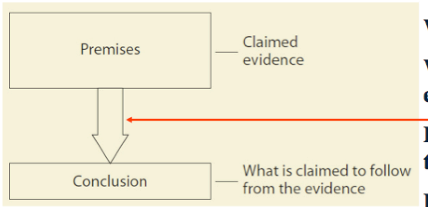
An Introduction to Classical Predicate Calculus

Jingde Cheng

Saitama University

Logic: What Is It?



[From P. J. Hurley, "A Concise Introduction to Logic"]

What entails what?

What follows from what?

Why? What are the evaluation criteria?

How to establish/define the evaluation criteria?

How to evaluate arguments/reasoning?

It is LOGIC to answer these fundamental questions.

The Fundamental Questions

What is (mathematical) logic?
What entails what?
What follows from what?
Why? What are the evaluation criteria?
How to establish/define the evaluation criteria?
How to evaluate arguments/reasoning?
What is Classical Predicate Calculus?

Note: This lecture note is an introduction to CFOPC for CS/IS students from the viewpoint of applications; it includes all of important meta-theorems of CFOPC but no proof is given.

An Introduction to Classical Predicate Calculus

The Limitations of Propositional Logic CPC: An Example

♣ Example

- All men are mortal — A: All *M* are *P*
  Socrates is a man — B: *S* is a *M*
  Therefore: — Therefore:
  Socrates is mortal — C: *S* is *P* (This form should be valid)
- However, there is no way in propositional logic **CPC** to represent "all are …" and "there is a …".
- $((A \wedge B) \rightarrow C)$ is not a tautology in propositional logic **CPC**.

♣ Example

- A: $\forall x(M(x) \rightarrow P(x))$
  B: $\exists s(M(s))$
  Therefore:
  C: $P(s)$
- $((A \wedge B) \rightarrow C)$ is a tautology in first-order predicate logic **CFOPC**.

The Limitations of Propositional Logic CPC: More Examples [Mendelson]

1. Any friend of Martin is a friend of John.
   Peter is not John's friend.
   Hence, Peter is not Martin's friend.
2. All human beings are rational.
   Some animals are human beings.
   Hence, some animals are rational.
3. The successor of an even integer is odd.
   2 is an even integer.
   Hence, the successor of 2 is odd.

$$\frac{(\forall x)(F(x,m) \Rightarrow F(x,j)) \quad \neg F(p,j)}{\neg F(p,m)} \qquad \frac{(\forall x)(H(x) \Rightarrow R(x)) \quad (\exists x)(A(x) \wedge H(x))}{(\exists x)(A(x) \wedge R(x))} \qquad \frac{(\forall x)(I(x) \wedge E(x) \Rightarrow D(s(x))) \quad I(b) \wedge E(b)}{D(s(b))}$$

The Limitations of Propositional Logic CPC: More Examples [Mendelson]

There are various kinds of logical inference that cannot be justified on the basis of the propositional calculus; for example:

1. Any friend of Martin is a friend of John.
   Peter is not John's friend.
   Hence, Peter is not Martin's friend.
2. All human beings are rational.
   Some animals are human beings.
   Hence, some animals are rational.
3. The successor of an even integer is odd.
   2 is an even integer.
   Hence, the successor of 2 is odd.

The correctness of these inferences rests not only upon the meanings of the truth-functional connectives, but also upon the meaning of such expressions as "any," "all," and "some," and other linguistic constructions.

In order to make the structure of complex sentences more transparent, it is convenient to introduce special notation to represent frequently occurring expressions. If $P(x)$ asserts that x has the property P, then $(\forall x)P(x)$ means that property P holds for all x or, in other words, that everything has the property P. On the other hand, $(\exists x)P(x)$ means that some x has the property P—that is, that there is at least one object having the property P. In $(\forall x)P(x)$, "$(\forall x)$" is called a *universal quantifier*; in $(\exists x)P(x)$, "$(\exists x)$" is called an *existential quantifier*. The study of quantifiers and related concepts is the principal subject of this chapter.

An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen's Natural Deduction System for **CFOPC**
- ♣ Gentzen's Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- {$\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\neg$, $\forall$, $\exists$, $\top$, $\bot$, (,),
  $x_1, x_2, \ldots, x_n, \ldots,$ (countable)
  $c_1, c_2, \ldots, c_n, \ldots,$ (countable)
  $f^1{}_1, \ldots, f^1{}_k, \ldots, f^2{}_1, \ldots, f^2{}_k, \ldots, f^n{}_1, \ldots, f^n{}_k, \ldots,$ (countable)
  $p^0{}_1, \ldots, p^0{}_k, \ldots, p^1{}_1, \ldots, p^1{}_k, \ldots, p^2{}_1, \ldots, p^2{}_k, \ldots, p^n{}_1, \ldots, p^n{}_k, \ldots$ (countable)}
- ***Logical (propositional) connectives***: $\rightarrow$ (***material implication***), $\wedge$ (***conjunction***), $\vee$ (***disjunction***), $\leftrightarrow$ (***equivalence***), $\neg$ (***negation***).
- ***Quantifiers***: $\forall$ (for all, the ***universal quantifier***) (a turned A), $\exists$ (there exists a, the ***existential quantifier***) (a flip-horizontal E).
- ***Logical constants***: $\top$ (***top***) and $\bot$ (***bottom***).
- Punctuation: left and right parentheses '(' and ')'.

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***Individual variables (variable symbols)***
  (**V**): $x_1, x_2, \ldots, x_n, \ldots$ (countable, not empty)
- Individual variables, which rang over the domain of discourse, act as placeholders in much the same way as pronouns act as placeholders in ordinary language.
- ***Individual constants (Names) (constant symbols, name symbols)***
  (**Con**): $c_1, c_2, \ldots, c_n, \ldots$ (countable, possibly empty)
- Individual constants play the role of "names" for objects (individuals) in the domain of discourse.
- Note: Let Saitama-University be the domain of discourse, **Con** may be the union of faculty-identifier set and student-identifier set.
- The individual constants, function symbols, and predicate symbols of a language L are called the ***non-logical constants*** of L.

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***(Individual) Functions (function symbols)***
  (**Fun**): $f^1{}_1, \ldots, f^1{}_k, \ldots, f^2{}_1, \ldots, f^2{}_k, \ldots, f^n{}_1, \ldots, f^n{}_k, \ldots,$ (countable, possibly empty)
- n and k are any positive integers.
- The superscript n indicates the number of arguments, whereas the subscript k is just an indexing number to distinguish different function symbols (letters) with the same number of arguments.
- Constants can be regarded as 0-ary functions because they are objects (individuals) that have no dependence on any inputs; they simply denote objects (individuals) of the domain of discourse.
- The individual constants, function symbols, and predicate symbols of a language L are called the ***non-logical constants*** of L.

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***(Individual) Predicates (Relations) (predicate/relation symbols)***
  (**Pre**): $p^0{}_1, \ldots, p^0{}_k, \ldots, p^1{}_1, \ldots, p^1{}_k, \ldots, p^2{}_1, \ldots, p^2{}_k, \ldots, p^n{}_1, \ldots, p^n{}_k, \ldots$ (countable, not empty)
- n and k are any positive integers.
- The superscript n indicates the number of arguments, whereas the subscript k is just an indexing number to distinguish different predicate symbols (letters) with the same number of arguments.
- 0-ary predicates can be regarded as propositions (sentences) because they are simply statements of facts independent of any individual variables.
- Unary predicates are simply properties of objects (individuals), binary predicates are relations between pairs of objects (individuals).
- In general n-ary predicates express relations among n-tuple of objects (individuals).

Formal (Object) Language (Syntax) of CFOPC: "First-Order"

♣ ***CFOPC***

- ***Classical First-Order Predicate Calculus***

♣ ***Individual quantifiers***

- In CFOPC, the quantifiers (universal quantifier and existential quantifier) are applied to only the individual variables.

♣ ***First-order***

- The adjective "first-order" is used to distinguish the languages we shall study here from those (i.e., high-order languages) in which there are predicates having other predicates or functions as arguments or in which predicate quantifiers or function quantifiers are permitted, or both.

♣ Sufficiency

- Most mathematical theories can be formalized within first-order languages, although there may be a loss of some of the intuitive content of those theories.

Formal (Object) Language (Syntax) of CFOPC: Terms

♣ ***Terms***

- (1) Every individual variable (symbol) is a term;
  (2) Every individual constant (symbol) is a term;
  (3) If f is an n-ary function (symbol) ($n = 1, 2, \ldots$) and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term;
  (4) Nothing else are terms.
- **Ter**: the set of all terms
- The functions applied to the individual variables and individual constants generate the terms.
- Terms correspond to what in ordinary languages are nouns and noun phrases.

♣ ***Closed terms***

- A term is ***closed***, called a ***variable-free term*** or ***ground term*** IFF it contains no individual variables.

Formal (Object) Language (Syntax) of CFOPC: Well-Formed Formulas

♣ ***Formulas (Well-formed formulas)***

- (1) If p is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$ is a formula (called an ***atomic formula***) (predicates applied to terms generate the atomic formulas); also T and $\perp$ are ***atomic formulas***;
  (2) If A and B are formulas and x is an individual variable, then so are $(\neg A)$, $(A \rightarrow B)$, $(A \wedge B)$, $(A \vee B)$, $(A \leftrightarrow B)$, $((\forall x)A)$, and $((\exists x)A)$;
  (3) Nothing else are formulas.
- **WFF**$_{\text{CFOPC}}$: the set of all formulas of **CFOPC** (**WFF** for short).
- $((\exists x)A) =_{\text{df}} (\neg((\forall x)\neg A))$, $((\forall x)A) =_{\text{df}} (\neg((\exists x)\neg A))$; A is called the ***scope*** of quantifiers $(\forall x)$ and $(\exists x)$.

♣ Open formulas

- An ***open formula*** is a formula without quantifiers.

♣ Notation

- Let $F \in$ **WFF** and x be an individual variable, we use $F(x)$ to mean that x appears in F.

Formal (Object) Language (Syntax) of CFOPC: Subformulas

♣ Immediate subformulas

- ***Immediate subformulas*** are defined as follows:
  (1) an atomic formula has no immediate subformula;
  (2) the only immediate subformula of $(\neg A)$, $((\forall x)A)$, and $((\exists x)A)$ is A;
  (3) for a binary connective $*$, the immediate subformulas of $(A*B)$ are A and B.

♣ Subformulas

- For any $A \in$ **WFF**, The set of ***subformulas*** of A is the smallest set S that contains A and contains, with each member, the immediate subformulas of that member. A is called an ***improper subformula*** of itself.

♣ Homework

- Try to develop an algorithm to check whether or not a string is a formula of **CFOPC**.

Formal (Object) Language (Syntax) of CFOPC: General Guidelines

♣ The sentences of the form "All As are Bs"

- $((\forall x_1)(p^1{}_1(x_1) \rightarrow p^1{}_2(x_1)))$ $\quad((\forall x)(p_A(x) \rightarrow p_B(x)))$ $\quad$ (difference?)
- Ex: "Every mathematician loves music" is translated as $(\forall x)(M(x) \rightarrow LM(x)))$ where $M(x)$ means "x is a mathematician" and $LM(x)$ means "x loves music."

♣ The sentences of the form "Some As are Bs"

- $((\exists x_1)(p^1{}_1(x_1) \wedge p^1{}_2(x_1)))$ $\quad((\exists x)(p_A(x) \wedge p_B(x)))$ $\quad$ (difference?)
- Ex: "Some New Yorkers are friendly" is translated as $(\exists x(NY(x) \wedge F(x)))$ where $NY(x)$ means "x is a New Yorker" and $F(x)$ means "x is friendly."

♣ The sentences of the form "No As are Bs"

- $((\forall x_1)(p^1{}_1(x_1) \rightarrow (\neg p^1{}_2(x_1))))$ $\quad((\forall x)(p_A(x) \rightarrow (\neg p_B(x))))$ (difference?)
- Ex: "No philosopher understands politics" is translated as $((\forall x)(P(x) \rightarrow \neg UP(x)))$ where $P(x)$ means "x is a philosopher" and $UP(x)$ means "x understands politics."

Indicating Universal and Existential Quantity [Kelley]

Universal		Existential	
Affirmative	**Negative**	**Affirmative**	**Negative**
All	No	Some are	Some are not
Any	None	There is (are)	There is (are) . . . not
A (A cat is a predator)	Not a (Not a creature was stirring)	A (A car is parked outside)	A (A student is not present)
Every, Everything	Nothing	Something	Something
Everyone (people)	No one	Someone	Someone
Always (time)	Never	Sometimes	Sometimes
Everywhere (place)	Nowhere	Somewhere	Somewhere

Types of Statements in First-Order Predicate Logic [Kelley]

SUMMARY **Types of Statements in Predicate Logic**

Type	Predicate Notation	Meaning	Example
Atomic singular statement	*Pa*	*a* is *P*	London is a city.
Truth-functional compounds of singular atomic statements	*Pa* ⊃ *Qa* *Pa* • *Qb*, etc.	If *a* is *P*, it is *Q* *a* is *P* and *b* is *Q*; etc.	If London is a city then it is large. Tom is healthy but Sue is sick.
Quantifier applied to atomic open sentence *Px*	(*x*)*Px* (∃*x*)*Px*	For all *x*, *x* is *P* There exists an *x* that is *P*	Everything is material. Something is on fire.
Truth-functional compounds of open sentences within the scope of a quantifier	(*x*)~*Px* (∃*x*)~*Px* (*x*)(*Px* ⊃ *Qx*) (∃*x*)(*Px* • *Qx*) (*x*)[*Px* ∨ (*Qx* • *Rx*)]	Nothing is *P* Something is not *P* For all *x*, if *x* is *P* then it is *Q* Some *x* is *P* and it is *Q* For all *x*, *x* is *P* or it is *Q* and *R*	Nothing is infinite. Something is not infinite. All men are mortal. Some birds are carnivores. Everything is either scary or warm and fuzzy.
Truth-functional compounds of quantified statements. Scope of quantifiers does not cover connective.	~(*x*)*Px* (*x*)*Px* ⊃ (*y*)*Qy* (*x*)*Px* ⊃ (∃*x*)*Qx* (∃*x*)*Px* ∨ (∃*y*)*Qy*, etc.	It is not the case that all *x* are *P* If all *x* are *P*, then all *y* are *Q* If all *x* are *P*, then some *x* is *Q* Either something is *P* or something is *Q*, etc.	It is not the case that everything is physical. If everything is physical, then everything has mass. If everything is created, then there is a God. There was either an explosion or a collision.

Representation Examples of First-Order Language [Smullyan]

- Let *h* stand for Holmes (Sherlock Holmes) and let *m* stand for Moriarty. Let us abbreviate "*x* can catch *y*" by "*x*C*y*"(*C*(*x*,*y*)). Give symbolic renditions of the following statements:

  (a) Holmes can catch anyone who can catch Moriarty. (∀*x*)(*x*C*m*→*h*C*x*)
  (b) Holmes can catch anyone whom Moriarty can catch. (∀*x*)(*m*C*x*→*h*C*x*)
  (c) Holmes can catch anyone who can be caught by Moriarty. (Same as (b))
  (d) If anyone can catch Moriarty, then Holmes can. (∃*x*)(*x*C*m*→*h*C*m*)
  (e) If everyone can catch Moriarty, then Holmes can. (∀*x*)(*x*C*m*→*h*C*m*)
  (f) Anyone who can catch Holmes can catch Moriarty. (∀*x*)(*x*C*h*→*x*C*m*)
  (g) No one can catch Holmes unless he can catch Moriarty. (Same as (f))
  (h) Everyone can catch someone who cannot catch Moriarty.
  (∀*x*)((∃*y*)(*x*C*y*∧(¬*y*C*m*)))
  (i) Anyone who can catch Holmes can catch anyone whom Holmes can catch. (∀*x*)(*x*C*h*→((∀*y*)(*h*C*y*→*x*C*y*))); or, (∀*x*)((∀*y*)((*x*C*h*∧*h*C*y*)→*x*C*y*))
- Homework: Try to represent (g) using negation.

Representation Examples of First-Order Language [Smullyan]

- Let us symbolize "*x* knows *y*" by "*x*K*y*"(*K*(*x*,*y*)).

  (a) Everyone knows someone. (∀*x*)((∃*y*)*x*K*y*)
  (b) Someone knows everyone. (∃*x*)((∀*y*)*x*K*y*)
  (c) Someone is known by everyone. (∃*x*)((∀*y*)*y*K*x*)
  (d) Every person *x* knows someone who doesn't know *x*.
  (∀*x*)((∃*y*)(*x*K*y*∧(¬*y*K*x*)))
  (e) There is someone *x* who knows everyone who knows *x*.
  (∃*x*)((∀*y*)(*y*K*x*→*x*K*y*))
- Let D*x* abbreviate "*x* can do it" and let "*b*" abbreviate Bernard. Let "*x* = *y*"(=(*x*,*y*)) abbreviate "*x* is identical with *y*".

  (a) Bernard, if anyone, can do it. (∃*x*)(D*x*→D*b*); or, (∀*x*)(D*x*→D*b*)
  (b) Bernard is the only one who can do it.
  D*b*∧((∀*x*)(D*x*→*x*=*b*)); or, (∀x)(D*x*↔*x*=*b*)

Representation Examples of First-Order Language [Smullyan]

- Here "number" shall mean natural number, i.e., 0 or some positive whole number. The usual abbreviation of "*x* is less than *y*" is "*x* < *y*"(<(*x*,*y*)), the abbreviation of "*x* is greater than *y*" is "*x* > *y*"(>(*x*,*y*)), and the abbreviation of "*x* is equivalent to *y*" is "*x* = *y*"(=(*x*,*y*)).

  (a) For every number there is a greater number. (∀*x*)((∃*y*)*y*>*x*)
  (b) Every number other than 0 is greater than some number.
  (∀*x*)[(¬*x*=0)→((∃*y*)*x*>*y*)]
  (c) 0 is the one and only number having the property that no number is less than it.
  {¬[(∃*y*)*y*<0]}∧{(∀*x*)(¬[(∃*z*)*z*<*x*])→*x*=0};
  or, (∀*x*){(¬[(∃*y*)*y*<*x*])↔*x*=0}

  Question: How about ¬((∃*y*)*y*<0) ?

Representation Examples of First-Order Language [Dalen]

$\exists x P(x)$ – there is an x with property P,

$\forall y P(y)$ – for all y P holds (all y have the property P),

$\forall x \exists y (x = 2y)$ – for all x there is a y such that x is two times y,

$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (\frac{1}{n} < \varepsilon))$ – for all positive ϵ there is an n such that $\frac{1}{n} < \varepsilon$,

$x < y \rightarrow \exists z (x < z \wedge z < y)$ – if $x < y$, then there is a z such that $x < z$ and $z < y$,

$\forall x \exists y (x.y = 1)$ – for each x there exists an inverse y.

Representation Examples of First-Order Language [Manin]

♣ "*x* is the direct product *y* X *z* of the set *y* and the set *z*"

- This means that the elements of *x* are the ordered pairs of elements of *y* and *z*, respectively.
- The definition of an unordered pair (a set that has two elements) y_1 and z_1 is obvious: the formula
  $\forall u(u \in x_0 \leftrightarrow (u = y_1 \vee u = z_1))$
  "means," or may be briefly written in the form, "$x_0 = \{y_1, z_1\}$".
- The ordered pair y_1 and z_1 is introduced using a device of Kuratowski and Wiener: this is the set of x_1 whose elements are the unordered pairs $\{y_1, y_1\}$ and $\{y_1, z_1\}$. Thus, we arrive at the formula
  $\exists y_2 \exists z_2$("$x_1 = \{y_2, z_2\}$" ∧ "$y_2 = \{y_1, y_1\}$" ∧ "$z_2 = \{y_1, z_1\}$")
  which will be abbreviated "$x_1 = <y_1, z_1>$" and will be read "x_1 is the ordered pair with first element y_1 and second element z_1".
- Finally, the statement "$x = y$ X z" may be written in the form
  $\forall x_1(x_1 \in x \leftrightarrow \exists y_1 \exists z_1(y_1 \in y \wedge z_1 \in z \wedge$ "$x_1 = <y_1, z_1>$"$))$

Representation Examples of First-Order Language [Manin]

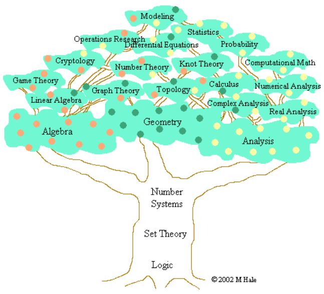
♣ "f is a (total) mapping from the set u to the set v."

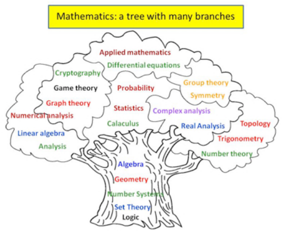
- The following formula imposes three conditions on (total) mapping f:
  (1) f is a subset of $u \times v$;
  (2) the projection of f onto u coincides with all of u;
  (3) each elements of u corresponds to exactly one element of v.
- $\forall z[z \in f \rightarrow \exists u_1 \exists v_1(u_1 \in u \wedge v_1 \in v \wedge \text{“}z=<u_1,v_1>\text{”})] \wedge$
  $\forall u_1[u_1 \in u \rightarrow \exists v_1 \exists z(v_1 \in v \wedge \text{“}z=<u_1,v_1>\text{”} \wedge z \in f)] \wedge$
  $\forall u_1 \forall v_1 \forall v_2[\exists z_1 \exists z_2(z_1 \in f \wedge z_2 \in f \wedge \text{“}z_1=<u_1,v_1>\text{”} \wedge \text{“}z_2=<u_1,v_2>\text{”}) \rightarrow v_1=v_2]$

♣ Homework

- Try to define/represent a partial mapping, an injective mapping, a surjective mapping, and a bijective mapping, as a first-order formula, respectively.

Logic as the Fundamental Basis for all Mathematics





First-Order Number Theory [Mendelson]

♣ Number theory as the foundation for mathematics

- Together with geometry, the theory of numbers is the most immediately intuitive of all branches of mathematics.
- It is obvious that attempts to formalize mathematics and to establish a rigorous foundation for mathematics should begin with number theory.

♣ Peano's postulates for number theory

- The first semi-axiomatic presentation of this subject was given by Dedekind in 1879 and, in a slightly modified form (by Peano), has come to be known as Peano's postulates.

First-Order Number Theory [Mendelson]: Peano's Postulates

♣ ***Peano's postulates for number theory***

- (P1) 0 is a natural number.
- (P2) If x is a natural number, there is another natural number denoted by x' (the ***successor*** of x).
- (P3) $0 \neq x'$ for every natural number x.
- (P4) If $x' = y'$, then $x = y$.
- (P5) (***mathematical induction principle***) If Q is a property that may or may not hold for any given natural number, and if
  (I) 0 has the property Q, and
  (II) whenever a natural number x has the property Q, then x' has the property Q,
  then all natural numbers have the property Q.

First-Order Number Theory [Mendelson]: The Language $\mathbf{L_A}$

♣ The language of the first-order number theory: $\mathbf{L_A}$ (***the language of arithmetic***)

- $\mathbf{L_A}$ has a single predicate symbol (letter) p^2_1. We shall abbreviate $p^2_1(t, s)$ by $t = s$, and $\neg p^2_1(t, s)$ by $t \neq s$.
- $\mathbf{L_A}$ has one individual constant symbol a_1. We shall use 0 as an alternative notation for a_1.
- $\mathbf{L_A}$ has three function symbols (letters) f^1_1, f^2_1, and f^2_2. We shall write (t') instead of $f^1_1(t)$, $(t + s)$ instead of $f^2_1(t, s)$, and $(t \cdot s)$ instead of $f^2_2(t, s)$. However, we shall write t', $t + s$, and $t \cdot s$ instead of (t'), $(t + s)$, and $(t \cdot s)$ whenever this will cause no confusion.

- Note
  - We do not give interpretations for the three function symbols but remain them to be defined by PA axioms.

First-Order Number Theory [Mendelson]: The Axioms

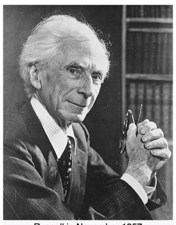
♣ ***The axioms of the first-order number theory: Peano Arithmetic (PA)***

- (PA1) $x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)$
- (PA2) $x_1 = x_2 \rightarrow x_1' = x_2'$
- (PA3) $0 \neq x_1'$
- (PA4) $x_1' = x_2' \rightarrow x_1 = x_2$
- (PA5) $x_1 + 0 = x_1$
- (PA6) $x_1 + x_2' = (x_1 + x_2)'$
- (PA7) $x_1 \cdot 0 = 0$
- (PA8) $x_1 \cdot (x_2)' = (x_1 \cdot x_2) + x_1$
- (PA9) $B(0) \rightarrow [(\forall x)(B(x) \rightarrow B(x')) \rightarrow (\forall x)B(x)]$ for any wff $B(x)$ of $\mathbf{L_A}$. (PA9) is called ***the principle of mathematical induction***.
- Notes: Axioms (PA1)-(PA8) are particular wffs, whereas (PA9) is an axiom schema providing an infinite number of axioms. **PA** is a first-order (**CFOPC**) theory with equality.

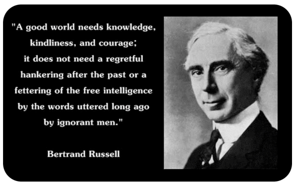
Russell's Paradox

♣ ***Russell's Paradox*** [Russell, 1901] [SEP]

- Russell's paradox is the most famous of the logical or set-theoretical paradoxes. Also known as the Russell-Zermelo paradox, the paradox arises within naïve set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself. Hence the paradox.



Russell in November 1957



Russell's Paradox

♣ A classification of all sets in Naïve Set Theory

- The first type: All sets that do not include itself as an element.
- The second type: All sets that include itself as an element.

♣ ***Russell's Paradox*** [Russell, 1901]

- Question: Let $M =_{df} \{x \mid x \notin x\}$. M is a set of the first type, or the second type?
- If M is a set of the first type, then because it does not include itself as an element, i.e., $x \notin x$, therefore, $M \in M$, it should be a set of the second type.
- If M is a set of the second type, then because it includes itself as an element, i.e., $x \in x$, therefore, $M \notin M$, it should be a set of the first type.

First-Order Theory NBG (Axiomatic Set Theory) [Mendelson]

♣ ***NBG axiomatic set theory*** [von Neumann (1925, 1928), Robinson (1937), Bernays (1937–1954), Gödel (1940)]

- **NBG** axiomatic set theory was originally proposed by J. von Neumann (1925, 1928) and later thoroughly revised and simplified by R. Robinson (1937), P. Bernays (1937–1954), and K. Gödel (1940).

♣ Purpose of presenting **NBG**

- **NBG** axiomatic set theory is the most fundamental theory of **CML** as well as mathematics.
- As an outstanding example, to show how to use the language of **CFOPC** to present a very useful important formal theory.

First-Order Theory NBG [Mendelson]: Predicates, Functions, Individual Constants, and Individual Variables

♣ Predicates and functions

- **NBG** has a single predefined predicate symbol (letter) $p^2{}_2$.
- We shall abbreviate $p^2{}_2\ (X, Y)$ by $X{\in}Y$, and $\neg p^2{}_2\ (X, Y)$ by $X{\notin}Y$.
- Intuitively, '$\in$' is to be thought of as the ***membership relation***.
- **NBG** has no predefined function symbol (letter) (some function symbols will be introduced and defined).

♣ Individual constants and individual variables

- **NBG** has no predefined individual constant symbols (letters) (some individual constant symbols will be introduced and defined).
- We shall use capital italic letters $X_1, X_2, X_3, \ldots$ as variable symbols instead of $x_1, x_2, x_3, \ldots$.
- As usual, we shall use meta-symbols $X, Y, Z, \ldots$ to represent variables.
- The values of the variables are to be thought of as ***classes***.

First-Order Theory NBG [Mendelson]: Classes

♣ ***Classes***

- ***Classes*** are certain collections of definite and distinguishable objects.
- Some properties determine classes, in the sense that a property P may determine a class of all those objects that possess that property.
- This "interpretation" is as imprecise as the notions of "collection" and "property."

♣ Note

- Recall that the definition of set in Naïve Set Theory is "A set S is a collection of definite and distinguishable objects (called elements or members of S), to be conceived as a whole."

First-Order Theory NBG [Mendelson]: Axioms

♣ ***Axioms***

- The ***axioms*** will reveal more about what we have in mind.
- They will provide us with the classes (sets) we need in mathematics and appear modest enough so that contradictions are not derivable from them.

First-Order Theory NBG [Mendelson]: Equality

♣ Definition (***equality***)

- $X=Y$ for $(\forall Z)(Z \in X \leftrightarrow Z \in Y)$
- '$\leftrightarrow$' is the logical connective equivalence.
- '$\in$' is used to represent the only predicate of the first-order theory **NBG**.
- '=' is defined as the ***equality relation***. We abbreviate $\neg(X=Y)$ by $X \neq Y$.
- This definition says "two classes are equal when and only when they have the same members."
- **NBG** is a first-order (**CFOPC**) theory with equality.
- Previously, 'p^2_1' was used for the equality, as a predefined predicate.

♣ Definition (***unique quantifier***)

- $(\exists_1 X)p(X)$ for $(\exists X)p(X) \wedge (\forall Y_1)(\forall Y_2)((p(Y_1) \wedge p(Y_2)) \rightarrow Y_1=Y_2)$

First-Order Theory NBG [Mendelson]: Inclusion

♣ Definitions (***inclusion*** and ***proper inclusion relation***)

- $X \subseteq Y$ for $(\forall Z)(Z \in X \rightarrow Z \in Y)$ (***inclusion relation***)
- $X \subset Y$ for $(X \subseteq Y) \wedge (X \neq Y)$ (***proper inclusion relation***)
- When $X \subseteq Y$, we say that X is a ***subclass*** of Y. When $X \subset Y$, we say that X is a ***proper subclass*** of Y.

♣ Propositions

- $\vdash_{NBG} X=Y \leftrightarrow (X \subseteq Y) \wedge (Y \subseteq X))$
- $\vdash_{NBG} X=X$
- $\vdash_{NBG} X=Y \rightarrow Y=X$
- $\vdash_{NBG} X=Y \rightarrow (Y=Z \rightarrow X=Z)$

First-Order Theory NBG [Mendelson]: Sets

♣ Definitions (***set*** and ***proper class***)

- $M(X)$ for $(\exists Y)(X \in Y)$ (***X is a set***)
- A class is a set if it is a member of some class.
- $Pr(X)$ for $\neg M(X)$ (***X is a proper class***)
- Those classes that are not sets are called proper classes.
- The usual derivations of the paradoxes now no longer lead to contradictions but only yield the results that various classes are proper classes, not sets. (Why?)
- The sets are intended to be those safe, comfortable classes that are used by mathematicians in their daily work, whereas proper classes are thought of as monstrously large collections that, if permitted to be sets (i.e., allowed to belong to other classes), would engender contradictions.

♣ Proposition

- $\vdash_{NBG} X \in Y \rightarrow M(X)$

First-Order Theory NBG [Mendelson]: Classes and Concrete Individuals

♣ Classes and concrete individuals

- The formal system **NBG** is designed to handle classes, not concrete individuals.
- The reason for this is that mathematics has no need for objects (such as cows and molecules); all mathematical objects and relations can be formulated in terms of classes alone.
- If non-classes are required for applications to other sciences, then the system **NBG** can be modified slightly so as to apply to both classes and non-classes alike.

First-Order Theory NBG [Mendelson]: Special Restricted Variables for Sets

♣ Special restricted variables for sets

- Let us introduce lower-case letters $x_1, x_2, x_3, \ldots$ as special restricted variables for sets.
- $(\forall x_j)B(x_j)$ stands for $(\forall X)(M(X) \rightarrow B(X))$, that is, B holds for all sets, and $(\exists x_j)B(x_j)$ stands for $(\exists X)(M(X) \wedge B(X))$, that is, B holds for some set.
- As usual, the variable X used in these definitions should be the first one that does not occur in $B(x_j)$.
- We shall use $x, y, z, \ldots$ to stand for arbitrary set variables.

First-Order Theory NBG [Mendelson]: The Extensionality Principle

♣ Proposition (the ***extensionality principle***)

- $\vdash_{NBG} X=Y \leftrightarrow (\forall z)(z \in X \leftrightarrow z \in Y)$
- Two classes are equal when and only when they contain the same sets as members.

First-Order Theory NBG [Mendelson]: Axioms T and P

♣ **Axiom T**
- $X_1=X_2 \rightarrow (X_1 \in X_3 \leftrightarrow X_2 \in X_3)$
- This axiom states that equal classes belong to the same classes.

♣ Proposition
- $\vdash_{NBG} (\mathrm{M}(Z) \wedge Z=Y) \rightarrow \mathrm{M}(Y)$

♣ **Axiom P** (***Pairing Axiom***)
- $(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (u=x \vee u=y))$
- This axiom states that for any sets x and y, there is a set z that has x and y as its only members.

♣ Propositions
- $\vdash_{NBG} (\forall X)(\mathrm{M}(X) \leftrightarrow (\exists y)(X \in y))$
- $\vdash_{NBG} (\exists X)\mathrm{Pr}(X) \rightarrow \neg(\forall Y)(\forall Z)(\exists W)(\forall U)(U \in W \leftrightarrow (U=Y \vee U=Z))$

First-Order Theory NBG [Mendelson]: Axiom N

♣ **Axiom N** (The ***Null (Empty) Set***)
- $(\exists x)(\forall y)(y \notin x)$
- This axiom states that there is a set that has no members.

♣ The empty set as an individual constant
- From axiom N and the extensionality principle, there is a unique set that has no members, i.e., $\vdash_{NBG} (\exists_1 x)(\forall y)(y \notin x)$.
- We can introduce a new individual constant $\varnothing$ by definition: $(\forall y)(y \notin \varnothing)$.

First-Order Theory NBG [Mendelson]: Unordered Pair

♣ Proposition (***unordered pair***)
- $\vdash_{NBG} (\forall x)(\forall y)(\exists_1 z)(\forall u)(u \in z \leftrightarrow (u=x \vee u=y))$
- There is a unique set z, called the ***unordered pair*** of x and y, such that z has x and y as its only members.
- We can introduce a new function symbol (letter) 'g', $g(x, y)$, to designate the unordered pair of x and y. In accordance with the traditional notation, we shall write $\{x, y\}$ instead of $g(x, y)$.
- We have to define a unique value for $\{X, Y\}$ for any classes X and Y, not only for sets x and y. We shall let $\{X, Y\}$ be $\varnothing$ whenever X is not a set or Y is not a set.

♣ Propositions
- $\vdash_{NBG} (\exists_1 Z)(\ [(\neg\mathrm{M}(X) \vee \neg\mathrm{M}(Y)) \wedge Z=\varnothing] \vee$
  $[\mathrm{M}(X) \wedge \mathrm{M}(Y) \wedge (\forall u)(u \in Z \leftrightarrow (u=X \vee u=Y))]\ )$
- $\vdash_{NBG} [\mathrm{M}(X) \wedge \mathrm{M}(Y)) \wedge (\forall u)(u \in \{X, Y\} \leftrightarrow (u=X \vee u=Y))] \vee$
  $[(\neg\mathrm{M}(X) \vee \neg\mathrm{M}(Y)) \wedge \{X, Y\}=\varnothing]$

First-Order Theory NBG [Mendelson]: Unordered Pair

♣ Propositions
- $\vdash_{NBG} (\forall x)(\forall y)(\forall u)(u \in \{x, y\} \leftrightarrow (u=x \vee u=y))$
- $\vdash_{NBG} (\forall X)(\forall Y)\mathrm{M}(\{X, Y\})$
- $\vdash_{NBG} \{X, Y\} \leftrightarrow \{Y, X\}$

First-Order Theory NBG [Mendelson]: Singleton

♣ Definition (***singleton***)
- $\{X\}$ for $\{X, X\}$
- For a set x, $\{x\}$ is called the ***singleton*** of x. It is a set that has x as its only member.

- Proposition
  - $\vdash_{NBG} (\forall x)(\forall y)(\{x\}=\{y\} \leftrightarrow x=y)$

First-Order Theory NBG [Mendelson]: Ordered Pair

♣ Definition (***ordered pair***) [Kuratowski, 1921]
- $<X, Y>$ for $\{\{X\}, \{X, Y\}\}$
- For sets x and y, $<x, y>$ is called the ***ordered pair*** of x and y.

♣ Proposition
- $\vdash_{NBG} (\forall x)(\forall y)(\forall u)(\forall v)(<x, y>=<u, v> \rightarrow (x=u \wedge y=v))$

♣ Definitions
- $<X> = X$
- $<X_1, \ldots, X_n, X_{n+1}> = <<X_1, \ldots, X_n>, X_{n+1}>$

♣ Proposition (the generalization of the above proposition)
- $\vdash_{NBG} (\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_n)(<x_1, \ldots, x_n>=<y_1, \ldots, y_n> \rightarrow (x_1=y_1 \wedge \ldots \wedge x_n=y_n))$

First-Order Theory NBG [Mendelson]: Axioms of Class Existence

♣ **Axioms of Class Existence**

- (B1) $(\exists X)(\forall u)(\forall v)(\langle u, v\rangle \in X \leftrightarrow u \in v)$ (∈-relation)
- (B2) $(\forall X)(\forall Y)(\exists Z)(\forall u)(u \in Z \leftrightarrow (u \in X \wedge u \in Y))$ (intersection)
- (B3) $(\forall X)(\exists Z)(\forall u)(u \in Z \leftrightarrow u \notin X)$ (complement)
- (B4) $(\forall X)(\exists Z)(\forall u)(u \in Z \leftrightarrow (\exists v)(\langle u, v\rangle \in X))$ (domain)
- (B5) $(\forall X)(\exists Z)(\forall u)(\forall v)(\langle u, v\rangle \in Z \leftrightarrow u \in X)$
- (B6) $(\forall X)(\exists Z)(\forall u)(\forall v)(\forall w)(\langle u, v, w\rangle \in Z \leftrightarrow \langle v, w, u\rangle \in X)$
- (B7) $(\forall X)(\exists Z)(\forall u)(\forall v)(\forall w)(\langle u, v, w\rangle \in Z \leftrightarrow \langle u, w, v\rangle \in X)$

First-Order Theory NBG [Mendelson]: Class Operations

♣ Propositions (Justifying the introduction of $X \cap Y$, X^C, and $Dom(X)$)

- From axioms (B2)-(B4) and the extensionality principle, we can obtain:
- $\vdash_{\text{NBG}} (\forall X)(\forall Y)(\exists_1 Z)(\forall u)(u \in Z \leftrightarrow (u \in X \wedge u \in Y))$
- $\vdash_{\text{NBG}} (\forall X)(\exists_1 Z)(\forall u)(u \in Z \leftrightarrow u \notin X)$
- $\vdash_{\text{NBG}} (\forall X)(\exists_1 Z)(\forall u)(u \in Z \leftrightarrow (\exists v)(\langle u, v\rangle \in X))$

♣ Definitions ($X \cap Y$, X^C, $Dom(X)$, $X \cup Y$, V, $X - Y$)

- $(\forall u)(u \in (X \cap Y) \leftrightarrow (u \in X \wedge u \in Y))$ (***intersection*** of X and Y)
- $(\forall u)(u \in X^C \leftrightarrow u \notin X)$ (***complement*** of X)
- $(\forall u)(u \in Dom(X) \leftrightarrow (\exists v)(\langle u, v\rangle \in X))$ (***domain*** of X)
- $X \cup Y = (X^C \cap Y^C)^C$ (***union*** of X and Y)
- $V = \varnothing^C$ (***universal class***) (V is a proper class)
- $X - Y = X \cap Y^C$ (***difference*** of X and Y)
- '$\cap$', '^{C}', 'Dom', '$\cup$', '$-$' are new function symbols (letters)

First-Order Theory NBG [Mendelson]: Class Operations

♣ Propositions

- $\vdash_{\text{NBG}} (\forall u)(u \in X \cup Y \leftrightarrow (u \in X \vee u \in Y))$
- $\vdash_{\text{NBG}} (\forall u)(u \in V)$ (V is the universal class)
- $\vdash_{\text{NBG}} (\forall u)(u \in X - Y \leftrightarrow (u \in X \wedge u \notin Y))$
- $\vdash_{\text{NBG}} X \cap X = X$, $\vdash_{\text{NBG}} X \cup X = X$
- $\vdash_{\text{NBG}} X \cap Y = Y \cap X$, $\vdash_{\text{NBG}} X \cup Y = Y \cup X$
- $\vdash_{\text{NBG}} (X \cap Y) \cap Z = X \cap (Y \cap Z)$, $\vdash_{\text{NBG}} (X \cup Y) \cup Z = X \cup (Y \cup Z)$
- $\vdash_{\text{NBG}} X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, $\vdash_{\text{NBG}} X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
- $\vdash_{\text{NBG}} X \subseteq Y \leftrightarrow (X \cap Y = X)$, $\vdash_{\text{NBG}} X \subseteq Y \leftrightarrow (X \cup Y = Y)$
- $\vdash_{\text{NBG}} X \cap \varnothing = \varnothing$, $\vdash_{\text{NBG}} X \cup \varnothing = X$, $\vdash_{\text{NBG}} X \cap V = X$, $\vdash_{\text{NBG}} X \cup V = V$
- $\vdash_{\text{NBG}} (X \cap Y)^C = X^C \cup Y^C$, $\vdash_{\text{NBG}} (X \cup Y)^C = X^C \cap Y^C$
- $\vdash_{\text{NBG}} (X^C)^C = X$, $\vdash_{\text{NBG}} V^C = \varnothing$, $\vdash_{\text{NBG}} X - X = \varnothing$, $\vdash_{\text{NBG}} V - X = X^C$
- $\vdash_{\text{NBG}} X - (X - Y) = X \cap Y$, $\vdash_{\text{NBG}} Y \subseteq X^C \rightarrow (X - Y) = X$

First-Order Theory NBG [Mendelson]: Class Existence Theorem

♣ Proposition (***Class Existence Theorem***)

- By a ***predicative formula*** we mean a formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ whose variables occur among $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ and in which only set variables are quantified (i.e., φ can be abbreviated in such a way that only set variables are quantified).
- Let $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ be a predicative formula. Then:
- $\vdash_{\text{NBG}} (\exists Z)(\forall x_1)\ldots(\forall x_n)(\langle x_1, \ldots, x_n\rangle \in Z \leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m))$

First-Order Theory NBG [Mendelson]: Cartesian (Direct) Product

♣ Propositions

- Let $\varphi(X, Y_1, Y_2)$ be $(\exists u)(\exists v)(X = \langle u, v\rangle \wedge u \in Y_1 \wedge v \in Y_2)$. The only quantifiers in φ involve set variables. Hence, by the class existence theorem, $\vdash_{\text{NBG}} (\exists Z)(\forall x)(x \in Z \leftrightarrow (\exists u)(\exists v)(x = \langle u, v\rangle \wedge u \in Y_1 \wedge v \in Y_2))$.
- By the extensionality principle,
- $\vdash_{\text{NBG}} (\exists_1 Z)(\forall x)(x \in Z \leftrightarrow (\exists u)(\exists v)(x = \langle u, v\rangle \wedge u \in Y_1 \wedge v \in Y_2))$.
- There is a unique class Z whose elements are ordered pairs from Y_1 and Y_2.

♣ Definition (***Cartesian (direct) product*** of Y_1 and Y_2)

- $(\forall x)(x \in (Y_1 \times Y_2) \leftrightarrow (\exists u)(\exists v)(x = \langle u, v\rangle \wedge u \in Y_1 \wedge v \in Y_2))$
- '$\times$' is a new function symbol (letter).

♣ Definitions (***Cartesian power (Cartesian exponentiation)*** and ***Relation***)

- Y^2 for $Y \times Y$
- Y^n for $Y^{n-1} \times Y$ when $n > 2$
- $Rel(X)$ for $X \subseteq Y^2$ (X is a ***birary relation*** defined on Y^2)

First-Order Theory NBG [Mendelson]: Power Class

♣ Proposition

- Let $\varphi(X, Y)$ be $X \subseteq Y$. By the class existence theorem and the extensionality principle, $\vdash_{\text{NBG}} (\exists_1 Z)(\forall x)(x \in Z \leftrightarrow x \subseteq Y)$.
- Thus, there is a unique class Z that has as its members all subsets of Y. Z is called the power class of Y and is denoted $P(Y)$.

♣ Definition (***Power class***)

- $(\forall x)(x \in P(Y) \leftrightarrow x \subseteq Y)$

First-Order Theory NBG [Mendelson]: Sum Class

♣ Proposition

- Let $\varphi(X, Y)$ be $(\exists v)(X \in v \wedge v \in Y))$. By the class existence theorem and the extensionality principle, $\vdash_{\mathrm{NBG}} (\exists_1 Z)(\forall x)(x \in Z \Leftrightarrow (\exists v)(x \in v \wedge v \in Y))$.
- Thus, there is a unique class Z that contains its members of members of Y. Z is called the sum class of Y and is denoted $\bigcup Y$.

♣ Definition (***sum class***)

- $(\forall x)(x \in \bigcup Y \Leftrightarrow (\exists v)(x \in v \wedge v \in Y))$

First-Order Theory NBG [Mendelson]: Identity Relation

♣ Proposition

- Let $\varphi(X)$ be $(\exists u)(X = \langle u, u \rangle)$. By the class existence theorem and the extensionality principle, there is a unique class Z such that $(\forall x)(x \in Z \Leftrightarrow (\exists u)(x = \langle u, u \rangle))$.
- Z is called the identity relation and is denoted I.

♣ Definition (***identity relation***)

- $(\forall x)(x \in I \Leftrightarrow (\exists u)(x = \langle u, u \rangle))$

First-Order Theory NBG [Mendelson]: Class of n-Tuples

♣ Corollary

- If $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ be a predicative formula, then $\vdash_{\mathrm{NBG}} (\exists_1 W)(W \subseteq V^n \wedge$
  $(\forall x_1)\ldots(\forall x_n)(\langle x_1, \ldots, x_n \rangle \in W \Leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)))$.

♣ Definition

- Given a predicative formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$, let
  $\{ \langle x_1, \ldots, x_n \rangle \mid \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m) \}$
  denotes the class of all ***n-tuples*** $\langle x_1, \ldots, x_n \rangle$ that satisfy $\varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)$; that is,
  $(\forall u)(u \in \{ \langle x_1, \ldots, x_n \rangle \mid \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m) \}$
  $\Leftrightarrow (\exists x_1)\ldots(\exists x_n)(u = \langle x_1, \ldots, x_n \rangle \wedge \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)))$.
- This definition is justified by the above corollary.
- When $n=1$, $\vdash_{\mathrm{NBG}} (\forall u)(u \in \{ x \mid \varphi(x, Y_1, \ldots, Y_m) \} \Leftrightarrow \varphi(u, Y_1, \ldots, Y_m)$.

First-Order Theory NBG [Mendelson]: Inverse Relation and Range

♣ Notation (***inverse relation***)

- Take φ to be $\langle x_2, x_1 \rangle \in Y$.
- Let $Y^{\smile}$ be an abbreviation for $\{ \langle x_1, x_2 \rangle \mid \langle x_2, x_1 \rangle \in Y \}$.
- $Y^{\smile} \subseteq V^2 \wedge (\forall x_1)(\forall x_2)(\langle x_1, x_2 \rangle \in Y^{\smile} \Leftrightarrow \langle x_2, x_1 \rangle \in Y)$
- $Y^{\smile}$ is called the ***inverse relation*** of Y.

♣ Notation (***range***)

- Take φ to be $(\exists v)(\langle v, x \rangle \in Y)$.
- Let $\mathrm{Ran}(Y)$ stand for $\{ x \mid (\exists v)(\langle v, x \rangle \in Y) \}$.
- $\vdash_{\mathrm{NBG}} (\forall u)(u \in \mathrm{Ran}(Y) \Leftrightarrow (\exists v)(\langle v, u \rangle \in Y))$
- $\vdash_{\mathrm{NBG}} \mathrm{Ran}(Y) = \mathrm{Dom}(Y^{\smile})$
- $\mathrm{Ran}(Y)$ is called the ***range*** of Y.

First-Order Theory NBG [Mendelson]: Axioms U, W, and S

♣ **Axiom U** (***Sum Set***)

- $(\forall x)(\exists y)(\forall u)(u \in y \Leftrightarrow (\exists v)(u \in v \wedge v \in x))$
- This axiom states that the sum class $\bigcup x$ of a set x is also a set, which we shall call the sum set of x, that is, $\vdash_{\mathrm{NBG}} (\forall x)\mathrm{M}(\bigcup x)$. The sum set $\bigcup x$ is usually referred to as the union of all the sets in the set x.

♣ **Axiom W** (***Power Set***)

- $(\forall x)(\exists y)(\forall u)(u \in y \Leftrightarrow u \subseteq x)$
- This axiom states that the power class $\mathrm{P}(x)$ of a set x is itself a set, that is, $\vdash_{\mathrm{NBG}} (\forall x)\mathrm{M}(\mathrm{P}(x))$.

♣ **Axiom S** (***Subsets***)

- $(\forall x)(\forall Y)(\exists z)(\forall u)(u \in z \Leftrightarrow (u \in x \wedge u \in Y))$
- This axiom states that there is a subset of a class such that its any member is a member of that class.

First-Order Theory NBG [Mendelson]: Intersection

♣ Corollary

- $\vdash_{\mathrm{NBG}} (\forall x)(\forall Y)\mathrm{M}(x \cap Y)$ (The intersection of a set and a class is a set.)
- $\vdash_{\mathrm{NBG}} (\forall x)(\forall Y)(Y \subseteq x \rightarrow \mathrm{M}(Y))$ (A subclass of a set is a set.)
- For any predicative formula $\mathrm{B}(y)$, $\vdash_{\mathrm{NBG}} (\forall x)\mathrm{M}(\{ y \mid y \in x \wedge \mathrm{B}(y) \})$

♣ Definition (***intersection***)

- Based on axiom S, we can show that the intersection of any non-empty class of sets is a set.
- $\bigcap X$ for $\{ y \mid (\forall x)(x \in X \rightarrow y \in x) \}$ (***intersection***)

♣ Propositions

- $\vdash_{\mathrm{NBG}} (\forall x)(x \in X \rightarrow \bigcap X \subseteq x)$
- $\vdash_{\mathrm{NBG}} X \neq \varnothing \rightarrow \mathrm{M}(\bigcap X)$
- $\vdash_{\mathrm{NBG}} \bigcap \varnothing = V$

First-Order Theory NBG [Mendelson]: Functions

♣ Definition (***function***)

- Fnc(X) for Rel(X) ∧ $(\forall x)(\forall y)(\forall z)((<x, y>\in X \wedge <x, z>\in X) \rightarrow (y=z))$ (X is a ***function***)
- X: $Y \rightarrow Z$ for Fnc(X) ∧ Dom(X)=Y ∧ Ran(X)=Z (X is a ***function from Y(Domain) into Z (Range)***)
- $R \leftarrow X$ for $X \cap (R \times V)$ (***restriction*** of function X to the domain R) (V is the universal class, a proper class)
- $\text{Fnc}_1(X)$ for Fnc(X) ∧ Fnc($X^{\smile}$) (X is a ***one-one function, bijection***)
- $X'Y = z$ if $(\forall u)(<Y, u>\in X \leftrightarrow u=z)$, or ∅ otherwise
- $X''R$ = Ran($R \leftarrow X$)
- If there is a unique z such that $<y, z>\in X$, then $z = X'y$; otherwise, $X'y=\emptyset$.
- If X is a function and y is a set in its domain, $X'y$ is the value of the function applied to y (i.e., in traditional set-theoretic notation, $X(y)$).
- If X is a function, $X''R$ is the range of X restricted to R (i.e., the image of R).

First-Order Theory NBG [Mendelson]: Axioms R and I

♣ **Axiom R** (***Replacement***)

- Fnc(Y) → $(\forall x)(\exists y)(\forall u)(u\in y \leftrightarrow (\exists v)(<v, u>\in Y \wedge v\in x))$
- Axiom R asserts that, if Y is a function and x is a set, then the class of second components of ordered pairs in Y whose first components are in x is a set (or, equivalently, Ran(x←Y) is a set).

♣ **Axiom I** (***Infinity***)

- $(\exists x)(\emptyset\in x \wedge (\forall u)(u\in x \rightarrow u\cup\{u\}\in x))$
- Axiom I states that there is a set x that contains ∅ and such that, whenever a set u belongs to x, then $u\cup\{u\}$ also belongs to x.
- Hence, for such a set x, $\{\emptyset\}\in x$, $\{\emptyset, \{\emptyset\}\}\in x$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\in x$, and so on.
- If we let 1 stand for {∅}, 2 for {∅, 1}, 3 for {∅, 1, 2}, …, and n for $\{\emptyset, 1, 2, \ldots, n-1\}$, etc., then, for all ordinary integers $n\geq 0$, $n\in x$, and $\emptyset\neq 1, \emptyset\neq 2, 1\neq 2, \emptyset\neq 3, 1\neq 3, 2\neq 3, \ldots$.

First-Order Theory NBG [Mendelson]: Axioms of NBG

♣ **Axioms of NBG**

- **NBG** has only a finite number of axioms -- namely,
  axiom T,
  axiom P (pairing),
  axiom N (null set),
  axiom U (sum set),
  axiom W (power set),
  axiom S (subsets),
  axiom R (replacement),
  axiom I (infinity), and
  the seven class existence axioms (B1)–(B7).

♣ Note

- Axiom S is provable from the other axioms; it has been included here because it is of interest in the study of certain weaker sub-theories of **NBG**.

First-Order Theory NBG [Mendelson]: Summary about Paradoxes

♣ Untenableness of Russell's paradox

- The usual argument for Russell's paradox does not hold in **NBG**.
- By the class existence theorem, there is a class $Y = \{x \mid x \notin x\}$.
- Then $(\forall x)(x \in Y \leftrightarrow x \notin x)$.
- In unabbreviated notation this becomes $(\forall X)(\text{M}(X) \rightarrow (X \in Y \leftrightarrow X \notin X))$.
- Assume M(Y). Then $Y\in Y \leftrightarrow Y\notin Y$, which, by the tautology $(A \leftrightarrow \neg A) \rightarrow (A \wedge \neg A)$, yields $Y\in Y \wedge Y\notin Y$.
- Hence, by the derived rule of proof by contradiction, we obtain $\vdash \neg\text{M}(Y)$.
- Thus, in **NBG**, the argument for Russell's paradox merely shows that Russell's class Y is a proper class, not a set.

♣ Untenableness of the paradoxes of Cantor and Burali-Forti

- The **NBG** will avoid the paradoxes of Cantor and Burali-Forti in a similar way.

An Introduction to Classical Predicate Calculus

♣ The Limitations of Propositional Logic **CPC**
♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
♣ Substitutions
♣ Semantics (Model Theory) of **CFOPC**
♣ Semantic (Model-theoretical, Logical) Consequence Relation
♣ Hilbert Style Formal Logic Systems for **CFOPC**
♣ Gentzen's Natural Deduction System for **CFOPC**
♣ Gentzen's Sequent Calculus System for **CFOPC**
♣ Semantic Tableau Systems for **CFOPC**
♣ Resolution Systems for **CFOPC**
♣ Classical Second-Order Predicate Calculus (**CSOPC**)