

# Lecture 3:

# Sorting Algorithms

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# Sorting Problem

- ◆ Sorting problem
  - ◆ Input: an array  $A[1..n]$  with  $n$  integers
  - ◆ Output: a sorted array  $A$  (in ascending order)
- ◆ Problem is:         $\text{sort } A[1..n]$
- ◆ Input:                 $| 8 | 6 | 1 | 3 | 7 | 2 | 5 | 4 |$
- ◆ Output:               $| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |$

# Our Roadmap

- ◆ Comparison-based Sorting
  - ◆ Quadratic Cost
    - ◆ Selection Sort, Insertion Sort, Bubble Sort
  - ◆  $O(n \log n)$  Cost
    - ◆ Merge Sort, Heap Sort (we will skip here)
  - ◆ Quick Sort
- ◆ Other sorting algorithms
  - ◆ Counting sort, radix sort, bucket sort

# Selection Sort

# Selection Sort

- ◆ Idea of a selection sort method
  - ◆ Start with empty hand, all cards on table
  - ◆ Pick the smallest card from table
  - ◆ Insert the card into the hand



	8
	5
	2
	6
	9
	3
	1
	4
	0
	7

# Selection Sort Algorithm

## ◆ SelectionSort

8	6	1	3	7	2	5	4
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- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order
- ◆ Selection-Sort (  $A[1..n]$  )

1. for integer  $i \leftarrow 1$  to  $n-1$

2.      $k \leftarrow i$

3.     for integer  $j \leftarrow i+1$  to  $n$

4.         if  $A[k] > A[j]$  then

5.              $k \leftarrow j$

6.     swap  $A[i]$  and  $A[k]$

1	6	8	3	7	2	5	4
---	---	---	---	---	---	---	---



*sorted*



*unsorted*

1	2	8	3	7	6	5	4
---	---	---	---	---	---	---	---



*sorted*



*unsorted*

# Selection Sort Time Complexity

## ◆ Selection Sort

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order
- ◆ Selection-Sort (  $A[1..n]$  )

1. for integer  $i \leftarrow 1$  to  $n-1$

Cost:  $n-1 = O(n)$

2.      $k \leftarrow i$

Cost:  $n-1 = O(n)$

3.     for integer  $j \leftarrow i+1$  to  $n$

Cost:  $n-1+n-2+\dots+1 = O(n^2)$

4.         if  $A[k] > A[j]$  then

Cost:  $O(n^2)$

5.              $k \leftarrow j$

Cost:  $O(n^2)$

6.     swap  $A[i]$  and  $A[k]$

Cost:  $O(n)$

## ◆ Selection sort total cost:

◆  $O(n) + O(n) + O(n^2) + O(n^2) + O(n^2) + O(n) = O(n^2)$

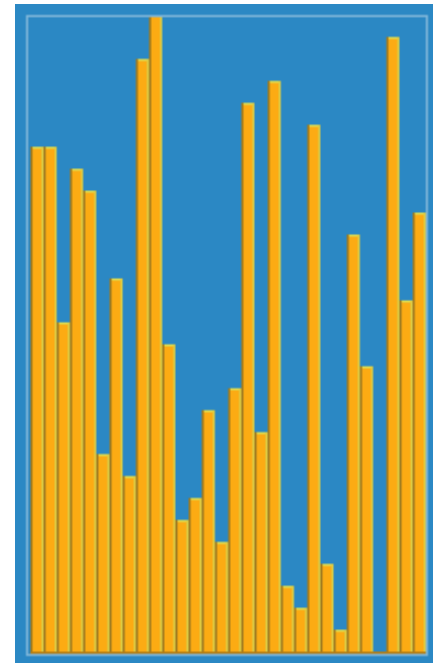
# Insertion Sort



# Insertion Sort

- ◆ Idea of a insertion sort method
  - ◆ One input each iteration, growing a sorted output list
  - ◆ Remove one element from input data
  - ◆ Find the location it belongs within the sorted list
  - ◆ Repeat until no input elements remain

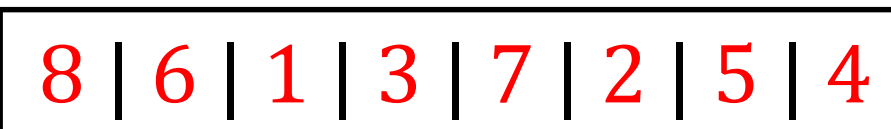
6 5 3 1 8 7 2 4



# Insertion Sort Algorithm

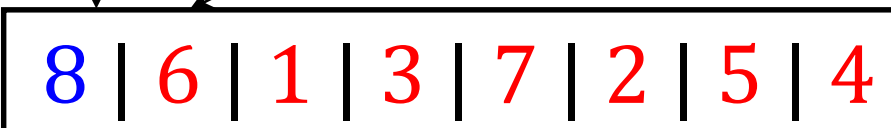
## ◆ InsertionSort

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order
- ◆ Insertion-Sort (  $A[1..n]$  )
  1. for integer  $i \leftarrow 1$  to  $n$
  2.   for integer  $j \leftarrow i$  to  $1$  with  $j > 1$
  3.       if  $A[j-1] > A[j]$  then
  4.        swap  $A[j-1]$  and  $A[j]$



*sorted*    *unsorted*

↓      ↙



*sorted*    *unsorted*

↑      ↑

↘      ↘



# Insertion Sort Time Complexity

## ◆ Insertion Sort

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order

### ◆ Insertion-Sort ( $A[1..n]$ )

1. for integer  $i \leftarrow 1$  to  $n$  Cost:  $n-1=O(n)$
2.   for integer  $j \leftarrow i$  to 1 with  $j > 1$  Cost:  $1+\dots+n-2=O(n^2)$
3.       if  $A[j-1] > A[j]$  then Cost:  $O(n^2)$
4.       swap  $A[j-1]$  and  $A[j]$  Cost:  $O(n^2)$

## ◆ Insertion sort total cost:

- ◆  $O(n)+O(n^2) +O(n^2) +O(n^2) =O(n^2)$

# Bubble Sort

# Bubble Sort

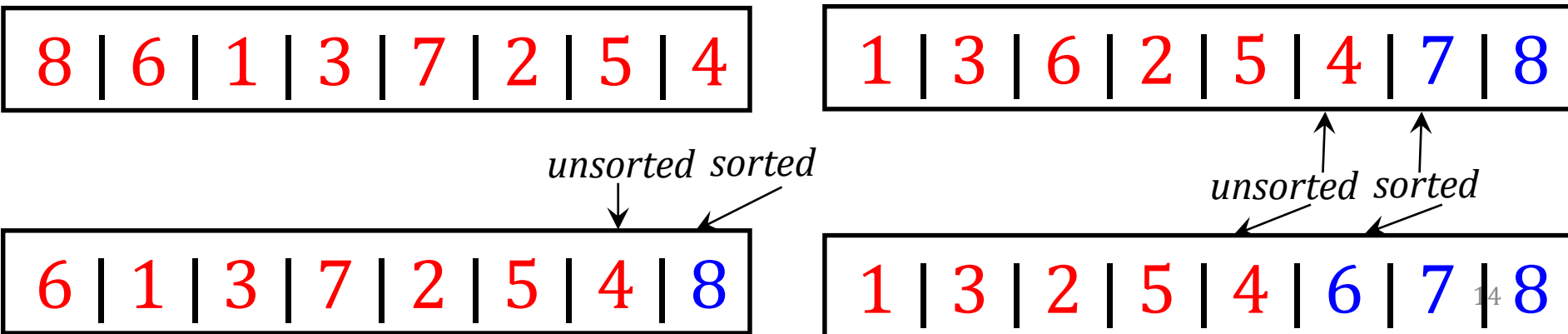
- ◆ Idea of a bubble sort method
  - ◆ For each pass
    - ◆ Compare the pair of adjacent item
    - ◆ Swap them if they are in the wrong order
  - ◆ Repeat the pass through until no swaps are needed

6 5 3 1 8 7 2 4

# Bubble Sort Algorithm

## ◆ BubbleSort (optimized version)

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order
- ◆ Bubble-Sort (  $A[1..n]$  )
  1. for integer  $i \leftarrow 1$  to  $n-1$
  2.     for integer  $j \leftarrow 2$  to  $n$
  3.         if  $A[j-1] > A[j]$  then
  4.             swap  $A[j-1]$  and  $A[j]$



# Bubble Sort Time Complexity

## ◆ Bubble Sort

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order

### ◆ Bubble-Sort ( $A[1..n]$ )

1. for integer  $i \leftarrow 1$  to  $n-1$  Cost:  $n-1=O(n)$
2.     for integer  $j \leftarrow 2$  to  $n$  Cost:  $n-1+\dots+1=O(n^2)$
3.         if  $A[j-1] > A[j]$  then Cost:  $O(n^2)$
4.         swap  $A[j-1]$  and  $A[j]$  Cost:  $O(n^2)$

## ◆ Bubble sort total cost:

- ◆  $O(n)+O(n^2) +O(n^2) +O(n^2) =O(n^2)$

# Pop Quiz

- ◆ We say a sorting algorithm is “stable” if it does not change the relative order of elements with equal keys, which of the following is/are stable ()  
A: Selection sort B: Insertion Sort C: Bubble Sort
- ◆ Watch a video:
  - ◆ 1) Which sorting algorithm is used in that video?
  - ◆ 2) TB is No. x, so  $x = ?$

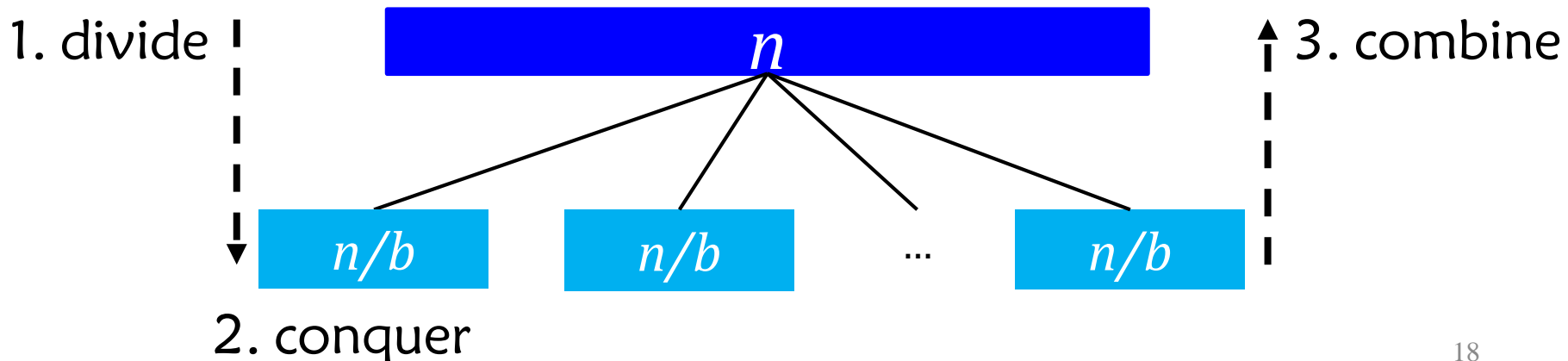


# Merge Sort

## (Divide-and-Conquer)

# Divide and Conquer

- ◆ Divide and Conquer: an algorithmic technique
  - ◆ **Divide:** divide the problem into smaller subproblems
  - ◆ **Conquer:** solve each subproblem recursively
  - ◆ **Combine:** combine the solution of subproblems into the solution of the original problem



# Example: Merge Sort

- ◆ Sorting problem
  - ◆ Input: an array  $A[1..n]$  with  $n$  integers
  - ◆ Output: a sorted array  $A$  (in ascending order)

- ◆ Original problem is:             $\text{sort } A[1..n]$

| 8 | 6 | 1 | 3 | 7 | 2 | 5 | 4 |

- ◆ What is a subproblem?

- ◆ Sort a subarray  $A[l..r]$

| 7 | 2 | 5 | 4 |

# Merge Sort

## Merge-Sort( $A, n$ )

### ◆ Merge Sort

- ◆ **Divide**: divide the array into two subarrays of  $n/2$  numbers each

- ◆ **Conquer**: sort two subarrays recursively

- ◆ **Combine**: merge two sorted subarrays into a sorted array

1. if  $n > 1$

2.  $p \leftarrow \lfloor n/2 \rfloor$

3.  $B[1..p] \leftarrow A[1..p]$

4.  $C[1..n-p] \leftarrow A[p+1..n]$

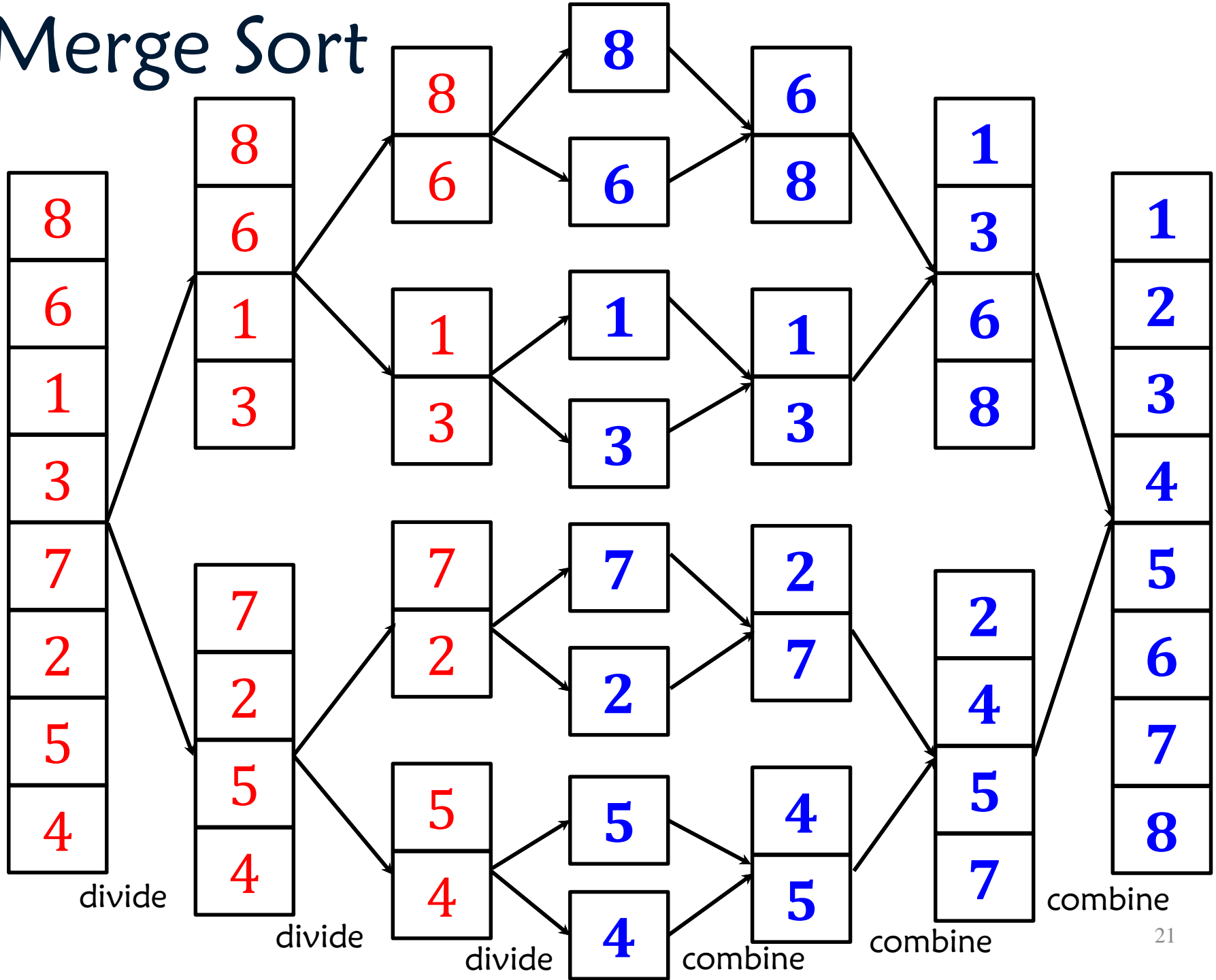
5. Merge-Sort( $B, p$ )

6. Merge-Sort( $C, n-p$ )

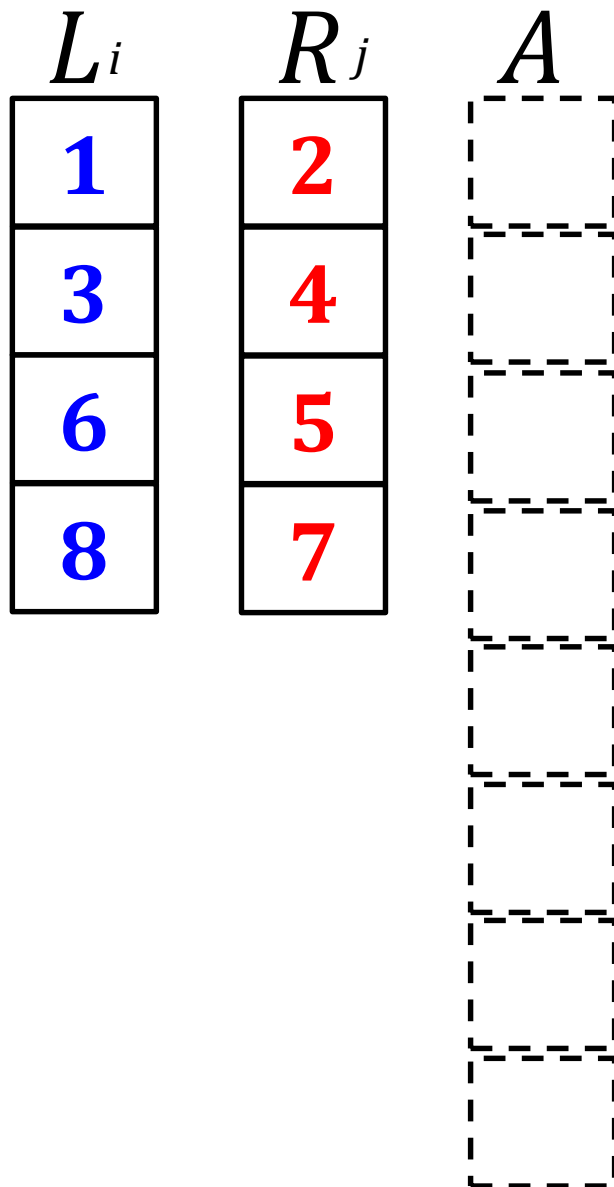
7.  $A[1..n] \leftarrow \text{Merge}(B, p, C, n-p)$

We'll discuss the Combine phase ("Merge" function) later

# Merge Sort



# Merge Sort: Combine Phase



Sorted arrays

Merge( $L, n_L, R, n_R$ )

1.  $n \leftarrow n_L + n_R$
2. let  $A[1..n]$  be a new array
3.  $i \leftarrow 1; j \leftarrow 1$
4. for  $k \leftarrow 1$  to  $n$
5.   if  $i \leq n_L$  and ( $j > n_R$  or  $L[i] \leq R[j]$ )
6.      $A[k] \leftarrow L[i]; i \leftarrow i + 1$
7.   else
8.      $A[k] \leftarrow R[j]; j \leftarrow j + 1$
9. return  $A$

# Running time of Merge

Sorted arrays

Merge( $L, n_L, R, n_R$ )

1.  $n \leftarrow n_L + n_R$
2. let  $A[1..n]$  be a new array
3.  $i \leftarrow 1; j \leftarrow 1$
4. for  $k \leftarrow 1$  to  $n$
5.   if  $i \leq n_L$  and ( $j > n_R$  or  $L[i] \leq R[j]$ )
6.      $A[k] \leftarrow L[i]; i \leftarrow i + 1$
7.   else
8.      $A[k] \leftarrow R[j]; j \leftarrow j + 1$
9. return  $A$

◆ Let  $n = n_L + n_R$  be the total number of items

◆ Time of merge:  $O(n)$  time

◆ Line 1:  $O(1)$

◆ Line 2:  $O(n)$

◆ Line 3:  $O(1)$

◆ Lines 4-8:  $O(n)$

# Running time of Merge Sort

## Merge-Sort( $A, n$ )

1. if  $n > 1$
2.  $p \leftarrow \lfloor n/2 \rfloor$
3.  $B[1..p] \leftarrow A[1..p]$
4.  $C[1..n-p] \leftarrow A[p+1..n]$
5. Merge-Sort( $B, p$ )
6. Merge-Sort( $C, n-p$ )
7.  $A[1..n] \leftarrow \text{Merge}(B, p, C, n-p)$

◆ Let  $T(n)$  be the running time of Merge Sort

- ◆ Lines 3, 4 take  $O(n)$  time
- ◆ Line 5 takes  $T(n/2)$  time
- ◆ Line 6 takes  $T(n/2)$  time
- ◆ Line 7 takes  $O(n)$  time

◆ Thus, we obtain the recurrence

$$T(n) = 2 T(n/2) + O(n)$$

◆ Solving it, we get:

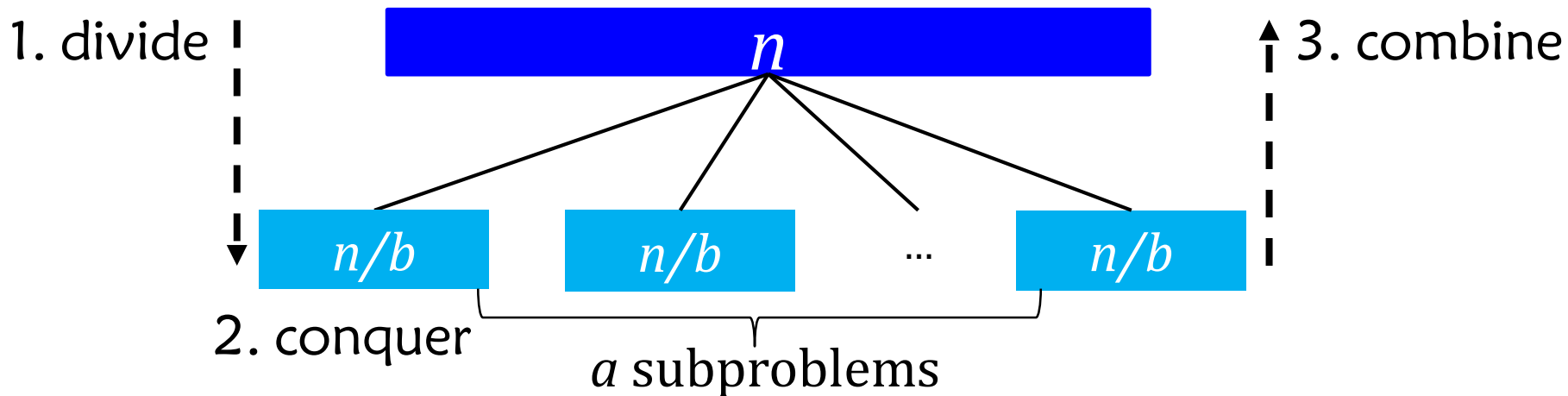
$$T(n) = O(n \log n)$$



# Time Complexity

- ◆  $T(n)$ : time complexity of algorithm at input size  $n$ 
  - ◆ Divide the problem into  $a$  subproblems
  - ◆ Size of each subproblem is  $n/b$
  - ◆ Combine phase takes  $f(n)$  time

Note:  $a$  and  $b$  can have different values



- ◆ Recurrence equation:  $T(n) = a T(n/b) + f(n)$
- ◆ E.g., Merge Sort:  $T(n) = 2 T(n/2) + O(n)$

# Methods for Solving Recurrences

- ◆ Recurrence equation:  $T(n) = a T(n/b) + f(n)$
- ◆ Two methods for solving recurrences
  - ◆ Master theorem
  - ◆ Substitution method
- ◆ Master theorem
  - ◆ It could be proved by carefully applying the “expansion method”, the details are tedious and omitted from this course
- ◆ Substitution method (we skip it here)
  - ◆ It is mathematical induction

# Master Theorem

- ◆ Recurrence equation:  $T(n) = a T(n/b) + f(n)$
- ◆ Let  $T(n)$  be a function that return a positive value for every integer  $n > 0$ . We know that:

- ◆  $T(1) = O(1)$

- ◆  $T(n) = \alpha T\left(\left\lceil \frac{n}{\beta} \right\rceil\right) + O(n^\gamma)$  for  $(n \geq 2)$

where  $\alpha \geq 1$ ,  $\beta > 1$ , and  $\gamma \geq 0$ . Then:

- ◆ If  $\log_\beta \alpha < \gamma$ , then  $T(n) = O(n^\gamma)$
- ◆ If  $\log_\beta \alpha = \gamma$ , then  $T(n) = O(n^\gamma \log n)$
- ◆ If  $\log_\beta \alpha > \gamma$ , then  $T(n) = O(n^{\log_\beta \alpha})$

# Master Theorem

- ◆ Consider the recurrence of **binary search**:
  - ◆  $T(1) \leq c_1$
  - ◆  $T(n) \leq T(n/2) + c_2$  (for  $n \geq 2$ )
  - ◆ Hence,  $\alpha = 1$ ,  $\beta = 2$ , and  $\gamma = 0$ . Since  $\log_{\beta} \alpha = \log_2 1 = 0 = \gamma$ , we know that  $T(n) = O(n^0 \log n) = O(\log n)$ .
- ◆ Consider the recurrence of **merge sort**:
  - ◆  $T(1) \leq c_1$
  - ◆  $T(n) = 2 T(n/2) + O(n) = 2 T(n/2) + c_2 n$  (for  $n \geq 2$ )
  - ◆ Hence,  $\alpha = 2$ ,  $\beta = 2$ , and  $\gamma = 1$ . Since  $\log_{\beta} \alpha = \log_2 2 = 1 = \gamma$ , we know that  $T(n) = O(n^1 \log n) = O(n \log n)$ .

# Quick Sort

## RAM with Randomization

# Deterministic & Randomized

- ◆ So far in CS203, all our algorithms are **deterministic**, namely, they do not involve any randomization.
- ◆ We will introduce **randomized** algorithms, e.g., quick sort in the sorting problem.
- ◆ Randomized algorithms play an important role in computer science, they often simpler, and sometimes can be provably faster as well.
- ◆ Recall the core of the RAM model is a set of atomic operations, we extend this set with:
  - ◆ **RANDOM(x, y)**: given integers  $x$  and  $y$  ( $x \leq y$ ), this operation returns an integer chosen uniformly at random in  $[x, y]$ , i.e.,  $x, x+1, \dots, y$  has the same probability of being returned.

# Randomized Algorithm Example

- ◆ Find-a-Zero: Given an array of integers with size  $n$ , among which there is at least 0. Design an algorithm to report an arbitrary position of  $A$  that contains a 0
- ◆ Suppose  $A = (9, 18, 0, 0, 15, 0)$ , an algorithm can report 3, 4 or 6, consider the following randomized algorithm
  - ◆ 1. do
  - ◆ 2.  $r \leftarrow \text{RANDOM}(1, n)$
  - ◆ 3. until  $A[r] = 0$
  - ◆ 4. return  $r$
- ◆ What is the cost of the algorithm? It depends
  - ◆ If all numbers in  $A$  are 0,  $O(1)$  time. If  $A$  has only one 0,  $O(n)$  expected time.
  - ◆ As before, we care about the worst expected time:  $O(n)$

# Quick Sort

- ❖ Idea of a quick sort method
  - ❖ **Randomly** pick an integer  $p$  in  $A$ , call it the **pivot**
  - ❖ Re-arrange the integers in an array  $A'$  such that
    - ◆ All the integers **smaller** than  $p$  are positioned **before**  $p$  in  $A'$
    - ◆ All the integers **larger** than  $p$  are positioned **after**  $p$  in  $A'$
  - ❖ Sort the part of  $A'$  before  $p$  recursively
  - ❖ Sort the part of  $A'$  after  $p$  recursively



# Quicksort

## ◆ Quick Sort

- ◆ Input: an **array**  $A$  of  $n$  numbers
- ◆ Output : an **array**  $A$  of  $n$  numbers in the ascending order
- ◆ Quicksort (  $A[1..n]$ ,  $lo=1$ ,  $hi=n$ )
  1.  $p \leftarrow \text{partition}(A, lo, hi)$
  2. Quicksort( $A, lo, p-1$ )
  3. Quicksort( $A, p+1, hi$ )

# Quicksort

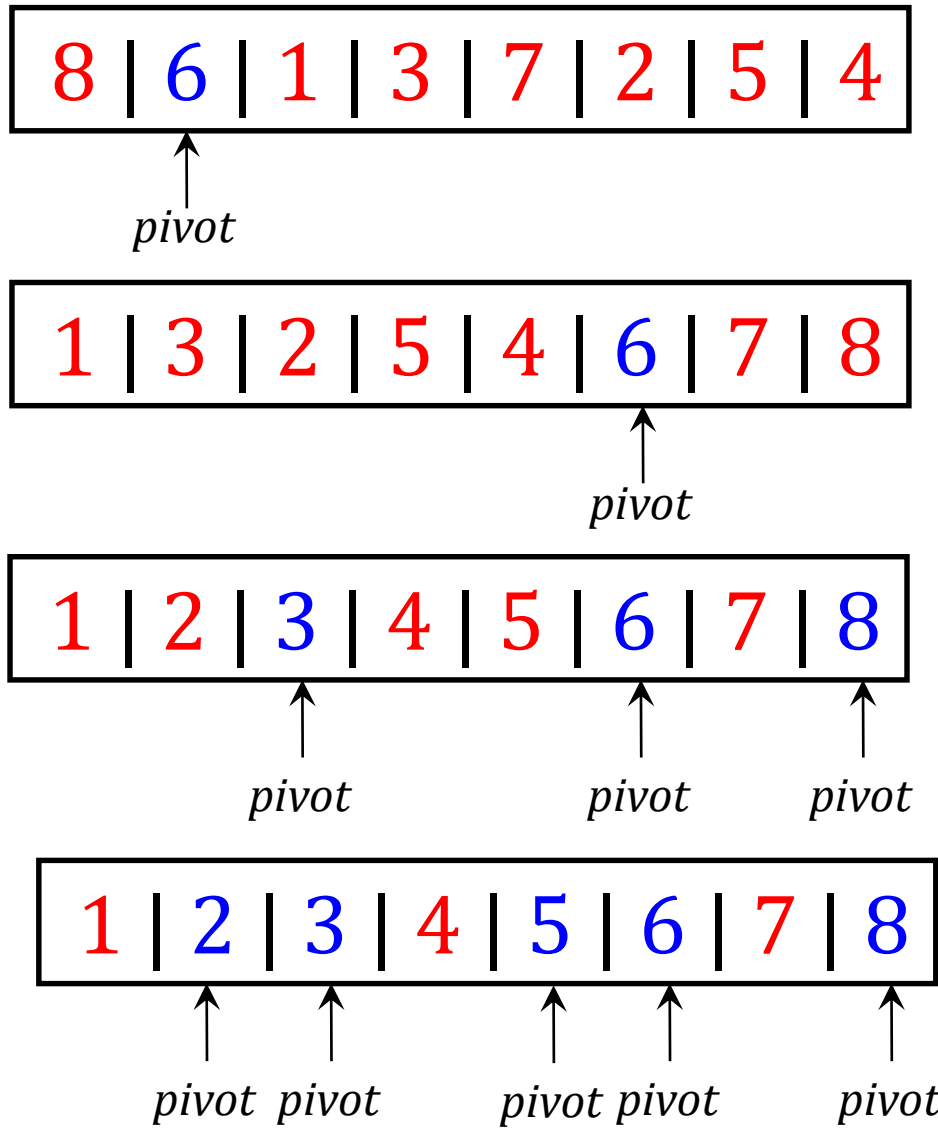
## ◆ Partition(A, lo, hi)

1.  $p \leftarrow \text{RANDOM}(\text{lo}, \text{hi}); \text{pivot} \leftarrow A[p];$
2.  $L \leftarrow \text{lo}, R \leftarrow \text{hi}$
3. for integer  $i$  from  $\text{lo}$  to  $\text{hi}$
4.     if( $i \neq p$ )
5.         if( $A[i] < \text{pivot}$ )  $A'[L++] \leftarrow A[i]$
6.         else  $A'[R--] \leftarrow A[i]$
7.  $A'[L++] \leftarrow \text{pivot}$
8.  $A[\text{lo}, \text{hi}] \leftarrow A'$
9. return  $L$ ;

## ◆ Question:

- ◆ If we set  $p \leftarrow \text{lo}$  or  $\text{hi}$  in Line 1, quick sort is still correct ?
- ◆ What are the difference between  $p \leftarrow \text{lo/hi}$  and  $p \leftarrow \text{RANDOM}(\text{lo}, \text{hi})$ ?

# Quicksort Example



# Quicksort Time Complexity

- ❖ Quicksort's running time is not attractive in the worst case: it is  $O(n^2)$  (why?) However, quick sort is fast in expectation, i.e.,  $O(n \log n)$ , remember this holds for every input array  $A$ .
- ❖ Whether quicksort has any advantage over merge sort? which guarantees  $O(n \log n)$  in the worst case.
- ❖ No in theory, but there is an advantage in practice
- ❖ Quicksort permits a faster implementation that leads to a smaller hidden constant compared to merge sort.  
(why?)

# Quicksort Time Complexity

- ◆ Let  $X$  be the number of comparisons in quicksort algorithm. The running is bounded by  $O(n+x)$ .
- ◆ We prove that  $E[X]=O(n \log n)$
- ◆ Denote  $e_i$  be the  $i$ -th smallest integer in  $A$ , consider  $e_i$  and  $e_j$  for any  $i,j$  such that  $i \neq j$
- ◆ What is the probability that quicksort compares  $e_i$  and  $e_j$ ?
  - ◆ Every element will be selected as pivot precisely once
  - ◆  $e_i$  and  $e_j$  are not compared, if any element between them gets selected as a pivot before them.
  - ◆ Therefore,  $e_i$  and  $e_j$  are compared if and only if either one is the first among  $e_i, e_{i+1}, \dots, e_j$  picked as a pivot
  - ◆ The probability is  $2/(j-i+1)$  (random pivot selection)

# Quicksort Time Complexity

- Define random variable  $X_{ij}$  to be 1, if  $e_i$  and  $e_j$  are compared. Otherwise,  $X_{ij}$  to be 0. Thus, we have
- $\Pr[X_{ij} = 1] = 2/(j-i+1)$ , that is  $E[X_{ij}] = 2/(j-i+1)$
- Since  $X = \sum_{i,j} X_{ij}$ , hence:
- $$E[X] = \sum_{i,j} E[X_{ij}] = \sum_{i,j} \frac{2}{j-i+1}$$
- $$= 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{1}{j-i+1}$$
- $$= 2 \sum_{i=1}^{n-1} O(\log(j-i+1)) \quad (1+1/2+\dots+1/n = O(\log n))$$
- $$= 2 \sum_{i=1}^{n-1} O(\log n)$$
- $$= O(n \log n)$$
- Harmonic series:  $1+1/2+\dots+1/n$ , which is frequently encountered in computer science.

# Summary

Sort	Average	Space	Stable
Selection	$O(n^2)$	$O(1)$	Yes
Insertion	$O(n^2)$	$O(1)$	Yes
Bubble	$O(n^2)$	$O(1)$	Yes
Heap	$O(n \log n)$	$O(1)$	No
Merge	$O(n \log n)$	Depends	Yes
Quick	$O(n \log n)$	$O(1)$	Yes

- ❖ Comparison lower bound of sorting algorithm:  $\Omega(n \log n)$
- ❖ We omit the proof here.

# Other Sorting Methods



# Other Sorting Algorithms

- ◆ Counting sort (Chapter 8.2)
  - ◆ it is applicable when each input is known to belong to a particular set,  $S$ , of possibilities. The algorithm runs in  $O(|S| + n)$  time and  $O(|S|)$  memory where  $n$  is the length of the input.
- ◆ Radix sort (Chapter 8.3)
  - ◆ radix sort is an algorithm that sorts numbers by processing individual digits.  $n$  numbers consisting of  $k$  digits each are sorted in  $O(n \cdot k)$  time
- ◆ Bucket sort (Chapter 8.4)
  - ◆ Bucket sort is a divide and conquer sorting algorithm that generalizes counting sort by partitioning an array into a finite number of buckets.

Thank You!