7.1

(a) $\vdash_L (\neg(\neg A)) \to A$

(1)

1. $(A \to (B \to A))$

...... {AS1 $(A \to (B \to A))$, $A = A$, $B = B$}

2. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$

...... {AS2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A = A$, $B = (B \to A)$, $C = A$}

3. $(A \to (B \to A)) \to (A \to A)$

...... {From 1 and 2 by MP}

4. $(A \to A)$

...... {From 1 and 3 by MP}

5. $((B \to C) \to ((A \to B) \to C)) \to ((A \to B) \to (A \to C)))) \to (((B \to C) \to (A \to (B \to C))) \to ((B \to C) \to ((A \to B) \to (A \to C))))$

...... {AS2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A = (B \to C)$, $B = (A \to B)$, $C = C$}

6. $((A \to (B \to C)) \to ((A \to B) \to (A \to C))) \to ((B \to C) \to ((A \to (B \to C) \to ((A \to B) \to (A \to C))))$

...... {AS1 $(A \to (B \to A))$, $A = (A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $B = (B \to C)$}

7. $(A \to (B \to C)) \to ((A \to B) \to (A \to C)))$

...... {AS2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A = A$, $B = B$, $C = C$}

8. $((B \to C) \to ((A \to (B \to C) \to ((A \to B) \to (A \to C))))$

...... {From 6 and 7 by MP}

9. $((B \to C) \to (A \to (B \to C))) \to ((B \to C) \to ((A \to B) \to (A \to C)))$

...... {From 5 and 8 by MP}

10. $(B \to C) \to (A \to (B \to C))$

...... {AS1 $(A \to (B \to A))$, $A = (B \to C)$, $B = A$}

11. $(B \to C) \to ((A \to B) \to (A \to C))$

...... {From 9 and 10 by MP}

12. $(((\neg B) \to (\neg A)) \to (A \to B)) \to (((\neg A) \to ((\neg B) \to (\neg A)) \to ((\neg A) \to (A \to B)))$

...... {From 11, $A = (\neg A)$, $B = ((\neg B) \to (\neg A))$, $C = (A \to B)$}

13. $(((\neg B) \to (\neg A)) \to (A \to B))$

...... {AS3 $(((\neg A) \to (\neg B)) \to (B \to A))$, $A = B$, $B = A$}

14. $((\neg A) \to ((\neg B) \to (\neg A)) \to ((\neg A) \to (A \to B))$

...... {From 12 and 13 by MP}

15. $((\neg A) \to ((\neg B) \to (\neg A)))$

...... {AS1 $(A \to (B \to A))$, $A = (\neg A)$, $B = (\neg B)$}

16. $(\neg A) \to (A \to B)$

...... {From 14 and 15 by MP}

17. $((\neg(\neg A)) \to ((\neg(\neg A)) \to A) \to (((\neg(\neg A)) \to (\neg(\neg A))) \to ((\neg(\neg A)) \to A))$

...... {AS2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A = (\neg(\neg A))$, $B = (\neg(\neg A))$, $C = A$}

18. $(((\neg A) \to (\neg(\neg(\neg A)))) \to ((\neg(\neg A)) \to A)) \to (((\neg(\neg A)) \to ((\neg A) \to (\neg(\neg(\neg A))))) \to ((\neg(\neg A)) \to ((\neg(\neg A)) \to A))$

...... {From 11, $A = (\neg(\neg A))$, $B = (\neg A) \to (\neg(\neg(\neg A)))$, $C = ((\neg(\neg A)) \to A)$ }

19. $(((\neg A) \to (\neg(\neg(\neg A)))) \to ((\neg(\neg A)) \to (\neg A)))$

...... {AS3 $(((\neg A) \to (\neg B)) \to (B \to A))$, $A = (\neg A)$, $B = (\neg(\neg(\neg A)))$}

20. $((\neg(\neg A)) \to ((\neg A) \to (\neg(\neg(\neg A))))) \to ((\neg(\neg A)) \to ((\neg(\neg A)) \to A))$

...... {From 18 and 19 by MP}

21. $((\neg(\neg A)) \to ((\neg(\neg A)) \to (\neg(\neg(\neg A))))$

...... {From 16, $A = (\neg(\neg A))$, $B = (\neg(\neg(\neg A)))$}

22. $(\neg(\neg A)) \to ((\neg(\neg A)) \to A)$

...... {From 20 and 21 by MP}

23. $((\neg(\neg A)) \to (\neg(\neg A))) \to ((\neg(\neg A)) \to A)$

...... {From 17 and 22 by MP}

24. $((\neg(\neg A)) \to (\neg(\neg A)))$

...... {From 4, $A = (\neg(\neg A))$}

25. $(\neg(\neg A)) \to A$

...... {From 23 and 24 by MP}

(2)

Show that $\{(\neg(\neg A))\} \vdash_L A$

1. $(\neg(\neg A))$

...... {Premise}

2. $(\neg(\neg A)) \to (A \to (\neg(\neg A)))$

...... {AS1 $(A \to (B \to A))$, $A = (\neg(\neg A))$, $B = A$}

3. $A \to (\neg(\neg A))$

...... {From 1 and 2 by MP}

4. $(\neg(\neg A)) \to A$

...... {From 3, $A = (\neg(\neg A))$}

5. A

...... {From 1 and 4 by MP}

By using the deduction theorem, we have:

$\vdash_L A \to (\neg(\neg A))$

(b) $\vdash_L A \to (\neg(\neg A))$

(1)

1. $(\neg(\neg(\neg A))) \to (\neg A)$

...... {From (a), $A = (\neg A)$}

2. $((\neg(\neg(\neg A))) \to (\neg A)) \to (A \to (\neg(\neg A)))$

...... {AS3 $(((\neg A) \to (\neg B)) \to (B \to A))$, $A = (\neg(\neg A))$, $B = (\neg A)$}

3. $A \to (\neg(\neg A))$

...... {From 1 and 2 by MP}

(2)

Show that $\{A\} \vdash_L (\neg(\neg A))$

1. A

...... {Premise}

2. $A \rightarrow ((\neg(\neg A)) \rightarrow A)$

…… {AS1 $(A \rightarrow (B \rightarrow A))$, $A = A$, $B = (\neg A)$}

3. $(\neg(\neg A)) \rightarrow A$

…… {From 1 and 2 by MP}

4. $A \rightarrow (\neg(\neg A))$

…… {From 3, $A = (\neg(\neg A))$}

5. $(\neg(\neg A))$

…… {From 1 and 4 by MP}

By using the deduction theorem, we have:
$\vdash_L A \rightarrow (\neg(\neg A))$

(c) $\vdash_L (\neg B) \rightarrow (B \rightarrow C)$

(1)
1. $(\neg B) \rightarrow ((\neg C) \rightarrow (\neg B))$

…… {AS1 $(A \rightarrow (B \rightarrow A))$, $A = (\neg B)$, $B = (\neg C)$}

2. $((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

…… {AS3 $((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A)$, $A = C$, $B = B$}

3. $(((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)) \rightarrow ((\neg B) \rightarrow (((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)))$

…… {AS1 $(A \rightarrow (B \rightarrow A))$, $A =$ …… {AS1 $(A \rightarrow (B \rightarrow A))$, $A = ((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$, $B = (\neg B)$}

4. $((\neg B) \rightarrow (((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)))$

…… {From 2 and 3 by MP}

5. $((\neg B) \rightarrow (((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C))) \rightarrow (((\neg B) \rightarrow ((\neg C) \rightarrow (\neg B))) \rightarrow ((\neg B) \rightarrow (B \rightarrow C)))$

…… {AS2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$, $A = (\neg B)$, $B = ((\neg C) \rightarrow (\neg B))$, $C = (B \rightarrow C)$}

6. $((\neg B) \rightarrow ((\neg C) \rightarrow (\neg B))) \rightarrow ((\neg B) \rightarrow (B \rightarrow C))$

…… {From 4 and 5 by MP}

7. $(\neg B) \rightarrow (B \rightarrow C)$

…… {From 1 and 6 by MP}

(2)
Show that $\{(\neg B), B\} \vdash_L C$
1. $(\neg B)$

…… {Premise}

2. B

…… {Premise}

3. $(\neg B) \rightarrow ((\neg C) \rightarrow (\neg B))$

…… {AS1 $(A \rightarrow (B \rightarrow A))$, $A = (\neg B)$, $B = (\neg C)$}

4. $(\neg C) \rightarrow (\neg B)$

…… {From 1 and 3 by MP}

5. $((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

…… {AS3 $((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A)$, $A = C$, $B = B$}

6. $B \rightarrow C$

…… {From 4 and 5 by MP}

7. C

…… {From 2 and 6 by MP}

By using the deduction theorem, we have:
$\vdash_L (\neg B) \rightarrow (B \rightarrow C)$

(d) $\vdash_L ((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

(1)
1. $((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

…… {AS3 $((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A)$, $A = C$, $B = B$}

(2)
Show that $\{((\neg C) \rightarrow (\neg B)), B\} \vdash_{HB} C$
1. $(\neg C) \rightarrow (\neg B)$

…… {Premise}

2. B

…… {Premise}

4. $(\neg(\neg A)) \rightarrow A$

…… {From (a)}

5. $((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

…… {AS3 $((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A)$, $A = C$, $B = B$}

6. $B \rightarrow C$

…… {From 1 and 5 by MP}

7. C

…… {From 2 and 6 by MP}

By using the deduction theorem, we have:
$\vdash_L ((\neg C) \rightarrow (\neg B)) \rightarrow (B \rightarrow C)$

(e) $\vdash_L (B \rightarrow C) \rightarrow ((\neg C) \rightarrow (\neg B))$

(1)
1. $(\neg(\neg A)) \rightarrow A$

…… {From (a)}

2. $(B \rightarrow C) \rightarrow ((\neg C) \rightarrow (\neg B))$

…… {AS3 $((\neg A) \rightarrow (\neg B)) \rightarrow (B \rightarrow A)$, $A = (\neg B)$, $B = (\neg C)$}

(2)
Show that $\{(B \rightarrow C), (\neg C)\} \vdash_{HB} (\neg B)$
1. $B \rightarrow C$

…… {Premise}

2. $\neg C$

…… {Premise}

4. $(\neg(\neg A)) \to A$

...... {From (a)}

5. $(B \to C) \to ((\neg C) \to (\neg B))$

...... {AS3 $((\neg A) \to (\neg B)) \to (B \to A)$, $A = \neg B$, $B = \neg C$}

6. $(\neg C) \to (\neg B)$

...... {From 1 and 5 by MP}

7. $\neg B$

...... {From 2 and 6 by MP}

By using the deduction theorem, we have:
$\vdash_L (B \to C) \to ((\neg C) \to (\neg B))$

(f) $\vdash_L B \to ((\neg C) \to (\neg(B \to C)))$

(1)

(2)
Show that $\{B, (\neg C)\} \vdash_L (\neg(B \to C))$
1. $\neg C$

...... {Premise}

2. B

...... {Premise}

3. $((B \to C) \to C) \to ((\neg C) \to (\neg(B \to C)))$

...... {AS3 $((\neg A) \to (\neg B)) \to (B \to A)$, $A = \neg (B \to C)$, $B = \neg C$}

4. $B \to ((B \to C) \to B)$

...... {AS1 $(A \to (B \to A))$, $A = B$, $B = (B \to C)$}

5. $(B \to C) \to B$

...... {From 2 and 4 by MP}

6. $((B \to C) \to (B \to C)) \to (((B \to C) \to B) \to ((B \to C) \to C))$

...... {AS2 $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$, $A = (B \to C)$, $B = B$, $C= C$}

7. $(X \to (B \to X))$

...... {AS1 $(A \to (B \to A))$, $A = X$, $B = B$}

8. $(X \to ((B \to X) \to X)) \to ((X \to (B \to X)) \to (X \to X))$

...... {AS2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, $A = X$, $B = (B \to X)$, $C = X$}

9. $(X \to (B \to X)) \to (X \to X)$

...... {From 7 and 8 by MP}

10. $(X \to X)$

...... {From 7 and 9 by MP}

11. $(B \to C) \to (B \to C)$

...... {From 10, $X = (B \to C)$}

12. $((B \to C) \to B) \to ((B \to C) \to C)$

...... {From 6 and 11 by MP}

13. $(B \to C) \to C$

...... {From 5 and 12 by MP}

14. $(\neg C) \to (\neg(B \to C))$

…… {From 3 and 13 by MP}

15. $\neg(B \to C)$

…… {From 1 and 14 by MP}

By using the deduction theorem, we have:
$\vdash_L B \to ((\neg C) \to (\neg(B \to C)))$

(g) $\vdash_L (B \to C) \to (((\neg B) \to C) \to C)$

(1)

(2)
Show that $\{((\neg B) \to C), (B \to C)\} \vdash_L C$
1. $((\neg B) \to C)$

…… {Premise}

2. $(B \to C)$

…… {Premise}

3. $(\neg B) \to ((\neg C) \to (\neg B))$

…… {AS1 $(A \to (B \to A))$, $A = (\neg B)$, $B = (\neg C)$}

4. $((\neg C) \to (\neg B)) \to (B \to C)$

…… {AS3 $((\neg A) \to (\neg B)) \to (B \to A)$, $A = C$, $B = B$}

5. $(\neg C) \to (\neg B)$

…… {From 1 and 3 by MP}

6. $((\neg C) \to (\neg B)) \to (B \to C)$

…… {AS3 $((\neg A) \to (\neg B)) \to (B \to A)$, $A = C$, $B = B$}

7. $B \to C$

…… {From 4 and 5 by MP}

8. C

…… {From 2 and 6 by MP}

By using the deduction theorem, we have:
$\vdash_L (B \to C) \to (((\neg B) \to C) \to C)$

7.2

(a) $\vdash_{HB} A \to (B \to (A \wedge B))$

(1)
1.

(2)
Show that $\{A , B\} \vdash_{HB} (A \wedge B)$
1. A

…… {Premise}

2. B

…… {Premise}

3. $A \rightarrow (A \rightarrow A)$

…… {AS $(A \rightarrow (B \rightarrow A))$, $A = A$, $B = A$}

4. $A \rightarrow A$

…… {From 1 and 3 by MP}

5. $(A \rightarrow A) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow (A \wedge B))$

…… {AS $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))$, $A = A$, $B = A$, $C = B$}

6. $(A \rightarrow B) \rightarrow (A \rightarrow (A \wedge B))$

…… {From 4 and 5 by MP}

7. $B \rightarrow (A \rightarrow B)$

…… {AS $(A \rightarrow (B \rightarrow A))$, $A = B$, $B = A$}

8. $A \rightarrow B$

…… {From 2 and 7 by MP}

9. $A \rightarrow (A \wedge B)$

…… {From 6 and 8 by MP}

10. $(A \wedge B)$

…… {From 1 and 9 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} A \rightarrow (B \rightarrow (A \wedge B))$

(b) $\vdash_{HB} ((A \wedge B) \leftrightarrow (B \wedge A))$

(1)

(2)
(i) Show that $\vdash_{HB} ((A \wedge B) \rightarrow (B \wedge A))$

1. $(A \wedge B)$

…… {Premise}

2. $(A \wedge B) \rightarrow B$

…… {AS $(A \wedge B) \rightarrow B$}

3. $(A \wedge B) \rightarrow A$

…… {AS $(A \wedge B) \rightarrow A$}

4. B

…… {From 1 and 2 by MP}

5. A

…… {From 1 and 3 by MP}

6. $B \rightarrow (A \rightarrow B)$

…… {AS $A \rightarrow (B \rightarrow A)$, $A = B$, $B = A$}

7. $A \rightarrow B$

…… {From 4 and 6 by MP}

8. $A \rightarrow (A \rightarrow A)$

…… {AS $(A \rightarrow (B \rightarrow A))$, $A = A$, $B = A$}

9. $A \rightarrow A$

…… {From 5 and 8 by MP}

10. $(A \rightarrow B) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow (B \wedge A))$

…… {AS $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))$, $A = A$, $B = B$, $C = A$}

11. $(A \rightarrow A) \rightarrow (A \rightarrow (B \wedge A)$

…… {From 7 and 10 by MP}

12. $A \rightarrow (B \wedge A)$

…… {From 9 and 11 by MP}

13. $(B \wedge A)$

…… {From 5 and 12 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} ((A \wedge B) \rightarrow (B \wedge A))$

(ii) Show that $\vdash_{HB} ((B \wedge A) \rightarrow (A \wedge B))$

1. $(B \wedge A)$

…… {Premise}

2. $(B \wedge A) \rightarrow B$

…… {AS $(B \wedge A) \rightarrow B$}

3. $(B \wedge A) \rightarrow A$

…… {AS $(B \wedge A) \rightarrow A$}

4. B

…… {From 1 and 2 by MP}

5. A

…… {From 1 and 3 by MP}

6. $B \rightarrow (A \rightarrow B)$

…… {AS $A \rightarrow (B \rightarrow A)$, $A = B$, $B = A$}

7. $A \rightarrow B$

…… {From 4 and 6 by MP}

8. $A \rightarrow (A \rightarrow A)$

…… {AS $(A \rightarrow (B \rightarrow A))$, $A = A$, $B = A$}

9. $A \rightarrow A$

…… {From 5 and 8 by MP}

10. $(A \rightarrow A) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow (A \wedge B))$

…… {AS $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))$, $A = A$, $B = A$, $C = B$}

11. $(A \to B) \to (A \to (A \wedge B)$

…… {From 9 and 10 by MP}

12. $A \to (A \wedge B)$

…… {From 7 and 11 by MP}

13. $(A \wedge B)$

…… {From 5 and 12 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} ((B \wedge A) \to (A \wedge B))$

(iii) Show that $\vdash_{HB} ((A \wedge B) \leftrightarrow (B \wedge A))$
1. $(A \wedge B) \to (B \wedge A)$

…… {From (i)}

2. $(B \wedge A) \to (A \wedge B)$

…… {From (ii)}

3. $((A \wedge B) \to (B \wedge A)) \to (((B \wedge A) \to (A \wedge B)) \to ((A \wedge B) \leftrightarrow (B \wedge A)))$

…… {AS $(A \to B) \to ((B \to A) \to (A \leftrightarrow B))$, $A = A \wedge B$, $B = B \wedge A$}

4. $((B \wedge A) \to (A \wedge B)) \to ((A \wedge B) \leftrightarrow (B \wedge A))$

…… {From 1 and 3 by MP}

5. $(A \wedge B) \leftrightarrow (B \wedge A)$

…… {From 2 and 4 by MP}

Above all, $\vdash_{HB} ((A \wedge B) \leftrightarrow (B \wedge A))$

(c) $\vdash_{HB} (A \vee B) \leftrightarrow (B \vee A)$

(1)

(2)

(i) Show that $\{(A \vee B)\} \vdash_{HB} (B \vee A)$

1. $(A \vee B)$

…… {Premise}

2. $A \to (B \vee A)$

…… {AS $B \to (A \vee B)$, $A = B$, $B = A$}

3. $B \to (B \vee A)$

…… {AS $A \to (A \vee B)$, $A = B$, $B = A$}

4. $(A \to (B \vee A)) \to ((B \to (B \vee A)) \to ((A \vee B) \to (B \vee A)))$

…… {AS $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$, $A = A$, $B = B$, $C = B \vee A$ }

5. $(B \to (B \vee A)) \to ((A \vee B) \to (B \vee A))$

…… {From 2 and 4 by MP}

6. $(A \lor B) \to (B \lor A)$

…… {From 3 and 5 by MP}

7. $(B \lor A)$

…… {From 1 and 6 by MP}

By using the deduction theorem, we have:

$\vdash_{HB} ((A \lor B) \to (B \lor A))$

(ii) Show that $\{(B \lor A)\} \vdash_{HB} (A \lor B)$

1. $(B \lor A)$

…… {Premise}

2. $A \to (A \lor B)$

…… {AS $A \to (A \lor B)$, A = A, B = B}

3. $B \to (A \lor B)$

…… {AS $B \to (A \lor B)$, A = A, B = B}

4. $(A \to (A \lor B)) \to ((B \to (A \lor B)) \to ((B \lor A) \to (A \lor B)))$

…… {AS $(A \to C) \to ((B \to C) \to ((A \lor B) \to C))$, A = A, B = B, C = $A \lor B$}

5. $(B \to (A \lor B)) \to ((B \lor A) \to (A \lor B))$

…… {From 2 and 4 by MP}

6. $(B \lor A) \to (A \lor B)$

…… {From 3 and 5 by MP}

7. $(A \lor B)$

…… {From 1 and 6 by MP}

By using the deduction theorem, we have:

$\vdash_{HB} ((B \lor A) \to (A \lor B))$

(iii) Show that $\vdash_{HB} ((A \lor B) \leftrightarrow (B \lor A))$

1. $(A \lor B) \to (B \lor A)$

…… {From (i)}

2. $(B \lor A) \to (A \lor B)$

…… {From (ii)}

3. $((B \lor A) \to (A \lor B)) \to (((A \lor B) \to (B \lor A)) \to ((B \lor A) \leftrightarrow (A \lor B)))$

…… {AS $(A \to B) \to ((B \to A) \to (A \leftrightarrow B))$, A = $B \lor A$, B = $A \lor B$}

4. $((A \lor B) \to (B \lor A)) \to ((B \lor A) \leftrightarrow (A \lor B))$

…… {From 2 and 3 by MP}

5. $(B \lor A) \leftrightarrow (A \lor B)$

…… {From 1 and 4 by MP}

Above all, $\vdash_{HB} ((A \lor B) \leftrightarrow (B \lor A))$

(d) $\vdash_{HB} (A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B))$

(1)
1.

(2)
Show that $\{(A \rightarrow B), (C \vee A)\} \vdash_{HB} (C \vee B)$

1. $(A \rightarrow B)$ …… {Premise}
2. $(C \vee A)$ …… {Premise}
3. $(A \rightarrow B) \rightarrow ((B \rightarrow (C \vee B)) \rightarrow (A \rightarrow (C \vee B)))$ …… {AS $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$, $A = A$, $B = B$, $C = C \vee B$}
4. $((B \rightarrow (C \vee B)) \rightarrow (A \rightarrow (C \vee B)))$ …… {From 1 and 3 by MP}
5. $B \rightarrow (C \vee B)$ …… {AS $B \rightarrow (A \vee B)$, $A = C$, $B = B$}
6. $A \rightarrow (C \vee B)$ …… {From 4 and 5 by MP}
7. $(C \rightarrow (C \vee B)) \rightarrow ((A \rightarrow (C \vee B)) \rightarrow ((C \vee A) \rightarrow (C \vee B)))$ …… {AS $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$, $A = C$, $B = A$, $C = C \vee B$ }
8. $C \rightarrow (C \vee B)$ …… {AS $A \rightarrow (A \vee B)$, $A = C$, $B = B$}
9. $(A \rightarrow (C \vee B)) \rightarrow ((C \vee A) \rightarrow (C \vee B))$ …… {From 7 and 8 by MP}
10. $(C \vee A) \rightarrow (C \vee B)$ …… {From 6 and 9 by MP}
11. $(C \vee B)$ …… {From 2 and 10 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} (A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B))$

(e) $\vdash_{HB} (A \vee (B \vee C)) \leftrightarrow ((A \vee B) \vee C))$

(1)
1.

(2)

(f) $\vdash_{HB} (A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$

(1)

1.

(2)

(g) $\vdash_{HB} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

(1)

1.

(2)

(i) Show that $\{(A \wedge (B \vee C))\} \vdash_{HB} ((A \wedge B) \vee (A \wedge C))$

1. $(A \wedge (B \vee C))$

…… {Premise}

2. $(A \wedge (B \vee C)) \rightarrow A$

…… {AS $(A \wedge B) \rightarrow A$, $A = A$, $B = B \vee C$}

3. $(A \wedge (B \vee C)) \rightarrow (B \vee C)$

…… {AS $(A \wedge B) \rightarrow B$, $A = A$, $B = B \vee C$}

4. A

…… {From 1 and 2 by MP}

5. $(B \vee C)$

…… {From 1 and 3 by MP}

6. $A \to (B \to A)$

…… {AS $A \to (B \to A)$, $A = A$, $B = B$}

7. $A \to (C \to A)$

…… {AS $A \to (B \to A)$, $A = A$, $B = C$}

8. $(B \to A)$

…… {From 4 and 6 by MP}

9. $(C \to A)$

…… {From 4 and 7 by MP}

10. $B \to (B \to B)$

…… {AS $A \to (B \to A)$, $A = B$, $B = B$}

11. $(B \to (B \to B)) \to (B \to B)$

…… {AS $(A \to (A \to B)) \to (A \to B)$, $A = B$, $B = B$}

12. $(B \to B)$

…… {From 10 and 11 by MP}

13. $C \to (C \to C)$

…… {AS $A \to (B \to A)$, $A = C$, $B = C$}

14. $(C \to (C \to C)) \to (C \to C)$

…… {AS $(A \to (A \to B)) \to (A \to B)$, $A = C$, $B = C$}

15. $(C \to C)$

…… {From 13 and 14 by MP}

16. $(B \to A) \to ((B \to B) \to (B \to (A \wedge B)))$

…… {AS $(A \to B) \to ((A \to C) \to (A \to (B \wedge C)))$, $A = B$, $B = A$, $C = B$}

17. $(C \to A) \to ((C \to C) \to (C \to (A \wedge C)))$

…… {AS $(A \to B) \to ((A \to C) \to (A \to (B \wedge C)))$, $A = C$, $B = A$, $C = C$}

18. $(B \to B) \to (B \to (A \wedge B))$

…… {From 8 and 16 by MP}

19. $(C \to C) \to (C \to (A \wedge C))$

…… {From 9 and 17 by MP}

20. $B \to (A \wedge B)$

…… {From 12 and 18 by MP}

21. $C \to (A \wedge C)$

…… {From 15 and 19 by MP}

22. a. $(A \wedge B)$

…… {From 5 and 20 by MP}

or b. $(A \wedge C)$

…… {From 5 and 21 by MP}

23. $(A \wedge B) \to ((A \wedge B) \vee (A \wedge C))$

…… {AS $(A \to (A \vee B))$, $A = A \wedge B$, $B = A \wedge C$}

24. $(A \wedge C) \to ((A \wedge B) \vee (A \wedge C))$

…… {AS $(B \to (A \vee B))$, $A = A \wedge B$, $B = A \wedge C$}

25. $(A \wedge B) \vee (A \wedge C)$

...... {From 22.a and 23 by MP}

or $(A \wedge B) \vee (A \wedge C)$

...... {From 22.b and 23 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} ((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)))$

(ii) Show that $\{((A \wedge B) \vee (A \wedge C))\} \vdash_{HB} (A \wedge (B \vee C))$

① First we can show that $\{(A \wedge B)\} \vdash_{HB} (B \vee C)$

1. $(A \wedge B)$

...... {premise}

2. $(A \wedge B) \rightarrow B$

...... {AS $(A \wedge B) \rightarrow B$, $A = A$, $B = B$}

3. B

...... {From 1 and 2 by MP}

4. $B \rightarrow (B \vee C)$

...... {AS $(A \rightarrow (A \vee B))$, $A = B$, $B = C$}

5. $(B \vee C)$

...... {From 3 and 4 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} ((A \wedge B) \rightarrow (B \vee C))$

② Then we can show that $\{((A \wedge B) \vee (A \wedge C))\} \vdash_{HB} (A \wedge (B \vee C))$

1. $(A \wedge B) \rightarrow (B \vee C)$

...... {From ①}

2. $((A \wedge B) \rightarrow A) \rightarrow (((A \wedge B) \rightarrow (B \vee C)) \rightarrow ((A \wedge B) \rightarrow (A \wedge (B \vee C))))$

...... {AS $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C)))$, $A = A \wedge B$, $B = A$, $C = B \vee C$}

3. $(A \wedge B) \rightarrow A$

...... {AS $(A \wedge B) \rightarrow A$, $A = A$, $B = B$}

4. $((A \wedge B) \rightarrow (B \vee C)) \rightarrow ((A \wedge B) \rightarrow (A \wedge (B \vee C)))$

...... {From 2 and 3 by MP}

5. $(A \wedge B) \rightarrow (A \wedge (B \vee C))$

...... {From 1 and 4 by MP}

6. $(A \wedge (B \vee C))$

...... {From 3 and 5 by MP}

By using the deduction theorem, we have:
$\vdash_{HB} (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$

(iii) Show that $\vdash_{HB} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

1. $((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)))$

...... {From (i)}

2. $(((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$

…… {From (ii)}

3. $((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \rightarrow ((((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C))) \rightarrow ((A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)))$

…… {AS $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$, $A = ((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)))$, $B = (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$}

4. $(((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C))) \rightarrow ((A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)))$

…… {From 1 and 3 by MP}

5. $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

…… {From 2 and 4 by MP}

Above all, $\vdash_{HB} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

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