An Introduction to Classical Predicate Calculus

- ♣ The Limitations of Propositional Logic **CPC**
- ♣ Formal (Object) Language (Syntax) of Classical First-Order Predicate Calculus (**CFOPC**)
- ♣ Substitutions
- ♣ Semantics (Model Theory) of **CFOPC**
- ♣ Semantic (Model-theoretical, Logical) Consequence Relation
- ♣ Hilbert Style Formal Logic Systems for **CFOPC**
- ♣ Gentzen's Natural Deduction System for **CFOPC**
- ♣ Gentzen's Sequent Calculus System for **CFOPC**
- ♣ Semantic Tableau Systems for **CFOPC**
- ♣ Resolution Systems for **CFOPC**
- ♣ Classical Second-Order Predicate Calculus (**CSOPC**)

Free and Bound Variables: A Motivation Example

♣ The Example

- $\Sigma^{n}_{k=1} x^k + k = (x^n + n) + (x^{n-1} + n-1) + \ldots + (x + 1)$

♣ Consider the following question about the above example

- $\Sigma^{n}_{k=1} x^k + k = \Sigma^{n}_{j=1} x^j + j$?
- $\Sigma^{n}_{k=1} x^k + k = \Sigma^{j}_{k=1} x^k + k$?
- $\Sigma^{n}_{k=1} x^k + k = \Sigma^{n}_{n=1} x^n + n$?
- $\Sigma^{n}_{k=1} x^k + k = \Sigma^{k}_{k=1} x^k + k$?
- $\Sigma^{n}_{k=1} x^k + k = \Sigma^{n}_{k=1} y^k + k$?

♣ Notes

- In the above example, n and x are free (occurrence) variables, and k is a bound (occurrence) variable.
- A bound (occurrence) variable can be replaced by other variables, expect free (occurrence) variables in the formula, without meaning change.
- The value of a formula is dependent on values of its free (occurrence) variables.

Free and Bound Occurrences of Individual Variables

♣ ***Free occurrences*** of individual variables

- (1) The ***free variable occurrences*** in an atomic formula are all the variable occurrences in the formula.
  (2) The free variable occurrences in $(\neg A)$ are the free variable occurrences in A.
  (3) The free variable occurrences in $(A*B)$ (* is a binary connective) are the free variable occurrences in A together with the free variable occurrences in B,
  (4) The free variable occurrences in $((\forall x)A)$ and $((\exists x)A)$ are the free variable occurrences in A, except for occurrences of x.

♣ ***Bound occurrences*** of individual variables

- A variable occurrence is ***bound*** IFF it is not free occurrence.

Free and Bound Occurrences of Individual Variables

♣ ***Bound occurrences*** of individual variables

- An occurrence of a variable x is said to be ***bound*** in a formula B if either it is the occurrence of x in a quantifier "$(\forall x)$" or "$(\exists x)$" in B or it lies within the scope of a quantifier "$(\forall x)(\ldots)$" or "$(\exists x)(\ldots)$" in B.

♣ ***Free occurrences*** of individual variables

- A variable occurrence is ***free*** in a formula IFF it is not bound occurrence.

♣ ***Free / Bound variables***

- A variable is said to be ***free*** (***bound***) in a formula B if it has a free (bound) occurrence in B.
- Thus, a variable may be both free and bound in the same formula.

♣ ***Closed formulas (Sentences)***

- A formula with no free (occurrence) variables (called a ***closed formula*** or ***sentence***) represents a proposition that must be true or false.

Free and Bound Occurrences of Individual Variables: Examples

♣ Example 1: $p^2_1(x_1,x_2)$

- In Example 1, the single occurrence of x_1 (or x_2) is free.

♣ Example 2: $p^2_1(x_1,x_2) \rightarrow (\forall x_1)p^1_1(x_1)$

- In Example 2, the first occurrence of x_1 in $p^2_1(x_1,x_2)$ is free, but the second and third occurrences of x_1 are bound.

♣ Example 3: $(\forall x_1)p^2_1(x_1,x_2) \rightarrow (\forall x_1)p^1_1(x_1)$

- In Example 3, all occurrences of x_1 are bound.

♣ Example 4: $(\exists x_1)p^2_1(x_1,x_2)$

- In Example 4, both occurrences of x_1 are bound.

♣ Notes

- In all four examples, every occurrence of x_2 is free.
- A variable may have both free and bound occurrences in the same formula; a variable may be bound in a formula but free in a subformula of the formula.

Terms being Free for Individual Variables [Mendelson]

♣ ***Terms being free for variables***

- If B is a formula and t is a term, then t is said to be free for x_i in B if no free occurrence of x_i in B lies within the scope of any quantifier $(\forall x_j)$ or $(\exists x_j)$, where x_j is a variable in t.
- Note: This concept of t being free for x_i in a formula $B(x_i)$ will have certain technical applications later on. It means that, if t is substituted for all free occurrences (if any) of x_i in $B(x_i)$, no occurrence of a variable in t becomes a bound occurrence in $B(t)$.

♣ Examples

- The term x_2 is free for x_1 in $p^1_1(x_1)$, but x_2 is not free for x_1 in $(\forall x_2)p^1_1(x_1)$.
- The term $f^2_1(x_1,x_3)$ is free for x_1 in $(\forall x_2)p^2_1(x_1,x_2) \rightarrow p^1_1(x_1)$ but is not free for x_1 in $(\exists x_3)(\forall x_2)p^2_1(x_1,x_2) \rightarrow p^1_1(x_1)$.

Terms being Free for Individual Variables [Mendelson]

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- If B is a formula and t is a term, then t is said to be free for x_i in B if no free occurrence of x_i in B lies within the scope of any quantifier $(\forall x_j)$ or $(\exists x_j)$, where x_j is a variable in t.

♣ Facts

- A term that contains no variables is free for any variable in any formulas.
- A term t is free for any variable in formula B if none of the variables of t is bound in B.
- x_i is free for x_i in any formula.
- Any term is free for x_i in formula B if B contains no free occurrences of x_i.

Substitutions of Variables

♣ ***Substitution*** of variable

- A formula may contain some free variables that can be replaced by other terms.
- A ***variable substitution*** is a mapping σ: **V**→**Ter** from the set of individual variables **V** to the set of terms **Ter**.
- We denote $\sigma[x]$ by $x\sigma$, to represent the result of applying the mapping σ to x.

♣ ***Substitution*** of variable on all terms

- Let σ: **V**→**Ter** be a variable substitution. It can be extended to all terms:
  (1) $c\sigma = c$ for any $c \in$ **Con**, $\top\sigma = \top$, $\bot\sigma = \bot$;
  (2) $[f(t_1, ..., t_n)]\sigma = f(t_1\sigma, ..., t_n\sigma)$ for any n-ary $f \in$ **Fun**.

♣ Examples

- Let a, b, c, x, y, z be variables and f, g, h, i, j, k be functions. Suppose $x\sigma = f(x, y)$, $y\sigma = h(a)$, and $z\sigma = g(c, h(x))$. Then $j(k(x), y)\sigma = j(k(f(x, y)), h(a))$.

Substitutions of Variables: Composition and Support

♣ ***Composition*** of substitutions

- Let σ and τ be substitutions. By the ***composition*** of σ and τ, we mean that substitution, which we denote by $\sigma\bullet\tau$, such that for each variable $x \in$ **V**, $x(\sigma\bullet\tau) = (x\sigma)\tau$.
- Theorem: For any term $t \in$ **Ter** and any two substitutions σ and τ, $t(\sigma\bullet\tau) = (t\sigma)\tau$.
- Note: The above theorem does not carry over to formulas.
- Theorem: Composition of substitutions is associative, i.e., for any substitutions σ_1, σ_2, and σ_3, $(\sigma_1\bullet\sigma_2)\bullet\sigma_3 = \sigma_1\bullet(\sigma_2\bullet\sigma_3)$.

♣ ***Support*** of substitution

- The ***support*** of a substitution σ is the set of variables x for which $x\sigma \neq x$. A substitution has ***finite support*** if its support set is finite.
- Theorem: The composition of two substitutions having finite support is a substitution having finite support.

Substitutions of Variables: Notation of Substitution and Composition

♣ Notation of substitution

- Suppose σ is a substitution having finite support; say $\{x_1, x_2, ..., x_n\}$ is the support, and for each $i = 1, ..., n$, $x_i\sigma = t_i$.
- Our notation for σ is: $[x_1/t_1, x_2/t_2, ..., x_n/t_n]$.
- In particular, our notation for the identity substitution is [ ].

♣ Notation of substitution composition

- Let $\sigma_1 = [x_1/t_1, ..., x_n/t_n]$ and $\sigma_2 = [y_1/u_1, ..., y_k/u_k]$ are two substitutions having finite support. Then $\sigma_1\bullet\sigma_2$ has notation:
  $[x_1/(t_1\sigma_2), ..., x_n/(t_n\sigma_2), z_1/(z_1\sigma_2), ..., z_m/(z_m\sigma_2)]$
  where $z_1, ..., z_m$ are those variables in the list $y_1, ..., y_k$ that are not also in the list $x_1, ..., x_n$.

♣ Examples

- Let Suppose $\sigma_1 = [x/f(x, y), y/h(a), z/g(c, h(x))]$ and $\sigma_2 = [x/b, y/g(a, x), w/z]$. Then $\sigma_1\bullet\sigma_2 = [x/f(b, g(a, x)), y/h(a), z/g(c, h(b)), w/z]$.

Substitutions of Variables on Terms and Formulas

♣ ***Substitution*** of variable on terms and formulas

- Let σ: **V**→**Ter** be a variable substitution. It can be extended to all terms and formulas as follows:
  (1) $c\sigma = c$ for any $c \in$ **Con**, $\top\sigma = \top$, $\bot\sigma = \bot$;
  (2) $x\sigma = x\sigma$ for any $x \in$ **V**;
  (3) $[f(t_1, ..., t_n)]\sigma = f(t_1\sigma, ..., t_n\sigma)$ for any n-ary $f \in$ **Fun**;
  (4) $[p(t_1, ..., t_n)]\sigma = p(t_1\sigma, ..., t_n\sigma)$ for any n-ary $p \in$ **Pre**;
  (5) $(\neg A)\sigma = (\neg(A\sigma))$ for any $A \in$ **WFF**;
  (6) $(A*B)\sigma = ((A\sigma)*(B\sigma))$ for a binary connective * and any $A, B \in$ **WFF**;
  (7) $((\forall x)A)\sigma = ((\forall x)(A\sigma_x))$ and $((\exists x)A)\sigma = ((\exists x)(A\sigma_x))$ for any $A \in$ **WFF**, where by σ_x we mean the substitution that is like σ except that it does not change x, i.e., $y\sigma_x = y\sigma$ if $y \neq x$ and $y\sigma_x = x$ if $y = x$.
- Note: The result of applying a substitution to a term always producers another term.

Substitutions of Variables on Terms and Formulas: Examples

♣ An example

- Let $\sigma = [x/a, y/b]$.
  $$\begin{aligned}((\forall x)R(x, y) \supset (\exists y)R(x, y))\sigma &= ((\forall x)R(x, y))\sigma \supset ((\exists y)R(x, y))\sigma \\ &= (\forall x)(R(x, y))\sigma_x \supset (\exists y)(R(x, y))\sigma_y \\ &= (\forall x)(R(x, b)) \supset (\exists y)(R(a, y))\end{aligned}$$

♣ An example

- Let $\sigma = [x/y]$ and $\tau = [y/c]$. Then $\sigma\bullet\tau = [x/c, y/c]$.
  If $A = ((\forall y)R(x, y))$, then $A\sigma = ((\forall y)R(y, y))$, so $(A\sigma)\tau = ((\forall y)R(y, y))$.
  But $A(\sigma\bullet\tau) = ((\forall y)R(c, y))$, which is different.
- The example shows that the fact about substitution in terms, for any term t, $(t\sigma)\tau = t(\sigma\bullet\tau)$, does not carry over to formulas.
- What is needed is some restriction that will ensure composition of substitutions behaves well.

Free Substitutions

♣ ***Free substitution***

- A substitution being ***free for a formula*** is characterized as follows:

(1) If $A \in \mathbf{WFF}$ is an atomic formula, then σ is free for A.

(2) For any $A \in \mathbf{WFF}$, σ is free for $\neg A$, if σ is free for A.

(3) For any $A, B \in \mathbf{WFF}$, σ is free for $(A*B)$, if σ is free for A and σ is free for B, where * is a binary connective.

(4) For any $A \in \mathbf{WFF}$, σ is free for $((\forall x)A)$ and $((\exists x)A)$ provided: σ_x is free for A, and if y is a free variable of A other than x, $y\sigma$ does not contain x.

♣ Theorem (***free substitution***)

- Suppose the substitution σ is free for the formula A, and the substitution τ is free for $A\sigma$. Then $(A\sigma)\tau = A(\sigma\bullet\tau)$.

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