Q4_1

1. Solution. $X_1 - X_2 \sim N(0, 2\sigma^2)$

$X_3 + 2X_4 \sim N(0, 5\sigma^2)$

Also, $X_1 - X_2$ and $X_3 + 2X_4$ independent.

Hence, $a = \frac{1}{2\sigma^2}$, $b = \frac{1}{5\sigma^2}$, $Y \sim \chi^2(2)$.

2. Solution. $D(X_i) = \frac{1}{12}$

$$D(\bar{X}) = \frac{1}{n} D(X_i) = \frac{1}{120}.$$

3. Solution. (1) $\mathbb{E}X = \frac{\theta}{\theta+1}$

MoM: $\bar{X}_n = \frac{\hat{\theta}}{\hat{\theta}+1}$, i.e. $\hat{\theta} = \frac{\bar{X}_n}{1-\bar{X}_n}$

(2) $L(\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \mathbb{1}_{\{0 < x_{(1)} < x_{(n)} < 1\}}$

$$\ell(\theta) = n\ln\theta + (\theta-1)\sum_{i=1}^{n} \ln x_i$$

MLE: $\ell'(\hat{\theta}) = 0$ i.e. $\hat{\theta} = \frac{n}{-\sum_{i=1}^{n} \ln x_i}$

Since $\ell''(\theta) = -\frac{n}{\theta^2} \leq 0$,

then $\hat{\theta}$ is MLE.

4. Solution. $1-\alpha=0.9$

$$\frac{|\overline{X}-\mu|}{1/5} \sim N(0,1)$$

μ 的 0.90 置信区间为:

$$\left(\overline{X}-\frac{1}{5}u_{1-\frac{\alpha}{2}},\ \overline{X}+\frac{1}{5}u_{1-\frac{\alpha}{2}}\right)$$

i.e.

$$\left(16-\frac{1}{5}u_{0.95},\ 16+\frac{1}{5}u_{0.95}\right)$$

5. Solution. $1-\alpha=0.95$

$$\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim N(0,1)$$

$\mu_1-\mu_2$ 的 0.95 置信区间,

$$\left(2-\sqrt{\frac{7}{15}}\,u_{0.975},\ 2+\sqrt{\frac{7}{15}}\,u_{0.975}\right)$$

Q 4-2

1. Solution. $X_2 - 2X_3 \sim N(-2, 5\times 4)$

$X_1 - a$, $X_2 - 2X_3 - C$ mutually independent.

$a = 2$, $b = \frac{1}{5\times4}$, $C = -2$, $Y \sim \chi^2(2)$

2. Solution $D(X_i) = \frac{1}{4}$.

$$\mathbb{E}(K) = \frac{9}{10}\,\mathbb{E}S^2 = \frac{9}{40}$$

3. Solution. (1) $\mathbb{E}X = \frac{1}{\lambda}$

MoM: $\overline{X}_n = \frac{1}{\hat{\lambda}}$, i.e. $\hat{\lambda} = \frac{1}{\overline{X}_n}$.

(2) $L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \mathbb{1}_{\{x_{(1)} > 0\}}$

$\ell(\lambda) = n\ln\lambda - n\lambda\overline{X}_n$

MLE: $\ell'(\lambda) = 0$, $\lambda = \frac{1}{\overline{X}_n}$.

Since $\ell''(\lambda) \le 0$, then MLE $\hat{\lambda} = \frac{1}{\overline{X}_n}$.

4. Solution. $1-\alpha=0.95$, $\bar{x}=20$.

$$\frac{\bar{X}-\mu}{2/\sqrt{16}} \sim t(15)$$

μ的0.95置信区间为:

$$\left(20-\frac{1}{2}t_{0.975}(15),\ 20+\frac{1}{2}t_{0.975}(15)\right)$$

5. Solution. $1-\alpha=0.95$, $\bar{x}-\bar{y}=5$

$$\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{4}{15}+\frac{1}{15}}} \sim N(0,1)$$

$\mu_1-\mu_2$的0.95置信区间为:

$$\left(5-\sqrt{\frac{7}{15}}u_{0.975},\ 5+\sqrt{\frac{7}{15}}u_{0.975}\right).$$