# Relevant Logics:

## Proof Theory and Model Theory

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## Relevant Logics: Proof Theory and Model Theory

- Formal Language of Relevant Logics
- Hilbert Style Axiomatic Systems of Relevant Logics
- Various Properties of Relevant Logics
- Model Theory for Relevant Logics
- Natural Deduction Systems of Relevant Logics
- Sequent Calculus Systems of Relevant Logics
- Semantic Tableau Systems of Relevant Logics
- Bibliography



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## Reference Books on Relevant Logics


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## Formal Language of Propositional Relevant Logics

- **Alphabet (Symbols)**

  $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, \top, \bot, (, ), p_1, p_2, ..., p_n, ... \}$
- **Extensional connectives**

  $\neg, \rightarrow, \wedge, \vee, \leftrightarrow$
- **Intensional connectives**

  $\Rightarrow, \otimes, \oplus, \Leftrightarrow$
- **Propositional constants (Propositional constant symbols)**

  $\top, \bot$
- **Propositional variables (Proposition symbols)**

  $p_1, p_2, ..., p_n, ...,$



## Formal Language of Predicate Relevant Logics

- **Alphabet (Symbols)**

  $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, \forall, \exists, \top, \bot, (, ),$
  $x_1, x_2, ..., x_n, ..., c_1, c_2, ..., c_n, ...,$
  $f^1{}_1, ..., f^1{}_n, ..., f^2{}_1, ..., f^2{}_n, ..., f^k{}_1, ..., f^k{}_n, ...,$
  $p^0{}_1, ..., p^0{}_n, ..., p^1{}_1, ..., p^1{}_n, ..., p^2{}_1, ..., p^2{}_n, ..., p^k{}_1, ..., p^k{}_n, ... \}$
- **Individual variables (Variable symbols)**

  $x_1, x_2, ..., x_n, ...,$
- **Individual Constants/Names (Constant/Name symbols)**

  $c_1, c_2, ..., c_n, ...,$
- **Individual Functions (Function symbols)**

  $f^1{}_1, ..., f^1{}_n, ..., f^2{}_1, ..., f^2{}_n, ..., f^k{}_1, ..., f^k{}_n, ...,$
- **Individual Predicates/Relations (Predicate/relation symbols)**

  $p^0{}_1, ..., p^0{}_n, ..., p^1{}_1, ..., p^1{}_n, ..., p^2{}_1, ..., p^2{}_n, ..., p^k{}_1, ..., p^k{}_n, ...$



## Formal Language of Relevant Logics

- **Primitive logical connectives**

  $\Rightarrow$ : entailment (primitive! And therefore, intensional!)
  $\neg$ : negation
  $\wedge$ : extensional conjunction
- **Defined logical connectives**

  $\otimes$ : intensional conjunction (fusion), $A \otimes B =_{df} \neg(A \Rightarrow \neg B)$
  $\oplus$ : intensional disjunction, $A \oplus B =_{df} \neg A \Rightarrow B$
  $\Leftrightarrow$ : intensional equivalence, $A \Leftrightarrow B =_{df} (A \Rightarrow B) \otimes (B \Rightarrow A)$
  $\vee$ : extensional disjunction, $A \vee B =_{df} \neg(\neg A \wedge \neg B)$
  $\rightarrow$ : material implication, $A \rightarrow B =_{df} \neg(A \wedge \neg B)$ or $\neg A \vee B$
  $\leftrightarrow$ : extensional equivalence, $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$
  $L$ : necessity operator, $LA =_{df} (A \Rightarrow A) \Rightarrow A$
- **Terms and formulas**
  - Similar to that of classical propositional/predicate calculus



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## Axiom Schemata on Entailment

- **Axiom schemata on entailment**

E1 $A \Rightarrow A$ (Self-Implication)
E2 $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$ (Prefixing)
E2′ $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ (Suffixing)
E3 $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (Contraction)
E3′ $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ (Self-Distribution)
E3″ $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$ (Permuted Self-Distribution)
E4 $(A \Rightarrow ((B \Rightarrow C) \Rightarrow D)) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow D))$ (Restricted Permutation)
E4′ $(A \Rightarrow B) \Rightarrow (((A \Rightarrow B) \Rightarrow C) \Rightarrow C)$ (Restricted Assertion)
E4″ $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$ (Specialized Assertion)
E4‴ $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (((A \Rightarrow C) \Rightarrow D) \Rightarrow D))$
E5 $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (Permutation)
E5′ $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ (Assertion)
E5″ $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$ (Demodalizer)

## Axiom Schemata on Conjunction and Disjunction

♣ **Axiom schemata on conjunction**

- C1 $(A\wedge B)\Rightarrow A$ (Conjunction Elimination)
- C2 $(A\wedge B)\Rightarrow B$ (Conjunction Elimination)
- C3 $((A\Rightarrow B)\wedge(A\Rightarrow C))\Rightarrow(A\Rightarrow(B\wedge C))$ (Conjunction Introduction)
- C4 $(LA\wedge LB)\Rightarrow L(A\wedge B)$, where $LA =_{df} (A\Rightarrow A)\Rightarrow A$ (Distribution of Necessity over Conjunction)

♣ **Axiom schemata on disjunction**

- D1 $A\Rightarrow(A\vee B)$ (Disjunction Introduction)
- D2 $B\Rightarrow(A\vee B)$ (Disjunction Introduction)
- D3 $((A\Rightarrow C)\wedge(B\Rightarrow C))\Rightarrow((A\vee B)\Rightarrow C)$ (Disjunction Elimination)

♣ **Distribution axiom schema**

DCD $(A\wedge(B\vee C))\Rightarrow((A\wedge B)\vee C)$ (Distribution of Conjunction over Disjunction)



## Axiom Schemata on Negation and Necessity

♣ **Axiom schemata on negation**

- N1 $(A\Rightarrow(\neg A))\Rightarrow(\neg A)$ (Reduction)
- N2 $(A\Rightarrow(\neg B))\Rightarrow(B\Rightarrow(\neg A))$ (Contraposition)
- N3 $(\neg(\neg A))\Rightarrow A$ (Double Negation)

♣ **Mingle axiom schemata**

- EM0 $(A\Rightarrow B)\Rightarrow((A\Rightarrow B)\Rightarrow(A\Rightarrow B))$
- RM0 $A\Rightarrow(A\Rightarrow A)$

♣ **Axiom schemata on necessity**

- $L1$ $LA\Rightarrow A$
- $L2$ $L(A\Rightarrow B)\Rightarrow(LA\Rightarrow LB)$
- $L3$ $(LA\wedge LB)\Rightarrow L(A\wedge B)$
- $L4$ $LA\Rightarrow LLA$
- $L5$ $LA\Rightarrow((LA\Rightarrow LA)\Rightarrow LA)$



## Axiom Schemata on Individual Quantification

♣ **Axiom schemata on individual quantification**

- IQ1 $\forall x(A\Rightarrow B)\Rightarrow(\forall xA\Rightarrow\forall xB)$
- IQ2 $(\forall xA\wedge\forall xB)\Rightarrow\forall x(A\wedge B)$
- IQ3 $\forall xAx\Rightarrow Ay$
- IQ4 $\forall x(A\Rightarrow B)\Rightarrow(A\Rightarrow\forall xB)$ ($x$ not free in $A$)
- IQ5 $\forall x(A\vee B)\Rightarrow(A\vee\forall xB)$ ($x$ not free in $A$)
- IQ6 $\forall x_1 \ldots \forall x_n (((A\Rightarrow A)\Rightarrow B)\Rightarrow B)$ $(n\geq 0)$ (for E and EM)
- IQ7 $Ay\Rightarrow\exists xAx$
- IQ8 $\forall x(A\Rightarrow B)\Rightarrow(\exists xA\Rightarrow B)$ ($x$ not free in $B$)
- IQ9 $(\exists xA\wedge B)\Rightarrow\exists x(A\wedge B)$ ($x$ not free in $B$)

Axiom clause: if $A$ is an axiom, so is $\forall xA$.



## Inference Rules

♣ **Inference rules**

- $\Rightarrow$E : From $A$ and $A\Rightarrow B$ to infer $B$ (Modus Ponens)
- $\wedge$I : From $A$ and $B$ to infer $A\wedge B$ (Adjunction)
- $L$I : If $A$ is a theorem, so is $LA$
- $\otimes$I : If $A\Rightarrow(B\Rightarrow C)$ is a theorem, so is $(A\otimes B)\Rightarrow C$
- $\otimes$E : If $(A\otimes B)\Rightarrow C$ is a theorem, so is $A\Rightarrow(B\Rightarrow C)$
- $T$I : If $A$ is a theorem, so is $T\Rightarrow A$
- $T$E : If $T\Rightarrow A$ is a theorem, so is $A$



## The Pure Entailment Fragments

♣ **The pure entailment (relevant implication) fragments**

$T_{\Rightarrow}$ = {E1, E2, E2′, E3 | E3″} + $\Rightarrow$E

$E_{\Rightarrow}$ = {E1, E2 | E2′, E3 | E3′, E4 | E4′} + $\Rightarrow$E
$E_{\Rightarrow}$ = {E2′, E3, E4″} + $\Rightarrow$E
$E_{\Rightarrow}$ = {E1, E3, E4‴} + $\Rightarrow$E
$E_{\Rightarrow}$ = $T_{\Rightarrow}$ + E4 [E4: $(A\Rightarrow((B\Rightarrow C)\Rightarrow D))\Rightarrow((B\Rightarrow C)\Rightarrow(A\Rightarrow D))$]
$E_{\Rightarrow}$ = $T_{\Rightarrow}$ + E4′ [E4′: $(A\Rightarrow B)\Rightarrow(((A\Rightarrow B)\Rightarrow C)\Rightarrow C)$]
$E_{\Rightarrow}$ = $T_{\Rightarrow}$ + E4″ [E4″: $((A\Rightarrow A)\Rightarrow B)\Rightarrow B$]

$R_{\Rightarrow}$ = {E1, E2 | E2′, E3 | E3′, E5 | E5′} + $\Rightarrow$E
$R_{\Rightarrow}$ = $E_{\Rightarrow}$ + E5″ [E5″: $A\Rightarrow((A\Rightarrow A)\Rightarrow A) = A\Rightarrow LA$]

♣ **Note**

"A | B" means that one can choose any one of the two axiom schemata A and B.



## The Positive (Negation-free) Fragments

♣ **The positive (negation-free) fragments**

$T_{+}$ = $T_{\Rightarrow}$ + {C1~C3, D1~D3, DCD} + $\wedge$I

$E_{+}$ = $E_{\Rightarrow}$ + {C1~C4, D1~D3, DCD} + $\wedge$I
$E_{+}$ = $T_{+}$ + {E4 | E4′ | E4″, C4}

$R_{+}$ = $R_{\Rightarrow}$ + {C1~C3, D1~D3, DCD} + $\wedge$I
$R_{+}$ = $E_{+}$ + E5″

## The Entailment with Negation Fragments

- The entailment (relevant implication) with negation fragments
  - $T_{\Rightarrow,\neg} = T_{\Rightarrow} + \{N1, N2, N3\}$
  - $E_{\Rightarrow,\neg} = E_{\Rightarrow} + \{N1, N2, N3\}$
  - $R_{\Rightarrow,\neg} = R_{\Rightarrow} + \{N2, N3\}$



## Hilbert Style Axiomatic Systems of Relevant Logics

- Propositional relevant logics
  - $T = T_{\Rightarrow,\neg} + \{C1\sim C3, D1\sim D3, DCD\} + \wedge I$
  - $E = E_{\Rightarrow,\neg} + \{C1\sim C4, D1\sim D3, DCD\} + \wedge I$
  - $E = T + \{ E4 \mid E4' \mid E4'', C4\}$
  - $R = R_{\Rightarrow,\neg} + \{C1\sim C3, D1\sim D3, DCD\} + \wedge I$
  - $R = E + A \Rightarrow LA$, $LR = R - DCD$
  - EM = E + EM0 (semi-relevant logic)
  - RM = R + RM0 (semi-relevant logic)
  - RM = EM + $A \Rightarrow LA$ (semi-relevant logic)
  - $E^L = E + \{L1\sim L5\} + LI$
  - $R^L = R + \{L1\sim L4\} + LI$



## Hilbert Style Axiomatic Systems of Relevant Logics

- Predicate relevant logics
  - $S^{\forall\exists x} = S + \{IQ1, IQ3, IQ4, IQ7, IQ8\}$
    where $S = T_{\neg}$ , $T_{\Rightarrow,\neg}$ , $R_{\neg}$ , $R_{\Rightarrow,\neg}$
  - $S^{\forall\exists x} = S + \{IQ1\sim IQ5, IQ7\sim IQ9\}$
    where $S = T_{+}$ , T , $R_{+}$ , R , RM
  - $S^{\forall\exists x} = S + \{IQ1, IQ3, IQ4, IQ6\sim IQ8\}$
    where $S = E_{\neg}$ , $E_{\Rightarrow,\neg}$
  - $S^{\forall\exists x} = S + \{IQ1\sim IQ9\}$
    where $S = E_{+}$ , E , EM



## Hilbert Style Axiomatic Systems of Relevant Logics

- Axiom schemata on conjunction
  - C5 $(A \wedge A) \Rightarrow A)$
  - C6 $(A \wedge B) \Rightarrow (B \wedge A)$
  - C7 $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
  - C8 $(A \wedge (A \Rightarrow B)) \Rightarrow B)$
  - C9: $\neg(A \wedge \neg A)$
  - C10: $A \Rightarrow (B \Rightarrow (A \wedge B))$
- Strong relevant logics
  - $Tc =_{df} T_{\Rightarrow,\neg} + \{C3, C5\sim C10\}$ $TcQ =_{df} Tc + \{IQ1\sim IQ5\} + \forall I$
  - $Ec =_{df} E_{\Rightarrow,\neg} + \{C3\sim C10\}$ $EcQ =_{df} Ec + \{IQ1\sim IQ5\} + \forall I$
  - $Rc =_{df} R_{\Rightarrow,\neg} + \{C3, C5\sim C10\}$ $RcQ =_{df} Rc + \{IQ1\sim IQ5\} + \forall I$



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## The Notion of Degree

- The degree of a formula with implicational connective
  - The *degree* of a formula with implicational connective is the largest number of nesting of implicational connective, e.g. $\Rightarrow$, that represents the notion of conditional within it.
- Zero degree formula
  - A formula is called a *zero degree formula* (zdf) if and only if there is no occurrence of $\Rightarrow$ in it.
- First degree conditional (entailment)
  - A formula of the form $A \Rightarrow B$ is called a *first degree conditional* (fdc) (*entailment*, fde) if and only if both $A$ and $B$ are zero degree formulas.

## The Notion of Degree

♣ **First degree formula**

- A formula $A$ is called a ***first degree formula*** **(fdf)** if and only if it satisfies the one of the following conditions:
  (1) $A$ is a first degree conditional,
  (2) $A$ is in the form $+B$ (+ is a one-place connective such as negation and so on) where $B$ is a first degree formula, and
  (3) $A$ is in the form $B*C$ (* is a non-implicational two-place connective such as conjunction or disjunction and so on) where both of $B$ and $C$ is first degree formulas, or one of $B$ and $C$ is a first degree formula and the another is a zero degree formula.



## The Notion of Degree

♣ **$k^{th}$ degree conditional (entailment)**

- Let $k$ be a natural number. A formula of the form $A \Rightarrow B$ is called a ***$k^{th}$ degree conditional*** **(kdc)** (***entailment***, **kde**) if and only if both $A$ and $B$ are $(k-1)^{th}$ degree formulas, or one of $A$ and $B$ is a $(k-1)^{th}$ degree formula and another is a $j^{th}$ $(j<k-1)$ degree formula.

♣ **$k^{th}$ degree formula**

- Let $k$ be a natural number. A formula $A$ is called a ***$k^{th}$ degree formula*** **(kdf)** if and only if it satisfies the one of the following conditions:
  (1) $A$ is a $k^{th}$ degree conditional,
  (2) $A$ is in the form $+B$ where $B$ is a $k^{th}$ degree formula, and
  (3) $A$ is in the form $B*C$ where both of $B$ and $C$ is $k^{th}$ degree formulas, or one of $B$ and $C$ is a $k^{th}$ degree formula and another is a $j^{th}$ $(j<k)$ degree formula.



## Zero Degree Fragments of Relevant Logics: CPC

♣ **Classical Propositional Calculus (CPC) is contained in E (R, T)** [A&B-E1-75]

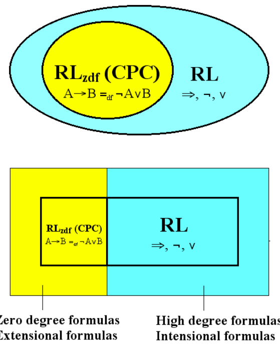
- Theorem: All tautologies (theorems) of CPC are provable in E (R, T), i.e., E (R, T) is complete with respect to CPC.
- Theorem: Only tautologies (theorems) of CPC among the zero degree formulas of E (R, T) are provable in E (R, T), i.e., E (R, T) is conservative with respect to CPC.

♣ **Classical Propositional Calculus (CPC) is the zero degree fragment of E (R, T)** [A&B-E1-75]

- CPC is in exactly the right sense contained in E (R, T), i.e., it is exactly the zero degree (extensional) fragment of E (R, T).



## Zero Degree Fragments of Relevant Logics: CPC






## Primitive Entailments

♣ **Atom**

- An ***atom*** is a propositional variable or the negate of such, i.e., an atom has either the form $p$ or the form $\neg p$.

♣ **Primitive conjunction and disjunction**

- A ***primitive conjunction*** is a conjunction $A_1 \wedge A_2 \wedge \ldots \wedge A_m$ $(m \geq 1)$ where each $A_i$ is an atom.
- A ***primitive disjunction*** is a disjunction $B_1 \vee B_2 \vee \ldots \vee B_n$ $(n \geq 1)$ where each $B_i$ is an atom.

♣ **Primitive entailment**

- $A \Rightarrow B$ is a ***primitive entailment*** if $A$ is a primitive conjunction and $B$ is a primitive disjunction, i.e., $A \Rightarrow B = (A_1 \wedge A_2 \wedge \ldots \wedge A_m) \Rightarrow (B_1 \vee B_2 \vee \ldots \vee B_n)$.



## Explicitly Tautological Entailments

♣ **Explicitly tautological entailments**

- A primitive entailment $A \Rightarrow B$ is said to be ***explicitly tautological*** if some (conjoined) atom of $A$ is identical with some (disjoined) atom of $B$, i.e., $A_i = B_j$, for some $i$ and $j$.
- Explicitly tautological is both necessary and sufficient for the (**weak-relevant!**) validity of a primitive entailment.

♣ **Note**

- Explicitly tautological entailments satisfy the von Wright-Geach-Smiley criterion for entailment: every explicitly tautological entailment answers to a material "implication" which is a substitution instance of a tautologous material "implication" with non-contradictory antecedent and non-tautologous consequent; and evidently we may ascertain the truth of the entailment without coming to know the truth of the consequent or the falsity of the antecedent.

## Explicitly Tautological Entailments in Normal Form

♣ Entailments in normal form

◆ An entailment $A \Rightarrow B$ is said to be *in normal form* if it has the form $(A_1 \vee A_2 \vee \ldots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \ldots \wedge B_n)$ $(m \geq 1, n \geq 1)$ where each $A_i$ is a primitive conjunction and each $B_j$ is a primitive disjunction.

◆ An entailment $A \Rightarrow B$ in normal form is (weak-relevantly!) valid just in case each $A_i \Rightarrow B_j$ $(m \geq i \geq 1, n \geq j \geq 1)$ is explicitly tautological.

♣ Explicitly tautological entailments in normal form

◆ An entailment $(A_1 \vee A_2 \vee \ldots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \ldots \wedge B_n)$ $(m \geq 1, n \geq 1)$ in normal form is said to be *explicitly tautological* if and only if for every $A_i$ and $B_j$, $A_i \Rightarrow B_j$ $(m \geq i \geq 1, n \geq j \geq 1)$ is explicitly tautological.



## Tautological Entailments

♣ Tautological entailments

◆ An entailment $(A_1 \vee A_2 \vee \ldots \vee A_m) \Rightarrow (B_1 \wedge B_2 \wedge \ldots \wedge B_n)$ $(m \geq 1, n \geq 1)$ in normal form is called a *tautological entailment* if and only if it is explicitly tautological, i.e., for every $A_i$ and $B_j$, $A_i \Rightarrow B_j$ $(m \geq i \geq 1, n \geq j \geq 1)$ is explicitly tautological.

◆ Explicitly tautological is both necessary and sufficient for the (weak-relevant!) validity of a first degree entailment.

♣ Fundamental question

◆ Can we construct a formal calculus of tautological entailments?

◆ Answer: YES



## Tautological Entailment Fragments of Relevant Logics

♣ $E_{fde}$ ($R_{fde}$, $T_{fde}$): First degree entailment fragment of E (R, T)

◆ Entailment:
Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$

◆ Conjunction:
Axiom schemata: $(A \wedge B) \Rightarrow A$ (C1), $(A \wedge B) \Rightarrow B$ (C2)
Inference rule: from $A \Rightarrow B$ and $A \Rightarrow C$ to infer $A \Rightarrow (B \wedge C)$

◆ Disjunction:
Axiom schemata: $A \Rightarrow (A \vee B)$ (D1), $B \Rightarrow (A \vee B)$ (D2)
Inference rule: from $A \Rightarrow C$ and $B \Rightarrow C$ to infer $(A \vee B) \Rightarrow C$

◆ Distribution:
Axiom schema: $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$ (DCD)

◆ Negation:
Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$



## First Degree Entailment Fragments of Relevant Logics

♣ $E_{fde}$ ($R_{fde}$, $T_{fde}$): First degree entailment fragment of E (R, T)

◆ Entailment:
Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$

◆ Conjunction:
Axiom schemata: $(A \wedge B) \Rightarrow A$ (C1), $(A \wedge B) \Rightarrow B$ (C2)
Inference rule: from $A \Rightarrow B$ and $A \Rightarrow C$ to infer $A \Rightarrow (B \wedge C)$

◆ Disjunction:
Axiom schemata: $A \Rightarrow (A \vee B)$ (D1), $B \Rightarrow (A \vee B)$ (D2)
Inference rule: from $A \Rightarrow C$ and $B \Rightarrow C$ to infer $(A \vee B) \Rightarrow C$

◆ Distribution:
Axiom schema: $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$ (DCD)

◆ Negation:
Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$



## First Degree Formula Fragments of Relevant Logics

♣ $E_{fdf}$ and tautological entailments

◆ All logical theorems of $E_{fdf}$ are first degree entailments.

◆ Every tautological entailment is provable in $E_{fdf}$.

◆ Only tautological entailments are provable in $E_{fdf}$.

◆ Therefore, $E_{fdf}$ is a formalization of tautological entailments.

♣ Perfect interpolation theorem

◆ If $A \Rightarrow C$ is provable in $E_{fdf}$, then there is an "interpolation formula" B such that (1) $A \Rightarrow B$ is provable in $E_{fdf}$, (2) $B \Rightarrow C$ is provable in $E_{fdf}$, and (3) $B$ has no variables not in both $A$ and $C$.



## Antecedent and Consequent Parts of Formulas

♣ Antecedent/Consequent parts of formulas

◆ $A$ is a consequent part of $A$.

◆ If $\neg B$ is a *consequent* {*antecedent*} *part* of $A$, then $B$ is an antecedent part {consequent part} of $A$.

◆ If $B \Rightarrow C$ is a consequent {antecedent} part of $A$, then $B$ is an antecedent {consequent} part of $A$, and $C$ is a consequent {antecedent} part of $A$.

◆ If either $B \wedge C$ or $B \vee C$ is a consequent {antecedent} part of $A$, then both $B$ and $C$ are consequent {antecedent} parts of $A$.

## Theorems of Variable-Sharing in Relevant Logics

- Variable-Sharing in $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$) [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$), then $A$ and $B$ share a sentential variable.
  - If $A$ is a theorem of $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$), then every sentential variable occurring in $A$ occurs at least once as an antecedent part and at least once as a consequent part of $A$.
- Variable-Sharing in $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$ ) [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$), then $A$ and $B$ share a sentential variable.
  - If A is a theorem of $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$), then every sentential variable occurring in $A$ occurs at least once as an antecedent part and at least once as a consequent part of $A$ (Note: This is NOT true for E (or for R, or for T).



## Theorems of Variable-Sharing in Relevant Logics

- Variable-Sharing in E (R, T) [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in E (R, T), then $A$ and $B$ share a sentential variable.
  - If $A \Rightarrow B$ is a theorem of E, then some sentential variable occurs as an antecedent part of both $A$ and $B$, or else as a consequent part of both $A$ and $B$.
  - If $A$ is a theorem of E containing no conjunctions as antecedent parts and no disjunctions as consequent parts, then every sentential variable in $A$ occurs at least once as an antecedent part and at least once as a consequent part [Maksimova, 1967].



## Theorems of Variable-Sharing in Relevant Logics

- Variable-Sharing in CPC [A&B-E1-75]
  - If $A \rightarrow B$ is provable in CPC, then either (1) $A$ and $B$ share a sentential variable or (2) either $\neg A$ or $B$ is provable in CPC.
- Variable-Sharing in RM [A&B-E1-75]
  - If $A \rightarrow B$ is provable in RM, then either (1) $A$ and $B$ share a sentential variable or (2) both $\neg A$ and $B$ are provable in RM.



## Facts in Relevant Logics

- Why the inference rule of adjunction?
  - $A \Rightarrow (B \Rightarrow (A \wedge B))$ is not a logical theorem of E (R, T).
  - $A \Rightarrow (B \Rightarrow (A \wedge B))$ is a familiar axiom for intuitionistic and classical logic, but it is only a hair's breadth away from positive paradox $A \Rightarrow (B \Rightarrow A)$, and indeed yields it given $(A \wedge B) \Rightarrow A$ and $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$.
- The disjunctive syllogism and the inference rule γ
  - $(A \wedge (\neg A \vee B)) \Rightarrow B$ (or $(\neg A \wedge (A \vee B)) \Rightarrow B$) is not a logical theorem of either E (R, T). This is the most notorious feature of relevant logic.
  - The inference rule γ, i.e., suppose $\vdash A$ and $\vdash \neg A \vee B$, then $\vdash B$, is admissible in E, R, and many others.



## The Relationship between Relevant Logics and Others

- The relationship between relevant logics and other logics

S5 ---> S4 ---> S3 ---> EM

CML --->                                   ---> E ---> T

RM -----------------------> R

(S3, S4, S5: Lewis's modal systems)

(---> means the inclusion relationship)

- The relationship between relevant logics and their first degree entailment fragments
  - If $A \Rightarrow B$ is provable in $E_{fde}$ ($R_{fde}$, $T_{fde}$), then it is also provable in E (R, T) [A&B-E1-75].



## Facts in Relevant Logics

- Conservative extensions
  - Let S′ is an extension of S in the sense that S′ has some new language components, e.g., connectives, or axioms, or inference rules. S′ is called to be a *conservative extension* of S if for any formula A in the notation of S, if A is provable in S′ then A is also provable in S.
- Conservative extensions in relevant logics
  - Both E and $E_{\Rightarrow,\neg}$ are conservative extensions of $E_{\Rightarrow}$
  - Both R and $R_{\Rightarrow,\neg}$ are conservative extensions of $R_{\Rightarrow}$

## Facts in Relevant Logics

**Conservative extensions in relevant logics**

- RM is not a conservative extension of $RM0_{\Rightarrow}$ ($RM0_{\Rightarrow} = R_{\Rightarrow} + RM0$)
- $RM0_{\Rightarrow}$ is not the pure implicational fragment of RM
- RM does not satisfy the relevance principle but it does satisfy the weaker relevance principle that $A \rightarrow B$ is a theorem of RM only if either $A$ and $B$ share a sentential variable or both $\neg A$ and $B$ are theorems.



## Facts in Relevant Logics

**Necessity**

- In E, the necessity operator $L$ can be defined as $LA =_{df} (A \Rightarrow A) \Rightarrow A$, but this is impossible in R because R has $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ as an axiom scheme.
- E is both a relevant logic and a modal logic but R is only a relevant logic.
- The rule of necessity (if $A$ is provable in E, then $LA$ is also provable in E) is naturally holds in E, and therefore no new logical primitives need be introduced to get the desired effect.



## Why R Is Interesting?

**Major reason 1: Age**

- The pure implicational fragment $R\Rightarrow$ of R, first considered by Moh Shaw-Kwei in 1950 and by A. Church in 1951, was regarded to be the oldest of the relevant logics [A&B-E1-75].
- The implication-negation fragment $R_{\Rightarrow,\neg}$ of R, given by I. E. Orlov in 1928, is the oldest of the relevant logics [Dosen, 1990].

**Major reason 2: Isolating relevance**

- In R one has an even clearer view of relevance than in E, just because of the absence of modal complications.

**Other reasons**

- Stability, Richness, Easy proof theory, Fragments, Applicability, Extensibility.



## Decision Problems in Relevant Logic

- Both $E_{\Rightarrow}$ and $R_{\Rightarrow}$ are decidable [Kripke, 1959].
- LR is decidable [Mayer, 1966].
- $E_{fde}$ ($R_{fde}$, $T_{fde}$) is decidable [Anderson and Belnap, 1975].
- $E_{fdf}$ is decidable [Anderson and Belnap, 1975].
- Both $E_{\Rightarrow,\neg}$ and $R_{\Rightarrow,\neg}$ are decidable [Anderson and Belnap, 1975].
- RM is decidable [Anderson and Belnap, 1975].
- $E_{+}$, $R_{+}$, and $T_{+}$ are undecidable [Urquhart, 1982].
- E, R, and T are undecidable [Urquhart, 1982].
- The decision problem of $T_{\Rightarrow}$ (or of $T_{\Rightarrow,\neg}$) is open.



## Relevant Logics: Proof Theory and Model Theory

- Formal Language of Relevant Logics
- Hilbert Style Axiomatic Systems of Relevant Logics
- Various Properties of Relevant Logics
- Model Theory for Relevant Logics
- Natural Deduction Systems of Relevant Logics
- Sequent Calculus Systems of Relevant Logics
- Semantic Tableau Systems of Relevant Logics
- Bibliography