9.1 Recall five you known properties about natural numbers (number theory) at first, then based on PA, use LA to formalize the properties (write formula and give its English or Chinese interpretations). You can define new predicates and functions if they are necessary.

① Commutative property of addition in natural number:

We can define the addition in natural number by the following:

Ⅰ. (PA5) $\forall m \in N, m + 0 = m$;

Ⅱ. (PA6) $\forall m, n \in N, n' + m = (n + m)'$

Now, we need to prove that:

q. $\forall m, n \in N, m + n = n + m.$

From I., it is obvious that q. is true for n = 0 (m + 0 = 0 + m). Similarly, true for n = 1 (m + 1 = 1 + m).

Assume that q. is also true for n (m + n = n + m), then for n'

1. $m + n' = m + (n + 0)' = m + (n + 0') = m + (n + 1) = (m + n) + 1$

(by definition)

2. $(m + n) + 1 = 1 + (m + n)$

(from m + 1 = 1 + m as formal said)

3. $1 + (m + n) = 1 + (n + m)$

(from m + n = n + m as formal assumption)

4. $1 + (n + m) = (1 + n) + m = (n + 1) + m = n' + m$

(from m + 1 = 1 + m as formal said and definition)

5. $m + n' = n' + m$

By the mathematical induction principle, we can see that the property holds for all natural numbers.

In L_A, $\forall$ m, n $\in$ N, m + n = n + m.

② Associative property of addition in natural number:

The addition is defined above.

Now we need to prove that:

q. $\forall m, n, k \in N, (m + n) + k = m + (n + k)$

It is obvious that q. is true for k = 0 by definition. ($(m + n) + 0 = m + n = m + (n + 0)$)

Assume that q. is also true for k, then for k'

1. $(m + n) + k' = ((m + n) + k)'$

(PA6)

2. $m + (n + k') = m + (n + k)' = (m + (n + k))'$

(PA6)

3. $(m + n) + k = m + (n + k)$

(by the assumption)

4. $((m + n) + k)' = (m + (n + k))'$

(from 3 by PA2)

5. $(m + n) + k' = m + (n + k')$

(from 1 and 2 by 4)

By the mathematical induction principle, we can see that the property holds for all natural numbers.

In L_A, $\forall$ m, n, k $\in$ N, (m + n) + k = m + (n + k)

③Commutative property of multiplication in natural number:

We can define the multiplication in natural number by the following:

Ⅰ. (PA7) $\forall m \in N, 0 \cdot m = 0;$

Ⅱ. (PA8) $\forall m, n \in N, n' \cdot m = (n \cdot m) + m$

Now we need to prove that

q. $\forall m, n \in N, m \cdot n = n \cdot m$

From I., it is obvious that q. is true for n = 0 ($m \cdot 0 = 0 \cdot m$).

Assume that q. is also true for n ($m \cdot n = n \cdot m$), then for n'

$m \cdot n' = m \cdot (n + 1) = m \cdot n + m = n \cdot m + m = n' \cdot m$

By the mathematical induction principle, we can see that the property holds for all natural numbers.

In L_A, $\forall$ m, n $\in$ N, $m \cdot n = n \cdot m$

④Associative property of multiplication in natural number:

The multiplication is defined above.

Now we need to prove that:

q. $\forall m, n, k \in N, (m \cdot n) \cdot k = m \cdot (n \cdot k)$

It is obvious that q. is true for m = 0 by definition. ($(0 \cdot n) \cdot k = 0 = 0 \cdot (n \cdot k)$)

Assume that q. is also true for m ($(m \cdot n) \cdot k = m \cdot (n \cdot k)$), then for m'

$(m' \cdot n) \cdot k = ((m + 1) \cdot n) \cdot k = (m \cdot n + n) \cdot k = (m \cdot n) \cdot k + (n \cdot k) = m \cdot (n \cdot k) + (n \cdot k) = (m + 1) \cdot (n \cdot k) = m' \cdot (n \cdot k)$

By the mathematical induction principle, we can see that the property holds for all natural numbers.

In L_A, $\forall$ m, n, k $\in$ N, $(m \cdot n) \cdot k = m \cdot (n \cdot k)$

⑤Cancellation property of addition in natural number:

The addition is defined above.

Now we need to prove that:

q. $\forall m, n, k \in N, (m + k = n + k) \Rightarrow (m = n)$

It is obvious that q. is true for k = 0 by definition. Similarly, true for k = 1

Assume that q. is also true for k, then for k'

$m + k' = n + k' \Rightarrow m + k + 1 = n + k + 1 \Rightarrow m + 1 = n + 1 \Rightarrow m = n$

By the mathematical induction principle, we can see that the property holds for all natural numbers.

In L_A, $\forall$ m, n, k $\in$ N, $(m + k = n + k) \Rightarrow (m = n)$

9.2 Based on NBG, formalize the following concepts/notions of set theory (write formula and give its English or Chinese interpretations):

(a) Reflexive relation

Let R be the reflexive relation, A be the source

In Naïve set theory: R: A → A, $(\forall a)\ (a \in A \Rightarrow (a, a) \in R)$

In NBG: $\vdash_{NBG} R \subseteq V^2 \wedge ((\forall x)\ (x \in A) \wedge (<x, x> \in R))$

(b) Irreflexive relation

Let R be the irreflexive relation, A be the source

In Naïve set theory: R: $A \to A$, $(\forall a)\ (a \in A \Rightarrow (a, a) \notin R)$

In NBG: $\vdash_{NBG} R \subseteq V^2 \land ((\forall x)\ (x \in A) \land (<x, x> \notin R))$

(c) Symmetric relation

Let R be the symmetric relation, A be the source

In Naïve set theory: R: $A \to A$, $(\forall a)\ (\forall b)\ ((a \in A \land b \in A) \Rightarrow ((a, b) \in R \Rightarrow (b, a) \in R))$

In NBG: $\vdash_{NBG} R \subseteq V^2 \land (\forall x_1)\ (\forall x_2)\ (((x_1 \in A) \land (x_2 \in A)) \to (<x_1, x_2> \in R \to < x_2, x_1> \in R))$

(d) Antisymmetric relation

Let R be the antisymmetric relation, A be the source

In Naïve set theory: R: $A \to A$, $(\forall a)\ (\forall b)\ ((a \in A \land b \in A) \Rightarrow (((a, b) \in R \land (b, a) \in R) \Rightarrow a = b))$

In NBG: $\vdash_{NBG} R \subseteq V^2 \land (\forall x_1)\ (\forall x_2)\ (((x_1 \in A) \land (x_2 \in A)) \to ((<x_1, x_2> \in R \to < x_2, x_1> \in R) \to x_1 = x_2))$

(e) Connected relation

Let R be the connected relation, A be the source

In Naïve set theory: R: $A \to A$, $(\forall a)\ (\forall b)\ ((a \in A \land b \in A) \Rightarrow (a \neq b \Rightarrow ((a, b) \in R \lor (b, a) \in R)))$

In NBG: $\vdash_{NBG} R \subseteq V^2 \land (\forall x_1)\ (\forall x_2)\ ((x_1 \in A \land x_2 \in A) \to (\neg\ (x_1 = x_2) \to (<x_1, x_2> \in R \lor <x_2, x_1> \in R)))$

(f) Transitive relation

Let R be the transitive relation, A be the source

In Naïve set theory: R: $A \to A$, $(\forall a)\ (\forall b)\ (\forall c)\ ((a \in A \land b \in A \land c \in A) \Rightarrow (((a, b) \in R \land (b, c) \in R) \Rightarrow (a, c) \in R))$

In NBG: $\vdash_{NBG} R \subseteq V^2 \land (\forall x_1)\ (\forall x_2)\ (\forall x_3)\ ((x_1 \in A \land x_2 \in A \land x_3 \in A) \to ((<x_1, x_2> \in R \land <x_2, x_3> \in R) \to <x_1, x_3> \in R))$

(g) Equivalence relation

Let R be the equivalence relation, A be the source

$\vdash_{NBG} R \subseteq V^2 \land ((\forall x)\ (x \in A) \land (<x, x> \in R)) \land (\forall x_1)\ (\forall x_2)\ (((x_1 \in A) \land (x_2 \in A)) \to (<x_1, x_2> \in R \to < x_2, x_1> \in R)) \land (\forall a)\ (\forall b)\ (\forall c)\ ((a \in A \land b \in A \land c \in A) \to ((<a, b> \in R \land <b, c> \in R) \to <a, c> \in R))$

(h) Partial order relation

Let R be the partial order relation, A be the source

$\vdash_{NBG} R \subseteq V^2 \land ((\forall x)\ (x \in A) \land (<x, x> \in R)) \land (\forall x_1)\ (\forall x_2)\ (((x_1 \in A) \land (x_2 \in A)) \to ((<x_1, x_2> \in R \to < x_2, x_1> \in R) \to x_1 = x_2)) \land (\forall a)\ (\forall b)\ (\forall c)\ ((a \in A \land b \in A \land c \in A) \to ((<a, b> \in R \land <b, c> \in R) \to <a, c> \in R))$

(i) Partial function/mapping

Let Dom(X) be the domain of X $((\forall u)\ (u \in Dom(X) \leftrightarrow (\exists v)\ (<u, v> \in X)))$, Ran(X) be the range

of X ((∀u) (u ∈ Ran(X) ↔ (∃v) (<v, u> ∈ X))), A be the resource of X;
X is a partial function/mapping if
$\vdash_{NBG}$ $X \subseteq V^2 \wedge (\forall u) (\exists v) (<u, v> \in X) \wedge (\forall x) (\forall y) (\forall z) ((x \in Dom(X) \wedge y \in Ran(X) \wedge z \in Ran(X) \wedge Dom(X) \subset A) \rightarrow ((<x, y> \in X \wedge <x, z> \in X) \rightarrow (y=z)))$

(j) Total function/mapping
Let Dom(X) be the domain of X ((∀u) (u ∈ Dom(X) ↔ (∃v) (<u, v> ∈ X))), Ran(X) be the range of X ((∀u) (u ∈ Ran(X) ↔ (∃v) (<v, u> ∈ X))), A be the resource of X;
X is a total function/mapping if
$\vdash_{NBG}$ $X \subseteq V^2 \wedge (\forall u) (\exists v) (<x, v> \in X) \wedge (\forall x) (\forall y) (\forall z) ((x \in Dom(X) \wedge y \in Ran(X) \wedge z \in Ran(X) \wedge Dom(X) = A) \rightarrow ((<x, y> \in X \wedge <x, z> \in X) \rightarrow (y=z)))$

(k) Injective function/mapping
Let Dom(X) be the domain of X ((∀u) (u ∈ Dom(X) ↔ (∃v) (<u, v> ∈ X))), Ran(X) be the range of X ((∀u) (u ∈ Ran(X) ↔ (∃v) (<v, u> ∈ X)));
X is an injective function/mapping if
$\vdash_{NBG}$ $X \subseteq V^2 \wedge (\forall u) (\exists v) (<u, v> \in X) \wedge (\forall x) (\forall y) (\forall z) (\forall t) ((x, y \in Dom(X) \wedge z, t \in Ran(X)) \rightarrow (((x, z) \in X \wedge (y, t) \in X \wedge x \neq y) \rightarrow z \neq t))$

(l) Surjective function/mapping
Let Dom(X) be the domain of X ((∀u) (u ∈ Dom(X) ↔ (∃v) (<u, v> ∈ X))), Ran(X) be the range of X ((∀u) (u ∈ Ran(X) ↔ (∃v) (<v, u> ∈ X)));
X is a surjective function/mapping if
$\vdash_{NBG}$ $X \subseteq V^2 \wedge (\exists v) (\forall u) (<v, u> \in X) \wedge (\forall y) (\exists x) ((x \in Dom(X) \wedge y \in Ran(X)) \rightarrow (<x, y> \in X)$

(m) Bijective function/mapping
Let Dom(X) be the domain of X ((∀u) (u ∈ Dom(X) ↔ (∃v) (<u, v> ∈ X))), Ran(X) be the range of X ((∀u) (u ∈ Ran(X) ↔ (∃v) (<v, u> ∈ X)));
X is a bijective function/mapping if
$\vdash_{NBG}$ $X \subseteq V^2 \wedge (\forall v) (\forall u) (<v, u> \in X) \wedge (\forall x) (\forall y) (\forall z) (\forall t) ((x, y \in Dom(X) \wedge z, t \in Ran(X)) \rightarrow (((x, z) \in X \wedge (y, t) \in X \wedge x \neq y) \rightarrow z \neq t)) \wedge (\forall q) (\exists p) ((p \in Dom(X) \wedge q \in Ran(X)) \rightarrow (<p, q> \in X)$