}{2\kappa L} \frac{\sin(\kappa L)}{\cos(\kappa L) - \cos\left(\frac{\omega}{V} L \right)}$$

d) Compute numerically the response (at $x = 0$) versus time (make sure you respect the sampling theorem related to the discrete Fourier transform). Interpret the result.

For computing and plotting, the following parameter values should be used:

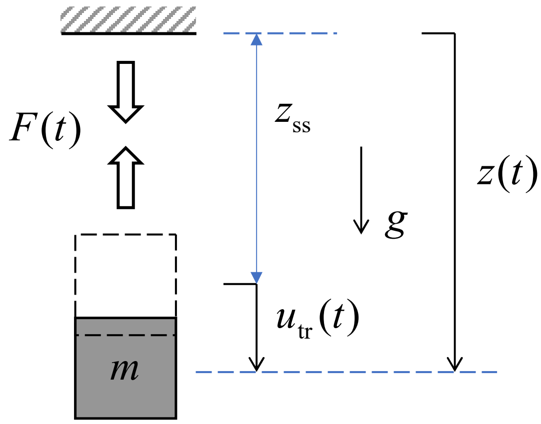
$$\rho A = 1.1 \text{ kg/m}$$
$$T = 15 \cdot 10^3 \text{ Nm}^2$$
$$L = 10 \text{ m}$$
$$k_s = 4 \cdot 10^3 \text{ N/m}$$
$$\eta_s = 0.5 \text{ Ns/m}$$
$$Q_0 = 55 \text{ N}$$
$$V = 28 \text{ m/s}$$

Make sure that that the time window is large enough to ensure that the signal is very close to zero at the edges of the time window. To ensure this, a large time window may be needed; one could also increase the damping to allow the time window to be smaller.

(another problem will be added soon)

Problem 7 (continuation of Canonical Problem 3): **for bonus points, not compulsory**

Consider the model of an electromagnetically suspended mass (m) from a rigid structure, as shown in the figure below. It is a simplified model of a Hyperloop vehicle, where the supporting structure is assumed to be infinitely stiff for simplicity.



The dynamics of the 1.5 DOF system is governed by the following set of equations:

$$m\ddot{z}(t) = -F(t) + mg$$
$$F(t) = C\frac{I^2}{z^2}$$
$$\dot{I} + \frac{z}{2C}\left(R - 2C\frac{\dot{z}}{z^2}\right)I = \frac{z}{2C}U(t)$$
$$U(t) = U_{ss} + K_p(z - z_{ss}) + K_d\dot{z}$$

Here, z denotes the position of the vehicle, g the gravitational constant, $F(t)$ the interaction force associated with the electromagnet, $I(t)$ its current, while C and R are constants; $U(t)$ denotes the voltage across the electromagnet, and U_{ss} is the steady-state voltage associated with the steady-state position of the vehicle z_{ss} (constant). K_p and K_d are parameters of the controller that is used to stabilize the system; if the position of the vehicle differs from z_{ss} and/or its velocity differs from $\dot{z}_{ss} = 0$, $U(t)$ is changed to modify $I(t)$ and thus alter $F(t)$ so as to push or pull the vehicle back to the equilibrium state (i.e., the steady state).

Linearize the above system of equations about the steady state by introducing small variations of in the field variables relative to the steady state:

$$z(t) = z_{ss} + u_{tr}(t)$$
$$F(t) = F_{ss} + F_{tr}(t)$$
$$I(t) = I_{ss} + I_{tr}(t)$$

Derive the three eigenvalues of the obtained system to conclude for what range of K_p values the system is stable (i.e., the free vibration of the system decays over time, implying that the steady state is reached when the system is perturbed).

The following parameter values should be used:

$$m = 7650 \text{ kg}$$
$$C = 0.05 \text{ Nm}^2/\text{A}^2$$
$$R = 9.71 \text{ V/A}$$
$$z_\mathrm{ss} = 0.015 \text{ m}$$
$$K_\mathrm{d} = 10000 \text{ Vs/m}$$