

Relevant Logics: Proof Theory and Model Theory

- ◆ Formal Language of Relevant Logics
- ◆ Hilbert Style Axiomatic Systems of Relevant Logics
- ◆ Various Properties of Relevant Logics
- ◆ Model Theory for Relevant Logics
- ◆ Natural Deduction Systems of Relevant Logics
- ◆ Sequent Calculus Systems of Relevant Logics
- ◆ Semantic Tableau Systems of Relevant Logics
- ◆ Bibliography

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Pioneers' Seminal / Primitive Works

- ◆ W. Ackermann, "Begründung Einer Strengen Implikation," The Journal of Symbolic Logic, Vol. 21, pp. 113-128, 1956 (in German).
- ◆ S-K. Moh, "The Deduction Theorems and Two New Logical Systems," Methodos, Vol. 2, pp. 56-75, 1950.
- ◆ A. Church, "The Weak Theory of Implication," in A. Menne, A. Wilhelmy, and H. Angsil (Eds.), "Kontrolliertes Denken, Untersuchungen zum Logikkalkül und zur Logik der Einzelwissenschaften," pp. 22-37, 1951.
- ◆ I. E. Orlov, "The Calculus of Compatibility of Propositions," Matematicheskii Sbornik, Vol. 35, pp. 263-286, 1928 (in Russian). (Known by the community of relevant logic from a report in 1990 by K. Dosen)

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Major Reference Books on Relevant Logics

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- ◆ E. D. Mares, "Relevant Logic: A Philosophical Interpretation," Cambridge University Press, Cambridge, 2004. [M-RL-04]
- ◆ S. Read, "Relevant Logic: A Philosophical Examination of Inference," Basil Blackwell, Oxford, 1988, 2012 (Corrected Edition). [R-RL-12]

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- ◆ R. Routley, V. Plumwood, R. K. Meyer, and R. T. Brady, "Relevant Logics and their Rivals, Part I, The Basic Philosophical and Semantical Theory," Ridgeview, Atascadero, California, 1982.
- ◆ J. Norman and R. Sylvan (Eds.), "Directions in Relevant Logic," Kluwer Academic, Dordrecht, 1989.
- ◆ D. M. Gabbay and J. Woods, "Agenda Relevance: A Study in Formal Pragmatics," Elsevier, Amsterdam, 2003.
- ◆ R. Brady (Ed.), "Relevant Logics and their Rivals, Volume II, A Continuation of the Work of Richard Sylvan, Robert Meyer, Val Plumwood, and Ross Brady," Ashgate Publishing, Farnham, 2003.
- ◆ G. Restall, "An Introduction to Substructural Logics," Routledge, London, 2000.
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- ◆ E. D. Mares and R. K. Meyer, "Relevant Logics," in L. Goble (Ed.), "The Blackwell Guide to Philosophical Logic," pp. 280-308, Blackwell, Oxford, 2001. [M&M-RL-01]
- ◆ J. M. Dunn and G. Restall, "Relevance Logic," in D. Gabbay and F. Guenther (Eds.), "Handbook of Philosophical Logic, 2nd Edition," Vol. 6, pp. 1-128, Kluwer Academic, Dordrecht, 2002. [D&R-RL-02]
- ◆ E. D. Mares, "Relevance Logic," in D. Jacquette (Ed.), "A Companion to Philosophical Logic," pp. 609-627, Blackwell, Oxford, 2002. [M-RL-02]
- ◆ K. Bimbo, "Relevance Logics," in D. Jacquette (Ed.), "Philosophy of Logic," pp. 723-789, Elsevier, Amsterdam, 2007. [B-RL-07]
- ◆ E. D. Mares, "Relevance Logic," in Stanford Encyclopedia of Philosophy, Center for the Study of Language and Information (CSLI), Stanford University, 2012. [M-RL-12]

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Formal Language of Propositional Relevant Logics

♣ Alphabet (Symbols)

 $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, \top, \perp, (,), p_1, p_2, \dots, p_n, \dots \}$

♣ Extensional connectives

 $\neg, \rightarrow, \wedge, \vee, \leftrightarrow$

♣ Intensional connectives

 $\Rightarrow, \otimes, \oplus, \Leftrightarrow$

♣ Propositional constants (Propositional constant symbols)

 \top, \perp

♣ Propositional variables (Proposition symbols)

 $p_1, p_2, \dots, p_n, \dots$

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Formal Language of Predicate Relevant Logics

♣ Alphabet (Symbols)

 $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, \forall, \exists, \top, \perp, (,), x_1, x_2, \dots, x_n, \dots, c_1, c_2, \dots, c_n, \dots, f^1_1, \dots, f^1_n, \dots, f^2_1, \dots, f^2_n, \dots, f^k_1, \dots, f^k_n, \dots, p^0_1, \dots, p^0_n, \dots, p^1_1, \dots, p^1_n, \dots, p^2_1, \dots, p^2_n, \dots, p^k_1, \dots, p^k_n, \dots \}$

♣ Individual variables (Variable symbols)

 $x_1, x_2, \dots, x_n, \dots$

♣ Individual Constants/Names (Constant/Name symbols)

 $c_1, c_2, \dots, c_n, \dots$

♣ Individual Functions (Function symbols)

 $f^1_1, \dots, f^1_n, \dots, f^2_1, \dots, f^2_n, \dots, f^k_1, \dots, f^k_n, \dots$

♣ Individual Predicates/Relations (Predicate/relation symbols)

 $p^0_1, \dots, p^0_n, \dots, p^1_1, \dots, p^1_n, \dots, p^2_1, \dots, p^2_n, \dots, p^k_1, \dots, p^k_n, \dots$

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Formal Language of Relevant Logics

♣ Primitive logical connectives

\Rightarrow : entailment (**primitive!** And therefore, **intensional!**)
 \neg : negation
 \wedge : extensional conjunction

♣ Defined logical connectives

\otimes : intensional conjunction (fusion), $A \otimes B =_{df} \neg(A \Rightarrow \neg B)$
 \oplus : intensional disjunction, $A \oplus B =_{df} \neg A \Rightarrow B$
 \Leftrightarrow : intensional equivalence, $A \Leftrightarrow B =_{df} (A \Rightarrow B) \otimes (B \Rightarrow A)$
 \vee : extensional disjunction, $A \vee B =_{df} \neg(\neg A \wedge \neg B)$
 \rightarrow : material implication, $A \rightarrow B =_{df} \neg(A \wedge \neg B)$ or $\neg A \vee B$
 \leftrightarrow : extensional equivalence, $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$
 L : necessity operator, $LA =_{df} (A \Rightarrow A) \Rightarrow A$

♣ Terms and formulas

- ◆ Similar to that of classical propositional/predicate calculus.

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Axiom Schemata on Entailment

♣ Axiom schemata on entailment

E1 $A \Rightarrow A$ (Self-Implication)
E2 $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$ (Prefixing)
E2' $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ (Suffixing)
E3 $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (Contraction)
E3' $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ (Self-Distribution)
E3'' $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$ (Permuted Self-Distribution)
E4 $(A \Rightarrow ((B \Rightarrow C) \Rightarrow D)) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow D))$ (Restricted Permutation)
E4' $(A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$ (Restricted Assertion)
E4'' $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$ (Specialized Assertion)
E4''' $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \Rightarrow C) \Rightarrow D) \Rightarrow D)$
E5 $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (Permutation)
E5' $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ (Assertion)
E5'' $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$ (Demodalizer)

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Axiom Schemata on Conjunction and Disjunction

♣ Axiom schemata on conjunction

- C1 $(A \wedge B) \Rightarrow A$ (Conjunction Elimination)
 C2 $(A \wedge B) \Rightarrow B$ (Conjunction Elimination)
 C3 $((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$ (Conjunction Introduction)
 C4 $(L A \wedge L B) \Rightarrow L(A \wedge B)$, where $L A =_{df} (A \Rightarrow A) \Rightarrow A$ (Distribution of Necessity over Conjunction)

♣ Axiom schemata on disjunction

- D1 $A \Rightarrow (A \vee B)$ (Disjunction Introduction)
 D2 $B \Rightarrow (A \vee B)$ (Disjunction Introduction)
 D3 $((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$ (Disjunction Elimination)

♣ Distribution axiom schema

- DCD $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$ (Distribution of Conjunction over Disjunction)

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Axiom Schemata on Negation and Necessity

♣ Axiom schemata on negation

- N1 $(A \Rightarrow (\neg A)) \Rightarrow (\neg A)$ (Reduction)
 N2 $(A \Rightarrow (\neg B)) \Rightarrow (B \Rightarrow (\neg A))$ (Contraposition)
 N3 $(\neg(\neg A)) \Rightarrow A$ (Double Negation)

♣ Mingle axiom schemata

- EM0 $(A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow B))$
 RM0 $A \Rightarrow (A \Rightarrow A)$

♣ Axiom schemata on necessity

- L1 $L A \Rightarrow A$
 L2 $L(A \Rightarrow B) \Rightarrow (L A \Rightarrow L B)$
 L3 $(L A \wedge L B) \Rightarrow L(A \wedge B)$
 L4 $L A \Rightarrow L L A$
 L5 $L A \Rightarrow ((L A \Rightarrow L A) \Rightarrow L A)$

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Axiom Schemata on Individual Quantification

♣ Axiom schemata on individual quantification

- IQ1 $\forall x(A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B)$
 IQ2 $(\forall x A \wedge \forall x B) \Rightarrow \forall x(A \wedge B)$
 IQ3 $\forall x A x \Rightarrow A y$
 IQ4 $\forall x(A \Rightarrow B) \Rightarrow (A \Rightarrow \forall x B)$ (x not free in A)
 IQ5 $\forall x(A \vee B) \Rightarrow (A \vee \forall x B)$ (x not free in A)
 IQ6 $\forall x_1 \dots \forall x_n (((A \Rightarrow A) \Rightarrow B) \Rightarrow B)$ (for E and EM) ($n \geq 0$)
 IQ7 $A y \Rightarrow \exists x A x$
 IQ8 $\forall x(A \Rightarrow B) \Rightarrow (\exists x A \Rightarrow B)$ (x not free in B)
 IQ9 $(\exists x A \wedge B) \Rightarrow \exists x(A \wedge B)$ (x not free in B)

Axiom clause: if A is an axiom, so is $\forall x A$.

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Inference Rules

♣ Inference rules

- $\Rightarrow E$: From A and $A \Rightarrow B$ to infer B (Modus Ponens)
 $\wedge I$: From A and B to infer $A \wedge B$ (Adjunction)
 LI: If A is a theorem, so is LA
 $\otimes I$: If $A \Rightarrow (B \Rightarrow C)$ is a theorem, so is $(A \otimes B) \Rightarrow C$
 $\otimes E$: If $(A \otimes B) \Rightarrow C$ is a theorem, so is $A \Rightarrow (B \Rightarrow C)$
 TI: If A is a theorem, so is $T \Rightarrow A$
 TE: If $T \Rightarrow A$ is a theorem, so is A

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The Pure Entailment Fragments

♣ The pure entailment (relevant implication) fragments

- $T_{\Rightarrow} = \{E1, E2, E2', E3 \mid E3''\} + \Rightarrow E$
 $E_{\Rightarrow} = \{E1, E2 \mid E2', E3 \mid E3', E4 \mid E4'\} + \Rightarrow E$
 $E_{\Rightarrow} = \{E2', E3, E4''\} + \Rightarrow E$
 $E_{\Rightarrow} = \{E1, E3, E4'''\} + \Rightarrow E$
 $E_{\Rightarrow} = T_{\Rightarrow} + E4$ [E4: $(A \Rightarrow ((B \Rightarrow C) \Rightarrow D)) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow D))$]
 $E_{\Rightarrow} = T_{\Rightarrow} + E4'$ [E4': $(A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow C) \Rightarrow C$]
 $E_{\Rightarrow} = T_{\Rightarrow} + E4''$ [E4'': $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$]
 $R_{\Rightarrow} = \{E1, E2 \mid E2', E3 \mid E3', E5 \mid E5'\} + \Rightarrow E$
 $R_{\Rightarrow} = E_{\Rightarrow} + E5''$ [E5'': $A \Rightarrow ((A \Rightarrow A) \Rightarrow A) = A \Rightarrow L A$]

♣ Note

“A | B” means that one can choose any one of the two axiom schemata A and B.

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The Positive (Negation-free) Fragments

♣ The positive (negation-free) fragments

- $T_+ = T_{\Rightarrow} + \{C1 \sim C3, D1 \sim D3, DCD\} + \wedge I$
 $E_+ = E_{\Rightarrow} + \{C1 \sim C4, D1 \sim D3, DCD\} + \wedge I$
 $E_+ = T_+ + \{E4 \mid E4' \mid E4'', C4\}$
 $R_+ = R_{\Rightarrow} + \{C1 \sim C3, D1 \sim D3, DCD\} + \wedge I$
 $R_+ = E_+ + E5''$

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The Entailment with Negation Fragments

♣ The entailment (relevant implication) with negation fragments

$$T_{\Rightarrow, \neg} = T_{\Rightarrow} + \{N1, N2, N3\}$$

$$E_{\Rightarrow, \neg} = E_{\Rightarrow} + \{N1, N2, N3\}$$

$$R_{\Rightarrow, \neg} = R_{\Rightarrow} + \{N2, N3\}$$

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Hilbert Style Axiomatic Systems of Relevant Logics

♣ Propositional relevant logics

$$T = T_{\Rightarrow, \neg} + \{C1 \sim C3, D1 \sim D3, DCD\} + \Lambda I$$

$$E = E_{\Rightarrow, \neg} + \{C1 \sim C4, D1 \sim D3, DCD\} + \Lambda I$$

$$E = T + \{E4 \mid E4' \mid E4'', C4\}$$

$$R = R_{\Rightarrow, \neg} + \{C1 \sim C3, D1 \sim D3, DCD\} + \Lambda I$$

$$R = E + A \Rightarrow LA, LR = R - DCD$$

$$EM = E + EM0 \text{ (semi-relevant logic)}$$

$$RM = R + RM0 \text{ (semi-relevant logic)}$$

$$RM = EM + A \Rightarrow LA \text{ (semi-relevant logic)}$$

$$E^L = E + \{L1 \sim L5\} + LI$$

$$R^L = R + \{L1 \sim L4\} + LI$$

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Hilbert Style Axiomatic Systems of Relevant Logics

♣ Predicate relevant logics

$$SV^{\neg} = S + \{IQ1, IQ3, IQ4, IQ7, IQ8\}$$

$$\text{where } S = T_{\neg}, T_{\Rightarrow, \neg}, R_{\neg}, R_{\Rightarrow, \neg}$$

$$SV^{\neg} = S + \{IQ1 \sim IQ5, IQ7 \sim IQ9\}$$

$$\text{where } S = T_{+}, T, R_{+}, R, RM$$

$$SV^{\neg} = S + \{IQ1, IQ3, IQ4, IQ6 \sim IQ8\}$$

$$\text{where } S = E_{\neg}, E_{\Rightarrow, \neg}$$

$$SV^{\neg} = S + \{IQ1 \sim IQ9\}$$

$$\text{where } S = E_{+}, E, EM$$

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Hilbert Style Axiomatic Systems of Relevant Logics

♣ Axiom schemata on conjunction

$$C5 \quad (A \wedge A) \Rightarrow A$$

$$C6 \quad (A \wedge B) \Rightarrow (B \wedge A)$$

$$C7 \quad ((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$$

$$C8 \quad (A \wedge (A \Rightarrow B)) \Rightarrow B$$

$$C9: \neg(A \wedge \neg A)$$

$$C10: A \Rightarrow (B \Rightarrow (A \wedge B))$$

♣ Strong relevant logics

$$Tc =_{df} T_{\Rightarrow, \neg} + \{C3, C5 \sim C10\} \quad TcQ =_{df} Tc + \{IQ1 \sim IQ5\} + \forall I$$

$$Ec =_{df} E_{\Rightarrow, \neg} + \{C3 \sim C10\} \quad EcQ =_{df} Ec + \{IQ1 \sim IQ5\} + \forall I$$

$$Rc =_{df} R_{\Rightarrow, \neg} + \{C3, C5 \sim C10\} \quad RcQ =_{df} Rc + \{IQ1 \sim IQ5\} + \forall I$$

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The Notion of Degree

♣ The degree of a formula with implicational connective

- ◆ The *degree* of a formula with implicational connective is the largest number of nesting of implicational connective, e.g. \Rightarrow , that represents the notion of conditional within it.

♣ Zero degree formula

- ◆ A formula is called a *zero degree formula (zdf)* if and only if there is no occurrence of \Rightarrow in it.

♣ First degree conditional (entailment)

- ◆ A formula of the form $A \Rightarrow B$ is called a *first degree conditional (fdc) (entailment, fde)* if and only if both A and B are zero degree formulas.

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The Notion of Degree

♣ First degree formula

- ◆ A formula A is called a **first degree formula (fdf)** if and only if it satisfies the one of the following conditions:
 - (1) A is a first degree conditional,
 - (2) A is in the form $+B$ ($+$ is a one-place connective such as negation and so on) where B is a first degree formula, and
 - (3) A is in the form B^*C ($*$ is a non-implicational two-place connective such as conjunction or disjunction and so on) where both of B and C is first degree formulas, or one of B and C is a first degree formula and the another is a zero degree formula.

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The Notion of Degree

♣ k^{th} degree conditional (entailment)

- ◆ Let k be a natural number. A formula of the form $A \Rightarrow B$ is called a **k^{th} degree conditional (kdc) (entailment, kde)** if and only if both A and B are $(k-1)^{\text{th}}$ degree formulas, or one of A and B is a $(k-1)^{\text{th}}$ degree formula and another is a j^{th} ($j < k-1$) degree formula.

♣ k^{th} degree formula

- ◆ Let k be a natural number. A formula A is called a **k^{th} degree formula (kdf)** if and only if it satisfies the one of the following conditions:
 - (1) A is a k^{th} degree conditional,
 - (2) A is in the form $+B$ where B is a k^{th} degree formula, and
 - (3) A is in the form B^*C where both of B and C is k^{th} degree formulas, or one of B and C is a k^{th} degree formula and another is a j^{th} ($j < k$) degree formula.

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Zero Degree Fragments of Relevant Logics: CPC

♣ Classical Propositional Calculus (CPC) is contained in $E(R, T)$ [A&B-E1-75]

- ◆ Theorem: All tautologies (theorems) of CPC are provable in $E(R, T)$, i.e., $E(R, T)$ is complete with respect to CPC.
- ◆ Theorem: Only tautologies (theorems) of CPC among the zero degree formulas of $E(R, T)$ are provable in $E(R, T)$, i.e., $E(R, T)$ is conservative with respect to CPC.

♣ Classical Propositional Calculus (CPC) is the zero degree fragment of $E(R, T)$ [A&B-E1-75]

- ◆ CPC is in exactly the right sense contained in $E(R, T)$, i.e., it is exactly the zero degree (extensional) fragment of $E(R, T)$.

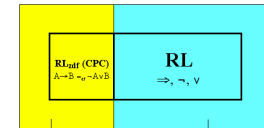
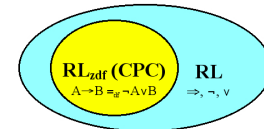
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Zero Degree Fragments of Relevant Logics: CPC

Zero degree formulas
Extensional formulasHigh degree formulas
Intensional formulas

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Primitive Entailments

♣ Atom

- ◆ An **atom** is a propositional variable or the negate of such, i.e., an atom has either the form p or the form $\neg p$.

♣ Primitive conjunction and disjunction

- ◆ A **primitive conjunction** is a conjunction $A_1 \wedge A_2 \wedge \dots \wedge A_m$ ($m \geq 1$) where each A_i is an atom.
- ◆ A **primitive disjunction** is a disjunction $B_1 \vee B_2 \vee \dots \vee B_n$ ($n \geq 1$) where each B_i is an atom.

♣ Primitive entailment

- ◆ $A \Rightarrow B$ is a **primitive entailment** if A is a primitive conjunction and B is a primitive disjunction, i.e., $A \Rightarrow B = (A_1 \wedge A_2 \wedge \dots \wedge A_m) \Rightarrow (B_1 \vee B_2 \vee \dots \vee B_n)$.

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Explicitly Tautological Entailments

♣ Explicitly tautological entailments

- ◆ A primitive entailment $A \Rightarrow B$ is said to be **explicitly tautological** if some (conjoined) atom of A is identical with some (disjoined) atom of B , i.e., $A_i = B_j$ for some i and j .
- ◆ Explicitly tautological is both necessary and sufficient for the (**weak-relevant!**) validity of a primitive entailment.

♣ Note

- ◆ Explicitly tautological entailments satisfy the von Wright-Geach-Smiley criterion for entailment: every explicitly tautological entailment answers to a material "implication" which is a substitution instance of a tautologous material "implication" with non-contradictory antecedent and non-tautologous consequent; and evidently we may ascertain the truth of the entailment without coming to know the truth of the consequent or the falsity of the antecedent.

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Explicitly Tautological Entailments in Normal Form

♣ Entailments in normal form

- ◆ An entailment $A \Rightarrow B$ is said to be **in normal form** if it has the form $(A_1 \vee A_2 \vee \dots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \dots \wedge B_n)$ ($m \geq 1, n \geq 1$) where each A_i is a primitive conjunction and each B_j is a primitive disjunction.
- ◆ An entailment $A \Rightarrow B$ in normal form is (**weak-relevantly!**) valid just in case each $A_i \Rightarrow B_j$ ($m \geq i \geq 1, n \geq j \geq 1$) is explicitly tautological.

♣ Explicitly tautological entailments in normal form

- ◆ An entailment $(A_1 \vee A_2 \vee \dots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \dots \wedge B_n)$ ($m \geq 1, n \geq 1$) in normal form is said to be **explicitly tautological** if and only if for every A_i and B_j , $A_i \Rightarrow B_j$ ($m \geq i \geq 1, n \geq j \geq 1$) is explicitly tautological.

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Tautological Entailments

♣ Tautological entailments

- ◆ An entailment $(A_1 \vee A_2 \vee \dots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \dots \wedge B_n)$ ($m \geq 1, n \geq 1$) in normal form is called a **tautological entailment** if and only if it is explicitly tautological, i.e., for every A_i and B_j , $A_i \Rightarrow B_j$ ($m \geq i \geq 1, n \geq j \geq 1$) is explicitly tautological.
- ◆ Explicitly tautological is both necessary and sufficient for the (**weak-relevant!**) validity of a first degree entailment.

♣ Fundamental question

- ◆ Can we construct a formal calculus of tautological entailments?
- ◆ Answer: YES

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Tautological Entailment Fragments of Relevant Logics

♣ E_{fde} (R_{fde} , T_{fde}): First degree entailment fragment of E (R, T)

- ◆ Entailment:
Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$
- ◆ Conjunction:
Axiom schemata: $(A \wedge B) \Rightarrow A$ (C1), $(A \wedge B) \Rightarrow B$ (C2)
Inference rule: from $A \Rightarrow B$ and $A \Rightarrow C$ to infer $A \Rightarrow (B \wedge C)$
- ◆ Disjunction:
Axiom schemata: $A \Rightarrow (A \vee B)$ (D1), $B \Rightarrow (A \vee B)$ (D2)
Inference rule: from $A \Rightarrow C$ and $B \Rightarrow C$ to infer $(A \vee B) \Rightarrow C$
- ◆ Distribution:
Axiom schema: $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$ (DCD)
- ◆ Negation:
Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$

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First Degree Entailment Fragments of Relevant Logics

♣ E_{fde} (R_{fde} , T_{fde}): First degree entailment fragment of E (R, T)

- ◆ Entailment:
Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$
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Inference rule: from $A \Rightarrow B$ and $A \Rightarrow C$ to infer $A \Rightarrow (B \wedge C)$
- ◆ Disjunction:
Axiom schemata: $A \Rightarrow (A \vee B)$ (D1), $B \Rightarrow (A \vee B)$ (D2)
Inference rule: from $A \Rightarrow C$ and $B \Rightarrow C$ to infer $(A \vee B) \Rightarrow C$
- ◆ Distribution:
Axiom schema: $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$ (DCD)
- ◆ Negation:
Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$

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First Degree Formula Fragments of Relevant Logics

♣ E_{fdr} and tautological entailments

- ◆ All logical theorems of E_{fdr} are first degree entailments.
- ◆ Every tautological entailment is provable in E_{fdr} .
- ◆ Only tautological entailments are provable in E_{fdr} .
- ◆ Therefore, E_{fdr} is a formalization of tautological entailments.

♣ Perfect interpolation theorem

- ◆ If $A \Rightarrow C$ is provable in E_{fdr} , then there is an "interpolation formula" B such that (1) $A \Rightarrow B$ is provable in E_{fdr} , (2) $B \Rightarrow C$ is provable in E_{fdr} , and (3) B has no variables not in both A and C .

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Antecedent and Consequent Parts of Formulas

♣ Antecedent/Consequent parts of formulas

- ◆ A is a consequent part of A .
- ◆ If $\neg B$ is a **consequent** {**antecedent**} **part** of A , then B is an antecedent part {consequent part} of A .
- ◆ If $B \Rightarrow C$ is a consequent {antecedent} part of A , then B is an antecedent {consequent} part of A , and C is a consequent {antecedent} part of A .
- ◆ If either $B \wedge C$ or $B \vee C$ is a consequent {antecedent} part of A , then both B and C are consequent {antecedent} parts of A .

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Theorems of Variable-Sharing in Relevant Logics

♣ Variable-Sharing in $E_{\Rightarrow}(R_{\Rightarrow}, T_{\Rightarrow})$ [A&B-E1-75]

- ◆ If $A \Rightarrow B$ is provable in $E_{\Rightarrow}(R_{\Rightarrow}, T_{\Rightarrow})$, then A and B share a sentential variable.
- ◆ If A is a theorem of $E_{\Rightarrow}(R_{\Rightarrow}, T_{\Rightarrow})$, then every sentential variable occurring in A occurs at least once as an antecedent part and at least once as a consequent part of A .

♣ Variable-Sharing in $E_{\Rightarrow, \neg}(R_{\Rightarrow, \neg}, T_{\Rightarrow, \neg})$ [A&B-E1-75]

- ◆ If $A \Rightarrow B$ is provable in $E_{\Rightarrow, \neg}(R_{\Rightarrow, \neg}, T_{\Rightarrow, \neg})$, then A and B share a sentential variable.
- ◆ If A is a theorem of $E_{\Rightarrow, \neg}(R_{\Rightarrow, \neg}, T_{\Rightarrow, \neg})$, then every sentential variable occurring in A occurs at least once as an antecedent part and at least once as a consequent part of A (Note: This is NOT true for E (or for R , or for T).

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Theorems of Variable-Sharing in Relevant Logics

♣ Variable-Sharing in $E(R, T)$ [A&B-E1-75]

- ◆ If $A \Rightarrow B$ is provable in $E(R, T)$, then A and B share a sentential variable.
- ◆ If $A \Rightarrow B$ is a theorem of E , then some sentential variable occurs as an antecedent part of both A and B , or else as a consequent part of both A and B .
- ◆ If A is a theorem of E containing no conjunctions as antecedent parts and no disjunctions as consequent parts, then every sentential variable in A occurs at least once as an antecedent part and at least once as a consequent part [Maksimova, 1967].

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Theorems of Variable-Sharing in Relevant Logics

♣ Variable-Sharing in CPC [A&B-E1-75]

- ◆ If $A \rightarrow B$ is provable in CPC, then either (1) A and B share a sentential variable or (2) either $\neg A$ or B is provable in CPC.

♣ Variable-Sharing in RM [A&B-E1-75]

- ◆ If $A \rightarrow B$ is provable in RM, then either (1) A and B share a sentential variable or (2) both $\neg A$ and B are provable in RM.

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Facts in Relevant Logics

♣ Why the inference rule of adjunction?

- ◆ $A \Rightarrow (B \Rightarrow (A \wedge B))$ is not a logical theorem of $E(R, T)$.
- ◆ $A \Rightarrow (B \Rightarrow (A \wedge B))$ is a familiar axiom for intuitionistic and classical logic, but it is only a hair's breadth away from positive paradox $A \Rightarrow (B \Rightarrow A)$, and indeed yields it given $(A \wedge B) \Rightarrow A$ and $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$.

♣ The disjunctive syllogism and the inference rule γ

- ◆ $(A \wedge (\neg A \vee B)) \Rightarrow B$ (or $(\neg A \wedge (A \vee B)) \Rightarrow B$) is not a logical theorem of either $E(R, T)$. This is the most notorious feature of relevant logic.
- ◆ The inference rule γ , i.e., suppose $\vdash A$ and $\vdash \neg A \vee B$, then $\vdash B$, is admissible in E, R , and many others.

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The Relationship between Relevant Logics and Others

♣ The relationship between relevant logics and other logics

$S5 \rightarrow S4 \rightarrow S3 \rightarrow EM$
 $CML \rightarrow \quad \quad \quad \rightarrow E \rightarrow T$
 $RM \rightarrow \quad \quad \quad R$
 (S3, S4, S5: Lewis's modal systems)
 (\rightarrow means the inclusion relationship)

♣ The relationship between relevant logics and their first degree entailment fragments

- ◆ If $A \Rightarrow B$ is provable in $E_{fde}(R_{fde}, T_{fde})$, then it is also provable in $E(R, T)$ [A&B-E1-75].

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Facts in Relevant Logics

♣ Conservative extensions

- ◆ Let S' is an extension of S in the sense that S' has some new language components, e.g., connectives, or axioms, or inference rules. S' is called to be a *conservative extension* of S if for any formula A in the notation of S , if A is provable in S' then A is also provable in S .

♣ Conservative extensions in relevant logics

- ◆ Both E and $E_{\Rightarrow, \neg}$ are conservative extensions of E_{\Rightarrow}
- ◆ Both R and $R_{\Rightarrow, \neg}$ are conservative extensions of R_{\Rightarrow}

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Facts in Relevant Logics

♣ Conservative extensions in relevant logics

- ◆ RM is not a conservative extension of $RM0_{\Rightarrow}$
($RM0_{\Rightarrow} = R_{\Rightarrow} + RM0$)
- ◆ $RM0_{\Rightarrow}$ is not the pure implicational fragment of RM
- ◆ RM does not satisfy the relevance principle but it does satisfy the weaker relevance principle that $A \rightarrow B$ is a theorem of RM only if either A and B share a sentential variable or both $\neg A$ and B are theorems.

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Facts in Relevant Logics

♣ Necessity

- ◆ In E, the necessity operator L can be defined as $LA =_{df} (A \Rightarrow A) \Rightarrow A$, but this is impossible in R because R has $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ as an axiom scheme.
- ◆ E is both a relevant logic and a modal logic but R is only a relevant logic.
- ◆ The rule of necessity (if A is provable in E, then LA is also provable in E) is naturally holds in E, and therefore no new logical primitives need be introduced to get the desired effect.

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Why R Is Interesting?

♣ Major reason 1: Age

- ◆ The pure implicational fragment $R \Rightarrow$ of R, first considered by Moh Shaw-Kwei in 1950 and by A. Church in 1951, was regarded to be the oldest of the relevant logics [A&B-E1-75].
- ◆ The implication-negation fragment $R_{\Rightarrow, \neg}$ of R, given by I. E. Orlov in 1928, is the oldest of the relevant logics [Dosen, 1990].

♣ Major reason 2: Isolating relevance

- ◆ In R one has an even clearer view of relevance than in E, just because of the absence of modal complications.

♣ Other reasons

- ◆ Stability, Richness, Easy proof theory, Fragments, Applicability, Extensibility.

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Decision Problems in Relevant Logic

- ◆ Both E_{\Rightarrow} and R_{\Rightarrow} are decidable [Kripke, 1959].
- ◆ LR is decidable [Mayer, 1966].
- ◆ E_{fde} (R_{fde} , T_{fde}) is decidable [Anderson and Belnap, 1975].
- ◆ E_{fdf} is decidable [Anderson and Belnap, 1975].
- ◆ Both $E_{\Rightarrow, \neg}$ and $R_{\Rightarrow, \neg}$ are decidable [Anderson and Belnap, 1975].
- ◆ RM is decidable [Anderson and Belnap, 1975].
- ◆ E_+ , R_+ , and T_+ are undecidable [Urquhart, 1982].
- ◆ E, R, and T are undecidable [Urquhart, 1982].
- ◆ The decision problem of T_{\Rightarrow} (or of $T_{\Rightarrow, \neg}$) is open.

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Relevant Logics: Proof Theory and Model Theory

- ◆ Formal Language of Relevant Logics
- ◆ Hilbert Style Axiomatic Systems of Relevant Logics
- ◆ Various Properties of Relevant Logics
- ◆ Model Theory for Relevant Logics
- ◆ Natural Deduction Systems of Relevant Logics
- ◆ Sequent Calculus Systems of Relevant Logics
- ◆ Semantic Tableau Systems of Relevant Logics
- ◆ Bibliography

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