Axiom schemata of **L:**

(L1) $p \to (q \to p)$

(L2) $\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)$

(L3) $(\neg p \to \neg q) \to (q \to p)$

Lemma 1.

$$\{p \to (q \to r)\} \vdash q \to (p \to r)$$

Deduction THM forbidden:

(1)$p \to (q \to r)$ $\quad Assumption$

(2)$\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)$ $\quad$ (L2)

(3)$(p \to q) \to (p \to r)$ $\quad (1),(2),\mathrm{MP}$

(4)$\left((p \to q) \to (p \to r)\right) \to \left(q \to \left((p \to q) \to (p \to r)\right)\right)$ $\quad$ (L1)

(5)$q \to \left((p \to q) \to (p \to r)\right)$ $\quad (3),(4),\mathrm{MP}$

(6)$\left(q \to \left((p \to q) \to (p \to r)\right)\right) \to \left(\left(q \to (p \to q)\right) \to \left(q \to (p \to r)\right)\right)$ $\quad$ (L2)

(7)$\left(q \to (p \to q)\right) \to \left(q \to (p \to r)\right)$ $\quad (5),(6),\mathrm{MP}$

(8)$q \to (p \to q)$ $\quad$ (L1)

(9)$q \to (p \to r)$ $\quad (7),(8),\mathrm{MP}$

■

Deduction used:

According to Deduction THM,

$$\{p \to (q \to r)\} \vdash q \to (p \to r) \Leftrightarrow \{p \to (q \to r), q, p\} \vdash r$$

The following steps is so easy that is omitted.

Lemma 2.

$$\vdash (q \to r) \to \left((p \to q) \to (p \to r)\right)$$

Deduction THM forbidden:

(1)$\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)$ $\quad$ (L2)

(2)$\left(\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)\right) \to \left((q \to r) \to \left(\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)\right)\right)$ $\quad$ (L1)

(3)$(q \to r) \to \left(\left(p \to (q \to r)\right) \to \left((p \to q) \to (p \to r)\right)\right)$ $\quad (1),(2),\mathrm{MP}$

$(4)\left((q \to r) \to \left((p \to (q \to r)) \to ((p \to q) \to (p \to r))\right)\right) \to \left(\left((q \to r) \to (p \to (q \to r))\right) \to\right.$

$\left.\left((q \to r) \to ((p \to q) \to (p \to r))\right)\right)$ (L2)

$(5)\left((q \to r) \to (p \to (q \to r))\right) \to \left((q \to r) \to ((p \to q) \to (p \to r))\right)$ (3), (4), MP

$(6)(q \to r) \to (p \to (q \to r))$ (L1)

$(7)(q \to r) \to ((p \to q) \to (p \to r))$ (5), (6), MP

■

Lemma 3.

$$\vdash (p \to q) \to ((q \to r) \to (p \to r))$$

Deduction THM forbidden:

$(1)(q \to r) \to ((p \to q) \to (p \to r))$ *lemma* 2

$(2)(p \to q) \to ((q \to r) \to (p \to r))$ (1), *lemma* 1

■

Lemma 4.

$$\{p \to q, q \to r\} \vdash p \to r$$

Deduction THM forbidden:

$(1)q \to r$ *Assumption*

$(2)(q \to r) \to (p \to (q \to r))$ (L1)

$(3)p \to (q \to r)$ (1), (2), MP

$(4)(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (L2)

$(5)(p \to q) \to (p \to r)$ (3), (4), MP

$(6)p \to q$ *Assumption*

$(7)p \to r$ (5), (6), MP

■

Or easily proved by *lemma* 2 or *lemma* 3, which is too easy to show.

Deduction used:

According to Deduction THM,

$$\{p \to q, q \to r\} \vdash p \to r \Leftrightarrow \{p \to q, q \to r, p\} \vdash r$$

The following steps is so easy that is omitted. The proves of *lemma* 2 or *lemma* 3 with Deduction THM used are same as this.

Rule **HS** is the usage of *lemma* 4 in the following.

7.1 (e)

$$\vdash (B \to C) \to (\neg C \to \neg B)$$

Deduction THM forbidden:

$(1)(\neg\neg B \to B) \to \big((B \to C) \to (\neg\neg B \to C)\big)$ $\quad$ *lemma* 3

$(2)\neg\neg B \to B$ $\quad$ (a)

$(3)(B \to C) \to (\neg\neg B \to C)$ $\quad$ (1), (2), MP

$(4)C \to \neg\neg C$ $\quad$ (b)

$(5)(C \to \neg\neg C) \to \big(\neg\neg B \to (C \to \neg\neg C)\big)$ $\quad$ (L1)

$(6)\neg\neg B \to (C \to \neg\neg C)$ $\quad$ (4), (5), MP

$(7)\big(\neg\neg B \to (C \to \neg\neg C)\big) \to \big((\neg\neg B \to C) \to (\neg\neg B \to \neg\neg C)\big)$ $\quad$ (L2)

$(8)(\neg\neg B \to C) \to (\neg\neg B \to \neg\neg C)$ $\quad$ (6), (7), MP

$(9)(B \to C) \to (\neg\neg B \to \neg\neg C)$ $\quad$ (3), (8), HS

$(10)(\neg\neg B \to \neg\neg C) \to (\neg C \to \neg B)$ $\quad$ $(L3)$

$(11)(B \to C) \to (\neg C \to \neg B)$ $\quad$ (9), (10), MP

■

Deduction used:

According to Deduction THM,

$$\vdash (B \to C) \to (\neg C \to \neg B) \Leftrightarrow \{B \to C\} \vdash \neg C \to \neg B \Leftrightarrow \{B \to C, \neg C\} \vdash \neg B$$

$(1)B \to C$ $\quad$ *Assumption*

$(2)\neg\neg B \to B$ $\quad$ (a)

$(3)\neg\neg B \to C$ $\quad$ (1), (2), HS

$(4)C \to \neg\neg C$ $\quad$ (b)

$(5)\ \neg\neg B \to \neg\neg C$ $\quad$ (3), (4), HS

$(6)(\neg\neg B \to \neg\neg C) \to (\neg C \to \neg B)$ $\quad$ $(L3)$

$(7)\neg C \to \neg B$ $\quad$ (5), (6), MP

$(8)\neg C$ $\quad$ *Assumption*

$(9)\neg B$ $\quad$ (7), (8), MP

$\blacksquare$

Part of, which means it would be surely used, Axiom schemata of **Hilbert and Bernays' system:**

(AS1) $A \to (A \vee B)$

(AS2) $B \to (A \vee B)$

(AS3) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C)$

(AS4) $(A \leftrightarrow B) \to (A \to B)$

(AS5) $(A \to B) \to ((B \to A) \to (A \leftrightarrow B))$

(AS6) $(A \to B) \to ((B \to C) \to (A \to C))$

(AS6) here is same as *lemma* 3 in **L**, which means the **HS** rule could be also used here.

7.2 (e)

$$\vdash (A \vee (B \vee C)) \leftrightarrow ((A \vee B) \vee C)$$

(1) $B \to (A \vee B)$ (AS2)

(2) $(A \vee B) \to ((A \vee B) \vee C)$ (AS1)

(3) $B \to ((A \vee B) \vee C)$ (1), (2), HS

(4) $\left(B \to ((A \vee B) \vee C)\right) \to \left(\left(C \to ((A \vee B) \vee C)\right) \to \left((B \vee C) \to ((A \vee B) \vee C)\right)\right)$ (AS3)

(5) $\left(C \to ((A \vee B) \vee C)\right) \to \left((B \vee C) \to ((A \vee B) \vee C)\right)$ (3), (4), MP

(6) $C \to ((A \vee B) \vee C)$ (AS2)

(7) $(B \vee C) \to ((A \vee B) \vee C)$ (5), (6), MP

(8) $\left((B \vee C) \to ((A \vee B) \vee C)\right) \to \left(\left(A \to ((A \vee B) \vee C)\right) \to \left(((B \vee C) \vee A) \to ((A \vee B) \vee C)\right)\right)$

(AS3)

(9) $\left(A \to ((A \vee B) \vee C)\right) \to \left(((B \vee C) \vee A) \to ((A \vee B) \vee C)\right)$ (7), (8), MP

(10) $A \to (A \vee B)$ (AS1)

(11) $(A \vee B) \to ((A \vee B) \vee C)$ (AS1)

(12) $A \to ((A \vee B) \vee C)$ (10), (11), MP

(13) $((B \vee C) \vee A) \to ((A \vee B) \vee C)$ (9), (12), MP

(14) $(A \vee (B \vee C)) \leftrightarrow ((B \vee C) \vee A)$ (c)

(15) $\left((A \vee (B \vee C)) \leftrightarrow ((B \vee C) \vee A)\right) \to \left((A \vee (B \vee C)) \to ((B \vee C) \vee A)\right)$ (AS4)

$(16)\big(A \vee (B \vee C)\big) \rightarrow \big((B \vee C) \vee A\big)$ $(14), (15), \mathrm{MP}$

$(17)\big(A \vee (B \vee C)\big) \rightarrow \big((A \vee B) \vee C\big)$ $(13), (16), \mathrm{HS}$

$(18)\big((A \vee B) \vee C\big) \rightarrow \big(A \vee (B \vee C)\big)$

(18) could be easily got by similar steps as (1)-(17).

$(19)\left(\big(A \vee (B \vee C)\big) \rightarrow \big((A \vee B) \vee C\big)\right) \rightarrow \left(\left(\big((A \vee B) \vee C\big) \rightarrow \big(A \vee (B \vee C)\big)\right) \rightarrow \left(\big(A \vee (B \vee C)\big) \leftrightarrow \big((A \vee B) \vee C\big)\right)\right)$ $(\mathrm{AS5})$

$(20)\left(\big((A \vee B) \vee C\big) \rightarrow \big(A \vee (B \vee C)\big)\right) \rightarrow \left(\big(A \vee (B \vee C)\big) \leftrightarrow \big((A \vee B) \vee C\big)\right)$ $(17), (19), \mathrm{MP}$

$(21)\big(A \vee (B \vee C)\big) \leftrightarrow \big((A \vee B) \vee C\big)$ $(18), (20), \mathrm{MP}$

■

Deduction used:

The benefits of usage of Deduction THM in this proof is not obvious as before, which means there is nothing special here, so the complete proof would not show.

However, what needs attention is when you want to prove $\vdash A \leftrightarrow B$ in this system, proving $\{A\} \vdash B$, $\{B\} \vdash A$, getting $\vdash A \rightarrow B$, $\vdash B \rightarrow A$ and using MP rule with $(\mathrm{AS5})(A \rightarrow B) \rightarrow \big((B \rightarrow A) \rightarrow (A \leftrightarrow B)\big)$ is one possible method. Moreover, do **not** confound the **metalanguage** with **formal logic language** on **double-headed arrow** while reading Deduction THM!