Lecture 2
Algorithm Analysis

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Several pages are based on the notes by Dr. Ken Yiu (PolyU) and Dr. Yufei Tao(CUHK)

Our Roadmap
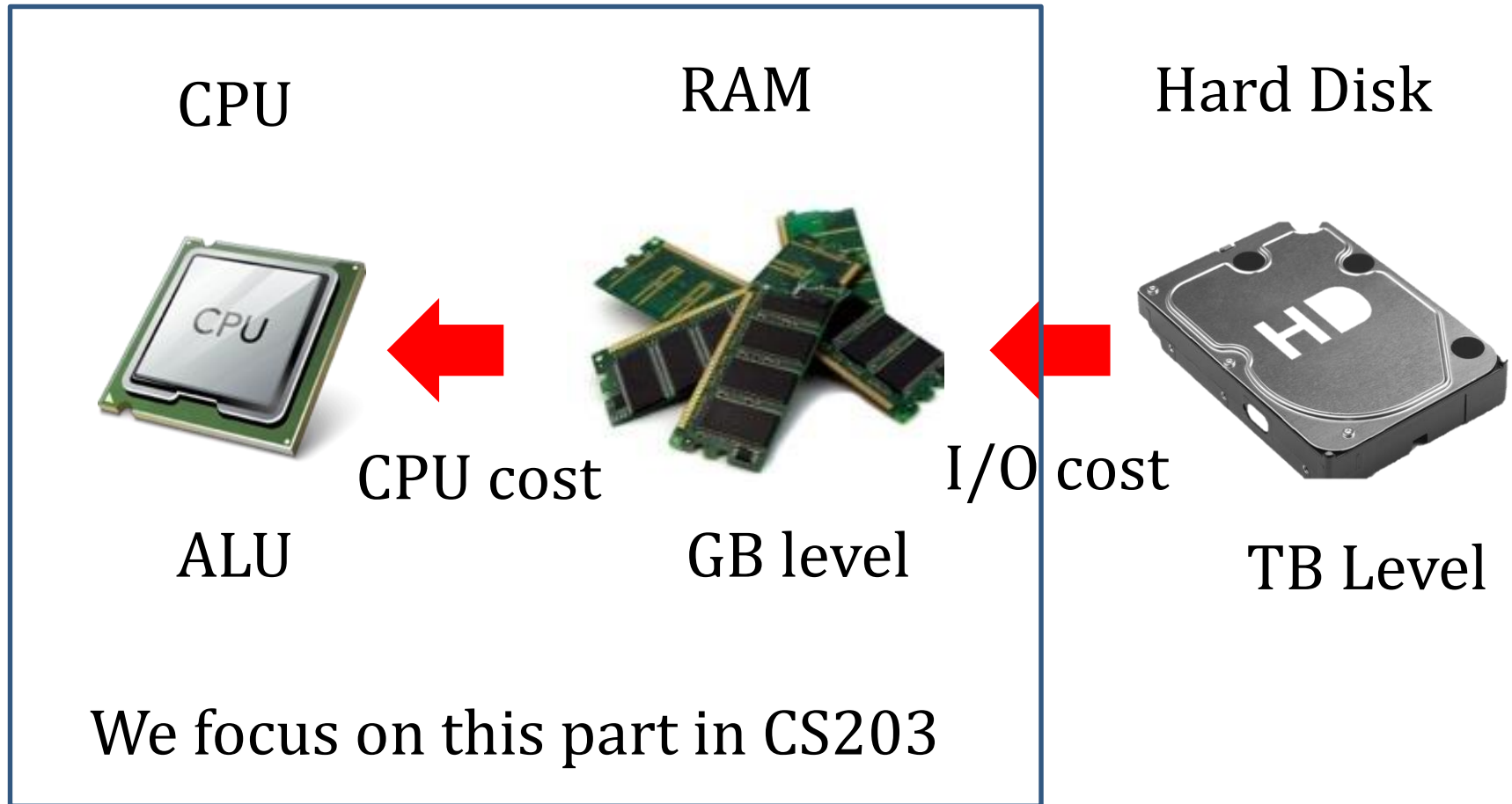
- RAM Computation Model

  - Memory, CPU, Algorithm
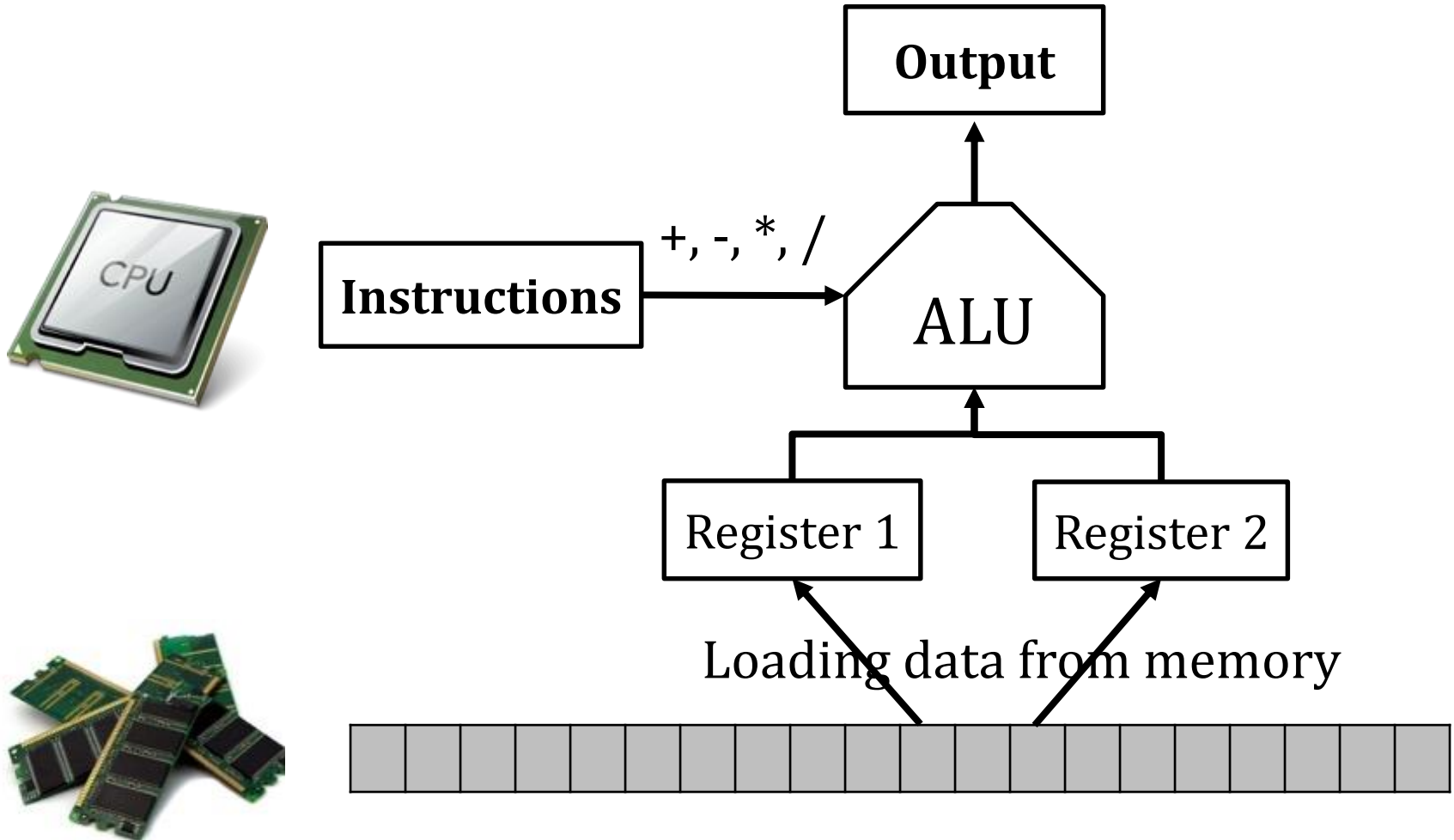
  - Algorithm, Pseudocode

- Worst Case Analysis

  - Binary Search Problem

  - Big O notation

RAM Computation Model

CPU

RAM

Hard Disk

CPU cost

I/O cost

ALU

GB level

TB Level

We focus on this part in CS203

RAM Computation Model

Output

CPU

Instructions $\rightarrow$ +, -, *, /

ALU

Register 1   Register 2

Loading data from memory

Memory

◈ A finite sequence of cells, each cell has the same number of bits.

◈ Every cell has an address: the first cell of memory has address 0, the second cell 1, and so on.

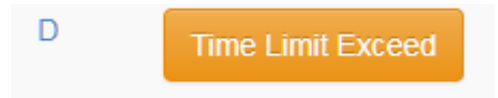◈ Store information for immediate use in a computer

◈ Computer hardware devices
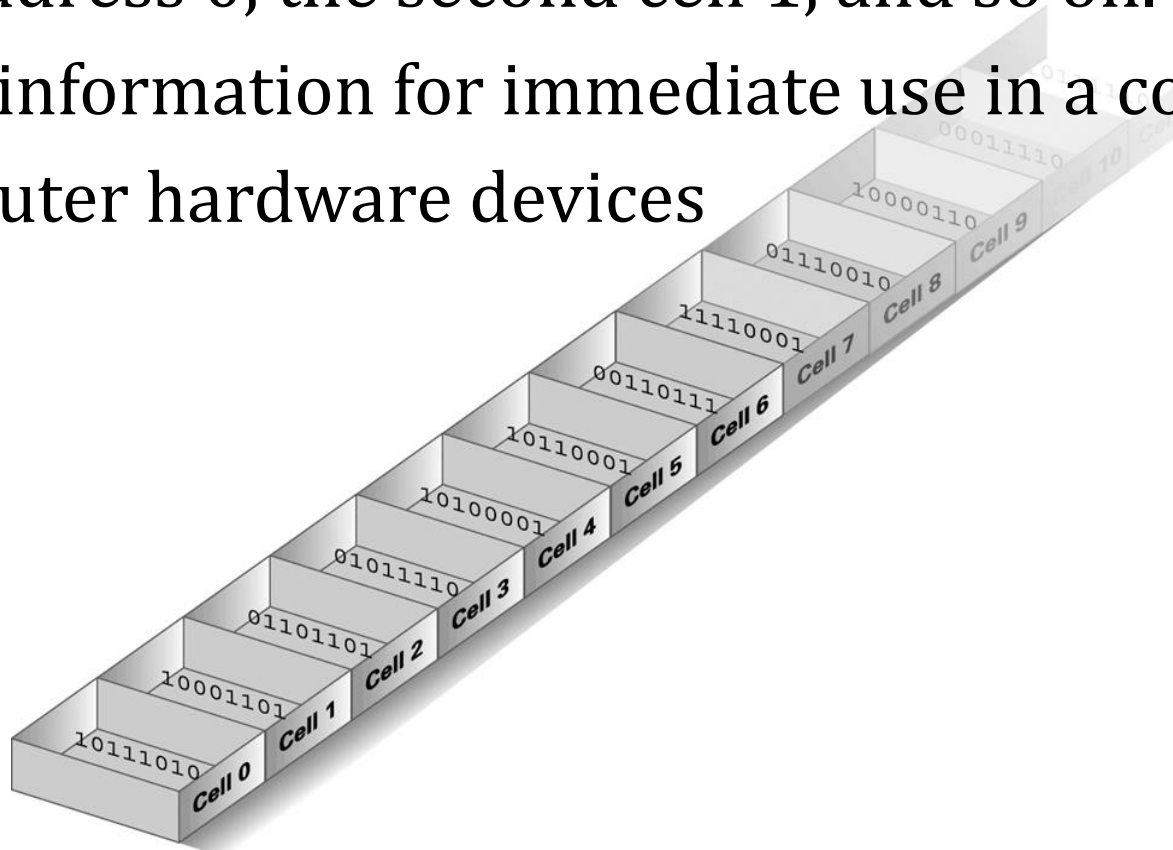
Center Process Unit (CPU)

◈ Contains a fixed number of registers

◈ Basic (atomic) operations

◈ **Initialization**
   - Set a register to a fixed values (e.g., 100, 1000, etc.)

◈ **Arithmetic** (ALU)
   - Take integers a, b stored in two registers, calculate one of {+, -, *, /} and store the result in a register

◈ **Comparison / Branching**
   - Take integers a, b stored in two registers, compare them, and learn which of {a<b, a=b, a>b} is true.

◈ **Memory Access**
   - Take a memory address A currently stored in a register, Do the READ (i.e., load data from memory) or WRITE (i.e., flush data to memory)operator

Algorithm Analysis

- **Algorithm**
  - A sequence of basic operations
- **Algorithm Analysis**
  - **Cost analysis**
    - Algorithm cost (running time) is the length of the sequences, i.e., the number of basic operations
    - My algorithm is correct, why my submission is **TLE**?
    - Is your algorithm fast?
      - Focus on the order of growth (how the running time grows for large n)

  - Unless otherwise stated, we refer algorithm analysis as cost analysis in CS203

Algorithm Correctness Analysis

- **Correctness** analysis
  - I have passed all test cases, why is still **WA**?
  - It is not enough even if you have tested your algorithm on many instances
    - Will your algorithm fail on some other instances?
  - Proof your algorithm is correct
  - Guarantee your implementation is correct

- Software testing is an individual course in other many Universities
  - We will not introduce software testing techniques in this course.

Example 1: Summation

- **Problem**: given integer n, calculate $1+2+3+\ldots+n$

- **Algorithm**:
    - Initialize variable a to 1, b to n, c to 0
    - Repeat the following until $a > b$:
        - Calculate c plus a, and store the result to c.
        - Calculate a plus 1, and store the result to a.

- **Cost** of the algorithm:
    - $3 + n + n + n = 3n + 3$

- Which atomic operations are performed?
- Algorithm is described by English words

Example I: Summation

◈ **Algorithm**:

```
1. load n from memory to register b
2. register a ← 1, c ← 0
3. repeat
4.        c ← c + a
5.        a ← a + 1
6. until a > b
7. return c
```

◈ The above is **pseudocode**, it serves the purpose of express (without **ambiguity**) how our algorithm runs.

◈ **Pseudocode** does not reply on any particular programming language

Example II: Summation

- **Problem**: given integer n, calculate 1+2+3+...+n

- **Cost** of the above algorithm: 3n + 3

- Can we make it faster?

- In our middle school math course:
$$1+2+3+...+n = (1+n)*n / 2$$

Example II: Summation

◈ **Algorithm**:

```
1. load n from memory to register b
2. register a ← 1
3. a ← a + b
4. a ← a * b
5. a ← a / 2
6. return a
```

◈ **Cost of the algorithm = 5**

   ◈ This is significantly faster than the previous algorithm

   ◈ The time of the previous algorithm increases linearly with n

   ◈ The time of this algorithm remains constant with n

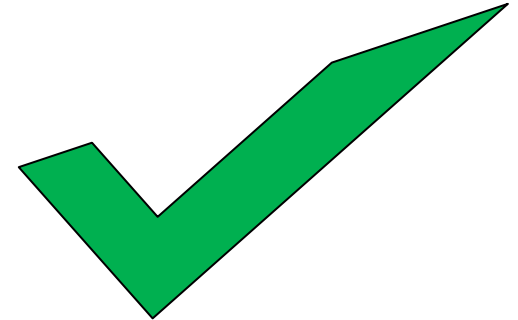# Our Roadmap

- RAM Computation Model

  - Memory, CPU, Algorithm

  - Algorithm, Pseudocode

- Worst Case Analysis

  - Binary Search Problem

  - Big O notation

Search Problem

◈ An array **A** of **n** integers have been sorted in ascending order. Design an algorithm to determine whether given value **t** exists in **A**.

◈ Example

A	5	8	10	13	16	19	27	46	51	86

◈ **t = 16,** the result is "TRUE"

◈ **t = 17**, the result is "FALSE"

Search Problem

◈ The First Algorithm

  ◈ Simply read the value of $\mathbf{A}[i]$ for each $i \in [1, n]$

  ◈ If any of those cell equals to $\boldsymbol{t}$, return "TRUE", otherwise return "FALSE"

◈ Pseudocode:

```
1. variable i ← 1
2. Repeat
3.    if A[i] = t then
4.            return "TRUE"
5.    i ← i + 1
6. until i > n
7. return "FALSE"
```

Running Time of the First Algorithm

A	5	8	10	13	16	19	27	46	51	86

- How much time does the algorithm require?
  - If t is 5, the algorithm has running time = 3
  - If t is 6, the algorithm has running time = $4n + 1 = 41$

- In computer science, it is an art to design algorithms with performance guarantees.

- What is the largest running time on the worst input with n integers?

Worst-Case Running Time

The worst-case running time (or worst case cost) of an algorithm under a problem size n, is defined to be the largest running time of the algorithm on all the inputs of the same size n.

Worst-Case Time of Search Problem

- Our algorithm has worst-case time

$$f(n) = 4n + 1$$

- In other words, the algorithm will terminates with a cost at most **4n+1**.

- This is a performance guarantee on every **n**

- Can we make it faster?
  - **Binary search algorithm**

Binary Search Algorithm

⬥ We utilize the fact that array **A** has been sorted in ascending order.

⬥ Let us compare t to the element **x** in the middle of **A** (i.e., **A[n/2]**)

  ⬥ If **t = A[n/2]**, we have found **t**, return "TRUE", terminate
  ⬥ If **t < A[n/2]**, we can ignore **A[n/2+1] to A[n]**
  ⬥ If **t > A[n/2]**, we can ignore **A[0] to A[n/2]**

⬥ In the 2nd and 3rd cases, we have at most **n/2** elements. Then repeat the above on these left elements.

Binary Search Algorithm

A | 5 | 8 | 10 | 13 | 16 | 19 | 27 | 46 | 51 | 86 | *t=27*

A | 5 | 8 | 10 | 13 | **16** | 19 | 27 | 46 | 51 | 86 | **< t**

A | | | | | | 19 | 27 | **46** | 51 | 86 | **> t**

A | | | | | | 19 | **27** | 46 | | | **= t**

Binary Search Algorithm

◈ Binary Search in Pseudocode

```
1.    left ← 1, right ← n
2.    repeat
3.     mid ← (left+right)/2
4.     if (t = A[mid]) then
5.           return TRUE
6.     else if (t < A[mid]) then
7.           right ← mid -1
8.     else
9.           left ← mid + 1
10.   until left > right
11.   return FLASE
```

Worst-Case Time of Binary Search

⬥ We call the elements from left to right as surviving elements

⬥ Line 1: initialization: 2 basic operations

⬥ Line 2 – 10: iteration, each iteration performs at most 9 basic operations

⬥ Line 11: termination

⬥ How many iterations in the algorithm?

Worst-Case Time of Binary Search

- How many iterations in the algorithm?
    - After the 1st iteration, the number of surviving elements is at most **n/2**
    - After the 2nd iteration, the number of surviving elements is at most **n/4**
    - In general, after *i-th* iteration, the number of surviving elements is at mots n / 2^i

    - Suppose that there are *h* iterations in total, it holds that *h* is the smallest integer satisfying (why?):
    $$n / 2^h < 1$$
    - Then, $h > log_2\ n$ ➔ $h = 1 + log_2\ n$

- Thus, the worst case time of binary search is at most:
    $$g(n) = 2 + 9h = 2 + 9(1 + log_2\ n)$$

- This is a performance guarantee that holds on all values of *n.*

Search Problem

◈ Running time of two algorithms, with input size *n*

  ◈ Algorithm 1: $f(n) = 4n + 1$ (operations)

  ◈ Algorithm 2: $g(n) = 9\log_2 n + 11$ (operations)

◈ Which algorithm is better?

  ◈ <u>Algorithm 2</u>.   Why?

  ◈ We care about the running time at *large input size*

  ◈ Constant factors do not affect *the order of growth*

Asymptotic Analysis

- Running time of two algorithms, with input size *n*
  - Algorithm 1: $f(n) = 4n + 1$ (operations)
  - Algorithm 2: $g(n) = 9\log_2 n + 11$ (operations)
- In computer science, we rarely calculate the time to such a level.
- We ignore all the constants, but only worry about the dominating term.
  - Why not constant? 10n VS. 5n? Which one is faster?
  - "it depends", 10n comparison, 5n multiplication
  - Why dominating term: 3n VS. $\log_2 n$? Which one is faster
  - "$\log_2 n$" is better than 3n in theoretical computer science

Big-O notation

- Let $f(n)$ and $g(n)$ be two functions of n.

- We say that $f(n)$ grows asymptotically no faster than $g(n)$ if there is a constant $c_1 > 0$ such that:
$$f(n) \leq c_1 \cdot g(n)$$

  holds for all $n \geq c_2$.

- We denote this by $f(n) = O(g(n))$

- We say that 5n is considered equally fast as on with 10n, why?

- Big-O capture this by having both of following true (can you prove that?):

$$10n = O(5n)$$

$$5n = O(10n)$$

Big-O example

◈ 10000log$_2$ n is considered better than n. Big-O capture this by having both of following true:

$$10000log_2\ n = O(n)$$

$$n \neq O(10000log_2\ n)$$

◈ Proof of **10000log$_2$ n = O(n)**

◈ There are constants $c_1 = 1, c_2 = 2^{20}$ such that

$$10000log_2\ n \leq c_1 n$$

holds for all $n \geq c_2$

Big-O example

- Proof of $n \neq O(10000 log_2\ n)$

- We can proof it by contradiction. Suppose that are constant c_1, c_2 such that

$$n \leq c_1 \cdot 10000 log_2\ n$$

holds for all $n \geq c_2$. The above can be rewritten as:

$$\frac{n}{log_2\ n} \leq c_1 \cdot 10000$$

however, $\frac{n}{log_2\ n}$ tends to be ∞ as n increases.

Therefore, the inequality cannot hold for all $n \geq c_2$

Exercise

◈ Is $(5n^2 + 3n) = O(n^2)$?
  ◈ Fix $c=6$ and $n_0=3$, then prove $f(n) \leq c\, g(n)$
    [note: other choices also possible]

◈ Is $(5n^2 + 3n) = O(n^3)$?

◈ Is $(5n^2 + 3n) = O(n)$?

◈ Proof the following statements:
$$10000 = O(1)$$
$$100\sqrt{n} + 10n = O(n)$$
$$1000n^{1.5} = O(n^2)$$
$$(\log_2 n)^3 = O(\sqrt{n})$$
$$\log_a n = O(\log_b n) \text{ for a>1, b>1}$$

Asymptotic Analysis

⬦ Henceforth, we will describe the running time of an algorithm only in the asymptotical (i.e., big-O) form, which is also called the algorithm's time complexity.

⬦ Instead of saying the running time of binary search is $g(n) = 8\log_2 n + 10$, we will say $g(n)=O(\log n)$, which captures the fastest-growing term in the running time. This is also the binary search's time complexity.

Worst-Case of Algorithms

Complexity		*Algorithm*
O(1)	Constant time	E.g., Compare two numbers
O($\log n$)	Logarithmic	E.g., Binary search (on a sorted array)
O(n)	Linear time	E.g., Search (on a unsorted array)
O($n \log n$)		E.g., Merge sort
O(n^2)	Quadratic	E.g., Selection sort
O(n^3)	Cubic	E.g., Matrix multiplication
O(2^n)	Exponential	E.g., Brute-force search on boolean satisfiability
O($n!$)	Factorial	E.g., Brute-force search on traveling salesman

Big-Ω notation

- Let *f(n)* and $g(n)$ be two functions of n.
- We say that *f(n)* grows asymptotically no slower than $g(n)$ if there is a constant $c_1 > 0$ such that:

$$f(n) \geq c_1 \cdot g(n)$$

  holds for all $n \geq c_2$.

- We denote this by *f(n)* $= \Omega\big(g(n)\big)$
- Examples:
  - $\log_2 n = \Omega(1)$
  - $0.001n = \Omega(\sqrt{n})$

Big-Θ notation

◈ Let *f(n)* and $g(n)$ be two functions of n.

◈ If $f(n) = O(g(n))$ and *f(n)* $= \Omega(g(n))$, then we define: *f(n)* $= \Theta(g(n))$ to indicate *f(n)* grows asymptotically as fast as $g(n)$

◈ Examples:

   ◈ $1000 + 30 \log n + 1.5\sqrt{n} = \Theta(\sqrt{n})$