Lecture 7: tree

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Our Roadmap

- Tree
  - Basic Concepts
  - Properties of Tree (focus on binary tree)
  - Binary Tree Traversal
- Binary Tree Applications
  - Algebraic expression
  - Huffman encoding

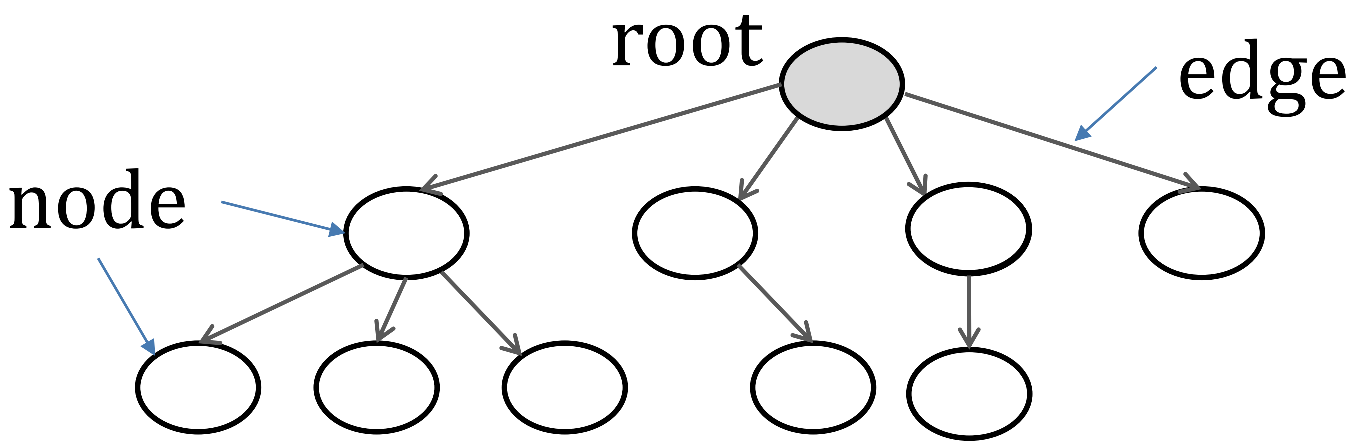
Tree

- This lecture provides a formal definition of ***trees***, which constitute an important approach to organize data in computer science. We will also prove some basic properties of trees that will be useful in computer science.

Motivation

- Data Dictionary: maintain a sorted collection of data
  - *search* for an item (with possibly delete it)
  - *insert* a new item
- A list implemented using an array
  - Searching for an item, O(log n)
  - Inserting an item, O(n)
- A list implemented using a linked list
  - Searching for an item, O(n)
  - Inserting an item, O(n)
- In the next few lectures, we will look at data structures (**trees**) that can be used for a more efficient data dictionary

What is a tree?

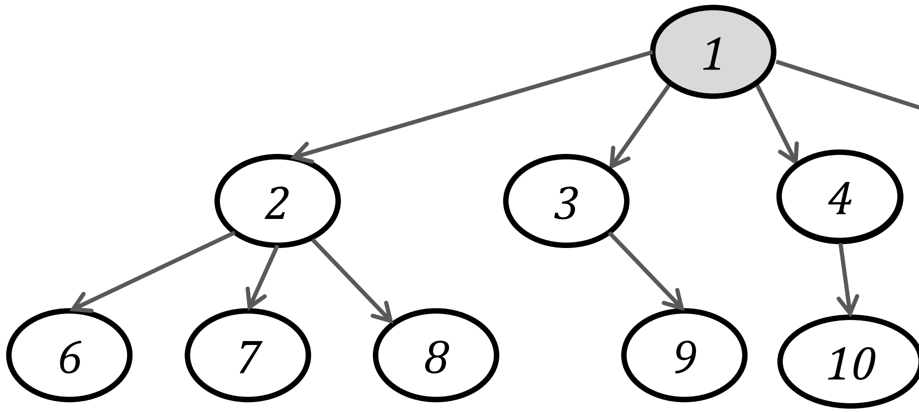


- A tree consist of:
  - A set of nodes,
  - A set of edges, each of which connects a pair of nodes
- Each node may have one or more data items
  - Key field = the field used when searching for a data item
  - Multiple data items with the same key are referred to as duplicates
- The node at the “top” of the tree is the “root” of the tree

Tree Property I

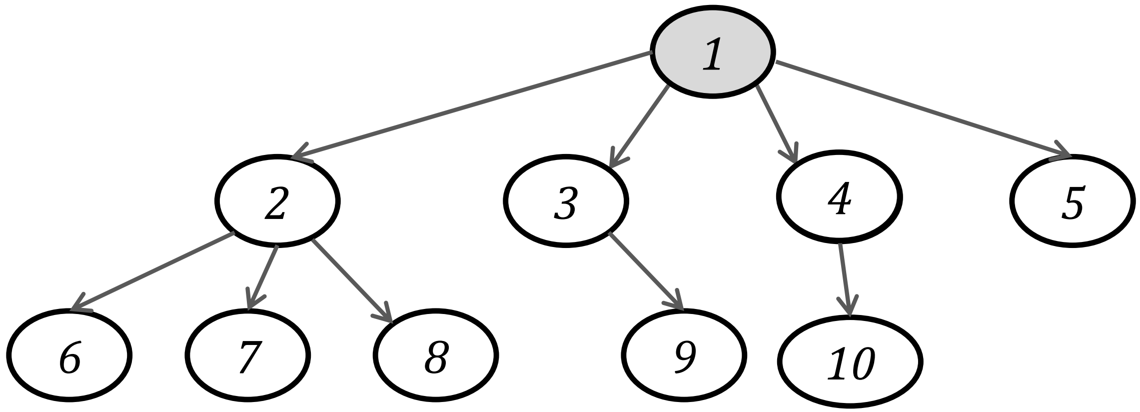
- A tree with n nodes with n-1 edges
- Proof?

Relationship between nodes



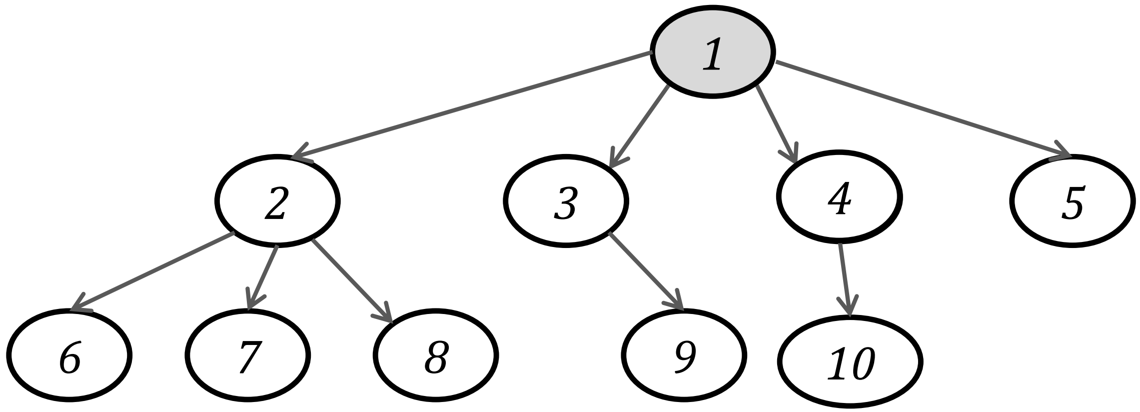
- Consider a tree T, let u and v be two nodes in T. We say that u is the *parent* of v if v is the node directly below u
- Accordingly, we say that v is a *child* of u.
  - e.g., node 1 is the parent of node 2, 3, 4, 5, and node 2 is the child of node 1.
- Each node is the child of at most one parent
- Node with the same parent are siblings
  - e.g., node 2, 3, 4, 5 are siblings

Relationship between nodes



- Consider a tree T, let u and v be two nodes in T. We say that u is an ancestor of v if one of the following holds:
  - u = v
  - u is the parent of v, or
  - u is the parent of an ancestor of v.
- Accordingly, we say that v is a descendant of u.
- In particular, if u != v, u is a proper ancestor of v, and likewise, v is a proper descendant of u.

Tree node types

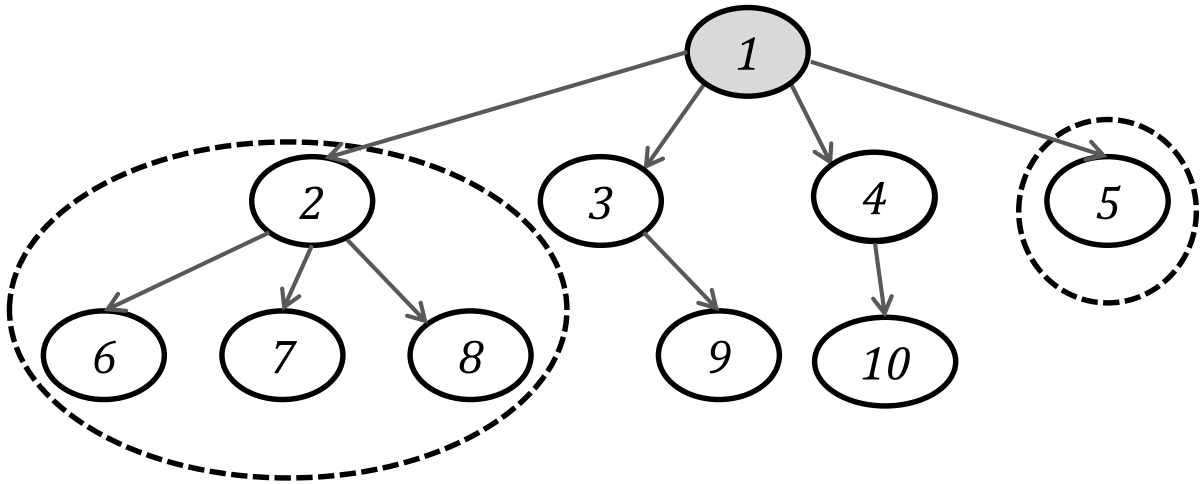


- A *leaf* node is a node without children
- An internal node is a node with one or more children
- E.g.,
  - Leaf nodes: 5, 6, 7, 8, 9, 10
  - Internal nodes: 1, 2, 3, 4

Tree Property II

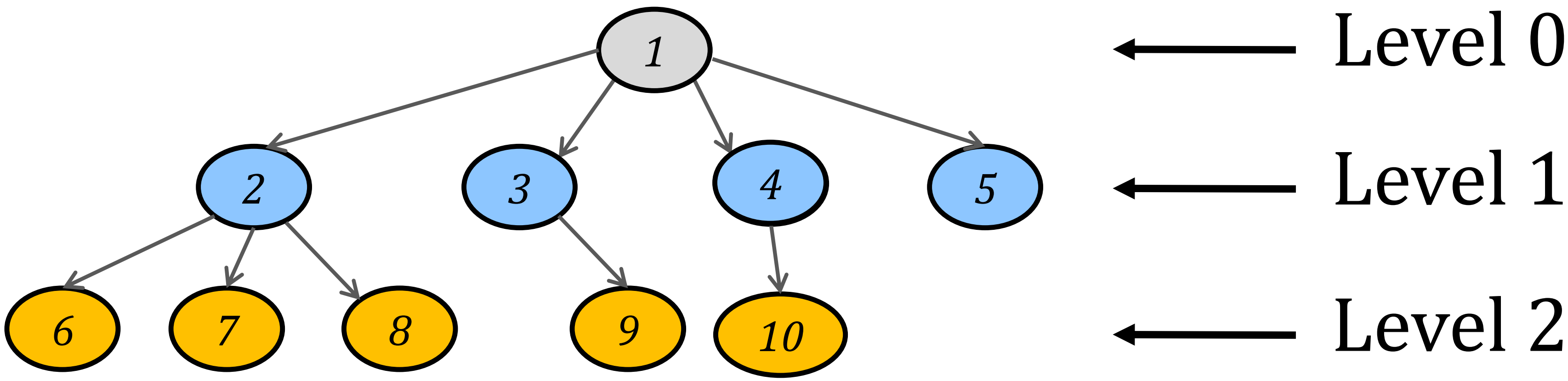
- Let T be a tree where every internal node has at least 2 child nodes. If m is the number of leaf nodes, then the number of internal nodes is at most m-1.

Tree: a Recursive Data Structure



- Each node in the tree is the root of a smaller tree!
  - Refer to such trees as subtrees to distinguish them from the tree as a whole
  - Example: node 2 is the root of the subtree circled above
  - Example: node 5 is the root of a subtree with only one node.
- We will see that tree algorithms often lend themselves to recursive implementations

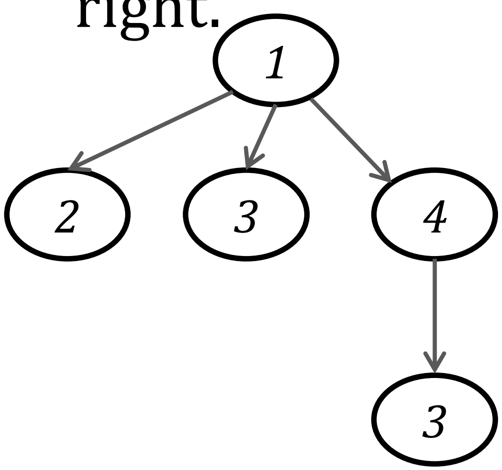
Path, Depth, Level, and Height



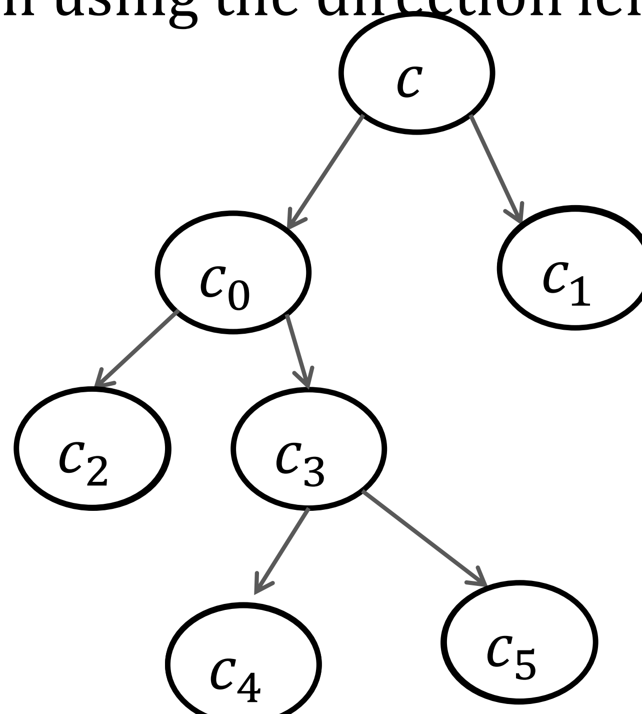
- There is exactly one path (one sequence of edges) connecting each node to the root.
- depth of a node = # of edges on the path from it to the root.
- Nodes with the same depth form a level of the tree
- The height of a tree is the maximum depth of its nodes: the tree above has a height of 2.

k-ary and Binary Tree

- A k-ary tree is a rooted where every internal node has at most k child nodes.
- A 2-ary tree is called a binary tree
- In a binary tree, nodes have at most two children.
  - Distinguish between them using the direction left and right.



3-ary



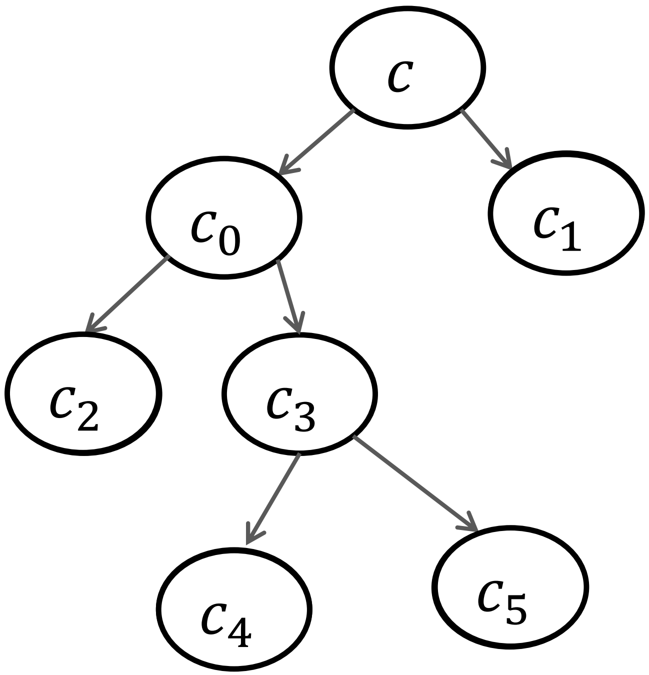
binary

Binary Tree Definition

- Binary tree recursive definition:
- A binary tree is either:
  - 1) empty or
  - 2) a node (the root of the tree) that has
    - one or more pieces of data (the key, and possibly others)
    - a *left subtree,* which is itself a binary tree
    - a *right subtree*, which is itself a binary tree
- A binary tree implies an ordering among the nodes at the same level.

Binary Tree: Full Level

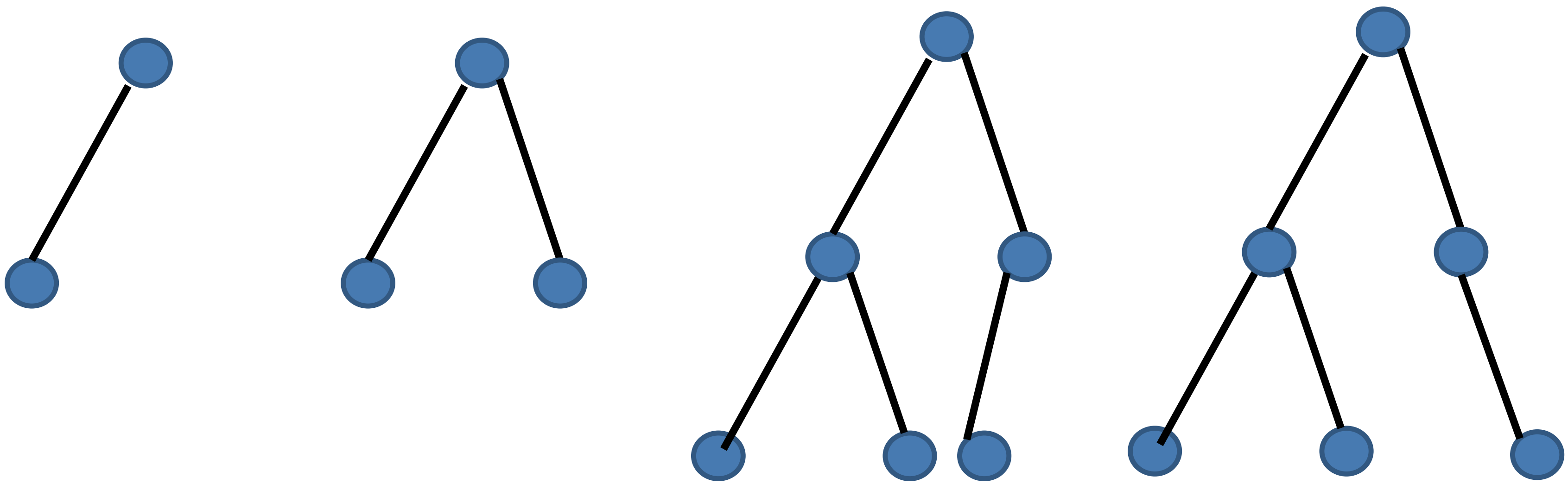
- Consider a binary tree with height h, its level l $(0 \leq l \leq h)$ is full if it contains 2^l nodes.



- Levels 0 and 1 are full, but levels 2 and 3 are not.

Binary Tree: Complete Binary Tree

- A binary tree of height h is complete if:
  - Level 0, 1, …, h-1 are all full
  - At level h, the leaf nodes are "as far left as possible"
    - This means that if you want to add a leaf node v at level h, v would need to be on the right of all the existing leaf nodes.



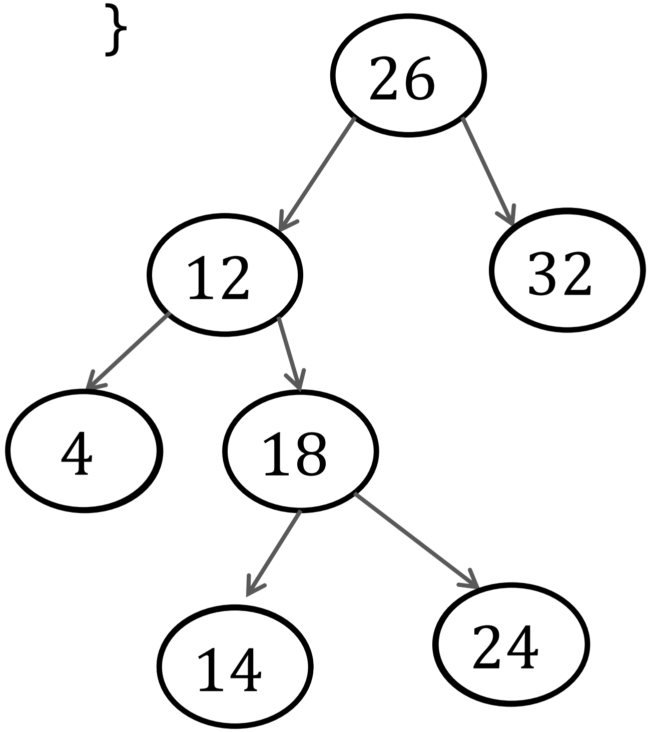
Tree Property III

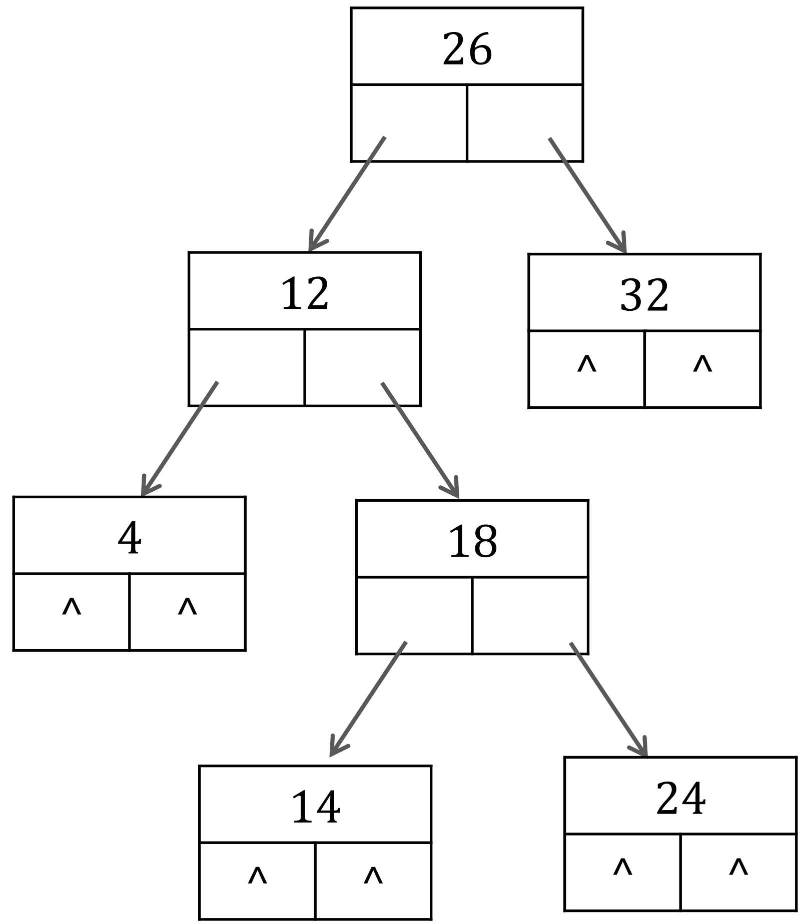
- A complete binary tree with $n \geq 2$ nodes has height O(log n)
- Proof?

Binary Tree Implementation

- Struct treeNode

```
{
    int key;
    treeNode left;
    treeNode right;
}
```



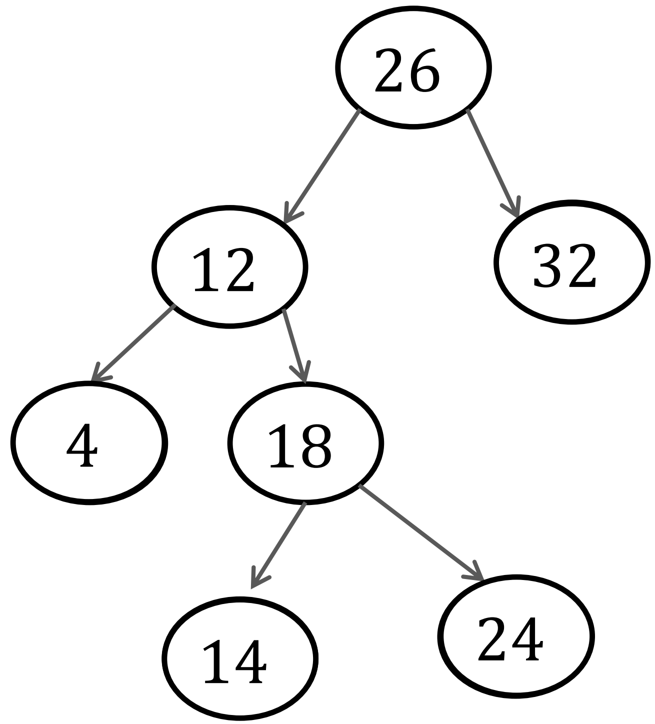


Traversing a Binary Tree

- Traversing a tree involves visiting all of the nodes in the tree.
  - Visiting a node = processing its data in some way
    - example: print the key
- We will look at four types of traversals. Each of them visits the nodes in a different order.
- To understand traversals, it helps to remember the recursive definition of a binary tree, in which every node is the root of a subtree.

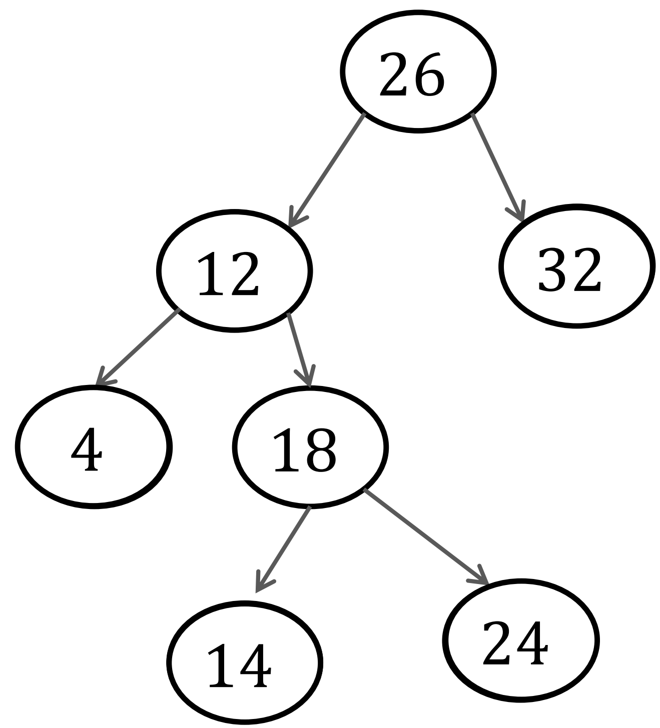
Preorder Traversal

- Preorder traversal of the tree whose root is N
  - visit the root N
  - Recursively perform a preorder traversal of N's left subtree
  - Recursively perform a preorder traversal of N's right subtree
- Preorder traversal
  - 26, 12, 4, 18, 14, 24, 32



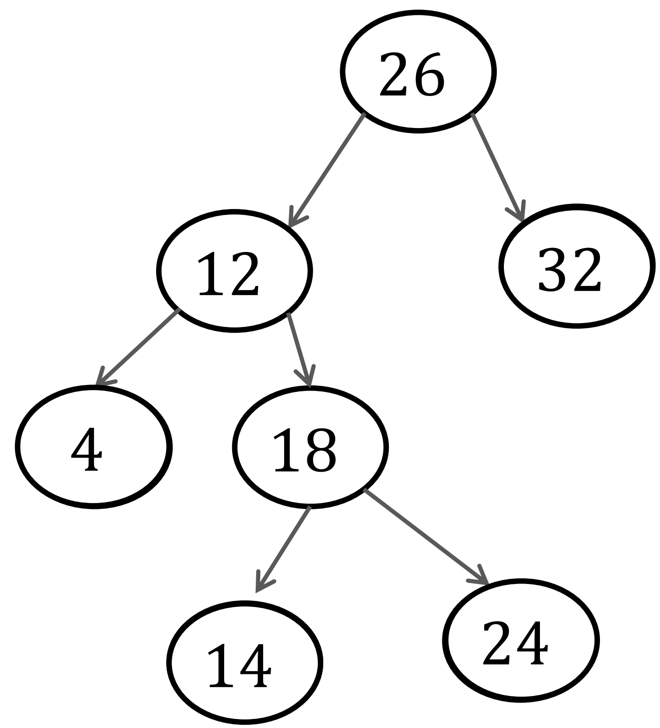
Postorder Traversal

- Postorder traversal of the tree whose root is N
  - Recursively perform a postorder traversal of N’s left subtree
  - Recursively perform a postorder traversal of N’s right subtree
  - visit the root N
- Postorder traversal
  - 4, 14, 24, 18, 12, 32, 26



Inorder Traversal

- Inorder traversal of the tree whose root is N
  - Recursively perform a inorder traversal of N's left subtree
  - Visit the root N
  - Recursively perform a inorder traversal of N's right subtree
- Inorder traversal
  - 4, 12, 14, 18, 24, 26, 32



Preorder Travesal

- Implementation:
  - Recursive Implementation? So easy?
  - `preorderprint(treeNode root):`

```
1. print(root)
2. if(root->left!=null)
3.    preorderprint(root->left)
4. if(root->right!=null)
5.    preorderprint(root->right)
```

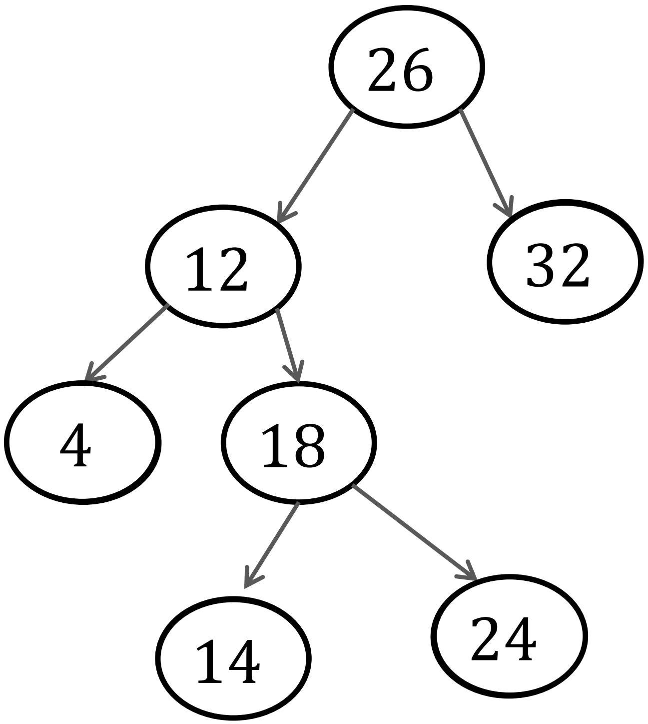
Preorder Travesal

- Implementation:
  - Iterative Implementation?
  - `preorderiterative(treeNode root)`:

```
treeNode stack s
s.push(root)
while(s!=empty)
    treeNode node= s.top()
    print(node)
    s.pop()
    if(node->right!=null)
        s.push(node->right)
    if(node->left!=null)
        s.push(node->left)
```

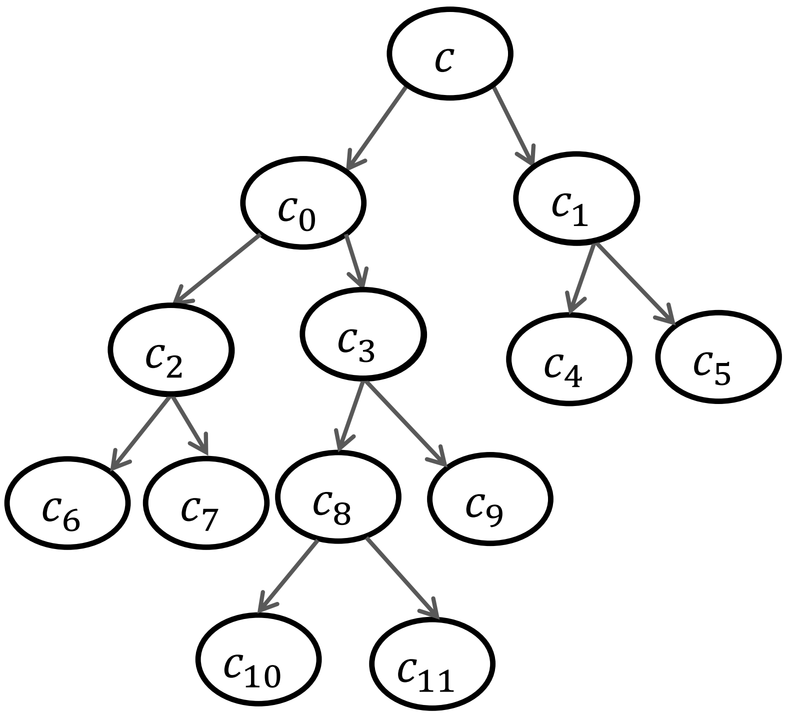
Level Traversal

- Visit the nodes one level at a time, from top to bottom, and left to right.
- Level-order of the tree:
  - 26, 12, 32, 4, 18, 14, 24
- How to implement?



Summary

- Preorder: root, left subtree, right subtree
- Postoder: left subtree, right subtree, root
- Inorder: left subtree, root, right subtree
- Level-order: top to bottom, left to right



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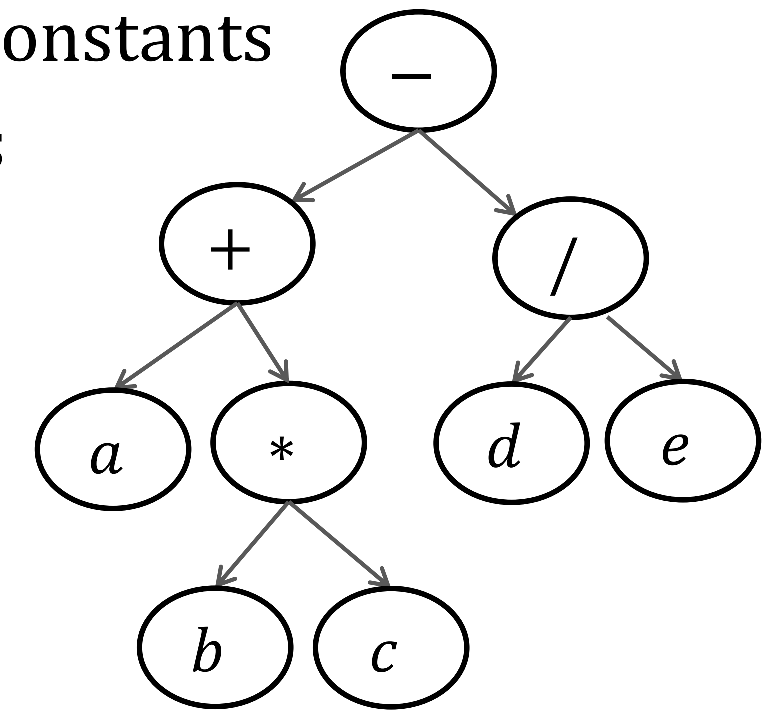
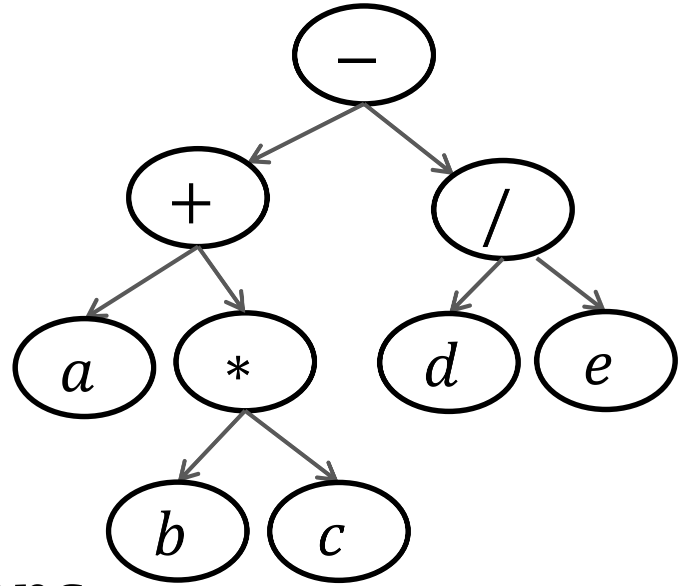
  - Binary Tree Traversal
- Binary Tree Applications
  - Algebraic expression
  - Huffman encoding

Algebraic Expression

- We only consider fully parenthesized expressions with binary operators: +, -, *, /
- Example expression: ((a+(b*c))-(d/e))
- Leaf nodes are variables or constants
- Internal nodes are operators
- Why is it a binary tree?
  - Binary operators



Algebraic-Expression Tree Traversal



- Inorder gives conventional algebraic notation
  - print "(" before visit left tree
  - print ")" after visit right tree
  - for tree at right: ((a+(b*c))-(d/e))
- Preorder gives functional notations
  - Print "(" and ")" as for inorder, and commas after visit left subtree
  - for tree above: subtr(add(a,mult(b,c)),divide(d,e))
- Postorder gives the order in which the computation must be carried out on a stack.
  - for tree above: push a, push b, push c, multiply, add, …

Character Encoding

- A character encoding maps each character to a number
- Computers usually use fixed-length character encodings
  - ASCII uses 8 bits per character
    - "bat" in computer: 01100010 01100001 01110100
  - Unicode uses 16 bits per character
    - ASCII codes are a subset
  - Fixed-length encoding are simple, because:
    - All character encodings have the same length
    - A given character always has the same encoding
  - Problem: fixed length encoding waste space
    - Solution: a variable-length encoding

Variable-Length Character Encodings

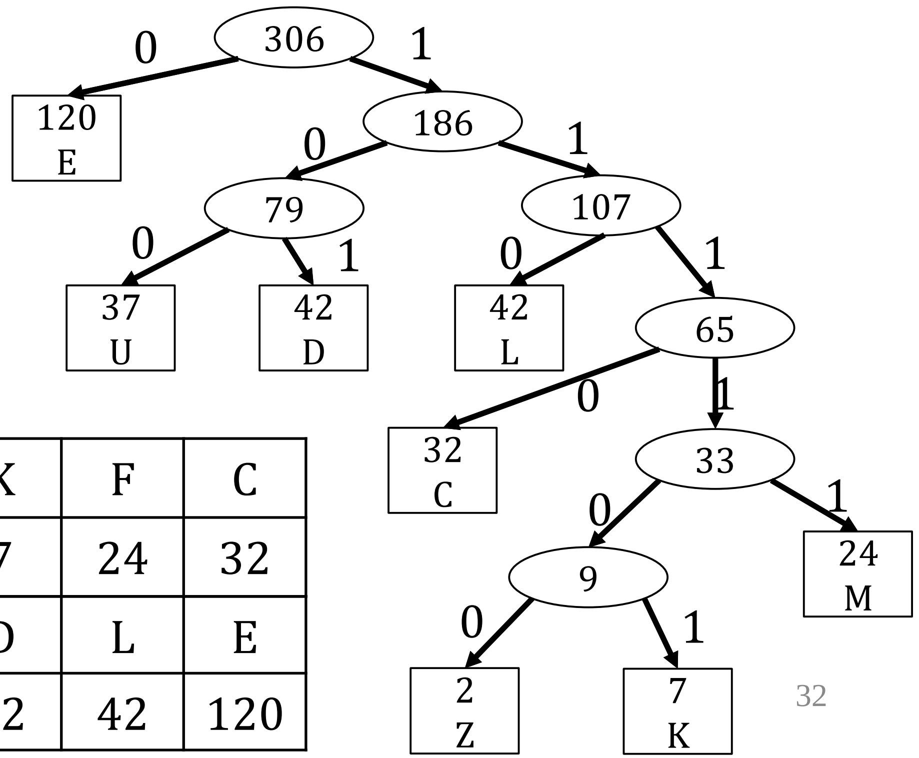
- Variable-length encoding
  - Use encodings of different lengths for different characters
  - Assign shorter encodings to frequently occurring characters
- Example: “test” would be encoded as:
  - 00 01 111 00 → 000111100

e	01	s	111
o	100	t	00

- Challenge: when decoding an encoded document, how do we determine the boundaries between characters?
  - For the above example, how do we know whether the next character is 2 bits or 3 bits
- Requirement: no character’s encoding can be the prefix of another character’s encoding (e.g., couldn’t have 00 and 001)

Huffman Encoding

- Huffman encoding is a type of variable-length encoding that is based on the actual character frequencies in a given document.
- Huffman encoding uses a binary tree:
  - to determine the encoding of each character
  - to decode an encoded file
- Example:
  - Leaf nodes are characters
  - 101 = 'D'
  - "000110" = "EEEL"



Z	K	F	C
2	7	24	32
U	D	L	E
37	42	42	120

Building a Huffman tree

- 1) Begin by reading through the text to determine the frequencies
- 2) Create a list of nodes that contain (character, frequency) pairs for each character that appears in text
- 3) Remove and "merge" the nodes with the two lowest frequencies, forming a new node that is their parent
- 4) Add the parent to the list of nodes
- 5) Repeat steps 3) and 4) until there is only a single node in the list, which will be the root of the Huffman tree.
- Example: build the Huffman tree for the following (character, frequency) pairs:

Z	K	F	C	U	D	L	E
2	7	10	12	27	30	43	65

Correctness of Huffman tree

- Given a Huffman tree, it includes at least 2 nodes, assume node u and node v have the top-2 lowest frequencies, then
  - 1) node u and v have the same parent node
  - 2) depth(u) and depth(v) >= depth(x), where node x is any leaf node in the Huffman tree.
    - proof?

- Huffman encoding is the optimum prefix code, i.e., the space cost is minimized.
  - Proof.
  - http://home.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL17.pdf

Thank You!

Tree Property I

- A tree with n nodes with n-1 edges
- Proof?
  - For each non-root node v, it has one and only one edge point to itself.
  - A tree with n nodes, thus the number of non-root nodes is n-1.
  - Thus, this tree has n-1 edges.

Tree Property II

- Let T be a tree where every internal node has at least 2 child nodes. If m is the number of leaf nodes, then the number of internal nodes is at most m-1.
  - Suppose internal node v has x_v child nodes
  - The average child nodes of each internal node is x
  - It has m leaf nodes, thus it has m/x parent nodes at most, i.e., they are parent of leaf nodes.
  - For m/x internal nodes, it has at most m/x^2 parents.
  - For m/x^2 internal nodes, it has at most m/x^3 parents.
  - …
  - The total number of internal nodes is

    $$m/x + m/x^2 + \ldots + 1$$
  - It is at most m-1.

Tree Property III

- A complete binary tree with $n \geq 2$ nodes has height O(log n)
- Proof?
  - Suppose the height is h.
  - The number of nodes at each level:
  - Level 0: $2^0 = 1$, Level 1: $2^1 = 2$
  - Level 2: $2^2 = 4$, Level 3: $2^3 = 8$
  - …
  - Level h-1: 2^(h-1), Level h: x (x >= 1)
  - Thus, $2^0 + 2^1 + \ldots 2^{h-1} + x = n$
  - $\rightarrow (1-2^{h-1})/(1-2) = n-x \rightarrow 2^{h-1} < n$
  - Thus, h = O(log n)