An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen's Natural Deduction System for **CFOPC**
- ♣ Gentzen's Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

Semantics (Model Theory) of CFOPC: The Fundamental Question

♣ **The fundamental question**

- Why the semantics (model theory) of **CFOPC** is indispensable?

♣ **The answer to the question**

- Well-formed formulas of **CFOPC** have meaning only when an interpretation is given for the symbols of **CFOPC**.
- The semantics (model theory) of **CFOPC** gives a ***truth-value (truth-functional) interpretation*** for the symbols/well-formed formulas of **CFOPC**.
- The semantics (model theory) of **CFOPC** provides a (philosophical and mathematical) fundamental basis for studying and using **CFOPC**.

Semantics (Model Theory) of CFOPC: Important Notes

♣ Important notes

- The semantics (model theory) of **CFOPC** is the most intrinsic foundation of **CFOPC**.
- Without a sound semantics, **CFOPC** is meaningless.
- The semantics (model theory) of **CFOPC** is only relatively correct/sound/satisfactory, i.e., it is correct/sound/satisfactory only because it is based on those fundamental assumptions/principles underlying **CML** (**C**lassical **M**athematical **L**ogic).

Fundamental Assumptions/Principles Underlying CML

♣ **The classical abstraction**

- The only properties of a proposition that matter to logic are its form and its truth-value.

♣ **The Fregean assumption / the principle of extensionality**

- The truth-value of a (composite) proposition depends only on its (composition) form and the truth-values of its constituents, not on their meaning.

♣ **The principle of bivalence**

- There are exactly two truth-values, "***TRUE***" and "***FALSE***". Every proposition has one or other, but not both, of these truth-values.

♣ **The classical account of validity (CAV)**

- An argument is valid if and only if it is impossible for all its premises to be true while its conclusion is false.

Semantics (Model Theory) of CFOPC: Models (Structures)

♣ ***Models*** (***Structures***) for first-order languages

- Let L(**Con**, **Fun**, **Pre**) be a first-order language determined by **Con**, **Fun**, and **Pre**.
- A ***model*** (***structure***) for L(**Con**, **Fun**, **Pre**) is an ordered pair $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ where $\boldsymbol{D}$ is a non-empty set of entities, called the ***domain*** or ***universe*** of $\boldsymbol{M}$ and $\boldsymbol{I}$ is a mapping, called an ***interpretation*** of $\boldsymbol{M}$ such that:

  for every constant symbol $c \in \mathbf{Con}$,

  $c^{\boldsymbol{I}}$ is an element (entity) of $\boldsymbol{D}$, $c^{\boldsymbol{I}} \in \boldsymbol{D}$;

  for every n-ary function symbol $f \in \mathbf{Fun}$,

  $f^{\boldsymbol{I}}$ is an n-ary function on $\boldsymbol{D}$, $f^{\boldsymbol{I}} : \boldsymbol{D}^n \to \boldsymbol{D}$;

  for every n-ary predicate symbol $p \in \mathbf{Pre}$,

  $p^{\boldsymbol{I}}$ is an n-ary relation on $\boldsymbol{D}$, $p^{\boldsymbol{I}} \subseteq \boldsymbol{D}^n$.

Semantics (Model Theory) of CFOPC: Models (Structures)

♣ Notes

- The domain $\boldsymbol{D}$ defines the application area of the language L, and the interpretation mapping $\boldsymbol{I}$ relates various symbols of L to entities and relationships among them in the application area $\boldsymbol{D}$.
- The interpretation mapping $\boldsymbol{I}$ relates each individual constant symbol c to an entity $c^{\boldsymbol{I}}$ in $\boldsymbol{D}$, each n-ary function symbol f to an n-ary function $f^{\boldsymbol{I}}$ in $\boldsymbol{D}$, and each n-ary predicate symbol p to an n-ary relation $p^{\boldsymbol{I}}$ in $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Assignments

♣ ***Assignments*** in a model

- An ***assignment*** ***Ass*** in a model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ for the first-order language L(**Con**, **Fun**, **Pre**) is a mapping from the set of individual variables **V** to the domain $\boldsymbol{D}$, $\boldsymbol{Ass}$: $\mathbf{V} \to \boldsymbol{D}$.
- The image of the individual variable x under the assignment ***Ass*** is denoted by x^{Ass}.
- The assignment $\boldsymbol{B}$ in the model $\boldsymbol{M}$ is an ***x-variant*** of the assignment $\boldsymbol{A}$ in the model $\boldsymbol{M}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ assign the same values to every individual variable in **V** except possibly x. (Note: An assignment may have many x-variants.)

♣ Notes

- ***Ass*** relates each individual variable x to an entity x^{ASS} in $\boldsymbol{D}$.
- A model may have many different assignments.
- Once a model (structure) $(\boldsymbol{D}, \boldsymbol{I})$ for the language L(**Con**, **Fun**, **Pre**) together with an assignment ***Ass*** is defined (given), various symbols of L have certain meaning (interpretation) in the application area $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Interpretations for Terms

♣ ***Interpretations*** for terms

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model.
- For every term $t \in$ **Ter**, its interpretation (a ***value*** in $\boldsymbol{D}$) is defined as follows:

(1) $c^{IA} = c^{I}$ for every $c \in$ **Con**, if $t = c$;

(2) $x^{IA} = x^{A}$ for every $x \in$ **V**, if $t = x$;

(3) $[f(t_1, ..., t_n)]^{IA} = f^{I}(t_1^{IA}, ..., t_n^{IA})$ for every $f \in$ **Fun**.

Semantics (Model Theory) of CFOPC: Interpretations for Terms

♣ Notes

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model.
- The interpretation mapping $\boldsymbol{I}$ relates each individual constant symbol c to an entity c^{I} in $\boldsymbol{D}$; each n-ary function symbol f to an n-ary function f^{I} in $\boldsymbol{D}$; each n-ary predicate symbol p to an n-ary relation p^{I} in $\boldsymbol{D}$.
- The assignment $\boldsymbol{A}$ relates each individual variable x to an entity x^{A} in $\boldsymbol{D}$.
- For every term $t \in$ **Ter** and every n-ary function symbol $f \in$ **Fun**, if $t = c$, t is interpreted as c^{I}, an entity in $\boldsymbol{D}$; if $t = x$, t is interpreted as x^{A}, also an entity in $\boldsymbol{D}$; and for n terms $t_1, ..., t_n \in$ **Ter** and an n-ary function f^{I} in $\boldsymbol{D}$, $f(t_1, ..., t_n)$ is interpreted as $f^{I}(t_1^{IA}, ..., t_n^{IA})$, its value is an entity in $\boldsymbol{D}$.
- The value of a closed term does not depend on the assignment $\boldsymbol{A}$.

Semantics (Model Theory) of CFOPC: Truth-Value of Formula

♣ ***Truth-value*** of a formula in a model

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model. For any $R \in$ **WFF**, its ***truth-value*** $v_f^{IA}(R)$ under $\boldsymbol{A}$ in $\boldsymbol{M}$ is defined by a ***truth valuation*** function v_f^{IA} : **WFF**$\to${**T**, **F**} as follows:

(1) for every atomic formula $p(t_1, ..., t_n) \in$ **WFF**,
$v_f^{IA}(p(t_1, ..., t_n)) = \mathbf{T}$ if $(t_1^{IA}, ..., t_n^{IA}) \in p^{I}$, and
$v_f^{IA}(p(t_1, ..., t_n)) = \mathbf{F}$ otherwise;

(2) for any $(\neg R)$, $(R*S) \in$ **WFF**, where $*$ is a binary connective,
$v_f^{IA}(\neg R)$, $v_f^{IA}(R*S)$ are the same as the definition of v_f of **CPC**;

(3) for any $((\forall x)R)$, $v_f^{IA}(((\forall x)R)) = \mathbf{T}$ if $v_f^{IB}(R) = \mathbf{T}$ for every assignment $\boldsymbol{B}$ in M that is an x-variant of $\boldsymbol{A}$, and $v_f^{IA}(((\forall x)R)) = \mathbf{F}$ otherwise;

(4) for any $((\exists x)R)$, $v_f^{IA}(((\exists x)R)) = \mathbf{T}$ if $v_f^{IB}(R) = \mathbf{T}$ for some assignment $\boldsymbol{B}$ in M that is an x-variant of $\boldsymbol{A}$, and $v_f^{IA}(((\forall x)R)) = \mathbf{F}$ otherwise.

Semantics (Model Theory) of CFOPC: Truth-Value of Formula

♣ Notes

- We use **T** and **F** to represent "***TRUE***" and "***FALSE***" respectively; they belong to our meta-language but not the object language of **CFOPC**.
- The truth-value of a closed formula (sentence) does not depend on the assignment $\boldsymbol{A}$.
- Recall: A formula with no free (occurrence) variables (called a closed formula or sentence) represents a proposition that must be true or false.
- Any atomic formula $p(t_1, ..., t_n)$ is valuated under $\boldsymbol{A}$ in $\boldsymbol{M}$ as **T** if and only if it is interpreted as a real relation instance of n-ary relation p^{I} in $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Satisfiability of Formula

♣ ***Satisfiability*** of a formula in a model

For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**,

- R is ***satisfiable*** in $\boldsymbol{M}$ or R ***may be true*** in $\boldsymbol{M}$ IFF there is some assignment $\boldsymbol{A}$ (called a ***satisfying assignment***) such that under A, $v_f^{IA}(R) = \mathbf{T}$;
- $\boldsymbol{M}$ ***satisfies*** R or R is ***true*** in $\boldsymbol{M}$, written as $\models_M R$, IFF $v_f^{IA}(R) = \mathbf{T}$ for any assignment $\boldsymbol{A}$;
- $\boldsymbol{M}$ ***does not satisfy*** R or R ***may be false*** in $\boldsymbol{M}$ IFF there is some assignment $\boldsymbol{A}$ such that under $\boldsymbol{A}$, $v_f^{IA}(R) = \mathbf{F}$;
- R is ***unsatisfiable*** in $\boldsymbol{M}$ or R is ***false*** in $\boldsymbol{M}$, written as $\models\!\!\!\!/_M R$, IFF $v_f^{IA}(R) = \mathbf{F}$ for any assignment $\boldsymbol{A}$.

♣ Note

- A formula with free variables may be satisfied (i.e., true) for some values in the domain and not satisfied (i.e., false) for the others.

Semantics (Model Theory) of CFOPC: Logical Validity of Formula

♣ ***Logical validity*** of a formula (***logical theorem***)

- For the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**, R is ***logically valid***, written as $\models_{\mathrm{CFOPC}} R$, IFF $\models_M R$ in any model M for the language (Ex: $R = (A \vee \neg A)$).

♣ ***Unsatisfiability*** of a formula

- For the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**, R is ***unsatisfiable***, written as $\not\models_{\mathrm{CFOPC}} R$, IFF $\not\models_M R$ in any model M for the language (Ex: $R = (A \wedge \neg A)$).
- For any $R \in$ **WFF**, R is logically valid IFF $\neg R$ is unsatisfiable, and R is satisfiable IFF $\neg R$ is not logically valid.

♣ ***The undecidability of CFOPC*** [A. Church, 1936, A. M. Turing, 1936]

- Theorem: The validity problem for **CFOPC**, i.e., whether a formula of **CFOPC** is valid or not, is undecidable.
- The undecidability of **CFOPC** is one of the fundamental results for logic as well as for computer science.

Semantics (Model Theory) of CFOPC: Tautologies, Contradictions, and Contingencies

♣ ***Tautologies***, ***contradictions***, and ***contingencies***

- A formula $A \in$ **WFF** is a ***tautology*** (***logical theorem***) of **CFOPC**, written as $\models_{\mathrm{CFOPC}} A$, IFF $\models_M A$ for any model M of **CFOPC**, i.e., A is logically valid;
  A formula $A \in$ **WFF** is a ***contradiction*** of **CFOPC**, written as $\not\models_{\mathrm{CFOPC}} A$, IFF $\not\models_M A$ for any model M of **CFOPC** (i.e., A is unsatisfiable);
  A formula is a ***contingency*** IFF it is neither a tautology nor a contradiction.
- A formula must be any one of tautology, contradiction, and contingency.
- The set of all tautologies (logical theorems) of **CFOPC** is denoted by **Th(CFOPC)**.

♣ Relationship between tautologies and contradictions

- Theorem: For any $A \in$ **WFF**, A is a tautology IFF $(\neg A)$ is a contradiction, and A is a contradiction IFF $(\neg A)$ is a tautology.

Semantics (Model Theory) of CFOPC: Models of Formulas

♣ Satisfiability of a set of formulas

- For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $\Gamma \subseteq$ **WFF**, Γ is ***satisfiable*** in M if there is some assignment $\boldsymbol{A}$ (called a ***satisfying assignment***) such that under $\boldsymbol{A}$, $v_f^{IA}(R) = \mathbf{T}$ for all $R \in \Gamma$.
- Theorem (***Compactness***): Let Γ be a set of sentences. If every finite subset of Γ is satisfiable in model $\boldsymbol{M}$, so is Γ.
- Note: Γ may be an infinite set.

♣ Models of a set of formulas

- For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $\Gamma \subseteq$ **WFF**, $\boldsymbol{M}$ is called a ***model*** of Γ IFF $\models_M R$ (i.e., $v_f^{IA}(R) = \mathbf{T}$ for any assignment $\boldsymbol{A}$) for any $R \in \Gamma$.
- The set of all models of Γ is denoted by $\boldsymbol{M}(\Gamma)$.

Semantics (Model Theory) of CFOPC: Models of Formulas

♣ ***Consistency (Satisfiability)*** of a set of formulas

- For any $\Gamma \subseteq$ **WFF**, Γ is ***semantically (model-theoretically, logically) consistent (satisfiable)*** IFF it has at least one model; Γ is ***semantically (model-theoretically, logically) inconsistent (unsatisfiable)*** IFF it has no model.

♣ Note

- Here, consistency says "has at least one model", and inconsistency says "has no model".

Some Tautologies of CFOPC

- $\models_{\mathrm{CFOPC}} B(t) \rightarrow (\exists x)B(x)$, if t is free for x in $B(x)$
- $\models_{\mathrm{CFOPC}} ((\forall x)B) \rightarrow (\exists x)B$
- $\models_{\mathrm{CFOPC}} ((\forall x)(\forall y)B) \rightarrow (\forall y)(\forall x)B$
- $\models_{\mathrm{CFOPC}} ((\forall x)B) \leftrightarrow \neg(\exists x)\neg B$
- $\models_{\mathrm{CFOPC}} ((\forall x)(B \rightarrow C)) \rightarrow (((\forall x)B) \rightarrow (\forall x)C)$
- $\models_{\mathrm{CFOPC}} (((\forall x)B) \wedge (\forall x)C) \leftrightarrow (\forall x)(B \wedge C)$
- $\models_{\mathrm{CFOPC}} (((\forall x)B) \vee (\forall x)C) \rightarrow (\forall x)(B \vee C)$
- $\models_{\mathrm{CFOPC}} ((\exists x)(\exists y)B) \leftrightarrow (\exists y)(\exists x)B$
- $\models_{\mathrm{CFOPC}} ((\exists x)(\forall y)B) \rightarrow (\forall y)(\exists x)B$

Uniform Notation of First-order Formulas

♣ Uniform notation of first-order formulas [R. M. Smullyan, 1968]

- Classify all quantified formulas and their negations into two categories, i.e., ***γ-formulas*** which act universally, and ***δ-formulas***, which act existentially.
- For each variety and for each term t, an instance is defined.

♣ Proposition

- Let S be a set of sentences (closed formulas), and γ and δ be sentences. If $S \cup \{\gamma\}$ is satisfiable, so is $S \cup \{\gamma, \gamma(t)\}$ for any closed term t. If $S \cup \{\delta\}$ is satisfiable, so is $S \cup \{\delta, \delta(p)\}$ for any constant symbol p that is new to S and δ.

Uniform Notation of First-order Formulas

- γ-formulas and δ-formulas and their instances

Universal		Existential	
γ	$\gamma(t)$	δ	$\delta(t)$
$(\forall x\Phi)$	$\Phi[x/t]$	$(\exists x\Phi)$	$\Phi[x/t]$
$\neg(\exists x\Phi)$	$\neg\Phi[x/t]$	$\neg(\forall x\Phi)$	$\neg\Phi[x/t]$

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Semantic (Model-theoretical) Logical Consequence Relation

- Semantic (Model-theoretical, Logical) consequence relation
  - For any $\Gamma \subseteq$ **WFF** and any $A \in$ **WFF**, Γ ***semantically (model-theoretically, logically) entails*** A, or A ***semantically (model-theoretically, logically) follows from*** Γ, or A is a ***semantic (model-theoretical, logical) consequence*** of Γ, written as $\Gamma \models_{\mathbf{CFOPC}} A$, IFF $\models_{M} A$ for any model M of Γ.
  - $\varnothing \models_{\mathbf{CFOPC}} A = \models_{\mathbf{CFOPC}} A$ and it means that A is a tautology (logical theorem) of **CFOPC**, $A \in$ **Th**(**CFOPC**).
- All semantic (model-theoretical, logical) consequences of premises
  - The set of all semantic (model-theoretical, logical) consequences of Γ is denoted by $C_{sem}(\Gamma)$.
- Note
  - The semantic (model-theoretical, logical) consequence relation of **CFOPC** is a semantic (model-theoretical) formalization of the notion that one proposition follows from another or others.

Semantic (Model-theoretical, Logical) Equivalence Relation

- Semantic (Model-theoretical, Logical) equivalence relation
  - For any $A, B \in$ **WFF**, A is ***semantically (model-theoretically, logically) equivalent*** to B in **CFOPC** IFF both $\{A\} \models_{\mathbf{CFOPC}} B$ and $\{B\} \models_{\mathbf{CFOPC}} A$.
  - Theorem: A is semantically (model-theoretically, logically) equivalent to B IFF $(A \leftrightarrow B)$ is a tautology.
- Properties of semantic (model-theoretical, logical) consequence relation
  - The same as those of **CPC**.

Semantic Deduction Theorems

- Semantic deduction theorems
  - ***Semantic (model-theoretical, logical) deduction theorem for CFOPC***: For any $A, B \in$ **WFF** and any $\Gamma \subseteq$ **WFF**,
    $\Gamma \cup \{A\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} (A \rightarrow B)$;
    $\{A\} \models_{\mathbf{CFOPC}} B$ IFF $\models_{\mathbf{CFOPC}} (A \rightarrow B)$.
  - ***Semantic (model-theoretical, logical) deduction theorem for CFOPC for finite consequences***: For any $A_1, \ldots, A_{n-1}, A_n, B \in$ **WFF** and any $\Gamma \subseteq$ **WFF**,
    $\Gamma \cup \{A_1, \ldots, A_{n-1}, A_n\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} (A_1 \rightarrow (\ldots(A_{n-1} \rightarrow (A_n \rightarrow B))\ldots))$;
    $\Gamma \cup \{A_1, \ldots, A_{n-1}, A_n\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} ((A_1 \wedge (\ldots(A_{n-1} \wedge A_n)\ldots)) \rightarrow B)$.
  - The semantic deduction theorems are intrinsically important meta-theorems of **CFOPC**.

Semantic Deduction Theorems

- Notes
  - As a special case of the above deduction theorems, $\{A\} \models_{\mathbf{CFOPC}} B$ IFF $\models_{\mathbf{CFOPC}} (A \rightarrow B)$, i.e., A semantically (model-theoretically, logically) entails B IFF $(A \rightarrow B)$ is a tautology.
  - In the framework of **CFOPC**, the semantic (model-theoretical, logical) consequence relation, which is a representation of the notion of entailment in the sense of meta-logic, is "equivalent" to the notion of material implication (denoted by '→' in **CFOPC**).
  - However, in semantics, the notion of material implication is NOT an accurate representation of the notion of entailment.

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