§3.5

1. a.

Y \ X	1	2	3	4	P(Y)
1	0.10	0.05	0.02	0.02	0.19
2	0.05	0.20	0.05	0.02	0.32
3	0.02	0.05	0.20	0.04	0.31
4	0.02	0.02	0.04	0.10	0.18
P(X)	0.19	0.32	0.31	0.18	1

b.

X	1	2	3	4
P(X|Y=1)	0.53	0.26	0.11	0.11

Y	1	2	3	4
P(Y|X=1)	0.53	0.26	0.11	0.11

8. a. (i) $P(X>Y)=\iint_{X>Y} f_{XY}(x,y)\,dx\,dy=\int_0^1\int_y^1 \frac{6}{7}(x+y)^2\,dx\,dy=\frac{6}{7}\int_0^1 \frac{1}{3}+y+y^2-\frac{7}{3}y^3\,dy$

$=\frac{1}{2}$

(ii) $P(X+Y\le 1)=\iint_{X+Y\le 1} f_{XY}(x,y)\,dx\,dy=\int_0^1\int_0^{1-y}\frac{6}{7}(x+y)^2\,dx\,dy=\int_0^1$

(iii) $P(X\le\frac{1}{2})=\iint_{X\le\frac{1}{2}} f_{XY}(x,y)\,dx\,dy=\int_0^1\int_0^{\frac{1}{2}}\frac{6}{7}(x+y)^2\,dx\,dy=\frac{6}{7}\int_0^1 \frac{1}{24}+\frac{1}{4}y+\frac{1}{2}y^2\,dy=\frac{2}{7}$

b. $f_X(x)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy=\int_0^1\frac{6}{7}(x+y)^2\,dy=\frac{6}{7}x^2+\frac{6}{7}x+\frac{2}{7}\quad(x\in[0,1])$.

$f_Y(y)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx=\int_0^1\frac{6}{7}(x+y)^2\,dx=\frac{6}{7}y^2+\frac{6}{7}y+\frac{2}{7}\quad(y\in[0,1])$.

c. $f_{X|Y}(x|y)=\dfrac{f_{XY}(x,y)}{f_Y(y)}=\dfrac{3(x+y)^2}{3x^2+3x+1}\quad(x\in[0,1],y\in[0,1])$

$f_{Y|X}(y|x)=\dfrac{f_{XY}(x,y)}{f_X(x)}=\dfrac{3(x+y)^2}{3y^2+y+1}\quad(x\in[0,1],y\in[0,1])$.

9. a. $\because \int_{-1}^1\int_0^{1-x^2}dy\,dx=\int_{-1}^1 1-x^2\,dx=\frac{4}{3}$

$\therefore f(x,y)=\begin{cases}\frac{3}{4}, & 0\le y\le 1-x^2,\ -1\le x\le 1\\ 0, & \text{其他}\end{cases}$

$f_X(x)=\begin{cases}\int_0^{1-x^2}\frac{3}{4}\,dy=\frac{3}{4}(1-x^2), & x\in[-1,1]\\ 0, & \text{其他}\end{cases}$

$f_Y(y)=\begin{cases}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\frac{3}{4}\,dx=\frac{3}{2}\sqrt{1-y}, & y\in[0,1]\\ 0, & \text{其他}\end{cases}$

b. $f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=\begin{cases}\dfrac{1}{2\sqrt{1-y}}, & x\in[-\sqrt{1-y},\sqrt{1-y}]\\ 0, & \text{其他}\end{cases}$

$f_{Y|X}(y|x)=\dfrac{f(x,y)}{f_X(x)}=\begin{cases}\dfrac{1}{1-x^2}, & y\in[0,1-x^2]\\ 0, & \text{其他}\end{cases}$

10. $f(x,y)=xe^{-x(y+1)}$, $x\in[0,+\infty)$, $y\in[0,+\infty)$. $x<0$, $f_X(x)=0$

a. $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\int_0^{+\infty}xe^{-x(y+1)}dy=\left[-e^{-x(y+1)}\right]_{y=0}^{y=+\infty}=e^{-x}\ (x\geq 0)$

$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\int_0^{+\infty}xe^{-x(y+1)}dx=\left[-\frac{x}{y+1}e^{-x(y+1)}\right]_{x=0}^{x=+\infty}+\int_0^{+\infty}\frac{1}{y+1}e^{-x(y+1)}dx$

$=\frac{1}{y+1}\left[-\frac{1}{y+1}e^{-x(y+1)}\right]_{x=0}^{x=+\infty}=\frac{1}{(y+1)^2}\ (y\geq 0)$ $y<0$, $f_Y(y)=0$

$\because f_X(x)f_Y(y)=\frac{e^{-x}}{(y+1)^2}\neq f(x,y)=xe^{-x(y+1)}\ (x\geq 0, y\geq 0)$

$\therefore X,Y$ 不独立

b. $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\begin{cases}x(y+1)^2e^{-x(y+1)} & (x\in[0,+\infty), y\in[0,+\infty))\\ 0, & \text{其他}\end{cases}$

$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\begin{cases}xe^{y+1} & (x\in[0,+\infty), y\in[0,+\infty))\\ 0, & \text{其他}\end{cases}$

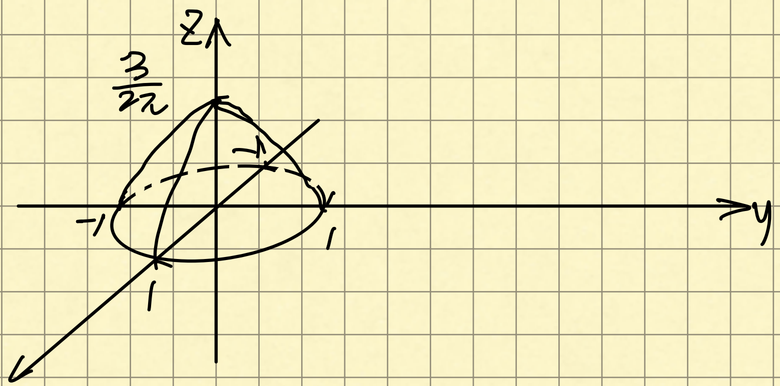
15. a. $\iint_D f(x,y)\,dxdy=\iint_{x^2+y^2\leq 1}f(x,y)\,dxdy$ $\therefore C=\frac{3}{2\pi}$

令 $x=\rho\cos\theta$, $y=\rho\sin\theta$. 则

$=\int_0^1 d\rho\int_{-\pi}^{\pi}C\rho\sqrt{1-\rho^2}\,d\theta$

$=2\pi C\cdot\left[-\frac{1}{3}(1-\rho^2)^{\frac{3}{2}}\right]_0^1=\frac{2}{3}\pi C=1$

b. 如图. $f(x,y)=\frac{3}{2\pi}\sqrt{1-x^2-y^2}$



c. $P(X^2+Y^2\leq\frac{1}{2})=\iint_{x^2+y^2\leq\frac{1}{2}}f_{XY}(x,y)\,dxdy=\iint_{x^2+y^2\leq\frac{1}{2}}\frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dxdy$

令 $x=\rho\cos\theta$, $y=\rho\sin\theta$, 则 $J=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\ \sin\theta & \rho\cos\theta\end{vmatrix}=\rho$

$P(X^2+Y^2\leq\frac{1}{2})=\int_0^{\frac{\sqrt{2}}{2}}\int_0^{2\pi}\frac{3}{2\pi}\rho\sqrt{1-\rho^2}\,d\theta d\rho=\int_0^{\frac{\sqrt{2}}{2}}3\rho\sqrt{1-\rho^2}\,d\rho=\left[-(1-\rho^2)^{\frac{3}{2}}\right]_0^{\frac{\sqrt{2}}{2}}=1-\frac{\sqrt{2}}{4}$

d. $f_X(x)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy=\begin{cases}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy=\frac{3}{4}(1-x^2),\ x\in[-1,1]\\ 0\ ,\ 其他\end{cases}$

$f_Y(y)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy=\begin{cases}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dx=\frac{3}{4}(1-y^2),\ y\in[-1,1].\\ 0\ ,\ 其他\end{cases}$

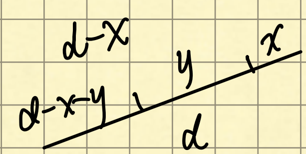
$\because f_{XY}(x,y)\neq f_X(x)f_Y(y)$ $\therefore X、Y$ 不独立

e. $f_{X|Y}(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\begin{cases}\frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{4}(1-y^2)}=\frac{2\sqrt{1-x^2-y^2}}{\pi(1-y^2)}\ \left(y\in[-1,1],\ x\in[-\sqrt{1-y^2},\sqrt{1-y^2}]\right)\\ 0\ ,\ 其他\end{cases}$

$f_{Y|X}(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}=\begin{cases}\frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{4}(1-x^2)}=\frac{2\sqrt{1-x^2-y^2}}{\pi(1-x^2)}\ \left(x\in[-1,1],\ y\in[-\sqrt{1-x^2},\sqrt{1-x^2}]\right)\\ 0\ ,\ 其他\end{cases}$

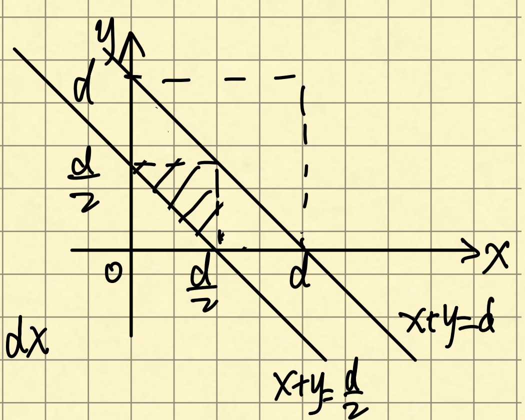
补充题:

1. 令三段中两段分别长为x,y,则第三段长为$(d-x-y)$.



构成三角形:

$\begin{cases}x+y>d-x-y\\ d-x-y+x>y\\ d-x-y+y>x\end{cases}\Rightarrow\begin{cases}2(x+y)>d\\ d>2y\\ d>2x\end{cases}\Rightarrow\begin{cases}x+y>\frac{d}{2}\\ x<\frac{d}{2}\\ y<\frac{d}{2}\end{cases}$



且有 $f_X(x)=\frac{1}{d}\ (0<x<d)$, $f_{Y|X}(y|x)=\frac{1}{d-x}\ (0<y<d-x)$

$\Rightarrow f_{XY}(x,y)=\frac{1}{d(d-x)}\ \left(x\in(0,d),\ y\in(0,d-x)\right).$

$P=\int_0^{\frac{d}{2}}\int_{\frac{d}{2}-x}^{\frac{d}{2}}\frac{1}{d(d-x)}\,dy\,dx=\int_0^{\frac{d}{2}}\frac{x}{d(d-x)}\,dx=\frac{1}{d}\int_0^{\frac{d}{2}}\left(\frac{d}{d-x}-1\right)dx$

$=\frac{1}{d}\left[-d\ln(d-x)-x\right]_0^{\frac{d}{2}}=\ln 2-\frac{1}{2}$

2. $X\sim U(0,1)$

(1) $f_X(x)=1\ (x\in(0,1))$. $f_{Y|X}(y|x)=\frac{1}{x}\ (y\in(0,x))$.

$f_{XY}(x,y)=f_X(x)f_{Y|X}(y|x)=\frac{1}{x}\ (0<y<x<1)$

(2) $f_Y(y)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx=\begin{cases}\int_y^1\frac{1}{x}\,dx=-\ln y\ ,\ y\in(0,1)\\ 0\ ,\ 其他\end{cases}$

(3) $P(X+Y>1)=\iint_{x+y>1}\frac{1}{x}\,dx\,dy=\int_{\frac{1}{2}}^{1}\int_{1-x}^{x}\frac{1}{x}\,dy\,dx=1-\ln 2$

3.(1) $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_x^{+\infty}e^{-y}dy=x & (x\in(0,+\infty))\\ 0, & 其他\end{cases}$

$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\begin{cases}\int_0^y e^{-y}dx=ye^{-y} & (y\in(0,+\infty)).\\ 0, & 其他\end{cases}$

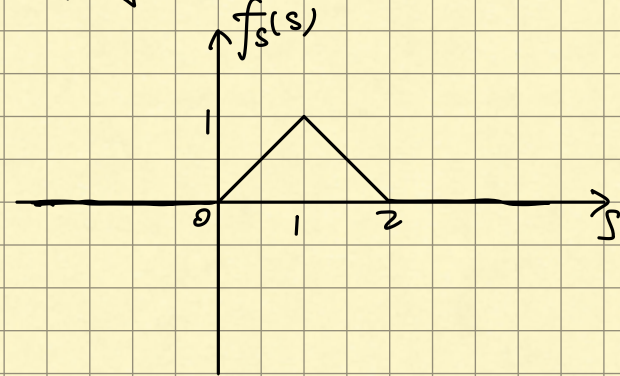
$\because f_{XY}(x,y)\neq f_X(x)f_Y(y)\quad\therefore X,Y$不独立

(2) $y>0$, $f_{X|Y}(x|y)=\dfrac{f_{XY}(x,y)}{f_Y(y)}=\begin{cases}\frac{1}{y}, & x\in(0,y)\\ 0, & 其他\end{cases}$

$x>0$, $f_{Y|X}(y|x)=\dfrac{f_{XY}(x,y)}{f_X(x)}=\begin{cases}e^{-(x+y)}, & y\in(x,+\infty)\\ 0, & 其他\end{cases}$

§3.6

43. $U_1\sim U(0,1)$, $U_2\sim U(0,1)$, $S=U_1+U_2$ 如图:



$f_{U_1}(u_1)=f_{U_2}(u_2)=\begin{cases}1, & u_i\in[0,1]\\ 0, & 其他\end{cases}\quad(i=1,2)$

$f_S(s)=\int_{-\infty}^{+\infty}f_{U_1}(x)f_{U_2}(u-x)dx$

$=\begin{cases}0, & u\le 0\\ \int_0^u dx=u, & u\in(0,1]\\ \int_{u-1}^1 dx=2-u, & u\in(1,2]\\ 0, & u>2\end{cases}$

44. $P\{X=0\}=P\{X=1\}=P\{X=2\}=\frac{1}{3}$　且有$\sum_{i=0}^{2}P\{K=i\}=1$ $(K=X或Y)$

$P\{Y=0\}=P\{Y=1\}=P\{Y=2\}=\frac{1}{3}$　故以上为X或Y的全部取值.

令$S=X+Y$,则S可取0,1,2,3,4. 由于X,Y独立,故有$P\{X=i,Y=j\}=P\{X=i\}P\{Y=j\}$ $(i,j=0,1,2)$.

$P\{S=0\}=P\{X=0,Y=0\}=P\{X=0\}P\{Y=0\}=\frac{1}{9}$

$P\{S=1\}=P\{X=0,Y=1\}+P\{X=1,Y=0\}=\frac{2}{9}$

$P\{S=2\}=P\{X=1,Y=1\}+P\{X=0,Y=2\}+P\{X=2,Y=0\}=\frac{1}{3}$

$P\{S=3\}=P\{X=1,Y=2\}+P\{X=2,Y=1\}=\frac{2}{9}$

$P\{S=4\}=P\{X=2,Y=2\}=\frac{1}{9}$

故$S=X+Y$的pmf为:

S	0	1	2	3	4
P	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$

51. $Z=XY$.

$F_Z(z)=P\{Z\le z\}=P\{XY\le z\}$

$$=\iint_{xy\le z} f(x,y)\,dx\,dy$$

$z>0$, $$=\int_0^{+\infty}dx\int_{-\infty}^{\frac{z}{x}} f(x,y)\,dy+\int_{-\infty}^{0}dx\int_{\frac{z}{x}}^{+\infty} f(x,y)\,dy$$

令 $t=xy$, 则 $dt=x\,dy$

$$=\int_0^{+\infty}dx\int_{-\infty}^{z} f(x,\tfrac{t}{x})\tfrac{1}{x}\,dt+\int_{-\infty}^{0}dx\int_{z}^{+\infty} f(x,\tfrac{t}{x})\tfrac{1}{x}\,dt$$

$$=\int_{-\infty}^{z}dt\left[\int_{-\infty}^{0}-\tfrac{1}{x}f(x,\tfrac{t}{x})\,dx+\int_0^{+\infty}\tfrac{1}{x}f(x,\tfrac{t}{x})\,dx\right]$$

$$=\int_{-\infty}^{z}dt\int_{-\infty}^{+\infty}\frac{1}{|x|}f(x,\tfrac{t}{x})\,dx$$

$z\le 0$, $$=\int_{-\infty}^{0}dx\int_{\frac{z}{x}}^{+\infty} f(x,y)\,dy+\int_{\infty}^{0}dx\int_{-\infty}^{\frac{z}{x}} f(x,y)\,dy$$

与上同理, 可得: $F_Z(z)=\int_{-\infty}^{z}dt\int_{-\infty}^{+\infty}\frac{1}{|x|}f(x,\frac{t}{x})dx$

故密度为 $f_Z(z)=F_Z'(z)=\int_{-\infty}^{+\infty}\frac{1}{|x|}f(x,\frac{z}{x})dx=\int_{-\infty}^{+\infty}\frac{1}{|y|}f(y,\frac{z}{y})dy$

52. $Z=\frac{X}{Y}$. $X\sim U(a,b)$, $Y\sim U(c,d)$

$$\begin{cases} Z=\frac{X}{Y}\\ Y=Y\end{cases}\Rightarrow\begin{cases} X=ZY\\ Y=Y\end{cases}\qquad J=\begin{vmatrix} y & z\\ 0 & 1\end{vmatrix}=y$$

$f_{ZY}(z,y)=|y|f_{XY}(zy,y)=|y|f_X(zy)f_Y(y)=|y|\ (zy\in(a,b),\ y\in(c,d))$

57. $Y_1\sim N(0,1)$, $Y_2\sim(0,2)$, $(Y_1,Y_2)\sim N(0,1;0,2;\frac{1}{\sqrt{2}})$.

$$\begin{bmatrix} x_1\\ x_2\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} y_1\\ y_2\end{bmatrix}\Rightarrow\begin{bmatrix} y_1\\ y_2\end{bmatrix}=\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12}\\ a_{21} & a_{11}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix}\Rightarrow|J|=\frac{1}{|a_{11}a_{22}-a_{21}a_{12}|}$$

令 $D=|a_{11}a_{22}-a_{12}a_{21}|$, 则

$$f_{X_1X_2}(x_1,x_2)=\frac{D}{2\pi}\exp\left\{-D^2\left[(a_{22}x_1-a_{12}x_2)^2-(a_{22}x_1-a_{12}x_2)(a_{21}x_1+a_{11}x_2)+\tfrac{1}{2}(a_{11}x_2-a_{21}x_1)^2\right]\right\}$$

$$=\frac{1}{2\pi}\exp\{-\tfrac{1}{2}x_1^2-\tfrac{1}{2}x_2^2\}$$

即有: $$\begin{cases} D=|a_{11}a_{22}-a_{12}a_{21}|=1\\ a_{22}^2+a_{22}a_{21}+\frac{1}{2}a_{21}^2=\frac{1}{2}\\ a_{12}^2+a_{12}a_{11}+\frac{1}{2}a_{11}^2=\frac{1}{2}\\ -2a_{22}a_{12}-a_{22}a_{11}-a_{12}a_{21}-a_{11}a_{21}=0\end{cases}\Rightarrow\begin{cases} a_{11}=1\\ a_{12}=0\\ a_{21}=-1\\ a_{22}=1\end{cases}\Rightarrow\begin{cases} X_1=Y_1\\ X_2=-Y_1+Y_2\end{cases}$$

补充题：

1. $X \sim N(0,1)$, $Y \sim N(0,1)$

$\begin{cases} U = X+Y \\ V = X-Y \end{cases} \Rightarrow \begin{cases} X = \frac{1}{2}(U+V) \\ Y = \frac{1}{2}(U-V) \end{cases}$, $J = \begin{vmatrix} \frac{1}{2}V & \frac{1}{2}U \\ \frac{1}{2}V & \frac{1}{2}U \end{vmatrix} = \frac{1}{2}UV$.

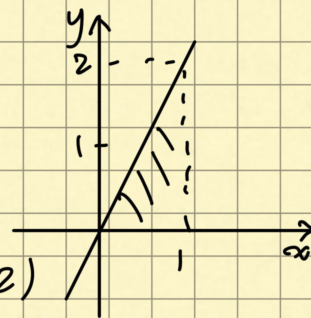
$f_{UV}(u,v) = \frac{1}{2} \cdot \frac{1}{2\pi} \cdot \exp\{-\frac{1}{2} \cdot \frac{1}{4}[(u+v)^2 + (u-v)^2]\} = \frac{1}{4\pi} \cdot \exp(-\frac{u^2}{4} - \frac{v^2}{4})$

由于u,v可分，故u,v独立，且有

$f_U(u) = \frac{1}{2\sqrt{\pi}} \exp(-\frac{u^2}{4})$, $f_V(v) = \frac{1}{2\sqrt{\pi}} \exp(-\frac{v^2}{4})$

2. (1) $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^{2x} f_{XY}(x,y)\,dy = 2x, & x \in (0,1) \\ 0, & \text{其他} \end{cases}$

$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_{\frac{y}{2}}^{1} f_{XY}(x,y)\,dx = 1 - \frac{y}{2}, & y \in (0,2) \\ 0, & \text{其他} \end{cases}$



(2) $F_Z(z) = P\{2X - Y \le z\} = \begin{cases} 0, & z \le 0 \\ \int_0^{\frac{z}{2}} \int_0^{2x} dy\,dx + \int_{\frac{z}{2}}^{1} \int_{2x-z}^{2x} dy\,dx = z - \frac{z^2}{4}, & z \in (0,2] \\ 1, & z > 2 \end{cases}$

因此，$f_Z(z) = F_Z'(z) = \begin{cases} 1 - \frac{z}{2}, & z \in (0,2] \\ 0, & \text{其他} \end{cases}$

(3) $P\{Y < \frac{1}{2}, X < \frac{1}{2}\} = \frac{1}{2} \times (\frac{1}{4} + \frac{1}{2}) \times \frac{1}{2} = \frac{3}{16}$

$P\{X < \frac{1}{2}\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P\{Y < \frac{1}{2} \mid X < \frac{1}{2}\} = \frac{P\{Y < \frac{1}{2}, X < \frac{1}{2}\}}{P\{X < \frac{1}{2}\}} = \frac{(\frac{3}{16})}{(\frac{1}{4})} = \frac{3}{4}$.

§3.7

70. $P(X_{(1)} > x, X_{(n)} \le y) = P(x < X_i \le y, i = 1, \cdots, n) = (F(y) - F(x))^n$

$P(X_{(1)} > x, X_{(n)} \le y) = P(X_{(n)} \le y) - P(X_{(1)} \le x, X_{(n)} \le y) = P(X_{(n)} \le y) - F(x,y) = (F(y))^n - F(x,y)$

故有 $(F(y) - F(x))^n = (F(y))^n - F(x,y)$

$\Rightarrow F(x,y) = (F(y))^n - (F(y) - F(x))^n \quad (x \le y)$

补充题：

1、如图表：

Y \ X	1	2	3	P(Y)
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
P(X)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Z \ X	1	2	3	P(Z)
1	0	0	0	0
2	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
P(X)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

2. $X \sim N(0,1),\ Y \sim N(0,1)$

$$F_Z(z) = P(\min(X,Y) \leq z) = 1 - P\{X \geq z, Y \geq z\}$$
$$= 1 - P\{X \geq z\} P\{Y \geq z\}$$
$$= 1 - (1 - P\{X \leq z\})(1 - P\{Y \leq z\})$$
$$= 1 - (1 - F_X(x))(1 - F_Y(y))$$
$$= F_X(x) + F_Y(y) - F_X(x) F_Y(y)$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) - \frac{1}{2\pi} \exp\left(-\frac{x^2}{2} - \frac{y^2}{2}\right)$$