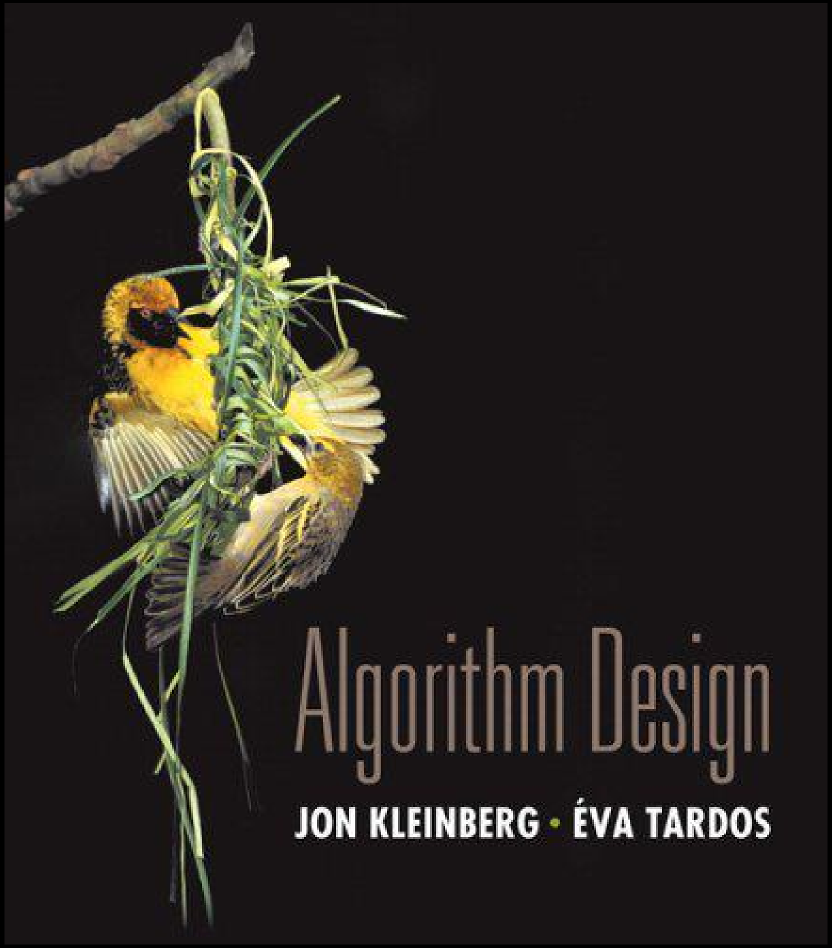


Chapter 4

Greedy Algorithms

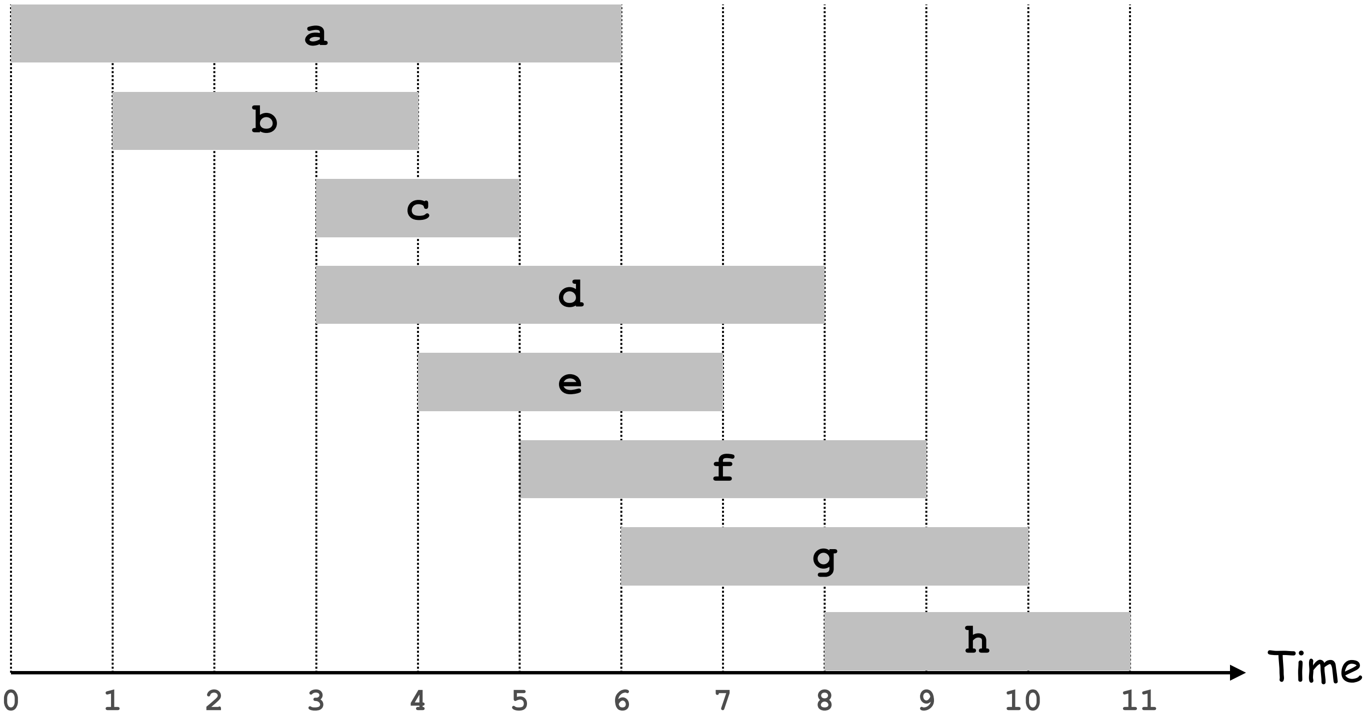
Slides by Kevin Wayne.

4.1.1 Interval Scheduling

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



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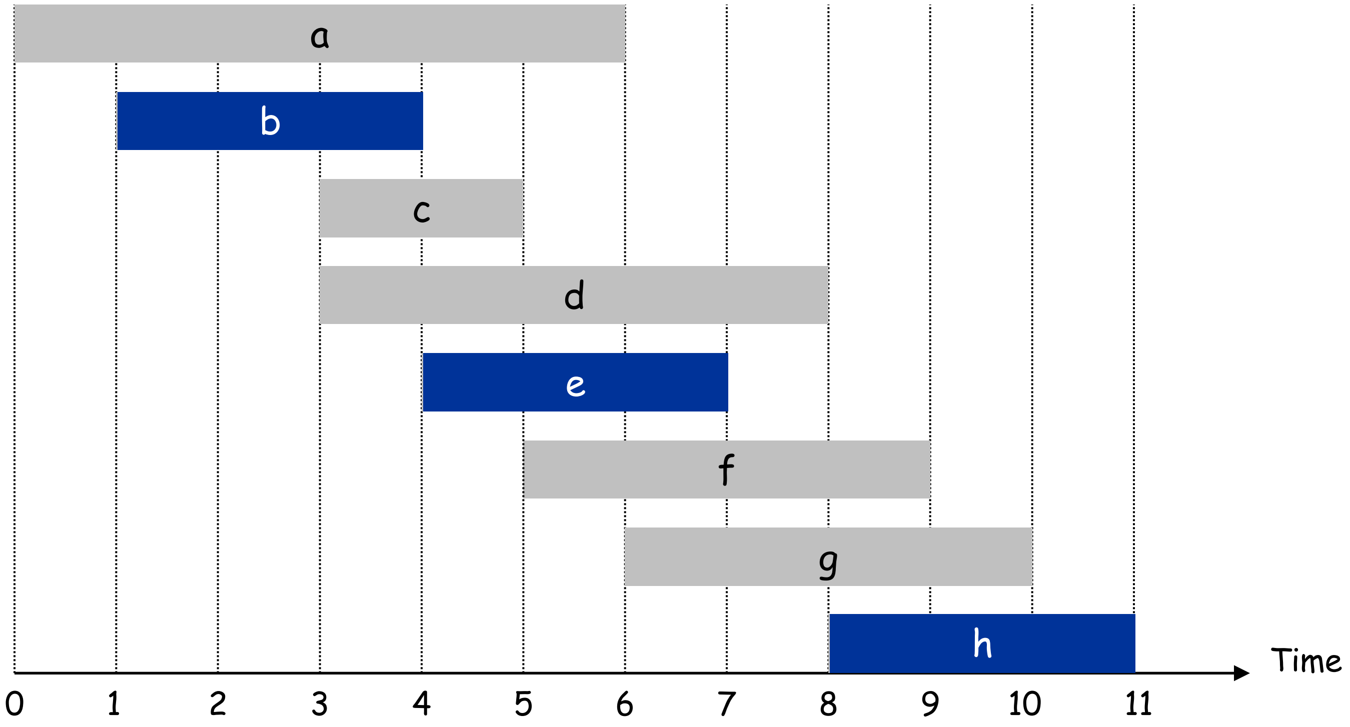


Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of s_j.
- [Earliest finish time] Consider jobs in ascending order of f_j.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

   set of jobs selected
$A \leftarrow \phi$
for j = 1 to n {
   if (job j compatible with A)
      $A \leftarrow A \cup \{j\}$
}
return A
```

Implementation. O(n log n).

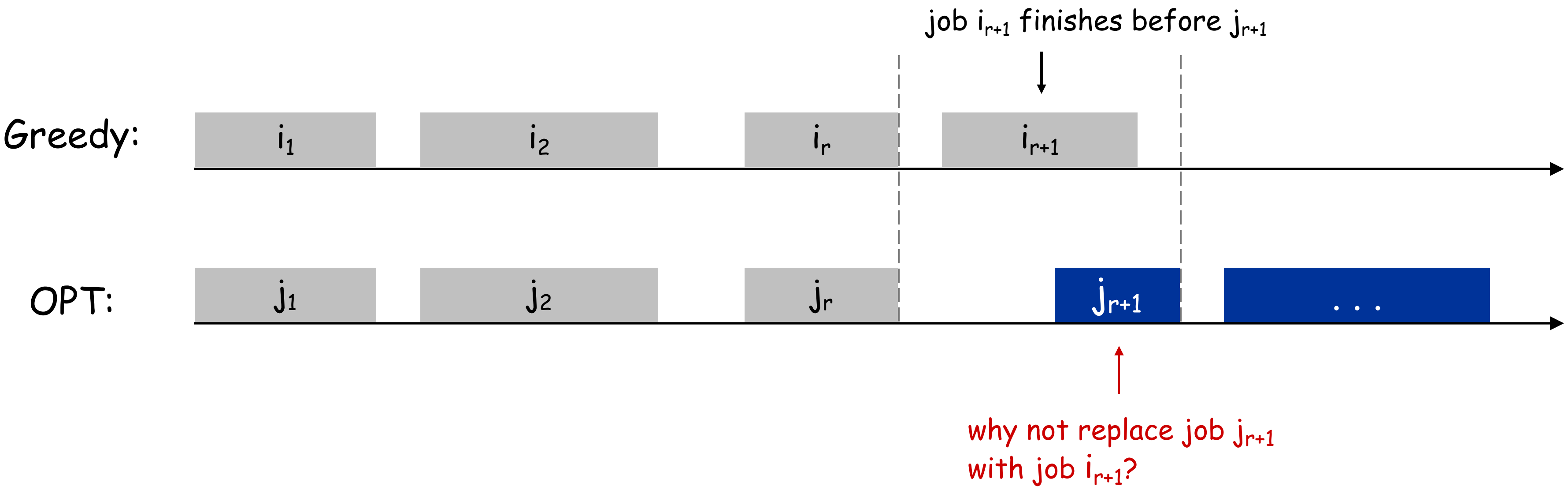
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.


job i_{r+1} finishes before j_{r+1}

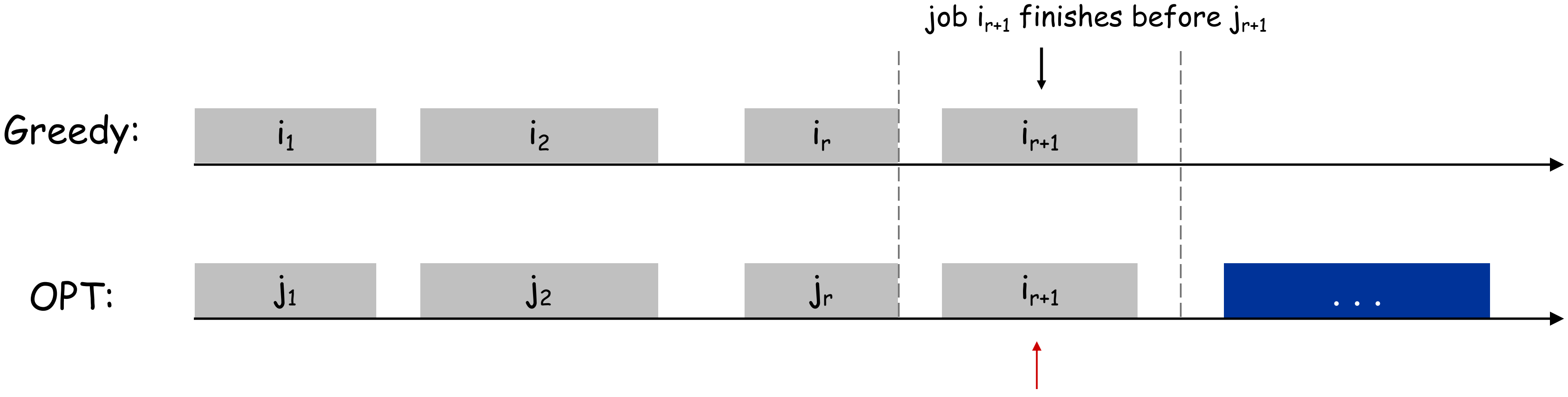
why not replace job j_{r+1} with job i_{r+1}?

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \dots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \dots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \dots, i_r = j_r$ for the largest possible value of r.



solution still feasible and optimal, but contradicts maximality of r.

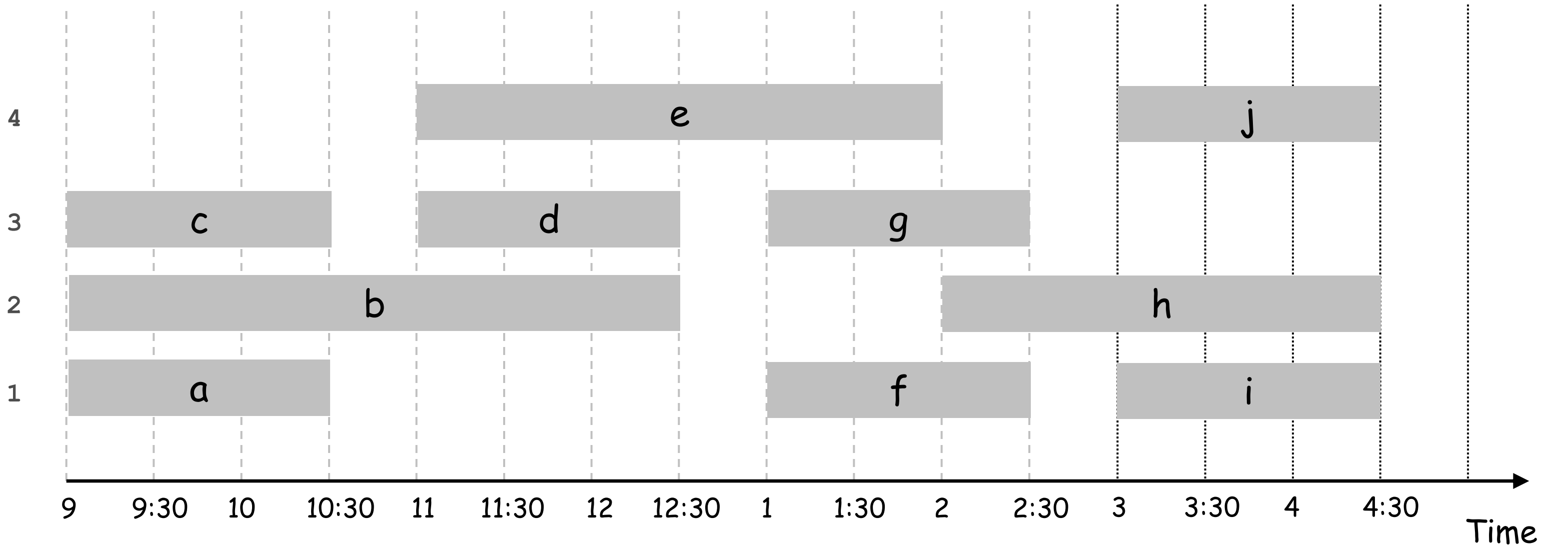
4.1.2 Interval Partitioning

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

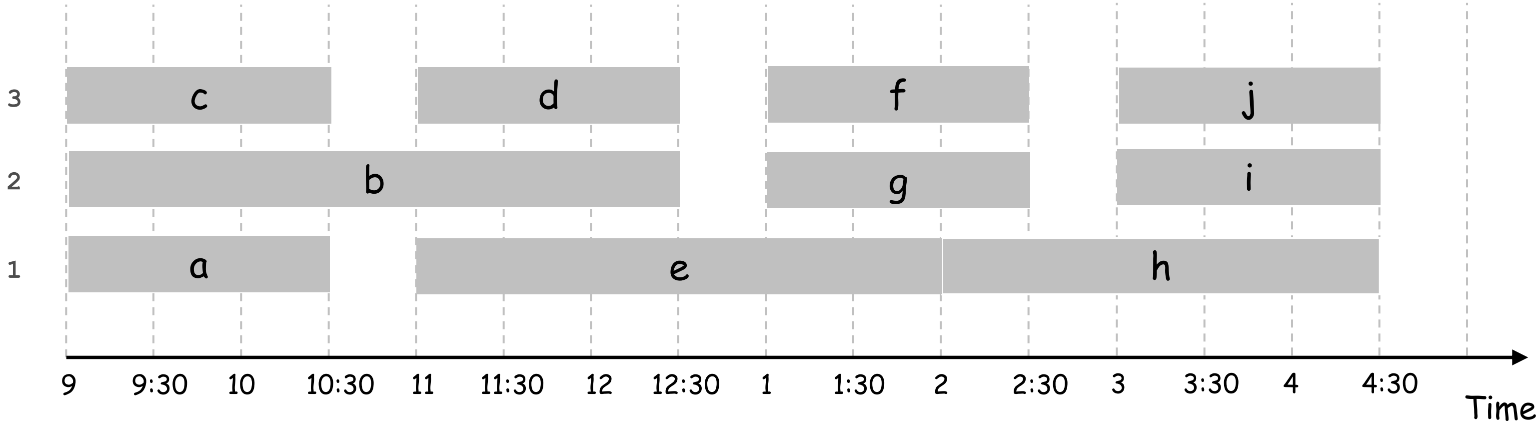


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

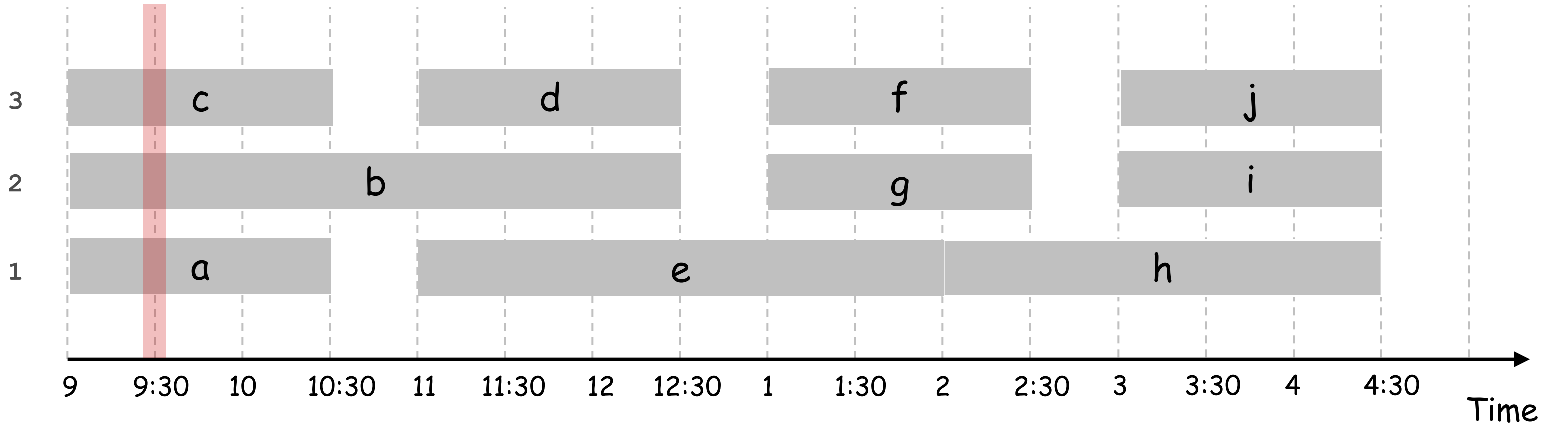
Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed ≥ depth.

Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \le s_2 \le ... \le s_n$.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.
Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- These d jobs (one from each classroom) each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- If d >depth, then there are d lectures overlapping which is impossible, $\Rightarrow$ d must be $\leq depth$
- Key observation $\Rightarrow$ all schedules use $\geq$ depth classrooms. ▪

4.2 Scheduling to Minimize Lateness

Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, \; f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

lateness = 2

lateness = 0

max lateness = 6

$d_3 = 9$ $d_2 = 8$ $d_6 = 15$ $d_1 = 6$ $d_5 = 14$ $d_4 = 9$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

- [Earliest deadline first] Consider jobs in ascending order of deadline d_j.

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

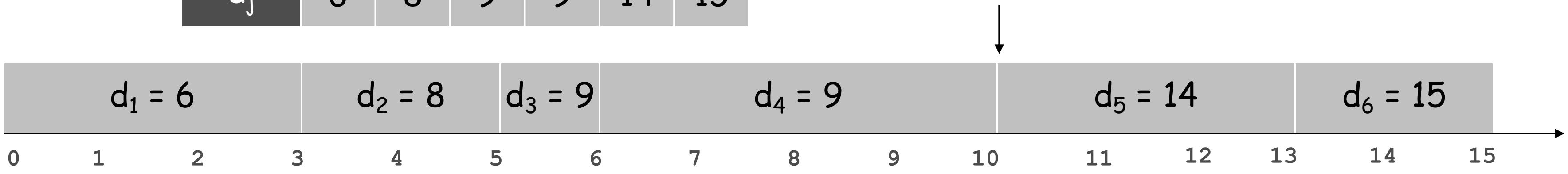
Rename all jobs as 1, 2, …, n according to their deadlines

```
Sort n jobs by deadline so that $d_1 \le d_2 \le \dots \le d_n$

$t \leftarrow 0$
for j = 1 to n
   Assign job j to interval $[t, t + t_j]$
   $s_j \leftarrow t$, $f_j \leftarrow t + t_j$
   $t \leftarrow t + t_j$
output intervals $[s_j, f_j]$
```

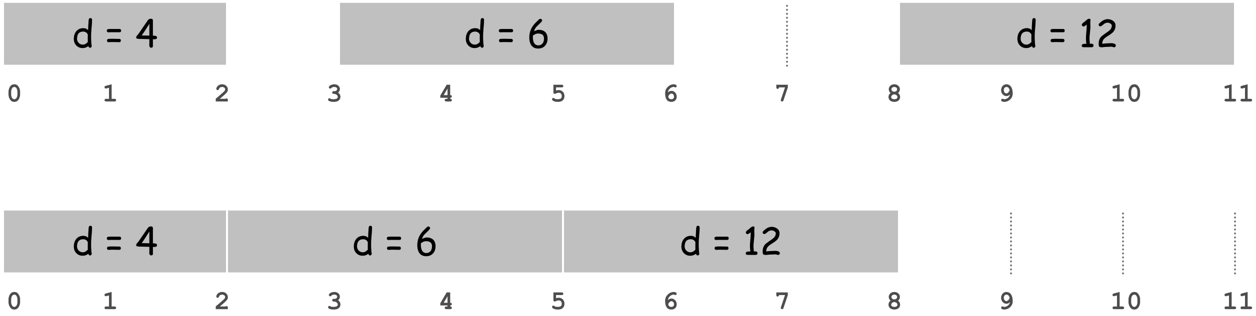
	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

max lateness = 1



Minimizing Lateness: No Idle Time

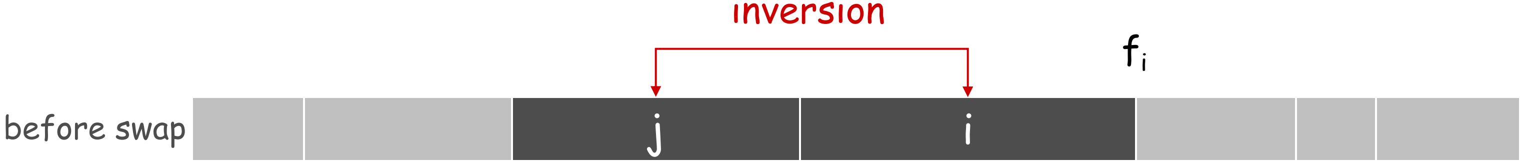
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



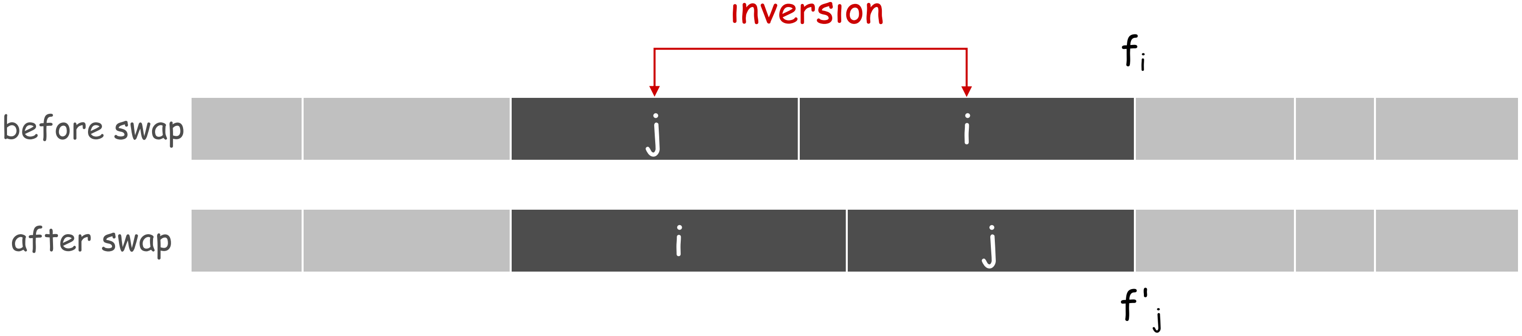
[as before, we assume jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$]

Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*. ← Greedy schedule has no inversions
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

Greedy Analysis Strategies

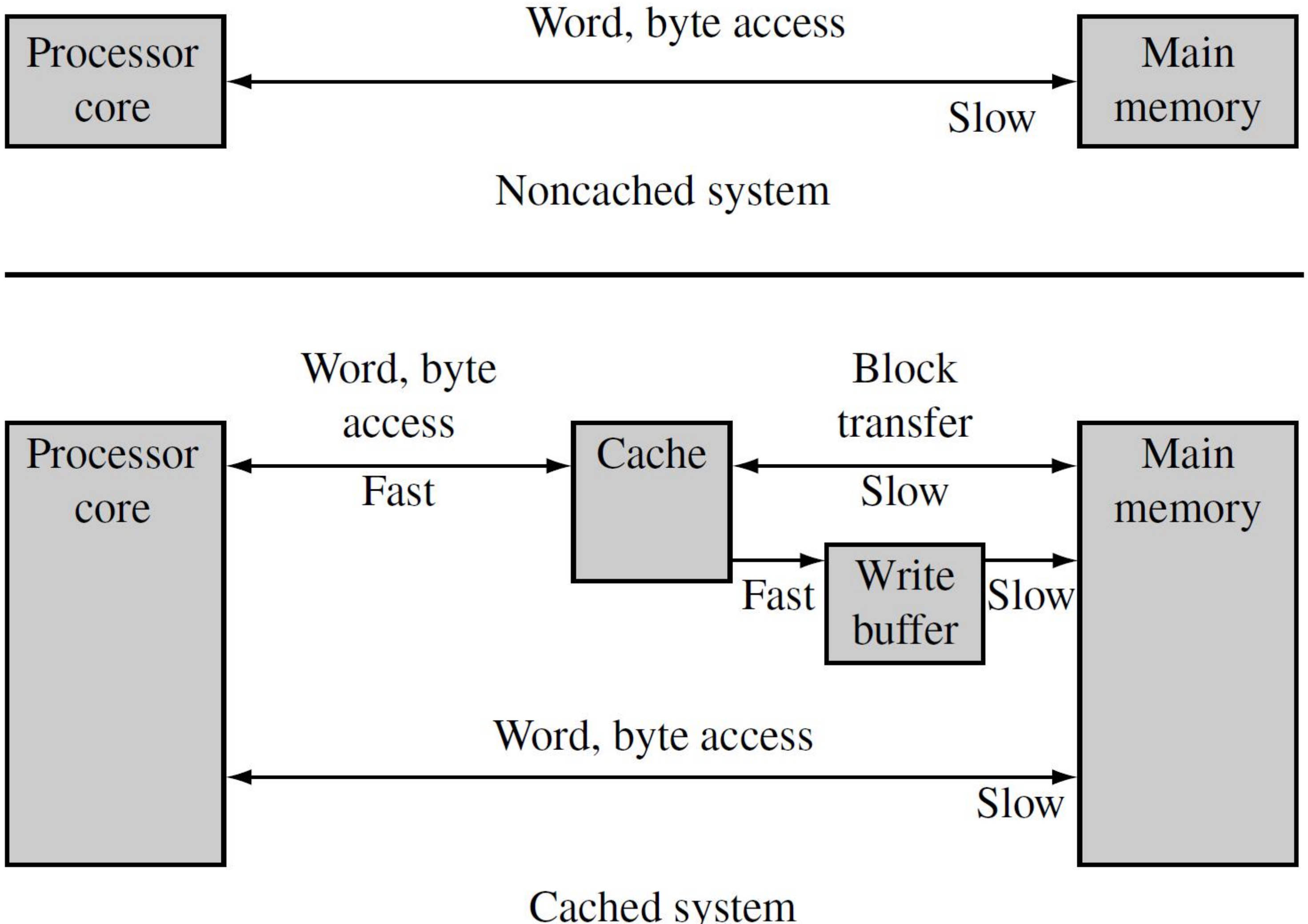
Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's. (Interval scheduling)

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound. (Interval Partitioning)

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality. (Scheduling to Minimize Lateness)

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, …

4.3 Optimal Caching



Optimal Offline Caching

Caching.

- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

to *evict* some other piece of data that is currently in the cache to make room for d_i .

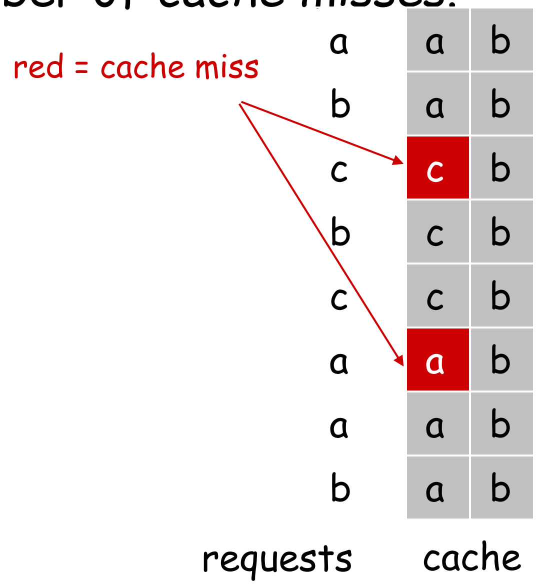
Goal. *Eviction schedule* that minimizes number of *cache misses*.

specifying which items should be evicted from the cache at which points in the sequence

Ex: k = 2, initial cache = ab,

requests: a, b, c, b, c, a, a, b.

Optimal eviction schedule: 2 cache misses.



red = cache miss

a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b
requests	cache	

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache: | a | b | c | d | e | f |

future queries: `g a b c e d a b b a c d e a f a d e f g h ...`

↑ cache miss (g)

↑ eject this one (f)

Theorem. [Bellady, 1960s] FF is an optimal eviction schedule.

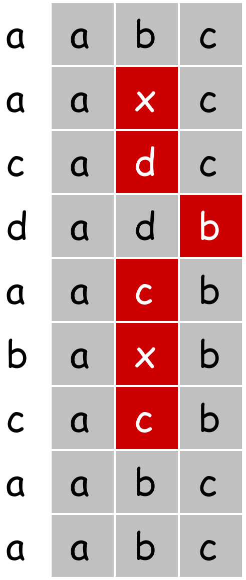
Pf. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

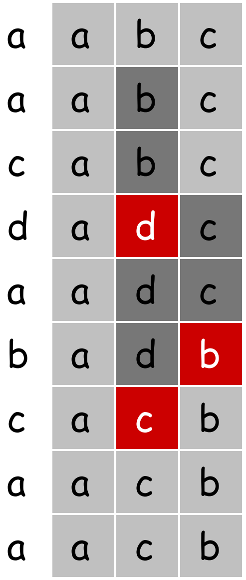
Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.

to evict some other piece of data that is currently in the cache to make room for data d_i .



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule



a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

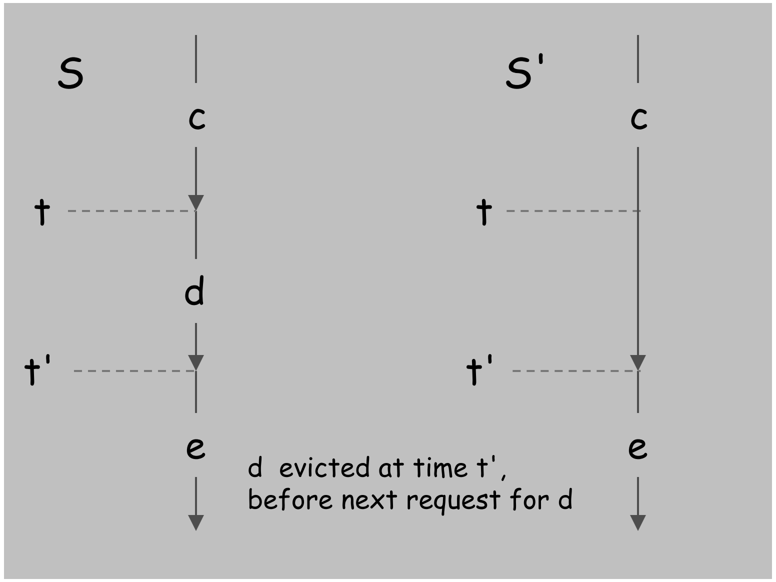
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

Pf. (by induction on number of unreduced items)

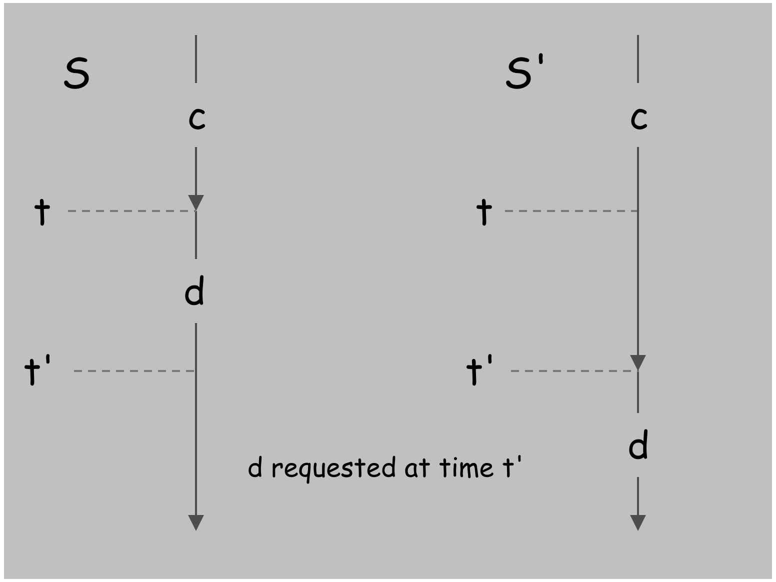
doesn't enter cache at requested time

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪



d evicted at time t', before next request for d

Case 1



d requested at time t'

Case 2

Farthest-In-Future: Analysis

Theorem. FF is an optimal eviction algorithm.

Pf. (by induction on number or requests j)

> Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first j+1 requests.

Let S be an optimal reduced schedule that satisfies invariant through j requests. We produce S' that satisfies invariant through the first j+1 requests.

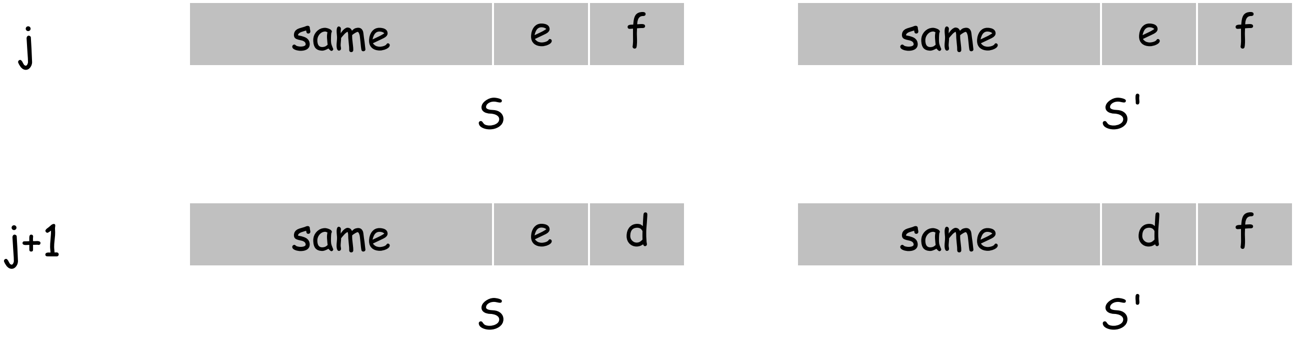
- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since S and S_{FF} have agreed up until now, they have the same cache contents before request j+1.
- *Case 1:* (d is already in the cache). S' = S satisfies invariant.
- *Case 2:* (d is not in the cache and S and S_{FF} evict the same element). S' = S satisfies invariant.

Farthest-In-Future: Analysis

Pf. (continued)

e is far away than f

- Case 3: (d is not in the cache; S_{FF} evicts e; S evicts $f \neq e$).
  - begin construction of S' from S by evicting e instead of f

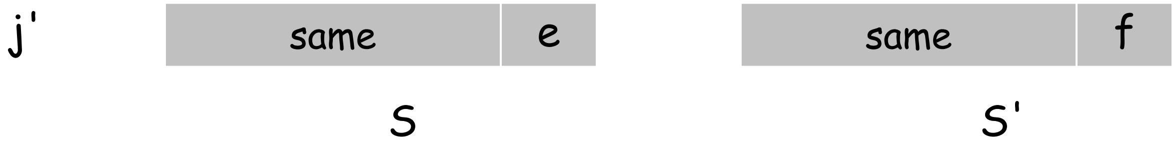


- now S' agrees with S_{FF} on first j+1 requests; we show that having element f in cache is no worse than having element e

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



- Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.

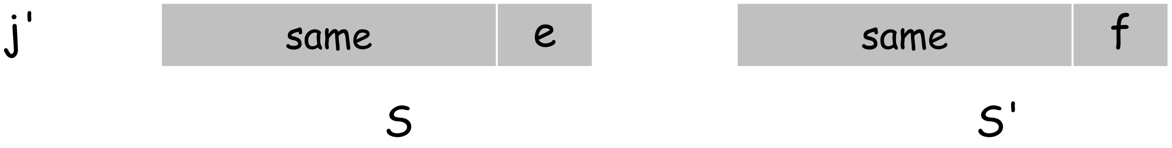
- Case 3b: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step j+1

Farthest-In-Future: Analysis

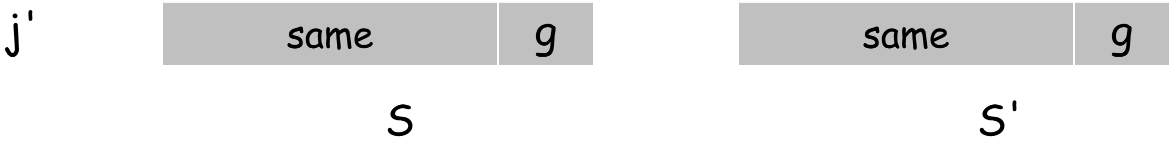
Let j' be the first time after $j+1$ that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



otherwise S' would take the same action

- Case 3c: $g \neq e, f$. S must evict e.
  Make S' evict f; now S and S' have the same cache. ▪



Hence, in all these cases, we have a new reduced schedule S' that agrees with S_{FF} through the first $j + 1$ items and incurs no more misses than S does.

Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

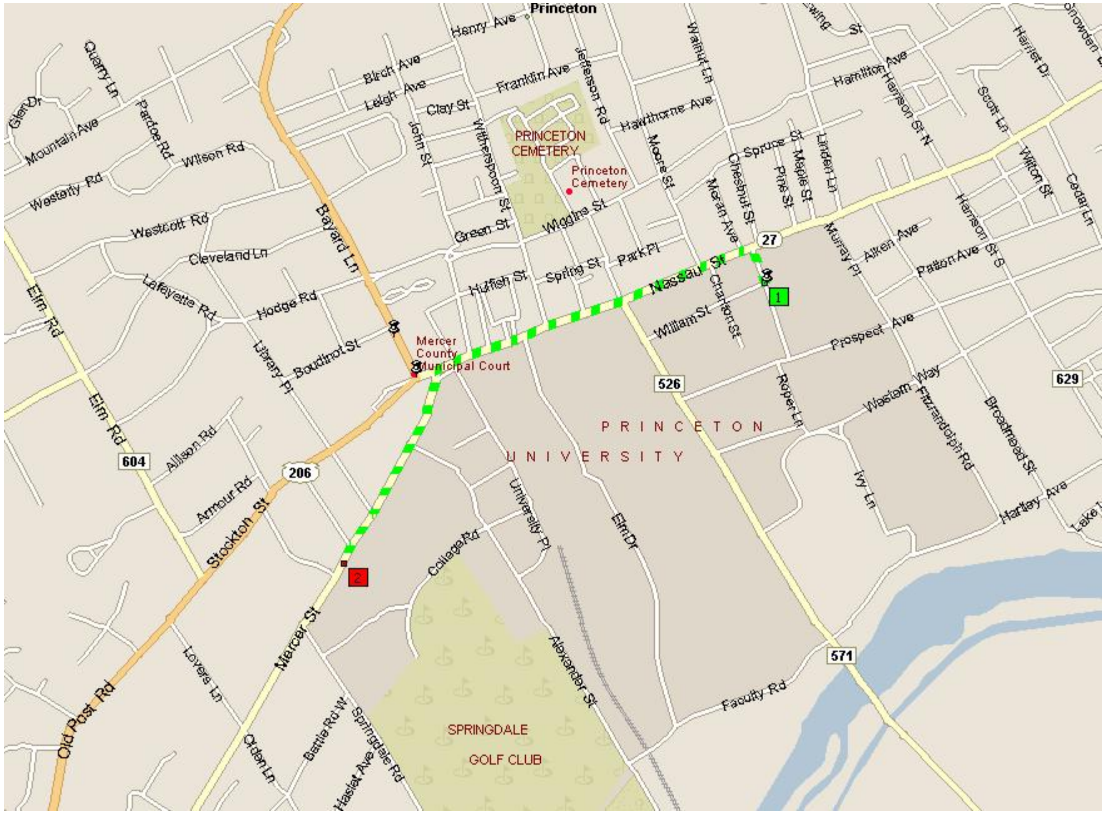
Least-Recently-Used (LRU). Evict page whose most recent access was earliest.

↑ FF with direction of time reversed!

Theorem. FF is an optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.

4.4 Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

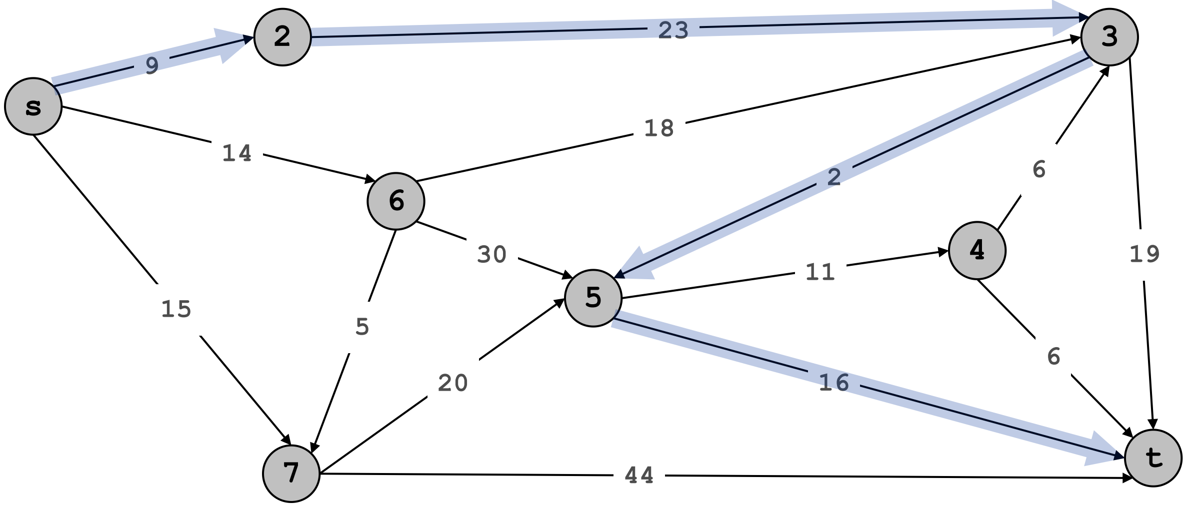
Shortest Path Problem

Shortest path network.

- Directed graph G = (V, E).
- Source s, destination t.
- Length ℓ_e = length (cost) of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 50.

Dijkstra's Algorithm

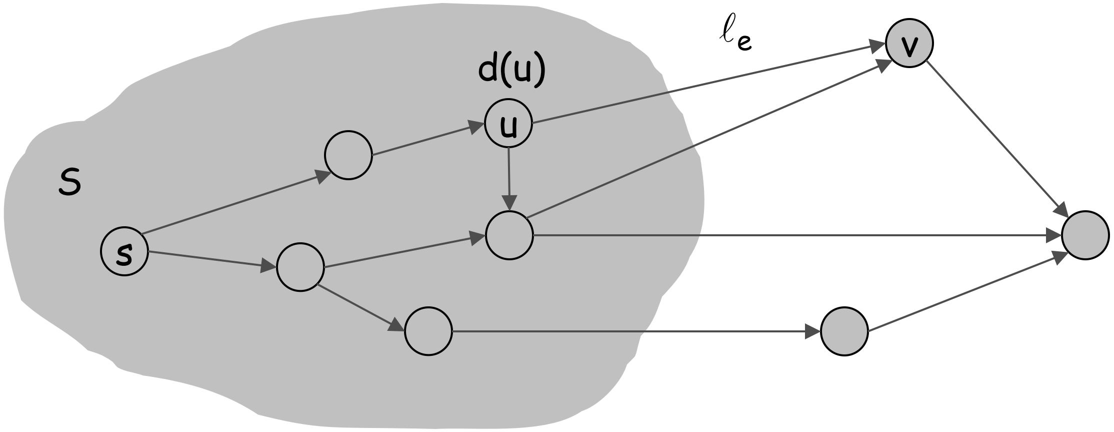
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) \,:\, u \in S} d(u) + \ell_e ,$$

add v to S, and set d(v) = π(v).

shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Algorithm

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- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
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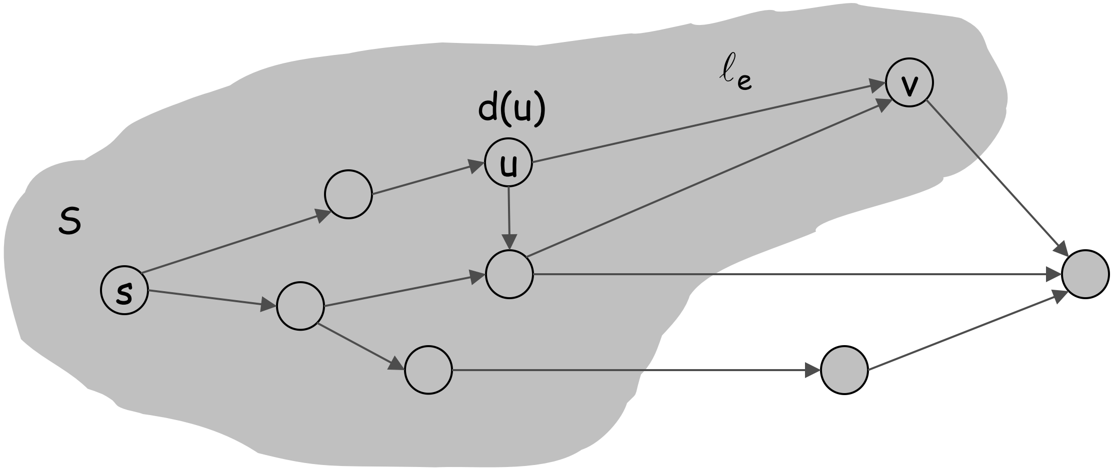
add v to S, and set d(v) = π(v).



Dijkstra's Algorithm: Proof of Correctness

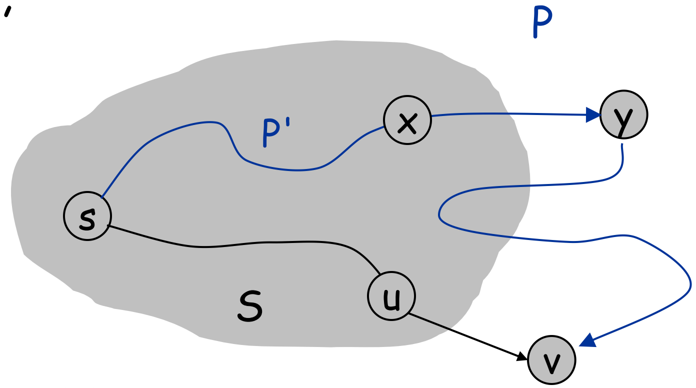
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s-u path.

Pf. (by induction on $|S|$)

Base case: $|S| = 1$ is trivial.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length $\pi(v)$.
- Consider any s-v path P. We'll see that it's no shorter than $\pi(v)$.
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S.



$$\ell(P) \underset{\substack{\uparrow \\ \text{nonnegative} \\ \text{weights}}}{\geq} \ell(P') + \ell(x,y) \underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{\geq} d(x) + \ell(x, y) \underset{\substack{\uparrow \\ \text{defn of } \pi(y)}}{\geq} \pi(y) \underset{\substack{\uparrow \\ \text{Dijkstra chose v} \\ \text{instead of y}}}{\geq} \pi(v)$$

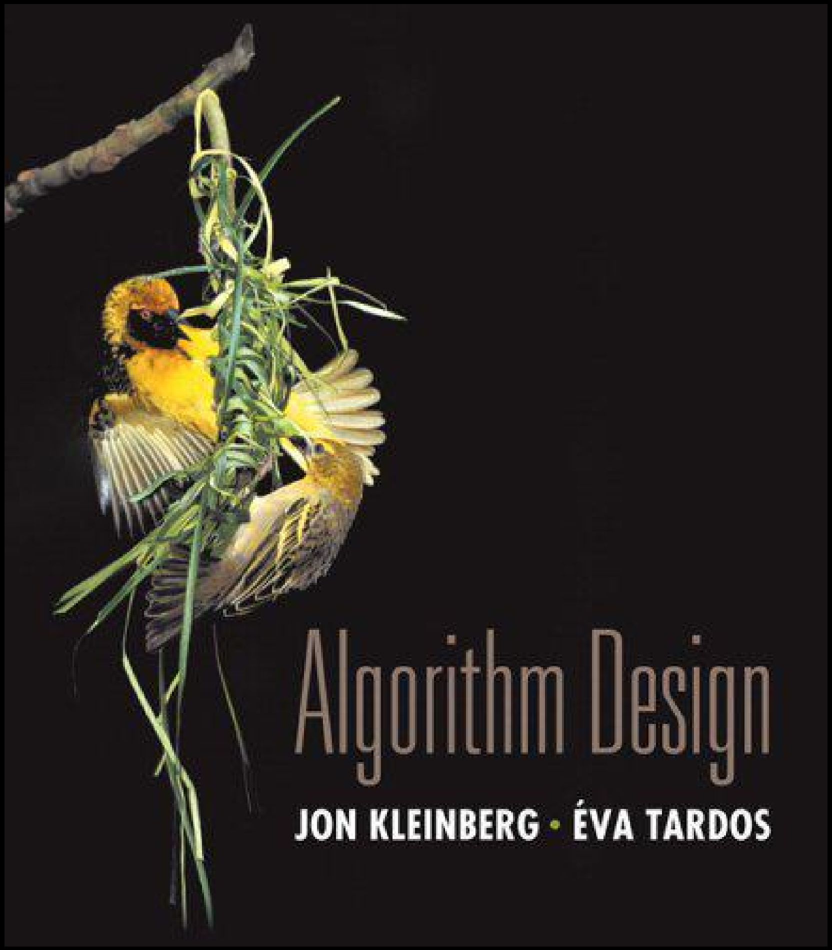
Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v)\,:\, u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \ \pi(w),\ \ \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.



Chapter 4

Greedy Algorithms

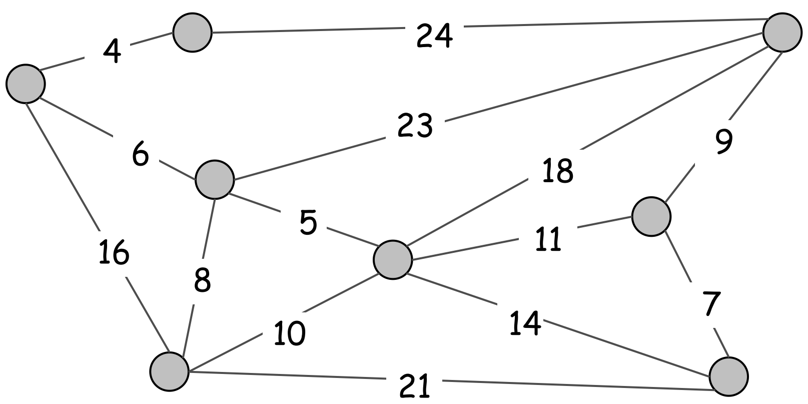
Slides by Kevin Wayne.

4.5 Minimum Spanning Tree

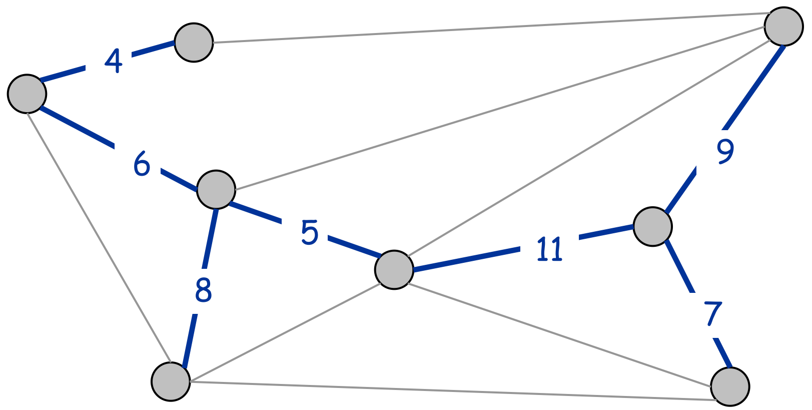
Minimum Spanning Tree

Minimum spanning tree. Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a *spanning tree* whose sum of edge weights is minimized.

if *(V, T)* is a tree



$G = (V, E)$



$T, \ \Sigma_{e \in T} c_e = 50$

Applications

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Greedy Algorithms

Kruskal's algorithm. Start with T = ϕ. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.

Reverse-Delete algorithm. Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Prim's algorithm. Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T.

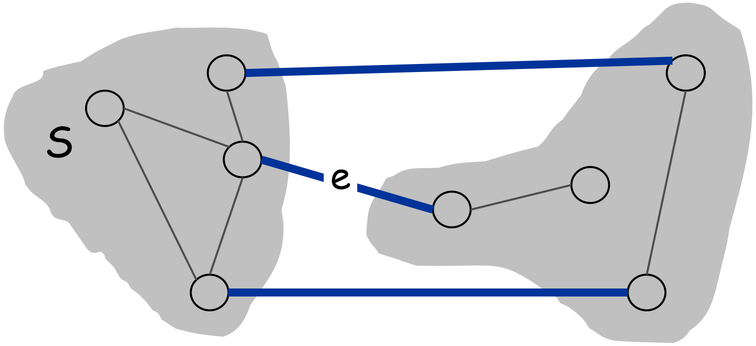
Remark. All three algorithms produce an MST.

Greedy Algorithms

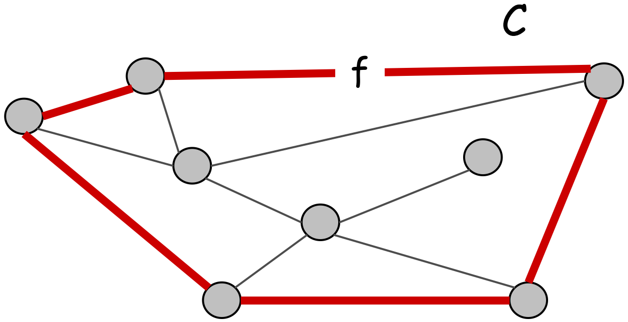
Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.



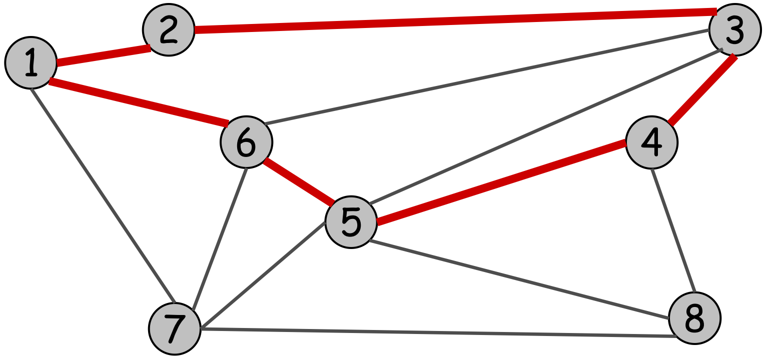
e is in the MST



f is not in the MST

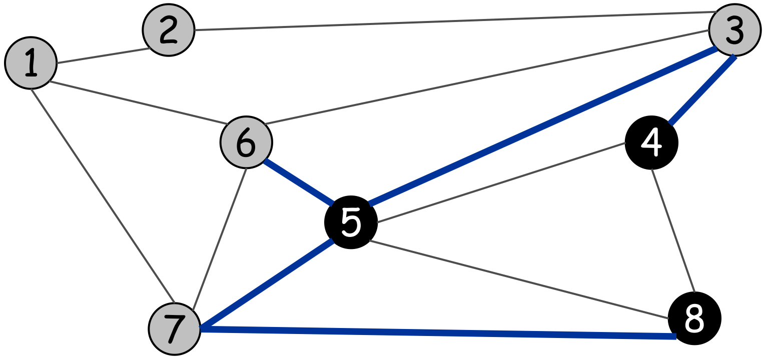
Cycles and Cuts

Cycle. Set of edges that form a-b, b-c, c-d, …, y-z, z-a.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1

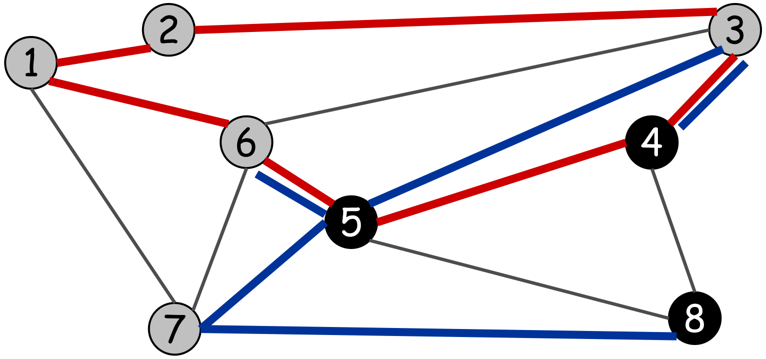
Cutset. A cut S is a subset of nodes. The corresponding cutset D is the subset of edges with exactly one endpoint in S.



Cut S = { 4, 5, 8 }
Cutset D = 5-6, 5-7, 3-4, 3-5, 7-8

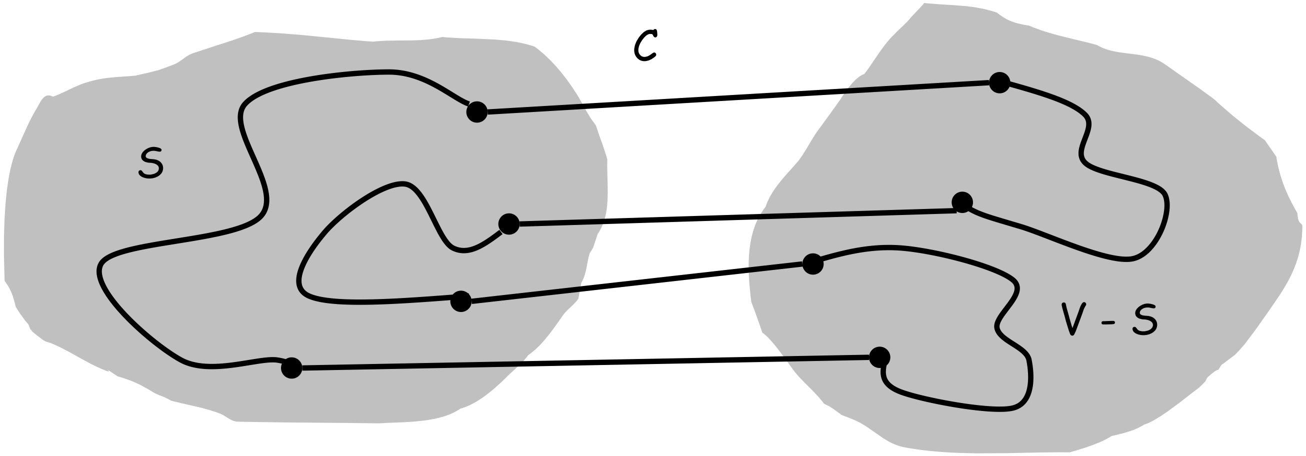
Cycle-Cut Intersection

Claim. A cycle and a cutset intersect in an even number of edges.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6
Cut S = {4, 5, 8}

Pf. (by picture)



Greedy Algorithms

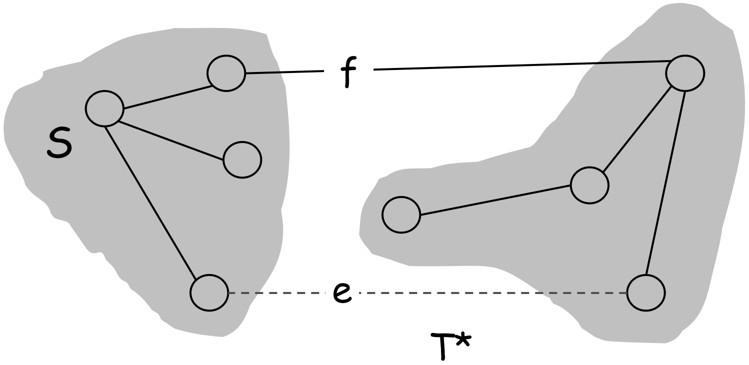
Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. (exchange argument)

- Suppose e does not belong to T*, and let's see what happens.
- Adding e to T* creates a cycle C in T*.
- Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say f, that is in both C and D.
- T' = T* $\cup$ { e } - { f } is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- This is a contradiction. ▪

Previous slide: an even number of edges



Greedy Algorithms

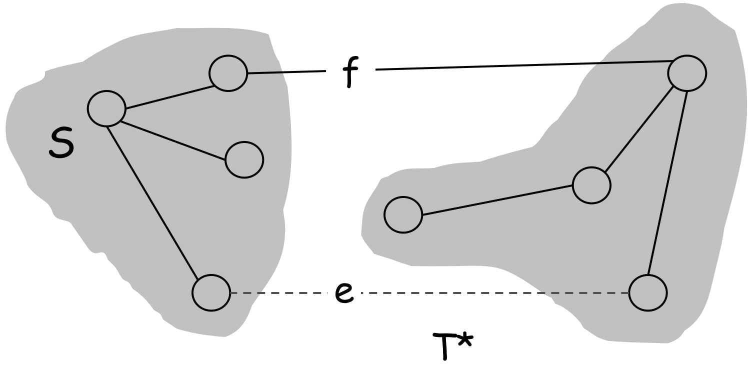
Simplifying assumption. All edge costs c_e are distinct.

Cycle property. Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (exchange argument)

- Suppose f belongs to T*, and let's see what happens.
- Deleting f from T* creates a cut S in T*.
- Edge f is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say e, that is in both C and D.
- T' = T* $\cup$ { e } - { f } is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- This is a contradiction. ▪

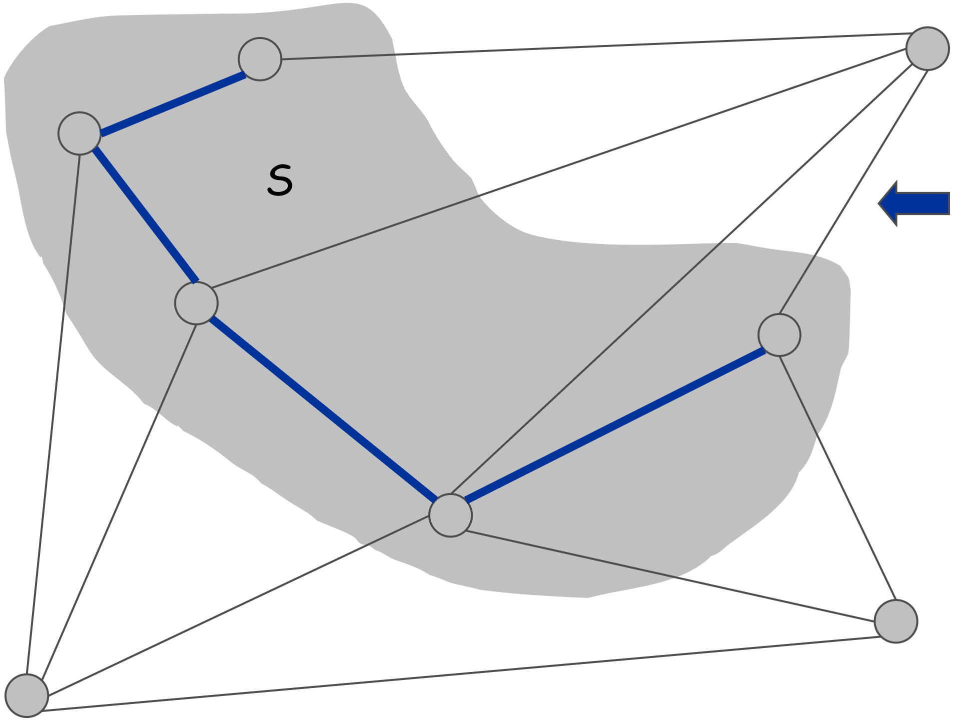
Previous 2 slides: an even number of edges



Prim's Algorithm: Proof of Correctness

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

- Initialize S = any node.
- Apply cut property to S.
- Add min cost edge in cutset corresponding to S to T, and add one new explored node u to S.



Implementation: Prim's Algorithm

Implementation. Use a priority queue ala Dijkstra.

- Maintain set of explored nodes S.
- For each unexplored node v, maintain attachment cost a[v] = cost of cheapest edge v to a node in S.
- O(n^2) with an array; O(m log n) with a binary heap.

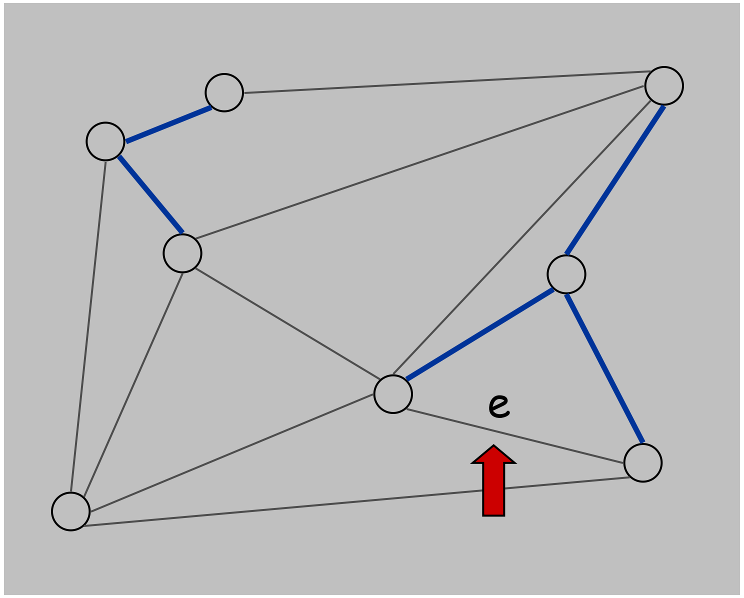
```
Prim(G, c) {
   foreach (v ∈ V) a[v] ← ∞
   Initialize an empty priority queue Q
   foreach (v ∈ V) insert v onto Q
   Initialize set of explored nodes S ← ϕ

   while (Q is not empty) {
      u ← delete min element from Q
      S ← S ∪ { u }
      foreach (edge e = (u, v) incident to u)
          if ((v ∉ S) and ($c_e$ < a[v]))
             decrease priority a[v] to $c_e$
}
```

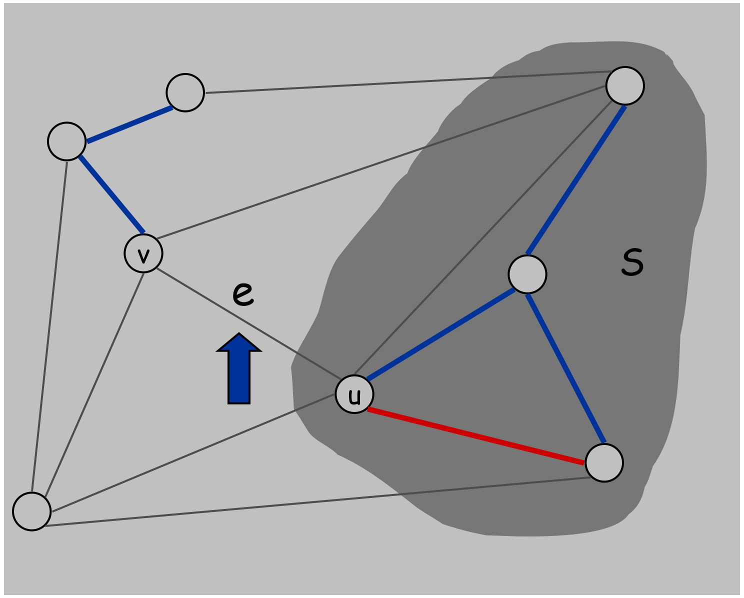
Kruskal's Algorithm: Proof of Correctness

Kruskal's algorithm. [Kruskal, 1956]

- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e according to cycle property.
- Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.



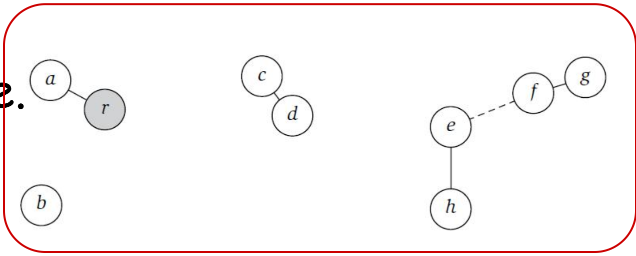
Case 1



Case 2

Implementation: Kruskal's Algorithm

Union(Find(u),Find(v))



Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain set for each connected component.
- $O(m \log n)$ for sorting and $O(m\, \underbrace{\alpha\,(m, n)}_{\text{essentially a constant}})$ for union-find.

$m \leq n^2 \Rightarrow \log m$ is $O(\log n)$

```
Kruskal(G, c) {
   Sort edges weights so that c₁ ≤ c₂ ≤ ... ≤ cₘ.
   T ← φ

   foreach (u ∈ V) make a set containing singleton u

   for i = 1 to m
      (u,v) = eᵢ
      if (u and v are in different sets) {
         T ← T ∪ {eᵢ}
         merge the sets containing u and v
      }
   return T
}
```

are u and v in different connected components?

merge two components

Lexicographic Tiebreaking

To remove the assumption that all edge costs are distinct: perturb all edge costs by tiny amounts to break any ties.

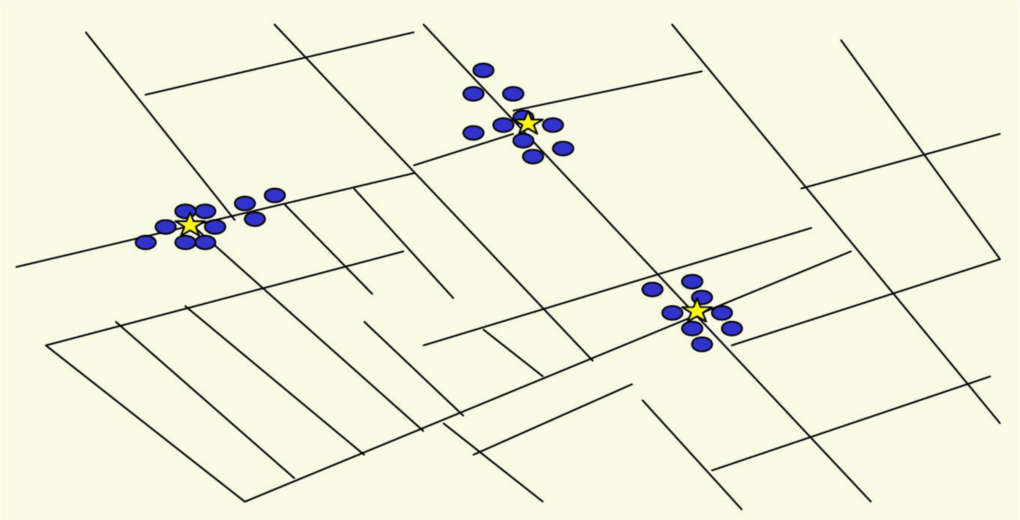
Impact. Kruskal and Prim only interact with costs via pairwise comparisons. If perturbations are sufficiently small, MST with perturbed costs is MST with original costs.

e.g., if all edge costs are integers,
perturbing cost of edge e_i by i / n^2

Implementation. Can handle arbitrarily small perturbations implicitly by breaking ties lexicographically, according to index.

```
boolean less(i, j) {
   if       (cost(e_i) < cost(e_j)) return true
   else if  (cost(e_i) > cost(e_j)) return false
   else if  (i < j)                 return true
   else                             return false
}
```

4.7 Clustering



Outbreak of cholera deaths in London in 1850s.
Reference: Nina Mishra, HP Labs

Clustering

Clustering. Given a set U of n objects labeled $p_1, \ldots, p_n$, classify into coherent groups.

↑ photos, documents. micro-organisms

Distance function. Numeric value specifying "closeness" of two objects.

↑ number of corresponding pixels whose intensities differ by some threshold

Fundamental problem. Divide into clusters so that points in different clusters are far apart.

- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

Clustering of Maximum Spacing

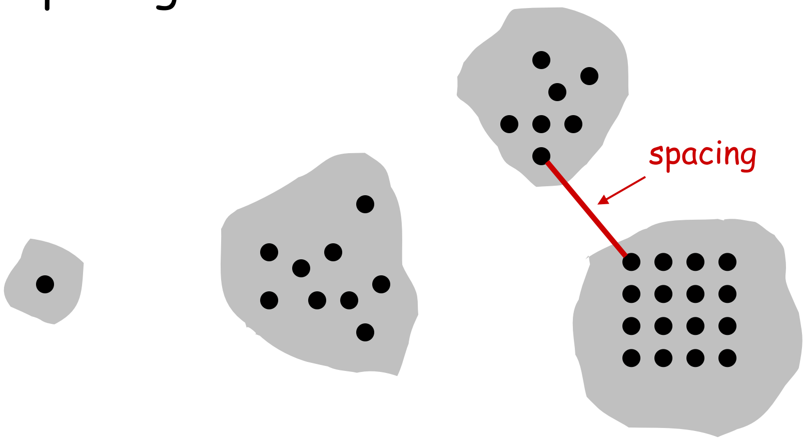
k-clustering. Divide objects into k non-empty groups.

Distance function. Assume it satisfies several natural properties.

- $d(p_i, p_j) = 0$ iff $p_i = p_j$ (identity of indiscernibles)
- $d(p_i, p_j) \geq 0$ (nonnegativity)
- $d(p_i, p_j) = d(p_j, p_i)$ (symmetry)

Spacing. Min distance between any pair of points in different clusters.

Clustering of maximum spacing. Given an integer k, find a k-clustering of maximum spacing.



k = 4

Greedy Clustering Algorithm

Single-link k-clustering algorithm.

- Form a graph on the vertex set U, corresponding to n clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them to merge these two clusters into a new cluster (one cluster less).
- Repeat n-k times until there are exactly k clusters.

Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

Remark. Equivalent to finding an MST and deleting the k-1 most expensive edges.

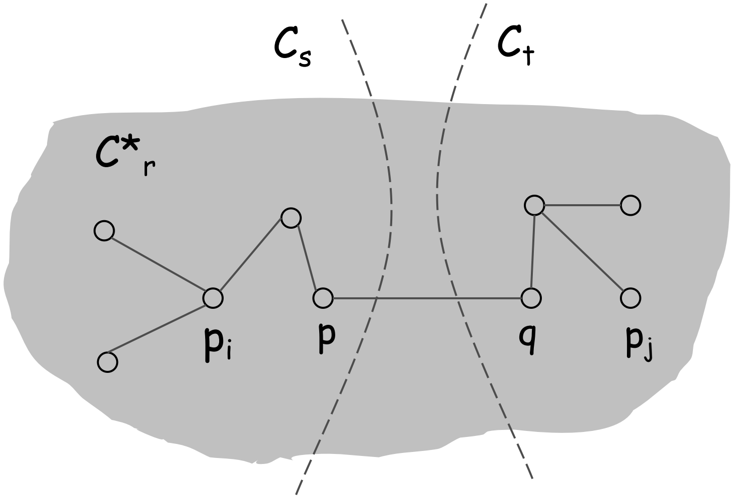
Greedy Clustering Algorithm: Analysis

Theorem. Let C^* denote the clustering $C^*_1, \ldots, C^*_k$ formed by deleting the k-1 most expensive edges of a MST. C^* is a k-clustering of max spacing.

Pf. Let C denote some other clustering $C_1, \ldots, C_k$.

- The spacing of C^* is the length d^* of the $(k-1)^{st}$ most expensive edge.
- Let p_i, p_j be in the same cluster in C^*, say C^*_r, but different clusters in C, say C_s and C_t (which must be possible, otherwise same)
- Some edge (p, q) on p_i-p_j path in C^*_r spans two different clusters in C.
- All edges on p_i-p_j path have length $\leq d^*$ since Kruskal chose them before the d^*.
- Spacing of C is $\leq d^*$ since p and q are in different clusters.
- C is not max spacing

➔ C^* is a k-clustering of max spacing



4.8 Huffman Codes

These lecture slides are supplied by Mathijs de Weerd

Data Compression

Q. Given a text that uses 32 symbols (26 different letters, space, and some punctuation characters), how can we encode this text in bits?
A. We can encode 2^5 different symbols using a fixed length of 5 bits per symbol. This is called fixed length encoding.

Q. Some symbols (e, t, a, o, i, n) are used far more often than others. How can we use this to reduce our encoding?
A. Encode these characters with fewer bits, and the others with more bits.

Q. How do we know when the next symbol begins?
A. Use a separation symbol (like the pause in Morse), or make sure that there is no ambiguity by ensuring that no code is a prefix of another one.

Ex. c(a) = 01
c(b) = 010
c(e) = 1

What is 0101?

Prefix Codes

Definition. A prefix code for a set S is a function c that maps each $x \in S$ to 1s and 0s in such a way that for $x,y \in S$, $x \neq y$, $c(x)$ is not a prefix of $c(y)$.

Ex. c(a) = 11
c(e) = 01
c(k) = 001
c(l) = 10
c(u) = 000

Q. What is the meaning of 1001000001 ?

A. "leuk"

Suppose frequencies are known in a text of 1G (characters):

f_a=0.4, f_e=0.2, f_k=0.2, f_l=0.1, f_u=0.1

Q. What is the size of the encoded text?

A. $2^*f_a + 2^*f_e + 3^*f_k + 2^*f_l + 3^*f_u = 2.3G$ (bits)

Optimal Prefix Codes

Definition. The average bits per letter of a prefix code c is the sum over all symbols of its frequency times the number of bits of its encoding:

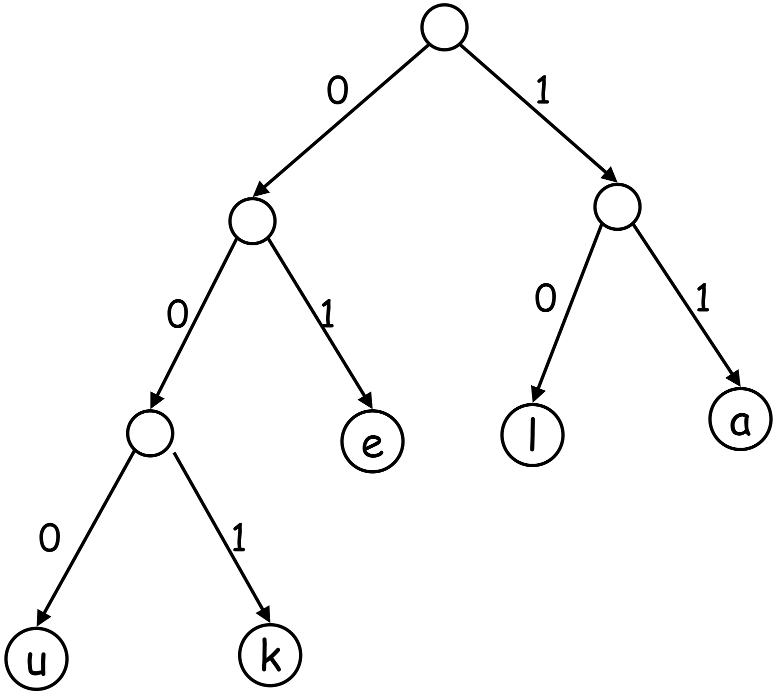
$$ABL(c) = \sum_{x \in S} f_x \cdot |c(x)|$$

We would like to find a prefix code that has the lowest possible average bits per letter.

Suppose we model a code in a binary tree…

Representing Prefix Codes using Binary Trees

Ex. c(a) = 11
c(e) = 01
c(k) = 001
c(l) = 10
c(u) = 000



Q. How does the tree of a prefix code look?

Representing Prefix Codes using Binary Trees

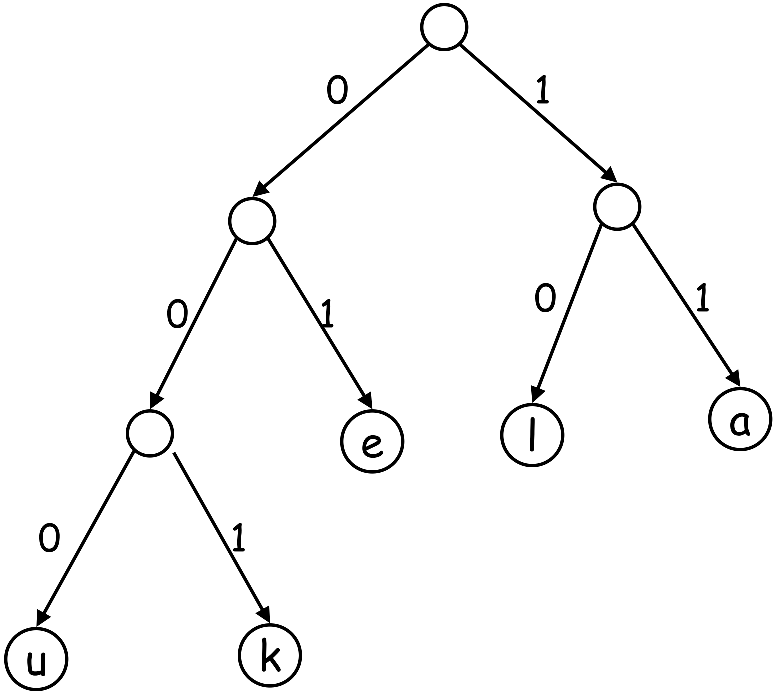
Ex. c(a) = 11

c(e) = 01

c(k) = 001

c(l) = 10

c(u) = 000



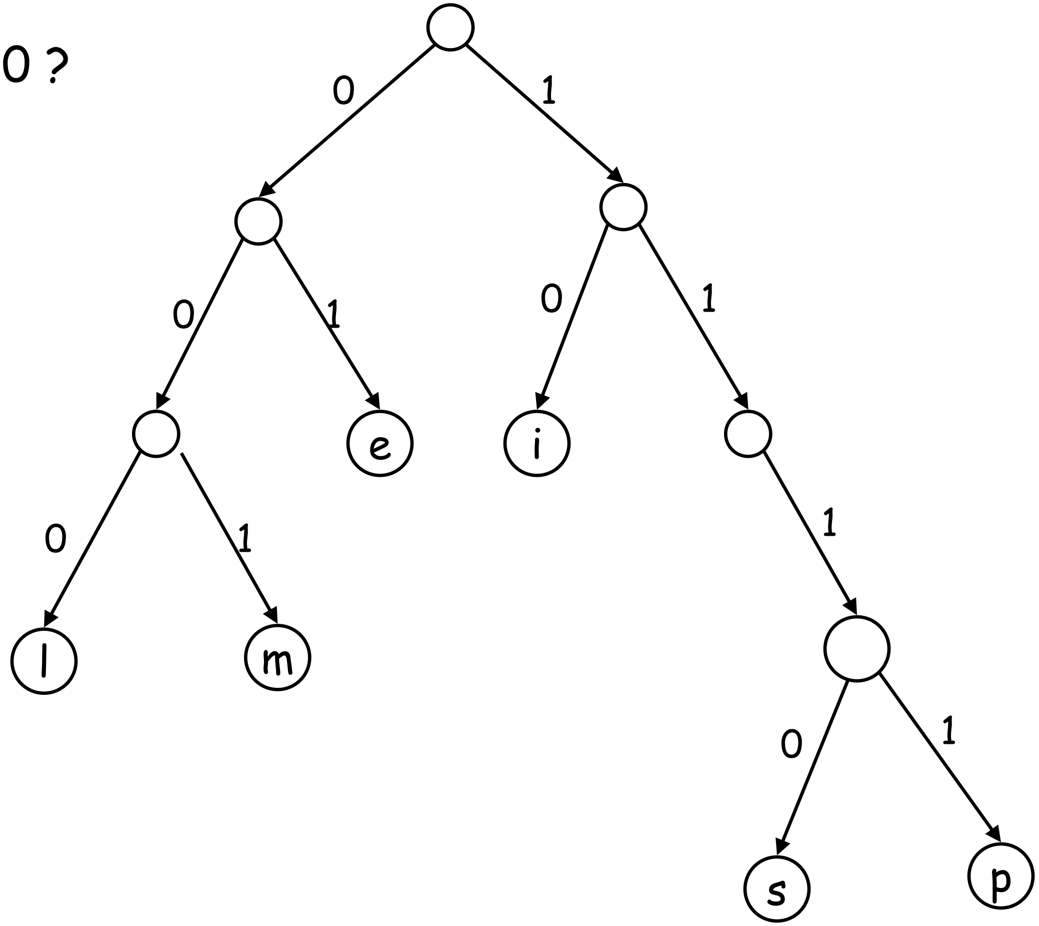
Q. How does the tree of a prefix code look?

A. Only the leaves have a label.

Pf. An encoding of x is a prefix of an encoding of y if and only if the path of x is a prefix of the path of y.

Representing Prefix Codes using Binary Trees

Q. What is the meaning of
111010001111101000 ?

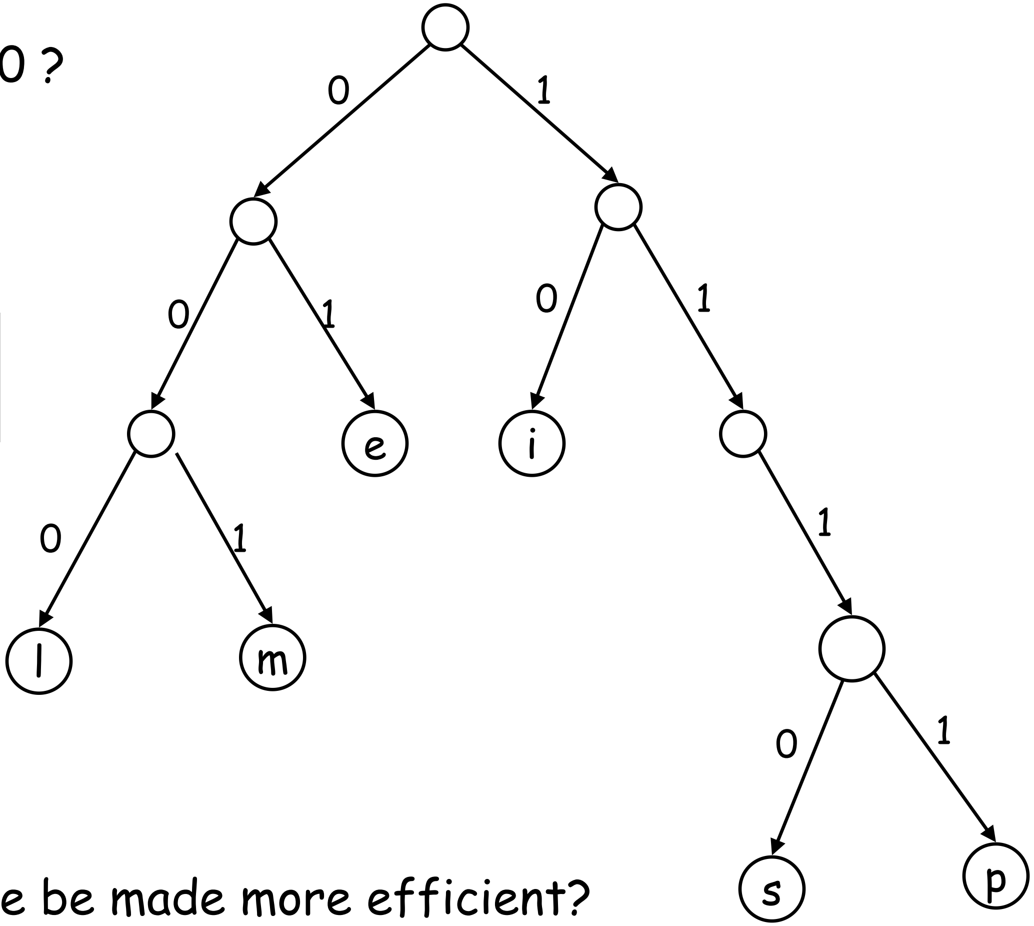


Representing Prefix Codes using Binary Trees

Q. What is the meaning of

111010001111101000 ?

A. "simpel"



$$ABL(T) = \sum_{x \in S} f_x \cdot \mathrm{depth}_T(x)$$

depth: the length of the path from the root to the leaf

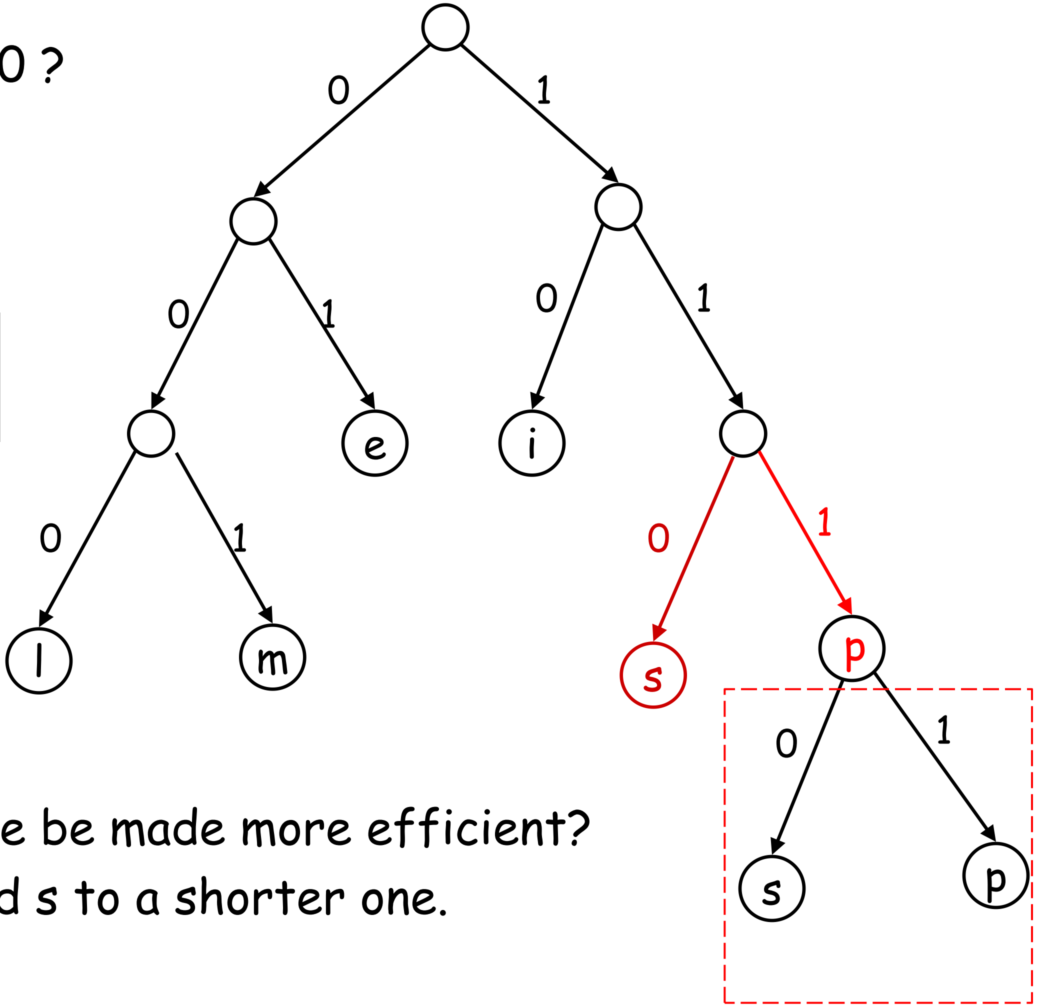
Q. How can this prefix code be made more efficient?

Representing Prefix Codes using Binary Trees

Q. What is the meaning of
111010001111101000 ?

A. "simpel"

$$ABL(T) = \sum_{x \in S} f_x \cdot \text{depth}_T(x)$$



Q. How can this prefix code be made more efficient?

A. Change encoding of p and s to a shorter one.

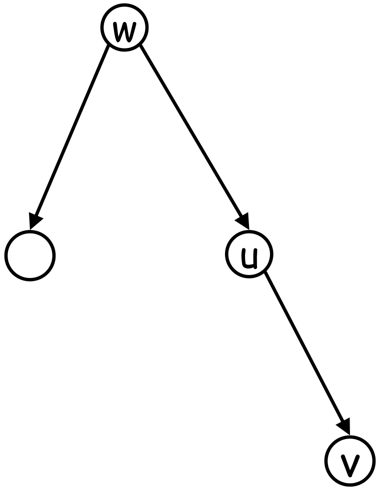
This tree is now full.

Representing Prefix Codes using Binary Trees

Definition. A tree is full if every node that is not a leaf has two children.

Claim. The binary tree corresponding to the optimal prefix code is full.

Pf.



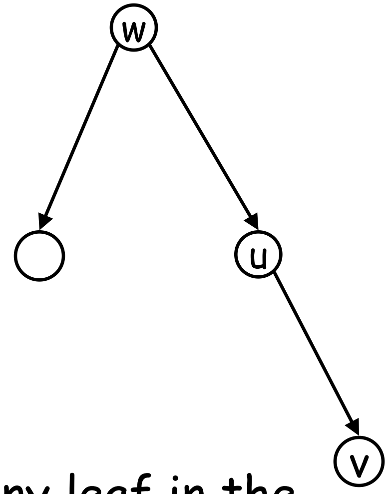
Representing Prefix Codes using Binary Trees

Definition. A tree is full if every node that is not a leaf has two children.

Claim. The binary tree corresponding to the optimal prefix code is full.

Pf. (by contradiction)

- Suppose T is binary tree of optimal prefix code and is not full.
- This means there is a node u with only one child v.
- Case 1: u is the root; delete u and use v as the root
- Case 2: u is not the root
  - let w be the parent of u
  - delete u and make v be a child of w in place of u



- In both cases the number of bits needed to encode any leaf in the subtree of v is decreased. The rest of the tree is not affected.
- Clearly this new tree T' has a smaller ABL than T. Contradiction.

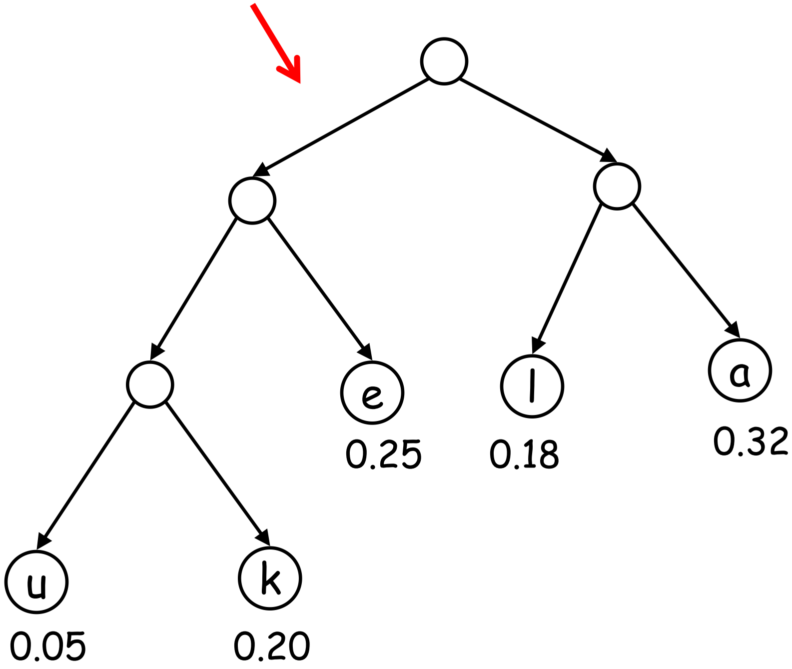
Optimal Prefix Codes: False Start

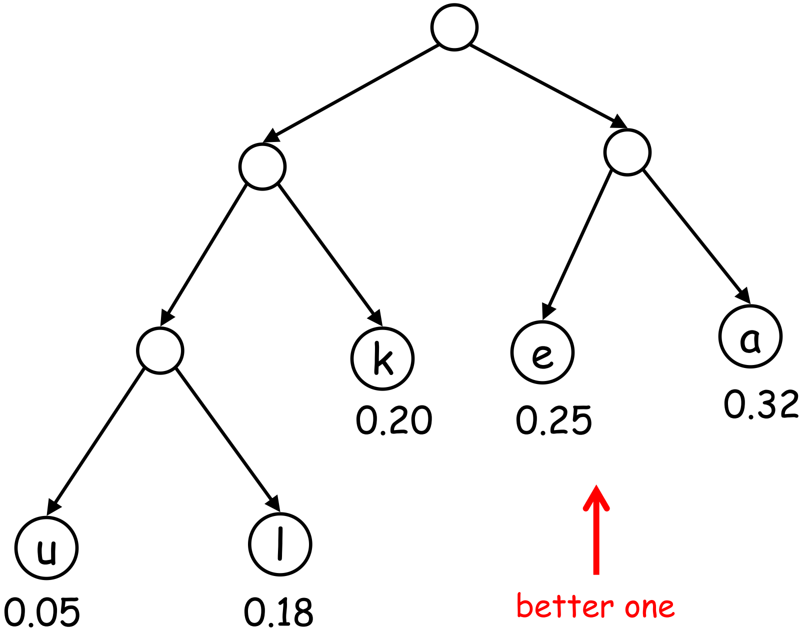
Q. Where in the tree of an optimal prefix code should letters be placed with a high frequency?

Optimal Prefix Codes: False Start

Q. Where in the tree of an optimal prefix code should letters be placed with a high frequency?
A. Near the top.

Greedy template. Create tree top-down, split S into two sets S_1 and S_2 with (almost) equal frequencies. Recursively build tree for S_1 and S_2.
[Shannon-Fano, 1949] f_a=0.32, f_e=0.25, f_k=0.20, f_l=0.18, f_u=0.05





better one

Optimal Prefix Codes: Huffman Encoding

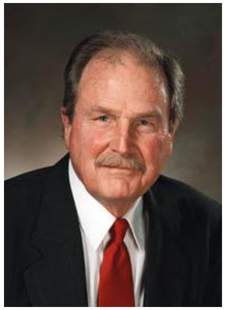
Observation. Lowest frequency items should be at the lowest level in tree of optimal prefix code.

Observation. For n > 1, the lowest level always contains at least two leaves.

Observation. The order in which items appear in a level does not matter.

Claim. There is an optimal prefix code with tree T* where the two lowest-frequency letters are assigned to leaves that are siblings in T*.

Greedy template. [Huffman, 1952] Create tree bottom-up.
Make two leaves for two lowest-frequency letters y and z.
Recursively build tree for the rest using a meta-letter for yz.



Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S|=2 {
      return tree with root and 2 leaves
   } else {
      let y and z be lowest-frequency letters in S
      S′ = S
      remove y and z from S′
      insert new letter ω in S′ with $f_\omega=f_y+f_z$
      T′ = Huffman(S′)
      T = add two children y and z to leaf ω from T′
      return T
   }
}
```

Q. What is the time complexity?

Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S|=2 {
      return tree with root and 2 leaves
   } else {
      let y and z be lowest-frequency letters in S
      S′ = S
      remove y and z from S′
      insert new letter ω in S′ with $f_\omega=f_y+f_z$
      T′ = Huffman(S′)
      T = add two children y and z to leaf ω from T′
      return T
   }
}
```

Q. What is the time complexity?

A. $T(n) = T(n-1) + O(n)$

so $O(n^2)$

Q. How to implement finding lowest-frequency letters efficiently?

A. Use priority queue for S: $T(n) = T(n-1) + O(\log n)$ so $O(n \log n)$

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. by induction, based on optimality of T' (y and z removed, ω added) (see next page)

Claim. ABL(T')=ABL(T)-f_ω

Pf.

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. by induction, based on optimality of T' (y and z removed, ω added) (see next page)

Claim. ABL(T')=ABL(T)-f_ω

Pf.

$$\begin{aligned}
\text{ABL}(T) &= \sum_{x \in S} f_x \cdot \text{depth}_T(x) \\
&= f_y \cdot \text{depth}_T(y) + f_z \cdot \text{depth}_T(z) + \sum_{x \in S, x \neq y,z} f_x \cdot \text{depth}_T(x) \\
&= (f_y + f_z) \cdot (1 + \text{depth}_T(\omega)) + \sum_{x \in S, x \neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega \cdot (1 + \text{depth}_T(\omega)) + \sum_{x \in S, x \neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega + \sum_{x \in S'} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + \text{ABL}(T')
\end{aligned}$$

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Pf. (by induction over n=|S|)

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Base: For n=2 there is no shorter code than root and two leaves.
Hypothesis: Suppose Huffman tree T' for S' of size n-1 with ω instead of y and z is optimal.
Step: (by contradiction)

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Hypothesis: Suppose Huffman tree T' for S' of size n-1 with ω instead of y and z is optimal. (IH)

Step: (by contradiction)

- *Idea of proof:*
  - Suppose other tree Z of size n is better.
  - Delete lowest frequency items y and z from Z creating Z'
  - Z' cannot be better than T' by IH.

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Step: (by contradiction)

- Suppose Huffman tree T for S is not optimal.
- So there is some tree Z such that ABL(Z) < ABL(T).
- Then there is also a tree Z for which leaves y and z exist that are siblings and have the lowest frequency (see observation).
- Let Z' be Z with y and z deleted, and their former parent labeled ω.
- Similar T' is derived from S' in our algorithm.
- We know that $ABL(Z')=ABL(Z)-f_{\omega}$, as well as $ABL(T')=ABL(T)-f_{\omega.}$
- But also ABL(Z) < ABL(T), so ABL(Z') < ABL(T').
- Contradiction with IH.