An Introduction to Classical Predicate Calculus

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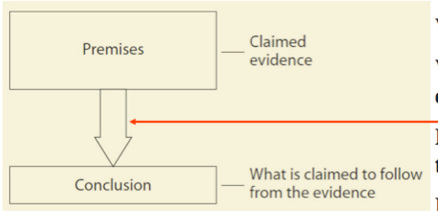
Logic: What Is It?



[From P. J. Hurley, "A Concise Introduction to Logic"]

What entails what?

What follows from what?

Why? What are the evaluation criteria?

How to establish/define the evaluation criteria?

How to evaluate arguments/reasoning?

It is LOGIC to answer these fundamental questions.

The Fundamental Questions

What is (mathematical) logic?
What entails what?
What follows from what?
Why? What are the evaluation criteria?
How to establish/define the evaluation criteria?
How to evaluate arguments/reasoning?
What is Classical Predicate Calculus?

Note: This lecture note is an introduction to CFOPC for CS/IS students from the viewpoint of applications; it includes all of important meta-theorems of CFOPC but no proof is given.

An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen's Natural Deduction System for **CFOPC**
- ♣ Gentzen's Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

The Limitations of Propositional Logic CPC: An Example

♣ Example

• All men are mortal	A: All M are P
Socrates is a man	B: S is a M
Therefore:	Therefore:
Socrates is mortal	C: S is P

- However, there is no way in propositional logic **CPC** to represent "all are …" and "there is a …".
- $((A \wedge B) \rightarrow C)$ is not a tautology in propositional logic **CPC**.

♣ Example

- A: $\forall x(M(x) \rightarrow P(x))$
  B: $\exists s(M(s))$
  Therefore:
  C: $P(s)$
- $((A \wedge B) \rightarrow C)$ is a tautology in first-order predicate logic **CFOPC**.

一阶谓词演算

The Limitations of Propositional Logic CPC: More Examples [Mendelson]

1. Any friend of Martin is a friend of John.
   Peter is not John's friend.
   Hence, Peter is not Martin's friend.
2. All human beings are rational.
   Some animals are human beings.
   Hence, some animals are rational.
3. The successor of an even integer is odd.
   2 is an even integer.
   Hence, the successor of 2 is odd.

$$\frac{(\forall x)(F(x,m) \Rightarrow F(x,j)) \quad \neg F(p,j)}{\neg F(p,m)} \qquad \frac{(\forall x)(H(x) \Rightarrow R(x)) \quad (\exists x)(A(x) \wedge H(x))}{(\exists x)(A(x) \wedge R(x))} \qquad \frac{(\forall x)(I(x) \wedge E(x) \Rightarrow D(s(x))) \quad I(b) \wedge E(b)}{D(s(b))}$$

The Limitations of Propositional Logic CPC: More Examples [Mendelson]

There are various kinds of logical inference that cannot be justified on the basis of the propositional calculus; for example:

1. Any friend of Martin is a friend of John.
   Peter is not John's friend.
   Hence, Peter is not Martin's friend.
2. All human beings are rational.
   Some animals are human beings.
   Hence, some animals are rational.
3. The successor of an even integer is odd.
   2 is an even integer.
   Hence, the successor of 2 is odd.

The correctness of these inferences rests not only upon the meanings of the truth-functional connectives, but also upon the meaning of such expressions as "any," "all," and "some," and other linguistic constructions.

In order to make the structure of complex sentences more transparent, it is convenient to introduce special notation to represent frequently occurring expressions. If $P(x)$ asserts that x has the property P, then $(\forall x)P(x)$ means that property P holds for all x or, in other words, that everything has the property P. On the other hand, $(\exists x)P(x)$ means that some x has the property P—that is, that there is at least one object having the property P. In $(\forall x)P(x)$, "$(\forall x)$" is called a *universal quantifier*; in $(\exists x)P(x)$, "$(\exists x)$" is called an *existential quantifier*. The study of quantifiers and related concepts is the principal subject of this chapter.

An Introduction to Classical Predicate Calculus

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Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols) 字符集

- {→, ∧, ∨, ⇔, ¬, ∀, ∃, T, ⊥, (,),
  $x_1, x_2, \ldots, x_n, \ldots,$ (countable) 个体变量
  $c_1, c_2, \ldots, c_n, \ldots,$ (countable) 常数符号
  $f^1{}_1, \ldots, f^1{}_k, \ldots, f^2{}_1, \ldots, f^2{}_k, \ldots, f^n{}_1, \ldots, f^n{}_k, \ldots,$ (countable) 函数符号
  $p^0{}_1, \ldots, p^0{}_k, \ldots, p^1{}_1, \ldots, p^1{}_k, \ldots, p^2{}_1, \ldots, p^2{}_k, \ldots, p^n{}_1, \ldots, p^n{}_k, \ldots$ (countable)} 谓词符号 (关系符号)
- ***Logical (propositional) connectives***: → (***material implication***), ∧ (***conjunction***), ∨ (***disjunction***), ↔ (***equivalence***), ¬ (***negation***).
- 量词 ***Quantifiers***: ∀ (for all, the ***universal quantifier***) (a turned A), 全称量词
  ∃ (there exists a, the ***existential quantifier***) (a flip-horizontal E). 存在量词
- 逻辑常数 ***Logical constants***: T and ⊥.
- Punctuation: left and right parentheses '(' and ')'.

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***Individual variables (variable symbols)*** x为论域中的事物，但具体内容不关心
  (**V**): $x_1, x_2, \ldots, x_n, \ldots$ (countable, not empty) 必为非空
- Individual variables, which rang over the domain of discourse, act as placeholders in much the same way as pronouns act as placeholders in ordinary language. 占位符
- ***Individual constants (Names) (constant symbols, name symbols)***
  (**Con**): $c_1, c_2, \ldots, c_n, \ldots$ (countable, possibly empty) C为
- Individual constants play the role of "names" for objects (individuals) in the domain of discourse.
- Note: Let Saitama-University be the domain of discourse, **Con** may be the union of faculty-identifier set and student-identifier set.
- The individual constants, function symbols, and predicate symbols of a language L are called the ***non-logical constants*** of L.

$f(t)=S$

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***(Individual) Functions (function symbols)***
  (**Fun**): $f^1{}_1, \ldots, f^1{}_k, \ldots, f^2{}_1, \ldots, f^2{}_k, \ldots, f^n{}_1, \ldots, f^n{}_k, \ldots,$ (countable, possibly empty)
- n and k are any positive integers. 上标表示推论数
- The superscript n indicates the number of arguments, whereas the subscript k is just an indexing number to distinguish different function symbols (letters) with the same number of arguments. 下标表示第几个函数(主要用于区分相同上标的函数)
- Constants can be regarded as 0-ary functions because they are objects (individuals) that have no dependence on any inputs; they simply denote objects (individuals) of the domain of discourse.
- The individual constants, function symbols, and predicate symbols of a language L are called the ***non-logical constants*** of L.

函数的自变量和因变量都必为空，输入和输出均不能为函数

个体 $\xrightarrow{映射}$ 个体

谓词可代替函数作用

$P(t,S)$

Formal (Object) Language (Syntax) of CFOPC: Alphabet (Symbols)

♣ ***Alphabet*** (Symbols)

- ***(Individual) Predicates (Relations) (predicate/relation symbols)***
  (**Pre**): $p^0{}_1, \ldots, p^0{}_k, \ldots, p^1{}_1, \ldots, p^1{}_k, \ldots, p^2{}_1, \ldots, p^2{}_k, \ldots, p^n{}_1, \ldots, p^n{}_k, \ldots$ (countable, not empty)
- n and k are any positive integers. 上标表示推论数
- The superscript n indicates the number of arguments, whereas the subscript k is just an indexing number to distinguish different predicate symbols (letters) with the same number of arguments. 下标表示第几个谓词
- 0-ary predicates can be regarded as propositions (sentences) because they are simply statements of facts independent of any individual variables.
- Unary predicates are simply properties of objects (individuals), binary predicates are relations between pairs of objects (individuals).
- In general n-ary predicates express relations among n-tuple of objects (individuals).

谓词描述的是一种关系

Formal (Object) Language (Syntax) of CFOPC: "First-Order"

♣ ***CFOPC***
- ***Classical First-Order Predicate Calculus***

♣ ***Individual quantifiers***
- In CFOPC, the quantifiers (universal quantifier and existential quantifier) are applied to only the individual variables. 量词符号之后一定为个体变量

♣ ***First-order***
- The adjective "first-order" is used to distinguish the languages we shall study here from those (i.e., high-order languages) in which there are predicates having other predicates or functions as arguments or in which predicate quantifiers or function quantifiers are permitted, or both.

♣ Sufficiency
- Most mathematical theories can be formalized within first-order languages, although there may be a loss of some of the intuitive content of those theories.

Formal (Object) Language (Syntax) of CFOPC: Terms

♣ ***Terms*** 项
- (1) Every individual variable (symbol) is a term; 个体变量和个体常量都是项
  (2) Every individual constant (symbol) is a term;
  (3) If f is an n-ary function (symbol) ($n = 1, 2, \ldots$) and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term; 函数应用的结果(也是变量/常量)
  (4) Nothing else are terms.
- **Ter**: the set of all terms
- The functions applied to the individual variables and individual constants generate the terms.
- Terms correspond to what in ordinary languages are nouns and noun phrases.

♣ ***Closed terms*** 封闭项：只有常量无变量（导致结果固定）
- A term is ***closed***, called a ***variable-free term*** or ***ground term***, IFF it contains no individual variables.

谓词里面的参数为项

公式的组成单位是谓词而非命题

Formal (Object) Language (Syntax) of CFOPC: Well-Formed Formulas

♣ ***Formulas (Well-formed formulas)***
- (1) If p is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$ is a formula (called an ***atomic formula***) (predicates applied to terms generate the atomic formulas); also $\top$ and $\perp$ are ***atomic formulas***; 原子公式
  (2) If A and B are formulas and x is an individual variable, then so are $(\neg A)$, $(A \to B)$, $(A \wedge B)$, $(A \vee B)$, $(A \leftrightarrow B)$, $((\forall x)A)$, and $((\exists x)A)$;
  (3) Nothing else are formulas.
- **WFF**$_{\text{CFOPC}}$: the set of all formulas of **CFOPC** (**WFF** for short).
- $((\exists x)A) =_{\text{df}} (\neg((\forall x)\neg A))$, $((\forall x)A) =_{\text{df}} (\neg((\exists x)\neg A))$; A is called the ***scope*** of quantifiers $(\forall x)$ and $(\exists x)$. 辖域

♣ Open formulas 开放公式
- An ***open formula*** is a formula without quantifiers. 无量词

♣ Notation
- Let $F \in$ **WFF** and x be an individual variable, we use $F(x)$ to mean that x appears in F.

Formal (Object) Language (Syntax) of CFOPC: Subformulas

♣ Immediate subformulas 直接子公式
- ***Immediate subformulas*** are defined as follows:
  (1) an atomic formula has no immediate subformula;
  (2) the only immediate subformula of $(\neg A)$, $((\forall x)A)$, and $((\exists x)A)$ is A;
  (3) for a binary connective $*$, the immediate subformulas of $(A*B)$ are A and B.

♣ Subformulas
- For any $A \in$ **WFF**, The set of ***subformulas*** of A is the smallest set S that contains A and contains, with each member, the immediate subformulas of that member. A is called an ***improper subformula*** of itself.

♣ Homework
- Try to develop an algorithm to check whether or not a string is a formula of **CFOPC**.

Formal (Object) Language (Syntax) of CFOPC: General Guidelines

♣ The sentences of the form "All As are Bs"
- $((\forall x_1)(p^1{}_1(x_1) \to p^1{}_2(x_1)))$ $((\forall x)(p_A(x) \to p_B(x)))$ (difference?)
- Ex: "Every mathematician loves music" is translated as $(\forall x)(M(x) \to LM(x)))$ where $M(x)$ means "x is a mathematician" and $LM(x)$ means "x loves music."

♣ The sentences of the form "Some As are Bs"
- $((\exists x_1)(p^1{}_1(x_1) \wedge p^1{}_2(x_1)))$ $((\exists x)(p_A(x) \wedge p_B(x)))$ (difference?)
- Ex: "Some New Yorkers are friendly" is translated as $(\exists x(NY(x) \wedge F(x)))$ where $NY(x)$ means "x is a New Yorker" and $F(x)$ "means x is friendly."

♣ The sentences of the form "No As are Bs" 全否定也是全称的一种
- $((\forall x_1)(p^1{}_1(x_1) \to (\neg p^1{}_2(x_1))))$ $((\forall x)(p_A(x) \to (\neg p_B(x))))$ (difference?)
- Ex: "No philosopher understands politics" is translated as $((\forall x)(P(x) \to (\neg UP(x))))$ where $P(x)$ means "x is a philosopher" and $UP(x)$ means "x understands politics."

Indicating Universal and Existential Quantity [Kelley]

Universal		Existential	
Affirmative	**Negative**	**Affirmative**	**Negative**
All	No	Some are	Some are not
Any	None	There is (are)	There is (are) . . . not
A (A cat is a predator)	Not a (Not a creature was stirring)	A (A car is parked outside)	A (A student is not present)
Every, Everything	Nothing	Something	Something
Everyone (people)	No one	Someone	Someone
Always (time)	Never	Sometimes	Sometimes
Everywhere (place)	Nowhere	Somewhere	Somewhere

Types of Statements in First-Order Predicate Logic [Kelley]

SUMMARY **Types of Statements in Predicate Logic**

Type	Predicate Notation	Meaning	Example
Atomic singular statement	Pa	a is P	London is a city.
Truth-functional compounds of singular atomic statements	$Pa \supset Qa$ $Pa \bullet Qb$, etc.	If a is P, it is Q a is P and b is Q; etc.	If London is a city then it is large. Tom is healthy but Sue is sick.
Quantifier applied to atomic open sentence Px	$(x)Px$ $(\exists x)Px$	For all x, x is P There exists an x that is P	Everything is material. Something is on fire.
Truth-functional compounds of open sentences within the scope of a quantifier	$(x){\sim}Px$ $(\exists x){\sim}Px$ $(x)(Px \supset Qx)$ $(\exists x)(Px \bullet Qx)$ $(x)[Px \lor (Qx \bullet Rx)]$	Nothing is P Something is not P For all x, if x is P then it is Q Some x is P and it is Q For all x, x is P or it is Q and R	Nothing is infinite. Something is not infinite. All men are mortal. Some birds are carnivores. Everything is either scary or warm and fuzzy.
Truth-functional compounds of quantified statements. Scope of quantifiers does not cover connective.	${\sim}(x)Px$ $(x)Px \supset (y)Qy$ $(x)Px \supset (\exists x)Qx$ $(\exists x)Px \lor (\exists y)Qy$, etc.	It is not the case that all x are P If all x are P, then all y are Q If all x are P, then some x is Q Either something is P or something is Q, etc.	It is not the case that everything is physical. If everything is physical, then everything has mass. If everything is created, then there is a God. There was either an explosion or a collision.

$(\forall x)(\neg P(x))$: 没有东西是P

$\neg((\forall x)(P(x)))$: 存在某个x是P.

Representation Examples of First-Order Language [Smullyan]

- Let h stand for Holmes (Sherlock Holmes) and let m stand for Moriarty. Let us abbreviate "x can catch y" by "xCy". Give symbolic renditions of the following statements:

(a) Holmes can catch anyone who can catch Moriarty. $((\forall x)(xCm \rightarrow hCx))$

(b) Holmes can catch anyone whom Moriarty can catch. $((\forall x)(mCx \rightarrow hCx))$

(c) Holmes can catch anyone who can be caught by Moriarty. (Same as (b))

(d) If anyone can catch Moriarty, then Holmes can. $((\exists x)(xCm) \rightarrow hCm))$

(e) If everyone can catch Moriarty, then Holmes can. $((\forall x)(xCm) \rightarrow hCm))$

(f) Anyone who can catch Holmes can catch Moriarty. $((\forall x)(xCh \rightarrow xCm))$

(g) No one can catch Holmes unless he can catch Moriarty. (Same as (f))

(h) Everyone can catch someone who cannot catch Moriarty.
$((\forall x)((\exists y)(xCy \wedge (\neg yCm))))$

(i) Anyone who can catch Holmes can catch anyone whom Holmes can catch. $((\forall x)(xCh \rightarrow ((\forall y)(hCy \rightarrow xCy))))$; or, $((\forall x)((\forall y)((xCh \wedge hCy) \rightarrow xCy)))$

Representation Examples of First-Order Language [Smullyan]

- Let us symbolize "x knows y" by "xKy".

(a) Everyone knows someone. $((\forall x)((\exists y)x\mathrm{K}y))$

(b) Someone knows everyone. $((\exists x)((\forall y)x\mathrm{K}y))$

(c) Someone is known by everyone. $((\exists x)((\forall y)y\mathrm{K}x))$

(d) Every person x knows someone who doesn't know x.
$((\forall x)((\exists y)(x\mathrm{K}y \wedge (\neg y\mathrm{K}x))))$

(e) There is someone x who knows everyone who knows x.
$((\exists x)((\forall y)(y\mathrm{K}x \rightarrow x\mathrm{K}y)))$

- Let Dx abbreviate "x can do it" and let "b" abbreviate Bernard. Let "$x = y$" abbreviate "x is identical with y".

(a) Bernard, if anyone, can do it. $((\exists x)(\mathrm{D}x \rightarrow \mathrm{D}b))$; or, $((\forall x)(\mathrm{D}x \rightarrow \mathrm{D}b))$

(b) Bernard is the only one who can do it.
$(\mathrm{D}b \wedge ((\forall x)(\mathrm{D}x \rightarrow (x=b))))$; or, $((\forall x)(\mathrm{D}x \leftrightarrow (x = b)))$

Representation Examples of First-Order Language [Smullyan]

- Here "number" shall mean natural number, i.e., 0 or some positive whole number. The usual abbreviation of "x is less than y" is "$x < y$", and the abbreviation of "x is greater than y" is "$x > y$".

(a) For every number there is a greater number. $((\forall x)((\exists y)y>x))$

(b) Every number other than 0 is greater than some number.
$((\forall x)(\neg(x=0) \rightarrow ((\exists y)x>y)))$

(c) 0 is the one and only number having the property that no number is less than it.
$(\neg(\exists y)(y<0 \wedge ((\forall x)(\neg(\exists y)(y<x \rightarrow x=0)))))$; or, $((\forall x)(\neg(\exists y)(y<x \leftrightarrow x=0)))$

"=": 等词　　最小　　唯一

$\neg((\exists y)y<0)$ 不能保证0是唯一的
只保证了0是最小的

Representation Examples of First-Order Language [Dalen]

$\exists x P(x)$	– there is an x with property P,
$\forall y P(y)$	– for all y P holds (all y have the property P),
$\forall x \exists y (x = 2y)$	– for all x there is a y such that x is two times y,
$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (\frac{1}{n} < \varepsilon))$	– for all positive ϵ there is an n such that $\frac{1}{n} < \varepsilon$,
$x < y \rightarrow \exists z (x < z \wedge z < y)$	– if $x < y$, then there is a z such that $x < z$ and $z < y$,
$\forall x \exists y (x.y = 1)$	– for each x there exists an inverse y.

x∧y

Representation Examples of First-Order Language [Manin]

♣ "x is the direct product y X z."

- This means that the elements of x are the ordered pairs of elements of y and z, respectively.
- The definition of an unordered pair is obvious: the formula
  $\forall u(u \in x \leftrightarrow (u = y_1 \vee u = z_1))$
  "means," or may be briefly written in the form, $x = \{y_1, z_1\}$.
- The ordered pair y_1 and z_1 is introduced using a device of Kuratowski and Wiener: this is the set of x_1 whose elements are the unordered pairs $\{y_1, y_1\}$ and $\{y_1, z_1\}$.
- Thus, we arrive at the formula
  $\exists y_2 \exists z_2$("$x_1 = \{y_2, z_2\}$" $\wedge$ "$y_2 = \{y_1, y_1\}$" $\wedge$ "$z_2 = \{y_1, z_1\}$")
  which will be abbreviated $x_1 = \langle y_1, z_1 \rangle$ and will be read "x_1 is the ordered pair with first element y_1 and second element z_1".
- Finally, the statement "$x = y$ X z" may be written in the form
  $\forall x_1(x_1 \in x \leftrightarrow \exists y_1 \exists z_1(y_1 \in y \wedge z_1 \in z \wedge$ "$x_1 = \langle y_1, z_1 \rangle$"$))$

Representation Examples of First-Order Language [Manin]

♣ "f is a mapping from the set u to the set v."

- The following formula successively imposes three conditions on mapping f:
  (1) f is a subset of u X v;
  (2) the projection of f onto u coincides with all of u;
  (3) each elements of u corresponds to exactly one element of v.
- $\forall z(z \in f \rightarrow \exists u_1 \exists v_1(u_1 \in u \wedge v_1 \in v \wedge \text{“}z=\langle u_1,v_1\rangle\text{”})) \wedge$
  $\forall u_1(u_1 \in u \rightarrow \exists v_1 \exists z(v_1 \in v \wedge \text{“}z=\langle u_1,v_1\rangle\text{”} \wedge z \in f)) \wedge$
  $\forall u_1 \forall v_1 \forall v_2(\exists z_1 \exists z_2(z_1 \in f \wedge z_2 \in f \wedge \text{“}z_1=\langle u_1,v_1\rangle\text{”} \wedge \text{“}z_2=\langle u_1,v_2\rangle\text{”}) \rightarrow v_1=v_2)$

♣ "x is a finite set."

- Dedekind's finiteness definition: "there does not exist a one-to-one mapping f of the set x onto a proper subset."
- $\neg \exists f\, [\ \text{“}f \text{ is a mapping from } x \text{ to } x\text{”} \wedge$
  $\forall u_1 \forall u_2 \forall v_1 \forall v_2(\ (\text{“}\langle u_1,v_1\rangle \in f\text{”} \wedge \text{“}\langle u_2,v_2\rangle \in f\text{”} \wedge \neg(u_1=u_2))$
  $\rightarrow \neg(v_1=v_2) \wedge \exists v_1(v_1 \in x \wedge \neg \exists u_1(\text{“}\langle u_1,v_1\rangle \in f\text{”}))\ )\ ]$

First-Order Number Theory [Mendelson]

♣ Number theory as the foundation for mathematics

- Together with geometry, the theory of numbers is the most immediately intuitive of all branches of mathematics.
- It is obvious that attempts to formalize mathematics and to establish a rigorous foundation for mathematics should begin with number theory.

♣ Peano's postulates for number theory

- The first semi-axiomatic presentation of this subject was given by Dedekind in 1879 and, in a slightly modified form, has come to be known as Peano's postulates.

First-Order Number Theory [Mendelson]: Peano's Postulates

♣ ***Peano's postulates for number theory***

- (P1) 0 is a natural number.
- (P2) If x is a natural number, there is another natural number denoted by x' (the successor of x).
- (P3) $0 \neq x'$ for every natural number x.
- (P4) If $x' = y'$, then $x = y$.
- (P5) If Q is a property that may or may not hold for any given natural number, and if (I) 0 has the property Q and (II) whenever a natural number x has the property Q, then x' has the property Q, then all natural numbers have the property Q (***mathematical induction principle***).

First-Order Number Theory [Mendelson]: The Language $\mathbf{L_A}$

♣ The language of the first-order number theory: $\mathbf{L_A}$ (***the language of arithmetic***)

- $\mathbf{L_A}$ has a single predicate symbol (letter) $p^2{}_1$. We shall abbreviate $p^2{}_1\,(t, s)$ by $t = s$, and $\neg p^2{}_1\,(t, s)$ by $t \neq s$.
- $\mathbf{L_A}$ has one individual constant symbol a_1. We shall use 0 as an alternative notation for a_1.
- $\mathbf{L_A}$ has three symbols (letters) $f^1{}_1, f^2{}_1$, and $f^2{}_2$. We shall write (t') instead of $f^1{}_1\,(t)$, $(t + s)$ instead of $f^2{}_1\,(t, s)$, and $(t \cdot s)$ instead of $f^2{}_2\,(t, s)$. However, we shall write t', $t + s$, and $t \cdot s$ instead of (t'), $(t + s)$, and $(t \cdot s)$ whenever this will cause no confusion.

First-Order Number Theory [Mendelson]: The Axioms

♣ ***The axioms of the first-order number theory: Peano Arithmetic (PA)***

- (PA1) $x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3)$
- (PA2) $x_1 = x_2 \rightarrow x_1' = x_2'$
- (PA3) $0 \neq x_1'$
- (PA4) $x_1' = x_2' \rightarrow x_1 = x_2$
- (PA5) $x_1 + 0 = x_1$
- (PA6) $x_1 + x_2' = (x_1 + x_2)'$
- (PA7) $x_1 \cdot 0 = 0$
- (PA8) $x_1 \cdot (x_2)' = (x_1 \cdot x_2) + x_1$
- (PA9) $B(0) \rightarrow ((\forall x)(B(x) \rightarrow B(x')) \rightarrow (\forall x)B(x))$ for any wff $B(x)$ of $\mathbf{L_A}$. (PA9) is called ***the principle of mathematical induction***.
- Notes: Axioms (PA1)-(PA8) are particular wffs, whereas (PA9) is an axiom schema providing an infinite number of axioms. **PA** is a first-order theory with equality.

First-Order Theory NBG (Axiomatic Set Theory) [Mendelson]

♣ ***NBG axiomatic set theory*** [von Neumann (1925, 1928), Robinson (1937), Bernays (1937–1954), Gödel (1940)]

- **NBG** axiomatic set theory was originally proposed by J. von Neumann (1925, 1928) and later thoroughly revised and simplified by R. Robinson (1937), P. Bernays (1937–1954), and K. Gödel (1940).

♣ Purpose of presenting NBG at here

- As an outstanding example, to show how to use the language of CFOPC to present a very useful important formal theory.

First-Order Theory NBG [Mendelson]: Predicates, Functions, Individual Constants, and Individual Variables

♣ Predicates and functions

- **NBG** has a single predicate symbol (letter) p^2_2.
- We shall abbreviate $p^2_2(X, Y)$ by $X \in Y$, and $\neg p^2_2(X, Y)$ by $X \notin Y$.
- Intuitively, '$\in$' is to be thought of as the ***membership relation***.
- **NBG** has no predefined function symbol (letter) (some function symbols will be introduced and defined).

♣ Individual constants and individual variables

- NBG has no predefined individual constant symbols (letters) (some individual constant symbols will be introduced and defined).
- We shall use capital italic letters $X_1, X_2, X_3, \ldots$ as variable symbols instead of $x_1, x_2, x_3, \ldots$.
- As usual, we shall use meta-symbols $X, Y, Z, \ldots$ to represent variables.
- The values of the variables are to be thought of as ***classes***.

First-Order Theory NBG [Mendelson]: Classes

♣ ***Classes***

- ***Classes*** are certain collections of objects.
- Some properties determine classes, in the sense that a property P may determine a class of all those objects that possess that property.
- This "interpretation" is as imprecise as the notions of "collection" and "property."

First-Order Theory NBG [Mendelson]: Axioms

♣ ***Axioms***

- The ***axioms*** will reveal more about what we have in mind.
- They will provide us with the classes we need in mathematics and appear modest enough so that contradictions are not derivable from them.

First-Order Theory NBG [Mendelson]: Equality

♣ Definition (***equality***)

- $X=Y$ for $(\forall Z)(Z \in X \leftrightarrow Z \in Y)$
- '$\leftrightarrow$' is the logical connective equivalence.
- '$\in$' is used to represent the only predicate of the first-order theory NBG.
- '$=$' is defined as the ***equality relation***. We abbreviate $\neg(X=Y)$ by $X \neq Y$.
- This definition says "two classes are equal when and only when they have the same members."
- **NBG** is a first-order theory with equality.
- Previously, 'p^2_1' was used for the equality, as a predefined predicate.

First-Order Theory NBG [Mendelson]: Inclusion

♣ Definitions (***inclusion*** and ***proper inclusion relation***)

- $X \subseteq Y$ for $(\forall Z)(Z \in X \rightarrow Z \in Y)$ (***inclusion relation***)
- $X \subset Y$ for $(X \subseteq Y) \wedge (X \neq Y)$ (***proper inclusion relation***)
- When $X \subseteq Y$, we say that X is a ***subclass*** of Y. When $X \subset Y$, we say that X is a ***proper subclass*** of Y.

♣ Propositions

- $\vdash_{\mathrm{NBG}} X=Y \leftrightarrow (X \subseteq Y) \wedge (Y \subseteq X))$
- $\vdash_{\mathrm{NBG}} X=X$
- $\vdash_{\mathrm{NBG}} X=Y \rightarrow Y=X$
- $\vdash_{\mathrm{NBG}} X=Y \rightarrow (Y=Z \rightarrow X=Z)$

First-Order Theory NBG [Mendelson]: Sets

♣ Definitions (***set*** and ***proper class***)

- $\mathrm{M}(X)$ for $(\exists Y)(X \in Y)$ (***X is a set***)
- $\mathrm{Pr}(X)$ for $\neg \mathrm{M}(X)$ (***X is a proper class***)
- The usual derivations of the paradoxes now no longer lead to contradictions but only yield the results that various classes are proper classes, not sets.
- The sets are intended to be those safe, comfortable classes that are used by mathematicians in their daily work, whereas proper classes are thought of as monstrously large collections that, if permitted to be sets (i.e., allowed to belong to other classes), would engender contradictions.

♣ Proposition

- $\vdash_{\mathrm{NBG}} X \in Y \rightarrow \mathrm{M}(X)$

First-Order Theory NBG [Mendelson]: Classes and Concrete Individuals

- Classes and concrete individuals
  - The formal system NBG is designed to handle classes, not concrete individuals.
  - The reason for this is that mathematics has no need for objects (such as cows and molecules); all mathematical objects and relations can be formulated in terms of classes alone.
  - If non-classes are required for applications to other sciences, then the system **NBG** can be modified slightly so as to apply to both classes and non-classes alike.

First-Order Theory NBG [Mendelson]: Special Restricted Variables for Sets

- Special restricted variables for sets
  - Let us introduce lower-case letters $x_1, x_2, \ldots$ as special restricted variables for sets.
  - $(\forall x_j)B(x_j)$ stands for $(\forall X)(M(X) \rightarrow B(X))$, that is, B holds for all sets, and $(\exists x_j)B(x_j)$ stands for $(\exists X)(M(X) \wedge B(X))$, that is, B holds for some set.
  - As usual, the variable X used in these definitions should be the first one that does not occur in $B(x_j)$.
  - We shall use $x, y, z, \ldots$ to stand for arbitrary set variables.

First-Order Theory NBG [Mendelson]: The Extensionality Principle

- Proposition (the ***extensionality principle***)
  - $\vdash_{NBG} X=Y \leftrightarrow (\forall z)(z\in X \leftrightarrow z\in Y)$
  - Two classes are equal when and only when they contain the same sets as members.

First-Order Theory NBG [Mendelson]: Axioms T, P, and N

- **Axiom T**
  - $X_1=X_2 \rightarrow (X_1\in X_3 \leftrightarrow X_2\in X_3)$
  - This axiom states that equal classes belong to the same classes.
- **Axiom P** (***Pairing Axiom***)
  - $(\forall x)(\forall y)(\exists z)(\forall u)(u\in z \leftrightarrow (u=x \vee u=y))$
  - This axiom states that for any sets x and y, there is a set z that has x and y as its only members.
- **Axiom N** (***Null (Empty) Set***)
  - $(\exists x)(\forall y)(y\notin x)$
  - This axiom states that there is a set that has no members.
- The empty set as an individual constant
  - From axiom N and the extensionality principle, there is a unique set that has no members, i.e., $\vdash_{NBG} (\exists_1 x)(\forall y)(y\notin x)$.
  - We can introduce a new individual constant $\varnothing$ by definition: $(\forall y)(y\notin\varnothing)$.

First-Order Theory NBG [Mendelson]: Unordered Pair

- Proposition (***unordered pair***)
  - $\vdash_{NBG} (\forall x)(\forall y)(\exists_1 z)(\forall u)(u\in z \leftrightarrow (u=x \vee u=y))$
  - There is a unique set z, called the ***unordered pair*** of x and y, such that z has x and y as its only members.
  - We have to define a unique value for $\{X, Y\}$ for any classes X and Y, not only for sets x and y.
  - We shall let $\{X, Y\}$ be $\varnothing$ whenever X is not a set or Y is not a set.
- Propositions
  - $\vdash_{NBG} (\exists_1 Z)(\ [(\neg M(X) \vee \neg M(Y)) \wedge Z=\varnothing] \vee [M(X) \wedge M(Y) \wedge (\forall u)(u\in Z \leftrightarrow (u=X \vee u=Y))]\ )$
  - $\vdash_{NBG} (\forall x)(\forall y)(\forall u)(u\in\{x, y\} \leftrightarrow (u=x \vee u=y))$
  - $\vdash_{NBG} (\forall X)(\forall Y)M(\{X, Y\})$
  - $\vdash_{NBG} \{X, Y\} \leftrightarrow \{Y, X\}$
  - $\vdash_{NBG} (\forall x)(\forall y)(\{x\}=\{y\}\leftrightarrow x=y)$

First-Order Theory NBG [Mendelson]: Ordered Pair

- Definition (***singleton***)
  - $\{X\}$ for $\{X, X\}$
  - For a set x, $\{x\}$ is called the ***singleton*** of x. It is a set that has x as its only member.
- Definition (***ordered pair***) [Kuratowski, 1921]
  - $<X, Y>$ for $\{\{X\}, \{X, Y\}\}$
  - For sets x and y, $<x, y>$ is called the ***ordered pair*** of x and y.
- Proposition
  - $\vdash_{NBG} (\forall x)(\forall y)(\forall u)(\forall v)(<x, y>=<u, v> \rightarrow (x=u \wedge y=v))$
- Definitions
  - $<X> = X$
  - $<X_1, \ldots, X_n, X_{n+1}>=<<X_1, \ldots, X_n>, X_{n+1}>$
- Proposition (the generalization of the above proposition)
  - $\vdash_{NBG} (\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_n)(<x_1, \ldots, x_n>=<y_1, \ldots, y_n> \rightarrow (x_1=y_1 \wedge \ldots \wedge x_n=y_n))$

First-Order Theory NBG [Mendelson]: Axioms of Class Existence

♣ **Axioms of Class Existence**

- (B1) $(\exists X)(\forall u)(\forall v)(<u, v>\in X \leftrightarrow u\in v)$ (∈-relation)
- (B2) $(\forall X)(\forall Y)(\exists Z)(\forall u)(u\in Z \leftrightarrow (u\in X \wedge u\in Y))$ (intersection)
- (B3) $(\forall X)(\exists Z)(\forall u)(u\in Z \leftrightarrow u\notin X)$ (complement)
- (B4) $(\forall X)(\exists Z)(\forall u)(u\in Z \leftrightarrow (\exists v)(<u, v>\in X))$ (domain)
- (B5) $(\forall X)(\exists Z)(\forall u)(\forall v)(<u, v>\in Z \leftrightarrow u\in X)$
- (B6) $(\forall X)(\exists Z)(\forall u)(\forall v)(\forall w)(<u, v, w>\in Z \leftrightarrow <v, w, u>\in X)$
- (B7) $(\forall X)(\exists Z)(\forall u)(\forall v)(\forall w)(<u, v, w>\in Z \leftrightarrow <u, w, v>\in X)$

First-Order Theory NBG [Mendelson]: Class Operations

♣ Propositions (***intersection***, ***complement***, and ***domain***)

- From axioms (B2)-(B4) and the extensionality principle, we can obtain:
- $|-_{NBG} (\forall X)(\forall Y)(\exists_1 Z)(\forall u)(u\in Z \leftrightarrow (u\in X \wedge u\in Y))$
- $|-_{NBG} (\forall X)(\exists_1 Z)(\forall u)(u\in Z \leftrightarrow u\notin X)$
- $|-_{NBG} (\forall X)(\exists_1 Z)(\forall u)(u\in Z \leftrightarrow (\exists v)(<u, v>\in X))$

♣ Definitions ($X\cap Y$, $!X$, $D(X)$, $X\cup Y$, V, $X-Y$)

- $(\forall u)(u\in(X\cap Y) \leftrightarrow (u\in X \wedge u\in Y))$ (***intersection*** of X and Y)
- $(\forall u)(u\in !X \leftrightarrow u\notin X)$ (***complement*** of X)
- $(\forall u)(u\in Dom(X) \leftrightarrow (\forall v)(<u, v>\in X))$ (***domain*** of X)
- $X\cup Y = !(!X\cap !Y)$ (***union*** of X and Y)
- $V = !\varnothing$ (***universal class***)
- $X-Y = X\cap !Y$ (***difference*** of X and Y)
- '∩', '!', 'D', '∪', '−' are new function symbols (letters)

First-Order Theory NBG [Mendelson]: Class Existence Theorem

♣ Proposition (***Class Existence Theorem***)

- By a ***predicative formula*** we mean a formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ whose variables occur among $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ and in which only set variables are quantified (i.e., φ can be abbreviated in such a way that only set variables are quantified).
- Let $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ be a predicative formula. Then:
- $|-_{NBG} (\exists Z)(\forall x_1)\ldots(\forall x_n)(<x_1, \ldots, x_n>\in Z \leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m))$

♣ Propositions

- Let $\varphi(X, Y_1, Y_2)$ be $(\exists u)(\exists v)(X=<u, v> \wedge u\in Y_1 \wedge v\in Y_2)$. The only quantifiers in φ involve set variables. Hence, by the class existence theorem, $|-_{NBG} (\exists Z)(\forall x)(x\in Z \leftrightarrow (\exists u)(\exists v)(X=<u, v> \wedge u\in Y_1 \wedge v\in Y_2))$.
- By the extensionality principle,
- $|-_{NBG} (\exists_1 Z)(\forall x)(x\in Z \leftrightarrow (\exists u)(\exists v)(x=<u, v> \wedge u\in Y_1 \wedge v\in Y_2))$.
- There is a unique class Z whose elements are ordered pairs from Y_1 and Y_2.

First-Order Theory NBG [Mendelson]: Cartesian (Direct) Product

♣ Definition (***Cartesian (direct) product*** of Y_1 and Y_2)

- $(\forall x)(x\in(Y_1 \times Y_2) \leftrightarrow (\exists u)(\exists v)(x=<u, v> \wedge u\in Y_1 \wedge v\in Y_2))$

♣ Definitions (***Cartesian power (Cartesian exponentiation)*** and ***Relation***)

- Y^2 for $Y \times Y$
- Y^n for $Y^{n-1} \times Y$ when $n > 2$
- $Rel(X)$ for $X\subseteq Y^2$ (X is a ***relation***)

♣ Proposition

- Let $\varphi(X, Y)$ be $X\subseteq Y$. By the class existence theorem and the extensionality principle, $|-_{NBG} (\exists_1 Z)(\forall x)(x\in Z \leftrightarrow x\subseteq Y)$.
- Thus, there is a unique class Z that has as its members all subsets of Y. Z is called the power class of Y and is denoted $P(Y)$.

♣ Definition (***Power class***)

- $(\forall x)(x\in P(Y) \leftrightarrow x\subseteq Y)$

First-Order Theory NBG [Mendelson]: Sum Class

♣ Proposition

- Let $\varphi(X, Y)$ be $(\exists v)(X\in v \wedge v\in Y)$. By the class existence theorem and the extensionality principle, $|-_{NBG} (\exists_1 Z)(\forall x)(x\in Z \leftrightarrow (\exists v)(X\in v \wedge v\in Y))$.
- Thus, there is a unique class Z that contains its members of members of Y. Z is called the sum class of Y and is denoted $\bigcup Y$.

♣ Definition (***sum class***)

- $(\forall x)(x\in \bigcup Y \leftrightarrow (\exists v)(x\in v \wedge v\in Y))$

♣ Proposition

- Let $\varphi(X)$ be $(\exists u)(X=<u, u>)$. By the class existence theorem and the extensionality principle, there is a unique class Z such that $(\forall x)(x\in Z \leftrightarrow (\exists u)(x=<u, u>))$. Z is called the identity relation and is denoted I.

♣ Definition (***identity relation***)

- $(\forall x)(x\in I \leftrightarrow (\exists v)(x=<u, u>))$

First-Order Theory NBG [Mendelson]: Class of n-Tuples

♣ Corollary

- If $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ be a predicative formula, then $|-_{NBG} (\exists_1 W)(W\subseteq V^n \wedge$
  $(\forall x_1)\ldots(\forall x_n)(<x_1, \ldots, x_n>\in W \leftrightarrow \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)))$.

♣ Definition

- Given a predicative formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$, let $\{ <x_1, \ldots, x_n> \mid \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m) \}$ denotes the class of all n-tuples $<x_1, \ldots, x_n>$ that satisfy $\varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)$; that is,
  $(\forall u)(u\in\{ <x_1, \ldots, x_n> \mid \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m) \}$
  $\leftrightarrow (\exists x_1)\ldots(\exists x_n)(u=<x_1, \ldots, x_n> \wedge \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)))$.
- This definition is justified by the above corollary.
- When n=1, $|-_{NBG} (\forall u)(u\in\{ x \mid \varphi(x, Y_1, \ldots, Y_m) \} \leftrightarrow \varphi(u, Y_1, \ldots, Y_m)$.

First-Order Theory NBG [Mendelson]: Inverse Relation and Range

♣ Notation (*inverse relation*)

- Take φ to be $<x_2, x_1>\in Y$.
- Let $\sim Y$ be an abbreviation for $\{ <x_1, x_2> \mid <x_2, x_1>\in Y \}$.
- $\sim Y \subseteq V^2 \wedge (\forall x_1)(\forall x_2)(<x_1, x_2>\in Y \leftrightarrow <x_2, x_1>\in Y)$
- $\sim Y$ is called the ***inverse relation*** of Y.

♣ Notation (*range*)

- Take φ to be $(\exists v)(<v, x>\in Y)$.
- Let Ran(Y) stand for $\{ x \mid (\exists v)(<v, x>\in Y) \}$.
- $\vdash_{\text{NBG}} (\forall u)(u\in \text{Ran}(Y) \leftrightarrow (\exists v)(<v, u>\in Y))$
- $\vdash_{\text{NBG}} \text{Ran}(Y) = \text{Dom}(\sim Y)$
- Ran(Y) is called the ***range*** of Y.

First-Order Theory NBG [Mendelson]: Axioms U, W, and S

♣ **Axiom U** (***Sum Set***)

- $(\forall x)(\exists y)(\forall u)(u\in y \leftrightarrow (\exists v)(u\in v \wedge v\in x))$
- This axiom states that the sum class $\bigcup x$ of a set x is also a set, which we shall call the sum set of x, that is, $\vdash_{\text{NBG}} (\forall x)\text{M}(\bigcup x)$. The sum set $\bigcup x$ is usually referred to as the union of all the sets in the set x.

♣ **Axiom W** (***Power Set***)

- $(\forall x)(\exists y)(\forall u)(u\in y \leftrightarrow u\subseteq x)$
- This axiom states that the power class P(x) of a set x is itself a set, that is, $\vdash_{\text{NBG}} (\forall x)\text{M}(\text{P}(x))$.

♣ **Axiom S** (***Subsets***)

- $(\forall x)(\forall Y)(\exists z)(\forall u)(u\in z \leftrightarrow (u\in x \wedge u\in Y))$
- This axiom states that there is a subset of a class such that its any member is a member of that class.

First-Order Theory NBG [Mendelson]: Intersection

♣ Corollary

- $\vdash_{\text{NBG}} (\forall x)(\forall Y)\text{M}(x\cap Y)$ (The intersection of a set and a class is a set.)
- $\vdash_{\text{NBG}} (\forall x)(\forall Y)(Y\subseteq x \rightarrow \text{M}(Y))$ (A subclass of a set is a set.)
- For any predicative formula B(y), $\vdash_{\text{NBG}} (\forall x)\text{M}(\{ y \mid y\in x \wedge \text{B}(y)\})$

♣ Definition (*intersection*)

- Based on axiom S, we can show that the intersection of any non-empty class of sets is a set.
- $\bigcap X$ for $\{ y \mid (\forall x)(x\in X \rightarrow y\in x) \}$ (***intersection***)

♣ Propositions

- $\vdash_{\text{NBG}} (\forall x)(x\in X \rightarrow \bigcap X\subseteq x)$
- $\vdash_{\text{NBG}} X\neq\varnothing \rightarrow \text{M}(\bigcap X)$
- $\vdash_{\text{NBG}} \bigcap\varnothing=V$

First-Order Theory NBG [Mendelson]: Functions

♣ Definition (*function*)

- Fnc(X) for Rel(X) $\wedge (\forall x)(\forall y)(\forall z)((<x, y>\in X \wedge <x, z>\in X) \rightarrow (y=z))$ (X is a ***function***)
- $X: Y \rightarrow Z$ for Fnc(X) $\wedge$ Dom(X)=Y $\wedge$ Ran(X)=Z (X is a ***function from Y(Domain) into Z (Range)***)
- $R\leftarrow X$ for $X\cap(R\times V)$ (***restriction*** of X to the domain R)
- $\text{Fnc}_1(X)$ for Fnc(X) $\wedge$ Fnc($\sim X$) (X is a ***one-one function***)
- $X'Y = z$ if $(\forall u)(<Y, u>\in X \leftrightarrow u=z)$, or $\varnothing$ otherwise
- $X''R$ = Ran($R\leftarrow X$)
- If there is a unique z such that $<y, z>\in X$, then $z= X'y$; otherwise, $X'y=\varnothing$.
- If X is a function and y is a set in its domain, $X'y$ is the value of the function applied to y.
- If X is a function, $X''R$ is the range of X restricted to R.

First-Order Theory NBG [Mendelson]: Axioms R and I

♣ **Axiom R** (***Replacement***)

- $\text{Fnc}(Y) \rightarrow (\forall x)(\exists y)(\forall u)(u\in y \leftrightarrow (\exists v)(<v, u>\in Y \wedge v\in x))$
- Axiom R asserts that, if Y is a function and x is a set, then the class of second components of ordered pairs in Y whose first components are in x is a set (or, equivalently, Ran(x←Y) is a set).

♣ **Axiom I** (***Infinity***)

- $(\exists x)(\varnothing\in x \wedge (\forall u)(u\in x \rightarrow u\cup\{u\}\in x))$
- Axiom I states that there is a set x that contains $\varnothing$ and such that, whenever a set u belongs to x, then $u\cup\{u\}$ also belongs to x.
- Hence, for such a set x, $\{\varnothing\}\in x$, $\{\varnothing, \{\varnothing\}\}\in x$, $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\in x$, and so on. If we let 1 stand for $\{\varnothing\}$, 2 for $\{\varnothing, 1\}$, 3 for $\{\varnothing, 1, 2\}$, …, n for $\{\varnothing, 1, 2, \ldots, n-1\}$, etc., then, for all ordinary integers $n\geq 0$, $n\in x$, and $\varnothing\neq 1, \varnothing\neq 2, 1\neq 2, \varnothing\neq 3, 1\neq 3, 2\neq 3, \ldots$.

First-Order Theory NBG [Mendelson]: Axioms of NBG

♣ **Axioms of NBG**

- **NBG** has only a finite number of axioms -- namely,
  axiom T,
  axiom P (pairing),
  axiom N (null set),
  axiom U (sum set),
  axiom W (power set),
  axiom S (subsets),
  axiom R (replacement),
  axiom I (infinity), and
  the seven class existence axioms (B1)–(B7).

♣ Note

- Axiom S is provable from the other axioms; it has been included here because it is of interest in the study of certain weaker sub-theories of **NBG**.

First-Order Theory NBG [Mendelson]: Summary about Paradoxes

♣ Untenableness of Russell's paradox

- The usual argument for Russell's paradox does not hold in **NBG**.
- By the class existence theorem, there is a class $Y = \{x \mid x \notin x\}$.
- Then $(\forall x)(x \in Y \leftrightarrow x \notin x)$.
- In unabbreviated notation this becomes $(\forall X)(M(X) \rightarrow (X \in Y \leftrightarrow X \notin X))$.
- Assume $M(Y)$. Then $Y \in Y \leftrightarrow Y \notin Y$, which, by the tautology $(A \leftrightarrow \neg A) \rightarrow (A \wedge \neg A)$, yields $Y \in Y \wedge Y \notin Y$.
- Hence, by the derived rule of proof by contradiction, we obtain $\vdash \neg M(Y)$.
- Thus, in **NBG**, the argument for Russell's paradox merely shows that Russell's class Y is a proper class, not a set.

♣ Untenableness of the paradoxes of Cantor and Burali-Forti

- The **NBG** will avoid the paradoxes of Cantor and Burali-Forti in a similar way.

An Introduction to Classical Predicate Calculus

♣ The Limitations of Propositional Logic **CPC**

♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)

♣ Substitutions

♣ Semantics (Model Theory) of **CFOPC**

♣ Semantic (Model-theoretical, Logical) Consequence Relation

♣ Hilbert Style Formal Logic Systems for **CFOPC**

♣ Gentzen's Natural Deduction System for **CFOPC**

♣ Gentzen's Sequent Calculus System for **CFOPC**

♣ Semantic Tableau Systems for **CFOPC**

♣ Resolution Systems for **CFOPC**

♣ Classical Second-Order Predicate Calculus (**CSOPC**)

Free and Bound Variables: A Motivation Example

♣ The Example

- $\Sigma^{n}_{k=1}\, x^k + k = (x^n + n) + (x^{n-1} + n-1) + \ldots + (x + 1)$

♣ Consider the following question about the above example

- $\Sigma^{n}_{k=1}\, x^k + k = \Sigma^{n}_{j=1}\, x^j + j$?
- $\Sigma^{n}_{k=1}\, x^k + k = \Sigma^{j}_{k=1}\, x^k + k$?
- $\Sigma^{n}_{k=1}\, x^k + k = \Sigma^{n}_{n=1}\, x^n + n$?
- $\Sigma^{n}_{k=1}\, x^k + k = \Sigma^{k}_{k=1}\, x^k + k$?
- $\Sigma^{n}_{k=1}\, x^k + k = \Sigma^{n}_{k=1}\, y^k + k$?

♣ Notes

- In the above example, n and x are free (occurrence) variables, and k is a bound (occurrence) variable.
- A bound (occurrence) variable can be replaced by other variables, expect free (occurrence) variables in the formula, without meaning change.
- The value of a formula is dependent on values of its free (occurrence) variables.

Free and Bound Occurrences of Individual Variables

♣ ***Free occurrences*** of individual variables

- (1) The ***free variable occurrences*** in an atomic formula are all the variable occurrences in the formula.
 (2) The free variable occurrences in $(\neg A)$ are the free variable occurrences in A.
 (3) The free variable occurrences in $(A*B)$ (* is a binary connective) are the free variable occurrences in A together with the free variable occurrences in B,
 (4) The free variable occurrences in $((\forall x)A)$ and $((\exists x)A)$ are the free variable occurrences in A, except for occurrences of x.

♣ ***Bound occurrences*** of individual variables

- A variable occurrence is ***bound*** IFF it is not free occurrence.

Free and Bound Occurrences of Individual Variables

♣ ***Bound occurrences*** of individual variables

- An occurrence of a variable x is said to be ***bound*** in a formula B if either it is the occurrence of x in a quantifier "$(\forall x)$" or "$(\exists x)$" in B or it lies within the scope of a quantifier "$(\forall x)(\ldots)$" or "$(\exists x)(\ldots)$" in B.

♣ ***Free occurrences*** of individual variables

- A variable occurrence is ***free*** in a formula IFF it is not bound occurrence.

♣ ***Free / Bound variables***

- A variable is said to be ***free*** (***bound***) in a formula B if it has a free (bound) occurrence in B.
- Thus, a variable may be both free and bound in the same formula.

♣ ***Closed formulas (Sentences)***

- A formula with no free (occurrence) variables (called a ***closed formula*** or ***sentence***) represents a proposition that must be true or false.

Free and Bound Occurrences of Individual Variables: Examples

♣ Example 1: $p^2{}_1(x_1, x_2)$

- In Example 1, the single occurrence of x_1 (or x_2) is free.

♣ Example 2: $p^2{}_1(x_1, x_2) \rightarrow (\forall x_1)p^1{}_1(x_1)$

- In Example 2, the first occurrence of x_1 in $p^2{}_1(x_1, x_2)$ is free, but the second and third occurrences of x_1 are bound.

♣ Example 3: $(\forall x_1)p^2{}_1(x_1, x_2) \rightarrow (\forall x_1)p^1{}_1(x_1)$

- In Example 3, all occurrences of x_1 are bound.

♣ Example 4: $(\exists x_1)p^2{}_1(x_1, x_2)$

- In Example 4, both occurrences of x_1 are bound.

♣ Notes

- In all four examples, every occurrence of x_2 is free.
- A variable may have both free and bound occurrences in the same formula; a variable may be bound in a formula but free in a subformula of the formula.

Terms being Free for Individual Variables [Mendelson]

♣ ***Terms being free for variables***

- If B is a formula and t is a term, then t is said to be free for x_i in B if no free occurrence of x_i in B lies within the scope of any quantifier $(\forall x_j)$ or $(\exists x_j)$, where x_j is a variable in t.
- Note: This concept of t being free for x_i in a formula $B(x_i)$ will have certain technical applications later on. It means that, if t is substituted for all free occurrences (if any) of x_i in $B(x_i)$, no occurrence of a variable in t becomes a bound occurrence in $B(t)$.

♣ Examples

- The term x_2 is free for x_1 in $p^1{}_1(x_1)$, but x_2 is not free for x_1 in $(\forall x_2)p^1{}_1(x_1)$.
- The term $f^2{}_1(x_1,x_3)$ is free for x_1 in $(\forall x_2)p^2{}_1(x_1,x_2) \to p^1{}_1(x_1)$ but is not free for x_1 in $(\exists x_3)(\forall x_2)p^2{}_1(x_1,x_2) \to p^1{}_1(x_1)$.

Terms being Free for Individual Variables [Mendelson]

♣ ***Terms being free for variables***

- If B is a formula and t is a term, then t is said to be free for x_i in B if no free occurrence of x_i in B lies within the scope of any quantifier $(\forall x_j)$ or $(\exists x_j)$, where x_j is a variable in t.

♣ Facts

- A term that contains no variables is free for any variable in any formulas.
- A term t is free for any variable in formula B if none of the variables of t is bound in B.
- x_i is free for x_i in any formula.
- Any term is free for x_i in formula B if B contains no free occurrences of x_i.

Substitutions of Variables

♣ ***Substitution*** of variable

- A formula may contain some free variables that can be replaced by other terms.
- A ***variable substitution*** is a mapping σ: **V**→**Ter** from the set of individual variables **V** to the set of terms **Ter.**
- We denote $\sigma[x]$ by $x\sigma$, to represent the result of applying the mapping σ to x.

♣ ***Substitution*** of variable on all terms

- Let σ: **V**→**Ter** be a variable substitution. It can be extended to all terms:
(1) $c\sigma = c$ for any $c \in$ **Con**, $\top\sigma = \top$, $\bot\sigma = \bot$;
(2) $[f(t_1, ..., t_n)]\sigma = f(t_1\sigma, ..., t_n\sigma)$ for any n-ary $f \in$ **Fun**.

♣ Examples

- Let a, b, c, x, y, z be variables and f, g, h, i, j, k be functions. Suppose $x\sigma = f(x, y)$, $y\sigma = h(a)$, and $z\sigma = g(c, h(x))$. Then $j(k(x), y)\sigma = j(k(f(x, y)), h(a))$.

Substitutions of Variables: Composition and Support

♣ ***Composition*** of substitutions

- Let σ and τ be substitutions. By the ***composition*** of σ and τ, we mean that substitution, which we denote by $\sigma\bullet\tau$, such that for each variable $x \in$ **V**, $x(\sigma\bullet\tau) = (x\sigma)\tau$.
- Theorem: For any term $t \in$ **Ter** and any two substitutions σ and τ, $t(\sigma\bullet\tau) = (t\sigma)\tau$.
- Note: The above theorem does not carry over to formulas.
- Theorem: Composition of substitutions is associative, i.e., for any substitutions σ_1, σ_2, and σ_3, $(\sigma_1\bullet\sigma_2)\bullet\sigma_3 = \sigma_1\bullet(\sigma_2\bullet\sigma_3)$.

♣ ***Support*** of substitution

- The ***support*** of a substitution σ is the set of variables x for which $x\sigma \neq x$. A substitution has ***finite support*** if its support set is finite.
- Theorem: The composition of two substitutions having finite support is a substitution having finite support.

Substitutions of Variables: Notation of Substitution and Composition

♣ Notation of substitution

- Suppose σ is a substitution having finite support; say $\{x_1, x_2, ..., x_n\}$ is the support, and for each $i = 1, ..., n$, $x_i\sigma = t_i$.
- Our notation for σ is: $[x_1/t_1, x_2/t_2, ..., x_n/t_n]$.
- In particular, our notation for the identity substitution is [ ].

♣ Notation of substitution composition

- Let $\sigma_1 = [x_1/t_1, ..., x_n/t_n]$ and $\sigma_2 = [y_1/u_1, ..., y_k/u_k]$ are two substitutions having finite support. Then $\sigma_1\bullet\sigma_2$ has notation:
$[x_1/(t_1\sigma_2), ..., x_n/(t_n\sigma_2), z_1/(z_1\sigma_2), ..., z_m/(z_m\sigma_2)]$
where $z_1, ..., z_m$ are those variables in the list $y_1, ..., y_k$ that are not also in the list $x_1, ..., x_n$.

♣ Examples

- Let Suppose $\sigma_1 = [x/f(x, y), y/h(a), z/g(c, h(x))]$ and $\sigma_2 = [x/b, y/g(a, x), w/z]$. Then $\sigma_1\bullet\sigma_2 = [x/f(b, g(a, x)), y/h(a), z/g(c, h(b)), w/z]$.

Substitutions of Variables on Terms and Formulas

♣ ***Substitution*** of variable on terms and formulas

- Let σ: **V**→**Ter** be a variable substitution. It can be extended to all terms and formulas as follows:
(1) $c\sigma = c$ for any $c \in$ **Con**, $\top\sigma = \top$, $\bot\sigma = \bot$;
(2) $x\sigma = x\sigma$ for any $x \in$ **V**;
(3) $[f(t_1, ..., t_n)]\sigma = f(t_1\sigma, ..., t_n\sigma)$ for any n-ary $f \in$ **Fun**;
(4) $[p(t_1, ..., t_n)]\sigma = p(t_1\sigma, ..., t_n\sigma)$ for any n-ary $p \in$ **Pre**;
(5) $(\neg A)\sigma = (\neg(A\sigma))$ for any $A \in$ **WFF**;
(6) $(A*B)\sigma = ((A\sigma)*(B\sigma))$ for a binary connective $*$ and any $A, B \in$ **WFF**;
(7) $((\forall x)A)\sigma = ((\forall x)(A\sigma_x))$ and $((\exists x)A)\sigma = ((\exists x)(A\sigma_x))$ for any $A \in$ **WFF**, where by σ_x we mean the substitution that is like σ except that it does not change x, i.e., $y\sigma_x = y\sigma$ if $y \neq x$ and $y\sigma_x = x$ if $y = x$.
- Note: The result of applying a substitution to a term always producers another term.

Substitutions of Variables on Terms and Formulas: Examples

♣ An example

- Let $\sigma = [x/a, y/b]$.
  $((\forall x)R(x, y) \supset (\exists y)R(x, y))\sigma = ((\forall x)R(x, y))\sigma \supset ((\exists y)R(x, y))\sigma$
  $= (\forall x)(R(x, y))\sigma_x \supset (\exists y)(R(x, y))\sigma_y$
  $= (\forall x)(R(x, b)) \supset (\exists y)(R(a, y))$

♣ An example

- Let $\sigma = [x/y]$ and $\tau = [y/c]$. Then $\sigma\bullet\tau = [x/c, y/c]$.
  If $A = ((\forall y)R(x, y))$, then $A\sigma = ((\forall y)R(y, y))$, so $(A\sigma)\tau = ((\forall y)R(y, y))$.
  But $A(\sigma\bullet\tau) = ((\forall y)R(c, y))$, which is different.
- The example shows that the fact about substitution in terms, for any term t, $(t\sigma)\tau = t(\sigma\bullet\tau)$, does not carry over to formulas.
- What is needed is some restriction that will ensure composition of substitutions behaves well.

Free Substitutions

♣ ***Free substitution***

- A substitution being ***free for a formula*** is characterized as follows:

(1) If $A \in$ **WFF** is an atomic formula, then σ is free for A.

(2) For any $A \in$ **WFF**, σ is free for $\neg A$, if σ is free for A.

(3) For any $A, B \in$ **WFF**, σ is free for $(A{*}B)$, if σ is free for A and σ is free for B, where * is a binary connective.

(4) For any $A \in$ **WFF**, σ is free for $((\forall x)A)$ and $((\exists x)A)$ provided: σ_x is free for A, and if y is a free variable of A other than x, $y\sigma$ does not contain x.

♣ Theorem (***free substitution***)

- Suppose the substitution σ is free for the formula A, and the substitution τ is free for $A\sigma$. Then $(A\sigma)\tau = A(\sigma\bullet\tau)$.

An Introduction to Classical Predicate Calculus

Semantics (Model Theory) of CFOPC: The Fundamental Question

♣ **The fundamental question**

- Why the semantics (model theory) of **CFOPC** is indispensable?

♣ **The answer to the question**

- Well-formed formulas of **CFOPC** have meaning only when an interpretation is given for the symbols of **CFOPC**.
- The semantics (model theory) of **CFOPC** gives a ***truth-value (truth-functional) interpretation*** for the symbols/well-formed formulas of **CFOPC**.
- The semantics (model theory) of **CFOPC** provides a (philosophical and mathematical) fundamental basis for studying and using **CFOPC**.

Semantics (Model Theory) of CFOPC: Important Notes

♣ Important notes

- The semantics (model theory) of **CFOPC** is the most intrinsic foundation of **CFOPC**.
- Without a sound semantics, **CFOPC** is meaningless.
- The semantics (model theory) of **CFOPC** is only relatively correct/sound/satisfactory, i.e., it is correct/sound/satisfactory only because it is based on those fundamental assumptions/principles underlying **CML** (**C**lassical **M**athematical **L**ogic).

Fundamental Assumptions/Principles Underlying CML

♣ **The classical abstraction**

- The only properties of a proposition that matter to logic are its form and its truth-value.

♣ **The Fregean assumption / the principle of extensionality**

- The truth-value of a (composite) proposition depends only on its (composition) form and the truth-values of its constituents, not on their meaning.

♣ **The principle of bivalence**

- There are exactly two truth-values, “***TRUE***” and “***FALSE***”. Every proposition has one or other, but not both, of these truth-values.

♣ **The classical account of validity (CAV)**

- An argument is valid if and only if it is impossible for all its premises to be true while its conclusion is false.

Semantics (Model Theory) of CFOPC: Models (Structures)

♣ ***Models*** (***Structures***) for first-order languages

- Let L(**Con**, **Fun**, **Pre**) be a first-order language determined by **Con**, **Fun**, and **Pre**. A ***model*** (***structure***) for L(**Con**, **Fun**, **Pre**) is an ordered pair $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ where $\boldsymbol{D}$ is a non-empty set of entities, called the ***domain*** or ***universe*** of $\boldsymbol{M}$ and $\boldsymbol{I}$ is a mapping, called an ***interpretation*** of $\boldsymbol{M}$ such that:

  for every constant symbol $c \in$ **Con**,
  $c^{\boldsymbol{I}}$ is an element (entity) of $\boldsymbol{D}$, $c^{\boldsymbol{I}} \in \boldsymbol{D}$;
  for every n-ary function symbol $f \in$ **Fun**,
  $f^{\boldsymbol{I}}$ is an n-ary function on $\boldsymbol{D}$, $f^{\boldsymbol{I}} : \boldsymbol{D}^n \rightarrow \boldsymbol{D}$;
  for every n-ary predicate symbol $p \in$ **Pre**,
  $p^{\boldsymbol{I}}$ is an n-ary relation on $\boldsymbol{D}$, $p^{\boldsymbol{I}} \subseteq \boldsymbol{D}^n$.

- An ***assignment*** ***Ass*** in a model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ is a mapping from the set of individual variables **V** to the domain $\boldsymbol{D}$, ***Ass***: $\mathbf{V} \rightarrow \boldsymbol{D}$. The image of the individual variable x under the assignment Ass is denoted by x^{Ass}.

Semantics (Model Theory) of CFOPC: Models (Structures)

♣ Notes

- A model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ for the first-order language L(**Con**, **Fun**, **Pre**) together with an assignment ***Ass*** in the model gives an interpretation for the language.
- The domain $\boldsymbol{D}$ defines the application area of the language L, and the interpretation mapping $\boldsymbol{I}$ relates various symbols of L to entities and relationships among them in the application area $\boldsymbol{D}$.
- The interpretation mapping $\boldsymbol{I}$ relates each individual constant symbol c to an entity $c^{\boldsymbol{I}}$ in $\boldsymbol{D}$, each n-ary function symbol f to an n-ary function $f^{\boldsymbol{I}}$ in $\boldsymbol{D}$, and each n-ary predicate symbol p to an n-ary relation $p^{\boldsymbol{I}}$ in $\boldsymbol{D}$.
- The assignment mapping ***Ass*** relates each individual variable x to an entity x^{ASS} in $\boldsymbol{D}$.
- As a result, once a model (structure) $(\boldsymbol{D}, \boldsymbol{I})$ for the language L(**Con**, **Fun**, **Pre**) together with an assignment ***Ass*** is defined (given), various symbols of L have certain meaning in the application area $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Interpretations for Terms

♣ ***Interpretations*** for terms

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model. For every term $t \in$ **Ter**, its interpretation (a ***value*** in $\boldsymbol{D}$) is defined as follows:

  (1) $c^{\boldsymbol{IA}} = c^{\boldsymbol{I}}$ for every $c \in$ **Con**, if $t = c$;

  (2) $x^{\boldsymbol{IA}} = x^{\boldsymbol{A}}$ for every $x \in$ **V**, if $t = x$;

  (3) $[f(t_1, ..., t_n)]^{\boldsymbol{IA}} = f^{\boldsymbol{I}}(t_1^{\boldsymbol{IA}}, ..., t_n^{\boldsymbol{IA}})$ for every $f \in$ **Fun**.

- Note: The value of a closed term does not depend on the assignment $\boldsymbol{A}$.

♣ ***Variant*** of assignment

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $x \in$ **V** be an individual variable. The assignment $\boldsymbol{B}$ in the model $\boldsymbol{M}$ is an ***x-variant*** of the assignment $\boldsymbol{A}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ assign the same values to every individual variable in **V** except possibly x.

Semantics (Model Theory) of CFOPC: Interpretations for Terms

♣ Notes

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model.
- The interpretation mapping $\boldsymbol{I}$ relates each individual constant symbol c to an entity $c^{\boldsymbol{I}}$ in $\boldsymbol{D}$; each n-ary function symbol f to an n-ary function $f^{\boldsymbol{I}}$ in $\boldsymbol{D}$; each n-ary predicate symbol p to an n-ary relation $p^{\boldsymbol{I}}$ in $\boldsymbol{D}$.
- The assignment $\boldsymbol{A}$ relates each individual variable x to an entity $x^{\boldsymbol{A}}$ in $\boldsymbol{D}$.
- For every term $t \in$ **Ter** and every n-ary function symbol $f \in$ **Fun**, if $t = c$, t is interpreted as $c^{\boldsymbol{I}}$, an entity in $\boldsymbol{D}$; if $t = x$, t is interpreted as $x^{\boldsymbol{A}}$, also an entity in $\boldsymbol{D}$; and for n terms $t_1, ..., t_n \in$ **Ter** and an n-ary function $f^{\boldsymbol{I}}$ in $\boldsymbol{D}$, $f(t_1, ..., t_n)$ is interpreted as $f^{\boldsymbol{I}}(t_1^{\boldsymbol{IA}}, ..., t_n^{\boldsymbol{IA}})$, its value is an entity in $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Truth-Value of Formula

♣ ***Truth-value*** of a formula in a model

- Let $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ be a model of the first-order language L(**Con**, **Fun**, **Pre**), and let $\boldsymbol{A}$ be an assignment in the model. For any $R \in$ **WFF**, its ***truth-value*** $v_f^{\boldsymbol{IA}}(R)$ under $\boldsymbol{A}$ in $\boldsymbol{M}$ is defined by a ***truth valuation*** function $v_f^{\boldsymbol{IA}}$: **WFF**→{**T**, **F**} as follows:

  (1) for every atomic formula $p(t_1, ..., t_n) \in$ **WFF**,
  $v_f^{\boldsymbol{IA}}(p(t_1, ..., t_n))$ = **T** if $(t_1^{\boldsymbol{IA}}, ..., t_n^{\boldsymbol{IA}}) \in p^{\boldsymbol{I}}$, and
  $v_f^{\boldsymbol{IA}}(p(t_1, ..., t_n))$ = **F** otherwise;
  (2) for any $(\neg R)$, $(R*S) \in$ **WFF**, where $*$ is a binary connective,
  $v_f^{\boldsymbol{IA}}(\neg R)$, $v_f^{\boldsymbol{IA}}(R*S)$ are the same as the definition of v_f of **CPC**;
  (3) for any $((\forall x)R)$, $v_f^{\boldsymbol{IA}}(((\forall x)R))$ = **T** if $v_f^{\boldsymbol{IB}}(R)$ = **T** for every assignment $\boldsymbol{B}$ in M that is an x-variant of $\boldsymbol{A}$, and $v_f^{\boldsymbol{IA}}(((\forall x)R))$ = **F** otherwise;
  (4) for any $((\exists x)R)$, $v_f^{\boldsymbol{IA}}(((\exists x)R))$ = **T** if $v_f^{\boldsymbol{IB}}(R)$ = **T** for some assignment $\boldsymbol{B}$ in M that is an x-variant of $\boldsymbol{A}$, and $v_f^{\boldsymbol{IA}}(((\forall x)R))$ = **F** otherwise.

Semantics (Model Theory) of CFOPC: Truth-Value of Formula

♣ Notes

- We use **T** and **F** to represent "***TRUE***" and "***FALSE***" respectively; they belong to our meta-language but not the object language of **CFOPC**.
- The truth-value of a closed formula (sentence) does not depend on the assignment $\boldsymbol{A}$.
- Recall: A formula with no free (occurrence) variables (called a closed formula or sentence) represents a proposition that must be true or false.
- Any atomic formula $p(t_1, ..., t_n)$ is valuated under $\boldsymbol{A}$ in $\boldsymbol{M}$ as **T** if and only if it is interpreted as a real relation instance of n-ary relation $p^{\boldsymbol{I}}$ in $\boldsymbol{D}$.

Semantics (Model Theory) of CFOPC: Satisfiability of Formula

♣ ***Satisfiability*** of a formula in a model

For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**,

- R is ***satisfiable*** in $\boldsymbol{M}$ or R ***may be true*** in $\boldsymbol{M}$ IFF there is some assignment $\boldsymbol{A}$ (called a ***satisfying assignment***) such that under A, $v_f^{IA}(R) = \mathbf{T}$;
- $\boldsymbol{M}$ ***satisfies*** R or R is ***true*** in $\boldsymbol{M}$, written as $|\!=_M R$, IFF $v_f^{IA}(R) = \mathbf{T}$ for any assignment $\boldsymbol{A}$;
- $\boldsymbol{M}$ ***does not satisfy*** R or R ***may be false*** in $\boldsymbol{M}$ IFF there is some assignment $\boldsymbol{A}$ such that under $\boldsymbol{A}$, $v_f^{IA}(R) = \mathbf{F}$;
- R is ***unsatisfiable*** in $\boldsymbol{M}$ or R is ***false*** in $\boldsymbol{M}$, written as $|\!\neq_M R$, IFF $v_f^{IA}(R) = \mathbf{F}$ for any assignment $\boldsymbol{A}$.
- Note: A formula with free variables may be satisfied (i.e., true) for some values in the domain and not satisfied (i.e., false) for the others.

Semantics (Model Theory) of CFOPC: Logical Validity of Formula

♣ ***Logical validity*** of a formula

- For the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**, R is ***logically valid***, written as $|\!=_{\mathbf{CFOPC}} R$, IFF $|\!=_M R$ in any model $\boldsymbol{M}$ for the language (Ex: $R = (A \lor \neg A)$).

♣ ***Unsatisfiability*** of a formula

- For the first-order language L(**Con**, **Fun**, **Pre**) and any $R \in$ **WFF**, R is ***unsatisfiable***, written as $|\!\neq_{\mathbf{CFOPC}} R$, IFF $|\!\neq_M R$ in any model $\boldsymbol{M}$ for the language (Ex: $R = (A \land \neg A)$).
- For any $R \in$ **WFF**, R is logically valid IFF $\neg R$ is unsatisfiable, and R is satisfiable IFF $\neg R$ is not logically valid.

♣ ***The undecidability of CFOPC*** [A. Church, 1936, A. M. Turing, 1936]

- Theorem: The validity problem for **CFOPC**, i.e., whether a formula of **CFOPC** is valid or not, is undecidable.
- The undecidability of **CFOPC** is one of the fundamental results for logic as well as for computer science.

Semantics (Model Theory) of CFOPC: Tautologies, Contradictions, and Contingencies

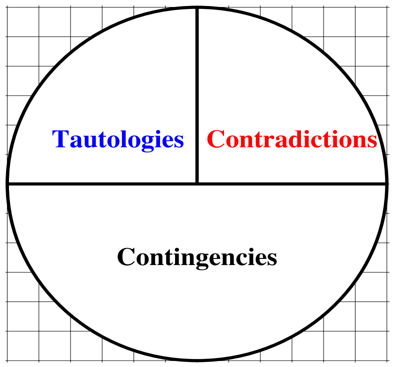
♣ ***Tautologies***, ***contradictions***, and ***contingencies***

- A formula $A \in$ **WFF** is a ***tautology*** of **CFOPC**, written as $|\!=_{\mathbf{CFOPC}} A$, IFF $|\!=_M A$ for any model M of **CFOPC** (i.e., A is logically valid);
  a formula $A \in$ **WFF** is a ***contradiction*** of **CFOPC**, written as $|\!\neq_{\mathbf{CFOPC}} A$, IFF $|\!\neq_M A$ for any model M of **CFOPC** (i.e., A is unsatisfiable);
  a formula is a ***contingency*** IFF it is neither a tautology nor a contradiction.
- A formula must be any one of tautology, contradiction, and contingency.
- The set of all tautologies of **CFOPC** is denoted by **Th(CFOPC)**.

♣ Relationship between tautologies and contradictions

- Theorem: For any $A \in$ **WFF**, A is a tautology IFF $(\neg A)$ is a contradiction, and A is a contradiction IFF $(\neg A)$ is a tautology.
- There is a bijection between tautologies and contradictions of **CFOPC**.

Semantics (Model Theory) of CFOPC: Tautologies, Contradictions, and Contingencies



Semantics (Model Theory) of CFOPC: Models of Formulas

♣ Satisfiability of formulas

- For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $\Gamma \subseteq$ **WFF**, Γ is ***satisfiable*** in M if there is some assignment $\boldsymbol{A}$ (called a ***satisfying assignment***) such that under $\boldsymbol{A}$, $v_f^{IA}(R) = \mathbf{T}$ for all $R \in \Gamma$.
- Theorem (***Compactness***): Let Γ be a set of sentences. If every finite subset of Γ is satisfiable in model $\boldsymbol{M}$, so is Γ.
- Note: Γ may be an infinite set.

♣ Models of formulas

- For any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ of the first-order language L(**Con**, **Fun**, **Pre**) and any $\Gamma \subseteq$ **WFF**, $\boldsymbol{M}$ is called a ***model*** of Γ IFF $|\!=_M R$ (i.e., $v_f^{IA}(R) = \mathbf{T}$ for any assignment $\boldsymbol{A}$) for any $R \in \Gamma$.
- The set of all models of Γ is denoted by $M(\Gamma)$.

Semantics (Model Theory) of CFOPC: Models of Formulas

♣ ***Consistency*** of formulas

- For any $\Gamma \subseteq$ **WFF**, Γ is ***semantically (model-theoretically, logically) consistent*** IFF it has at least one model; Γ is ***semantically (model-theoretically, logically) inconsistent*** IFF it has no model.
- Note: Here, consistency says "has at least one model", and inconsistency says "has no model".

Some Tautologies of CFOPC

- $\models_{\mathbf{CFOPC}} B(t) \rightarrow (\exists x)B(x)$, if t is free for x in $B(x)$
- $\models_{\mathbf{CFOPC}} ((\forall x)B) \rightarrow (\exists x)B$
- $\models_{\mathbf{CFOPC}} ((\forall x)(\forall y)B) \rightarrow (\forall y)(\forall x)B$
- $\models_{\mathbf{CFOPC}} ((\forall x)B) \leftrightarrow \neg(\exists x)\neg B$
- $\models_{\mathbf{CFOPC}} ((\forall x)(B\rightarrow C)) \rightarrow (((\forall x)B)\rightarrow(\forall x)C)$
- $\models_{\mathbf{CFOPC}} (((\forall x)B)\wedge(\forall x)C) \leftrightarrow (\forall x)(B\wedge C)$
- $\models_{\mathbf{CFOPC}} (((\forall x)B)\vee(\forall x)C) \rightarrow (\forall x)(B\vee C)$
- $\models_{\mathbf{CFOPC}} ((\exists x)(\exists y)B) \leftrightarrow (\exists y)(\exists x)B$
- $\models_{\mathbf{CFOPC}} ((\exists x)(\forall y)B) \rightarrow (\forall y)(\exists x)B$

Uniform Notation of First-order Formulas

♣ Uniform notation of first-order formulas [R. M. Smullyan, 1968]

- Classify all quantified formulas and their negations into two categories, i.e., ***γ-formulas*** which act universally, and ***δ-formulas***, which act existentially.
- For each variety and for each term t, an instance is defined.

♣ Proposition

- Let S be a set of sentences (closed formulas), and γ and δ be sentences. If $S \cup \{\gamma\}$ is satisfiable, so is $S \cup \{\gamma, \gamma(t)\}$ for any closed term t. If $S \cup \{\delta\}$ is satisfiable, so is $S \cup \{\delta, \delta(p)\}$ for any constant symbol p that is new to S and δ.

Uniform Notation of First-order Formulas

♣ γ-formulas and δ-formulas and their instances

Universal		Existential	
γ	$\gamma(t)$	δ	$\delta(t)$
$(\forall x\Phi)$	$\Phi[x/t]$	$(\exists x\Phi)$	$\Phi[x/t]$
$\neg(\exists x\Phi)$	$\neg\Phi[x/t]$	$\neg(\forall x\Phi)$	$\neg\Phi[x/t]$

An Introduction to Classical Predicate Calculus

Semantic (Model-theoretical) Logical Consequence Relation

♣ Semantic (Model-theoretical, Logical) consequence relation

- For any $\Gamma \subseteq \mathbf{WFF}$ and any $A \in \mathbf{WFF}$,
  Γ ***semantically (model-theoretically, logically) entails*** A, or
  A ***semantically (model-theoretically, logically) follows from*** Γ, or
  A is a ***semantic (model-theoretical, logical) consequence*** of Γ,
  written as $\Gamma \models_{\mathbf{CFOPC}} A$, IFF $\models_{M} A$ for any model $\boldsymbol{M}$ of Γ.

♣ All semantic (model-theoretical, logical) consequences of premises

- The set of all semantic (model-theoretical, logical) consequences of Γ is denoted by $C_{sem}(\Gamma)$.
- $\models_{\mathbf{CFOPC}} A =_{\mathrm{df}} \phi \models_{\mathbf{CFOPC}} A$ and it means that $\models_{M} A$ for any model $\boldsymbol{M}$, i.e., A is a tautology.

♣ Note

- The semantic (model-theoretical, logical) consequence relation of **CFOPC** is a semantic (model-theoretical) formalization of the notion that one proposition follows from another or others.

Semantic (Model-theoretical, Logical) Equivalence Relation

♣ Semantic (Model-theoretical, Logical) equivalence relation

- For any $A, B \in \mathbf{WFF}$, A is ***semantically (model-theoretically, logically) equivalent*** to B in **CFOPC** IFF both $\{A\} \models_{\mathbf{CFOPC}} B$ and $\{B\} \models_{\mathbf{CFOPC}} A$.
- Theorem: A is semantically (model-theoretically, logically) equivalent to B IFF $(A\leftrightarrow B)$ is a tautology.

♣ Properties of semantic (model-theoretical, logical) consequence relation

- The same as those of **CPC**.

Semantic Deduction Theorems

♣ Semantic deduction theorems

- ***Semantic (model-theoretical, logical) deduction theorem for CFOPC***: For any $A, B \in$ **WFF** and any $\Gamma \subseteq$ **WFF**,
  $\Gamma \cup \{A\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} (A \rightarrow B)$.
- ***Semantic (model-theoretical, logical) deduction theorem for CFOPC for finite consequences***: For any $A_1, \ldots, A_{n-1}, A_n, B \in$ **WFF** and any $\Gamma \subseteq$ **WFF**,
  $\Gamma \cup \{A_1, \ldots, A_{n-1}, A_n\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} (A_1 \rightarrow (\ldots(A_{n-1} \rightarrow (A_n \rightarrow B))\ldots))$;
  $\Gamma \cup \{A_1, \ldots, A_{n-1}, A_n\} \models_{\mathbf{CFOPC}} B$ IFF $\Gamma \models_{\mathbf{CFOPC}} ((A_1 \wedge (\ldots(A_{n-1} \wedge A_n)\ldots)) \rightarrow B)$.

Semantic Deduction Theorems

♣ Notes

- As a special case of the above deduction theorems, $\{A\} \models_{\mathbf{CFOPC}} B$ IFF $\models_{\mathbf{CFOPC}} (A \rightarrow B)$, i.e., A semantically (model-theoretically, logically) entails B IFF $(A \rightarrow B)$ is a tautology.
- In the framework of **CFOPC**, the semantic (model-theoretical, logical) consequence relation, which is a representation of the notion of entailment in the sense of meta-logic, is “equivalent” to the notion of material implication (denoted by ‘→’ in **CFOPC**).
- However, in semantics, the notion of material implication is NOT an accurate representation of the notion of entailment.

An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen’s Natural Deduction System for **CFOPC**
- ♣ Gentzen’s Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

L: A Hilbert Style Formal System for CFOPC

♣ Axiom schemata of **L** ($A, B, C \in$ **WFF**)

- AS1 $(A \rightarrow (B \rightarrow A))$
- AS2 $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
- AS3 $(((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A))$ [or $(((\neg A) \rightarrow (\neg B)) \rightarrow ((\neg A) \rightarrow B) \rightarrow A))$]
- AS4 $(((\forall x)A) \rightarrow A)$, if x does not occur free in A
- AS5 $(((\forall x)A) \rightarrow A[x/t])$, if x may appear free in A and t is free for x in A (i.e., free variables of t do not occur bound in A)
- AS6 $(((\forall x)(A \rightarrow B)) \rightarrow (A \rightarrow ((\forall x)B)))$, if x does not occur free in A

♣ Inference rules of **L** ($A, B \in$ **WFF**)

- Modus Ponens (MP) for material implication: From $A \rightarrow B$ and A to infer B.
- ***Generalization (Gen)***: From A to infer $((\forall x)A)$.

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS4

- AS4 $(((\forall x)A) \rightarrow A)$, if x does not occur free in A
- Relaxation of the restriction that x not be free in A would lead to the following disaster.
- If x occurs free in A, the following unpleasant result would arise.
- Let A be $\neg(\forall y)p(x, y)$. Notice that x occurs free in A.
- Consider the following pseudo-instance of axiom AS4:
  as4: $((\forall x)(\neg(\forall y)p(x, y))) \rightarrow \neg(\forall y)p(x, y)$
- Now take as interpretation any domain with at least two different members and let p stand for the identity relation.
- Then the antecedent of as4 is true and the consequent may be false for some $x=y$. Thus, as4 is false for this interpretation.

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS5

- AS5 $(((\forall x)A) \rightarrow A[x/t])$, if x may appear free in A and t is free for x in A (i.e., free variables of t do not occur bound in A)
- Relaxation of the restriction that x not be free in A would lead to the following disaster.
- If t were not free for x in A, the following unpleasant result would arise.
- Let A be $\neg(\forall y)p(x, y)$ and let t be y. Notice that t is not free for x in A.
- Consider the following pseudo-instance of axiom AS5:
  as5: $((\forall x)(\neg(\forall y)p(x, y))) \rightarrow \neg(\forall y)p(y, y)$
- Now take as interpretation any domain with at least two different members and let p stand for the identity relation.
- Then the antecedent of as5 is true and the consequent false. Thus, as5 is false for this interpretation.

L: A Hilbert Style Formal System for CFOPC

♣ Important note on AS6

- AS6 $(((\forall x)(A \to B)) \to (A \to ((\forall x)B)))$, if x does not occur free in A
- Relaxation of the restriction that x not be free in A would lead to the following disaster.
- Let A and B both be $p(x)$. Thus, x is free in A.
- Consider the following pseudo-instance of axiom AS6: as6: $((\forall x)(p(x) \to p(x))) \to (p(x) \to ((\forall x)p(x)))$
- The antecedent of as6 is valid. Now take as domain the set of integers and let $p(x)$ mean that x is even. Then $(\forall x)p(x)$ is false.
- So, any sequence $s = (s_1, s_2, \ldots)$ for which s_1 is even does not satisfy the consequent of as6. Hence, as6 is not true for this interpretation.

Properties of L: Deduction Theorem

♣ ***Syntactic (proof-theoretical) deduction theorems for* L**

- For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and no use was made of Generalization involving a free variable of A, then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and no application of Generalization to a formula that depends upon A has as its quantified variable a free variable of B. Then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- Corollary: For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and the deduction involves no application of Generalization of which the quantified variables is free in B, then $\Gamma \vdash_{\mathbf{L}} A \to B$.
- Corollary: For any $A, B \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$, if $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, and A is a closed, then $\Gamma \vdash_{\mathbf{L}} A \to B$.

♣ Note

- The deduction theorem for the propositional calculus cannot be carried over without modification to the first-order predicate calculus.

Properties of L: Deduction Theorem

♣ Notes

- The deduction theorem for the propositional calculus cannot be carried over without modification to the first-order predicate calculus.
- For example, for any $A \in \mathbf{WFF}$, by Generalization, $A \vdash_{\mathbf{L}} (\forall x)A$; but it is not always the case that $\vdash_{\mathbf{L}} A \to (\forall x)A$.
- Consider a domain $\boldsymbol{D}$ containing at least two elements c and d.
- Let A be $p^1{}_1(x_1)$. Interpret $p_1{}^1$ as a property that holds only for c.
- Then $p^1{}_1(x_1)$ is satisfied by any model $\boldsymbol{M} = (\boldsymbol{D}, \boldsymbol{I})$ where $x_1{}^{\boldsymbol{I}A} = x_1{}^A = c$ and $c \in p^1{}_1{}^{\boldsymbol{I}}$ (i.e., $v_f^{\boldsymbol{I}A}(p^1{}_1(x_1) = \mathbf{T})$, but $(\forall x_1)p^1{}_1(x_1)$ is satisfied by no $(\boldsymbol{D}, \boldsymbol{I})$ at all because $d \in \boldsymbol{D}$.
- Hence, $p^1{}_1(x_1) \to (\forall x_1)p^1{}_1(x_1)$ is not true in some models, and therefore it is not logically valid and should not be a logical theorem of **L**.

Properties of L: Deduction Theorem

♣ ***Dependence in deduction***

- Let $\Gamma \subseteq \mathbf{WFF}$ and $B \in \Gamma$. Assume that we are given a deduction $d_1, \ldots, d_n$ from Γ, together with justification for each step in the deduction.
- We say that d_i ***depends upon*** B in this deduction **IFF**: (1) d_i is B and the justification for d_i is that it belongs to Γ, or (2) d_i is justified as a direct consequence by Modus Ponens or Generalization of some preceding formulas of the sequence, where at least one of these preceding formulas depends upon B.
- If B does not depend upon A in a deduction $\Gamma \cup \{A\} \vdash_{\mathbf{L}} B$, then $\Gamma \vdash_{\mathbf{L}} B$.

♣ Example: $(\forall x_1)B \to C \vdash_{\mathbf{L}} (\forall x_1)C$

- d_1. B { Premise }
  d_2. $(\forall x_1)B$ { Apply Generalization to d_1 }
  d_3. $(\forall x_1)B \to C$ { Premise }
  d_4. C { Follow from d_2, and d_3 by MP }
  d_5. $(\forall x_1)C$ { Apply Generalization to d_4 }
- Here, d_1 depends upon B, d_2 depends upon B, d_3 depends upon $(\forall x_1)B \to C$, d_4 depends upon B and $(\forall x_1)B \to C$, and d_5 depends upon B and $(\forall x_1)B \to C$.

Properties of L: Consistency, Soundness, and Completeness

♣ ***Consistency of* L**

- For any $A \in \mathbf{WFF}$, not both $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} \neg A$, i.e., **L** is ***consistent***.

♣ ***Soundness theorems for* L**

- Theorem (***soundness***): If $\vdash_{\mathbf{L}} A$ then $\models_{\mathbf{CFOPC}} A$, for any $A \in \mathbf{WFF}$.
- Theorem (***strong soundness***): If $\Gamma \vdash_{\mathbf{L}} A$ then $\Gamma \models_{\mathbf{CFOPC}} A$, for any $A \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$.

♣ ***Completeness theorems for* L** (Gödel, 1930)

- Theorem (***completeness***): If $\models_{\mathbf{CFOPC}} A$ then $\vdash_{\mathbf{L}} A$, for any $A \in \mathbf{WFF}$.
- Theorem (***strong completeness***): If $\Gamma \models_{\mathbf{CFOPC}} A$ then $\Gamma \vdash_{\mathbf{L}} A$, for any $A \in \mathbf{WFF}$ and any $\Gamma \subseteq \mathbf{WFF}$.

♣ **CFOPC vs. L**

- **Th(CFOPC)** = **Th(L)**.

Properties of L: Completeness [Mendelson]

♣ ***Similar formulas***

- If x_i and x_j are distinct, then $B(x_i)$ and $B(x_j)$ are said to be ***similar*** IFF x_j is free for x_i in $B(x_i)$ and B (x_i) has no free occurrences of x_j.
- It is assumed here that $B(x_j)$ arises from $B(x_i)$ by substituting x_j for all free occurrences of x_i.
- Ex: $(\forall x_3)(p^2{}_1(x_1, x_3) \vee p^1{}_1(x_1))$ and $(\forall x_3)(p^2{}_1(x_2, x_3) \vee p^1{}_1(x_2))$ are similar.

♣ Lemmas

- If $B(x_i)$ and $B(x_j)$ are similar, then $\vdash_{\mathbf{L}} (\forall x_i)B(x_i) \leftrightarrow (\forall x_j)B(x_j)$.
- If a closed formula $\neg B$ is not provable in **L**, i.e., $\vdash_{\mathbf{L}} \neg B$ does not hold, and if **L′** is the **L**-theory obtained from **L** by adding B as a new axiom, then **L′** is consistent.
- Corollary: If a closed formula B is not provable in **L**, i.e., $\vdash_{\mathbf{L}} B$ does not hold, and if **L′** is the **L**-theory obtained from **L** by adding $\neg B$ as a new axiom, then **L′** is consistent.

Properties of L: Completeness [Mendelson]

♣ Definitions

- A **L**-theory **K** is said to be ***complete*** if, for every closed formula B of **K**, either $\vdash_{\mathbf{K}} B$ or $\vdash_{\mathbf{K}} \neg B$.
- A **L**-theory **K′** is said to be an ***extension*** of a **L**-theory **K** if every theorem of **K** is a theorem of **K′**. (We also say in such a case that K is a ***sub-theory*** of **K′**.)

♣ ***Lindenbaum's lemma***

- If **K** is a consistent **L**-theory, then there is a consistent, complete extension of **K**.

Properties of L: Completeness [Mendelson]

♣ Definitions

- A ***closed term*** is a term without variables.
- A **L**-theory **K** is a ***scapegoat theory*** if, for any $B(x)$ that has x as its only free variable, there is a closed term t such that $\vdash_{\mathbf{K}} (\exists x)\neg B(x) \rightarrow \neg B(t)$.

♣ Lemmas

- Every consistent **L**-theory **K** has a consistent extension **K′** such that **K′** is a scapegoat theory and **K′** contains denumerably many closed terms.
- Let **J** be a consistent, complete scapegoat theory. Then **J** has a model ***M*** whose domain is the set ***D*** of closed terms of **J**.

♣ Proposition

- Every consistent **L**-theory **K** has a denumerable model.
- Corollary: Any logically valid formula of a **L**-theory **K** is a theorem of **K**.
- Corollary (***Gödel's Completeness Theorem***, 1930): The theorems of **L** are precisely the logically valid formulas.

Properties of L: Completeness [Mendelson]

♣ Corollary

- Let **K** be any **L**-theory.
- a. A formula B is true in every denumerable model of **K** IFF $\vdash_{\mathbf{K}} B$.
- b. If, in every model of **K**, every sequence that satisfies all formulas in a set Γ of formulas also satisfies a formula B, then $\Gamma \vdash_{\mathbf{K}} B$.
- c. If a formula B of **K** is a logical consequence of a set Γ of formulas of **K**, then $\Gamma \vdash_{\mathbf{K}} B$.
- d. If a formula B of **K** is a logical consequence of a formula C of **K**, then $C \vdash_{\mathbf{K}} B$.

♣ ***Skolem–Löwenheim Theorem*** (1920, 1915)

- Any **L**-theory that has a model has a denumerable model.

Some Derived Rules in L (CFOPC) [Mendelson]

♣ **Particularization rule**

- If t is free for x in $A(x)$, then $(\forall x)A(x) \vdash_{\mathrm{L}} A(t)$.

♣ **Existential rule**

- Let t be a term that is free for x in a $A(x, t)$, and let $A(t, t)$ arise from $A(x, t)$ by replacing all free occurrences of x by t. (Note: $A(x, t)$ may or may not contain occurrences of t.) Then, $A(t, t) \vdash_{\mathrm{L}} (\exists x)A(x, t)$.

♣ **Proof by contradiction**

- If a proof of $\Gamma, \neg B \vdash_{\mathrm{L}} C\wedge\neg C$ involves no application of Generalization using a variable free in B, then $\Gamma \vdash_{\mathrm{L}} B$.
  Similarly, one obtains $\Gamma \vdash_{\mathrm{L}} \neg B$ from $\Gamma, B \vdash_{\mathrm{L}} C\wedge\neg C$.

Some Derived Rules in L (CFOPC) [Mendelson]

♣ **Rule C** ("C" for "choice")

- A rule C deduction in a first-order theory K, $\Gamma \vdash_{\mathbf{K\text{-}rule\text{-}C}} B$, is defined as follows:
- $\Gamma \vdash_{\mathbf{K\text{-}rule\text{-}C}} B$ IFF there is a sequence of formulas $D_1, \ldots, D_n$ such that D_n is B and the following four conditions hold:
- 1. For each $i<n$, either
  a. D_i is an axiom of K, or
  b. D_i is in Γ, or
  c. D_i follows by MP or Gen from preceding formulas in the sequence, or
  d. there is a preceding formula $(\exists x)C(x)$ such that D_i is $C(d)$, where d is a new individual constant (rule C).
- 2. As axioms in condition 1(a), we also can use all logical axioms that involve the new individual constants already introduced in the sequence by applications of rule C.
- 3. No application of Gen is made using a variable that is free in some $(\exists x)C(x)$ to which rule C has been previously applied.
- 4. B contains none of the new individual constants introduced in the sequence in any application of rule C.

Some Derived Rules in L (CFOPC) [Mendelson]

- Negation elimination: $\neg\neg B \vdash_{\mathrm{L}} B$
- Negation introduction: $B \vdash_{\mathrm{L}} \neg\neg B$
- Conjunction elimination: $B\wedge C \vdash_{\mathrm{L}} B$; $B\wedge C \vdash_{\mathrm{L}} C$; $\neg(B\wedge C) \vdash_{\mathrm{L}} \neg B\vee\neg C$
- Conjunction introduction: $B, C \vdash_{\mathrm{L}} B\wedge C$
- Disjunction elimination: $B\vee C, \neg B \vdash_{\mathrm{L}} C$; $B\vee C, \neg C \vdash_{\mathrm{L}} B$; $\neg(B\vee C) \vdash_{\mathrm{L}} \neg B\wedge\neg C$; $B\rightarrow D, C\rightarrow D, B\vee C \vdash_{\mathrm{L}} D$
- Disjunction introduction: $B \vdash_{\mathrm{L}} B\vee C$; $C \vdash_{\mathrm{L}} B\vee C$
- Conditional elimination: $B\rightarrow C, \neg C \vdash_{\mathrm{L}} \neg B$; $B\rightarrow\neg C, C \vdash_{\mathrm{L}} \neg B$; $\neg B\rightarrow C, \neg C \vdash_{\mathrm{L}} B$; $\neg B\rightarrow\neg C, C \vdash_{\mathrm{L}} B$; $\neg(B\rightarrow C) \vdash_{\mathrm{L}} B$; $\neg(B\rightarrow C) \vdash_{\mathrm{L}} \neg C$
- Conditional introduction: $B, \neg C \vdash_{\mathrm{L}} \neg(B\rightarrow C)$
- Conditional contrapositive: $B\rightarrow C \vdash_{\mathrm{L}} \neg C\rightarrow\neg B$; $\neg C\rightarrow\neg B \vdash_{\mathrm{L}} B\rightarrow C$
- Biconditional elimination: $B\leftrightarrow C, B \vdash_{\mathrm{L}} C$; $B\leftrightarrow C, \neg B \vdash_{\mathrm{L}} \neg C$; $B\leftrightarrow C, C \vdash_{\mathrm{L}} B$; $B\leftrightarrow C, \neg C \vdash_{\mathrm{L}} \neg B$; $B\leftrightarrow C \vdash_{\mathrm{L}} B\rightarrow C$; $B\leftrightarrow C \vdash_{\mathrm{L}} C\rightarrow B$
- Biconditional introduction: $B\rightarrow C, C\rightarrow B \vdash_{\mathrm{L}} B\leftrightarrow C$
- Biconditional negation: $B\leftrightarrow C \vdash_{\mathrm{L}} \neg B\leftrightarrow\neg C$; $\neg B\leftrightarrow\neg C \vdash_{\mathrm{L}} B\leftrightarrow C$

H: A Hilbert Style Formal System for CFOPC [Ben-Ari]

Definition 8.4 The axioms of the Hilbert system $\mathcal{H}$ for first-order logic are:

Axiom 1 $\vdash (A \rightarrow (B \rightarrow A))$,

Axiom 2 $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

Axiom 3 $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$,

Axiom 4 $\vdash \forall x A(x) \rightarrow A(a)$,

Axiom 5 $\vdash \forall x (A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x))$.

- In Axioms 1, 2 and 3, A, B and C are any formulas of first-order logic.
- In Axiom 4, $A(x)$ is a formula with a free variable x.
- In Axiom 5, $B(x)$ is a formula with a free variable x, while x is *not* a free variable of the formula A.

The rules of inference are ***modus ponens*** and ***generalization***:

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B}, \qquad \frac{\vdash A(a)}{\vdash \forall x A(x)}.$$

A Hilbert Style Formal System for CFOPC [R&C]

Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$ be sentences of the language. Then:

P1) $(\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{S}))$

P2) $((\mathcal{S} \rightarrow (\mathcal{T} \rightarrow \mathcal{R})) \rightarrow ((\mathcal{S} \rightarrow \mathcal{T}) \rightarrow (\mathcal{S} \rightarrow \mathcal{R})))$

P3) $(((\neg\mathcal{S}) \rightarrow (\neg\mathcal{T})) \rightarrow (\mathcal{T} \rightarrow \mathcal{S}))$

P4) $(\forall x_i \mathcal{S} \rightarrow \mathcal{S})$, if x_i does not occur free in $\mathcal{S}$

P5) $(\forall x_i \mathcal{S} \rightarrow \mathcal{S}[t/\,x_i])$, if $\mathcal{S}$ is a formula of the language in which x_i may appear free and t is free for x_i in $\mathcal{S}$

P6) $(\forall x_i(\mathcal{S} \rightarrow \mathcal{T}) \rightarrow (\mathcal{S} \rightarrow \forall x_i \mathrm{T}))$, if x_i does not occur free in $\mathcal{S}$.

Rules of deduction

1) Modus Ponens (MP), from $\mathcal{S}$ and $(\mathcal{S} \rightarrow \mathcal{T})$ deduce $\mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are any formulas of the language.

2) Generalization, from $\mathcal{S}$ deduce $\forall x_i \mathcal{S}$, where $\mathcal{S}$ is any formula of the language.

An Example of Deduction in L

♣ $\{B, (\forall x_1)B \rightarrow C\} \mid\!-_{\mathrm{L}} (\forall x_1)C$? (Omit the outermost brackets)

1. B { Premise }
2. $(\forall x_1)B$ { Apply Gen to 1 }
3. $(\forall x_1)B \rightarrow C$ { Premise }
4. C { Follow from 2 and 3 by MP }
5. $(\forall x_1)C$ { Apply Gen to 4 }

An Example of Theorem Proof in L

♣ $\mid\!-_{\mathrm{L}} ((\forall x_1)(\forall x_2)B) \rightarrow (\forall x_2)(\forall x_1)B$? (Omit the outermost brackets)

1. $(\forall x_1)(\forall x_2)B$ { Premise }
2. $(\forall x_1)(\forall x_2)B \rightarrow (\forall x_2)B$ { AS $(\forall x)A) \rightarrow A$ }
3. $(\forall x_2)B$ { Follow from 1 and 2 by MP }
4. $(\forall x_2)B \rightarrow B$ { AS $(\forall x)A) \rightarrow A$ }
5. B { Follow from 3 and 4 by MP }
6. $(\forall x_1)B$ { Apply Gen to 5 }
7. $(\forall x_2)(\forall x_1)B$ { Apply Gen to 6 }

Therefore, $(\forall x_1)(\forall x_2)B \mid\!-_{\mathrm{L}} (\forall x_2)(\forall x_1)B$.

In the above deduction, no application of Gen has as a quantified variable a free variable of $(\forall x_1)(\forall x_2)B$.

Hence, by the deduction theorem, $\mid\!-_{\mathrm{L}} ((\forall x_1)(\forall x_2)B) \rightarrow (\forall x_2)(\forall x_1)B$.

An Example of Theorem Proof (by Propositional Deduction) in H [Ben-Ari]

Example 8.5

$$\vdash \forall x p(x) \rightarrow (\exists y \forall x q(x, y) \rightarrow \forall x p(x))$$

is an instance of Axiom 1 in first-order logic and:

$$\frac{\vdash \forall x p(x) \rightarrow (\exists y \forall x q(x, y) \rightarrow \forall x p(x)) \qquad \vdash \forall x p(x)}{\vdash \exists y \forall x q(x, y) \rightarrow \forall x p(x)}$$

uses the rule of inference *modus ponens*.

Some Logical Theorems of L

- $\mid\!-_{\mathrm{L}} ((\forall x)(\forall y)A) \rightarrow (\forall y)(\forall x)A$
- $\mid\!-_{\mathrm{L}} (\neg(\forall x)A) \rightarrow (\exists x)\neg A$
- $\mid\!-_{\mathrm{L}} ((\forall x)(A \rightarrow B)) \rightarrow ((\forall x)A \rightarrow (\forall x)B)$
- $\mid\!-_{\mathrm{L}} ((\forall x)(A \leftrightarrow B)) \rightarrow ((\forall x)A \leftrightarrow (\forall x)B)$
- $\mid\!-_{\mathrm{L}} ((\forall x)(A \rightarrow B)) \rightarrow ((\exists x)A \rightarrow (\exists x)B)$
- $\mid\!-_{\mathrm{L}} ((\forall x)(A \wedge B)) \leftrightarrow (\forall x)A) \wedge (\forall x)B$
- $\mid\!-_{\mathrm{L}} ((\forall y_1) \ldots (\forall y_n)A) \rightarrow A$
- $\mid\!-_{\mathrm{L}} ((\forall x)(\forall y)p(x, y)) \rightarrow (\forall x)p(x, x)$
- $\mid\!-_{\mathrm{L}} ((\forall x)B) \vee (\forall x)C)) \rightarrow (\forall x)(B \vee C)$
- $\mid\!-_{\mathrm{L}} (\neg(\exists x)B) \rightarrow (\forall x)\neg B$
- $\mid\!-_{\mathrm{L}} ((\forall x)B) \rightarrow (\forall x)(B \vee C)$
- $\mid\!-_{\mathrm{L}} ((\forall x)(\forall y)(p(x, y) \rightarrow \neg p(y, x))) \rightarrow (\forall x)\neg p(x, x)$
- $\mid\!-_{\mathrm{L}} ((\exists x)B \rightarrow (\forall x)C) \rightarrow (\forall x)(B \rightarrow C)$

Some Logical Theorems of L

- $\vdash_L ((\forall x)(B\lor C)) \to (((\forall x)B)\lor(\exists x)C)$
- $\vdash_L (\forall x)(p(x, x) \to (\exists y)p(x, y))$
- $\vdash_L ((\forall x)(B\to C)) \to ((\forall x)\neg C \to (\forall x)\neg B)$
- $\vdash_L (\exists y)(p(y) \to (\forall y)p(y))$
- $\vdash_L ((\exists x)p(x, x)) \to (\exists x)(\exists y)p(x, y)$
- $\vdash_L [((\forall x)(\forall y)(p(x, y) \to p(y, x)))\land$
  $(\forall x)(\forall y)(\forall z)((p(x, y)\land p(y, z)) \to p(x, z))]$
  $\to (\forall x)(\forall y)(p(x, y) \to p(x, x))$
- $\vdash_L B \to (\forall x)B$, x is not free in B
- $\vdash_L ((\exists x)B) \to B$, x is not free in B
- $\vdash_L (B \to (\forall x)C) \leftrightarrow (\forall x)(B\to C)$, x is not free in B
- $\vdash_L (((\exists x)C) \to B) \leftrightarrow (\forall x)(C\to B)$, x is not free in B
- $\vdash_L ((\forall x)(p_1(x) \leftrightarrow p_2(x))) \to (((\exists x)p_1(x)) \leftrightarrow (\exists x)p_2(x))$

Some Logical Theorems of L

- $\vdash_L ((\exists x)\neg A) \leftrightarrow (\neg(\forall x)A)$
- $\vdash_L ((\forall x)A) \leftrightarrow \neg(\exists x)\neg A$
- $\vdash_L ((\exists x)(B\to\neg(C\lor D))) \to (\exists x)(B\to(\neg C\land\neg D))$
- $\vdash_L ((\forall x)(\exists y)(B\to C)) \leftrightarrow (\forall x)(\exists y)(\neg B\lor C)$
- $\vdash_L ((\forall x)(B\to\neg C)) \leftrightarrow \neg(\exists x)(B\land C)$
- $\vdash_L ((\exists x)(B(x)\to C(x))) \to (((\forall x)B(x))\to(\exists x)C(x))$
- $\vdash_L \neg(\exists y)(\forall x)(p(x, y)) \leftrightarrow \neg p(x, x))$
- $\vdash_L [(\forall x)(p_1(x)\to(p_2(x)\lor p_3(x)))\land\neg(\forall x)(p_1(x)\to p_2(x))] \to (\exists x)(p_1(x)\land p_3(x))$
- $\vdash_L (((\exists x)B(x))\land(\forall x)C(x)) \to (\exists x)(B(x)\land C(x))$
- $\vdash_L ((\exists x)C(x)) \to (\exists x)(B(x)\lor C(x))$
- $\vdash_L ((\exists x)(\exists y)B(x, y)) \leftrightarrow (\exists y)(\exists x)B(x, y)$
- $\vdash_L ((\exists x)(\forall y)B(x, y)) \to (\forall y)(\exists x)B(x, y)$
- $\vdash_L ((\exists x)(B(x)\land C(x))) \to ((\exists x)(B(x))\land(\exists x)C(x)$

Examples of First-Order Theories Based on L: Partially Ordered Structure [Mendelson]

♣ Predicates, functions, and individual constants
- A single predicate $p^2{}_2$. We shall write $x_i < x_j$ instead of $p^2{}_2(x_i, x_j)$.
- No functions and individual constants.

♣ Empirical premises/axioms
- $(\forall x_1)(\neg(x_1 < x_1))$ (***irreflexivity***)
- $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 < x_2 \land x_2 < x_3) \to x_1 < x_3))$ (***transitivity***)

♣ ***Partially ordered structure***
- A model of the above first-order theory is called a ***partially ordered structure***.

Examples of First-Order Theories Based on L: Group Theory [Mendelson]

♣ Predicates, functions, and individual constants
- A single predicate $p^2{}_1$. We shall write $x_i = x_j$ instead of $p^2{}_1(x_i, x_j)$.
- A single function $f^2{}_1$. We shall write $x_i + x_j$ instead of $f^2{}_1(x_i, x_j)$.
- A single individual constant c_1. We shall write 0 instead of c_1.

♣ Empirical premises/axioms
- a. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$ (***associativity***)
- b. $(\forall x_1)((0 + x_1) = x_1)$ (***identity***)
- c. $(\forall x_1)(\exists x_2)((x_2 + x_1) = 0)$ (***inverse***)
- d. $(\forall x_1)(x_1 = x_1)$ (***reflexivity*** of =)
- e. $(\forall x_1)(\forall x_2)((x_1 = x_2) \to (x_2 = x_1))$ (***symmetry*** of =)
- f. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 = x_2 \land x_2 = x_3) \to x_1 = x_3))$ (***transitivity*** of =)
- g. $(\forall x_1)(\forall x_2)(\forall x_3)((x_2 = x_3) \to ((x_1 + x_2) = (x_1 + x_3) \land (x_2 + x_1) = (x_3 + x_1)))$ (***substitutivity*** of =)

Examples of First-Order Theories Based on L: Group Theory [Mendelson]

♣ ***Group theory***
- A model for the above first-order theory, in which the interpretation of predicate "=" is the ***identity relation***, is called a ***group***.
- A group is said to be ***Abelian*** if, in addition, the following is true.
- h. $(\forall x_1)(\forall x_2)((x_1 + x_2) = (x_2 + x_1))$

L-Based First-Order Theories with Equality [Mendelson]

♣ ***L-based first-order theories with equality***
- Let **K** be a theory that has as one of its predicate symbols $p^2{}_1$. We write $t{=}s$ as an abbreviation for $p^2{}_1(t, s)$, and $t{\neq}s$ as an abbreviation for $\neg\, p^2{}_1(t, s)$.
- **K** is called a ***L-based first-order theory with equality*** if the following are theorems (empirical premises/axioms) of **K**:
  $(\forall x_1)(x_1 = x_1)$ (***reflexivity of equality***)
  $(x = y) \to (B(x, x) \to B(x, y))$ (***substitutivity of equality***)
  where x and y are any two individual variables, $B(x, x)$ is any formula, and $B(x, y)$ arises from $B(x, x)$ by replacing some, but not necessarily all, free occurrences of x by y, with the proviso that y is free for x in $B(x, x)$. Thus, $B(x, y)$ may or may not contain free occurrences of x.

♣ Propositions on the relation of equality
- $\vdash_K t = t$ for any term t.
- $\vdash_K t = s \to s = t$ for any terms t and s.
- $\vdash_K t = s \to (s = r \to t = r)$ for any terms t, s, and r.

L-Based First-Order Theories with Equality [Mendelson]

♣ The equivalence relation in **L**-based first-order theories with equality

- By the above proposition on the relation of equality, in any model for a **L**-based first-order theory **K** with equality, the relation E in the model corresponding to the predicate symbol "=" is an equivalence relation.

♣ Various **L**-based first-order theories with equality

- There are various L-based first-order theories with equality that include predicates other than equality and empirical premises/axioms other than the two axioms of reflexivity of equality and substitutivity of equality.
- **NBG** axiomatic set theory is a L-based first-order theory with equality.

Some Theorems of L-Based First-Order Theories with Equality [Mendelson]

- Let **K** be a **L**-based first-order theory with equality, then the following theorems hold.
- $\vdash_{\mathbf{K}} (\forall x)(B(x) \leftrightarrow (\exists y)(x{=}y \wedge B(y)))$, if y does not occur in $B(x)$
- $\vdash_{\mathbf{K}} (\forall x)(B(x) \leftrightarrow (\forall y)(x{=}y \rightarrow B(y)))$, if y does not occur in $B(x)$
- $\vdash_{\mathbf{K}} (\forall x)(\exists y)x{=}y$
- $\vdash_{\mathbf{K}} x{=}y \rightarrow f(x){=}f(y)$, where f is any function symbol of one argument
- $\vdash_{\mathbf{K}} B(x) \wedge x{=}y \rightarrow B(y)$, if y is free for x in $B(x)$
- $\vdash_{\mathbf{K}} B(x) \wedge \neg B(y) \rightarrow x{\neq}y$, if y is free for x in $B(x)$

Some Theorems of L-Based First-Order Theories with Equality [Mendelson]

- Let **K** be a **L**-based first-order theory with equality, then the following theorems hold.
- $\vdash_{\mathbf{K}} (x_1{=}y_1 \wedge ... \wedge x_n{=}y_n) \rightarrow t(x_1, \ldots, x_n){=}t(y_1, \ldots, y_n)$, where $t(y_1, \ldots, y_n)$ arises from the term $t(x_1, \ldots, x_n)$ by substitution of $y_1, \ldots, y_n$ for $x_1, \ldots, x_n$ respectively.
- $\vdash_{\mathbf{K}} (x_1{=}y_1 \wedge ... \wedge x_n{=}y_n) \rightarrow (B(x_1, \ldots, x_n) \leftrightarrow B(y_1, \ldots, y_n))$, where $B(y_1, \ldots, y_n)$ is obtained by substituting $y_1, \ldots, y_n$ for one or more occurrences of $x_1, \ldots, x_n$ respectively, in the formula $B(x_1, \ldots, x_n)$, and $y_1, \ldots, y_n$ are free for $x_1, \ldots, x_n$, respectively, in the formula $B(x_1, \ldots, x_n)$.

L-Based First-Order Theories with Equality [Mendelson]

♣ ***Unique quantifier***

- In **L**-based first-order theories with equality it is possible to define in the following way phrases that use the expression "There exists one and only one x such that … ."
- Definition: $(\exists_1 x)A(x) =_{\mathrm{df}} (\exists x)A(x) \wedge (\forall x)(\forall y)(A(x) \wedge A(y) \rightarrow x = y)$

♣ Properties of unique quantifier

- In any **L**-based first-order theory **K** with equality, the following hold:
  a. $\vdash_{\mathbf{K}} (\forall x)(\exists_1 y)(x = y)$
  b. $\vdash_{\mathbf{K}} (\exists_1 x)A(x) \leftrightarrow (\exists x)(\forall y)(x = y \leftrightarrow A(y))$
  c. $\vdash_{\mathbf{K}} (\forall x)(A(x) \leftrightarrow B(x)) \rightarrow ((\exists_1 x)A(x) \leftrightarrow (\exists_1 x)B(x))$
  d. $\vdash_{\mathbf{K}} (\exists_1 x)(A \vee B) \rightarrow ((\exists_1 x)A \vee (\exists_1 x)B)$
  e. $\vdash_{\mathbf{K}} (\exists_1 x)A(x) \leftrightarrow (\exists x)(A(x) \wedge (\forall y)(A(y) \rightarrow y = x))$

L-Based First-Order Theories with Equality: Normal Models [Mendelson]

♣ ***The normal models of first-order theories of equality***

- In any model for a **L**-based first-order theory **K** with equality, the relation E in the model corresponding to the predicate symbol "=" is an equivalence relation.
- If this relation E is the identity relation in the domain of the model, then the model is said to be ***normal***.
- Note: In general, an equivalence relation in a domain is not necessarily the identity relation in the domain.
- Any model M for **K** can be contracted to a normal model M^* for **K** by taking the domain D^* of M^* to be the set of equivalence classes determined by the relation E in the domain D of M.

♣ ***The normal models of first-order theories of equality*** (Gödel, 1930)

- Any consistent **L**-based first-order theory **K** with equality has finite or countable normal models.

Examples of L-Based First-Order Theories with Equality: The Pure First-Order Theory of Equality [Mendelson]

♣ ***The pure first-order theory of equality***

- Let $\mathbf{K}_1$ be a **L**-based first-order theory whose language has only = as a predicate symbol and no function symbols or individual constants.
- Let its empirical premises/axioms be:
  a. $(\forall x_1)(x_1 = x_1)$
  b. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow x_2 = x_1)$
  c. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \rightarrow (x_2 = x_3 \rightarrow x_1 = x_3))$
- $\mathbf{K}_1$ is a **L**-based first-order theory with equality called ***the pure first-order theory of equality***.

Examples of L-Based First-Order Theories with Equality: The Theory of Densely Ordered Sets with neither First nor Last Element [Mendelson]

♣ ***The theory of densely ordered sets with neither first nor last element***

- Let $\mathbf{K_2}$ be a **L**-based first-order theory whose language has only = and < as predicate symbols and no function symbols or individual constants.
- Let its empirical premises/axioms be:
  a. $(\forall x_1)(x_1 = x_1)$
  b. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow x_2 = x_1)$
  c. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \rightarrow (x_2 = x_3 \rightarrow x_1 = x_3))$
  d. $(\forall x_1)(\exists x_2)(\exists x_3)(x_1 < x_2 \wedge x_3 < x_1)$
  e. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 < x_2 \wedge x_2 < x_3 \rightarrow x_1 < x_3)$
  f. $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow \neg x_1 < x_2)$
  g. $(\forall x_1)(\forall x_2)(x_1 < x_2 \vee x_1 = x_2 \vee x_2 < x_1)$
  h. $(\forall x_1)(\forall x_2)(x_1 < x_2 \rightarrow (\exists x_3)(x_1 < x_3 \wedge x_3 < x_2))$
- $\mathbf{K_2}$ is a **L**-based first-order theory with equality called ***the theory of densely ordered sets with neither first nor last element***.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Group theory* G**

- If we regard the ***identity relation*** "=" in the group theory (denoted by **G**) and the Abelian group theory (denoted by Abelian **G**) of **L**-based first-order theory as "equality", then **G** and Abelian **G** are a **L**-based first-order theory with equality, respectively.

♣ ***Field theory* F**

- A single predicate $p^2{}_1$. We shall write $x_i = x_j$ instead of $p^2{}_1(x_i, x_j)$.
- Two functions $f^2{}_1$ and $f^2{}_2$. We shall write $x_i + x_j$ instead of $f^2{}_1(x_i, x_j)$ and write $x_i \bullet x_j$ instead of $f^2{}_2(x_i, x_j)$.
- Two individual constants c_1 and c_2. We shall write 0 instead of c_1 and write 1 instead of c_2.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Field theory* F**

- Empirical premises/axioms include all empirical premises/axioms of Abelian group theory, (a)-(h), and plus the following:
- i. $(\forall x_1)(\forall x_2)(\forall x_3)((x_2 = x_3) \rightarrow ((x_1 \bullet x_2) = (x_1 \bullet x_3) \wedge (x_2 \bullet x_1) = (x_3 \bullet x_1)))$
- j. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 \bullet x_3)) = ((x_1 \bullet x_2) \bullet x_3))$
- k. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 + x_3)) = ((x_1 \bullet x_2) + (x_1 \bullet x_3)))$
- l. $(\forall x_1)(\forall x_2)((x_1 \bullet x_2) = (x_2 \bullet x_1))$
- m. $(\forall x_1)((x_1 \bullet 1) = x_1)$
- n. $(\forall x_1)((x_1 \neq 0) \rightarrow (\exists x_2)((x_1 \bullet x_2) = 1))$
- o. $0 \neq 1$
- **F** is a **L**-based first-order theory with equality.

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Commutative ring with unit* $\mathbf{R}_C$**

- Empirical premises/axioms (a)–(m) of **F** define the **L**-based first-order theory $\mathbf{R}_C$ of commutative rings with unit.
- a. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$
- b. $(\forall x_1)((0 + x_1) = x_1)$
- c. $(\forall x_1)(\exists x_2)((x_2 + x_1) = 0)$
- d. $(\forall x_1)(x_1 = x_1)$
- e. $(\forall x_1)(\forall x_2)((x_1 = x_2) \rightarrow (x_2 = x_1))$
- f. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 = x_2 \wedge x_2 = x_3) \rightarrow x_1 = x_3))$
- g. $(\forall x_1)(\forall x_2)(\forall x_3)((x_2 = x_3) \rightarrow ((x_1 + x_2) = (x_1 + x_3) \wedge (x_2 + x_1) = (x_3 + x_1)))$
- h. $(\forall x_1)(\forall x_2)((x_1 + x_2) = (x_2 + x_1))$

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Commutative ring with unit* $\mathbf{R}_C$**

- i. $(\forall x_1)(\forall x_2)(\forall x_3)((x_2 = x_3) \rightarrow ((x_1 \bullet x_2) = (x_1 \bullet x_3) \wedge (x_2 \bullet x_1) = (x_3 \bullet x_1)))$
- j. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 \bullet x_3)) = ((x_1 \bullet x_2) \bullet x_3))$
- k. $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 \bullet (x_2 + x_3)) = ((x_1 \bullet x_2) + (x_1 \bullet x_3)))$
- l. $(\forall x_1)(\forall x_2)((x_1 \bullet x_2) = (x_2 \bullet x_1))$
- m. $(\forall x_1)((x_1 \bullet 1) = x_1)$

Examples of L-Based First-Order Theories with Equality: Abstract Algebra Theories [Mendelson]

♣ ***Ordered field* $\mathbf{F}_<$**

- If we add to **F** the predicate symbol $p^2{}_2$, abbreviate $p^2{}_2(t, s)$ by "$t < s$", and add the following empirical axioms (the first three are (e)-(g) of the theory of densely ordered sets with neither first nor last element):
- $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 < x_2 \wedge x_2 < x_3 \rightarrow x_1 < x_3)$
- $(\forall x_1)(\forall x_2)(x_1 = x_2 \rightarrow \neg x_1 < x_2)$
- $(\forall x_1)(\forall x_2)(x_1 < x_2 \vee x_1 = x_2 \vee x_2 < x_1)$
- $(\forall x_1)(\forall x_2)(\forall x_3)((x_1 < x_2) \rightarrow (x_1 + x_3) < (x_2 + x_3))$
- $(\forall x_1)(\forall x_2)(\forall x_3)(((x_1 < x_2) \wedge (0 < x_3)) \rightarrow (x_1 \bullet x_3) < (x_2 \bullet x_3))$
- Then the obtained new theory $\mathbf{F}_<$ is called the **L**-based first-order theory with equality of ordered fields.

Examples by a L-Based First-Order Language with Equality

- Defined predicates
  - Let $x, y, z \in$ CS (the set of professors, administrative staffs, and students of CS department).
  - Let '$P(x)$', '$AS(x)$', and '$S(x)$' be the predicates 'x is a professor', 'x is an administrative staff', and 'x is a student', respectively.
  - Let '$C(x)$', '$VC(x)$', and '$D(x)$' be the predicates 'x is the department chair, 'x is a department vice-chair', and 'x is the office director', respectively.
  - Let $M(x, y)$, '$T(x, y)$', '$A(x, y)$', and '$TA(x, y)$' be the predicates 'x manages y', 'x teaches y', 'x advises y', and 'x has y as a TA', respectively.
- Sentence examples
  - (a) CS-dept has only one professor as the department chair.
    $(\exists x)((P(x)\wedge C(x))\wedge(\forall x)(\forall y)(P(x)\wedge C(x)\wedge(P(y)\wedge C(y)\rightarrow x=y))$
    or $(\exists_1 x)((P(x)\wedge C(x))$
    or $(\exists x)(P(x)\wedge C(x)\wedge(\forall y)(\neg(y=x)\rightarrow\neg C(y)))$

Examples by a L-Based First-Order Language with Equality

- Sentence examples
  - (b) CS-dept has at least one professor as a department vice-chair.
    $(\exists x)(P(x)\wedge VC(x))$
  - (c) CS-dept has only one administrative staff as the office director.
    $(\exists x)((AS(x)\wedge D(x))\wedge(\forall x)(\forall y)(AS(x)\wedge D(x)\wedge(AS(y)\wedge D(y)\rightarrow x=y))$
    or $(\exists_1 x)((AS(x)\wedge D(x))$
    or $(\exists x)(AS(x)\wedge D(x)\wedge(\forall y)(\neg(y=x)\rightarrow\neg D(y)))$
  - (d) The department chair and vice-chair(s) manage all professors and administrative staffs.
    $(\exists x)(\exists y)(\forall z)((P(x)\wedge C(x)\wedge P(y)\wedge VC(y)\wedge(P(z)\vee AS(z))) \rightarrow (M(x, z)\wedge M(y, z)))$
  - (e) Each professor teaches all students.
    $(\forall x)(\forall y)((P(x)\wedge S(y)) \rightarrow T(x, y))$

Examples by a L-Based First-Order Language with Equality

- Sentence examples
  - (f) There is no professor who does not teach any student.
    $\neg(\exists x)(\forall y)(P(x)\wedge S(y)\wedge(\neg T(x, y)))$
    or $(\forall x)(\exists y)(P(x)\wedge S(y)\wedge T(x, y))$
  - (g) Each professor, as an advisor, advises at least one student.
    $(\forall x)(\exists y)(P(x)\wedge S(y)\wedge A(x, y))$
    or $\neg(\exists x)(\forall y)(P(x)\wedge S(y)\wedge(\neg A(x, y)))$
  - (h) There is no student who has no advisor.
    $\neg(\exists x)(\forall y)((S(x)\wedge P(y)\wedge\neg A(y, x))$
    or $(\forall x)(\exists y)(S(x)\wedge P(y)\wedge A(y, x))$
  - (i) A professor may have some students as teaching assistants.
    $(\exists x)(\exists y)(\exists z)(P(x)\wedge S(y)\wedge S(z)\wedge(\neg(y=z))\wedge TA(x, y)\wedge TA(x, z))$
  - (j) A student may be a teaching assistant to more than one professor.
    $(\exists x)(\exists y)(\exists z)(P(x)\wedge P(y)\wedge(\neg(y=x))\wedge S(z)\wedge TA(x, z)\wedge TA(y, z))$

An Introduction to Classical Predicate Calculus

- The Limitations of Propositional Logic **CPC**
- Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- Substitutions
- Semantics (Model Theory) of **CFOPC**
- Semantic (Model-theoretical, Logical) Consequence Relation
- Hilbert Style Formal Logic Systems for **CFOPC**
- Gentzen's Natural Deduction System for **CFOPC**
- Gentzen's Sequent Calculus System for **CFOPC**
- Semantic Tableau Systems for **CFOPC**
- Resolution Systems for **CFOPC**
- Classical Second-Order Predicate Calculus (**CSOPC**)