§6.3

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3、$X_i\ (i=1,\cdots,16)$ 为独立同分布于 $N(0,1)$ 的随机变量.

$\bar{X}=\frac{1}{16}\sum_{i=1}^{16}X_i\sim N(0,\frac{1}{16})$

$\because P(|\bar{X}|<C)=0.5$

$\therefore P(\frac{\bar{X}}{(\frac{1}{4})}>4C)=\frac{1}{4}$，即 $1-\Phi(4C)=\frac{1}{4}\Rightarrow\Phi(4C)=\frac{3}{4}\Rightarrow C=0.17$

6. $\because T\sim t_n$ $\therefore T=\frac{X}{\sqrt{Y/n}}$，其中 $X\sim N(0,1)$, $Y\sim\chi^2(n)$, X,Y 独立.

$\therefore T^2=\frac{X^2}{Y/n}$

又 $X^2\sim\chi^2(1)$ $\therefore T^2\sim F(1,n)$

8. $X\sim EXP(1)$, $Y\sim EXP(1)$, $Z=\frac{X}{Y}$

故 $f_Z(z)=\begin{cases}\frac{1}{(1+z)^2}, & z>0\\ 0, & z\le 0.\end{cases}$

由于F-分布的密度函数如下：$f(x)=\begin{cases}\frac{\Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)}n_1^{n_1/2}n_2^{n_2/2}\frac{x^{\frac{n_1}{2}-1}}{(n_1x+n_2)^{\frac{n_1+n_2}{2}}}, & x>0\\ 0, & x\le 0\end{cases}$

取 $n_1=n_2=2$ 时，F-分布的密度函数形式等同于 $f_Z(z)$

故 $Z=\frac{X}{Y}\sim F(2,2)$

补充题：

1. $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ $\because X_i\sim N(\mu,16)$ $\therefore\bar{X}\sim N(\mu,\frac{16}{n})$

由切比雪夫不等式：$P(|\bar{X}-\mu|<1)=1-P(|\bar{X}-\mu|\ge 1)\ge 1-\frac{16}{n}$

又 $P(|\bar{X}-\mu|<1)\ge 0.95$

故有 $1-\frac{16}{n}\ge 0.95$，即 $n\ge 320$

故当n至少为320时，不等式才成立.

2. $\bar{X}=\frac{1}{36}\sum_{i=1}^{36}X_i\sim(240,(\frac{20}{6})^2)$, $\bar{Y}=\frac{1}{49}\sum_{i=1}^{49}Y_i\sim N(240,(\frac{20}{7})^2)$

$\bar{X}-\bar{Y}\sim N(0,(\frac{20}{6})^2+(\frac{20}{7})^2)$

$P(|\bar{X}-\bar{Y}|\le 10)=1-2P(\bar{X}-\bar{Y}<-10)$

$=1-2P\{\frac{(\bar{X}-\bar{Y})}{\sqrt{(\frac{20}{6})^2+(\frac{20}{7})^2}}<\frac{-10}{\sqrt{(\frac{20}{6})^2+(\frac{20}{7})^2}}\}$

$=1-2\Phi(-\frac{21}{\sqrt{85}})=1-2(1-\Phi(\frac{21}{\sqrt{85}}))=2\Phi(\frac{21}{\sqrt{85}})-1\approx 0.9774$

3. $X_i\sim N(0,0.3^2)\ (i=1,\cdots,10)$，则 $\frac{X_i}{0.3}\sim N(0,1)\ (i=1,\cdots,10)$.

$P\{\sum_{i=1}^{10}X_i^2\le C\}=P\{\sum_{i=1}^{10}\frac{X_i^2}{0.3^2}\le\frac{C}{0.3^2}\}=0.95\Rightarrow\frac{C}{0.3^2}=\chi^2_{0.95}(10)\Rightarrow C=1.6479$

4. $X_1 \sim N(0,\sigma^2)$, $X_2 \sim N(0,\sigma^2)$

(1) 令 $\begin{cases} U = X_1 - X_2 \\ V = X_1 + X_2 \end{cases}$，则 $\begin{cases} X_1 = \frac{1}{2}(V+U) \\ X_2 = \frac{1}{2}(V-U) \end{cases}$

$f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2} \exp\left[\left(-\frac{u^2}{4\sigma^2}\right) + \left(-\frac{v^2}{4\sigma^2}\right)\right]$

故U, V独立. $X_1 - X_2 \sim N(0, 2\sigma^2)$，$X_1 + X_2 \sim (0, 2\sigma^2)$

$\frac{(X_1-X_2)^2}{(X_1+X_2)^2} = \frac{\left(\frac{X_1-X_2}{\sqrt{2}\sigma}\right)^2}{\left(\frac{X_1+X_2}{\sqrt{2}\sigma}\right)^2} \sim F(1,1)$

(2) $P\left\{\frac{(X_1+X_2)^2}{(X_1+X_2)^2+(X_1-X_2)^2} > k\right\} = P\left\{\frac{(X_1+X_2)^2+(X_1-X_2)^2}{(X_1+X_2)^2} < \frac{1}{k}\right\} = P\left\{\frac{(X_1-X_2)^2}{(X_1+X_2)^2} < \frac{1}{k} - 1\right\} = 0.10$

故 $\frac{1}{k} - 1 = F_{0.1}(1,1)$

$\therefore k \approx 0.9755$

5. S_n与$\bar{X}_n$和X_{n+1}都独立，故S_n与$X_{n+1} - \bar{X}_n$独立.

$\because \frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$，$X_{n+1} - \bar{X}_n \sim N(0, \frac{n+1}{n}\sigma^2)$

$\therefore c\frac{\sqrt{\frac{n+1}{n}\sigma^2}}{\sqrt{\sigma^2}} = 1$ $\quad \therefore c = \sqrt{\frac{n}{n+1}}$

$\therefore t_c \sim t(n-1)$ 自由度为$n-1$.