§4.1

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习题课：周三晚9、10节

6. a. $E(X)=\int_{-\infty}^{+\infty} xf(x)dx=\int_0^1 2x^2dx=\frac{2}{3}[x^3]_0^1=\frac{2}{3}$

b. $Y=X^2 \Rightarrow X=\sqrt{y}\ (y\in[0,1])$

$f_Y(y)=\begin{cases} |h_1'(y)|f_X(h_1(y))=1\ (y\in[0,1]) \\ 0,\ 其他 \end{cases}$

$E(Y)=\int_{-\infty}^{+\infty} yf(y)dy=\int_0^1 ydy=\frac{1}{2}[y^2]_0^1=\frac{1}{2}$

c. $\because Y=X^2\in[0,1]$.

$\therefore E(Y)=E(X^2)=\int_{-\infty}^{+\infty} x^2f(x)dx=\int_0^1 2x^3dx=\frac{1}{2}[x^4]_0^1=\frac{1}{2}$.

d. 方差定义计算：$var(X)=E\{[X-E(X)]^2\}$.

$E(X)=\frac{2}{3}$

$Var(X)=E[(X-\frac{2}{3})^2]=\int_0^1(x-\frac{2}{3})^2\cdot 2xdx=\int_0^1 2x^3-\frac{8}{3}x^2+\frac{8}{9}xdx=\frac{1}{18}$

定理计算：$Var(X)=E(X^2)-[E(X)]^2$

$Var(X)=E(X^2)-[E(X)]^2=\frac{1}{2}-(\frac{2}{3})^2=\frac{1}{18}$

15. 设m=(奖金数额)，两种彩票各只有一张中奖.

①一种彩票中买两张：

$E(X_1)=m\cdot\frac{C_{n-1}^1}{C_n^2}=\frac{2m}{n}$

$\because E(X_1)=E(X_2)$

$\therefore$ 一样

②两种彩票中各买一张：

$E(X_2)=m\cdot\frac{1}{n}\cdot\frac{n-1}{n}+m\cdot\frac{n-1}{n}\cdot\frac{1}{n}+2m\cdot\frac{1}{n^2}=\frac{2m}{n}$

20. $X\sim P(\lambda)$,

$E[\frac{1}{X+1}]=\sum_{k=0}^{\infty}\frac{1}{k+1}\cdot\frac{\lambda^k}{k!}e^{-\lambda}=\frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!}=\frac{e^{-\lambda}}{\lambda}(e^{\lambda}-1)=\frac{1}{\lambda}-\frac{e^{-\lambda}}{\lambda}$

21. 设正方形边长为X. $X\sim U[0,1]$，则面积$Y=X^2$

$E(Y)=E(X^2)=\int_0^1 x^2dx=\frac{1}{3}$

31. $X\sim U[1,2]$.

$E(\frac{1}{X})=\int_1^2\frac{1}{x}dx=\ln 2$

$E(X)=\int_1^2 xdx=\frac{1}{2}\Rightarrow\frac{1}{E(X)}=2\neq E(\frac{1}{X})=\ln 2$

补充题：

1. (1) $E(Y)=E(2X)=2E(X)=2\int_0^{\infty}xe^{-x}dx=2$

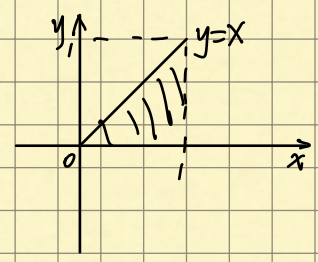
(2) $E(Y)=E(e^{-2X})=\int_0^{\infty}e^{-2x}e^{-x}dx=\int_0^{\infty}e^{-3x}dx=\frac{1}{3}$.

2. $E(X)=\int_0^1\int_0^x x\cdot 12y^2dydx=\int_0^1 4x^4dx=\frac{4}{5}$

$E(Y)=\int_0^1\int_0^x y\cdot 12y^2dydx=\int_0^1 3x^4dx=\frac{3}{5}$

$E(XY)=\int_0^1\int_0^x xy\cdot 12y^2dydx=\int_0^1 3x^5dx=\frac{1}{2}$

$E(X^2+Y^2)=\int_0^1\int_0^x(x^2+y^2)\cdot 12y^2dydx=\int_0^1 4x^5+\frac{12}{5}x^5dx=\frac{16}{15}$



§4.2

49、a. $E(Z)=E(\alpha X+(1-\alpha)Y)=\alpha E(X)+(1-\alpha)E(Y)=\alpha\mu+(1-\alpha)\mu=\mu$

b. $Var(Z)=\alpha^2\sigma_X^2+(1-\alpha)^2\sigma_Y^2=(\sigma_X^2+\sigma_Y^2)\alpha^2-2\sigma_Y^2\alpha+\sigma_Y^2$

当 $\alpha=\dfrac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}\in[0,1]$ 时，有最小的 $Var(Z)$.

c. 令 $Z=\dfrac{(X+Y)}{2}$，则 $\alpha=\frac{1}{2}$，此时 $Var(Z)=\frac{1}{4}(\sigma_X^2+\sigma_Y^2)$

若 $Var(Z)\le\min(\sigma_X^2,\sigma_Y^2)$，则 $\begin{cases}\sigma_Y^2\le3\sigma_X^2\\ \sigma_X^2\le3\sigma_Y^2\end{cases}$ 此时 $\frac{X+Y}{2}$ 优于单独使用X或Y.

50、$E(\bar{X})=E(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n}\sum_{i=1}^{n}E(X_i)=\frac{1}{n}\sum_{i=1}^{n}\mu=\frac{1}{n}\cdot n\mu=\mu$

$Var(\bar{X})=Var(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)=\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2=\frac{1}{n^2}\cdot\frac{1}{n}\sigma^2=\frac{\sigma^2}{n}$

55、$E(T)=E(\sum_{k=1}^{n}kX_k)=\sum_{k=1}^{n}kE(X_k)=\sum_{k=1}^{n}k\mu=\mu\sum_{k=1}^{n}k=\frac{n(n+1)}{2}\mu.$

$Var(T)=Var(\sum_{k=1}^{n}kX_k)=\sum_{k=1}^{n}k^2Var(X_k)=\sigma^2\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}\sigma^2$

补充题：

1、$E(Z)=E(5X-2Y+15)=5E(X)-2E(Y)+15=28$

$D(Z)=D(5X-2Y+15)=25D(X)+4D(Y)=136$

2、$E(Z)=E(2X_1-2X_2+3X_3-\frac{1}{2}X_4)=2E(X_1)-E(X_2)+3E(X_3)-\frac{1}{2}E(X_4)=14$

$D(Z)=D(2X_1-2X_2+3X_3-\frac{1}{2}X_4)=4D(X_1)+4D(X_2)+9D(X_3)+\frac{1}{4}D(X_4)=\frac{185}{4}$

§4.3

54、$Cov(U,V)=Cov(Z+X,Z+Y)=Cov(Z,Z)+Cov(Z,Y)+Cov(X,Z)+Cov(X,Y)$

$\because X,Y,Z$ 不相关 $\therefore Cov(X,Z)=Cov(X,Y)=Cov(Y,Z)=0$

又 $Cov(Z,Z)=D(Z)$

故 $Cov(U,V)=D(Z)=\sigma_Z^2$

$\rho_{UV}=\dfrac{Cov(U,V)}{\sqrt{D(U)}\sqrt{D(V)}}$ 又 $D(U)=D(Z+X)=D(Z)+D(X)=\sigma_Z^2+\sigma_X^2$ (Z,X,Y 独立)

$D(V)=D(Z+Y)=D(Z)+D(Y)=\sigma_Z^2+\sigma_Y^2$

故 $\rho_{UV}=\dfrac{\sigma_Z^2}{\sqrt{\sigma_Z^2+\sigma_X^2}\sqrt{\sigma_Z^2+\sigma_Y^2}}$

60、$E(Y)=\int_{-\infty}^{+\infty}yf_Y(y)dy=0$，$E(X)=E(SY)=E(S)E(Y)=0$

$Cov(X,Y)=Cov(SY,Y)=E(SY^2)-E(SY)E(Y)=E(S)E(Y^2)=0$

但 $f_X(x)=\frac{1}{2}f_Y(y)+\frac{1}{2}f_Y(-y)$，$P(X=y|Y=y)=P(X=-y|Y=y)=\frac{1}{2}$

因此 X,Y 不独立

补充题：

1、$E(X)=\int_0^2\int_0^2 x\cdot\frac{1}{8}(x+y)dxdy=\frac{1}{8}\int_0^2(\frac{8}{3}+2y)dy=\frac{7}{6}$，$E(X^2)=\int_0^2\int_0^2 x^2\frac{1}{8}(x+y)dxdy=\frac{1}{8}\int_0^2(4+\frac{8}{3}y)dy=\frac{5}{3}$

$E(Y)=\int_0^2\int_0^2 y\cdot\frac{1}{8}(x+y)dydx=\frac{1}{8}\int_0^2(\frac{8}{3}+2x)dx=\frac{7}{6}$，$E(Y^2)=\int_0^2\int_0^2 y^2\frac{1}{8}(x+y)dydx=\frac{1}{8}\int_0^2(4+\frac{8}{3}x)dx=\frac{5}{3}$

$E(XY)=\int_0^2\int_0^2 xy\cdot\frac{1}{8}(x+y)dydx=\frac{1}{8}\int_0^2 2x^2+\frac{8}{3}x\,dx=\frac{4}{3}$

$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{4}{3}-\frac{7}{6}\times\frac{7}{6}=-\frac{1}{36}$

$D(X)=E(X^2)-[E(X)]^2=\frac{5}{3}-(\frac{7}{6})^2=\frac{11}{36}$

$D(Y)=E(Y^2)-[E(Y)]^2=\frac{5}{3}-(\frac{7}{6})^2=\frac{11}{36}$

$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=\frac{(-\frac{1}{36})}{\sqrt{\frac{11}{36}}\cdot\sqrt{\frac{11}{36}}}=-\frac{1}{11}$

$D(X+Y)=D(X)+D(Y)+2E(X-E(X))(Y-E(Y))$
$=D(X)+D(Y)+2[E(XY)-E(X)E(Y)]$
$=\frac{11}{18}+2\times[\frac{4}{3}-(\frac{7}{6})^2]$
$=\frac{5}{9}$

2、$Cov(Z,W)=Cov(\alpha X+\beta Y,\alpha X-\beta Y)=Cov(\alpha X,\alpha X)+Cov(\alpha X,-\beta Y)+Cov(\beta Y,\alpha X)+Cov(\beta Y,-\beta Y)$
$=\alpha^2D(X)-\alpha\beta Cov(X,Y)+\alpha\beta Cov(Y,X)-\beta^2D(Y)$
$=(\alpha^2+\beta^2)\sigma^2$

$D(Z)=D(\alpha X+\beta Y)=\alpha^2D(X)+\beta^2D(Y)=(\alpha^2+\beta^2)\sigma^2$ （X,Y独立）

$D(W)=D(\alpha X-\beta Y)=\alpha^2D(X)+\beta^2D(Y)=(\alpha^2+\beta^2)\sigma^2$

$\rho_{ZW}=\frac{Cov(Z,W)}{\sqrt{D(Z)}\sqrt{D(W)}}=1$

3、(1) $E(X)=\int_{-\infty}^{+\infty}x\cdot\frac{1}{2}e^{-|x|}dx=\frac{1}{2}(\int_0^{+\infty}xe^{-x}dx+\int_{-\infty}^0xe^xdx)=\frac{1}{2}(1-1)=0$

$E(X^2)=\int_{-\infty}^{+\infty}x^2\cdot\frac{1}{2}e^{-|x|}dx=\frac{1}{2}(\int_0^{+\infty}x^2e^{-x}dx+\int_{-\infty}^0x^2e^xdx)=\frac{1}{2}(2+2)=2$

$D(X)=E(X^2)-[E(X)]^2=2$

(2)(3)

$E(|X|)=\int_{-\infty}^{+\infty}|x|\frac{1}{2}e^{-|x|}dx=\frac{1}{2}(\int_0^{+\infty}xe^{-x}dx-\int_{-\infty}^0xe^xdx)=\frac{1}{2}(1+1)=1$

$E(X|X|)=\int_{-\infty}^{+\infty}x|x|\frac{1}{2}e^{-|x|}dx=\frac{1}{2}(\int_0^{+\infty}x^2e^{-x}dx-\int_{-\infty}^0x^2e^xdx)=\frac{1}{2}(2-2)=0$

$Cov(X,|X|)=E(X|X|)-E(X)E(|X|)=0-0\cdot1=0$

故X与|X|不相关.

$\forall a\in R$且$a>0$，$0<P\{X\le a\}=\frac{1}{2}\int_{-\infty}^a e^{-|x|}dx<1$，有

$P\{X\le a,|X|\le a\}=P\{|X|\le a\}>P\{X\le a\}P\{|X|\le a\}$

故X与|X|不独立.

§4.4

67. $X \sim U(0,1)$, $Y|X=x \sim U(0,x)$

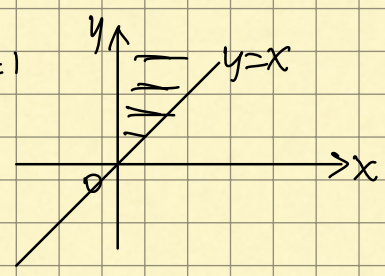
周长 $L=2(X+Y)$，面积 $S=XY$.

$E(L)=2E(X)+2E(Y)=1+2E(E(Y|X))=1+2E(\frac{X}{2})=\frac{3}{2}$

$E(S)=E(XY)=E(E(XY|X))=E(xE(Y|X))=E(x\cdot\frac{x}{2})=\frac{1}{2}E(x^2)=\frac{1}{6}$

77.

a. $E(X)=\int_0^{+\infty}dy\int_0^y xf(x,y)dx=\int_0^{+\infty}e^{-y}dy\int_0^y xdx=\frac{1}{2}\int_0^{+\infty}y^2e^{-y}dy=1$



$E(Y)=\int_0^{+\infty}dx\int_x^{+\infty}yf(x,y)dy=\int_0^{+\infty}dx\int_x^{+\infty}ye^{-y}dy=\int_0^{+\infty}(x+1)e^{-x}dx$

$=2$

$E(XY)=\int_0^{+\infty}xdx\int_x^{+\infty}ye^{-y}dy=\int_0^{+\infty}x(x+1)e^{-x}dx=3$

$Cov(X,Y)=E(XY)-E(X)E(Y)=3-2\times1=1$

$E(X^2)=\int_0^{+\infty}dy\int_0^y x^2f(x,y)dx=\int_0^{+\infty}e^{-y}dy\int_0^y x^2dx=\int_0^{+\infty}\frac{1}{3}y^3e^{-y}dy=2$

$E(Y^2)=\int_0^{+\infty}dx\int_x^{+\infty}y^2f(x,y)dy=\int_0^{+\infty}dx\int_x^{+\infty}y^2e^{-y}dy=\int_0^{+\infty}x^2e^{-x}+2xe^{-x}+2e^{-x}dx=6$

$D(X)=E(X^2)-[E(X)]^2=2-1=1$, $D(Y)=E(Y^2)-[E(Y)]^2=6-4=2$

$$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\cdot\sqrt{D(Y)}}=\frac{1}{1\times\sqrt{2}}=\frac{\sqrt{2}}{2}$$

b. $f(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\int_x^{+\infty}e^{-y}dy=e^{-x}$, $f(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\int_0^y e^{-y}dx=ye^{-y}$

$f_{Y|X}(y|x)=\frac{f(x,y)}{f(x)}=e^{x-y}$. $f_{X|Y}(x|y)=\frac{f(x,y)}{f(y)}=\frac{1}{y}$.

$E(X|Y=y)=\int_0^y x\cdot f_{X|Y}(x|y)dx=\int_0^y\frac{x}{y}dx=\frac{1}{2}y\quad(y>0)$

$E(Y|X=x)=\int_x^{+\infty}yf_{Y|X}(y|x)dy=\int_x^{+\infty}ye^{x-y}dy=x+1\quad(x>0)$.

c. 令 $U=E(X|Y)$, $V=E(Y|X)$

又 $\begin{cases}E(X|Y=y)=\frac{1}{2}y\ (y>0)\\E(Y|X=x)=1+x\ (x>0)\end{cases}$，则 $\begin{cases}U=\frac{1}{2}Y\\V=1+X\end{cases}$

故 $f_U(u)=\begin{cases}4u\cdot e^{-2u}, & u>0\\0, & u\le0\end{cases}$, $f_V(v)=\begin{cases}e^{1-v}, & v>1\\0, & v\le1\end{cases}$