Wolfgang Rautenberg

UNIVERSITEXT

A Concise Introduction to Mathematical Logic

Third Edition

Springer

Foreword

by Lev Beklemishev, Moscow

The field of mathematical logic—evolving around the notions of logical validity, provability, and computation—was created in the first half of the previous century by a cohort of brilliant mathematicians and philosophers such as Frege, Hilbert, Gödel, Turing, Tarski, Malcev, Gentzen, and some others. The development of this discipline is arguably among the highest achievements of science in the twentieth century: it expanded mathematics into a novel area of applications, subjected logical reasoning and computability to rigorous analysis, and eventually led to the creation of computers.

The textbook by Professor Wolfgang Rautenberg is a well-written introduction to this beautiful and coherent subject. It contains classical material such as logical calculi, beginnings of model theory, and Gödel's incompleteness theorems, as well as some topics motivated by applications, such as a chapter on logic programming. The author has taken great care to make the exposition readable and concise; each section is accompanied by a good selection of exercises.

A special word of praise is due for the author's presentation of Gödel's second incompleteness theorem, in which the author has succeeded in giving an accurate and simple proof of the derivability conditions and the provable Σ_1-completeness, a technically difficult point that is usually omitted in textbooks of comparable level. This work can be recommended to all students who want to learn the foundations of mathematical logic.

Introduction

Traditional logic as a part of philosophy is one of the oldest scientific disciplines. It can be traced back to the Stoics and to Aristotle[2] and is the root of what is nowadays called philosophical logic. Mathematical logic, however, is a relatively young discipline, having arisen from the endeavors of Peano, Frege, and Russell to reduce mathematics entirely to logic. It steadily developed during the twentieth century into a broad discipline with several subareas and numerous applications in mathematics, computer science, linguistics, and philosophy.

One feature of modern logic is a clear distinction between object language and metalanguage. The first is formalized or at least formalizable. The latter is, like the language of this book, a kind of a colloquial language that differs from author to author and depends also on the audience the author has in mind. It is mixed up with semiformal elements, most of which have their origin in set theory. The amount of set theory involved depends on one's objectives. Traditional semantics and model theory as essential parts of mathematical logic use stronger set-theoretic tools than does proof theory. In some model-theoretic investigations these are often the strongest possible ones. But on average, little more is assumed than knowledge of the most common set-theoretic terminology, presented in almost every mathematical course or textbook for beginners. Much of it is used only as a *façon de parler.*

The language of this book is similar to that common to almost all mathematical disciplines. There is one essential difference though. In mathematics, metalanguage and object language strongly interact with each other, and the latter is semiformalized in the best of cases. This method has proved successful. Separating object language and metalanguage is relevant only in special context, for example in axiomatic set theory, where formalization is needed to specify what certain axioms look like. Strictly formal languages are met more often in computer science. In analyzing complex software or a programming language, as in logic, formal linguistic entities are the central objects of consideration.

[2] The Aristotelian syllogisms are easy but useful examples for inferences in a first-order language with unary predicate symbols. One of these syllogisms serves as an example in Section **4.6** on logic programming.

The way of arguing about formal languages and theories is traditionally called the *metatheory*. An important task of a metatheoretic analysis is to specify procedures of logical inference by so-called *logical calculi*, which operate purely syntactically. There are many different logical calculi. The choice may depend on the formalized language, on the logical basis, and on certain aims of the formalization. Basic metatheoretic tools are in any case the naive natural numbers and inductive proof procedures. We will sometimes call them proofs by *metainduction*, in particular when talking about formalized object theories that speak about natural numbers. Induction can likewise be carried out on certain sets of strings over a fixed alphabet, or on the system of rules of a logical calculus.

The logical means of the metatheory are sometimes allowed or even explicitly required to be different from those of the object language. But in this book the logic of object languages, as well as that of the metalanguage, are classical, two-valued logic. There are good reasons to argue that classical logic is the logic of common sense. Mathematicians, computer scientists, linguists, philosophers, physicists, and others are using it as a common platform for communication.

It should be noticed that logic used in the sciences differs essentially from logic used in everyday language, where logic is more an art than a serious task of saying what follows from what. In everyday life, nearly every utterance depends on the context. In most cases logical relations are only alluded to and rarely explicitly expressed. Some basic assumptions of two-valued logic mostly fail, in particular, a context-free use of the logical connectives. Problems of this type are not dealt with here. To some extent, many-valued logic or Kripke semantics can help to clarify the situation, and sometimes intrinsic mathematical methods must be used in order to solve such problems. We shall use Kripke semantics here for a different goal, though, the analysis of self-referential sentences in Chapter **7**.

Let us add some historical remarks, which, of course, a newcomer may find easier to understand *after* and not *before* reading at least parts of this book. In the relatively short period of development of modern mathematical logic in the twentieth century, some highlights may be distinguished, of which we mention just a few. Many details on this development can be found in the excellent biographies [Daw] and [FF] on Gödel and Tarski, the leading logicians in the last century.

The first was the axiomatization of set theory in various ways. The most important approaches are those of Zermelo (improved by Fraenkel and von Neumann) and the theory of types by Whitehead and Russell. The latter was to become the sole remnant of Frege's attempt to reduce mathematics to logic. Instead it turned out that mathematics can be based entirely on set theory as a first-order theory. Actually, this became more salient after the rest of the hidden assumptions by Russell and others were removed from axiomatic set theory around 1915; see [Hei]. For instance, the notion of an ordered pair, crucial for reducing the notion of a function to set theory, is indeed a set-theoretic and not a logical one.

Right after these axiomatizations were completed, Skolem discovered that there are countable models of the set-theoretic axioms, a drawback to the hope for an axiomatic characterization of a set. Just then, two distinguished mathematicians, Hilbert and Brouwer, entered the scene and started their famous quarrel on the foundations of mathematics. It is described in a comprehensive manner for instance in [Kl2, Chapter IV] and need therefore not be repeated here.

As a next highlight, Gödel proved the completeness of Hilbert's rules for predicate logic, presented in the first modern textbook on mathematical logic, [HA]. Thus, to some extent, a dream of Leibniz became real, namely to create an *ars inveniendi* for mathematical truth. Meanwhile, Hilbert had developed his view on a foundation of mathematics into a program. It aimed at proving the consistency of arithmetic and perhaps the whole of mathematics including its nonfinitistic set-theoretic methods by finitary means. But Gödel showed by his incompleteness theorems in 1931 that Hilbert's original program fails or at least needs thorough revision.

Many logicians consider these theorems to be the top highlights of mathematical logic in the twentieth century. A consequence of these theorems is the existence of consistent extensions of Peano arithmetic in which true and false sentences live in peaceful coexistence with each other, called "dream theories" in **7.3**. It is an intellectual adventure of holistic beauty to see wisdom from number theory known for ages, such as the Chinese remainder theorem, simple properties of prime numbers, and Euclid's characterization of coprimeness (page 249), unexpectedly assuming pivotal positions within the architecture of Gödel's proofs. Gödel's methods were also basic for the creation of recursion theory around 1936.

Church's proof of the undecidability of the tautology problem marks another distinctive achievement. After having collected sufficient evidence by his own investigations and by those of Turing, Kleene, and some others, Church formulated his famous thesis (see **6.1**), although in 1936 no computers in the modern sense existed nor was it foreseeable that computability would ever play the basic role it does today.

Another highlight of mathematical logic has its roots in the work of Tarski, who proved first the undefinability of truth in formalized languages as explained in **6.5**, and soon thereafter started his fundamental work on decision problems in algebra and geometry and on model theory, which ties logic and mathematics closely together. See Chapter **5**.

As already mentioned, Hilbert's program had to be revised. A decisive step was undertaken by Gentzen, considered to be another groundbreaking achievement of mathematical logic and the starting point of contemporary proof theory. The logical calculi in **1.4** and **3.1** are akin to Gentzen's calculi of natural deduction.

We further mention Gödel's discovery that it is not the axiom of choice (AC) that creates the consistency problem in set theory. Set theory with AC and the continuum hypothesis (CH) is consistent, provided set theory without AC and CH is. This is a basic result of mathematical logic that would not have been obtained without the use of strictly formal methods. The same applies to the independence proof of AC and CH from the axioms of set theory by Cohen in 1963.

The above indicates that mathematical logic is closely connected with the aim of giving mathematics a solid foundation. Nonetheless, we confine ourself to logic and its fascinating interaction with mathematics, which characterizes mathematical logic. History shows that it is impossible to establish a programmatic view on the foundations of mathematics that pleases everybody in the mathematical community. Mathematical logic is the right tool for treating the technical problems of the foundations of mathematics, but it cannot solve its epistemological problems.

Notation

We assume that the reader is familiar with the most basic mathematical terminology and notation, in particular with the *union*, *intersection*, and *complementation* of sets, denoted by $\cup$, $\cap$, and $\setminus$, respectively. Here we summarize only some notation that may differ slightly from author to author or is specific for this book. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural numbers including 0, integers, rational, and real numbers, respectively, and $\mathbb{N}_+, \mathbb{Q}_+, \mathbb{R}_+$ the sets of positive members of the corresponding sets. n, m, i, j, k always denote natural numbers unless stated otherwise. Hence, extended notation like $n \in \mathbb{N}$ is mostly omitted.

In the following, M, N denote sets, $M \subseteq N$ denotes inclusion, while $M \subset N$ means proper inclusion (i.e., $M \subseteq N$ and $M \neq N$). As a rule, we write $M \subset N$ only if the circumstance $M \neq N$ has to be emphasized. If M is fixed in a consideration and N varies over subsets of M, then $M \setminus N$ may also be symbolized by $\setminus N$ or $\neg N$.

$\emptyset$ denotes the *empty set*, and $\mathfrak{P}M$ the *power set* (= set of all subsets) of M. If one wants to emphasize that all elements of a set S are sets, S is also called a *system* or *family* of sets. $\bigcup S$ denotes the union of S, that is, the set of elements belonging to at least one $M \in S$, and $\bigcap S$ stands for the intersection of a nonempty system S, the set of elements belonging to all $M \in S$. If $S = \{M_i \mid i \in I\}$ then $\bigcup S$ and $\bigcap S$ are mostly denoted by $\bigcup_{i \in I} M_i$ and $\bigcap_{i \in I} M_i$, respectively.

A *relation* between M and N is a subset of $M \times N$, the set of ordered pairs (a, b) with $a \in M$ and $b \in N$. A precise definition of (a, b) is given on page 114. Such a relation, f say, is said to be a *function* or *mapping from M to N* if for each $a \in M$ there is precisely one $b \in N$ with $(a, b) \in f$. This b is denoted by $f(a)$ or fa or a^f and called the *value of f at a*. We denote a function f from M to N also by $f\colon M \to N$, or by $f\colon x \mapsto t(x)$, provided $f(x) = t(x)$ for some term t (see **2.2**). $\mathit{ran}\, f = \{fx \mid x \in M\}$ is called the *range* of f, and $\mathit{dom}\, f = M$ its *domain*. id_M denotes the *identical function* on M, that is, $\mathrm{id}_M(x) = x$ for all $x \in M$.

$f\colon M \to N$ is *injective* if $fx = fy \Rightarrow x = y$, for all $x, y \in M$, *surjective* if $\mathit{ran}\, f = N$, and *bijective* if f is both injective and surjective. The reader should basically be familiar with this terminology. The phrase "let f be a function from M to N" is sometimes shortened to "let $f\colon M \to N$."

The set of all functions from a set I to a set M is denoted by M^I. If f, g are functions with $ran\, g \subseteq dom\, f$ then $h\colon x \mapsto f(g(x))$ is called their *composition* (or *product*). It will preferably be written as $h = f \circ g$.

Let I and M be sets, $f\colon I \to M$, and call I the *index set*. Then f will often be denoted by $(a_i)_{i \in I}$ and is named, depending on the context, an (indexed) *family*, an *I-tuple*, or a *sequence*. If 0 is identified with $\emptyset$ and $n > 0$ with $\{0, 1, \dots, n-1\}$, as is common in set theory, then M^n can be understood as the set of n-tuples $(a_i)_{i<n} = (a_0, \dots, a_{n-1})$ *of length n* whose members belong to M. In particular, $M^0 = \{\emptyset\}$. Also the set of sequences $(a_1, \dots, a_n)$ with $a_i \in M$ will frequently be denoted by M^n. In concatenating finite sequences, which has an obvious meaning, the *empty sequence* (i.e., $\emptyset$), plays the role of a neutral element. $(a_1, \dots, a_n)$ will mostly be denoted by $\vec{a}$. Note that this is the empty sequence for $n = 0$, similar to $\{a_1, \dots, a_n\}$ for $n = 0$ always being the empty set. $f\vec{a}$ means $f(a_1, \dots, a_n)$ throughout.

If A is an *alphabet*, i.e., if the elements $\mathsf{s} \in \mathsf{A}$ are symbols or at least named symbols, then the sequence $(\mathsf{s}_1, \dots, \mathsf{s}_n) \in \mathsf{A}^n$ is written as $\mathsf{s}_1 \cdots \mathsf{s}_n$ and called a *string* or a *word* over A. The empty sequence is called in this context the *empty string*. A string consisting of a single symbol s is termed an *atomic string*. It will likewise be denoted by s, since it will be clear from the context whether s means a symbol or an atomic string.

Let $\xi\eta$ denote the concatenation of the strings ξ and η. If $\xi = \xi_1\eta\xi_2$ for some strings ξ_1, ξ_2 and $\eta \neq \emptyset$ then η is called a *segment* (or *substring*) of ξ, termed a *proper* segment in case $\eta \neq \xi$. If $\xi_1 = \emptyset$ then η is called an *initial*, if $\xi_2 = \emptyset$, a *terminal* segment of ξ.

Subsets $P, Q, R, \ldots \subseteq M^n$ are called *n-ary predicates of M* or n-ary *relations*. A unary predicate will be identified with the corresponding subset of M. We may write $P\vec{a}$ for $\vec{a} \in P$, and $\neg P\vec{a}$ for $\vec{a} \notin P$. Metatheoretical predicates (or properties) cast in words will often be distinguished from the surrounding text by single quotes, for instance, if we speak of the syntactic predicate 'The variable x occurs in the formula α'. We can do so since quotes inside quotes will not occur in this book. Single-quoted properties are often used in induction principles or reflected in a theory, while ordinary ("double") quotes have a stylistic function only.

An n-ary *operation of M* is a function $f\colon M^n \to M$. Since $M^0 = \{\emptyset\}$, a 0-ary operation of M is of the form $\{(\emptyset, c)\}$, with $c \in M$; it is denoted by

c for short and called a *constant*. Each operation $f: M^n \to M$ is uniquely described by the *graph of* f, defined as

$$\operatorname{graph} f := \{(a_1, \dots, a_{n+1}) \in M^{n+1} \mid f(a_1, \dots, a_n) = a_{n+1}\}.$$[1]

Both f and graph f are essentially the same, but in most situations it is more convenient to distinguish between them.

The most important operations are binary ones. The corresponding symbols are mostly written between the arguments, as in the following listing of properties of a binary operation $\circ$ on a set A. $\circ: A^2 \to A$ is

commutative	if	$a \circ b = b \circ a$ for all $a, b \in A$,
associative	if	$a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in A$,
idempotent	if	$a \circ a = a$ for all $a \in A$,
invertible	if	for all $a, b \in A$ there are $x, y \in A$ with $a \circ x = b$ and $y \circ a = b$.

If H, Θ (read *eta, theta*) are expressions of our metalanguage, $\mathrm{H} \Leftrightarrow \Theta$ stands for 'H *iff* Θ' which abbreviates 'H *if and only if* Θ'. Similarly, $\mathrm{H} \Rightarrow \Theta$ and H & Θ mean '*if* H *then* Θ' and 'H *and* Θ', respectively, and $\mathrm{H} \vee \Theta$ is to mean 'H *or* Θ.' This notation does not aim at formalizing the metalanguage but serves improved organization of metatheoretic statements. We agree that $\Rightarrow$, $\Leftrightarrow$, ... separate stronger than linguistic binding particles such as "there is" or "for all." Therefore, in the statement

'$X \vdash \alpha \Leftrightarrow X \vDash \alpha$, *for all* X *and all* α' (Theorem 1.4.6)

the comma should not be dropped; otherwise, some serious misunderstanding may arise: '$X \vDash \alpha$ *for all* X *and all* α' is simply false.

$\mathrm{H} :\Leftrightarrow \Theta$ means that the expression H is defined by Θ. When integrating formulas in the colloquial metalanguage, one may use certain abbreviating notation. For instance, '$\alpha \equiv \beta$ and $\beta \equiv \gamma$' is occasionally shortened to $\alpha \equiv \beta \equiv \gamma$. ('the formulas α, β, and β, γ are equivalent'). This is allowed, since in this book the symbol $\equiv$ will never belong to the formal language from which the formulas α, β, γ are taken. W.l.o.g. or w.l.o.g. is a colloquial shorthand of "without loss of generality" used in mathematics.

[1] This means that the left-hand term graph f is *defined* by the right-hand term. A corresponding meaning has $:=$ throughout, except in programs and flow diagrams, where $x := t$ means the allocation of the value of the term t to the variable x.