

Probability and Statistics

Tutorial 9

Siyi Wang

Southern University of Science and Technology

11951002@mail.sustech.edu.cn

November 18, 2020

Outline

1 Review

2 Homework

3 Supplement Exercises

Review

1. Expectation

- Discrete Case:

- (DEF) $EX = \sum_{i=1}^{\infty} n_i P(X = n_i)$.
- (Property) $Eh(X) = \sum_{i=1}^{\infty} h(n_i)P(X = n_i)$.

- Continuous Case:

- (DEF) $EX = \int_{-\infty}^{\infty} xf_X(x)dx$.
- (Property) $Eh(X) = \int_{-\infty}^{\infty} h(x)f_X(x)dx$.
- (2-dim case) $E(h(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)f_{X,Y}(x, y)dxdy$

- Properties

- (Linearity) $E(aX + bY) = aE(X) + bE(Y)$.
- $E(aX + b) = aE(X) + b$.
- If $X \geq Y$, then $E(X) \geq E(Y)$.
- If $b \geq X \geq a$, then $b \geq E(X) \geq a$.
- If X and Y are independent, then $E(XY) = E(X)E(Y)$. (Warning: the converse is not true in general)
- If $A \subset \Omega$, then $E(1_A) = P(A)$.
- If $A \subset \mathbb{R}$, then $E(1_A(X)) = P(X \in A)$.

Review

2. Variance

- (DEF) $\text{Var}(X) = E(X - E(X))^2$
- $\text{Var}(X) = E(X^2) - (E(X))^2$
- $E(X^2) \geq (E(X))^2$ that is, $\text{Var}(X) \geq 0$.
- $\text{Var}(aX + b) = a^2 \text{Var}(X)$.
- $\text{Var}(X) = 0$ if and only if $X = c$ a.e.
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$.
- If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.
- $E(X) = \min_c \{E[(X - c)^2]\}$.

3. Several Inequalities

- $|E(X)| \leq E|X|$.
- $(E(X))^2 \leq E(X^2)$.
- For any nonnegative random variable X , we have $P(X \geq t) \leq \frac{E(X)}{t}$.
- (The above implies) for any r.v. X , we have $P(|X| \geq t) \leq \frac{E(|X|)}{t}$.
- For any random variable X , we have $P(|X - E(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}$.
- $\text{Var}(X) = E(X - EX)^2 \leq E(X - c)^2$ for any c .

Homework

6. 令 X 是连续型随机变量, 具有概率密度函数为 $f(x) = 2x, 0 \leq x \leq 1$.
- a. 计算 $E(X)$.
 - b. 令 $Y = X^2$, 计算 Y 的概率质量函数, 并由其计算 $E(Y)$.
 - c. 利用 4.1.1 节的定理 4.1.1.1 计算 $E(X^2)$, 并与 b 中的答案进行比较.
 - d. 根据 4.2 节方差的定义计算 $\text{Var}(x)$, 同时利用 4.2 节的定理 4.2.2 计算 $\text{Var}(x)$.

Homework

Solution

6. Solution.

$$(1) \mathbb{E}(X) = \int_0^1 x \cdot 2x dx = \frac{2}{3}.$$

$$(2) f_Y(y) = \begin{cases} |(\sqrt{y})'| \cdot 2\sqrt{y} = 1, & y \in (0,1) \\ 0, & \text{other} \end{cases}$$

$$\mathbb{E}(Y) = \frac{1}{2}.$$

$$(3) \mathbb{E}(X^2) = \int_0^1 x^2 \cdot 2x dx = \frac{1}{2}.$$

$$(4) \text{Var}(X) = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}.$$

Homework

D. 期望收入 $E(X)$ 与奖金 A .

15. 假设有两种彩票，每种彩票有 n 个可能的数字和相同的奖金。根据期望收入判断下列购买方式哪种更好：
从一种彩票中购买两张，或从两种彩票中各买一张？

Homework

Solution

15. Solution. Let $m =$ 奖金的数额.

$X_i =$ earning of i -th plan, $i=1,2$.

$$\mathbb{E}(X_1) = m \cdot \frac{n-1}{C_n^2} = \frac{2m}{n}$$

$$\begin{aligned}\mathbb{E}(X_2) &= m \cdot \frac{1}{n} \cdot \frac{n-1}{n} + m \cdot \frac{n-1}{n} \cdot \frac{1}{n} + 2m \cdot \frac{1}{n^2} \\ &= \frac{2m(n-1) + 2m}{n^2} = \frac{2m}{n}\end{aligned}$$

Then, $\mathbb{E}(X_1) = \mathbb{E}(X_2)$.

Homework

20. 计算 $E[1/(X + 1)]$, 其中 X 是泊松随机变量.

Homework

Solution

20. Solution.

$$\begin{aligned} \mathbb{E}\left(\frac{1}{X+1}\right) &= \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\ &= \frac{1}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k+1}}{(k+1)!} \\ &= \frac{1}{\lambda} (1 - e^{-\lambda}). \end{aligned}$$

Homework

21. 随机正方形的边长是 $[0, 1]$ 上的均匀随机变量. 计算正方形的期望面积.

Homework

Solution

21. Solution. $X \sim U(0,1)$

$$E(S) = E(X^2) = \int_0^1 x^2 dx = \frac{1}{3}.$$

Homework

31. 令 X 均匀分布在区间 $[1, 2]$ 上, 计算 $E(1/X) \cdot E(1/X) = 1/E(X)$ 吗?

Homework

Solution

31. Solution. $\mathbb{E} X = \frac{3}{2}$.

$$\mathbb{E}\left(\frac{1}{X}\right) = \int_1^2 \frac{1}{x} dx = \ln 2 \neq \frac{1}{\mathbb{E} X}.$$

Homework

1. 设随机变量 X 的概率密度为

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

求 (1) $Y = 2X$; (2) $Y = e^{-2X}$ 的数学期望.

Homework

Solution

Solution.

$$(1) \mathbb{E} Y = \mathbb{E}(2X) = 2\mathbb{E} X = 2$$

$$(2) \mathbb{E} Y = \mathbb{E}(e^{-2X}) = \int_0^{\infty} e^{-2x} e^{-x} dx \\ = \frac{1}{3}.$$

Homework

2. 设随机变量 (X, Y) 的概率密度为

$$f(x, y) = \begin{cases} 12y^2, & 0 \leq y \leq x \leq 1, \\ 0, & \text{其他.} \end{cases}$$

求 $E(X), E(Y), E(XY), E(X^2 + Y^2)$.

Homework

Solution

Solution.



$$\textcircled{1} \quad \mathbb{E}X = \int_0^1 \int_0^x x \cdot 12y^2 dy dx = \frac{4}{5}$$

$$\textcircled{2} \quad \mathbb{E}Y = \int_0^1 \int_0^x y \cdot 12y^2 dy dx = \frac{3}{5}$$

$$\textcircled{3} \quad \mathbb{E}(XY) = \int_0^1 \int_0^x (xy) \cdot 12y^2 dy dx = \frac{1}{2}$$

$$\textcircled{4} \quad \mathbb{E}(X^2 + Y^2) = \int_0^1 \int_0^x (x^2 + y^2) 12y^2 dy dx = \frac{16}{15}$$

Homework

49. X 和 Y 是取自数量 μ 的两个独立测量值. $E(X) = E(Y) = \mu$, σ_X 和 σ_Y 不相等. 利用加权平均组合两个测度值, 即

$$Z = \alpha X + (1 - \alpha)Y$$

其中 α 是个标量, 且 $0 \leq \alpha \leq 1$.

- 证明 $E(Z) = \mu$.
- 根据 σ_X 和 σ_Y , 寻找最小化 $\text{Var}(Z)$ 的 α .
- 在何种情况下, 使用平均数 $(X + Y)/2$ 优于单独使用 X 或 Y ?

n

Homework

Solution

49. Solution.

a. $E(Z) = \alpha E(X) + (1-\alpha)E(Y) = \mu.$

b. $\text{Var}(Z) = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2$
 $= (\sigma_X^2 + \sigma_Y^2)\alpha^2 - 2\sigma_Y^2\alpha + \sigma_Y^2$
 $\alpha_{\min} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$

c. When $\alpha = \frac{1}{2}$, we have

$$\text{Var}(Z) = \frac{1}{4}(\sigma_X^2 + \sigma_Y^2).$$

If $\text{Var}(Z) \leq \min(\sigma_X^2, \sigma_Y^2)$, then

$$\begin{cases} \sigma_Y^2 \leq 3\sigma_X^2 \\ \sigma_X^2 \leq 3\sigma_Y^2 \end{cases}.$$

Homework

50. 假设 X_i ($i = 1, \dots, n$)，是独立的随机变量，具有 $E(X_i) = \mu$, $\text{Var}(X_i) = \sigma^2$. 令 $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. 证明 $E(\bar{X}) = \mu$, $\text{Var}(\bar{X}) = \sigma^2/n$

Homework

Solution

50. Solution.

$$\textcircled{1} \quad E(\bar{X}) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \mu.$$

$$\textcircled{2} \quad \text{Var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{\sigma^2}{n}.$$

Homework

55. 令 $T = \sum_{k=1}^n kX_k$, 其中 X_k 为独立的随机变量, 具有均值 μ , 方差 σ^2 . 计算 $E(T)$ 和 $\text{Var}(T)$.

Homework

Solution

55. Solution.

$$\textcircled{1} \quad E(T) = \sum_{k=1}^n k E(X_k) = \mu \sum_{k=1}^n k = \frac{1}{2} n(n+1)\mu.$$

$$\begin{aligned}\textcircled{2} \quad Var(T) &= \sum_{k=1}^n k^2 Var(X_k) = \sigma^2 \sum_{k=1}^n k^2 \\ &= \frac{1}{6} (2n+1)(n+1)n\sigma^2.\end{aligned}$$

Homework

1. 设 X, Y 是互相独立的随机变量，且有 $E(X) = 3, E(Y) = 1, D(X) = 4, D(Y) = 9$. 令 $Z = 5X - 2Y + 15$, 求 $E(Z)$ 和 $D(Z)$.

Homework

Solution

Solution.

$$\begin{aligned} \textcircled{1} \quad \mathbb{E} Z &= 5\mathbb{E} X - 2\mathbb{E} Y + 15 = 28 \\ \textcircled{2} \quad \text{Var}(Z) &= 25\text{Var}(X) + 4\text{Var}(Y) = 136. \end{aligned}$$

Homework

2. 设随机变量 X_1, X_2, X_3, X_4 互相独立, 且有 $E(X_i) = 2i$, $D(X_i) = 5 - i$, 其中 $i = 1, 2, 3, 4$. 令 $Z = 2X_1 - X_2 + 3X_3 - \frac{1}{2}X_4$, 求 $E(Z)$ 和 $D(Z)$.

Homework

Solution

Solution.

$$\begin{aligned} \textcircled{1} \quad \mathbb{E}(Z) &= 2\mathbb{E}X_1 - \mathbb{E}X_2 + 3\mathbb{E}X_3 - \frac{1}{2}\mathbb{E}X_4 \\ &= 14 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad D(Z) &= 4D(X_1) + D(X_2) + 9D(X_3) + \frac{1}{4}D(X_4) \\ &= \frac{149}{4}. \end{aligned}$$

Exercise 1

8. 证明：如果 X 为离散型随机变量，且取值正整数，那么 $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$. 利用此结论计算几何随机变量的期望值.

Supplement Exercises

Solution

$$\begin{aligned}\text{Proof. RHS} &= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \mathbb{P}(X=j) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^j \mathbb{P}(X=j) = \sum_{j=1}^{\infty} j \mathbb{P}(X=j) \\ &= \mathbb{E} X.\end{aligned}$$

Exercise 2

13. 如果 X 是非负的连续型随机变量, 证明:

$$E(X) = \int_0^{\infty} [1 - F(x)]dx$$

应用这个结论计算指数分布的均值.

Supplement Exercises

Solution

$$\begin{aligned}\text{Proof. RHS} &= \int_0^\infty P(X > x) dx \\ &= \int_0^\infty \int_x^\infty f_X(u) du dx \\ &= \int_0^\infty \int_0^u f_X(u) dx du = \int_0^\infty u f_X(u) du \\ &= \mathbb{E} X. \square\end{aligned}$$

Supplement Exercises

Exercise 3

16. 如果 X 是连续型随机变量, 密度函数关于某个点 ξ 对称, 证明: 只要 $E(X)$ 存在, 就有 $E(X) = \xi$.

Supplement Exercises

Solution

1b. Proof. $E(X) = \int_{-\infty}^{+\infty} xf_X(x) dx$

$y = x - \frac{1}{2}$ $\int_{-\infty}^{+\infty} (y + \frac{1}{2}) f_X(y + \frac{1}{2}) dy$

$= \frac{1}{2} \int_{-\infty}^{+\infty} f_X(y + \frac{1}{2}) dy \quad (f_X(y + \frac{1}{2}) = f_X(-y + \frac{1}{2}))$

$= \frac{1}{2} \int_{-\infty}^{+\infty} f_X(x) dx = \frac{1}{2}$. \square

Exercise 4

9. 试证: 对任意的常数 $c \neq E(X)$, 有

$$\text{Var}(X) = E(X - E(X))^2 < E(X - c)^2.$$

Supplement Exercises

Solution

证

$$E(X - E(X))^2 = E[(X - c) - (E(X) - c)]^2 = E(X - c)^2 - (E(X) - c)^2,$$

由于 $c \neq E(X)$, 所以 $(E(X) - c)^2 > 0$, 由此得

$$\text{Var}(X) = E(X - E(X))^2 < E(X - c)^2.$$

Exercise 5

12. 从一个装有 m 个白球、 n 个黑球的袋中进行有返回地摸球，直到摸到白球时停止。试求取出黑球数的期望。

Supplement Exercises

Solution

解 令 X 为取到白球时已取出的黑球数, 则 $Y = X + 1$ 服从几何分布 $Ge(m/(n+m))$, 所以 $E(Y) = (n+m)/m = n/m + 1$, 由此得 $E(X) = E(Y) - 1 = n/m$.

Exercise 6

20. 设随机变量 $X \sim b(n, p)$, 试证明:

$$E\left(\frac{1}{X+1}\right) = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$

Supplement Exercises

Solution

证

$$\begin{aligned} E\left(\frac{1}{X+1}\right) &= \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \frac{n!}{(k+1)! (n-k)!} p^k (1-p)^{n-k} \\ &= \frac{1}{(n+1)p} \sum_{k=0}^n \frac{(n+1)!}{(k+1)! [n+1-(k+1)]!} p^{k+1} (1-p)^{n+1-(k+1)} \\ (\text{令 } k' = k + 1) &= \frac{1}{(n+1)p} \sum_{k'=1}^{n+1} \binom{n+1}{k'} p^{k'} (1-p)^{n+1-k'} \\ &= \frac{1}{(n+1)p} [1 - (1-p)^{n+1}]. \end{aligned}$$

Exercise 7

21. 掷一枚不均匀硬币，一直掷到正、反面都出现为止. 记出现正面的概率为 $p(0 < p < 1)$ ，试求平均抛掷次数.

Supplement Exercises

Solution

解 记 X 为直到正、反面都出现时的抛掷次数，则 X 可取值 $2, 3, \dots$ ，且有

$$P(X = k) = (1 - p)^{k-1}p + p^{k-1}(1 - p), \quad k = 2, 3, \dots.$$

可以验证：这是一个分布列。由此得 X 的数学期望为

$$\begin{aligned} E(X) &= \sum_{k=2}^{+\infty} k[(1 - p)^{k-1}p + p^{k-1}(1 - p)] \\ &= \sum_{k=1}^{+\infty} k(1 - p)^{k-1}p - p + \sum_{k=1}^{+\infty} kp^{k-1}(1 - p) - (1 - p) \\ &= \frac{1}{p} + \frac{1}{1 - p} - 1 = \frac{1}{p(1 - p)} - 1. \end{aligned}$$

Exercise 8

30. 设随机变量 $X \sim N(\mu, \sigma^2)$, 求 $E|X - \mu|$.

Supplement Exercises

Solution

解 利用变换 $t = (x - \mu) / \sigma$ 及偶函数性质可得

$$E |X - \mu| = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} |x - \mu| \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} dx$$

4

第二章 随机变量及其分布

$$= \sigma \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \exp\left\{-\frac{t^2}{2}\right\} d\left(\frac{t^2}{2}\right) = \sigma \sqrt{\frac{2}{\pi}}.$$

Exercise 9

2. 求掷 n 颗骰子出现点数之和的数学期望与方差.

Supplement Exercises

Solution

解 记 X_i 为第 i 颗骰子出现的点数, $i = 1, 2, \dots, n$. 则 X_1, X_2, \dots, X_n 独立同分布, 其共同的分布列为

X_i	1	2	3	4	5	6
P	1/6	1/6	1/6	1/6	1/6	1/6

所以

$$E(X_i) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2};$$

$$\text{Var}(X_i) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - \frac{49}{4} = \frac{35}{12}.$$

由此得

$$E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i) = \frac{7n}{2}; \quad \text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) = \frac{35n}{12}.$$

Exercise 10

3. 从数字 $0, 1, \dots, n$ 中任取两个不同的数字, 求这两个数字之差的绝对值的数学期望.

Supplement Exercises

Solution

解 记 X 与 Y 分别为第 1 次和第 2 次取出的数字，则

$$P(X = i, Y = j) = \frac{1}{(n+1)n}, \quad i, j = 0, 1, \dots, n, \quad i \neq j.$$

所以

$$\begin{aligned} E|X - Y| &= \frac{1}{(n+1)n} \sum_{i=0}^n \left\{ \sum_{j=0}^i (i-j) + \sum_{j=i+1}^n (j-i) \right\} \\ &= \frac{1}{(n+1)n} \sum_{i=0}^n \left\{ \frac{i(i+1)}{2} + \frac{(n-i)[(n-i)+1]}{2} \right\} \\ &= \frac{1}{(n+1)n} \sum_{i=0}^n \left\{ i^2 + \frac{n(n+1)}{2} - in \right\} \\ &= \frac{2n+1}{6} + \frac{n+1}{2} - \frac{n}{2} = \frac{n+2}{3}. \end{aligned}$$

Exercise 11

4. 设在区间 $(0,1)$ 上随机地取 n 个点, 求相距最远的两点间的距离的数学期望.

Supplement Exercises

Solution

解 解法一：分别记此 n 个点为 X_1, X_2, \dots, X_n ，则 X_1, X_2, \dots, X_n 相互独立，且都服从区间 $(0, 1)$ 上的均匀分布 $U(0, 1)$ 。我们的目的是求

$$E(\max\{X_1, X_2, \dots, X_n\} - \min\{X_1, X_2, \dots, X_n\}).$$

而 $Z = \max\{X_1, X_2, \dots, X_n\}$ 和 $T = \min\{X_1, X_2, \dots, X_n\}$ 的密度函数分别为

$$p_Z(z) = \begin{cases} nz^{n-1}, & 0 < z < 1, \\ 0, & \text{其他.} \end{cases} \quad p_T(t) = \begin{cases} n(1-t)^{n-1}, & 0 < t < 1, \\ 0, & \text{其他.} \end{cases}$$

又因为

$$E(Z) = \int_0^1 z nz^{n-1} dz = \frac{n}{n+1}; \quad E(T) = \int_0^1 tn(1-t)^{n-1} dt = \frac{1}{n+1},$$

所以

$$E(\max\{X_1, X_2, \dots, X_n\} - \min\{X_1, X_2, \dots, X_n\}) = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$$

Supplement Exercises

Solution

解法二： n 个点把区间 $(0,1)$ 分成 $n+1$ 段，它们的长度依次记为 Y_1, Y_2, \dots, Y_{n+1} 。因为此 n 个点是随机取的，所以 Y_1, Y_2, \dots, Y_{n+1} 具有相同的分布，从而有相同的数学期望。而 $Y_1 + Y_2 + \dots + Y_{n+1} = 1$ ，因此

$$E(Y_1) = E(Y_2) = \dots = E(Y_{n+1}) = \frac{1}{n+1}.$$

而相距最远的两点间的距离为 $Y_2 + Y_3 + \dots + Y_n$ ，因此所求期望为

$$E(Y_2 + Y_3 + \dots + Y_n) = \frac{n-1}{n+1}.$$

Supplement Exercises

Exercise 12

6. 设随机变量(X, Y)的联合分布列为

		Y	0	1
X	0	0.1	0.15	
	1	0.25	0.2	
		2	0.15	0.15

Supplement Exercises

Solution

解

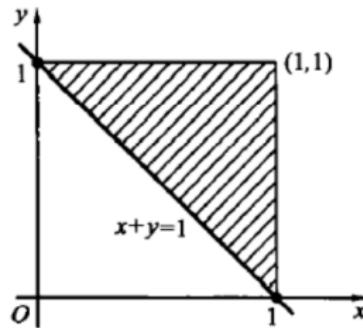
$$\begin{aligned}E(Z) &= 0.1\sin 0 + 0.15\sin \frac{\pi}{2} + 0.25\sin \frac{\pi}{2} + 0.2\sin \pi + 0.15\sin \pi + 0.15\sin \frac{3\pi}{2} \\&= 0.15 \times 1 + 0.25 \times 1 + 0.15 \times (-1) = 0.25.\end{aligned}$$

Supplement Exercises

Exercise 13

7. 随机变量 (X, Y) 服从以点 $(0,1), (1,0), (1,1)$ 为顶点的三角形区域上的均匀分布, 试求 $E(X + Y)$ 和 $\text{Var}(X + Y)$.

解 记此三角形区域为 D (如图 3.15 阴影部分).



Supplement Exercises

Solution

由于 X 与 Y 不独立, 所以先计算

$$E(XY) = \int_0^1 \int_{1-x}^1 2xy dy dx = \frac{5}{12}.$$

由此得

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{5}{12} - \frac{4}{9} = -\frac{1}{36}, \quad (\text{负相关})$$

最后得

$$E(X + Y) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3},$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = \frac{1}{18} + \frac{1}{18} - \frac{2}{36} = \frac{1}{18}.$$

Exercise 14

8. 设 X, Y 均为 $(0, 1)$ 上独立的均匀随机变量, 试证:

$$E(|X - Y|^\alpha) = \frac{2}{(\alpha + 1)(\alpha + 2)}, \alpha > 0.$$

Supplement Exercises

Solution

证 因为 (X, Y) 的联合密度函数为

$$p(x, y) = \begin{cases} 1, & 0 < x, y < 1, \\ 0 & \text{其他.} \end{cases}$$

所以

$$\begin{aligned} E|X - Y|^\alpha &= \int_0^1 \int_0^1 |x - y|^\alpha dx dy \\ &= \int_0^1 \int_0^y (y - x)^\alpha dx dy + \int_0^1 \int_y^1 (x - y)^\alpha dx dy \\ &= \frac{1}{(\alpha + 1)(\alpha + 2)} + \frac{1}{(\alpha + 1)(\alpha + 2)} \\ &= \frac{2}{(\alpha + 1)(\alpha + 2)}. \end{aligned}$$

Supplement Exercises

Exercise 15

EXAMPLE 1.3(A) *The Matching Problem.* At a party n people put their hats in the center of a room where the hats are mixed together. Each person then randomly selects one. We are interested in the mean and variance of X —the number that select their own hat.

Supplement Exercises

Solution

mean and variance of X — the number that select their own hat

To solve, we use the representation

$$X = X_1 + X_2 + \dots + X_n,$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person selects his or her own hat} \\ 0 & \text{otherwise} \end{cases}$$

Now, as the i th person is equally likely to select any of the n hats, it follows that $P\{X_i = 1\} = 1/n$, and so

$$E[X_i] = 1/n,$$

$$\text{Var}(X_i) = \frac{1}{n} \left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}$$

Supplement Exercises

Solution

Also

$$\text{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j].$$

Now,

$$X_i X_j = \begin{cases} 1 & \text{if the } i\text{th and } j\text{th party goers both select their own hats} \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$\begin{aligned} E[X_i X_j] &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\} \\ &= \frac{1}{n} \frac{1}{n-1}. \end{aligned}$$

Supplement Exercises

Solution

Hence,

$$\text{Cov}(X_i, X_j) = \frac{1}{n(n-1)} - \left(\frac{1}{n}\right)^2 = \frac{1}{n^2(n-1)}$$

Therefore, from (1.3.3) and (1.3.4),

$$E[X] = 1$$

and

$$\begin{aligned}\text{Var}(X) &= \frac{n-1}{n} + 2 \binom{n}{2} \frac{1}{n^2(n-1)} \\ &= 1.\end{aligned}$$

Exercise 16

26. 单位棒断裂成两段. 计算较长一段与较短一段长度的期望比.

Supplement Exercises

Solution

Solution. $\underline{x, 1-x}$, $X \sim U(0,1)$

$$Y = \max\{X, 1-X\}$$

$$P(Y \leq y) = \begin{cases} 0, & y \leq \frac{1}{2} \\ P(Y \in [1-y, y]) = 2y-1, & y \in (\frac{1}{2}, 1) \\ 1, & y \geq 1 \end{cases}$$

$$\text{Then, } f_Y(y) = \begin{cases} 2, & y \in (\frac{1}{2}, 1) \\ 0, & \text{other.} \end{cases}$$

$$\mathbb{E} Y = \frac{3}{4}. \quad \mathbb{E}[1-Y] = \frac{1}{4}.$$

$$\text{Hence, } \frac{\mathbb{E} Y}{\mathbb{E}[1-Y]} = 3.$$

Thank you!