§ 2.3

54. $X\sim N(0,\sigma^2)$, $Y=|X|=\begin{cases}-x, & x<0\\ x, & x\geq 0.\end{cases}$

反函数 $h_1(y)=-y$, $h_2(y)=y$. 导数 $h_1'(y)=-1$, $h_2'(y)=1$

Y的密度为

$$f_Y(y)=\sum_{i=1}^{2}|h_i'(y)|f_X(h_i(y))$$

$$=f_X(y)+f_X(-y)$$

$\because X\sim N(0,\sigma^2)$ $\therefore f_X(x)$ 关于 $x=0$ 对称，且 $f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$

$\therefore f_Y(y)=\frac{\sqrt{2}}{\sqrt{\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}$ $\therefore f_Y(y)=\begin{cases}0, & y<0\\ \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sigma}e^{-\frac{y^2}{2\sigma^2}}, & y\geq 0\end{cases}$

59. $U\sim U(-1,1)$, $Y=U^2$.

$f_U(u)=\begin{cases}\frac{1}{2}, & -1\leq u\leq 1\\ 0, & 其它\end{cases}$

$u\in[0,1]$, $P(Y\leq u)=P(-\sqrt{u}\leq U\leq\sqrt{u})=\sqrt{u}$

因此，$f_Y(y)=\begin{cases}\frac{1}{2\sqrt{y}}, & 0<y<1\\ 0, & 其它\end{cases}$

64. 证：$\because Y=aX+b$ $\therefore X=\frac{Y-b}{a}$, $X'=\frac{1}{a}$

$$f_Y(y)=|h'(y)|f_X(h(y))=\frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$

补充题：

1. $\because X=-2,-1,0,1,2$, $\therefore Y=X^2=0,1,4.$

故Y的频率函数为

Y	0	1	4
P(Y)	$\frac{1}{5}$	$\frac{2}{30}$	$\frac{17}{30}$

2. $Y=\sin X\Rightarrow x=\begin{cases}h_1(y)=\arcsin y.\\ h_2(y)=\pi-\arcsin y\end{cases}$ $\Rightarrow\begin{cases}h_1'(y)=\frac{1}{\sqrt{1-y^2}}\\ h_2'(y)=\frac{-1}{\sqrt{1-y^2}}\end{cases}$

$f_X(x)=\begin{cases}\frac{2x}{\pi^2}, & 0<x<\pi\\ 0, & 其它\end{cases}$

Y的概率密度为 $f_Y(y)=\sum_{i=1}^{2}|h_i'(y)|f_X(h_i(y))=\begin{cases}\frac{2}{\pi\sqrt{1-y^2}}, & 0<y<1\\ 0, & 其它\end{cases}$

3. $Y=\begin{cases}1, & x=2m\\ -1, & x=2m+1\end{cases}$ $(m\in Z)$, $P(X=k)=\left(\frac{1}{2}\right)^k$, $k=1,2,\cdots$.

$P_1=P\{Y=-1\}=\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{1}{2}\times\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right]=\frac{1}{2}\times(1+P_2)$

$P_2=P\{Y=1\}=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+\cdots=\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots$

即 $P_1=\frac{1}{2}\times(1+P_2)$ 又 $P_1+P_2=1$，故 $P_1=P\{Y=-1\}=\frac{2}{3}$, $P_2=P\{Y=1\}=\frac{1}{3}$

4. $X \sim U(1,2)$，则 X 的密度为 $f_X(x)=\begin{cases}1, & 1<x<2\\ 0, & \text{otherwise}\end{cases}$

$Y=e^{2X}>0$，则 $X=h(y)=\frac{1}{2}\ln y$，$h'(y)=\frac{1}{2y}$.

令 $1<\frac{1}{2}\ln y<2$，则 $y\in(e^2,e^4)$

$$f_Y(y)=|h'(y)|f_X(h(y))=\begin{cases}\frac{1}{2y}, & e^2<y<e^4\\ 0, & \text{otherwise}\end{cases}$$

§3.2

3. 由多项分布得：

$$P\{X_1=i, X_2=j, X_3=k\}=\frac{10!}{i!j!k!}\left(\frac{1}{3}\right)^{10} \quad (i+j+k=10)$$

补充题：

$P\{X=0, Y=3\}=P\{Y=3|X=0\}\cdot P\{X=0\}=1\cdot C_3^0\left(\frac{1}{2}\right)^3=\frac{1}{8}$

$P\{X=1, Y=1\}=P\{Y=1|X=1\}\cdot P\{X=1\}=1\cdot C_3^1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2=\frac{3}{8}$

$P\{X=2, Y=1\}=P\{Y=1|X=2\}\cdot P\{X=2\}=1\cdot C_3^2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)=\frac{3}{8}$

$P\{X=3, Y=3\}=P\{Y=3|X=3\}\cdot P\{X=3\}=1\cdot C_3^3\left(\frac{1}{2}\right)^3=\frac{1}{8}$

其余组合概率为0. X,Y的联合频率函数为

Y \ X	0	1	2	3	P(Y)
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{3}{4}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{1}{4}$
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	1