Maximum-Flow Problem

YAO ZHAO

OUTLINE

- Ford-Fulkerson Algorithm
- Edmonds-Karp Algorithm
- Dinic Algorithm
- A Question

Ford-Fulkerson Review

```
Augment(f, c, P) {
    b ← bottleneck(P)
    foreach e ∈ P {
        if (e ∈ E) f(e) ← f(e) + b
        else       f(e^R) ← f(e^R) - b
    }
    return f
}
```

```
Ford-Fulkerson(G, s, t, c) {
    foreach e ∈ E  f(e) ← 0
    G_f ← residual graph

    while (there exists augmenting path P) {
        f ← Augment(f, c, P)
        update G_f
    }
    return f
}
```

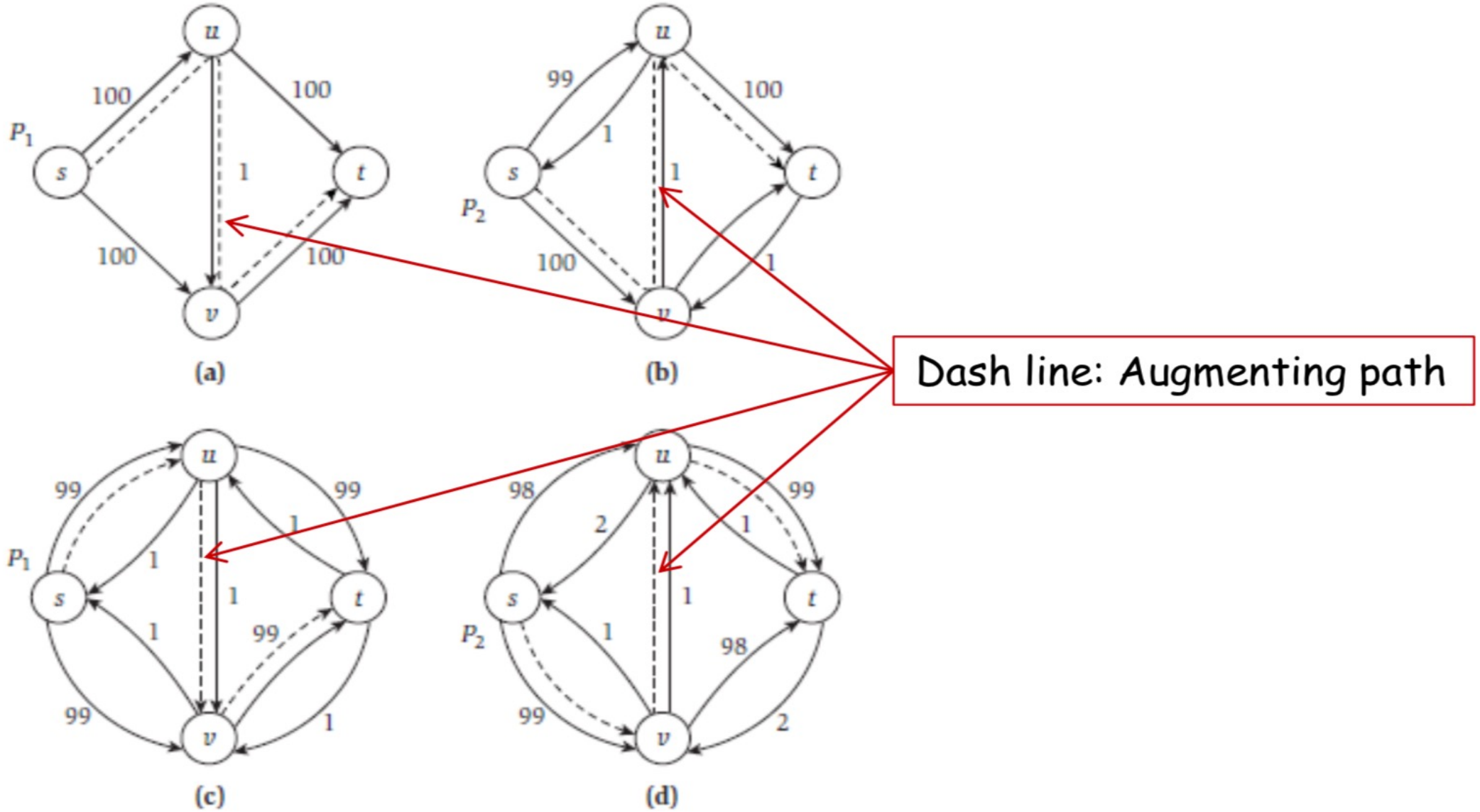


Figure Parts (a) through (d) depict four iterations of the Ford-Fulkerson Algorithm using a bad choice of augmenting paths: The augmentations alternate between the path P_1 through the nodes s, u, v, t in order and the path P_2 through the nodes s, v, u, t in order.

Edmonds-Karp (Shortest augmenting path)

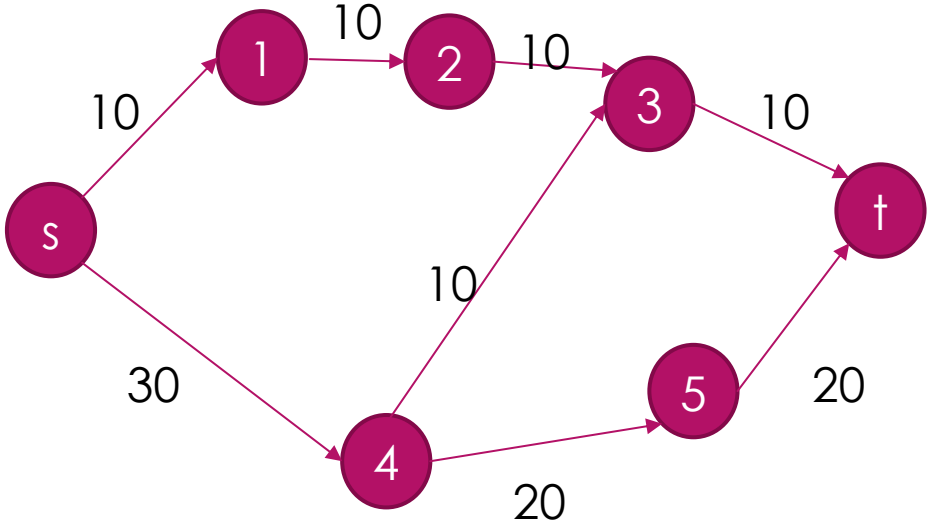
- Always find the shortest augmenting path
- Complexity improved to $O(|V||E|^2)$

BFS(G_f) and can find a path P from s to t

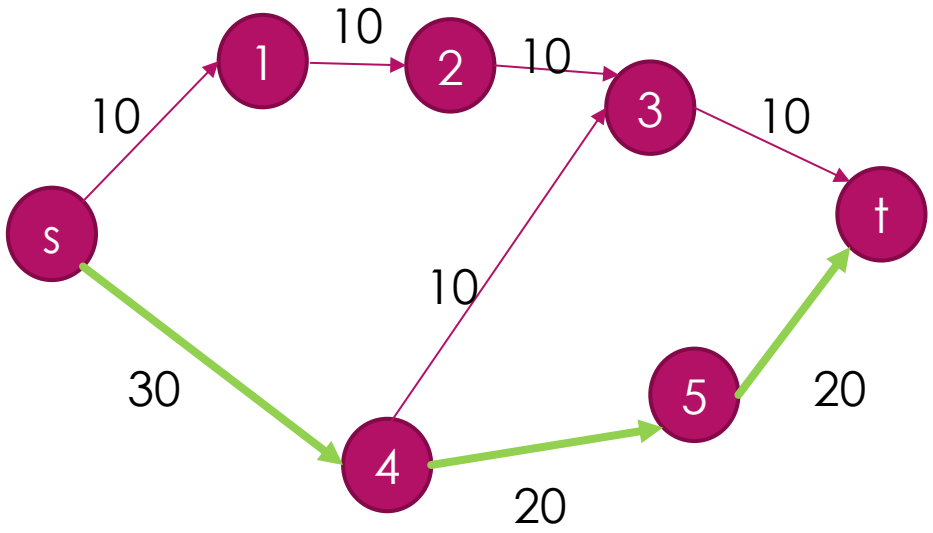
```
Ford-Fulkerson(G, s, t, c) {
    foreach e ∈ E   f(e) ← 0
    G_f ← residual graph

    while (there exists augmenting path P) {
        f ← Augment(f, c, P)
        update G_f
    }
    return f
}
```

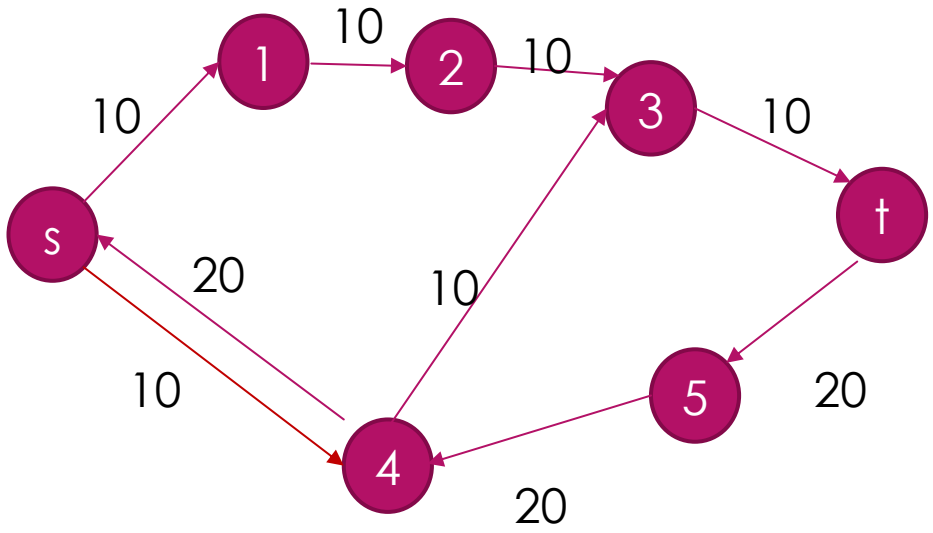
Edmonds-Karp process(1)



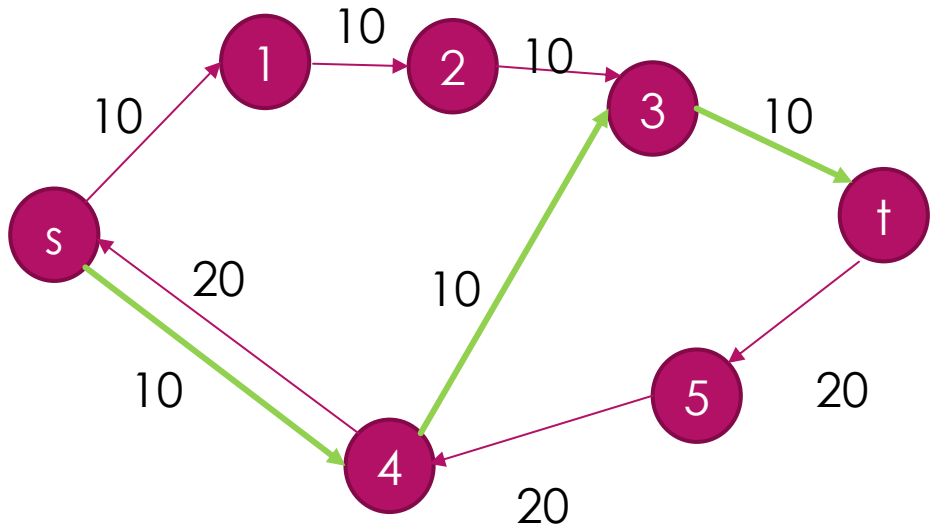
BFS first time



Edmonds-Karp process(2)

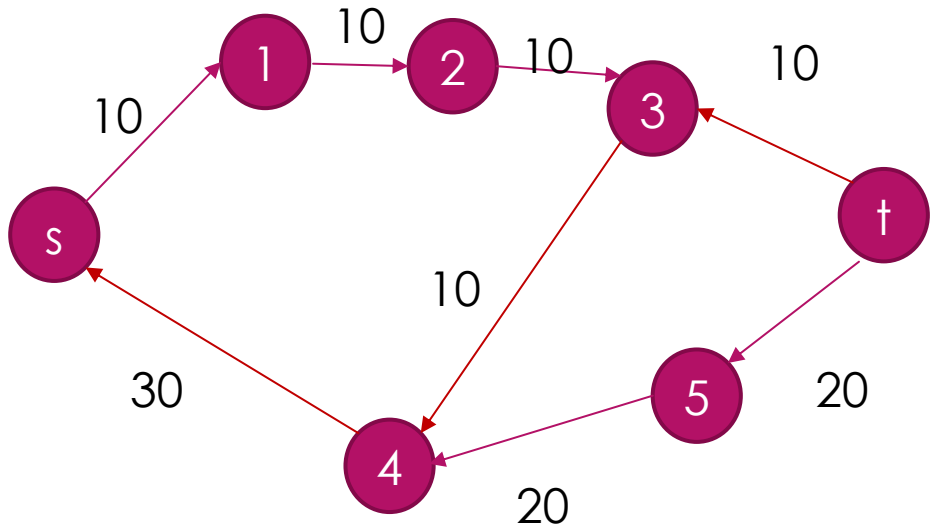


BFS second time



Edmonds-Karp process(3)

Stop
Max flow is 30



Every time BFS starts from s to t -- a waste of time?

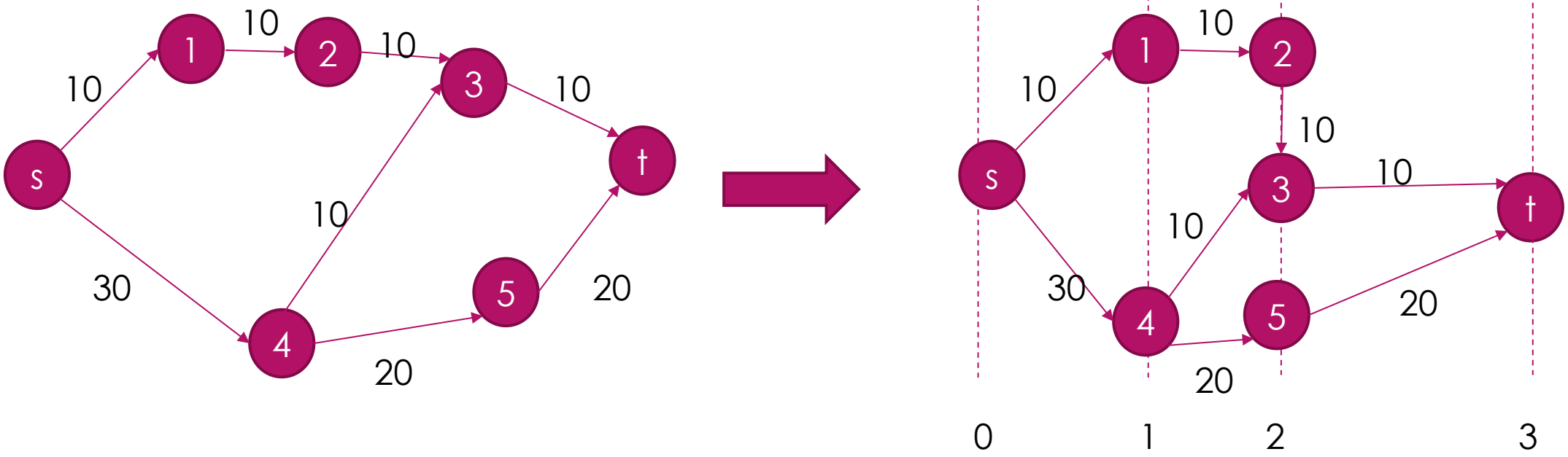
Dinic

- Further optimized Edmonds-Karp algorithm
- DFS is used to find augmented paths (In a hierarchical network, DFS always find a shortest path)

Dinic process(1)

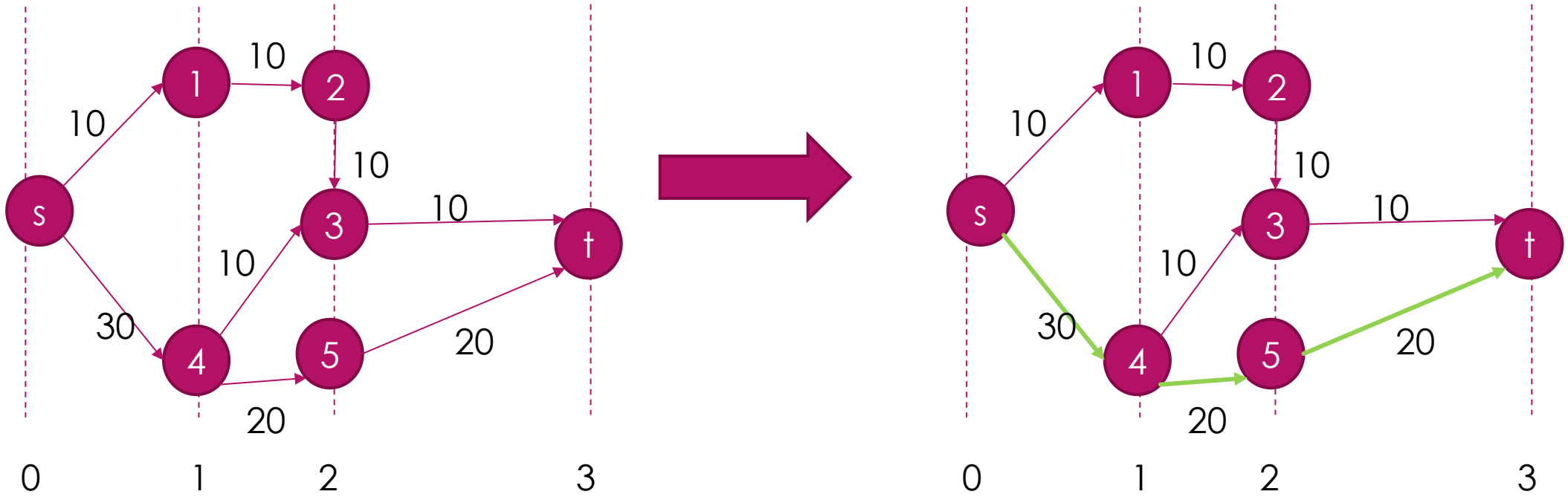
Initial residual graph

1、construct residual graph and hierarchical network， If the sink is not in the hierarchical network, stop; Otherwise, proceed to step 2.



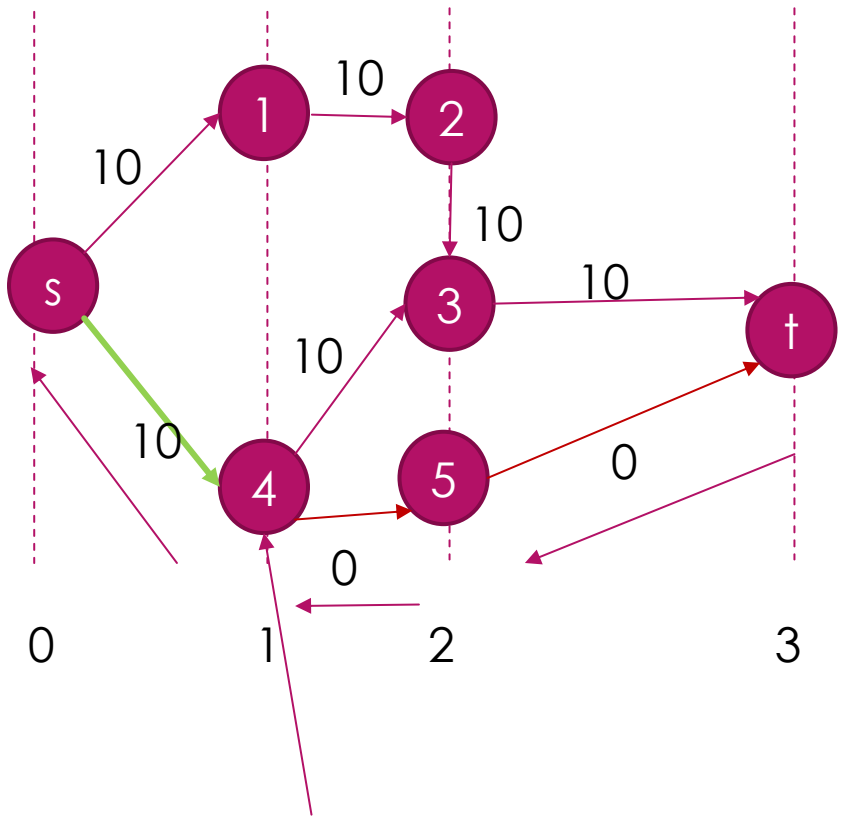
Dinic process(2)

2、 Using DFS algorithm, proceed from the vertex of layer i to the vertex of layer i +1, if reach the sink point t, means find an augmented path. If cannot reach t, Go back to step 1 （P.11）

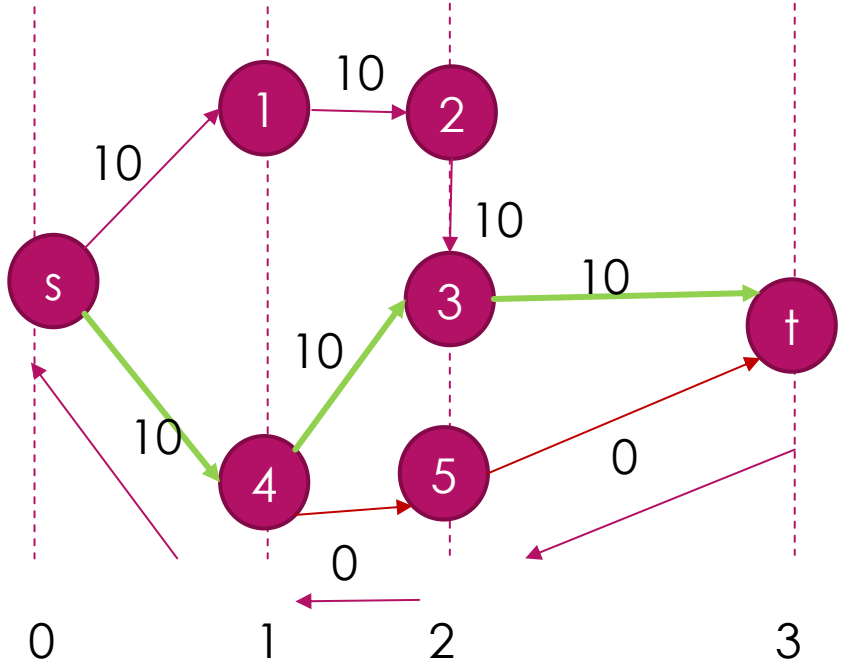


Dinic process(3)

3、Go back and update the capacity value. When return to layer l, and you find that you have another edge can reach layer l + 1. If the path reach t, you find another augmented path.



DFS return here，update the capacity of edge.

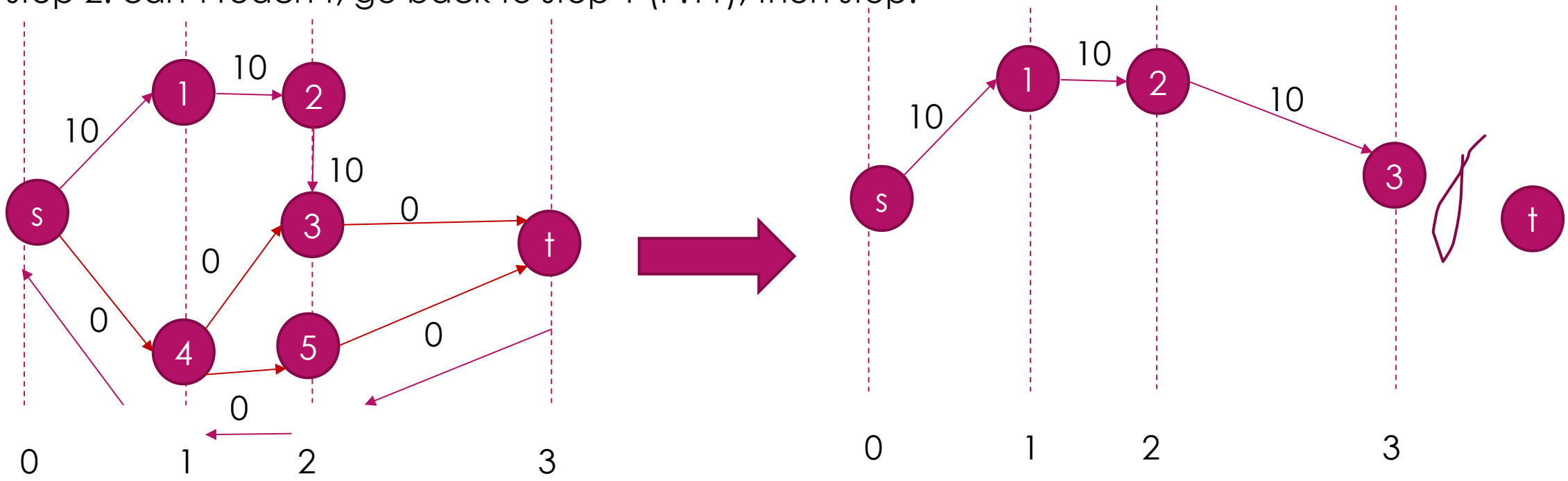


Find another path from the layer 1.

Dinic process(4)

4、 Step back, update the capacity value, and if no new augmented path can be found until step back to point s, the DFS process ends. Go back to step 2 (P.12).

In step 2: can't reach t, go back to step 1 (P.11), then stop.



Construct a new hierarchical network

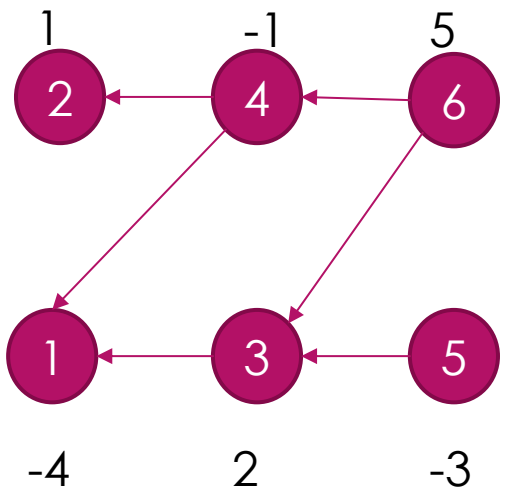
The sink point doesn't join to the new hierarchical network, stop.

A Quesion

- A software company is preparing a plan for its future development. There is a plan sheet, which contains many projects. The plan sheet shows the required investment or estimated profit for each project. Since the implementation of some projects must depend on the results of other projects, now please help the company choose a portion of these projects that will maximize the profit of the software company.
- Input：The count of projects is N, Investment or profit amount of each project is c[i](c[i]>=0, profit; c[i]<0, investment)and projects list P[i] which it depends on.
- Output: maximum profit of the company.

Sample Input and output

Input	Output
6 -4 1 2 1 -1 1 2 -3 3 5 3 4	3 1 2 3 4 6



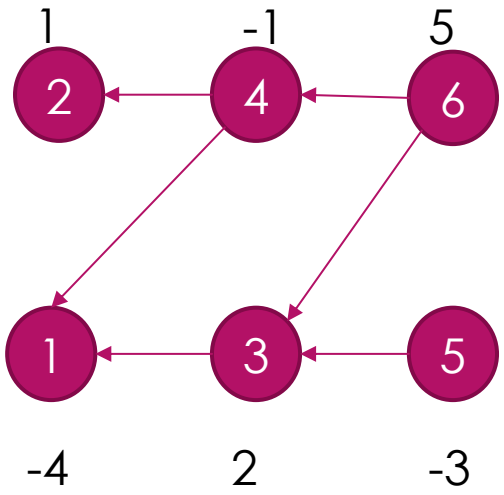
Key point

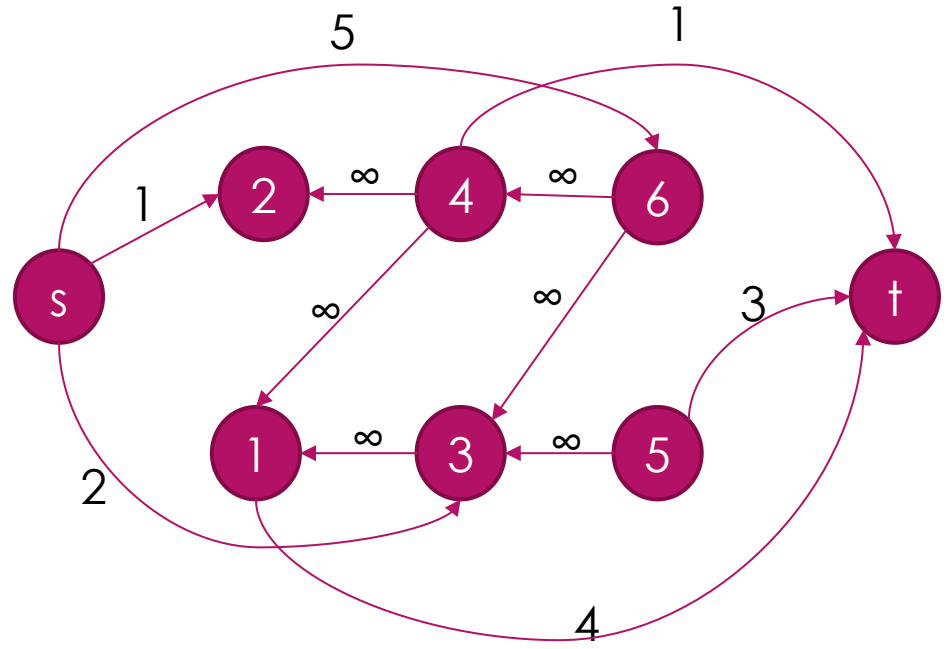
- The key point is how to construct a graph, convert to a familiar question to you.

Solution

- Create N vertices to represent N items, add the source point s and the sink point t.
- If the c[i] is positive, add an edge (weight is c[i]) from source s to vertex i;
- If the c[i] is negative, add an edge (weight is -c[i]) from vertex i to the sink t.

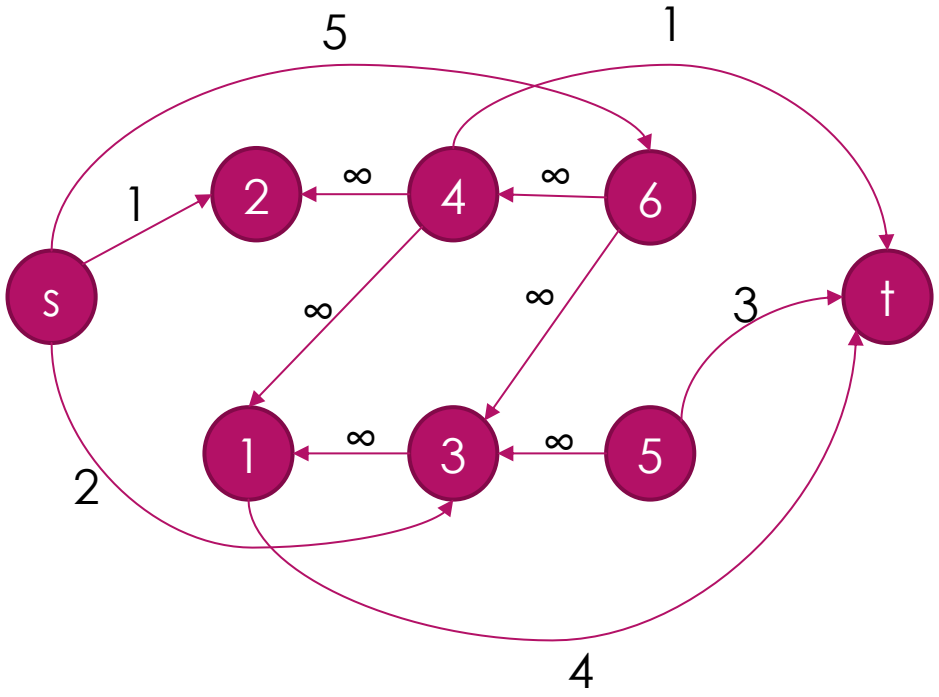
```
6
-4
1
2 1
-1 1 2
-3 3
5 3 4
```





Analysis

- The problem convert to the minimum cut problem of the network G'.Assumes that the minimum cut vertices set is (A',B'), A'- {s} is the optimal project set.Assume the minimum cut is c, while C=$\sum_{i \in P: p_i > 0} p_i$, the maximum profit is C-c。



C is the sum of all profitable items.

C = 5+1+2
c is the minimum cut in G
c = 5
Maximum profit = C-c = 8 – 5 = 3

c is the cost of all selected profitable projects + the abandoned profitable projects