Formal Logic Systems and Formal Theories

Jingde Cheng

Saitama University

Logic: What Is It and Why Study It ?

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"Logic is the science of sciences, and the art of arts."

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-- John Duns Scotus, 13th century.

"Nothing can be more important than the art of formal reasoning according to true logic."

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-- Gottfried Wilhelm Leibniz

"Logic is the basis for all other sciences"

- "There is a special discipline, called logic, which is considered to be the basis for all other sciences."
  "Logic evolved into an independent science long ago, earlier even than arithmetic and geometry."
  -- A. Tarski, 1941.
- "Mathematical Logic, it is a science prior to all others, which contains the ideas and principles underlying all sciences."
  -- K. Gödel, 1944.

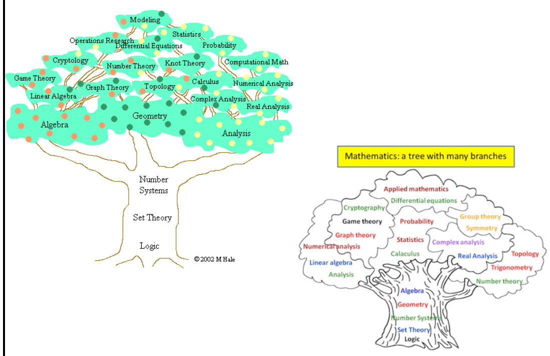
"Logic is the basis for all other sciences"

- "The development of Western Science has been based on two great achievements, the invention of the formal logical system (in Euclidean geometry) by the Greek philosophers, and the discovery of the possibility of finding out causal relationships by systematic experiment (at the Renaissance)."
  -- A. Einstein, 1953.

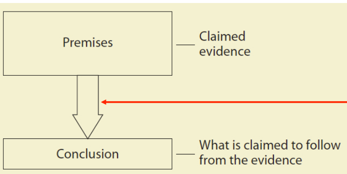
"Fields of Science and Technology" by UNESCO

- "Proposed International Standard Nomenclature for Fields of Science and Technology," UNESCO/ NS/ROU/257 rev.1, 1988.
- 11. Logic, 12. Mathematics
- 21. Astronomy and Astrophysics, 22. Physics, 23. Chemistry, 24. Life Sciences, 25. Earth and Space Science
- 31. Agricultural Sciences, 32. Medical Sciences, 33. Technological Sciences
- 1203. Computer Science
  3304. Computer Technology

Logic as the Fundamental Basis for all Mathematics



Logic: What Is It?



[From P. J. Hurley, "A Concise Introduction to Logic"]

What entails what?

What follows from what?

Why? What are the evaluation criteria?

How to establish/define the evaluation criteria?

How to evaluate arguments/reasoning?

It is LOGIC to answer these fundamental questions.

Major Reference Books on Mathematical Logic

- E. Mendelson, "Introduction to Mathematical Logic," Chapman & Hall, 1964, 1979, 1987, 1997, 2010, 2015 (6th Edition). [Mendelson]
- R. M. Smullyan, "A Beginner's Guide to Mathematical Logic," Dover Publications, 2014; "A Beginner's Further Guide to Mathematical Logic," World Scientific, 2017. [Smullyan]
- D. van Dalen, "Logic and Structure," Springer, 1980, 1983, 1994, 2004, 2013 (5th Edition). [Dalen]
- W. Routenberg, "A Concise Introduction to Mathematical Logic," Springer, 1979, 2006, 2010 (3rd Edition). [Routenberg]
- G. S. Boolos, J. P. Burgess, and R. C. Jeffrey, "Computability and Logic," Cambridge University Press, 1974, 1980, 1990, 2002, 2007 (5th Edition). [B&B&J]
- H. B. Enderton, "A Mathematical Introduction to Logic," Academic Press, 1972, 2001 (2nd Edition). [Enderton]
- H.-D. Ebbinghaus, J. Flum, and W. Thomas, "Mathematical Logic," Springer, 1978, 1994 (2nd Edition). [E&F&T]
- A. G. Hamilton, "Logic for Mathematicians," Cambridge University Press, 1978, 1989 (2nd Revised Edition). [Hamilton]

Major Reference Books on Mathematical Logic for CS

- J. H. Gallier, "Logic for Computer Science -- Foundations of Automatic Theorem Proving," Harper & Row, 1986, 2003 (Revised Online Edition), Dover Publications, 2015 (2nd Edition). [Gallier]
- M. Ben-Ari, "Mathematical Logic for Computer Science," Springer, 1993, 2001, 2012 (3rd Edition). [Ben-Ari]
- M. Huth and M. Ryan, "Logic in Computer Science: Modelling and Reasoning about Systems," 2004 (2nd Edition). [H&R]
- S. Reeves and M. Clarke, "Logic for Computer Science," Addison-Wesly, 1990-2003. [R&C]
- A. Nerode and R. A. Shore, "Logic for Applications," Springer, 1993, 1997 (2nd Edition). [N&S]
- M. Fitting, "First-Order Logic and Automated Theorem Proving," Springer, 1990, 1996 (2nd Edition). [Fitting]

Classical/Important Reference Books on Mathematical Logic

- B. Meltzer (translation) and R. B. Braithwaite (Introduction), K. Gödel, "On formally undecidable propositions of Principia Mathematica and related systems I," Basic Books, 1962, Dover Publications, 1992. [Gödel]
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- H. DeLong, "A Profile of Mathematical Logic," Addison-Wesley, 1970, Dover Publications, 2004.
- J. R. Shoenfield, "Mathematical Logic," Association for Symbolic Logic / Addison-Wesley, 1967.
- P. H. Nidditch, "The Development of Mathematical Logic," Thoemmes, 1962.

Formal Logic Systems and Formal Theories

- **Formal Logic Systems**
- **Formal Theories**
- **Model (Semantic) Theory**
- **Proof (Syntactic) Theory**
- **The Limitations of Formal Logic Systems/ Theories (Gödel's Incompleteness Theorems)**

Formal Logic Systems: What Are They ?

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Formal Logic Systems: What are They?

♣ **Formal logic systems as formalized logical validity criteria**

- **A *formal logic system* formally defines a *logical correctness/ validity criterion* of formal reasoning.**

♣ **Basic requirements for formal logic systems**

- **Different formal logic systems may have different formal languages; different formal languages may have different symbol sets.**
- **A formal logic system must define a formal language explicitly and unambiguously.**
- **Different formal logic systems must define different logical consequence relations.**
- **A formal logic system must define a logical consequence relation explicitly and unambiguously.**

Formal Logic Systems: What are They?

♣ **Two indispensable parts of any formal logic system**

- **A *formal language* (call the '*object language*') for representing various things formally.**
- **A *logical consequence relation* defined among the formulas of the formal language for defining the logical correctness/ validity of formal reasoning.**

♣ **Notes**

- **Different formal logic systems may have different formal languages; different formal languages may have different symbol sets.**
- **Different formal logic systems must define different logical consequence relations.**
- **A formal logic system may define its logical consequence relation in some different ways.**

Formal Logic Systems: Formal Definition

♣ **Formal logic system**

- ***Formal logic system*** **$L =_{df} (F(L), \vdash_L)$ where $\vdash_L =_{df} 2^{F(L)} \rightarrow F(L)$.**
- **$F(L)$: The *formal language* (call the '*object language*') of L, which is the set of all *well-formed formulas* of L.**
- **$\vdash_L$: The *logical consequence relation* defined among the formulas of $F(L)$, such that for $P \subseteq F(L)$ as the premises and $C \in F(L)$ as a conclusion, $P \vdash_L C$ means that within the framework of L, C *validly follows from* P, or equivalently, P *validly entails* C.**

♣ **Notes**

- **'$\vdash_L$' is read as 'the turnstile with subscript L'.**
- **Both $F(L)$ and $\vdash_L$ have to be defined in detail.**

Formal Logic Systems: Various Representations

♣ **Representations of formal logic systems**

- **A formal logic system can be represented by different forms and styles.**
- **Hilbert style axiomatic systems**
- **Gentzen natural deduction systems**
- **Gentzen sequent calculus systems**
- **Semantic tableau systems**
- **Resolution Systems**
- **……**

♣ **A fundamental question**

- **The equivalence between different representation forms of the same logic system?**

Formal Logic Systems: Logic Theorems

♣ Logic theorems

- For a formal logic system L = (F(L), $\vdash_L$), a *logical theorem t* of L is a formula such that $\varnothing \vdash_L t$ where $\varnothing$ is the empty set.
- A logical theorem of a formal logic system validly holds, without premises, in that logic system.
- *Th*(L): the set of all logical theorems of L.

♣ Notes

- *Th*(L) is completely determined by $\vdash_L$.
- It is *Th*(L) that characterizes the logic system L, i.e., if *Th*(L) = *Th*(L′), then we consider the L and L′ to be the same logic system.

Explosive and Paraconsistent Logic Systems

♣ Explosive logic systems

- A formal logic system L = (*F*(L), $\vdash_L$) is said to be *explosive* IFF $\{A, \neg A\} \vdash_L B$ for any $A, B \in F(L)$ and $A \neq B$.
- In an *explosive logic system*, anything is validly follows from premises with a contradiction, or equivalently, premises with a contradiction validly entails anything.
- Note: An explosive logic system is useful only if it is used to deal with those reasoning with contradiction-free premises.

♣ Paraconsistent logic systems

- A formal logic system L is said to be *paraconsistent* IFF it is not explosive, i.e., for some premises with a contradiction, there is at least one formula that does NOT follow from the premises.

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Formal Theories: What Are They ?

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Formal Theories: What are They?

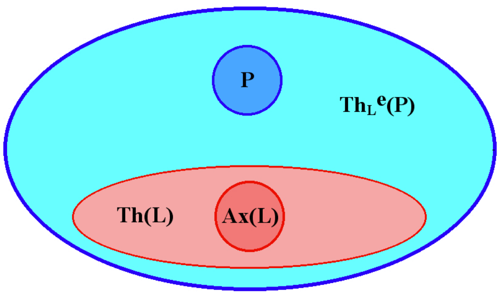
♣ Formal theory as a formal logical representation/model

- A *formal theory* is a *formal logical representation/model* of an area in the real world characterized by empirical premises *P* in a symbolic logic world characterized by a formal logic system L.

♣ Formal theories based on different logic systems

- An area in the real world characterized by empirical premises *P* may be represented/modeled by different formal theories characterized by different logic systems.
- A formal logic system L can underlie different areas in the real world characterized by different empirical premises *P*, and therefore, can underlie different formal theories.

Formal Theories



Formal Theories: Formal Definition

♣ L-theory with premises *P*

- Let L = (*F*(L), $\vdash_L$) be a formal logic system and $P \subseteq F(L)$ be a non-empty set of *propositions / closed well-formed formulas*.
- A *formal theory* with premises *P* based on L, called a *L-theory with premises P* and denoted by $T_L(P)$, is defined as

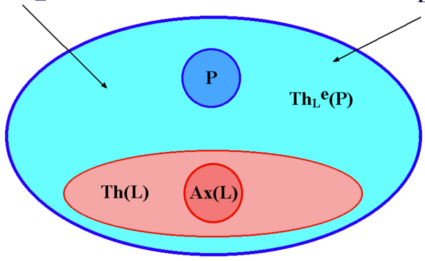
$$T_L(P) =_{df} Th(L) \cup Th_L^e(P)$$
$$Th_L^e(P) =_{df} \{ et \mid P \vdash_L et \text{ and } et \notin Th(L)\}$$

- *P*: The *empirical premises / axioms*.
- *Th*(L): The *logical part* of the formal theory, any element of *Th*(L) is called a *logical theorem* of that formal theory.
- $Th_L^e(P)$: The *empirical part* of the formal theory, any element of $Th_L^e(P)$ is called an *empirical theorem* of that formal theory.

Reasoning Empirical Theorems vs Proving Empirical Theorems

For given *P* For given *P* and *t*,

$? \in Th_L^e(P)$ $t \in Th_L^e(P)$?



Consistent and Inconsistent Formal Theories

♣ Inconsistent formal theories

- $T_L(P)$ is said to be *directly/explicitly inconsistent* IFF there exists a formula $A \in F(L)$ such that both $A \in P$ and $\neg A \in P$.
- $T_L(P)$ is said to be *indirectly/implicitly inconsistent* IFF it is not directly inconsistent but there exists a formula $A \in F(L)$ such that both $A \in T_L(P)$ and $\neg A \in T_L(P)$.

♣ Consistent formal theories

- $T_L(P)$ is said to be *consistent* IFF it is neither directly/explicitly inconsistent nor indirectly/implicitly inconsistent.

♣ Notes

- The consistency of *P* cannot ensure/guarantee the consistency of $T_L(P)$; a formal theory may be directly consistent but is indirectly inconsistent.

Explosive and Paraconsistent Formal Theories

♣ Explosive formal theories

- $T_L(P)$ is said to be *explosive* IFF $A \in T_L(P)$ for arbitrary formula $A \in F(L)$.
- If L is explosive, then any directly/explicitly or indirectly/implicitly inconsistent $T_L(P)$ must be explosive.
- Any explosive formal theory is NOT useful at all.
- If L is explosive, then only consistent $T_L(P)$ may be useful.

♣ Paraconsistent formal theories

- $T_L(P)$ is said to be *paraconsistent* IFF it is not explosive, i.e., there is at least one formula $B \in F(L)$ such that $B \notin T_L(L)$.
- Note: Any useful formal theory should be paraconsistent.

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Model (Semantic) Theory: What Is It ?

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Model (Semantic) Theory vs Proof (Syntactic) Theory

- **Two ways to study logic**
  - Logic can be studied and/or represented in two ways: *philosophical or model-theoretical way*, and *formal or proof-theoretical way*.
- **Model (Semantic) theory: What is it?**
  - Model theory is the study of formal languages/systems and their philosophical/mathematical interpretations.
  - Before CML was established in 1930s, logic was studied and interpreted with natural language, and after CML, now logic is usually studied and interpreted with mathematical structure.
- **Proof (Syntactic) Theory: What is it?**
  - Proof theory is the study of formal axiomatic systems and formal deduction structures.

Model (Semantic) Theory: What is It?

- **Proposition/sentence and its interpretation**
  - Sometimes we write or speak a proposition/sentence p that expresses nothing either true or false, because some crucial information is missing about what the words mean.
  - If we go on to add this information, so that p comes to express a true or false statement, we are said to interpret p, and the added information is called an *interpretation* of p.
- **Examples**
  - (1) All A are B. (2) C is a A. Therefore, (3) C is B.
  - $(A \wedge (A \Rightarrow B)) \Rightarrow B$: A and $A \Rightarrow B$, therefore, B.
  - All humans are mortal. Socrates is a human. Therefore, Socrates is mortal.
  - A is true, and $A \Rightarrow B$ is true, therefore, B is true.

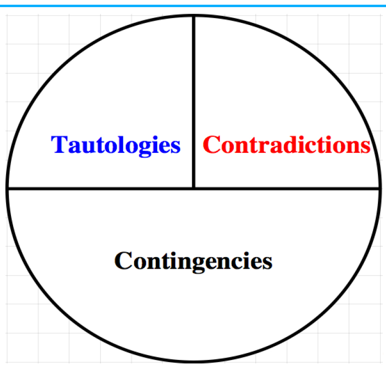
Model (Semantic) Theory: What is It?

- **Models of a formula**
  - If the interpretation M happens to make formula/ proposition/sentence p state something *true*, we say that M is a *model* of p, or that M *satisfies* p.
  - For a model M and a formula A, M *satisfies* A or A is interpreted to be *true* in M, is denoted as $\models_M A$; M *does not satisfy* A or A is interpreted to be *false* in M , is denoted as $\not\models_M A$. (Note: There may be some models satisfying a formula.)
- **Models of a set of formulas**
  - For a model M and a set Γ of formulas, if M *satisfies* every formula of Γ or every formula of Γ is interpreted to be *true* in M, then we say that M is a *model* of Γ and is denoted as $\models_M \Gamma$.

Model (Semantic) Theory: Tautologies, Contradictions, and Contingencies

- **Tautologies, contradictions, and contingencies**
  - A formula $A \in F(\mathrm{L})$ is a *tautology* (*universal truth*) of logic L, written as $\models_{\mathrm{L}} A$, IFF $\models_M A$ for any model M of L; a formula $A \in F(\mathrm{L})$ is a *contradiction* (*universal falsehood*) of logic L, written as $\not\models_{\mathrm{L}} A$, IFF $\not\models_M A$ for any model M of L; a formula is a *contingency* of logic L, IFF it is neither a tautology nor a contradiction of L.
- **Notes**
  - A formula must be any one of tautology (universal truth), contradiction (universal falsehood), and contingency.
  - A good/important characteristic of a formal logic system: The set of all tautologies of L is just the set of logical theorems of L.

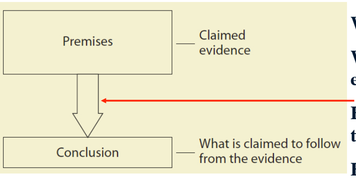
Model (Semantic) Theory: Tautologies, Contradictions, and Contingencies



Logical Semantics

- **Relations between formula expressions**
  - Relations between formula expressions in the formal language of a formal logic system can be determined in terms of properties that reach beyond language, through the expressions' interpretation and/or evaluation.
  - In general, these properties concerns the model (semantic) theory of that logic.
- **Logical semantics**
  - To speak of the validity of a deductive argument in terms of the relationship of possible interpretations of the premises and the conclusion is to describe/define validity semantically.

Logic: What Is It?



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(Semantic) Logical Consequence Relation

♣ **(Semantic) logical consequence relation**

- **To define the *(semantic) logical consequence relation* among formula expressions, we usually describe a model, which is a structure in terms of which formulas are interpreted/ evaluated, and rules for their interpretation/evaluation in a model.**
- **Then one might say that a formula, *C*, is a (semantic) logical consequence of others, *P*, or that the inference from *P* to *C* is semantically valid, just in case every model that satisfies every formulas in *P* also satisfies *C*.**

Semantic/Model-theoretical Logical Consequence Relation

♣ **Semantic/model-theoretical logical consequence relation**

- **For $P \subseteq F(\mathrm{L})$, $C \in F(\mathrm{L})$, *P semantically/model-theoretically/logically entails C*, or *C semantically/model-theoretically/logically follows from P*, or *C* is a *semantic/model-theoretical logical consequence* of *P*, written as $P \models_{\mathrm{L}} C$, IFF $\models_{M_{\mathrm{L}}} C$ for any model M_{L} of *P*.**
- **$\varnothing \models_{\mathrm{L}} C$ means that $\models_{M_{\mathrm{L}}} C$ for any model M_{L}, i.e., *C* is a tautology of L.**

♣ **Note**

- **The semantic/model-theoretical logical consequence relation is a model-theoretical formalization of the notion that one proposition follows from another or others.**

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Proof (Syntactic) Theory: What Is It ?

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♣ **Proof (Syntactic) theory**

- **Proof theory is the study of the general structure of deductive proofs, and of arguments with demonstrative force as encountered in logic.**
- **The idea of such demonstrative arguments, i.e., ones the conclusion of which follows necessarily from the assumptions made, is central in Aristotle's Analytica Posteriora: a deductive science is organized around a number of basic concepts that are assumed understood without further explanation, and a number of basic truths or axioms that are seen as true immediately.**

♣ **Proofs of theorems**

- **A theorem is deduced from axioms, as the last formula of a FINITE sequence of deduction.**

Proof (Syntactic) Theory: Deductive Logical Consequence Relation

♣ **Syntax**

- **Relationships between formula expressions in the formal language of a formal logic system can be characterized in terms of the syntatical structure of the expressions themselves.**

♣ **Syntactic (deductive) logical consequence relation**

- **The most important relation, *syntactic (deductive) logical consequence relation*, is the relation of *derivability* or *deducibility*, i.e., an expression can be drawn from others.**
- **A formula, *C*, is derivable from a set of formulas, *P*, if there is a *deductive proof* of *C* from *P* where a deductive proof is a structure of formulas that meets specific conditions.**
- **Here "proof", and hence "derivability", must be understood as relative to a particular deductive system or calculus that determines those conditions.**

Proof (Syntactic) Theory: Formal Logic (Deductive) System

♣ **Formal logic (deductive) system**

- **A *formal logic (deductive) system* has the following components:**
  (1) *alphabet*: a non-empty set of symbols,
  (2) *grammar*: a finite set of rules for forming formulas,
  (3) *axioms*: a set of formulas as start points for deduction,
  (4) *deduction (inference) rules*: a finite set of rules for generating a "new" formula (the consequence) from some "old" formulas (the premises and/or hypotheses).

Proof (Syntactic) Theory: Formal Logic (Deductive) System

♣ **Notes**

- **The alphabet and grammar of a formal logic (deductive) system defines the formal language of that logic system.**
- **The definition of a formal logic (deductive) system does not require the axioms to be finite but require the deduction rules to be.**
- **The above definition just define an abstract formal logic (deductive) system.**

Proof (Syntactic) Theory: Formal Proof and Theorem

♣ **Formal proof**

- **A *formal proof* of f_n in a formal logic (deductive) system is a FINITE sequence of formulas $f_1, ..., f_n$ such that, for all i ($i \leq n$), (1) f_i is an axiom, or (2) there are some members $f_{j1}, ..., f_{jm}$ ($j1, ..., jm<i$) of the sequence, which have f_i as the result of applying one of the deduction rules to $f_{j1}, ..., f_{jn}$.**
- **Note: The definition of a formal proof requires that it must be a FINITE (very important!) sequence.**

♣ **Formal theorem**

- **If $f_1, ..., f_n$ is a formal proof in a formal logic (deductive) system, then f_n is called a *formal theorem* of the logic system and said to be *provable* in that logic system.**
- **The set of all formal theorems of a formal logic (deductive) system L is denoted by Th(L).**

Proof (Syntactic) Theory: Formal Deduction

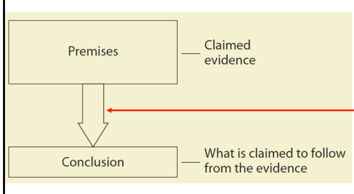
♣ **Formal deduction**

- **Let *P* be a set of formulas in a formal logic (deductive) system. A *formal deduction (proof) from P* in the system is a FINITE sequence of formulas $f_1, ..., f_n$ such that, for all i ($i \leq n$), (1) f_i is an axiom, or (2) $f_i \in P$, or (3) there are some members $f_{j1}, ..., f_{jm}$ ($j1, ..., jm<i$) of the sequence, which have f_i as the result of applying one of the deduction rules to $f_{j1}, ..., f_{jm}$.**
- **Note: The definition of a formal deduction requires that it must be a FINITE (very important!) sequence.**

♣ **Premises and consequence**

- **If $f_1, ..., f_n$ is a formal deduction (proof) from *P* in a formal logic (deductive) system, then *P* is called the *premises* of the deduction and f_n is called the *consequence* and said to be *deducible from P* based on the logic system.**

Logic: What Is It?



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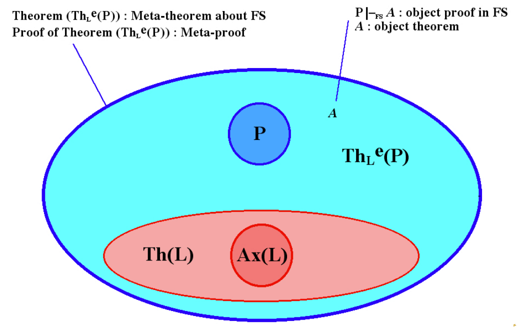
Proof (Syntactic) Theory: Logical Consequence Relation

- Syntactic/proof-theoretical/deductive logical consequence relation
  - In a formal logic (deductive) system L, for any set *P* of formulas and any formula *C*,
    P syntactically/proof-theoretically/deductively entails C, or
    C syntactically/proof-theoretically/deductively follows from P, or
    C is a *syntactic/proof-theoretical/deductive logical consequence* of *P*,
    written as $P \vdash_L C$ ("$\vdash$" is read as "turnstile"),
    IFF *C* is deducible from *P* based on L.
- Note
  - The syntactic consequence relation is a proof-theoretical formalization of the notion that one proposition follows from another or others.

Proof (Syntactic) Theory: Object Theorem and Meta-Theorem



Properties of Proof-theoretical Logical Consequence Relation

- Post-consistence and Post-completeness
  - Γ is *(syntactically/proof-theoretically/deductively) Post-consistent* in L IFF there is some formula *A* such that $\Gamma \vdash_L A$ does not hold.
  - Γ is *(syntactically/proof-theoretically/deductively) Post-complete* in L IFF for every formula *B* not in Γ and for every *C*, $\Gamma \cup \{B\} \vdash_L C$.
- Classical-consistence and Classical-completeness
  - Γ is *(syntactically/proof-theoretically/deductively) classically consistent* in L IFF for every formula *A*, NOT both $\Gamma \vdash_L A$ and $\Gamma \vdash_L (\neg A)$.
  - Γ is *(syntactically/proof-theoretically/deductively) classically complete* in L IFF for every formula *A*, at least one of *A* and $(\neg A)$ in Γ.

Model (Semantic) Theory vs Proof (Syntactic) Theory

The Relationship between Model (Semantic) Theory and Proof (Syntactic) Theory of a Formal Logic System ?

Model (Semantic) Theory vs Proof (Syntactic) Theory

- The fundamental question
  - What is the relationship between these two approaches to study/represent a formal logic system, Model (Semantic) Theory and Proof (Syntactic) Theory?
  - Or in other words, what is the relationship between the semantic/model-theoretical logical consequence relation and the syntactic/proof-theoretical/deductive logical consequence relation of a formal logic system?
- Two questions
  - If $P \vdash_L C$, then how about $P \models_L C$?
  - If $P \models_L C$, then how about $P \vdash_L C$?

Soundness and Completeness

- Soundness and completeness
  - The purpose of logic is to characterize the difference between valid and invalid arguments.
  - A formal logic system should be *sound*, i.e., every argument proven using the rules and axioms of the logic system is in fact valid.
  - A formal logic system should be *complete*, i.e., every valid argument has a formal proof in the logic system.
  - Demonstrating *soundness* and *completeness* of formal logic systems is a logician's central concern and task.
- Two questions
  - If $P \vdash_L C$, then how about $P \models_L C$?
  - If $P \models_L C$, then how about $P \vdash_L C$?

Soundness

Soundness

- Given a semantic specification of validity and a syntactically defined formal logic (deductive) system L, we say that L is ***sound*** or ***consistent*** or ***correct*** with respect to that semantics if whenever a formula C is deducible from a set of formulas P in L, then C is a semantic logical consequence of P.
- This means that the rules of the logic never yield invalid deduction, that they do not prove too much.

Soundness theorems

- Theorem (*soundness*): If $\vdash_{\mathrm{L}} C$ then $\models_{\mathrm{L}} C$, for any $C \in F(\mathrm{L})$.
- Theorem (*strong soundness*): If $P \vdash_{\mathrm{L}} C$ then $P \models_{\mathrm{L}} C$, for any $C \in F(\mathrm{L})$ and any $P \subseteq F(\mathrm{L})$.

Completeness

Completeness

- Given a semantic specification of validity and a syntactically defined formal logic (deductive) system L, if whenever a formal deduction from a set of formulas P to a formula C is semantically valid, C can be deducible from P in L, then we say that L is ***complete*** with respect to that semantics.
- This means that L has captured all that is contained in the structure of the models, that is does not prove too little.

Completeness theorems

- Theorem (*completeness*): If $\models_{\mathrm{L}} C$ then $\vdash_{\mathrm{L}} C$, for any $C \in F(\mathrm{L})$.
- Theorem (*strong completeness*): If $P \models_{\mathrm{L}} C$ then $P \vdash_{\mathrm{L}} C$, for any $C \in F(\mathrm{L})$ and any $P \subseteq F(\mathrm{L})$.

Formal Logic Systems and Formal Theories

- Formal Logic Systems
- Formal Theories
- Model (Semantic) Theory
- Proof (Syntactic) Theory
- **The Limitations of Formal Logic Systems/ Theories (Gödel's Incompleteness Theorems)**

The Limitations of Formal Logic Systems / Theories?

The Limitations of Formal Logic Systems/Theories?

The Limitations of Formal Logic Systems/Theories?

- Is there a universal formal logic system that can underlie any formal theory of any target area satisfactorily?
- Is there the best formal logic system that can underlie a formal theory of a certain area satisfactorily?
- For a certain area, is there a formal theory including all empirical theorems (i.e., those should be included) of the target area?

Hilbert's Problems [Hilbert, 1900]

David Hilbert

David Hilbert (1912)

Hilbert's Problems

- Hilbert's problems are 23 problems in mathematics published by German mathematician David Hilbert (1862–1943) in 1900.
- Hilbert presented ten of the problems (1, 2, 6, 7, 8, 13, 16, 19, 21, and 22) at the Paris conference of the International Congress of Mathematicians, speaking on August 8 in the Sorbonne.
- The complete list of 23 problems was published later, most notably in English translation in 1902 by Mary Frances Winston Newson in the Bulletin of the American Mathematical Society.

The second problem

- Prove that the axioms of arithmetic are consistent.

Hilbert’s Program [Zach]

♣Hilbert’s Program [1920s]

- David Hilbert (1862–1943) put forward a new proposal for the foundation of classical mathematics which has come to be known as Hilbert’s Program.
- It calls for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent.
- The consistency proof itself was to be carried out using only what Hilbert called *“finitary” methods*.
- The special epistemological character of finitary reasoning then yields the required justification of classical mathematics.

Hilbert’s Program [Wiki]

- The main goal of Hilbert’s program was to provide secure foundations for all mathematics. In particular this should include:
  - A formulation of all mathematics: in other words all mathematical statements should be written in a precise formal language, and manipulated according to well defined rules.
  - Completeness: a proof that all true mathematical statements can be proved in the formalism.
  - Consistency: a proof that no contradiction can be obtained in the formalism of mathematics. This consistency proof should preferably use only “finitistic” reasoning about finite mathematical objects.
  - Conservation: a proof that any result about “real objects” obtained using reasoning about “ideal objects” (such as uncountable sets) can be proved without using ideal objects.
  - Decidability: there should be an algorithm for deciding the truth or falsity of any mathematical statement.

Hilbert’s Program [王宪钧]

希尔伯特方案包括有几个步骤:

(i)把古典数学的某一基本理论如初等数论、集论或者数学分析严格形式化,加上逻辑演算,并把这两部分综合起来,整理为一形式公理学.然后再进一步形式化,构成一相当于以上公理系统的形式语言系统.

(ii)从不假定实无穷的有穷观点出发,建立一逻辑系统作为研究上述形式语言系统的工具.由于研究形式语言的逻辑性质需要用数论,因之也要建立一个不假定实无穷的初等数论.这样建立起来的逻辑和数论可以称为“元数学”或者“有穷逻辑”.

(iii)用元数学来研究形式语言系统的逻辑性质,特别是其中的证明,这就是所谓的“证明论”.证明论的目的是论证某一形式语言系统不包含逻辑矛盾.如果这目的达到,就可以保证那语言系统所表达的数学理论不会产生矛盾.

Hilbert’s Program [王宪钧]

(2)有穷观点 古典数学的一致性问题由于实无穷引起,古典的逻辑演算也假定了实无穷,因之在论证古典数学无矛盾时,不能应用以实无穷为前提的思想方法或工具,而只能依赖直观上明显可靠的,与古典逻辑和一般数论不同的方法,否则就有循环论证的错误.这是所谓的有穷观点.与此相应的方法称为有穷方法.对于有穷方法,希尔伯特没有给出一个精确完全的说明.在他和贝奈斯(P. Bernays, 1888－1977)合著的《数学基础》卷一(1934)里有较详细的讨论.他说,有穷观点是柯朗尼克先提出的,布劳维尔的直觉主义方法则是有穷方法的扩充(前书卷一,页 42－43).他的有穷方法有以下特征.

(甲)每一步骤只考虑确定的有穷数量的对象,承认潜无穷,而不处理任何包括无穷多对象的完成了的整体.

(乙)“所涉及的讨论、判断或者定义都必须满足其对象可以彻底给出并且其过程可以彻底进行的要求”(前书页 32).

(丙)全称命题只能在假言的意义下理解,亦即,它是对任一给定对象的断定.全称命题表达一规律,此规律对每一具体对象都必然可以得到验证.

(丁)存在判断必须能够直接给出某一特定对象,或者能够给出一个其步骤有特定界限的方法以得到那个对象.

Hilbert’s Program [王宪钧]

(戊)排中律.根据有穷观点,排中律对于某些形式的命题不能适用.由于有穷观点承认潜无穷,当我们否定一普遍命题时,我们就进入了超穷的领域.例如命题:有一数 a,使得

$$a+1 \neq 1+a.$$

这实际是一无穷析取.从有穷观点考虑,断定此命题真,至少要给出一具体数字;断定此命题假,要证明其不可能.有时既不能给出满足要求的具体数字,又不能得到一不可能性证明,因之对于这种类型的命题,排中律无效.

(己)数学归纳法.彭加勒曾批评在元数学里用数学归纳法为循环论证.希尔伯特指出,彭加勒未能区别两种数学归纳法.一是元数学里关于数学的具体构造的“内容的归纳法”,另一种是基于归纳公理的“形式归纳法”,“只有通过它,数学变元才开始在形式系统中发生作用”.(《数学基础》,1927,范海金诺《从弗雷格到歌德尔》,页 473)

综上所述,有穷方法是一种所谓的能行方法,它较一般递归为狭.王浩教授说:“这个不明确的概念的最可能的解释是,它们大约相当于原始递归算术(自然没有量词)的一个弱扩张.”(《数理逻辑通俗讲话》,1981,北京科学出版社,页 151,注 1)

Hilbert’s Program [王宪钧]

(范海金诺,页 489,编者注 3).1931 年歌德尔发表了著名的关于“形式不可判定命题”的文章,其中严格地证明了,如果一个包括初等数论的形式系统是一致的,那么其一致性不能用有穷方法甚至不能用狭谓词演算和初等数论的方法证明.歌德尔这个定理给希尔伯特方案以沉重的打击.希尔伯特得知以后颇为震惊,他们随即作出决定把有穷方法加以扩充,再增加超穷归纳法作为证明论的工具.此后不数年,甘岑(G. Gentzen, 1909－1945)于 1936年就用超穷归纳法证明了纯粹数论的一致性.当然这已经不是严格意义的有穷方法了.关于他的定理对希尔伯特方案的影响,歌德尔在答复希尔伯特传记的作者瑞德(Constance Reid)时说:“从纯粹数学的观点考虑,用恰当选择的较强的元数学假设为基础来作一致性证明(正如甘岑和其它人所给出的)还是同样有趣,并且这种证明也使我们对于数学的证明论结构能得到高度重要的理解.此外,下面这问题还并没有解决,那就是:能否或者在什么程度上能够,根据形式主义的观点,去‘构造地’证明古典数学的一致性;这也就是:能否用对于我们心灵活动的洞察理解去代替关于客观的柏拉图领域内的抽象事物的一些数学公理.”(瑞德:《希尔伯特》,1970,页 217－218)

The Gödel's Work Related to the Hilbert's Program

♣ The Gödel's work [Rathjen]

- Gödel's goal was to prove the consistency of analysis.
- According to Wang (1981), his idea was "to prove the consistency of analysis by number theory, where one can assume the truth of number theory, not only the consistency".
- The plan for establishing the consistency of analysis relative to number theory did not work out, instead Gödel found that sufficiently strong formal theories like Principia Mathematica and Zermelo-Fraenkel set theory are (syntactically) incomplete.
- Note: The most intrinsically important restriction is using only Hilbert's finitary methods.

Classical/Important References on Hilbert's Program

- D. Hilbert, "Mathematische Probleme," Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse, 253–297. Lecture given at the International Congress of Mathematicians, Paris, 1900.
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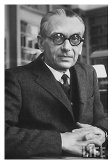
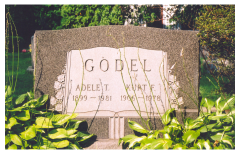
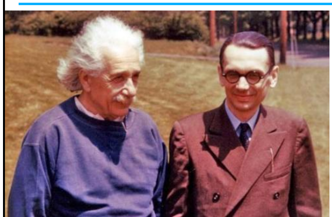
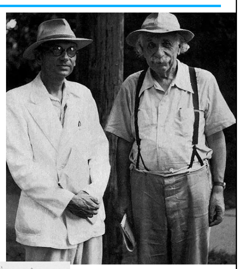
- H. Wang, "Popular Lectures on Mathematical Logic," Van Nostrand Reinhold, 1981, Dover Publications, 1993.

Kurt Gödel

- Kurt Friedrich Gödel,
- Born: April 28, 1906
  Birthplace: Brünn, Austria-Hungary (now Brno, Czech Republic)
- Died: January 14, 1978 (aged 71)
  Location of death: Princeton, New Jersey, U.S.
- Citizenship: Austria, United States

Kurt Gödel and Albert Einstein in Princeton

Gödel's Incompleteness Theorems: What are They?

♣ Gödel's original paper

- K. Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," Monatshefte für Mathematik Physik, Vol. 38, pp. 173–198, 1931. (The summary of the results of this work, published in Anzeiger der Akad. D. Wiss. In Wien (math.-naturw. Kl.) 1930, No. 19.)

♣ Translation

- B. Meltzer (translation) and R. B. Braithwaite (Introduction), K. Gödel, "On formally undecidable propositions of Principia Mathematica and related systems I," Basic Books, 1962, Dover Publications, 1992.

♣ Important notes

- The so-called "incompleteness theorems" are not named by Gödel himself, but by others. Many misunderstandings about Gödel's work stem from misunderstandings about "incompleteness".
- In fact, the "formally undecidable propositions" are more intrinsically important.

Gödel's Incompleteness Theorems: What are They?

- **Gödel's incompleteness theorems show the limitations of formal deductive systems**
  - **Gödel's incompleteness theorems show the limitations of formal deductive systems: the way of formal deductive proof with the restriction of finitude has not enough power to deal with infinite objects and their relationships in target areas of formal deductive systems.**
- **Gödel's incompleteness theorems are meta-mathematical**
  - **Gödel's incompleteness theorems are results which belong not to mathematics but to mate-mathematics, i.e., they talk about general methodologies used in mathematics.**
  - **Gödel's incompleteness theorems have nothing to do with many areas of mathematics, general philosophy, as well as almost areas of CS, in particular, AI.**

Gödel's Incompleteness Theorems

- **Gödel's first incompleteness theorem**
  - **In a certain class of arithmetical formal deductive systems formalized based on classical predicate calculus (i.e., the formal deductive system P, obtained by superimposing on the Peano axioms), if a formal system is consistent, then there are formally undecidable (i.e., neither provable nor disprovable by the way of formal proof in that system) propositions.**
  - **Notes: classical predicate calculus, arithmetical, consistent, undecidable (i.e., neither provable nor disprovable by the way of formal proof)**
- **Gödel's second incompleteness theorem**
  - **The consistency of formal deductive system P is unprovable in P by the way of formal proof.**

Gödel's Incompleteness Theorems

- **Gödel's first incompleteness theorem [Rosser, 1936]**
  - **"Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of which can neither be proved nor disproved in."**
- **Gödel's second incompleteness theorem [Raatikainen, 2020]**
  - **"For any consistent system within which a certain amount of elementary arithmetic can be carried out, the consistency of cannot be proved in itself. "**

Gödel's Incompleteness Theorems

- **Gödel's first incompleteness theorem [Raatikainen, 2020]**

Gödel's First Incompleteness Theorem

Assume F is a formalized system which contains Robinson arithmetic $\mathbf{Q}$. Then a sentence G_F of the language of F can be mechanically constructed from F such that:

i. If F is consistent, then $F \nvdash G_F$.
ii. If F is 1-consistent, then $F \nvdash \neg G_F$

Such an independent, or "undecidable" (that is, neither provable nor refutable in F) statement G_F in F is often called "the Gödel sentence" of F.

- **Gödel's second incompleteness theorem [Raatikainen, 2020]**

Gödel's second incompleteness theorem
Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \nvdash Cons(F)$.

Gödel Numbering and Self-reference Statement [Hofstadter]

- **Gödel numbering**
  - **"One day, while studying the extremely austere patterns of symbols in these volumes (Principia Mathematica), he (Gödel) had a flash that those patterns were so much like number patterns that he could in fact replace each symbol by a number and reperceive all of Principia Mathematica not as symbol shunting but as number crunching."**
  - **"This new way of looking at things had an astounding wraparound effect: since the subject matter of Principia Mathematica was numbers, and since Gödel had turned the mediun of volumes also into numbers, this showed that Principia Mathematica was its own subject matter, or in other words, that the patterned formulas of Russell and Whitehead's system could seen as saying things about each other, or possible even about themselves."**

Gödel Numbering and Self-reference Statement [Hofstadter]

- **Gödel's self-statement about PM**
  - **"Gödel realized that, in principle, he could write down a formula of Principia Mathematica that perversely said about itself, 'This formula is unrovable by the rules of Principia Mathematica.' "**
  - **The self-undermining Gödelian formula had to be dealt with, and Gödel did so most astutely, showing that although it resembled a paradox, it differed subtly from one. In particular, it was revealed to be a true statement that could not be proven using the rules of the system – indeed, a true statement whose unprovability resulted precisely from its truth."**

Premises of Gödel's First Incompleteness Theorem

- The target formal deductive systems [Gödel]
  - "The systems of PM" (Principia Mathematica (2nd Ed.), Whitehead & Russell, 1925), but "reckon among the axioms of PM the axioms of infinity, and the axioms of reducibility and choice".
  - "The axiom system for set theory of Zermelo-Fraenkel (later extended by J. v. Neumann)"
  - Note: The two systems are based on classical predicate calculus.
- Why the two systems? [Gödel]
  - "There two systems are so extensive that all methods of proof used in mathematics today have been formalized in them, i.e., reduced to a few axioms and rules of inference."
  - "It may therefore be surmised that these axioms and rules of inference are also sufficient to decide all mathematical questions which can in any way at all be expressed formally in the systems concerned."

Premises of Gödel's First Incompleteness Theorem

- The formal system P [Gödel]
  - "The formal system P, for which we seek to demonstrate the existence of undecidable propositions."
  - "P is essentially the system obtained by superimposing on the Peano axioms the logic of PM (numbers as individuals, relation of successor as undefined basic concept)."
- The basic signs of the system P [Gödel]
  - "I. Constants: "~" (not), "∨" (or), "Π" (for all), "0" (nought), "*f*" (the successor of), "(", ")" (brackets).
  - II. Variables of first type (for individuals, i.e. natural numbers including 0): "x_1", "y_1", "z_1", …
    Variables of second type (for classes of individuals): "x_2", "y_2", "z_2", …
    Variables of third type (for classes of classes of individuals): "x_3", "y_3", "z_3", …
    and so on for every natural number as type."

Premises of Gödel's First Incompleteness Theorem

- The conditions for the first incompleteness theorem [Wang, 1981]
  - "A method is given of constructing, for any formal (formalized axiom) system S of mathematics, a question of number theory undecidable in the system."
  - "This is a general result which depends on essentially only on three conditions: (a) The axiom system S is truly formal; (b) The system S is rich enough for developing a moderate amount of number theory; (c) S is consistent."
- Notes
  - "only on the three conditions" is problematic.
  - There are should another condition: The system S is formalized based on classical predicate calculus.

Conclusion of Gödel's First Incompleteness Theorem

- What is shown by the conclusion of Gödel's first incompleteness theorem? [Gödel]
  - "this is not the case." (the case: "to decide all mathematical questions which can in any way at all be expressed formally in the systems concerned.")
  - "in both the systems mentioned there are in fact relatively simple problems in the theory of ordinary whole numbers which cannot be decided from the axioms."
  - "more precisely, there are undecidable propositions in which, beside the logical constants ¬ (not), ∨ (or), (x) (for all) and = (identical with), there are no other concepts beyond + (addition) and . (multiplication), both referred to natural numbers, and where the prefixes (x) can also refer only to natural numbers."

Conclusion of Gödel's First Incompleteness Theorem

- What is shown by the conclusion of Gödel's first incompleteness theorem? [Gödel]
  - "This situation is not due in some way to the special nature of the systems set up, but holds for a very extensive class of formal system, including, in particular, all those arising from the addition of a finite number of axioms to the two systems mentioned, provided that thereby no false propositions of the kind described in footnote 4 become provable."
  - footnote 4: "more precisely, …… to natural numbers."

Conclusion of Gödel's First Incompleteness Theorem

- The conclusion of Gödel's first incompleteness theorem [Gödel]
  - "There are undecidable problems of the restricted predicate calculus."
  - Note: predicate calculus, CML!
  - Undecidable: "neither universal validity nor the existence of a counter-example is provable."
  - Note: "provable" means the finite formal proof method!
- Note
  - A method of constructing such undecidable problems is given by Gödel in his proof (meta-proof!) for the first incompleteness theorem.

Gödel's Second Incompleteness Theorem

- **Premises of Gödel's second incompleteness theorem**
  - *c* is a given recursive, consistent class of formulas.
- **Conclusion of Gödel's second incompleteness theorem [Gödel]**
  - "the propositional formula which states that *c* is consistent is not c-provable;"
  - "the consistency of P is unprovable in P, it being assumed that P is consistent (if not, of course, every statement is provable)."
  - Note: "provable" means the finite formal proof method!
- **Note**
  - The reason of "if not, of course, every statement is provable" is that all formal systems discussed by Gödel's work are formalized based on classical predicate calculus, where any arbitrary formula can be deduced from a contradiction.

The Heart of Gödel's Proof [N&N]

- (i) Gödel showed how to construct a formula G of PM, that represents the meta-mathematical statement: 'The formula G is not demonstrable using the rules of PM'. This formula thus ostensibly says of itself that it is not demonstrable.
- (ii) Gödel also showed that G is demonstrable if, and only if, its formal negation ¬G is demonstrable.
- (iii) Gödel then showed that, though G is not formally demonstrable, it nevertheless is a true arithmetical formula. G is true in the sense that it claims that a certain arithmetical property defined by Gödel is possessed by no integer – and indeed, no integer possesses the property, as Gödel shows.
- (iv) The realization that since G is both true and formally undecidable (within PM), PM must be incomplete. Moreover, Gödel established that PM is essentially incomplete.
- (v) Gödel described how to construct a formula A of PM that represents the meta-mathematical statement: 'PM is consistent'; and he showed that the formula 'A → G' is formally demonstrable inside PM. Finally, he showed that the formula A is not demonstrable inside PM.

Decidable and Complete Formal Theories [Raatikainen]

- **Complete number theories**
  - The theory of only addition of natural numbers but without multiplication (often called "*Presburger arithmetic*"), is complete and decidable. [Presburger, 1929]
  - The theory of multiplication of the positive integers is complete and decidable. [Skolem, 1930]
  - The natural first-order theory of arithmetic of real numbers (with both addition and multiplication), the so-called *theory of real closed fields*, is complete and decidable. [Tarski, 1948]
- **Complete other theories**
  - The the first-order theory of Euclidean geometry is complete and decidable. [Tarski, 1948]

Proving the Consistency of P by Transfinite Induction

- **Proving the consistency of P by transfinite induction**
  - Gentzen showed that the consistency of P can be proved if the transfinite induction principle is assumed. [Gentzen, 1935-1936]
  - Note: Because of the second incompleteness theorem, the principle itself cannot be provable in P.
- **Transfinite induction**
  - *Transfinite induction* is an extension of mathematical induction (that only can applied to natural numbers) to any well-ordered set.

Transfinite Induction [Wolfram MathWorld]

Transfinite Induction

Transfinite induction, like regular induction, is used to show a property $P(n)$ holds for all numbers n. The essential difference is that regular induction is restricted to the natural numbers $\mathbb{Z}^*$, which are precisely the finite ordinal numbers. The normal inductive step of deriving $P(n+1)$ from $P(n)$ can fail due to limit ordinals.

Let A be a well ordered set and let $P(x)$ be a proposition with domain A. A proof by transfinite induction uses the following steps (Gleason 1991, Hajnal 1999):

1. Demonstrate $P(0)$ is true.
2. Assume $P(b)$ is true for all $b < a$.
3. Prove $P(a)$, using the assumption in (2).
4. Then $P(a)$ is true for all $a \in A$.

To prove various results in point-set topology, Cantor developed the first transfinite induction methods in the 1880s. Zermelo (1904) extended Cantor's method with a "proof that every set can be well-ordered," which became the axiom of choice or Zorn's Lemma (Johnstone 1987). Transfinite induction and Zorn's lemma are often used interchangeably (Reid 1995), or are strongly linked (Beachy 1999). Hausdorff (1906) was the first to explicitly name transfinite induction (Grattan-Guinness 2001).

Jingde Cheng's Blogs on Gödel's Incompleteness Theorems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哥德尔不完全性定理的内容和有效范围 (1) 哥德尔断定了什么？

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2017-5-21 17:55

如题。 哥德尔不完全性定理的内容和有效范围 (3) 各种典型误解实例 (http://blog.sciencenet.cn/blog-2371919-1053033.html)

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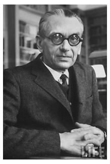
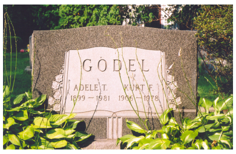
Kurt Gödel

- **Kurt Friedrich Gödel,**
- **Born: April 28, 1906**
  Birthplace: Brünn, Austria-Hungary (now Brno, Czech Republic)
- **Died: January 14, 1978 (aged 71)**
  Location of death: Princeton, New Jersey, U.S.
- **Citizenship: Austria, United States**

Gödel's Incompleteness Theorems: Original Paper

Gödel’s original paper

- **K. Gödel, “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,” Monatshefte für Mathematik Physik, Vol. 38, pp. 173–198, 1931. (The summary of the results of this work, published in Anzeiger der Akad. D. Wiss. In Wien (math.-naturw. Kl.) 1930, No. 19.)**

Translation

- **B. Meltzer (translation) and R. B. Braithwaite (Introduction), K. Gödel, “On formally undecidable propositions of Principia Mathematica and related systems I,” Basic Books, 1962, Dover Publications, 1992.**

Gödel's Incompleteness Theorems: Original Statements

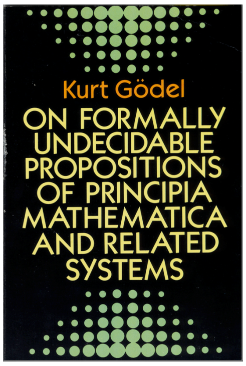
Gödel’s first incompleteness theorem

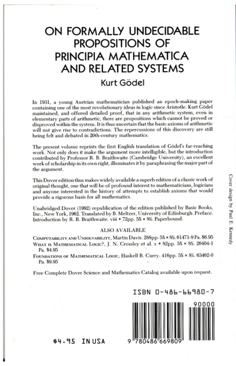
Proposition IX: In all the formal systems referred to in Proposition VI[53] there are undecidable problems of the restricted predicate calculus[54] (i.e. formulae of the restricted predicate calculus for which neither universal validity nor the existence of a counter-example is provable).[55]

Gödel’s second incompleteness theorem

Proposition XI: If c be a given recursive, consistent class[63] of *formulae*, then the *propositional formula* which states that c is consistent is not *c-provable*; in particular, the consistency of P is unprovable in P,[64] it being assumed that P is consistent (if not, of course, every statement is provable).

Gödel's Incompleteness Theorems: References





ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS

Kurt Gödel

In 1931, a young Austrian mathematician published an epoch-making paper containing one of the most revolutionary ideas in logic since Aristotle. Kurt Gödel maintained, and offered detailed proof, that in any arithmetic system, even in elementary parts of arithmetic, there are propositions which cannot be proved or disproved within the system. It is thus uncertain that the basic axioms of arithmetic will not give rise to contradictions. The repercussions of this discovery are still being felt and debated in 20th-century mathematics.

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On Formally Undecidable Propositions Of Principia Mathematica And Related Systems

KURT GÖDEL

Translated by
B. MELTZER

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Gödel's Incompleteness Theorems: References

PREFACE

Kurt Gödel's astonishing discovery and proof, published in 1931, that even in elementary parts of arithmetic there exist propositions which cannot be proved or disproved within the system, is one of the most important contributions to logic since Aristotle. Any formal logical system which disposes of sufficient means to compass the addition and multiplication of positive integers and zero is subject to this limitation, so that one must consider this kind of incompleteness an inherent characteristic of formal mathematics as a whole, which was before this customarily considered *the* unequivocal intellectual discipline *par excellence*.

No English translation of Gödel's paper, which occupied twenty-five pages of the *Monatshefte für Mathematik und Physik*, has been generally available, and even the original German text is not everywhere easily accessible. The argument, which used a notation adapted from that of Whitehead and Russell's *Principia Mathematica*, is a closely reasoned one and the present translation—besides being a long overdue act of piety—should make it more easily intelligible and much more widely read. In the former respect the reader will be greatly aided by the Introduction contributed by the Knightbridge Professor of Moral Philosophy in the University of Cambridge; for this is an excellent work of scholarship in its own right, not only pointing out the significance of Gödel's work, but illuminating it by a paraphrase of the major part of the whole great argument.

I proposed publishing a translation after a discussion meeting on "Gödel's Theorem and its bearing on the philosophy of science", held in 1959 by the Edinburgh Philosophy

vii

viii PREFACE

of Science Group. I wish to thank this society for providing the stimulus, the publishers for their ready co-operation on the proposal, and Professor Braithwaite not only for the Introduction but also for meticulous assistance in translation and proof-reading of a typographically intricate text. It may be noted here that the pagination of the original article is shown in the margins of the translation, while the footnotes retain their original numbers.

B. MELTZER

University of Edinburgh
January, 1962

Gödel's Incompleteness Theorems: References

INTRODUCTION

by

R. B. BRAITHWAITE

Every system of arithmetic contains arithmetical propositions, by which is meant propositions concerned solely with relations between whole numbers, which can neither be proved nor be disproved within the system. This epoch-making discovery by Kurt Gödel, a young Austrian mathematician, was announced by him to the Vienna Academy of Sciences in 1930 and was published, with a detailed proof, in a paper in the *Monatshefte für Mathematik und Physik* Volume 38 pp. 173-198 (Leipzig: 1931). This paper, entitled "*Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*" ("On formally undecidable propositions of *Principia Mathematica* and related systems I"), is translated in this book. Gödel intended to write a second part to the paper but this has never been published.

Gödel's Theorem, as a simple corollary of Proposition VI (p. 57) is frequently called, proves that there are arithmetical propositions which are undecidable (i.e. neither provable nor disprovable) within their arithmetical system, and the proof proceeds by actually specifying such a proposition, namely the proposition *g* expressed by the formula to which "17 Gen *r*" refers (p. 58). *g* is an arithmetical proposition; but the proposition that *g* is undecidable within the system is not an arithmetical proposition, since it is concerned with provability within an arithmetical system, and this is a meta-arithmetical and not an arithmetical notion. Gödel's Theorem is thus a result which belongs not to mathematics

1

2 INTRODUCTION

but to metamathematics, the name given by Hilbert to the study of rigorous proof in mathematics and symbolic logic.

METAMATHEMATICS. Gödel's paper presupposes some knowledge of the state of metamathematics in 1930, which therefore I shall briefly explain. Following on the work of Frege and Peano, Whitehead and Russell's *Principia Mathematica* (1910-13) had exhibited the fundamental parts of mathematics, including arithmetic, as a *deductive system* starting from a limited number of axioms, in which each theorem is shown to follow logically from the axioms and theorems which precede it according to a limited number of rules of inference. And other mathematicians had constructed other deductive systems which included arithmetic (see p. 37, n. 3). In order to show that in a deductive system every theorem follows from the axioms according to the rules of inference it is necessary to consider the *formulae* which are used to express the axioms and theorems of the system, and to represent the rules of inference by rules (Gödel calls them "mechanical" rules, p. 37) according to which from one or more formulae another formula may be obtained by a manipulation of symbols. Such a representation of a deductive system will consist of a sequence of formulae (a *calculus*) in which the initial formulae express the axioms of the deductive system and each of the other formulae, which express the theorems, are obtained from the initial formulae by a chain of symbolic manipulations. The chain of symbolic manipulations in the calculus corresponds to and represents the chain of deductions in the deductive system.

But this correspondence between calculus and deductive system may be viewed in reverse, and by looking at it the other way round Hilbert originated metamathematics. Here a calculus is constructed, independently of any inter-

Gödel's Incompleteness Theorems: References

INTRODUCTION 3

pretation of it, as a sequence of formulae which starts with a few initial formulae and in which every other formula is obtained from preceding formulae by symbolic manipulations. The calculus can then be interpreted as representing a deductive system if the initial formulae can be interpreted as expressing the axioms of the system and if the rules of symbolic manipulation can be interpreted as representing the logical rules of inference of the system. If this can be done, a proof that a formula (other than one of the initial formulae) occurs in the sequence of formulae of the calculus yields a proof that the proposition which is the interpretation of this formula is a theorem of the deductive system, i.e. can be deduced from the axioms of the system by the system's rules of inference. Metamathematicians in the 1920's established many important results about deductive systems by converting proofs of what formulae can be obtained by symbolic manipulations within a calculus into proofs of what theorems can be proved within a deductive system which could be represented by the calculus. Frequently consideration of symbolic manipulations provided a "decision procedure" by which whole classes of theorems could actually be proved. Thus Presburger in 1930 published a decision procedure applicable to every proposition of a mutilated system of arithmetic which uses the operation of addition but not that of multiplication; he proved that every one of its propositions is decidable, i.e. either provable or disprovable, within this system.

Gödel's paper established the opposite of this for an arithmetical system which uses multiplication as well as addition—"the theory of ordinary whole numbers" (p. 38). And this is the piece of mathematics which is oldest in the history of civilization and which is of such practical importance that we make all our children learn a great deal of it at an early age. Gödel was the first to prove any unprov-

4 INTRODUCTION

ability theorem for arithmetic, and his way of proof was subtler and deeper than the metamathematical methods previously employed. Either of these facts would have ranked this paper high in the development of metamathematics. But it was the fact that it was a proposition of whole-number arithmetic which he showed to be undecidable that created such a scandal.

GÖDEL'S 'FORMAL SYSTEM' P. In order rigorously to prove the undecidability of some arithmetical propositions it is necessary to be precise about the exact deductive system of arithmetic which is being considered. As is indicated in the title of his paper, Gödel takes for his arithmetical deductive system that part of the system of *Principia Mathematica* required to establish the theorems of whole-number arithmetic. Since his proof is metamathematical he is concerned with a calculus representing his arithmetical system: what he proves in Proposition VI (p. 57) is a result about the calculus and not about what the calculus represents, for what it directly establishes is that neither of two particular formulae—the first referred to by "17 Gen *r*", the second by "Neg (17 Gen *r*)" (p. 59)—can be obtained from the initial formulae of the calculus by the rules of symbolic manipulation of the calculus. If the calculus is interpreted (as it can be interpreted) so that it represents the arithmetical part of the *Principia Mathematica* deductive system, with the second formula expressing the contradictory of the arithmetical proposition expressed by the first formula, then the theorem about the deductive system which corresponds to the calculus-theorem states that the proposition *g* to which "17 Gen *r*" refers is such that neither it nor its contradictory is provable within the system. Hence within the system *g* is neither provable nor disprovable. An unprovability theorem for the arithmetical deductive

Gödel's Incompleteness Theorems: Axioms

I.
1. $\sim (fx_1 = 0)$
2. $fx_1 = fy_1 \supset x_1 = y_1$
3. $x_2(0) \,.\, x_1 \Pi (x_2(x_1) \supset x_2(fx_1)) \supset x_1 \Pi (x_2(x_1))$

II. Every formula derived from the following schemata by substitution of any formulae for p, q and r.

1. $p \vee p \supset p$
2. $p \supset p \vee q$
3. $p \vee q \supset q \vee p$
4. $(p \supset q) \supset (r \vee p \supset r \vee q)$

III. Every formula derived from the two schemata

1. $v \Pi (a) \supset \text{Subst}\, a \binom{v}{c}$
2. $v \Pi (b \vee a) \supset b \vee v \Pi (a)$

by making the following substitutions for a, v, b, c (and carrying out in 1 the operation denoted by "Subst"): for a any given formula, for v any variable, for b any formula in which v does not appear free, for c a sign of the same type as v, provided that c contains no variable which is bound in a at a place where v is free.[23]

IV. Every formula derived from the schema

1. $(Eu)(v \Pi (u(v) \equiv a))$

on substituting for v or u any variables of types n or $n+1$ respectively, and for a a formula which does not contain u free. This axiom represents the axiom of reducibility (the axiom of comprehension of set theory).

V. Every formula derived from the following by type-lift (and this formula itself):

1. $x_1 \Pi (x_2(x_1) \equiv y_2(x_1)) \supset x_2 = y_2$.

Gödel's Incompleteness Theorems: Propositions I - IV

I. *Every function (or relation) derived from recursive functions (or relations) by the substitution of recursive functions in place of variables is recursive; so also is every function derived from recursive functions by recursive definition according to schema* (2).

II. *If R and S are recursive relations, then so also are* $\overline{R}$, $R \vee S$ (*and therefore also* $R \,\&\, S$).

III. *If the functions* $\phi(\mathfrak{x})$ *and* $\psi(\mathfrak{y})$ *are recursive, so also is the relation:* $\phi(\mathfrak{x}) = \psi(\mathfrak{y})$.[30]

IV. *If the function* $\phi(\mathfrak{x})$ *and the relation* $R(x, \mathfrak{y})$ *are recursive, so also then are the relations S, T*

$$S(\mathfrak{x}, \mathfrak{y}) \sim (Ex)\,[x \leqq \phi(\mathfrak{x}) \,\&\, R(x, \mathfrak{y})]$$
$$T(\mathfrak{x}, \mathfrak{y}) \sim (x)\,[x \leqq \phi(\mathfrak{x}) \rightarrow R(x, \mathfrak{y})]$$

and likewise the function ψ

$$\psi(\mathfrak{x}, \mathfrak{y}) = \varepsilon x\,[x \leqq \phi(\mathfrak{x}) \,\&\, R(x, \mathfrak{y})],$$

where $\varepsilon x\, F(x)$ means: the smallest number x for which $F(x)$ holds and 0 if there is no such number.

Gödel's Incompleteness Theorems: Proposition V

The following proposition is an exact expression of a fact which can be vaguely formulated in this way: every recursive relation is definable in the system P (interpreted as to content), regardless of what interpretation is given to the formulae of P:

Proposition V: To every recursive relation $R(x_1 \ldots x_n)$ there corresponds an n-place *relation-sign* r (with the *free variables*[38] $u_1, u_2, \ldots u_n$) such that for every n-tuple of numbers $(x_1 \ldots x_n)$ the following hold:

$$R(x_1 \ldots x_n) \rightarrow \mathrm{Bew}\left[Sb\left(r \begin{matrix} u_1 & \ldots u_n \\ Z(x_1) & \ldots Z(x_n) \end{matrix}\right)\right] \qquad (3)$$

$$\bar{R}(x_1 \ldots x_n) \rightarrow \mathrm{Bew}\left[\mathrm{Neg}\; Sb\left(r \begin{matrix} u_1 & \ldots & u_n \\ Z(x_1) & \ldots & Z(x_n) \end{matrix}\right)\right] \qquad (4)$$

Gödel's Incompleteness Theorems: Proposition VI

Let c be any class of *formulae*. We denote by Flg (c) (set of consequences of c) the smallest set of *formulae* which contains all the *formulae* of c and all *axioms*, and which is closed with respect to the relation "*immediate consequence of*". c is termed ω-consistent, if there is no *class-sign* a such that:

$$(n)\,[Sb\left(a \begin{matrix} v \\ Z(n) \end{matrix}\right) \,\varepsilon\, \mathrm{Flg}(c)] \;\&\; [\mathrm{Neg}(v\,\mathrm{Gen}\,a)] \,\varepsilon\, \mathrm{Flg}(c)$$

where v is the *free variable* of the *class-sign* a.

Every ω-consistent system is naturally also consistent. The converse, however, is not the case, as will be shown later.

The general result as to the existence of undecidable propositions reads:

Proposition VI: To every ω-consistent recursive class c of *formulae* there correspond recursive *class-signs* r, such that neither v Gen r nor Neg (v Gen r) belongs to Flg (c) (where v is the *free variable* of r).

Gödel's Incompleteness Theorems: Proposition VII

From Proposition VI we now obtain further consequences and for this purpose give the following definition:

A relation (class) is called **arithmetical**, if it can be defined solely by means of the concepts +, . [addition and multiplication, applied to natural numbers][49] and the logical constants $\vee$, $\overline{\ }$, (x), $=$, where (x) and $=$ are to relate only to natural numbers.[50] The concept of "arithmetical proposition" is defined in a corresponding way. In particular the relations "greater" and "congruent to a modulus" are arithmetical, since

$$x > y \sim \overline{(Ez)}\,[y = x + z]$$
$$x \equiv y\,(\mathrm{mod}\,n) \sim (Ez)\,[x = y + z\,.\,n \vee y = x + z\,.\,n]$$

We now have:

Proposition VII: Every recursive relation is arithmetical.

Gödel's Incompleteness Theorems: Proposition VIII

According to Proposition VII there corresponds to every problem of the form $(x)\,F(x)$ (F recursive) an equivalent arithmetical problem, and since the whole proof of Proposition VII can be formalized (for every specific F) within the system P, this equivalence is provable in P. Hence:

Proposition VIII: In every one of the formal systems[53] referred to in Proposition VI there are undecidable arithmetical propositions.

Gödel's Incompleteness Theorems: Propositions IX & X

The same holds (in virtue of the remarks at the end of Section 3) for the axiom system of set theory and its extensions by ω-consistent recursive classes of axioms.

We shall finally demonstrate the following result also:

Proposition IX: In all the formal systems referred to in Proposition VI[53] there are undecidable problems of the restricted predicate calculus[54] (i.e. formulae of the restricted predicate calculus for which neither universal validity nor the existence of a counter-example is provable).[55]

This is based on

Proposition X: Every problem of the form $(x)\,F(x)$ (F recursive) can be reduced to the question of the satisfiability of a formula of the restricted predicate calculus (i.e. for every recursive F one can give a formula of the restricted predicate calculus, the satisfiability of which is equivalent to the validity of $(x)\,F(x)$).

Gödel's Incompleteness Theorems: Proposition XI

From the conclusions of Section 2 there follows a remarkable result with regard to a consistency proof of the system P (and its extensions), which is expressed in the following proposition:

Proposition XI: If c be a given recursive, consistent class[63] of *formulae*, then the *propositional formula* which states that c is consistent is not *c-provable*; in particular, the consistency of P is unprovable in P,[64] it being assumed that P is consistent (if not, of course, every statement is provable).

Formal Logic Systems and Formal Theories

- Formal Logic Systems
- Formal Theories
- Model (Semantic) Theory
- Proof (Syntactic) Theory
- The Limitations of Formal Logic Systems/ Theories (Gödel’s Incompleteness Theorems)