An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen's Natural Deduction System for **CFOPC**
- ♣ Gentzen's Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

L: A Hilbert Style Formal System for CFOPC [Mendelson]

♣ Axiom schemata of **L** ($A, B, C \in$ **WFF**)

- AS1 $(A \to (B \to A))$
- AS2 $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$
- AS3 $(((\neg A) \to (\neg B)) \to (B \to A))$ [or $(((\neg A) \to (\neg B)) \to ((\neg A) \to B) \to A))$]
- AS4 $(((\forall x)A) \to A[x/t])$, if x may appear free in A and t is free for x in A (i.e., free variables of t do not occur bound in A)
- AS4' $(((\forall x)A) \to A)$, if x does not occur free in A (a special case of AS4 when t=x)
- AS5 $(((\forall x)(A \to B)) \to (A \to ((\forall x)B)))$, if x does not occur free in A

♣ Inference rules of **L** ($A, B \in$ **WFF**)

- Modus Ponens (MP) for material implication: From $A \to B$ and A to infer B.
- ***Generalization (Gen)***: From A to infer $((\forall x)A)$.

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS4

- AS4 $((\forall x)A) \to A[x/t]$, if x may appear free in A and t is free for x in A (i.e., free variables of t do not occur bound in A)
- Relaxation of the restriction that t is free for x in A would lead to the following disaster.
- If t were not free for x in A, the following unpleasant result would arise.
- Let A be $\neg[(\forall y)p(x, y)]$ and let t be y. Notice that t is not free for x in A.
- Consider the following pseudo-instance of axiom AS4: as4: $((\forall x)(\neg[(\forall y)p(x, y)])) \to \neg[(\forall y)p(y, y)]$
- Now take as an interpretation such that any domain with at least two different members and let p stand for the identity relation.
- Then the antecedent of as4 is true and the consequent false. Thus, as4 is false for this interpretation.
- Note: $(\forall x)(\neg[(\forall y)p(x, y)]) \to \neg[(\forall y)p(x, y)]$
  $\equiv (\forall z)(\neg[(\forall y)p(z, y)]) \to \neg[(\forall y)p(x, y)]$

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS4'

- AS4' $((\forall x)A) \to A$, if x does not occur free in A
- Relaxation of the restriction that x not be free in A would lead to the following disaster.
- If x occurs free in A, the following unpleasant result would arise.
- Let A be $\neg[(\forall y)p(x, y)]$. Notice that x occurs free in A.
- Consider the following pseudo-instance of axiom AS4': as4': $(\forall x)(\neg[(\forall y)p(x, y)]) \to \neg[(\forall y)p(x, y)]$
- Now take as an interpretation such that any domain with at least two different members and let p stand for the identity relation.
- Then the antecedent of as4' is true and the consequent may be false for some x^A=y^A. Thus, as4' is false for this interpretation.
- Note: $(\forall x)(\neg[(\forall y)p(x, y)]) \to \neg[(\forall y)p(x, y)]$
  $\equiv (\forall z)(\neg[(\forall y)p(z, y)]) \to \neg[(\forall y)p(x, y)]$

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS5

- AS5 $((\forall x)(A \to B)) \to (A \to ((\forall x)B))$, if x does not occur free in A
- Relaxation of the restriction that x not be free in A would lead to the following disaster.
- Let A and B both be $p(x)$. Thus, x is free in A.
- Consider the following pseudo-instance of axiom AS5: as5: $((\forall x)(p(x) \to p(x))) \to (p(x) \to ((\forall x)p(x)))$
- The antecedent of as5 is valid. Now take as domain the set of integers and let $p(x)$ mean that x is even. Then $(\forall x)p(x)$ is false.
- So, any x^A which is even does not satisfy the consequent of as5. Hence, as5 is not true for this interpretation.

An Example of Deduction in L

♣ $\{B, ((\forall x_1)B) \to C\} \vdash_L (\forall x_1)C$? (Omit the outermost brackets)

1. B { Premise }
2. $(\forall x_1)B$ { Apply Gen to 1 }
3. $((\forall x_1)B) \to C$ { Premise }
4. C { Follow from 2 and 3 by MP }
5. $(\forall x_1)C$ { Apply Gen to 4 }

♣ Question

- $\vdash_L B \to [(((\forall x_1)B) \to C) \to ((\forall x_1)C)]$?
- Does the deduction theorem (if $\Gamma \cup \{A\} \vdash_L B$, then $\Gamma \vdash_L A \to B$) hold without condition for **L** (CFOPC)?

Properties of L: Deduction Theorem

♣ ***Dependence in deduction***

- Let $\Gamma \subseteq \mathbf{WFF}$ and $B \in \Gamma$. Assume that we are given a deduction $d_1, \ldots, d_n$ from Γ, together with justification for each step in the deduction.
- We say that d_i ***depends upon*** B in this deduction IFF
  (1) d_i is B and the justification for d_i is that it belongs to Γ, or
  (2) d_i is justified as a direct consequence by MP or Gen of some preceding formulas of the sequence, where at least one of these preceding formulas depends upon B.

♣ Example: $\{B, ((\forall x_1)B) \to C\} \vdash_{\mathbf{L}} (\forall x_1)C$

- d_1. B { Premise }
  d_2. $(\forall x_1)B$ { Apply Gen to d_1 }
  d_3. $((\forall x_1)B) \to C$ { Premise }
  d_4. C { Follow from d_2, and d_3 by MP }
  d_5. $(\forall x_1)C$ { Apply Gen to d_4 }
- Here, d_1 depends upon B, d_2 depends upon B, d_3 depends upon $((\forall x_1)B) \to C$, d_4 depends upon B and $((\forall x_1)B) \to C$, and d_5 depends upon B and $((\forall x_1)B) \to C$.

Properties of L: Deduction Theorem

♣ ***Syntactic (proof-theoretical) deduction theorem (DT) for*** **L** [Mendelson]

- For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and no application of Gen to a formula C that depends upon A has, as its quantified variable, a free variable of A. Then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- Corollary: For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and the deduction involves no application of Gen of which the quantified variables is free in A, then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- Corollary: For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and A is a closed, then $\Gamma \vdash_{\mathbf{L}} A \to B$.

♣ ***Syntactic (proof-theoretical) deduction theorem (DT) for*** **L** [Hamilton]

- For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and the deduction contains no application of Gen involving a variable which occurs free in A, then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- Corollary (HS): For any $A, B, C \in \mathbf{WFF}$, $\{A \to B, B \to C\} \vdash_{\mathbf{L}} A \to C$.

Properties of L: Deduction Theorem

♣ Notes

- The DT for the propositional calculus cannot be carried over without modification to the first-order predicate calculus.
- For example, for any $A \in \mathbf{WFF}$, by Gen, $A \vdash_{\mathbf{L}} (\forall x)A$; but it is not always the case that $\vdash_{\mathbf{L}} A \to (\forall x)A$.
- Consider a domain $\boldsymbol{D}$ containing at least two elements c and d.
- Let A be $p^1{}_1(x_1)$. Interpret $p_1{}^1$ as a property that holds only for c.
- Then $p^1{}_1(x_1)$ is satisfied by any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ where $x_1{}^{IA} = x_1{}^A = c$ and $c \in p^1{}_1{}^{I}$ (i.e., $v_f{}^{IA}(p^1{}_1(x_1) = \mathbf{T})$, but $(\forall x_1)p^1{}_1(x_1)$ is satisfied by no $(\boldsymbol{D}, \boldsymbol{I})$ at all because $d \in \boldsymbol{D}$.
- Hence, $p^1{}_1(x_1) \to (\forall x_1)p^1{}_1(x_1)$ is not true in some models, and therefore it is not logically valid and should not be a logical theorem of **L**.

An Example of Theorem Proof in L

♣ $\vdash_{\mathbf{L}} (\forall x_1)((\forall x_2)B) \to (\forall x_2)((\forall x_1)B)$? (Omit the outermost brackets)

1. $(\forall x_1)((\forall x_2)B)$ { Premise }
2. $(\forall x_1)((\forall x_2)B) \to (\forall x_2)B$ { AS4' $(\forall x)A) \to A$ }
3. $(\forall x_2)B$ { Follow from 1 and 2 by MP }
4. $((\forall x_2)B) \to B$ { AS4' $(\forall x)A) \to A$ }
5. B { Follow from 3 and 4 by MP }
6. $(\forall x_1)B$ { Apply Gen to 5 }
7. $(\forall x_2)((\forall x_1)B)$ { Apply Gen to 6 }

Therefore, $(\forall x_1)((\forall x_2)B) \vdash_{\mathbf{L}} (\forall x_2)((\forall x_1)B)$.

In the above deduction, no application of Gen has, as a quantified variable, a free variable of $(\forall x_1)((\forall x_2)B)$.

Hence, by the DT, $\vdash_{\mathbf{L}} (\forall x_1)((\forall x_2)B) \to (\forall x_2)((\forall x_1)B)$.

An Example of Theorem Proof in L

♣ $\vdash_{\mathbf{L}} (\forall x_1)((\forall x_2)B) \to (\forall x_2)((\forall x_3)B)$? (Omit the outermost brackets)

1. $(\forall x_1)((\forall x_2)B)$ { Premise }
2. $(\forall x_1)((\forall x_2)B) \to (\forall x_2)B$ { AS4' $(\forall x)A) \to A$ }
3. $(\forall x_2)B$ { Follow from 1 and 2 by MP }
4. $((\forall x_2)B) \to B$ { AS4' $(\forall x)A) \to A$ }
5. B { Follow from 3 and 4 by MP }
6. $(\forall x_3)B$ { Apply Gen to 5 }
7. $(\forall x_2)((\forall x_3)B)$ { Apply Gen to 6 }

Therefore, $(\forall x_1)((\forall x_2)B) \vdash_{\mathbf{L}} (\forall x_2)((\forall x_3)B)$.

Now, whether the DT can be applied or not depends on whether x_3 is a free variable of B (Note: B depends on $(\forall x_1)((\forall x_2)B)$). Let us apply the DT, then we have: $\vdash_{\mathbf{L}} (\forall x_1)((\forall x_2)B) \to (\forall x_2)((\forall x_3)B)$.

However, if we let B be $p(x_1, x_2, x_3) = (x_1 > x_2 > x_3)$, then $(\forall x_1)((\forall x_2)p(x_1, x_2, x_3)) \to (\forall x_2)((\forall x_3)p(x_1, x_2, x_3))$ is not a valid theorem.

An Example of Theorem Proof in L

♣ $\vdash_{\mathbf{L}} (A \to (\forall x_1)B) \to (\forall x_1)(A \to B)$? (Omit the outermost brackets)

1. $A \to (\forall x_1)B$ { Premise }
2. $((\forall x_1)B) \to B$ { AS4' $(\forall x)A) \to A$ }
3. $A \to B$ { Follow from 1 and 2 by HS }
4. $(\forall x_1)(A \to B)$ { Apply Gen to 3}

Therefore, $A \to (\forall x_1)B \vdash_{\mathbf{L}} (\forall x_1)(A \to B)$.

Now, in the above deduction, Gen is used, but only using the variable x_1. Therefore, whether the DT can be applied or not depends on whether x_1 occurs free in A or not.

Hence, if x_1 does not occur free in A, then by the DT, $\vdash_{\mathbf{L}} (A \to (\forall x_1)B) \to (\forall x_1)(A \to B)$, provided that x_1 does not occur free in A.

On the other hand, if x_1 occurs free in A (Note: $A \to B$ depends $A \to (\forall x_1)B$), say we let A be $p(x_1)$, then $(p(x_1) \to (\forall x_1)B) \to (\forall x_1)(p(x_1) \to B)$ is not a valid theorem.

An Example of Theorem Proof in L

♣ $\vdash_{\mathbf{L}} ((\forall x_1)(A \to B)) \to [((\exists x_1)A) \to (\exists x_1)B]$? (Omit the outermost brackets)

1. $(\forall x_1)(A \to B)$	{ Premise }
2. $(\forall x_1)\neg B$	{ Premise }
3. $((\forall x_1)(A \to B)) \to (A \to B)$	{ AS4' $(\forall x)A) \to A$ }
4. $A \to B$	{ Follow from 1 and 3 by MP }
5. $(A \to B) \to ((\neg B) \to (\neg A))$	{ Logical theorem }
6. $(\neg B) \to \neg A$	{ Follow from 4 and 5 by MP }
7. $((\forall x_1)\neg B) \to \neg B$	{ AS4' $(\forall x)A) \to A$ }
8. $\neg B$	{ Follow from 2 and 7 by MP }
9. $\neg A$	{ Follow from 6 and 8 by MP }
10. $(\forall x_1)\neg A$	{ Apply Gen to 9 }

Therefore, the above shows a deduction $\{(\forall x_1)(A \to B), (\forall x_1)\neg B\} \vdash_{\mathbf{L}} (\forall x_1)\neg A$.

Because x_1 does not occur free in $(\forall x_1)\neg B$, by the DT, we get:
$(\forall x_1)(A \to B) \vdash_{\mathbf{L}} ((\forall x_1)\neg B) \to ((\forall x_1)\neg A)$.

By logical theorem $(A \to B) \to ((\neg B) \to (\neg A))$, we have:
$\vdash_{\mathbf{L}} [((\forall x_1)\neg B) \to ((\forall x_1)\neg A)] \to [\neg((\forall x_1)\neg A) \to \neg((\forall x_1)\neg B)]$.
By MP, we have $(\forall x_1)(A \to B) \vdash_{\mathbf{L}} \neg((\forall x_1)\neg A) \to \neg((\forall x_1)\neg B)$, i.e., $(\forall x_1)(A \to B) \vdash_{\mathbf{L}} ((\exists x_1)A) \to (\exists x_1)B$.
Because x_1 does not occur free in $(\forall x_1)(A \to B)$, by the DT, we get:
$\vdash_{\mathbf{L}} (\forall x_1)(A \to B) \to [((\exists x_1)A) \to ((\exists x_1)B)]$.

Properties of L: Consistency, Soundness, and Completeness

♣ ***Consistency of* L**

- For any $A \in \mathbf{WFF}$, not both $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} \neg A$, i.e., **L** is ***consistent***.

♣ ***Soundness theorems for* L**

- Theorem (***soundness***): If $\vdash_{\mathbf{L}} A$ then $\models_{\mathbf{CFOPC}} A$, for any $A \in \mathbf{WFF}$.
- Theorem (***strong soundness***): If $\Gamma \vdash_{\mathbf{L}} A$ then $\Gamma \models_{\mathbf{CFOPC}} A$, for any $A \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$.

♣ ***Completeness theorems for* L** (Gödel, 1930)

- Theorem (***completeness***): If $\models_{\mathbf{CFOPC}} A$ then $\vdash_{\mathbf{L}} A$, for any $A \in \mathbf{WFF}$.
- Theorem (***strong completeness***): If $\Gamma \models_{\mathbf{CFOPC}} A$ then $\Gamma \vdash_{\mathbf{L}} A$, for any $A \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$.

♣ **CFOPC vs. L**

- **Th**(**CFOPC**) = **Th**(**L**).

Properties of L: Completeness [Mendelson]

♣ ***Similar formulas***

- If x_i and x_j are distinct, then $B(x_i)$ and $B(x_j)$ are said to be ***similar*** IFF x_j is free for x_i in $B(x_i)$ and B (x_i) has no free occurrences of x_j. It is assumed here that $B(x_j)$ arises from $B(x_i)$ by substituting x_j for all free occurrences of x_i.
- If $B(x_i)$ and $B(x_j)$ are similar, then x_i is free for x_j in $B(x_j)$ and B (x_j) has no free occurrences of x_i. Thus, if $B(x_i)$ and $B(x_j)$ are similar, then $B(x_j)$ and $B(x_i)$ are similar.
- Intuitively, $B(x_i)$ and $B(x_j)$ are similar IFF $B(x_i)$ and $B(x_j)$ are the same except that $B(x_i)$ has free occurrences of x_i in exactly those places where $B(x_j)$ has free occurrences of x_j.
- Ex: $(\forall x_3)(p^2_1(x_1, x_3) \vee p^1_1(x_1))$ and $(\forall x_3)(p^2_1(x_2, x_3) \vee p^1_1(x_2))$ are similar.

♣ Lemmas

- If $B(x_i)$ and $B(x_j)$ are similar, then $\vdash_{\mathbf{L}} (\forall x_i)B(x_i) \leftrightarrow (\forall x_j)B(x_j)$.
- If $B(x_i)$ and $B(x_j)$ are similar, then $\vdash_{\mathbf{L}} (\exists x_i)B(x_i) \leftrightarrow (\exists x_j)B(x_j)$.

Properties of L: Completeness [Mendelson]

♣ Lemmas

- If a closed formula $\neg B$ is not provable in a **L**-theory **K**, i.e., $\vdash_{\mathbf{K}} \neg B$ does not hold, and if **K′** is the **L**-theory obtained from **K** by adding B as a new axiom, then **K′** is consistent.
- If a closed formula B is not provable in a **L**-theory **K**, i.e., $\vdash_{\mathbf{K}} B$ does not hold, and if **K′** is the **L**-theory obtained from **K** by adding $\neg B$ as a new axiom, then **K′** is consistent.
- The set **WFF** of formulas of a first-order language L is countable/denumerable. Hence, the same is true of the set **Ter** of terms and the set of closed wffs.

Properties of L: Completeness [Mendelson]

♣ Definitions

- A **L**-theory **K** is said to be ***complete*** if, for every closed formula B of **K**, either $\vdash_{\mathbf{K}} B$ or $\vdash_{\mathbf{K}} \neg B$ (but not both).
- A **L**-theory **K′** is said to be an ***extension*** of a **L**-theory **K** if every theorem of **K** is a theorem of **K′**. (We also say in such a case that **K** is a ***sub-theory*** of **K′**.)

♣ ***Lindenbaum's lemma***

- If **K** is a consistent **L**-theory, then there is a consistent and complete extension of **K**.

Properties of L: Completeness [Mendelson]

♣ Definition

- A **L**-theory **K** is a ***scapegoat theory*** if, for any $B(x)$ that has x as its only free variable, there is a closed term t such that $\vdash_{\mathbf{K}} (\exists x)\neg B(x) \to \neg B(t)$.

♣ Lemmas

- Every consistent **L**-theory **K** has a consistent extension **K′** such that **K′** is a scapegoat theory and **K′** contains countably/denumerably many closed terms.
- Let **J** be a consistent and complete scapegoat theory. Then **J** has a model ***M*** whose domain is the set ***D*** of closed terms of **J**.

♣ Proposition (Gödel, 1930)

- Every consistent **L**-theory **K** has a countable/denumerable model (whose domain is a countable/denumerable set of closed terms).
- Corollary: Any logically valid formula of a **L**-theory **K** is a theorem of **K**.
- Corollary (***Gödel's Completeness Theorem***, 1930): The theorems of **L** are precisely the logically valid formulas.

Properties of L: Completeness [Mendelson]

♣ Corollary

- Let **K** be any **L**-theory.
- a. A formula B is true in every countable/denumerable model of **K** IFF $\vdash_{\mathbf{K}} B$.
- b. If, in every model of **K**, every sequence that satisfies all formulas in a set Γ of formulas also satisfies a formula B, then $\Gamma \vdash_{\mathbf{K}} B$.
- c. If a formula B of **K** is a logical consequence of a set Γ of formulas of **K**, then $\Gamma \vdash_{\mathbf{K}} B$.
- d. If a formula B of **K** is a logical consequence of a formula C of **K**, then $C \vdash_{\mathbf{K}} B$.

♣ ***Skolem–Löwenheim Theorem*** (1920, 1915)

- Any **L**-theory that has a model has a countable/denumerable model.

Some Derived Rules in L (CFOPC) [Mendelson]

♣ **Particularization rule**

- If t is free for x in $A(x)$, then $(\forall x)A(x) \vdash_{\mathrm{L}} A(t)$.

♣ **Existential rule**

- Let t be a term that is free for x in a formula $A(x, t)$, and let $A(t, t)$ arise from $A(x, t)$ by replacing all free occurrences of x by t. (Note: $A(x, t)$ may or may not contain occurrences of t.) Then, $A(t, t) \vdash_{\mathrm{L}} (\exists x)A(x, t)$.

♣ **Proof by contradiction**

- If a deduction/proof of $\Gamma\cup\{\neg B\} \vdash_{\mathrm{L}} C\wedge\neg C$ involves no application of Gen using a variable free in B, then $\Gamma \vdash_{\mathrm{L}} B$.
  Similarly, one obtains $\Gamma \vdash_{\mathrm{L}} \neg B$ from $\Gamma\cup\{B\} \vdash_{\mathrm{L}} C\wedge\neg C$.

Some Derived Rules in L (CFOPC) [Mendelson]

♣ **Rule C** ("C" for "choice")

- A ***rule C deduction*** in a first-order **L**-theory **K**, $\Gamma \vdash_{\textbf{K-rule-C}} B$, is defined as follows: $\Gamma \vdash_{\textbf{K-rule-C}} B$ IFF there is a sequence of formulas $D_1, \ldots, D_n$ such that D_n is B and the following four conditions hold:
- 1. For each $i<n$, either
  a. D_i is an axiom of **K**, or
  b. D_i is in Γ, or
  c. D_i follows by MP or Gen from preceding formulas in the sequence, or
  d. there is a preceding formula $(\exists x)C(x)$ such that D_i is $C(d)$, where d is a new individual constant (rule C).
- 2. As axioms in condition 1(a), we also can use all logical axioms that involve the new individual constants already introduced in the sequence by applications of rule C.
- 3. No application of Gen is made using a variable that is free in some $(\exists x)C(x)$ to which rule C has been previously applied.
- 4. B contains none of the new individual constants introduced in the sequence in any application of rule C.

Some Derived Rules in L (CFOPC) [Mendelson]

- Negation elimination: $\neg\neg B \vdash_{\mathrm{L}} B$
- Negation introduction: $B \vdash_{\mathrm{L}} \neg\neg B$
- Conjunction elimination: $B\wedge C \vdash_{\mathrm{L}} B$; $B\wedge C \vdash_{\mathrm{L}} C$; $\neg(B\wedge C) \vdash_{\mathrm{L}} \neg B\vee\neg C$
- Conjunction introduction: $B, C \vdash_{\mathrm{L}} B\wedge C$
- Disjunction elimination: $B\vee C, \neg B \vdash_{\mathrm{L}} C$; $B\vee C, \neg C \vdash_{\mathrm{L}} B$; $\neg(B\vee C) \vdash_{\mathrm{L}} \neg B\wedge\neg C$;
  $B\rightarrow D, C\rightarrow D, B\vee C \vdash_{\mathrm{L}} D$
- Disjunction introduction: $B \vdash_{\mathrm{L}} B\vee C$; $C \vdash_{\mathrm{L}} B\vee C$
- Conditional elimination: $B\rightarrow C, \neg C \vdash_{\mathrm{L}} \neg B$; $B\rightarrow\neg C, C \vdash_{\mathrm{L}} \neg B$; $\neg B\rightarrow C, \neg C \vdash_{\mathrm{L}} B$;
  $\neg B\rightarrow\neg C, C \vdash_{\mathrm{L}} B$; $\neg(B\rightarrow C) \vdash_{\mathrm{L}} B$; $\neg(B\rightarrow C) \vdash_{\mathrm{L}} \neg C$
- Conditional introduction: $B, \neg C \vdash_{\mathrm{L}} \neg(B\rightarrow C)$
- Conditional contrapositive: $B\rightarrow C \vdash_{\mathrm{L}} \neg C\rightarrow\neg B$; $\neg C\rightarrow\neg B \vdash_{\mathrm{L}} B\rightarrow C$
- Biconditional elimination: $B\leftrightarrow C, B \vdash_{\mathrm{L}} C$; $B\leftrightarrow C, \neg B \vdash_{\mathrm{L}} \neg C$; $B\leftrightarrow C, C \vdash_{\mathrm{L}} B$;
  $B\leftrightarrow C, \neg C \vdash_{\mathrm{L}} \neg B$; $B\leftrightarrow C \vdash_{\mathrm{L}} B\rightarrow C$; $B\leftrightarrow C \vdash_{\mathrm{L}} C\rightarrow B$
- Biconditional introduction: $B\rightarrow C, C\rightarrow B \vdash_{\mathrm{L}} B\leftrightarrow C$
- Biconditional negation: $B\leftrightarrow C \vdash_{\mathrm{L}} \neg B\leftrightarrow\neg C$; $\neg B\leftrightarrow\neg C \vdash_{\mathrm{L}} B\leftrightarrow C$

A Hilbert Style Formal System for CFOPC [R&C]

Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$ be sentences of the language. Then:

P1) $(\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{S}))$

P2) $((\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{R})) \rightarrow ((\mathcal{S} \rightarrow \mathcal{T}) \rightarrow (\mathcal{S} \rightarrow \mathcal{R})))$

P3) $(((\neg\mathcal{S}) \rightarrow (\neg\mathcal{T})) \rightarrow (\mathcal{T} \rightarrow \mathcal{S}))$

P4) $(\forall x_i\mathcal{S} \rightarrow \mathcal{S})$, if x_i does not occur free in $\mathcal{S}$

P5) $(\forall x_i\mathcal{S} \rightarrow \mathcal{S}[t/x_i])$, if $\mathcal{S}$ is a formula of the language in which x_i may appear free and t is free for x_i in $\mathcal{S}$

P6) $(\forall x_1(\mathcal{S} \rightarrow \mathcal{T}) \rightarrow (\mathcal{S} \rightarrow \forall x_1\mathrm{T}))$, if x_1 does not occur free in $\mathcal{S}$.

<u>Rules of deduction</u>

1) Modus Ponens (MP), from $\mathcal{S}$ and $(\mathcal{S} \rightarrow \mathcal{T})$ deduce $\mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are any formulas of the language.

2) Generalization, from $\mathcal{S}$ deduce $\forall x_1\mathcal{S}$, where $\mathcal{S}$ is any formula of the language.

Theorem

Let $\mathcal{S}$ and $\mathcal{T}$ be any sentences and let G be any set of sentences.
Then, if $G, \mathcal{S} \vdash \mathcal{T}$, and no use was made of Generalization involving a free variable of $\mathcal{S}$, it is the case that $G \vdash \mathcal{S} \rightarrow \mathcal{T}$.

K: A Hilbert Style Formal System for CFOPC [Hamilton]

Axioms

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be any *wf*s. of $\mathscr{L}$. The following are axioms of $K_{\mathscr{L}}$.

(K1) $(\mathscr{A} \rightarrow (\mathscr{B} \rightarrow \mathscr{A}))$.

(K2) $(\mathscr{A} \rightarrow (\mathscr{B} \rightarrow \mathscr{C})) \rightarrow ((\mathscr{A} \rightarrow \mathscr{B}) \rightarrow (\mathscr{A} \rightarrow \mathscr{C}))$.

(K3) $(\sim\mathscr{A} \rightarrow \sim\mathscr{B}) \rightarrow (\mathscr{B} \rightarrow \mathscr{A})$.

(K4) $((\forall x_i)\mathscr{A} \rightarrow \mathscr{A})$, if x_i does not occur free in $\mathscr{A}$.

(K5) $((\forall x_i)\mathscr{A}(x_i) \rightarrow \mathscr{A}(t))$, if $\mathscr{A}(x_i)$ is a *wf*. of $\mathscr{L}$ and t is a term in $\mathscr{L}$ which is free for x_i in $\mathscr{A}(x_i)$.

(K6) $(\forall x_i)(\mathscr{A} \rightarrow \mathscr{B}) \rightarrow (\mathscr{A} \rightarrow (\forall x_i)\mathscr{B})$, if $\mathscr{A}$ contains no free occurrence of the variable x_i.

Notice that these are axiom schemes, each with infinitely many instances.

Rules

(1) *Modus ponens*, i.e. from $\mathscr{A}$ and $(\mathscr{A} \rightarrow \mathscr{B})$ deduce $\mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are any *wf*s. of $\mathscr{L}$.

(2) Generalisation, i.e. from $\mathscr{A}$ deduce $(\forall x_i)\mathscr{A}$, where $\mathscr{A}$ is any *wf*. of $\mathscr{L}$ and x_i is any variable.

K: A Hilbert Style Formal System for CFOPC [Hamilton]

Proposition 4.8 (The Deduction Theorem for K)

Let $\mathscr{A}$ and $\mathscr{B}$ be *wfs*. of $\mathscr{L}$ and let Γ be a set (possibly empty) of *wfs*. of $\mathscr{L}$. If $\Gamma \cup \{\mathscr{A}\} \vdash_K \mathscr{B}$, and the deduction contains no application of Generalisation involving a variable which occurs free in $\mathscr{A}$, then $\Gamma \vdash_K (\mathscr{A} \to \mathscr{B})$.

▷ This is the most useful version of the Deduction Theorem for K. It is possible to weaken the additional condition concerning the use of Generalisation (see p. 61 in Mendelson), but we shall not need to. Strengthening this additional condition gives the following Corollary, which is often useful.

Corollary 4.9

If $\Gamma \cup \{\mathscr{A}\} \vdash_K \mathscr{B}$, and $\mathscr{A}$ is a closed *wf*., then $\Gamma \vdash_K (\mathscr{A} \to \mathscr{B})$.

In spite of the apparently less general form of the Deduction Theorem for K, we can still apply it usefully in showing that certain *wfs*. are theorems, just as we did for L.

H: A Hilbert Style Formal System for CFOPC [Ben-Ari]

Definition 8.4 The axioms of the Hilbert system $\mathscr{H}$ for first-order logic are:

Axiom 1 $\vdash (A \to (B \to A))$,

Axiom 2 $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

Axiom 3 $\vdash (\neg B \to \neg A) \to (A \to B)$,

Axiom 4 $\vdash \forall x A(x) \to A(a)$,

Axiom 5 $\vdash \forall x (A \to B(x)) \to (A \to \forall x B(x))$.

- In Axioms 1, 2 and 3, A, B and C are any formulas of first-order logic.
- In Axiom 4, $A(x)$ is a formula with a free variable x.
- In Axiom 5, $B(x)$ is a formula with a free variable x, while x is *not* a free variable of the formula A.

The rules of inference are ***modus ponens*** and ***generalization***:

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B}, \qquad \frac{\vdash A(a)}{\vdash \forall x A(x)}.$$

An Example of Theorem Proof (by Propositional Deduction) in H [Ben-Ari]

Example 8.5

$$\vdash \forall x p(x) \to (\exists y \forall x q(x,y) \to \forall x p(x))$$

is an instance of Axiom 1 in first-order logic and:

$$\frac{\vdash \forall x p(x) \to (\exists y \forall x q(x,y) \to \forall x p(x)) \qquad \vdash \forall x p(x)}{\vdash \exists y \forall x q(x,y) \to \forall x p(x)}$$

uses the rule of inference *modus ponens*.

Some Logical Theorems of L

- $\vdash_L ((\forall x)(\forall y)A) \to (\forall y)(\forall x)A$
- $\vdash_L (\neg(\forall x)A) \to (\exists x)\neg A$
- $\vdash_L ((\forall x)(A \to B)) \to ((\forall x)A \to (\forall x)B)$
- $\vdash_L ((\forall x)(A \leftrightarrow B)) \to ((\forall x)A \leftrightarrow (\forall x)B)$
- $\vdash_L ((\forall x)(A \to B)) \to ((\exists x)A \to (\exists x)B)$
- $\vdash_L ((\forall x)(A \wedge B)) \leftrightarrow (\forall x)A) \wedge (\forall x)B$
- $\vdash_L ((\forall y_1) \ldots (\forall y_n)A) \to A$, if $y_1, \ldots, y_n$ does not occur free in A
- $\vdash_L ((\forall x)(\forall y)p(x,y)) \to (\forall x)p(x,x)$
- $\vdash_L ((\forall x)B) \vee (\forall x)C)) \to (\forall x)(B \vee C)$
- $\vdash_L (\neg(\exists x)B) \to (\forall x)\neg B$
- $\vdash_L ((\forall x)B) \to (\forall x)(B \vee C)$
- $\vdash_L ((\forall x)(\forall y)(p(x,y) \to \neg p(y,x))) \to (\forall x)\neg p(x,x)$
- $\vdash_L ((\exists x)B \to (\forall x)C) \to (\forall x)(B \to C)$

Some Logical Theorems of L

- $\vdash_L ((\forall x)(B \vee C)) \to (((\forall x)B) \vee (\exists x)C)$
- $\vdash_L (\forall x)(p(x,x) \to (\exists y)p(x,y))$
- $\vdash_L ((\forall x)(B \to C)) \to ((\forall x)\neg C \to (\forall x)\neg B)$
- $\vdash_L (\exists y)(p(y) \to (\forall y)p(y))$
- $\vdash_L ((\exists x)p(x,x)) \to (\exists x)(\exists y)p(x,y)$
- $\vdash_L [((\forall x)(\forall y)(p(x,y) \to p(y,x))) \wedge$
  $(\forall x)(\forall y)(\forall z)((p(x,y) \wedge p(y,z)) \to p(x,z))]$
  $\to (\forall x)(\forall y)(p(x,y) \to p(x,x))$
- $\vdash_L B \to (\forall x)B$, if x is not free in B
- $\vdash_L ((\exists x)B) \to B$, if x is not free in B
- $\vdash_L (B \to (\forall x)C) \leftrightarrow (\forall x)(B \to C)$, if x is not free in B
- $\vdash_L (((\exists x)C) \to B) \leftrightarrow (\forall x)(C \to B)$, if x is not free in B
- $\vdash_L ((\forall x)(p_1(x) \leftrightarrow p_2(x))) \to (((\exists x)p_1(x)) \leftrightarrow (\exists x)p_2(x))$

Some Logical Theorems of L

- $\vdash_L ((\exists x)\neg A) \leftrightarrow (\neg(\forall x)A)$
- $\vdash_L ((\forall x)A) \leftrightarrow \neg(\exists x)\neg A$
- $\vdash_L ((\exists x)(B \to \neg(C \vee D))) \to (\exists x)(B \to (\neg C \wedge \neg D))$
- $\vdash_L ((\forall x)(\exists y)(B \to C)) \leftrightarrow (\forall x)(\exists y)(\neg B \vee C)$
- $\vdash_L ((\forall x)(B \to \neg C)) \leftrightarrow \neg(\exists x)(B \wedge C)$
- $\vdash_L ((\exists x)(B(x) \to C(x))) \to (((\forall x)B(x)) \to (\exists x)C(x))$
- $\vdash_L \neg(\exists y)(\forall x)(p(x,y)) \leftrightarrow \neg p(x,x))$
- $\vdash_L [(\forall x)(p_1(x) \to (p_2(x) \vee p_3(x))) \wedge \neg(\forall x)(p_1(x) \to p_2(x))] \to (\exists x)(p_1(x) \wedge p_3(x))$
- $\vdash_L (((\exists x)B(x)) \wedge (\forall x)C(x)) \to (\exists x)(B(x) \wedge C(x))$
- $\vdash_L ((\exists x)C(x)) \to (\exists x)(B(x) \vee C(x))$
- $\vdash_L ((\exists x)(\exists y)B(x,y)) \leftrightarrow (\exists y)(\exists x)B(x,y)$
- $\vdash_L ((\exists x)(\forall y)B(x,y)) \to (\forall y)(\exists x)B(x,y)$
- $\vdash_L ((\exists x)(B(x) \wedge C(x))) \to ((\exists x)(B(x)) \wedge (\exists x)C(x)$

Examples of First-Order Theories Based on L: Partially Ordered Structure [Mendelson]

♣ Predicates, functions, and individual constants
- A single predicate p^2_3. We shall write $x_i < x_j$ instead of $p^2_3(x_i, x_j)$.
- No functions and individual constants.

♣ Empirical premises/axioms
- $(\forall x_1)(\neg\, x_1 < x_1)$ (***irreflexivity***)
- $(\forall x_1)(\forall x_2)(\forall x_3)[(x_1 < x_2 \land x_2 < x_3) \rightarrow x_1 < x_3]$ (***transitivity***)

♣ ***Partially ordered structure (Strict order structure)***
- A model of the above first-order theory is called a ***partially ordered structure (Strict order structure)***.

♣ Note
- Here the partially ordered ('<') structure with irreflexivity and transitivity is different from the so-called "partially order" ('≤') with reflexivity, antisymmetricity, and transitivity.

Examples of First-Order Theories Based on L: Group Theory [Mendelson]

♣ Predicates, functions, and individual constants
- A single predicate p^2_1. We shall write $x_i = x_j$ instead of $p^2_1(x_i, x_j)$.
- A single function f^2_1. We shall write $x_i + x_j$ instead of $f^2_1(x_i, x_j)$.
- A single individual constant c_1. We shall write 0 instead of c_1.

♣ Empirical premises/axioms
- a. $(\forall x_1)(\forall x_2)(\forall x_3)[(x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3)]$ (***associativity***)
- b. $(\forall x_1)[(0 + x_1) = x_1]$ (***identity***)
- c. $(\forall x_1)(\exists x_2)[(x_2 + x_1) = 0]$ (***inverse***)
- d. $(\forall x_1)(x_1 = x_1)$ (***reflexivity*** of =)
- e. $(\forall x_1)(\forall x_2)[(x_1 = x_2) \rightarrow (x_2 = x_1)]$ (***symmetry*** of =)
- f. $(\forall x_1)(\forall x_2)(\forall x_3)[(x_1 = x_2 \land x_2 = x_3) \rightarrow x_1 = x_3]$ (***transitivity*** of =)
- g. $(\forall x_1)(\forall x_2)(\forall x_3)[(x_1 = x_2) \rightarrow ((x_1 + x_3) = (x_2 + x_3) \land (x_3 + x_1) = (x_3 + x_2))]$ (***substitutivity*** of =)

Examples of First-Order Theories Based on L: Group Theory [Mendelson]

♣ ***Group theory***
- A model for the above first-order theory, in which the interpretation of predicate "=" is the ***identity relation***, is called a ***group***.

♣ ***Abelian group theory***
- A group is said to be ***Abelian*** if, in addition, the following is true: h. $(\forall x_1)(\forall x_2)[(x_1 + x_2) = (x_2 + x_1)]$

L-Based First-Order Theories with Equality [Mendelson]

♣ ***L-based first-order theories with equality***
- Let **K** be a **L**-theory that has as one of its predicate symbols p^2_1. We write $t{=}s$ as an abbreviation for $p^2_1(t, s)$, and $t{\neq}s$ as an abbreviation for $\neg\, p^2_1(t, s)$.
- **K** is called a ***L-based first-order theory with equality*** if the following are theorems (empirical premises/axioms) of **K**:
  $(\forall x_1)(x_1 = x_1)$ (***reflexivity of equality***)
  $(x_1 = x_2) \rightarrow (B(x_1, x_1) \rightarrow B(x_1, x_2))$ (***substitutivity of equality***)
  where $B(x_1, x_1)$ is any formula, and $B(x_1, x_2)$ arises from $B(x_1, x_1)$ by replacing some, but not necessarily all, free occurrences of x_1 by x_2, with the proviso that x_2 is free for x_1 in $B(x_1, x_1)$. Thus, $B(x_1, x_2)$ may or may not contain free occurrences of x_1.

♣ Propositions on the relation of equality
- $\vdash_{\mathbf{K}} t = t$ for any term t.
- $\vdash_{\mathbf{K}} t = s \rightarrow s = t$ for any terms t and s.
- $\vdash_{\mathbf{K}} t = s \rightarrow (s = r \rightarrow t = r)$ for any terms t, s, and r.

L-Based First-Order Theories with Equality [Mendelson]

♣ The equivalence relation in **L**-based first-order theories with equality
- By the above proposition on the relation of equality, in any model for a **L**-based first-order theory **K** with equality, the relation E in the model corresponding to the predicate symbol "=" is an equivalence relation.

♣ Various **L**-based first-order theories with equality
- There are various L-based first-order theories with equality that include predicates other than equality and empirical premises/axioms other than the two axioms of reflexivity of equality and substitutivity of equality.
- **NBG** axiomatic set theory is a L-based first-order theory with equality.

Some Theorems of L-Based First-Order Theories with Equality [Mendelson]

- Let **K** be a **L**-based first-order theory with equality, then the following theorems hold.
- $\vdash_{\mathbf{K}} (\forall x)(B(x) \leftrightarrow (\exists y)(x{=}y \land B(y)))$, if y does not occur in $B(x)$
- $\vdash_{\mathbf{K}} (\forall x)(B(x) \leftrightarrow (\forall y)(x{=}y \rightarrow B(y)))$, if y does not occur in $B(x)$
- $\vdash_{\mathbf{K}} (\forall x)(\exists y)x{=}y$
- $\vdash_{\mathbf{K}} x{=}y \rightarrow f(x){=}f(y)$, where f is any function symbol of one argument
- $\vdash_{\mathbf{K}} B(x) \land x{=}y \rightarrow B(y)$, if y is free for x in $B(x)$
- $\vdash_{\mathbf{K}} B(x) \land \neg B(y) \rightarrow x{\neq}y$, if y is free for x in $B(x)$

Some Theorems of L-Based First-Order Theories with Equality [Mendelson]

- Let **K** be a **L**-based first-order theory with equality, then the following theorems hold.
- $\vdash_{\mathbf{K}} (x_1=y_1\wedge...\wedge x_n=y_n) \rightarrow t(x_1, \ldots, x_n)=t(y_1, \ldots, y_n)$, where $t(y_1, \ldots, y_n)$ arises from the term $t(x_1, \ldots, x_n)$ by substitution of $y_1, \ldots, y_n$ for $x_1, \ldots, x_n$ respectively.
- $\vdash_{\mathbf{K}} (x_1=y_1\wedge...\wedge x_n=y_n) \rightarrow (B(x_1, \ldots, x_n)\leftrightarrow B(y_1, \ldots, y_n))$, where $B(y_1, \ldots, y_n)$ is obtained by substituting $y_1, \ldots, y_n$ for one or more occurrences of $x_1, \ldots, x_n$ respectively, in the formula $B(x_1, \ldots, x_n)$, and $y_1, \ldots, y_n$ are free for $x_1, \ldots, x_n$, respectively, in the formula $B(x_1, \ldots, x_n)$.

L-Based First-Order Theories with Equality [Mendelson]

♣ ***Unique quantifier***

- In **L**-based first-order theories with equality it is possible to define in the following way phrases that use the expression "There exists one and only one x such that … ."
- Definition: $(\exists_1 x)A(x) =_{\mathrm{df}} (\exists x)A(x)\wedge(\forall x)(\forall y)(A(x)\wedge A(y) \rightarrow x = y)$

♣ Properties of unique quantifier

- In any **L**-based first-order theory **K** with equality, the following hold:
  a. $\vdash_{\mathbf{K}} (\forall x)(\exists_1 y)(x = y)$
  b. $\vdash_{\mathbf{K}} (\exists_1 x)A(x) \leftrightarrow (\exists x)(\forall y)(x = y \leftrightarrow A(y))$
  c. $\vdash_{\mathbf{K}} (\forall x)(A(x) \leftrightarrow B(x)) \rightarrow ((\exists_1 x)A(x) \leftrightarrow (\exists_1 x)B(x))$
  d. $\vdash_{\mathbf{K}} (\exists_1 x)(A \vee B) \rightarrow ((\exists_1 x)A \vee (\exists_1 x)B)$
  e. $\vdash_{\mathbf{K}} (\exists_1 x)A(x) \leftrightarrow (\exists x)(A(x) \wedge (\forall y)(A(y) \rightarrow y = x))$

L-Based First-Order Theories with Equality: Normal Models [Mendelson]

♣ ***The normal models of first-order theories of equality***

- In any model for a **L**-based first-order theory **K** with equality, the relation E in the model corresponding to the predicate symbol "=" is an equivalence relation. If this relation E is the identity relation in the domain of the model, then the model is said to be ***normal***.
- Note: In general, an equivalence relation in a domain is not necessarily the identity relation in the domain.
- Any model $\boldsymbol{M}$ for **K** can be contracted to a normal model $\boldsymbol{M}^*$ for **K** by taking the domain $\boldsymbol{D}^*$ of $\boldsymbol{M}^*$ to be the set of equivalence classes determined by the relation E in the domain $\boldsymbol{D}$ of $\boldsymbol{M}$.

L-Based First-Order Theories with Equality: Normal Models [Mendelson]

♣ ***The normal models of first-order theories of equality*** (Gödel, 1930)

- Any consistent **L**-based first-order theory **K** with equality has a countable/denumerable model (whose domain is a countable/denumerable set of closed terms).
- This is an extension of Gödel's Completeness Theorem.

♣ Corollary

- Any **L**-based first-order theory **K** that has an infinite normal model has a countable/denumerable model.
- This is an extension of Skolem–Löwenheim Theorem.

Examples of L-Based First-Order Theories with Equality: The Pure First-Order Theory of Equality [Mendelson]

♣ ***The pure first-order theory of equality***

- Let $\mathbf{K_1}$ be a **L**-based first-order theory whose language has only = as a predicate symbol and no function symbols or individual constants.
- Let its empirical premises/axioms be:
  a. $(\forall x_1)(x_1 = x_1)$
  b. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow x_2 = x_1)$
  c. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \rightarrow (x_2 = x_3 \rightarrow x_1 = x_3))$
- $\mathbf{K_1}$ is a **L**-based first-order theory with equality called ***the pure first-order theory of equality***.

Examples of L-Based First-Order Theories with Equality: The Theory of Densely Ordered Sets with neither First nor Last Element [Mendelson]

♣ ***The theory of densely ordered sets with neither first nor last element***

- Let $\mathbf{K_2}$ be a **L**-based first-order theory whose language has only = and < as predicate symbols and no function symbols or individual constants.
- Let its empirical premises/axioms be:
  a. $(\forall x_1)(x_1 = x_1)$
  b. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow x_2 = x_1)$
  c. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \rightarrow (x_2 = x_3 \rightarrow x_1 = x_3))$
  d. $(\forall x_1)(\exists x_2)(\exists x_3)(x_1 < x_2 \wedge x_3 < x_1)$
  e. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 < x_2 \wedge x_2 < x_3 \rightarrow x_1 < x_3)$
  f. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow \neg x_1 < x_2)$
  g. $(\forall x_1)(\forall x_2)(x_1 < x_2 \vee x_1 = x_2 \vee x_2 < x_1)$
  h. $(\forall x_1)(\forall x_2)(x_1 < x_2 \rightarrow (\exists x_3)(x_1 < x_3 \wedge x_3 < x_2))$
- $\mathbf{K_2}$ is a **L**-based first-order theory with equality called ***the theory of densely ordered sets with neither first nor last element***.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Group theory* G**

- If we regard the identity relation "=" in the group theory (denoted by **G**) and the Abelian group theory (denoted by Abelian **G**) of **L**-based first-order theory as "equality", then **G** and Abelian **G** are a **L**-based first-order theory with equality, respectively.

♣ ***Field theory* F**

- A single predicate $p^2{}_1$. We shall write $x_i = x_j$ instead of $p^2{}_1(x_i, x_j)$.
- Two functions $f^2{}_1$ and $f^2{}_2$. We shall write $x_i + x_j$ instead of $f^2{}_1(x_i, x_j)$ and write $x_i \bullet x_j$ instead of $f^2{}_2(x_i, x_j)$.
- Two individual constants c_1 and c_2. We shall write 0 instead of c_1 and write 1 instead of c_2.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Field theory* F**

- Empirical premises/axioms include all empirical premises/axioms of Abelian group theory, (a)-(h), and plus the following:
- i. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 = x_2) \rightarrow [(x_1 \bullet x_3) = (x_2 \bullet x_3) \land (x_3 \bullet x_1) = (x_3 \bullet x_2)])$
- j. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 \bullet x_3)) = ((x_1 \bullet x_2) \bullet x_3))$
- k. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 + x_3)) = ((x_1 \bullet x_2) + (x_1 \bullet x_3)))$
- l. $(\forall x_1)(\forall x_2)((x_1 \bullet x_2) = (x_2 \bullet x_1))$
- m. $(\forall x_1)((x_1 \bullet 1) = x_1)$
- n. $(\forall x_1)((x_1 \neq 0) \rightarrow (\exists x_2)((x_1 \bullet x_2) = 1))$
- o. $0 \neq 1$
- **F** is a **L**-based first-order theory with equality.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Commutative ring* $\mathbf{R_C}$ *with unit***

- Empirical premises/axioms (a)–(m) of **F** define the ***L-based first-order theory* $\mathbf{R_C}$ *of commutative rings with unit***.
- a. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$
- b. $(\forall x_1)((0 + x_1) = x_1)$
- c. $(\forall x_1)(\exists x_2)((x_2 + x_1) = 0)$
- d. $(\forall x_1)(x_1 = x_1)$
- e. $(\forall x_1)(\forall x_2)((x_1 = x_2) \rightarrow (x_2 = x_1))$
- f. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 = x_2 \land x_2 = x_3) \rightarrow x_1 = x_3))$
- g. $(\forall x_1)(\forall x_2)(\forall x_3)((x_2 = x_3) \rightarrow ((x_1 + x_2) = (x_1 + x_3) \land (x_2 + x_1) = (x_3 + x_1)))$
- h. $(\forall x_1)(\forall x_2)((x_1 + x_2) = (x_2 + x_1))$

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Commutative ring* $\mathbf{R_C}$ *with unit***

- i. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 = x_2) \rightarrow [(x_1 \bullet x_3) = (x_2 \bullet x_3) \land (x_3 \bullet x_1) = (x_3 \bullet x_2)])$
- j. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 \bullet x_3)) = ((x_1 \bullet x_2) \bullet x_3))$
- k. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 + x_3)) = ((x_1 \bullet x_2) + (x_1 \bullet x_3)))$
- l. $(\forall x_1)(\forall x_2)((x_1 \bullet x_2) = (x_2 \bullet x_1))$
- m. $(\forall x_1)((x_1 \bullet 1) = x_1)$

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Ordered Field* $\mathbf{F}_<$**

- If we add to **F** the predicate symbol $p^2{}_3$, abbreviate $p^2{}_3(t, s)$ by "$t < s$", and add the following empirical axioms (the first three are (e)-(g) of the theory of densely ordered sets with neither first nor last element):
- $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 < x_2 \land x_2 < x_3 \rightarrow x_1 < x_3)$
- $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow \neg x_1 < x_2)$
- $(\forall x_1)(\forall x_2)(x_1 < x_2 \lor x_1 = x_2 \lor x_2 < x_1)$
- $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 < x_2) \rightarrow (x_1 + x_3) < (x_2 + x_3))$
- $(\forall x_1)(\forall x_2)(\forall x_3)(((x_1 < x_2) \land (0 < x_3)) \rightarrow (x_1 \bullet x_3) < (x_2 \bullet x_3))$
- Then the obtained new theory $\mathbf{F}_<$ is called ***the L-based first-order theory with equality of ordered fields***.

First-Order Number Theory [Mendelson]: The Language $\mathbf{L_A}$

♣ The language of the first-order number theory: $\mathbf{L_A}$ (*the language of arithmetic*)

- $\mathbf{L_A}$ has a single predicate symbol (letter) $p^2{}_1$. We shall abbreviate $p^2{}_1(t, s)$ by $t = s$, and $\neg p^2{}_1(t, s)$ by $t \neq s$.
- $\mathbf{L_A}$ has one individual constant symbol a_1. We shall use 0 as an alternative notation for a_1.
- $\mathbf{L_A}$ has three function symbols (letters) $f^1{}_1$, $f^2{}_1$, and $f^2{}_2$. We shall write (t') instead of $f^1{}_1(t)$, $(t + s)$ instead of $f^2{}_1(t, s)$, and $(t \cdot s)$ instead of $f^2{}_2(t, s)$. However, we shall write t', $t + s$, and $t \cdot s$ instead of (t'), $(t + s)$, and $(t \cdot s)$ whenever this will cause no confusion.

♣ ***The first-order number theory* N**

- If we regard the identity relation "=" in $\mathbf{L_A}$ as "equality", take $(\forall x_1)(x_1 = x_1)$ (reflexivity of equality), $(x_1 = x_2) \rightarrow (B(x_1, x_1) \rightarrow B(x_1, x_2))$ (substitutivity of equality), and (PA1)-(PA9) as the axioms of **N**, then **N** is a **L**-based first-order theory with equality.

First-Order Number Theory [Mendelson]: The Arithmetic Axioms

♣ ***The arithmetic axioms of the first-order number theory (Peano Arithmetic, PA)***

- (PA1) $(\forall x_1)(\forall x_2)(\forall x_3)[x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)]$
- (PA2) $(\forall x_1)(\forall x_2)[x_1 = x_2 \rightarrow x_1' = x_2']$
- (PA3) $(\forall x_1)[0 \neq x_1']$
- (PA4) $(\forall x_1)(\forall x_2)[x_1' = x_2' \rightarrow x_1 = x_2]$
- (PA5) $(\forall x_1)[x_1 + 0 = x_1]$
- (PA6) $(\forall x_1)(\forall x_2)[x_1 + x_2' = (x_1 + x_2)']$
- (PA7) $(\forall x_1)[x_1 \cdot 0 = 0]$
- (PA8) $(\forall x_1)(\forall x_2)[x_1 \cdot (x_2)' = (x_1 \cdot x_2) + x_1]$
- (PA9) $B(0) \rightarrow [(\forall x)(B(x) \rightarrow B(x')) \rightarrow (\forall x)B(x)]$ for any wff $B(x)$ of $\mathbf{L_A}$.
  (PA9) is called ***the principle of mathematical induction***.

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***ZF axiomatic set theory*** [E. Zermelo (1905), A. Fraenkel (1920)]

- **ZF** axiomatic set theory was originally formulated by Zermelo (1905) and later modified by Fraenkel (1920).

♣ Predicates

- **ZF** has two predefined predicate symbols (letters) p^2_1 and p^2_2.
- We shall abbreviate $p^2_1(x_1, x_2)$ by $x_1 = x_2$, and $\neg p^2_1(x_1, x_2)$ by $x_1 \neq x_2$.
- We shall abbreviate $p^2_2(x_1, x_2)$ by $x_1 \in x_2$, and $\neg p^2_2(x_1, x_2)$ by $x_1 \notin x_2$.
- **ZF** has no predefined function symbols and individual constant symbols.

♣ Axioms of **ZF**

- $(\forall x_1)(x_1 = x_1)$ (reflexivity of equality),
  $(x_1 = x_2) \rightarrow (B(x_1, x_1) \rightarrow B(x_1, x_2))$ (substitutivity of equality),
  and the following axioms (ZF1)-(ZF8).

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***The Axiom of Extensionality***

- (ZF1) $(\forall x_1)(\forall x_2)[x_1 = x_2 \leftrightarrow (\forall x_3)(x_3 \in x_1 \leftrightarrow x_3 \in x_2)]$
- The intended meaning is that two sets are equal IFF they have the same elements.

♣ ***The Null Set Axiom***

- (ZF2) $(\exists x_1)(\forall x_2)\neg(x_2 \in x_1)$
- This guarantees the existence, in the intended interpretation, of a set with no members.
- We can thus introduce into the language the symbol $\varnothing$, to act as an individual constant, i.e., $(\forall x_2)\neg(x_2 \in \varnothing)$.

♣ Notation

- We introduce the symbol $\subseteq$ as an abbreviation as follows:
  $t_1 \subseteq t_2$ stands for $(\forall x_1)(x_1 \in t_1 \rightarrow x_1 \in t_2)$ where t_1 and t_2 are any terms.

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***The Axiom of Pairing***

- (ZF3) $(\forall x_1)(\forall x_2)(\exists x_3)(\forall x_4)[x_4 \in x_3 \leftrightarrow (x_4 = x_1 \vee x_4 = x_2)]$
- The intended meaning is that given any sets x_1 and x_2 there is a set z whose members are x_1 and x_2.
- $\{x_1, x_2\}$ will be regarded as a term, i.e., $x_4 \in \{x_1, x_2\} \leftrightarrow (x_4 = x_1 \vee x_4 = x_2)$

♣ ***The Axiom of Unions***

- (ZF4) $(\forall x_1)(\exists x_2)(\forall x_3)[x_3 \in x_2 \leftrightarrow (\exists x_4)(x_4 \in x_1 \wedge x_3 \in x_4)]$
- The intended meaning is that given any set x_1 there is a set x_2 which has as its members all members of members of x_1.

♣ Notation

- We denote by the symbol $\cup$, i.e., $\cup x_1$ the object whose existence is asserted in (ZF4).
- $\cup$ acts as a one-place function symbol.
- We can introduce $\cup$ by: $t_1 \cup t_2$ stands for $\cup\{t_1, t_2\}$.

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***The Power Set Axiom***

- (ZF5) $(\forall x_1)(\exists x_2)(\forall x_3)(x_3 \in x_2 \leftrightarrow x_3 \subseteq x_1)$
- The intended meaning is that given any set x_1 there is a set x_2 which has as its members all the subsets of x_1.

♣ ***The Axiom Scheme of Replacement***

- (ZF6) $(\forall x_1)(\exists_1 x_2)A(x_1, x_2) \rightarrow$
  $(\forall x_3)(\exists x_4)(\forall x_5)[x_5 \in x_4 \leftrightarrow (\exists x_6)(x_6 \in x_3 \wedge A(x_6, x_5))]$
  For every formula $A(x_1, x_2)$ in which x_1 and x_2 occur free (and in which, we may suppose without loss of generality, the quantifiers $(\forall x_5)$ and $(\forall x_6)$ do not appear).
- The intended meaning is that if the formula A determines a function, then for any set x_1, there is a set x_2 which has as its members all the images of members of x_1 under this function.

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***The Axiom of Infinity***

- (ZF7) $(\exists x_1)[\varnothing \in x_1 \wedge (\forall x_2)(x_2 \in x_1 \rightarrow x_2 \cup \{x_2, x_2\} \in x_1)]$
- This asserts the existence, in any model, of an infinite set.
- If this axiom were not included amongst the axioms there would be no way of ensuring that the formal theory had any relevance to intuitive set theory which includes infinite sets.

♣ ***The Axiom of Foundation***

- (ZF8) $(\forall x_1)(\neg x_1 = \varnothing \rightarrow (\exists x_2)[x_2 \in x_1 \wedge \neg(\exists x_3)(x_3 \in x_2 \wedge x_3 \in x_1)])$
- The intended meaning is that every non-empty set x contains a member which is disjoint from x.
- This is a technical axiom which is included in order to avoid anti-intuitive anomalies such as the possibility of a set being a member of itself.

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ Important facts

- On the assumption that it is a consistent system, we know that a normal model exists. It can be shown that in any such model there are sets with all the usual properties of the number systems.
- The number systems of integers, rationals, reals, and complex numbers can be constructed, starting from the natural numbers, by algebraic procedures. All of these procedures can be carried out within **ZF**.

♣ ***The Axiom of Choice***

- (**AC**) For any non-empty set x there is a set y which has precisely one element in common with each member of x.
- **ZFC** = **ZF** + **AC**

First-Order Theory ZF (Axiomatic Set Theory) [Hamilton]

♣ ***The Continuum Hypothesis***

- (**CH**) Each infinite set of real numbers either is countable or has the same cardinal number as the set of all real numbers.
- (**CH**) $\aleph_1 = \aleph = c(P(N))$? (There is no $\aleph'$ such that $\aleph_0 < \aleph' < \aleph$)
- (**GCH**) If $c(A) = \aleph_\omega$, then $\aleph_{\omega+1} = c(P(A))$?

♣ Questions about the fundamental principles of set theory

- Can (**AC**) and (**CH**) be deduced as theorems of **ZF**?
- If they can not, would it be consistent to include one or both as additional axioms?
- (**AC**) and (**CH**) are consistent with **ZF**. [Gödel, 1938]
- (**AC**) and (**CH**) cannot be deduced as theorems of **ZF**. [Cohen, 1963]

Examples by a L-Based First-Order Language with Equality

♣ Defined predicates

- Let $x, y, z \in$ CS (the set of professors, administrative staffs, and students of CS department).
- Let '$P(x)$', '$AS(x)$', and '$S(x)$' be the predicates 'x is a professor', 'x is an administrative staff', and 'x is a student', respectively.
- Let '$C(x)$', '$VC(x)$', and '$D(x)$' be the predicates 'x is the department chair, 'x is a department vice-chair', and 'x is the office director', respectively.
- Let $M(x, y)$, '$T(x, y)$', '$A(x, y)$', and '$TA(x, y)$' be the predicates 'x manages y', 'x teaches y', 'x advises y', and 'x has y as a TA', respectively.

♣ Sentence examples

- (a) CS-dept has only one professor as the department chair.
  $(\exists x)((P(x)\wedge C(x))\wedge(\forall x)(\forall y)(P(x)\wedge C(x)\wedge(P(y)\wedge C(y)\rightarrow x=y))$
  or $(\exists_1 x)((P(x)\wedge C(x))$
  or $(\exists x)(P(x)\wedge C(x)\wedge(\forall y)(\neg(y=x)\rightarrow\neg C(y)))$

Examples by a L-Based First-Order Language with Equality

♣ Sentence examples

- (b) CS-dept has at least one professor as a department vice-chair.
  $(\exists x)(P(x)\wedge VC(x))$
- (c) CS-dept has only one administrative staff as the office director.
  $(\exists x)((AS(x)\wedge D(x))\wedge(\forall x)(\forall y)(AS(x)\wedge D(x)\wedge(AS(y)\wedge D(y)\rightarrow x=y))$
  or $(\exists_1 x)((AS(x)\wedge D(x))$
  or $(\exists x)(AS(x)\wedge D(x)\wedge(\forall y)(\neg(y=x)\rightarrow\neg D(y)))$
- (d) The department chair and vice-chair(s) manage all professors and administrative staffs.
  $(\exists x)(\exists y)(\forall z)((P(x)\wedge C(x)\wedge P(y)\wedge VC(y)\wedge(P(z)\vee AS(z))) \rightarrow (M(x, z)\wedge M(y, z)))$
- (e) Each professor teaches all students.
  $(\forall x)(\forall y)((P(x)\wedge S(y)) \rightarrow T(x, y))$

Examples by a L-Based First-Order Language with Equality

♣ Sentence examples

- (f) There is no professor who does not teach any student.
  $\neg(\exists x)(\forall y)(P(x)\wedge S(y)\wedge(\neg T(x, y)))$
  or $(\forall x)(\exists y)(P(x)\wedge S(y)\wedge T(x, y))$
- (g) Each professor, as an advisor, advises at least one student.
  $(\forall x)(\exists y)(P(x)\wedge S(y)\wedge A(x, y))$
  or $\neg(\exists x)(\forall y)(P(x)\wedge S(y)\wedge(\neg A(x, y)))$
- (h) There is no student who has no advisor.
  $\neg(\exists x)(\forall y)((S(x)\wedge P(y)\wedge\neg A(y, x))$
  or $(\forall x)(\exists y)(S(x)\wedge P(y)\wedge A(y, x))$
- (i) A professor may have some students as teaching assistants.
  $(\exists x)(\exists y)(\exists z)(P(x)\wedge S(y)\wedge S(z)\wedge(\neg(y=z))\wedge TA(x, y)\wedge TA(x, z))$
- (j) A student may be a teaching assistant to more than one professor.
  $(\exists x)(\exists y)(\exists z)(P(x)\wedge P(y)\wedge(\neg(y=x))\wedge S(z)\wedge TA(x, z)\wedge TA(y, z))$

An Introduction to Classical Predicate Calculus

♣ The Limitations of Propositional Logic **CPC**
♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
♣ Substitutions
♣ Semantics (Model Theory) of **CFOPC**
♣ Semantic (Model-theoretical, Logical) Consequence Relation
♣ Hilbert Style Formal Logic Systems for **CFOPC**
♣ Gentzen's Natural Deduction System for **CFOPC**
♣ Gentzen's Sequent Calculus System for **CFOPC**
♣ Semantic Tableau Systems for **CFOPC**
♣ Resolution Systems for **CFOPC**
♣ Classical Second-Order Predicate Calculus (**CSOPC**)