An Introduction to Classical Propositional Calculus (CPC)

- Formal (Object) Language (Syntax) of **CPC**
- Principles of Structural Induction and Structural Recursion
- Semantics (Model Theory) of **CPC**
- Semantic (Model-theoretical, Logical) Consequence Relation
- Normal Forms and Uniform Notation of Formulas
- Deduction System (Proof Theory) of **CPC**
- Syntactic (Proof-theoretical, Deductive) Consequence Relation
- Hilbert Style Formal Logic Systems for **CPC**
- Gentzen's Natural Deduction System for **CPC**
- Gentzen's Sequent Calculus System for **CPC**
- Semantic Tableau System for **CPC**
- Resolution System for **CPC**
- Forward Deduction and Backward Deduction

The Basic Idea of Natural Deduction Systems

- Why natural deduction systems?
  - Hilbert style formal logic systems are suitable for meta-logical considerations because the small number of deduction rules making it easier to prove meta-theorems about object logics.
  - Hilbert style formal logic systems are difficult to construct proofs; they do not reflect the way mathematicians proceed when proving theorems.
  - What we need is a natural formalization of the way mathematicians do in informal arguments.
- The basic idea of natural deduction systems
  - The idea of ***subordinate proof***: derive conclusions from premises, and then discharge those premises to produce assumption-free results.
  - Mathematicians make assumptions, prove new results from these assumptions, and finally eliminate these assumptions.

NK: Gentzen's Natural Deduction System for CPC

- **NK**
  - ***Gentzen's natural deduction system for CPC*** [G. Gentzen, 1935]
  - Gentzen thought that **NK** gets at the heart of logical reasoning.
  - There are many natural deduction systems different from **NK**.
- Judgments
  - A string of the form $\Gamma \Rightarrow H$ called a ***judgment*** where Γ is a set of formulas and H is a formula, and its intuitive meaning is that H holds under the assumption that each member of Γ holds respectively.
  - Note: '$\Rightarrow$' is a meta-language(logic) symbol but not an object language symbol of **CPC**.
  - Note: Γ can be regarded as the "background knowledge" that H is based on.
  - The judgment $\Rightarrow H$ means that H holds under no assumption.

Deduction Rules of NK

- ***Deduction rules*** and their naive interpretation
  - $\dfrac{H_1 \quad \dots \quad H_n}{H}$
  - Meaning: We can conclude a judgment $\Gamma \Rightarrow H$ from the judgments $\Gamma_1 \Rightarrow H_1, \dots, \Gamma_n \Rightarrow H_n$ if, for all $i = 1, \dots, n$, each member of Γ_i is also a member of Γ.
  - $\dfrac{H_1 \quad \dots \quad \begin{array}{c}[A_i]\\ \vdots \\ H_i\end{array} \quad \dots \quad H_n}{H}$
  - Meaning: The same as the above except that only those members of Γ_i which are different from A_i are required to be in Γ (i.e., do not require A_i to be in Γ).

Deduction Rules of NK

- Implication

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A\to B}\ \to\text{I (Introduction)} \qquad \dfrac{A \quad A\to B}{B}\ \to\text{E (Elimination)}$$

$$(\ \{A\} \models_{CPC} B \ \text{ IFF } \ \models_{CPC} (A\to B)\ ) \qquad (\ \{A, (A\to B)\} \models_{CPC} B\ )$$

- Notes
  - If we can derive B from A (as a hypothesis), then we may conclude $A\to B$ (without the hypothesis A).
  - This agrees with the intuitive meaning of implication: $A\to B$ means "B follows from A".
  - If A is given and we know that B follows from A, then we have also B.

Deduction Rules of NK

- Contradiction and Negation

$$\dfrac{\bot}{A}\ \bot \qquad \text{where } \bot =_{df} A\wedge(\neg A) \text{ and } \neg A =_{df} A\to\bot$$

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\ \neg\text{I} \qquad \dfrac{A \quad \neg A}{\bot}\ \neg\text{E} \qquad \dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}\ \text{RAA (reductio ad absurdum)}$$

- Notes
  - The $\bot$ rule is called the *falsum rule*.
  - It expresses that from an absurdity we can derive everything (*ex falso sequitur quodlibet*).
  - The *reductio ad absurdum (RAA) rule*, is a formulation of ***the principle of proof by contradiction***: if one derives a contradiction from the hypothesis $\neg A$, then one has a derivation of A (without the hypothesis $\neg A$, of course).

Deduction Rules of NK

♣ Conjunction

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\wedge\mathrm{E}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{E}_2$$

($\{A, B\} \models_{\mathbf{CPC}} (A \wedge B)$)   ($\{(A \wedge B)\} \models_{\mathbf{CPC}} A$, $\{(A \wedge B)\} \models_{\mathbf{CPC}} B$)

♣ Disjunction

$$\frac{A}{A \vee B}\vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\vee\mathrm{I}_2 \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\vee\mathrm{E}$$

($\{A\} \models_{\mathbf{CPC}} (A \vee B)$, $\{B\} \models_{\mathbf{CPC}} (A \vee B)$)   ($\{(A \vee B), (A \rightarrow C), (B \rightarrow C)\} \models_{\mathbf{CPC}} C$)

♣ Note

- A common and important principle behind the organization of natural deduction systems is that deduction rules has been paired: one for introducing the connective and one for using it, in effect, eliminating it.

An Example of Theorem Proof in NK

♣ $\vdash_{\mathbf{NK}} (A \rightarrow (B \rightarrow (A \wedge B)))$?

$$\frac{{}_{2}\dfrac{{}_{1}\dfrac{{}^{2}A \quad {}^{1}B}{A \wedge B}\wedge\mathrm{I}}{B \rightarrow (A \wedge B)}\rightarrow\mathrm{I}}{A \rightarrow (B \rightarrow (A \wedge B))}\rightarrow\mathrm{I}$$

An Example of Deduction in NK

♣ $\{A, B\} \vdash_{\mathbf{NK}} (A \wedge B)$?

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{I}$$

NK

♣ Axioms

- $A \Rightarrow A$ $\Rightarrow A \vee (\neg A)$

♣ Deduction rules

- As the above.

♣ Proofs in **NK**

- A ***proof*** in **NK** is a finite sequence of judgments such that each member of the sequence is either an axiom or obtained from previous members of the sequence by application of a deduction rule.
- A ***proof of a formula A*** in **NK** is a derivation whose last member is the judgment $\Rightarrow A$.

♣ Theorems of NK

- A formula A is called a ***theorem*** of **NK** if the judgment $\Rightarrow A$ has a proof in **NK**.
- We can write $\vdash_{\mathbf{NK}} A$ to indicate that A as a theorem has a proof in **NK**.

Properties of NK

♣ The soundness of **NK**

- If $\vdash_{\mathbf{NK}} A$, then $\models_{\mathbf{CPC}} A$, for any $A \in \mathbf{WFF}$.

♣ The completeness of **NK**

- If $\models_{\mathbf{CPC}} A$, $\vdash_{\mathbf{NK}} A$, for any $A \in \mathbf{WFF}$.

♣ The relationship between **NK** and **L**

- The theorem set of **NK** is the same as that of **L**.
- **Th**(**NK**) = **Th**(**CPC**) = **Th**(**L**)

♣ Note

- **NK** and **L** just are two different representations of the same propositional logic system **CPC**.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Deduction Rules

INTRODUCTION RULES   ELIMINATION RULES

$$(\wedge I)\ \frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge I \qquad (\wedge E)\ \frac{\varphi \wedge \psi}{\varphi}\wedge E \quad \frac{\varphi \wedge \psi}{\psi}\wedge E$$

$$(\rightarrow I)\ \frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}\rightarrow I \qquad (\rightarrow E)\ \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\rightarrow E$$

Primitive connectives are ∧ and →.

$$(\bot)\ \frac{\bot}{\varphi}\bot \qquad (\mathrm{RAA})\ \frac{\begin{array}{c}[\neg\varphi]\\ \vdots \\ \bot\end{array}}{\varphi}\mathrm{RAA}$$

'$\neg\varphi$' is used here as an abbreviation for '$\varphi \rightarrow \bot$'.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Deduction Rules

♣ The rules for conjunction

- If we have φ and ψ, we may conclude φ∧ψ, and if we have φ∧ψ, we may conclude φ (or ψ).

♣ The rules for implication

- If we can derive ψ from φ (as a hypothesis), then we may conclude φ→ψ (without the hypothesis φ).
- This agrees with the intuitive meaning of implication: φ→ψ means "ψ follows from φ". We have written the rule (→I) in the above form to suggest a derivation. The notation will become clearer after we have defined derivations.
- The rule (→E) is also evident on the meaning of implication. If φ is given and we know that ψ follows from φ, then we have also ψ.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Deduction Rules

♣ The falsum rule

- The falsum rule, (⊥), expresses that from an absurdity we can derive everything (ex falso sequitur quodlibet).
- Note: This rue describes an intrinsic property of **CML**.

♣ The RAA rule

- The reductio ad absurdum rule, (RAA), is a formulation of the principle of proof by contradiction: if one derives a contradiction from the hypothesis ¬φ, then one has a derivation of φ (without the hypothesis ¬φ, of course).
- Note: This rue describes an intrinsic property of **CML**.

♣ The *cancellation of hypotheses*

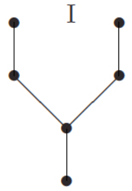
- In both (→I) and (RAA) hypotheses disappear, this is indicated by the striking out of the hypothesis. We say that such a hypothesis is cancelled.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Examples

I

$$\dfrac{\dfrac{\dfrac{[\varphi\wedge\psi]^1}{\psi}\wedge E \qquad \dfrac{[\varphi\wedge\psi]^1}{\varphi}\wedge E}{\psi\wedge\varphi}\wedge I}{\varphi\wedge\psi\to\psi\wedge\varphi}\to I_1$$

I



A Calculus of Natural Deduction for (CND) CPC [Dalen]: Examples

II

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\varphi\to\bot]^1}{\bot}\to E}{(\varphi\to\bot)\to\bot}\to I_1}{\varphi\to((\varphi\to\bot)\to\bot)}\to I_2$$

II

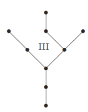
II′

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\neg\varphi]^1}{\bot}\to E}{\neg\neg\varphi}\to I_1}{\varphi\to\neg\neg\varphi}\to I_2$$

A Calculus of Natural Deduction for (CND) CPC [Dalen]: Examples

III

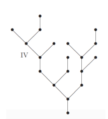
$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\wedge\psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi\wedge\psi]^1}{\varphi}\wedge E \qquad [\varphi\to(\psi\to\sigma)]^2}{\psi\to\sigma}\to E}{\sigma}\to E}{\varphi\wedge\psi\to\sigma}\to I_1}{(\varphi\to(\psi\to\sigma))\to(\varphi\wedge\psi\to\sigma)}\to I_2$$



A Calculus of Natural Deduction (CND) for CPC [Dalen]: Examples

IV

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \qquad \dfrac{[\varphi\leftrightarrow\neg\varphi]^3}{\varphi\to\neg\varphi}\wedge E}{\neg\varphi}\to E \qquad [\varphi]^1}{\bot}\to E}{\neg\varphi}\to I_1 \qquad \dfrac{[\varphi\leftrightarrow\neg\varphi]^3}{\neg\varphi\to\varphi}\wedge E}{\varphi}\to E \qquad \dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad \dfrac{[\varphi\leftrightarrow\neg\varphi]^3}{\varphi\to\neg\varphi}\wedge E}{\neg\varphi}\to E \qquad [\varphi]^2}{\bot}\to E}{\neg\varphi}\to I_2}{\bot}\to E}{\neg(\varphi\leftrightarrow\neg\varphi)}\to I_3$$



A Calculus of Natural Deduction for (CND) CPC [Dalen]: Derivation

♣ Notation of derivation

Notation

if $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array}$ are derivations with conclusions φ, φ', then $\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\end{array}}{\psi}, \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\psi}$

are derivations obtained by applying a derivation rule to φ (and φ and φ').

The cancellation of a hypothesis is indicated as follows: if $\begin{array}{c}\psi\\ \mathcal{D}\\ \varphi\end{array}$ is a derivation

with hypothesis ψ, then $\dfrac{\begin{array}{c}[\psi]\\ \mathcal{D}\\ \varphi\end{array}}{\sigma}$ is a derivation with ψ cancelled.

A Calculus of Natural Deduction (CND) for CPC [Dalen: Derivation

♣ Notes

- With respect to the cancellation of hypotheses, we do not necessarily cancel all occurrences of such a proposition ψ.
- This clearly is justified, as one feels that adding hypotheses does not make a proposition underivable (irrelevant information may always be added!).
- It is a matter of prudence, however, to cancel as much as possible. Why carry more hypotheses than necessary?
- Furthermore one may apply (→I) if there is no hypothesis available for cancellation, e.g., $\dfrac{\varphi}{\psi \to \varphi} \to I$ is a correct derivation, using just (→I).
- In summary, given a derivation tree of ψ (or ⊥), we obtain a derivation tree of φ→ψ (or φ) at the bottom of the tree and striking out some (or all) occurrences, if any, of φ (or ¬φ) on top of a tree.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Practical Use

♣ On the practical use of natural deduction

- If you want to give a derivation for a proposition it is advisable to devise some kind of strategy.
- Suppose that you want to show (φ∧ψ→σ)→(φ→(ψ→σ)) (the converse of Example III), then (since the proposition is an implicational formula) the rule (→I) suggests itself. So try to derive φ→(ψ→σ) from φ∧ψ→σ.
- Now we know where to start and where to go to.
- To make use of φ∧ψ→σ we want φ∧ψ (for (→E)), and to get φ→(ψ→σ) we want to derive ψ→σ from φ. So we may add φ as a hypothesis and look for a derivation of ψ→σ.
- Again, this asks for a derivation of σ from ψ, so add ψ as a hypothesis and look for a derivation of σ. By now we have the following hypotheses available: φ∧ψ→σ, φ and ψ.
- Keeping in mind that we want to eliminate φ∧ψ it is evident what we should do.
- After making a number of derivations one gets the practical conviction that one should first take propositions apart from the bottom upwards, and then construct the required propositions by putting together the parts in a suitable way.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: RAA

♣ A particular point which tends to confuse novices

- $\dfrac{\begin{array}{c}[\varphi]\\ \vdots\\ \bot\end{array}}{\neg\varphi} \to I$ and $\dfrac{\begin{array}{c}[\neg\varphi]\\ \vdots\\ \bot\end{array}}{\varphi}\ \mathrm{RAA}$ look very much alike. Are they not both cases of Reductio ad absurdum (RAA)?
- The left derivation tells us (informally) that the assumption of φ leads to a contradiction, so φ cannot be the case. This is in our terminology the meaning of "not φ".
- The right derivation tells us that the assumption of ¬φ leads to a contradiction, hence (by the same reasoning) ¬φ cannot be the case. So, on account of the meaning of negation, we only would get ¬¬φ. It is by no means clear that ¬¬φ is equivalent to φ (indeed, this is denied by the intuitionists), so it is an extra property of our logic. (This is confirmed in a technical sense: ¬¬φ→φ is not derivable in the system without RAA.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Derivations

♣ Definition: The set of derivations
(Note: **PROP** = **WFF**$_{\mathbf{CPC}}$)

Definition 1.4.1 *The set of derivations is the smallest set X such that*
(1) The one element tree φ belongs to X for all φ ∈PROP.

(2∧) If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array} \in X$, *then* $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\varphi \wedge \varphi'} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array} \in X$, *then* $\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array}}{\varphi}, \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\wedge\psi\end{array}}{\psi} \in X$.

(2→) If $\begin{array}{c}\varphi\\ \mathcal{D}\\ \psi\end{array} \in X$, *then* $\dfrac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi\to\psi} \in X$.

If $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}, \begin{array}{c}\mathcal{D}'\\ \varphi\to\psi\end{array} \in X$, *then* $\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi\to\psi\end{array}}{\psi} \in X$.

(2⊥) If $\begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X$, *then* $\dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

If $\begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X$, *then* $\dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi} \in X$.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Derivations

♣ Definition: The set of derivations

- The bottom formula of a derivation is called its conclusion.

♣ The principle of induction on derivations

- We have a principle of induction on *D*: Let *A* be a property. If *A*(*D*) holds for one element derivations and *A* is preserved under the clauses (2∧), (2→) and (2⊥), then *A*(*D*) holds for all derivations.

♣ Exercise (1.4.9): The size of a derivation

- The size, denoted by *s*(*D*), of a derivation is the number of proposition occurrences in *D*.
- Give an inductive definition of *s*(*D*).
- Show that one can prove properties of derivations by it induction on the size.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Syntactic (Proof-theoretical, Deductive) Consequence Relation

♣ ***The syntactic (proof-theoretical, deductive) consequence relation***

Definition 1.4.2 *The relation $\Gamma \vdash \varphi$ between sets of propositions and propositions is defined by: there is a derivation with conclusion φ and with all (uncancelled) hypotheses in Γ. (See also exercise 6).*

- The consequence relation $\Gamma \mid\!-_{\mathrm{CND}} \varphi$ (φ is derivable from Γ) between sets of propositions (the power set of propositions) and propositions (the set of propositions) is defined by: there is a derivation with conclusion φ and with all (uncancelled) hypotheses in Γ.
- If $\Gamma = \phi$, we write $\mid\!-_{\mathrm{CND}} \varphi$, and we say that φ is a theorem.

♣ Notes

- By definition, Γ may contain many superfluous "hypotheses".
- We could have avoided the notion of 'derivation' and taken instead the notion of 'derivability' as fundamental.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Syntactic (Proof-theoretical, Deductive) Consequence Relation

♣ Exercise (1.4.6): A stricter definition

- Give a recursive definition of the function *Hyp* which assigns to each derivation *D* its set of hypotheses *Hyp*(*D*) (this is a bit stricter than the notion in definition 1.4.2, since it the smallest set of hypotheses, i.e. hypotheses without 'garbage').

♣ Exercise (1.4.7): Substitution operator for derivations

- $D[\varphi/p]$ is obtained by replacing each occurrence of p in each proposition in D by φ.
- Give a recursive definition of $D[\varphi/p]$.
- Show that $D[\varphi/p]$ is a derivation if D is one, and that if $\Gamma \mid\!-_{\mathrm{CND}} \sigma$ then $\Gamma[\varphi/p] \mid\!-_{\mathrm{CND}} \sigma[\varphi/p]$.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Syntactic (Proof-theoretical, Deductive) Consequence Relation

♣ Consequence relation examples

Lemma 1.4.3
(a) $\Gamma \vdash \varphi$ *if* $\varphi \in \Gamma$,
(b) $\Gamma \vdash \varphi, \Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi$,
(c) $\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$ *and* $\Gamma \vdash \psi$,
(d) $\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \to \psi$,
(e) $\Gamma \vdash \varphi, \Gamma' \vdash \varphi \to \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi$,
(f) $\Gamma \vdash \bot \Rightarrow \Gamma \vdash \varphi$,
(g) $\Gamma \cup \{\neg\varphi\} \vdash \bot \Rightarrow \Gamma \vdash \varphi$.

Proof. Immediate from the definition of derivation.

- Note: "$A \Rightarrow B$" means "if A then B".

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Syntactic (Proof-theoretical, Deductive) Consequence Relation

♣ Theorem examples ($\neg$ and $\leftrightarrow$ are used as abbreviations)

Theorem 1.4.4
(1) $\vdash \varphi \to (\psi \to \varphi)$,
(2) $\vdash \varphi \to (\neg\varphi \to \psi)$,
(3) $\vdash (\varphi \to \psi) \to [(\psi \to \sigma) \to (\varphi \to \sigma)]$,
(4) $\vdash (\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$,
(5) $\vdash \neg\neg\varphi \leftrightarrow \varphi$,
(6) $\vdash [\varphi \to (\psi \to \sigma)] \leftrightarrow [\varphi \wedge \psi \to \sigma]$,
(7) $\vdash \bot \leftrightarrow (\varphi \wedge \neg\varphi)$.

Proof.

1. $$\dfrac{\dfrac{[\varphi]^1}{\psi \to \varphi} \to I}{\varphi \to (\psi \to \varphi)} \to I_1$$

2. $$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \quad [\neg\varphi]^1}{\bot} \to E}{\psi} \bot}{\neg\varphi \to \psi} \to I_1}{\varphi \to (\neg\varphi \to \psi)} \to I_2$$

3. $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi \to \psi]^3}{\psi} \to E \qquad [\psi \to \sigma]^2}{\sigma} \to E}{\varphi \to \sigma} \to I_1}{(\psi \to \sigma) \to (\varphi \to \sigma)} \to I_2}{(\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))} \to I_3$$

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Syntactic (Proof-theoretical, Deductive) Consequence Relation

♣ Theorem examples ($\neg$ and $\leftrightarrow$ are used as abbreviations)

4. For one direction, substitute $\bot$ for σ in 3, then $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.
Conversely:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi]^1 \quad [\neg\psi \to \neg\varphi]^3}{\neg\varphi} \to E \qquad [\varphi]^2}{\bot} \to E}{\psi} \mathrm{RAA}_1}{\varphi \to \psi} \to I_2}{(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)} \to I_3$$

So now we have $$\dfrac{\begin{matrix}\mathcal{D} \\ (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)\end{matrix} \qquad \begin{matrix}\mathcal{D}' \\ (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)\end{matrix}}{(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)} \wedge I$$

5. We already proved $\varphi \to \neg\neg\varphi$ as an example. Conversely:

$$\dfrac{\dfrac{\dfrac{[\neg\varphi]^1 \quad [\neg\neg\varphi]^2}{\bot} \to E}{\varphi} \mathrm{RAA}_1}{\neg\neg\varphi \to \varphi} \to I_2$$

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Summary

♣ A calculus of natural deduction

- The deductive system is called the "calculus of natural deduction" for a good reason because its manner of making inferences corresponds to the reasoning we intuitively use.
- The rules present means to take formulas apart, or to put them together.
- A derivation then consists of a skilful manipulation of the rules, the use of which is usually suggested by the form of the formula we want to prove.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: An example to illustrate the general strategy of building derivations

♣ The converse of the previous example III ((φ∧ψ→σ)→(φ→(ψ→σ)))

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \quad [\psi]^1}{\varphi\wedge\psi}\wedge I \qquad [\varphi\wedge\psi\to\sigma]^3}{\sigma}\to E}{\psi\to\sigma}\to I_1}{\varphi\to(\psi\to\sigma)}\to I_2}{(\varphi\wedge\psi\to\sigma)\to(\varphi\to(\psi\to\sigma))}\to I_3 \qquad \mathbf{III} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi\wedge\psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi\wedge\psi]^1}{\varphi}\wedge E \quad [\varphi\to(\psi\to\sigma)]^2}{\psi\to\sigma}\to E}{\sigma}\to E}{\varphi\wedge\psi\to\sigma}\to I_1}{(\varphi\to(\psi\to\sigma))\to(\varphi\wedge\psi\to\sigma)}\to I_2$$

- To prove (φ∧ψ→σ)→(φ→(ψ→σ)) there is just one initial step: assume φ∧ψ→σ and try to derive φ→(ψ→σ).
- Now we can either look at the assumption or at the desired result. Let us consider the latter one first: to show φ→(ψ→σ), we should assume φ and derive ψ→σ, but for the latter we should assume ψ and derive σ. So, altogether we may assume φ∧ψ→σ and φ and ψ. Now the procedure suggests itself: derive φ∧ψ from φ and ψ, and σ from φ∧ψ and φ∧ψ→σ.
- Had we considered φ∧ψ→σ first, then the only way to proceed is to add φ∧ψ and apply →E. Now φ∧ψ either remains an assumption, or it is obtained from something else. It immediately occurs to the reader to derive φ∧ψ from φ and ψ. But now he will build up the derivation we obtained above.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Properties

♣ Definition: The ***consistency*** of a set of propositions (formulas)

- A set Γ ⊆ **WFF** is consistent if Γ $|\!-_{\mathbf{CND}}$ ⊥ does not hold, Γ $|\!/\!-_{\mathbf{CND}}$ ⊥.

♣ Lemmas

- Lemma: The following three conditions are equivalent: (1) Γ is consistent; (2) For no φ ∈ **WFF**, Γ $|\!-_{\mathbf{CND}}$ φ, and Γ $|\!-_{\mathbf{CND}}$ ¬φ; (3) There is at least one φ such that Γ $|\!/\!-_{\mathbf{CND}}$ φ.
- Lemma: If there is a model ***M*** = (v_a, v_f) such that v_f(ψ) = **T** for all ψ ∈ Γ, then Γ is consistent.
- Lemma: (1) Γ ∪ {¬φ} is inconsistent, then Γ $|\!-_{\mathbf{CND}}$ φ; (2) Γ ∪ {φ} is inconsistent, then Γ $|\!-_{\mathbf{CND}}$ ¬φ.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Properties

♣ Definition: The ***maximally consistency*** of a set of propositions (formulas)

- A set Γ ⊆ **WFF** is maximally consistent IFF (1) Γ is consistent; (2) For any Γ ⊆ Γ', if Γ' is consistent then Γ = Γ'.

♣ Lemmas and corollaries

- L: Each consistent set Γ is contained in a maximally consistent set Γ*.
- Lemma: If Γ is maximally consistent, then Γ is closed under derivability, i.e., if Γ $|\!-_{\mathbf{CND}}$ φ, then φ ∈ Γ.
- L: Le Γ be maximally consistent, then (1) For all φ ∈ **WFF**, either φ ∈ Γ or ¬φ ∈ Γ; (2) For all φ, ψ ∈ **WFF**, φ→ψ ∈ Γ IFF if φ ∈ Γ then ψ ∈ Γ.
- Corollary: If Γ is maximally consistent, then φ ∈ Γ IFF ¬φ ∉ Γ.
- Lemma: If Γ consistent, then there is a model ***M*** = (v_a, v_f) such that v_f(ψ) = **T** for all ψ ∈ Γ.
- Corollary: Γ $|\!-_{\mathbf{CND}}$ φ does not hold, Γ $|\!/\!-_{\mathbf{CND}}$ φ, IFF there is a model ***M*** = (v_a, v_f) such that v_f(ψ) = **T** for all ψ ∈ Γ and v_f(φ) = **F**.

A Calculus of Natural Deduction (CND) for CPC [Dalen]: Soundness and Completeness

♣ A fundamental question about **CND** and **CPC**

- What is the relationship between **Th**(**CND**) and **Th**(**CPC**)?

♣ ***Soundness theorem for CND***

- If Γ $|\!-_{\mathbf{CND}}$ φ then Γ $|\!=_{\mathbf{CPC}}$ φ, for any φ ∈ **WFF** and any Γ ⊆ **WFF**.

♣ ***Completeness theorem for CND***

- If Γ $|\!=_{\mathbf{CPC}}$ φ then Γ $|\!-_{\mathbf{CND}}$ φ, for any φ ∈ **WFF** and any Γ ⊆ **WFF**.

♣ **CND** as a formal logic system of **CPC**

- The theorem set of **CND** is the same as that of **CPC**.
- **Th**(**CND**) = **Th**(**CPC**) = **Th**(**L**) = **Th**(**NK**)

Natural Deduction for CPC [Fitting]: Notation

♣ The notation for displaying subordinate proofs in a natural system

- A subordinate proof are written in boxes, with the first line inside a box being the particular assumption made in that subordinate proof, and the first line below the box being the result of discharging the assumption.

♣ A deduction rule for implication

A typical rule of many natural deduction systems follows: If one can derive Y from X as an assumption, then one can discharge the assumption X and conclude that one has proved $X \supset Y$. This is given schematically in Figure 4.1.

$$\boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}}$$

$$X \supset Y$$

- FIGURE 4.1. A Natural Deduction Rule for Implication

Natural Deduction for CPC [Fitting]: Active Formulas

♣ Definition: ***Active formulas***

- The formulas active at a stage in a proof are those occurring in boxes that have not closed by this stage.

♣ The ***Modus Ponens rule***

- From *X* and *X*→*Y*, conclude *Y*, provided both *X* and *X*→*Y* are active.

Natural Deduction for CPC [Fitting]: A Deduction Example

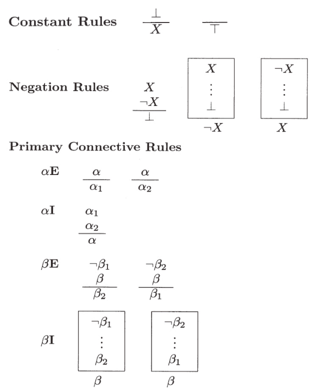
1. $P \supset (Q \supset R)$
2. Q
3. P
4. $Q \supset R$
5. R
6. $P \supset R$
7. $Q \supset (P \supset R)$
8. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

- **FIGURE 4.2.** Proof of $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$
- 1 through 3 are assumptions, each starting a subordinate proof; 4 is from 1 and 3 by Modus Ponens (note that at this point no boxes have closed, so 1 and 3 are both active); and 5 is from 2 and 4, again by Modus Ponens.
- Now a box is closed, assumption 3 is discharged, to conclude 6.
- Note that formulas 3 through 5 are no longer active.
- Two more assumption discharges produce 7 and 8.

α-Formulas and β-Formulas and Their Components

Conjunctive			Disjunctive		
α	α_1	α_2	β	β_1	β_2
$X \wedge Y$	X	Y	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$
$\neg(X \vee Y)$	$\neg X$	$\neg Y$	$X \vee Y$	X	Y
$\neg(X \rightarrow Y)$	X	$\neg Y$	$X \rightarrow Y$	$\neg X$	Y
$\neg(X \leftarrow Y)$	$\neg X$	Y	$X \leftarrow Y$	X	$\neg Y$
$\neg(X \neg\wedge Y)$	X	Y	$X \neg\wedge Y$	$\neg X$	$\neg Y$
$X \neg\vee Y$	$\neg X$	$\neg Y$	$\neg(X \neg\vee Y)$	X	Y
$X \neg\rightarrow Y$	X	$\neg Y$	$\neg(X \neg\rightarrow Y)$	$\neg X$	Y
$X \neg\leftarrow Y$	$\neg X$	Y	$\neg(X \neg\leftarrow Y)$	X	$\neg Y$

Natural Deduction for CPC [Fitting]: Deduction Rules



Natural Deduction for CPC [Fitting]: Derived Rules

♣ Definition: ***Derived rules***

- A deduction rule is derived if its addition does not changes the strength of the system; or more precisely, a deduction rule is derived if we can translate any use of it away.

♣ Two useful derived rules: Double negation rules

- $\dfrac{\neg\neg X}{X} \qquad \dfrac{X}{\neg\neg X}$

$$\begin{array}{ll} & \vdots \\ (n) & \neg\neg X \\ (n+1) & \neg X \\ (n+2) & \perp \\ (n+3) & X \end{array}$$

- To show the first of these rules is derived, suppose line (n) in a proof is the formula $\neg\neg X$. Then, without using the proposed new negation rules, we may proceed as above.
- Here line n+1 is an assumption. Line n+2 follows from n and n+1 by the first of the official negation rules. Then line n+3 follows using the third negation rule. Thus, we have added X without using any rules other than the basic ones; consequently, a rule allowing us to add X directly when $\neg\neg X$ is present is a derived rule.

Natural Deduction for CPC [Fitting]: Using Double Negation Rules

♣ Modus Ponens with double negation and ***Modus Tollens***

- In the βE rules, suppose we take β to be $X \rightarrow Y$, so that $\beta 1 = \neg X$ and $\beta 2 = Y$. Then the rules become the following:

$$\begin{array}{c} \neg\neg X \\ X \supset Y \\ \hline Y \end{array} \qquad \begin{array}{c} \neg Y \\ X \supset Y \\ \hline \neg X \end{array}$$

- The first is almost Modus Ponens and has the effect of it when used in conjunction with a double negation rule. The second of these is a common rule, called Modus Tollens.

Natural Deduction for CPC [Fitting]: Using Double Negation Rules

♣ The rule for introducing implication with double negation and the principle of contraposition

- The βI rules can become the following:

$$\begin{array}{c} \boxed{\begin{array}{c} \neg\neg X \\ \vdots \\ Y \end{array}} \\ X \supset Y \end{array} \qquad \begin{array}{c} \boxed{\begin{array}{c} \neg Y \\ \vdots \\ \neg X \end{array}} \\ X \supset Y \end{array}$$

- The first of these is almost the rule for introducing implication, but a double negation rule is also needed.
- The second embodies the principle of contraposition.

Natural Deduction for CPC [Fitting]: A Deduction Example

1. $\neg(P \wedge Q)$
2. $\neg\neg P$
3. $\neg Q$
4. $\neg P \vee \neg Q$
5. $\neg(P \wedge Q) \supset (\neg P \vee \neg Q)$

- FIGURE 4.3. A Natural Deduction Proof of $\neg(P \wedge Q) \supset (\neg P \vee \neg Q)$
- 1 and 2 are assumptions. Taking β to be $\neg(P \wedge Q)$, the first βE rule says; from $\neg\neg P$ and $\neg(P \wedge Q)$, conclude $\neg Q$. Using this, 3 follows from 1 and 2.
- Now, taking β to be $\neg P \vee \neg Q$, the first βI rule says; conclude $\neg P \vee \neg Q$ from a derivation of $\neg Q$ from $\neg\neg P$. This allows us to derive 4.
- Finally, 5 follows, again using βI.

Natural Deduction for CPC [Fitting]: Derivation

♣ Definition: ***Derivation***

- A natural deduction derivation of X from a set Γ of formulas meets the conditions for being a proof of X but also allows the following additional rule: At any stage, any member of Γ may be used as a line.
- We write $\Gamma \mid\!-_{\mathbf{PND}} X$ to indicate there is a derivation of X from Γ in the propositional natural deduction system.

♣ ***Soundness theorem for PND***

- If $\Gamma \mid\!-_{\mathbf{PND}} X$ then $\Gamma \models_{\mathbf{CPC}} X$, for any $X \in$ **WFF** and any $\Gamma \subseteq$ **WFF**.

♣ ***Completeness theorem for PND***

- If $\Gamma \models_{\mathbf{CPC}} X$ then $\Gamma \mid\!-_{\mathbf{PND}} X$, for any $X \in$ **WFF** and any $\Gamma \subseteq$ **WFF**.

♣ **PND** as a formal logic system of **CPC**

- The theorem set of **PDG** is the same as that of **CPC**.
- **Th**(**PND**) = **Th**(**CPC**) = **Th**(**L**) = **Th**(**NK**) = **Th**(**CND**)

Natural Deduction for CPC [R&C]: A Deduction Rule Example

♣ Natural deduction rule for introducing implication

Suppose that, on the assumption that some statement P is true, Q can be shown to hold, possibly via some intervening proof steps. Since, given P, Q holds we can conclude (using the truth table for $\rightarrow$) that P $\rightarrow$ Q holds (given nothing). We can represent this sort of reasoning with a diagram

$$\frac{\begin{array}{c}\cancel{P}\\ \vdots\\ Q\end{array}}{P \rightarrow Q}$$

where P is crossed through to remind us that, once P $\rightarrow$ Q has been derived, P need no longer be considered an assumption, P $\rightarrow$ Q is true outright. We say that the original assumption P has been discharged in the process of going from Q to P $\rightarrow$ Q.

Natural Deduction for CPC [R&C]: Deduction Rules

♣ Natural deduction rules

$$\wedge\mathrm{I}\ \frac{S \quad T}{S \wedge T} \qquad \wedge\mathrm{E}\ \frac{S \wedge T}{S} \qquad \wedge\mathrm{E}\ \frac{S \wedge T}{T}$$

$$\vee\mathrm{I}\ \frac{S}{S \vee T} \qquad \vee\mathrm{I}\ \frac{T}{S \vee T} \qquad \vee\mathrm{E}\ \frac{S \vee T \quad \begin{array}{c}\cancel{S}\\ \vdots\\ R\end{array} \quad \begin{array}{c}\cancel{T}\\ \vdots\\ R\end{array}}{R}$$

$$\rightarrow\mathrm{I}\ \frac{\begin{array}{c}\cancel{S}\\ \vdots\\ T\end{array}}{S \rightarrow T} \qquad \rightarrow\mathrm{E}\ \frac{S \rightarrow T \quad S}{T} \qquad \bot\ \frac{\bot}{S} \qquad \mathrm{C}\ \frac{\begin{array}{c}\cancel{\neg S}\\ \vdots\\ \bot\end{array}}{S}$$

-
- Note: '$\neg S$' is defined as '$S \rightarrow \bot$'.

Natural Deduction for CPC [R&C]: Deduction Rules with Negation

♣ Defining negation in terms of '⊥'

- We have chosen to define negation in terms of '⊥' which is the symbol for a proposition that is false in every valuation.
- $P \rightarrow \bot$ is false when P is true and true when P is false, so $P \rightarrow \bot$ and $\neg P$ are logically equivalent.
- For clarity and brevity we will allow negated formulas to appear in natural deduction proofs on the understanding that $\neg$P stands at all points for $P \rightarrow \bot$.

$$\frac{\begin{array}{c}\cancel{P}\\ \vdots\\ \bot\end{array}}{P \rightarrow \bot} \qquad \frac{P \quad P \rightarrow \bot}{\bot}$$

♣ Natural deduction rules with negation

- There are implicit introduction and elimination rules for negation because we have as instances of →I and →E the rules.
- Our rule ⊥ comes from the fact that an inconsistent set of formulas entails any formula. Note the difference between this and the rule C. The latter allows you to discharge the formula $\neg$P, the former does not.

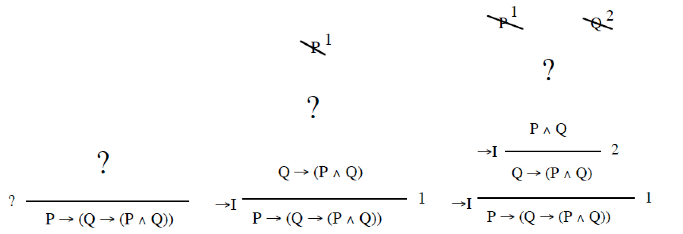
Natural Deduction for CPC [R&C]: A Deduction Example

♣ $\mid\!-$ P→(Q→(P∧Q))

$$\rightarrow\mathrm{I}\ \dfrac{\rightarrow\mathrm{I}\ \dfrac{\wedge\mathrm{I}\ \dfrac{\cancel{P}^{2} \qquad \cancel{Q}^{1}}{P \wedge Q}}{Q \rightarrow (P \wedge Q)}\,1}{P \rightarrow (Q \rightarrow (P \wedge Q))}\,2$$

-
- The application of a rule is indicated by a horizontal line, on the left of which is the name of the rule used.
- For instance, in the example above, the first rule applied is ∧I (which in discussing it with someone else you would pronounce "and introduction").
- Assumptions are identified with numbers as they are introduced and when they are discharged, i.e. used in a rule such as →I which brings down the assumption as the antecedent of an introduced implication, then the corresponding number is written to the right of the discharging rule line and the assumption is crossed through to show that it has been discharged.
- Note that the rule names and numbers are not part of the proof; they are just annotations to make it easier to check that the rules have been followed.

Natural Deduction for CPC [R&C]: A Deduction Example



- We know that we have to end up with $P\to(Q\to(P\wedge Q))$ as the bottom line, so what could the rule application that gives this be? For the moment we are working backwards. We start by writing down as left above.
- Now ∧E and ∨E are not very likely because you would be coming from a more complicated formula that nevertheless still contains the formula you want. *C* is a possibility, but we know by experience that this is something to try when all else fails (as we see several times later). The obvious choice is the introduction rule for the principal connective, namely →I. So we write down as centre above.
- Now we are in the same position again and the same argument gives. We see that things have worked out well for us because we have all the ingredients for an instance of ∧I, so we complete the proof shown as right above.

Natural Deduction for CPC [R&C]: Fitch Box Representation of Deduction

♣ Fitch Box Representation

- The idea is reminiscent of Chinese boxes, boxes-within boxes.
- When you make an assumption you put a box round it. When you discharge the assumption the box is closed and the derived formula is written outside the box.
- You can imagine the box, if you like, as a machine, literally a black box, which verifies the formula and "outputs" it.
- The box is part of the world outside, so you can copy into the box any formula from outside it, but no formula from inside the box, which of course may depend on assumptions local to the box, may be copied to the outside.

P	Assumption
Q	Assumption
P	From outer box
P ∧ Q	∧I
Q → (P ∧ Q)	→I
P → (Q → (P ∧ Q))	→I

α-Formulas and β-Formulas and Their Components [Ben-Ari]

α	α_1	α_2	β	β_1	β_2
$\neg\neg A$	A				
$\neg(A_1 \wedge A_2)$	$\neg A_1$	$\neg A_2$	$B_1 \wedge B_2$	B_1	B_2
$A_1 \vee A_2$	A_1	A_2	$\neg(B_1 \vee B_2)$	$\neg B_1$	$\neg B_2$
$A_1 \to A_2$	$\neg A_1$	A_2	$\neg(B_1 \to B_2)$	B_1	$\neg B_2$
$A_1 \uparrow A_2$	$\neg A_1$	$\neg A_2$	$\neg(B_1 \uparrow B_2)$	B_1	B_2
$\neg(A_1 \downarrow A_2)$	A_1	A_2	$B_1 \downarrow B_2$	$\neg B_1$	$\neg B_2$
$\neg(A_1 \leftrightarrow A_2)$	$\neg(A_1 \to A_2)$	$\neg(A_2 \to A_1)$	$B_1 \leftrightarrow B_2$	$B_1 \to B_2$	$B_2 \to B_1$
$A_1 \oplus A_2$	$\neg(A_1 \to A_2)$	$\neg(A_2 \to A_1)$	$\neg(B_1 \oplus B_2)$	$B_1 \to B_2$	$B_2 \to B_1$

Fig. 3.1 Classification of α- and β-formulas

- Note: The classification into α-formulas and β-formulas is the dual of the classification of Fig. 2.8.

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Definition

Definition 3.2 (Gentzen system $\mathscr{G}$) An *axiom* of $\mathscr{G}$ is a set of literals U containing a complementary pair. *Rule of inference* are used to infer a set of formulas U from one or two other sets of formulas U_1 and U_2; there are two types of rules, defined with reference to Fig. 3.1:

- Let $\{\alpha_1, \alpha_2\} \subseteq U_1$ and let $U_1' = U_1 - \{\alpha_1, \alpha_2\}$. Then $U = U_1' \cup \{\alpha\}$ can be inferred.
- Let $\{\beta_1\} \subseteq U_1$, $\{\beta_2\} \subseteq U_2$ and let $U_1' = U_1 - \{\beta_1\}$, $U_2' = U_2 - \{\beta_2\}$. Then $U = U_1' \cup U_2' \cup \{\beta\}$ can be inferred.

The set or sets of formulas U_1, U_2 are the *premises* and set of formulas U that is inferred is the *conclusion*. A set of formulas U that is an axiom or a conclusion is said to be *proved*, denoted $\vdash U$. The following notation is used for rules of inference:

$$\frac{\vdash U_1' \cup \{\alpha_1, \alpha_2\}}{\vdash U_1' \cup \{\alpha\}} \qquad \frac{\vdash U_1' \cup \{\beta_1\} \qquad \vdash U_2' \cup \{\beta_2\}}{\vdash U_1' \cup U_2' \cup \{\beta\}}.$$

Braces can be omitted with the understanding that a sequence of formulas is to be interpreted as a set (with no duplicates). ■

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Examples

Example 3.3 The following set of formulas is an axiom because it contains the complementary pair $\{r, \neg r\}$:

$$\vdash p \wedge q, q, r, \neg r, q \vee \neg r.$$

The disjunction rule for $A_1 = q$, $A_2 = \neg r$ can be used to deduce:

$$\frac{\vdash p \wedge q, q, r, \neg r, q \vee \neg r}{\vdash p \wedge q, r, q \vee \neg r, q \vee \neg r}.$$

Removing the duplicate formula $q \vee \neg r$ gives:

$$\frac{\vdash p \wedge q, q, r, \neg r, q \vee \neg r}{\vdash p \wedge q, r, q \vee \neg r}.$$

Note that the premises $\{q, \neg r\}$ are no longer elements of the conclusion.

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Examples

Example 3.4 Prove $\vdash (p \vee q) \to (q \vee p)$ in $\mathscr{G}$.

Proof

1.	$\vdash \neg p, q, p$	Axiom
2.	$\vdash \neg q, q, p$	Axiom
3.	$\vdash \neg(p \vee q), q, p$	$\beta\vee$, 1, 2
4.	$\vdash \neg(p \vee q), (q \vee p)$	$\alpha\vee$, 3
5.	$\vdash (p \vee q) \to (q \vee p)$	$\alpha\to$, 4

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Examples

Example 3.5 Prove $\vdash p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$ in $\mathscr{G}$.

Proof

1.	$\vdash \neg p, p, q$	Axiom
2.	$\vdash \neg p, (p \vee q)$	$\alpha\vee$, 1
3.	$\vdash \neg p, p, r$	Axiom
4.	$\vdash \neg p, (p \vee r)$	$\alpha\vee$, 3
5.	$\vdash \neg p, (p \vee q) \wedge (p \vee r)$	$\beta\wedge$, 2, 4
6.	$\vdash \neg q, \neg r, p, q$	Axiom
7.	$\vdash \neg q, \neg r, (p \vee q)$	$\alpha\vee$, 6
8.	$\vdash \neg q, \neg r, p, r$	Axiom
9.	$\vdash \neg q, \neg r, (p \vee r)$	$\alpha\vee$, 8
10.	$\vdash \neg q, \neg r, (p \vee q) \wedge (p \vee r)$	$\beta\wedge$, 7, 9
11.	$\vdash \neg (q \wedge r), (p \vee q) \wedge (p \vee r)$	$\alpha\wedge$, 10
12.	$\vdash \neg (p \vee (q \wedge r)), (p \vee q) \wedge (p \vee r)$	$\beta\vee$, 5, 11
13.	$\vdash p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$	$\alpha\rightarrow$, 12

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Notation of Proofs

♣ The notation of proofs

- A proof can be written as a sequence of sets of formulas, which are numbered for convenient reference.
- On the right of each line is its justification: either the set of formulas is an axiom, or it is the conclusion of a rule of inference applied to a set or sets of formulas earlier in the sequence.
- A rule of inference is identified by the rule used for the α - or β -formula on the principal operator of the conclusion and by the number or numbers of the lines containing the premises.

The Natural Deduction System $\mathscr{G}$ for CPC [Ben-Ari]: Properties

♣ The soundness of $\mathscr{G}$

- For any $A \in$ **WFF**, if A is a theorem of $\mathscr{G}$, $\vdash_{\mathscr{G}} A$, then A is a tautology, $\models_{\mathbf{CPC}} A$.

♣ The completeness of $\mathscr{G}$

- For any $A \in$ **WFF**, if A is a tautology, $\models_{\mathbf{CPC}} A$, then A is a theorem of **G**, $\vdash_{\mathscr{G}} A$.

♣ $\mathscr{G}$ as a formal logic system of **CPC**

- The theorem set of $\mathscr{G}$ is the same as that of **CPC**.
- **Th**($\mathscr{G}$) = **Th**(**CPC**) = **Th**(**L**) = **Th**(**NK**) = **Th**(**CND**) = **Th**(**PND**)

An Introduction to Classical Propositional Calculus (CPC)

♣ Formal (Object) Language (Syntax) of **CPC**
♣ Principles of Structural Induction and Structural Recursion
♣ Semantics (Model Theory) of **CPC**
♣ Semantic (Model-theoretical, Logical) Consequence Relation
♣ Normal Forms and Uniform Notation of Formulas
♣ Deduction System (Proof Theory) of **CPC**
♣ Syntactic (Proof-theoretical, Deductive) Consequence Relation
♣ Hilbert Style Formal Logic Systems for **CPC**
♣ Gentzen's Natural Deduction System for **CPC**
♣ Gentzen's Sequent Calculus System for **CPC**
♣ Semantic Tableau System for **CPC**
♣ Resolution System for **CPC**
♣ Forward Deduction and Backward Deduction

The Basic Idea of Sequent Calculus Systems

♣ Why sequent calculus systems?

- What we need is a less pictorial, more formal, notation for manipulating deduction.
- One can imagine the sequent calculus as providing a kind of "specification language" for a natural deduction system.

♣ The basic idea of sequent calculus systems

- Study the structure of deductions
- Sequents play the role that formulas play in Hilbert style formal logic systems and that judgments play in natural deduction systems.

♣ **LK**

- ***Gentzen's sequent calculus system for CPC*** [G. Gentzen, 1935]
- Gentzen thought that **NK** gets at the heart of logical reasoning, and used **LK** only as a convenient tool for proving his results about **NK**.
- Historically, both a natural deduction system and the sequent calculus can be found in Gentzen's fundamental paper [G. Gentzen, 1935].

Sequents and Their Naive Interpretations

♣ Sequent

- A ***sequent*** is an expression $\Gamma \mapsto \Delta$ where Γ and Δ are finite (may be empty) sequences of formulas $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.
- Note: '$\mapsto$' is a meta-language(logic) symbol but not an object language symbol of **CPC**.

♣ The naive interpretation of sequents

- $A_1, \ldots, A_n \mapsto B_1, \ldots, B_m$: Any of $B_1, \ldots,$ and B_m is deducible from $A_1, \ldots,$ and A_n
- $\phi \mapsto B_1, \ldots, B_m$: The disjunction of the B_i ($1 \leq i \leq m$)
- $\phi \mapsto B_1$: B_1
- $A_1, \ldots, A_n \mapsto \phi$: The negation of the conjunction of the A_j ($1 \leq j \leq n$)
- $\phi \mapsto \phi$: Contradiction

Structural Deduction Rules of LK

♣ The ***exchange rules***

$$\frac{\Gamma, A, B, \Pi \mapsto \Delta}{\Gamma, B, A, \Pi \mapsto \Delta}\,\text{EL} \qquad \frac{\Gamma \mapsto \Delta, A, B, \Sigma}{\Gamma \mapsto \Delta, B, A, \Sigma}\,\text{ER}$$

♣ The ***weakening rules***

$$\frac{\Gamma \mapsto \Delta}{A, \Gamma \mapsto \Delta}\,\text{WL} \qquad \frac{\Gamma \mapsto \Delta}{\Gamma \mapsto \Delta, A}\,\text{WR}$$

♣ The ***contraction rules***

$$\frac{A, A, \Gamma \mapsto \Delta}{A, \Gamma \mapsto \Delta}\,\text{CL} \qquad \frac{\Gamma \mapsto \Delta, A, A}{\Gamma \mapsto \Delta, A}\,\text{CR}$$

♣ The ***cut rule***

$$\frac{\Gamma \mapsto \Delta, A \quad A, \Pi \mapsto \Sigma}{\Gamma, \Pi \mapsto \Delta, \Sigma}\,\text{Cut (where } A \text{ is call the cut formula)}$$

Logical Deduction Rules of LK

♣ Implication

$$\frac{\Gamma \mapsto \Delta, A \quad B, \Pi \mapsto \Sigma}{A \rightarrow B, \Gamma, \Pi \mapsto \Delta, \Sigma}\rightarrow\text{L} \qquad \frac{A, \Gamma \mapsto \Delta, B}{\Gamma \mapsto \Delta, A \rightarrow B}\rightarrow\text{R}$$

♣ Negation

$$\frac{\Gamma \mapsto \Delta, A}{\neg A, \Gamma \mapsto \Delta}\neg\text{L} \qquad \frac{A, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta, \neg A}\neg\text{R}$$

♣ Conjunction

$$\frac{A, \Gamma \mapsto \Delta}{A \wedge B, \Gamma \mapsto \Delta}\wedge\text{L}_1 \qquad \frac{B, \Gamma \mapsto \Delta}{A \wedge B, \Gamma \mapsto \Delta}\wedge\text{L}_2 \qquad \frac{\Gamma \mapsto \Delta, A \quad \Gamma \mapsto \Delta, B}{\Gamma \mapsto \Delta, A \wedge B}\wedge\text{R}$$

♣ Disjunction

$$\frac{A, \Gamma \mapsto \Delta \quad B, \Gamma \mapsto \Delta}{A \vee B, \Gamma \mapsto \Delta}\vee\text{L} \qquad \frac{\Gamma \mapsto \Delta, A}{\Gamma \mapsto \Delta, A \vee B}\vee\text{R}_1 \qquad \frac{\Gamma \mapsto \Delta, B}{\Gamma \mapsto \Delta, A \vee B}\vee\text{R}_2$$

An Example of Theorem Proof in LK

$$\vdash_{\mathbf{LK}} (A \rightarrow (B \rightarrow (A \wedge B)))\ ?$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \mapsto A}{B, A \mapsto A}\text{WL}}{A, B \mapsto A}\text{EL} \qquad \dfrac{B \mapsto B}{A, B \mapsto B}\text{WL}}{A, B \mapsto A \wedge B}\wedge\text{R}}{A \mapsto B \rightarrow (A \wedge B)}\rightarrow\text{R}}{\mapsto A \rightarrow (B \rightarrow (A \wedge B))}\rightarrow\text{R}$$

An Example of Deduction in LK

$$\{A, B\} \vdash_{\mathbf{LK}} (A \wedge B)\ ?$$

$$\dfrac{\dfrac{\dfrac{A \mapsto A}{B, A \mapsto A}\text{WL}}{A, B \mapsto A}\text{EL} \qquad \dfrac{B \mapsto B}{A, B \mapsto B}\text{WL}}{A, B \mapsto A \wedge B}\wedge\text{R}$$

LK

♣ ***Initial sequent***

$A \mapsto A$

♣ Deduction rules

Structural deduction rules and logical deduction rules

♣ ***Proof figure*** and ***end sequent***

(1) Initial sequent is a proof figure and itself is the end sequent of the proof figure;

(2) if $P1$ and $P2$ are proof figures and $S1$ and $S2$ are the end sequents of the proof figures respectively, and $\frac{S1}{S}$ or $\frac{S1 \quad S2}{S}$ is a inference rule, then $\frac{P1}{S}$ or $\frac{P1 \quad P2}{S}$ is a proof figure and S is its end sequent;

(3) Nothing else are proof figures and their end sequents.

LK

♣ ***Provable sequent***

- Sequent S is said to be ***provable*** at **LK**, denoted by $\vdash_{\mathbf{LK}} S$, if and only if there is a proof figure such that S is its end sequent.

♣ ***Cut elimination theorem***

- If sequent S is provable at **LK**, then there is a proof figure with S as its end sequent such that the cut rule is never used.

* ***Subformula property*** of the cut-free sequent calculus: every formula occurring in a premise of a rule instance is a subformula of a formula occurring in the conclusion of this rule instance.

♣ Theorems of **LK**

- Formula A is called a ***theorem*** of **LK** if and only if there is a proof figure such that $\phi \mapsto A$ is its end sequent.

Properties of LK

- The soundness of **LK**
  - If A is a theorem of **LK**, then A is a tautology, $\models_{CPC} A$, for any $A \in$ **WFF**.
- The completeness of **LK**
  - If A is a tautology, $\models_{CPC} A$, then A is a theorem of **LK**, for any $A \in$ **WFF**.
- The relationship between **LK** and **L**
  - The theorem set of **LK** is the same as that of **L**.
  - **Th(LK)** = **Th(CPC)** = **Th(L)** = **Th(NK)**
- Note
  - **LK** and **L** just are two different representations of the same propositional logic system **CPC**.

A Sequent Calculus G for CPC [Routenberg]

- The calculus $\vdash_G$ (called ***Gentzen type calculus***)
  - The deduction rules of the calculus $\vdash_G$ are given with respect to pairs (X, α) of formulas X and formulas α (Gentzen's name for (X, α), ***Sequenz***, is translated as ***sequent***).
  - We do not require that X be finite; our particular goals here make such a restriction dispensable.
  - If $\vdash_G$ applies to the pair (X, α) then we write $X \vdash_G \alpha$ and say that ***α is derivable or provable from X***; otherwise we write $X \nvdash_G \alpha$.
  - The calculus is formulated in terms of functional complete connectives $\{\wedge, \neg\}$ and encompasses the six deduction rules called the basic deduction rules.
- Note
  - Other standard connectives can be introduced by the definitions.

A Sequent Calculus G for CPC [Routenberg]: Basic Deduction Rules

Each of the basic rules below has certain premises and a conclusion. Only (IS) has no premises. It allows the derivation of all sequents $\alpha \vdash \alpha$. These are called the *initial* sequents, because each derivation must start with these. (MR), the *monotonicity rule*, could be weakened. It becomes even provable if all pairs (X, α) with $\alpha \in X$ are called initial sequents.

$$\text{(IS)}\ \frac{}{\alpha \vdash \alpha}\ \text{(initial sequent)} \qquad \text{(MR)}\ \frac{X \vdash \alpha}{X' \vdash \alpha}\ (X' \supseteq X),$$

$$(\wedge 1)\ \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \qquad (\wedge 2)\ \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$$

$$(\neg 1)\ \frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} \qquad (\neg 2)\ \frac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta}$$

Here and in the following $X \vdash \alpha, \beta$ is to mean $X \vdash \alpha$ and $X \vdash \beta$. This convention is important, since $X \vdash \alpha, \beta$ has another meaning in Gentzen calculi that operate with pairs of sets of formulas. The rules $(\wedge 1)$ and $(\neg 1)$ actually have two premises, just like $(\neg 2)$. Note further that $(\wedge 2)$ really consists of two subrules corresponding to the conclusions $X \vdash \alpha$ and $X \vdash \beta$. In $(\neg 2)$, X, α means $X \cup \{\alpha\}$, and this abbreviated form will always be used when there is no risk of misunderstanding.

A Sequent Calculus G for CPC [Routenberg]

- The calculus $\vdash_G$ (called ***Gentzen type calculus***)
  - $X \vdash_G \alpha$ (read "***from X is provable or derivable α***") means that the sequent (X, α) can be obtained after a stepwise application of the basic rules.
  - We can make this idea of "stepwise application" of the basic rules rigorous and formally precise in the following way: a derivation is to mean a finite sequence $(S_0; \ldots; S_n)$ of sequents such that every S_i is either an initial sequent or is obtained through the application of some basic rule to preceding elements in the sequence.
  - Thus, $X \vdash_G \alpha$ if there is a derivation $(S_0; \ldots; S_n)$ with $S_n = (X, \alpha)$.
- A derivation example with the end sequent $\alpha, \beta \vdash_G \alpha\wedge\beta$ $(\{\alpha, \beta\}, \alpha\wedge\beta)$
  - $(\alpha \vdash_G \alpha\ ;\ \alpha, \beta \vdash_G \alpha\ ;\ \beta \vdash_G \beta\ ;\ \alpha, \beta \vdash_G \beta\ ;\ \alpha, \beta \vdash_G \alpha\wedge\beta)$
  - Here (MR) was applied twice, followed by an application of $(\wedge 1)$. Not shorter would be complete derivation of the sequent $(\phi, \top)$, i.e., a proof of $\vdash_G \top$. In this example both $(\neg 1)$ and $(\neg 2)$ are essentially involved.

A Sequent Calculus G for CPC [Routenberg]: Additional Deduction Rules

- The derivation of additional rules
  - Useful for shortening lengthy derivations is the derivation of additional rules.

$\dfrac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$		*proof*	*applied*
($\neg$-elimination)	1	$X, \alpha \vdash \alpha$	(IS), (MR)
	2	$X, \neg\alpha \vdash \alpha$	supposition
	3	$X \vdash \alpha$	$(\neg 2)$

- The following second example, a generalization of the first one above, is the often-used proof method reductio ad absurdum: α is proved from X by showing that the assumption $\neg\alpha$ leads to a contradiction.

A Sequent Calculus G for CPC [Routenberg]: Additional Deduction Rules

$\dfrac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$		*proof*	*applied*
(reductio ad absurdum)	1	$X, \neg\alpha \vdash \beta, \neg\beta$	supposition
	2	$X, \neg\alpha \vdash \alpha$	$(\neg 1)$
	3	$X \vdash \alpha$	$\neg$-elimination

$\dfrac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta}$			
($\rightarrow$-elimination)	1	$X, \alpha, \neg\beta \vdash \alpha, \neg\beta$	(IS), (MR)
	2	$X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta$	$(\wedge 1)$
	3	$X \vdash \neg(\alpha \wedge \neg\beta)$	supposition
	4	$X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)$	(MR)
	5	$X, \alpha, \neg\beta \vdash \beta$	$(\neg 1)$ on 2 and 4
	6	$X, \alpha \vdash \beta$	$\neg$-elimination

A Sequent Calculus G for CPC [Routenberg]: Additional Deduction Rules

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$

(cut rule)

1	$X, \neg\alpha \vdash \alpha$	supposition, (MR)
2	$X, \neg\alpha \vdash \neg\alpha$	(IS), (MR)
3	$X, \neg\alpha \vdash \beta$	($\neg$1)
4	$X, \alpha \vdash \beta$	supposition
5	$X \vdash \beta$	($\neg$2) on 4 and 3

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \rightarrow \beta}$$

($\rightarrow$-introduction)

1	$X, \alpha \wedge \neg\beta, \alpha \vdash \beta$	supposition, (MR)
2	$X, \alpha \wedge \neg\beta \vdash \alpha$	(IS), (MR), ($\wedge$2)
3	$X, \alpha \wedge \neg\beta \vdash \beta$	cut rule
4	$X, \alpha \wedge \neg\beta \vdash \neg\beta$	(IS), (MR), ($\wedge$2)
5	$X, \alpha \wedge \neg\beta \vdash \alpha \rightarrow \beta$	($\neg$1)
6	$X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \rightarrow \beta$	(IS), (MR)
7	$X \vdash \alpha \rightarrow \beta$	($\neg$2) on 5 and 6

A Sequent Calculus G for CPC [Routenberg]: Additional Deduction Rules

♣ Modus Ponens

A simple application of the $\rightarrow$-elimination and the cut rule is a proof of the *detachment rule*

$$\frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta}.$$

Indeed, the premise $X \vdash \alpha \rightarrow \beta$ yields $X, \alpha \vdash \beta$ by $\rightarrow$-elimination, and since $X \vdash \alpha$, it follows $X \vdash \beta$ by the cut rule. Applying detachment on $X = \{\alpha, \alpha \rightarrow \beta\}$, we obtain $\alpha, \alpha \rightarrow \beta \vdash \beta$. This collection of sequents is known as *modus ponens*. It will be more closely considered in 1.6.

♣ Note

- The inference rule of Modus Ponens in the Hilbert style formal logic systems (Modus Ponens for material implication): from ($A \Rightarrow B$) and A to infer B.

A Sequent Calculus G for CPC [Routenberg]: Properties

♣ The derivation of additional rules

- Many properties of $|-_G$ are proved through rule induction.
- We identify a property ε of sequents with the set of all pairs (X, α) to which ε applies.

All the rules considered here are of the form

$$R: \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

and are referred to as Gentzen-style rules. We say that $\mathcal{E}$ is *closed under* R when $\mathcal{E}(X_1, \alpha_1), \ldots, \mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$. For a rule without premises, i.e., $n = 0$, this is just to mean $\mathcal{E}(X, \alpha)$. For instance, consider

A Sequent Calculus G for CPC [Routenberg]: Properties

♣ ***Principle of rule induction***

- Let ε be a property closed under all basic rules of $|-_G$. If $X |-_G \alpha$ then $\varepsilon(X, \alpha)$.

♣ Soundness theorem ($|-_G$ is semantically sound: $X |-_G \alpha \Rightarrow X |=_{CPC} \alpha$)

- The logical consequence relation $|=_{CPC}$ is the property that applies to all pairs (X, α) with $X |=_{CPC} \alpha$.
- This property is closed under each basic rule of $|-_G$. In detail this means $\alpha |=_{CPC} \alpha$, $X |=_{CPC} \alpha \Rightarrow X' |=_{CPC} \alpha$ for $X' \supseteq X$, $X |=_{CPC} \alpha, \beta \Rightarrow X |=_{CPC} \alpha \wedge \beta$, etc.
- If $X |-_G \alpha$ then $X |=_{CPC} \alpha$ for all X, α.
- A question: Can we represent the property of soundness as If $X |-_G \alpha$ then $X |=_{CPC} \alpha$ for all (X, α) ?

♣ Finiteness theorem for $|-_G$

- If $X |-_G \alpha$ then there is a finite subset $X_0 \subseteq X$ such that $X_0 |-_G \alpha$.

A Sequent Calculus G for CPC [Routenberg]: Properties

♣ Definition: Inconsistency

- $X \subseteq \Gamma$ is called ***inconsistent*** (in the calculus $|-_G$) if $X |-_G \alpha$ for all $\alpha \in \Gamma$, and otherwise ***consistent***.
- X is called ***maximally consistent*** if X is consistent but each $Y \supset X$ is inconsistent.

♣ The inconsistency of a set of formulas

- The inconsistency of X can be identified by the derivability of a single formula, namely $\perp$ ($= p_1 \wedge \neg p_1$), because $X |-_G \perp$ implies $X |-_G p_1, \neg p_1$ by ($\wedge$2), hence $X |-_G \alpha$ for all α by ($\neg$1). Conversely, when X is inconsistent then in particular $X |-_G \perp$. Thus, $X |-_G \perp$ may be read as 'X is inconsistent', and $X |\!/\!-_G \perp$ as 'X is consistent'.
- From this it easily follows that X is maximally consistent IFF either $\alpha \in X$ or $\neg\alpha \in X$ for each α. The latter is necessary, for if $\alpha, \neg\alpha \notin X$ then both $X, \alpha |-_G \perp$ and $X, \neg\alpha |-_G \perp$, hence $X |-_G \perp$ by ($\neg$2). This contradicts the consistency of X.

A Sequent Calculus G for CPC [Routenberg]: Properties

♣ Lemma

- The derivability relation $|-_G$ has the following properties: C^+: $X |-_G \alpha$ IFF $X, \neg\alpha |-_G \perp$, C^-: $X |-_G \neg\alpha$ IFF $X, \alpha |-_G \perp$.

♣ ***Lindenbaum's theorem***

- Every consistent set $X \subseteq \Gamma$ can be extended to a maximally consistent set $X' \supseteq X$.

♣ Lemma

- A maximally consistent set $X \subseteq \Gamma$ has the following property: $X |-_G \neg\alpha$ IFF $X |\!/\!-_G \alpha$ for arbitrary α.

♣ Completeness theorem ($|-_G$ is complete: $X |=_{CPC} \alpha \Rightarrow X |-_G \alpha$)

- If $X |=_{CPC} \alpha$ then $X |-_G \alpha$ for all X, α.

A Sequent Calculus G for CPC [Routenberg]: Properties

♣ The soundness of **G**

- If $\vdash_{\mathbf{G}} A$, then $\models_{\mathbf{CPC}} A$, for any $A \in \mathbf{WFF}$.

♣ The completeness of **G**

- If $\models_{\mathbf{CPC}} A$, then $\vdash_{\mathbf{G}} A$, for any $A \in \mathbf{WFF}$.

♣ **G** as a formal logic system of **CPC**

- The theorem set of **G** is the same as that of **CPC**.
- **Th**(**G**) = **Th**(**CPC**) = **Th**(**L**) = **Th**(**LK**)

The Sequent Calculus for CPC [Fitting]: Sequents

♣ The connectives

- It will be simplest if we limit the binary connectives to those actually considered by Gentzen: $\wedge$, $\vee$, and $\rightarrow$. Therefore, formulas are limited to these binary connectives (and, of course, $\neg$, $\top$, and $\bot$).

♣ Definition: ***Sequents***

- *A* sequent is a pair $<\Gamma, \Delta>$ of finite sets of formulas.

♣ Notation of sequents

- In practice we use the suggestive notation introduced by Gentzen: the sequent $<\Gamma, \Delta>$ will be written $\Gamma \,|\!\!\rightarrow \Delta$.
- Note: ‘$|\!\!\rightarrow$’ is a meta-language(logic) symbol but not an object language symbol of **CPC**.
- The arrow suggests a kind of implication, “follows from”, and that is, indeed, the intention.
- A sequent as asserting the following: If all the formulas on the left of the arrow are true, then at least one of the formulas on the right is also true.

The Sequent Calculus for CPC [Fitting]: Sequents

♣ Notation of sequents

- Instead of writing $\{A_1, ..., A_n\} \,|\!\!\rightarrow \{B_1, ..., B_k\}$, we will write $A_1, ..., A_n \,|\!\!\rightarrow B_1, ..., B_k$.
- For a single formula X, and sets Γ and Δ of formulas, we will write $\Gamma, X \,|\!\!\rightarrow \Delta$ instead of $\Gamma \cup \{X\} \,|\!\!\rightarrow \Delta$, and so on.
- Likewise, we will write $|\!\!\rightarrow \Delta$ for $\phi \,|\!\!\rightarrow \Delta$, and similarly for other occurrences of ϕ.
- Generally, we follow the convention that capital Latin letters stand for formulas, while capital Greek letters stand for finite sets of formulas.

♣ Definition: Extending truth valuations to sequents (under model $\boldsymbol{M} = (v_a, v_f)$)

- $v_f(\Gamma \,|\!\!\rightarrow \Delta) = \mathbf{T}$, if $v_f(X) = \mathbf{F}$ for some $X \in \Gamma$ or $v_f(Y) = \mathbf{T}$ for some $Y \in \Delta$.
- Note: Under this definition, $v_f(\,|\!\!\rightarrow\,) = \mathbf{F}$ and $v_f(\,|\!\!\rightarrow X\,) = v_f(X)$.
- Note: As an example to show ‘$|\!\!\rightarrow$’ is a meta-logic symbol but not a connective, $X \,|\!\!\rightarrow (Y \rightarrow Z)$ is legal, but $X \rightarrow (Y \rightarrow Z)$ is not (why?).

The Sequent Calculus for CPC [Fitting]: Axioms and Rules

Axioms $X \rightarrow X$

$\bot \rightarrow$

$\rightarrow \top$ Note: Here ‘$\rightarrow$’ is ‘$|\!\!\rightarrow$’ and ‘$\supset$’ is ‘$\rightarrow$’ in our notation.

Structural Rule, Thinning If $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$ then

$$\frac{\Gamma_1 \rightarrow \Delta_1}{\Gamma_2 \rightarrow \Delta_2}$$

Negation Rules

$$\frac{\Gamma \rightarrow \Delta, X}{\Gamma, \neg X \rightarrow \Delta} \qquad \frac{\Gamma, X \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg X}$$

Conjunction Rules

$$\frac{\Gamma, X, Y \rightarrow \Delta}{\Gamma, X \wedge Y \rightarrow \Delta} \qquad \frac{\Gamma \rightarrow \Delta, X \quad \Gamma \rightarrow \Delta, Y}{\Gamma \rightarrow \Delta, X \wedge Y}$$

Disjunction Rules

$$\frac{\Gamma, X \rightarrow \Delta \quad \Gamma, Y \rightarrow \Delta}{\Gamma, X \vee Y \rightarrow \Delta} \qquad \frac{\Gamma \rightarrow \Delta, X, Y}{\Gamma \rightarrow \Delta, X \vee Y}$$

Implication Rules

$$\frac{\Gamma \rightarrow \Delta, X \quad \Gamma, Y \rightarrow \Delta}{\Gamma, X \supset Y \rightarrow \Delta} \qquad \frac{\Gamma, X \rightarrow \Delta, Y}{\Gamma \rightarrow \Delta, X \supset Y}$$

The Sequent Calculus for CPC [Fitting]: Proofs and Theorems

♣ Definition: Proofs and theorems

- A ***proof*** is a tree labeled with sequents (generally written with the root at the bottom) meeting the following conditions.
- If node N is labeled with $\Gamma \,|\!\!\rightarrow \Delta$, then if N is a leaf node, $\Gamma \,|\!\!\rightarrow \Delta$ must be an axiom; and if N has children, their labels must be the premises from which $\Gamma \,|\!\!\rightarrow \Delta$ follows by one of the sequent calculus rules.
- The label on the root node is the sequent that is proved.
- A formula X is a ***theorem*** of the sequent calculus if the sequent $|\!\!\rightarrow X$ has a proof.

The Sequent Calculus for CPC [Fitting]: A Proof Example

1. $P \rightarrow P$
2. $Q \rightarrow Q$
3. $P, Q \rightarrow P$
4. $P, Q \rightarrow Q$
5. $P, Q \rightarrow P \wedge Q$
6. $Q \rightarrow P \wedge Q, \neg P$
7. $\rightarrow P \wedge Q, \neg P, \neg Q$
8. $\neg(P \wedge Q) \rightarrow \neg P, \neg Q$
9. $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$
10. $\rightarrow \neg(P \wedge Q) \supset (\neg P \vee \neg Q)$

FIGURE 4.4. Sequent Calculus Proof of $\neg(P \wedge Q) \supset (\neg P \vee \neg Q)$

- The proof arranged as a tree with 10 as root and 1 and 2 as leaves.
- In it, 1 and 2 are axioms, from which 3 and 4 follow, respectively, by thinning; 5 follows from 3 and 4 using a conjunction rule.
- Then 6 through 8 follow by negation rules.
- Finally 9 follows from 8 using a disjunction rule and 10 from 9 by an implication rule.

The Sequent Calculus for CPC [Fitting]: Properties

♣ Soundness theorem

- If X is a theorem of the sequent calculus, then X is a tautology, $\models_{CPC} X$.

♣ Definition: Associated sequents

- Let S be a finite set of formulas. An ***associated sequent*** for S is a sequent $\Gamma \mapsto \neg\Delta$ ($\neg\Delta =_{df} \{\neg X \mid X \in \Delta\}$), where Γ, Δ is a partition of S ($\Gamma \cup \Delta = S$ and $\Gamma \cap \Delta = \phi$).

♣ Lemma

- If any associated sequent for S has a proof, every associated sequent does.

♣ Definition: Consistency

- A finite set S of formulas is ***sequent inconsistent*** if any associated sequent has proof. S is ***sequent consistent*** if is not sequent inconsistent.

♣ Completeness theorem

- If X is a tautology, $\models_{CPC} X$, then X is a theorem of the sequent calculus.

The Sequent Calculus for CPC [R&C]: Sequents

♣ Sequents

- A sequent is an expression of the form $G \mapsto H$ where G and H are sets of formulas.
- For cases in which H is a single formula, we write $G \mapsto \{A\}$ as $G \mapsto A$.
- We shall also for brevity write $G \cup \{A\} \mapsto B$ as $G, A \mapsto B$ and adopt the convention that $G \mapsto \{\}$, $G \mapsto$ and $G \mapsto \bot$ are all the same.

♣ Sequents as formulas in the observer's (meta-) language

- The use of the '$\mapsto$', instead of the '$\vdash$', makes an important distinction clear.
- If you assert $G \vdash B$ you are saying, in the observer's language, that B can be derived from the set of assumptions G in some system of deduction, whereas $G \mapsto B$ is just a formula in the observer's language.
- We are going to put forward a set of rules that say how formulas involving '$\mapsto$' can manipulated and we will draw conclusions of the form "if $G \mapsto A$ can be obtained by using the rules then $G \vdash$ and/or $\models A$ in **CPC**".

The Sequent Calculus for CPC [R&C]: The Natural Deduction Rules Expressed in Sequent Notation

$$\text{Assumption}\ \dfrac{}{\mathcal{S} \Rightarrow \mathcal{S}}$$

$$\text{Thin}\ \dfrac{H \Rightarrow \mathcal{S}}{G,H \Rightarrow \mathcal{S}}$$

$$\wedge I\ \dfrac{G \Rightarrow \mathcal{S} \quad H \Rightarrow \mathcal{T}}{G,H \Rightarrow \mathcal{S} \wedge \mathcal{T}} \qquad \wedge E\ \dfrac{G \Rightarrow \mathcal{S} \wedge \mathcal{T}}{G \Rightarrow \mathcal{S}} \qquad \wedge E\ \dfrac{G \Rightarrow \mathcal{S} \wedge \mathcal{T}}{G \Rightarrow \mathcal{T}}$$

$$\vee I\ \dfrac{G \Rightarrow \mathcal{S}}{G \Rightarrow \mathcal{S} \vee \mathcal{T}} \qquad \vee I\ \dfrac{G \Rightarrow \mathcal{T}}{G \Rightarrow \mathcal{S} \vee \mathcal{T}} \qquad \bot\ \dfrac{G \Rightarrow \bot}{G \Rightarrow \mathcal{S}}$$

$$\vee E\ \dfrac{G \Rightarrow \mathcal{S} \vee \mathcal{T} \quad \mathcal{S},H \Rightarrow \mathcal{R} \quad \mathcal{T},F \Rightarrow \mathcal{R}}{G,H,F \Rightarrow R}$$

$$\to I\ \dfrac{\mathcal{S},G \Rightarrow \mathcal{T}}{G \Rightarrow \mathcal{S} \to \mathcal{T}} \qquad \to E\ \dfrac{G \Rightarrow \mathcal{S} \to \mathcal{T} \quad H \Rightarrow \mathcal{S}}{G,H \Rightarrow \mathcal{T}}$$

- Note: Here '$\Rightarrow$' is '$\mapsto$' in our notation and the first two are added.

The Sequent Calculus for CPC [R&C]: An Example

$$\to I\ \dfrac{\to I\ \dfrac{\wedge I\ \dfrac{\text{Assumption}\ \dfrac{}{P \Rightarrow P} \quad \text{Assumption}\ \dfrac{}{Q \Rightarrow Q}}{P,Q \Rightarrow P \wedge Q}}{P \Rightarrow Q \to (P \wedge Q)}}{\Rightarrow P \to (Q \to (P \wedge Q))}$$

$$\to I\ \dfrac{\to I\ \dfrac{\wedge I\ \dfrac{\cancel{P}^{2} \quad \cancel{Q}^{1}}{P \wedge Q}}{Q \to (P \wedge Q)}\ 1}{P \to (Q \to (P \wedge Q))}\ 2$$

- Note: Here '$\Rightarrow$' is '$\mapsto$' in our notation.

The Sequent Calculus for CPC [R&C]: Gentzen's Classical Calculus

♣ Gentzen's Hauptsatz

- Sequents were introduced by Gentzen who proposed a calculus in which elimination rules are replaced by introduction rules on the left of the '$\mapsto$'.
- Gentzen proved the equivalence of his sequent calculus to natural deduction by adding the Cut Rule to his system, which makes relating it to natural deduction a lot easier.
- Then in a famous result known as Gentzen's Hauptsatz, he showed that any derivation involving the Cut rule could be converted to one that was ***Cut-free***.
- Gentzen's sequent calculus idea is a very powerful one for studying derivations and relating them to other systems such as the tableau method.
- A further generalization allows more than one formula on the right of the '$\mapsto$'. Doing this, and bringing in rules for $\neg$, gives the following symmetrical system shown which again can be shown equivalent to the natural deduction system for **CPC**.

The Sequent Calculus for CPC [R&C]: Gentzen's Classical Calculus

$$\text{Assumption}\ \dfrac{}{\mathcal{S} \Rightarrow \mathcal{S}}$$

$$\text{Thin} \Rightarrow \dfrac{G \Rightarrow H}{\mathcal{S},G \Rightarrow H} \qquad \Rightarrow \text{Thin}\ \dfrac{G \Rightarrow H}{G \Rightarrow H,\mathcal{T}}$$

$$\Rightarrow\wedge\ \dfrac{G \Rightarrow H,\mathcal{S} \quad G' \Rightarrow H',\mathcal{T}}{G,G' \Rightarrow H,H',\mathcal{S}\wedge\mathcal{T}} \qquad \wedge\Rightarrow\ \dfrac{\mathcal{S},\mathcal{T},G \Rightarrow H}{\mathcal{S}\wedge\mathcal{T},G \Rightarrow H}$$

$$\Rightarrow\vee\ \dfrac{G \Rightarrow H,\mathcal{S}}{G \Rightarrow H,\mathcal{S}\vee\mathcal{T}} \qquad \Rightarrow\vee\ \dfrac{G \Rightarrow H,\mathcal{T}}{G \Rightarrow H,\mathcal{S}\vee\mathcal{T}} \qquad \vee\Rightarrow\ \dfrac{\mathcal{S},G \Rightarrow H \quad \mathcal{T},G' \Rightarrow H'}{\mathcal{S}\vee\mathcal{T},G,G' \Rightarrow H,H'}$$

$$\Rightarrow\to\ \dfrac{\mathcal{S},G \Rightarrow H,\mathcal{T}}{G \Rightarrow H,\mathcal{S}\to\mathcal{T}} \qquad \to\Rightarrow\ \dfrac{G \Rightarrow H,\mathcal{S} \quad \mathcal{T},G' \Rightarrow H'}{,\mathcal{S}\to\mathcal{T},G,G' \Rightarrow H,H'}$$

$$\Rightarrow\neg\ \dfrac{\mathcal{S},G \Rightarrow H}{G \Rightarrow H,\neg\mathcal{S}} \qquad \neg\Rightarrow\ \dfrac{G \Rightarrow H,\mathcal{S}}{\neg\mathcal{S},G \Rightarrow H}$$

- Note: Here '$\Rightarrow$' is '$\mapsto$' in our notation.

The Sequent Calculus $\mathcal{S}$ for CPC [Ben-Ari]: Definitions

- ♣ Smullyan's Gentzen system [1968]
  - We now present a deductive system similar to the one that Gentzen originally proposed; this system is taken from Smullyan's book "First-Order Logic" [1968].
- ♣ Sequents
  - Definition: If U and V are (possibly empty) sets of formulas, then $U \mapsto V$ is a sequent.
  - Intuitively, a sequent represents 'provable from' in the sense that the formulas in U are assumptions for the set of formulas V that are to be proved.
- ♣ Axioms
  - Definition: Axioms in the Gentzen sequent system $\mathcal{S}$ are sequents of the form: $U \cup \{A\} \mapsto V \cup \{A\}$.

The Sequent Calculus $\mathcal{S}$ for CPC [Ben-Ari]: Inference Rules

op	Introduction into consequent	Introduction into antecedent
$\wedge$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \wedge B\}}$	$\dfrac{U \cup \{A, B\} \Rightarrow V}{U \cup \{A \wedge B\} \Rightarrow V}$
$\vee$	$\dfrac{U \Rightarrow V \cup \{A, B\}}{U \Rightarrow V \cup \{A \vee B\}}$	$\dfrac{U \cup \{A\} \Rightarrow V \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \vee B\} \Rightarrow V}$
$\rightarrow$	$\dfrac{U \cup \{A\} \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \rightarrow B\}}$	$\dfrac{U \Rightarrow V \cup \{A\} \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \rightarrow B\} \Rightarrow V}$
$\neg$	$\dfrac{U \cup \{A\} \Rightarrow V}{U \Rightarrow V \cup \{\neg A\}}$	$\dfrac{U \Rightarrow V \cup \{A\}}{U \cup \{\neg A\} \Rightarrow V}$

Fig. 3.2 Rules of inference for sequents

- Note: Here '$\Rightarrow$' is '$\mapsto$' in our notation.

The Sequent Calculus $\mathcal{S}$ for CPC [Ben-Ari]: Semantics

- ♣ Semantics
  - Definition: Let $S = U \mapsto V$ be a sequent where $U = \{U_1, ..., U_n\}$ and $V = \{V_1, ..., V_m\}$, and let $\boldsymbol{M} = (v_a, v_f)$ be a model. Then $v_f(S) = \mathbf{T}$ IFF $v_f(U_1) = ... = v_f(U_n) = \mathbf{T}$ implies that for some i, $v_f(V_i) = \mathbf{T}$.
  - This definition relates sequents to formulas: Given a model $\boldsymbol{M} = (v_a, v_f)$, $v_f(U \mapsto V) = \mathbf{T}$ IFF $v_f(\bigwedge U \rightarrow \bigvee V) = \mathbf{T}$.

An Introduction to Classical Propositional Calculus (CPC)

- ♣ Formal (Object) Language (Syntax) of **CPC**
- ♣ Principles of Structural Induction and Structural Recursion
- ♣ Semantics (Model Theory) of **CPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Normal Forms and Uniform Notation of Formulas
- ♣ Deduction System (Proof Theory) of **CPC**
- ♣ Syntactic (Proof-theoretical, Deductive) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CPC**
- ♣ Gentzen's Natural Deduction System for **CPC**
- ♣ Gentzen's Sequent Calculus System for **CPC**
- ♣ Semantic Tableau System for **CPC**
- ♣ Resolution System for **CPC**
- ♣ Forward Deduction and Backward Deduction