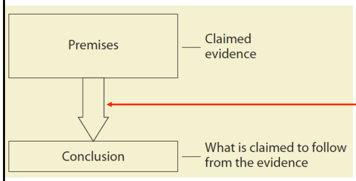
An Elementary Introduction to Modal Logics

Jingde Cheng

Saitama University

Logic: What Is It?



[From P. J. Hurley, "A Concise Introduction to Logic"]

What entails what?

What follows from what?

Why? What are the evaluation criteria?

How to establish/define the evaluation criteria?

How to evaluate arguments/reasoning?

It is LOGIC to answer these fundamental questions.

Major Reference Books on Modal Logics

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- J. Garson, "Modal Logic," in "Stanford Encyclopedia of Philosophy," 2000, 2018.

Major Reference Books on Various Modal Logics

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- J. P. Burgess, "Basic Tense Logic," in D. M. Gabbay and F. Guenthner (Eds.), "Handbook of Philosophical Logic, 2nd Edition," Vol. 7, pp. 1-42, Kluwer Academic, 2002.
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- P. Ohrstrom and P. F. V. Hasle, "Modern Temporal Logic: The Philosophical Background," In D. M. Gabbay and J. Woods (Eds.), "Handbook of the History of Logic," Vol. 7, pp. 447-498, Elsevier, 2006.
- V. Goranko and A. Rrumberg, "Temporal Logic," in "Stanford Encyclopedia of Philosophy," 1999, 2020.

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- L. Aqvist, "Deontic Logic," in D. M. Gabbay and F. Guenthner (Eds.), "Handbook of Philosophical Logic, 2nd Edition," Vol. 8, pp. 147-264, Kluwer Academic, 2002.
- P. McNamara, "Deontic Logic," In D. M. Gabbay and J. Woods (Eds.), "Handbook of the History of Logic," Vol. 7, pp. 197-288, Elsevier, 2006.
- P. McNamara, "Deontic Logic," in "Stanford Encyclopedia of Philosophy," 2006, 2010.
- J.-J. Ch. Meyer and W. van der Hoek, "Epistemic Logic for AI and Computer Science," Cambridge University Press, 1995.
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- P. Gochet and P. Gribomont, "Epistemic Logic," In D. M. Gabbay and J. Woods (Eds.), "Handbook of the History of Logic," Vol. 7, pp. 99-195, Elsevier, 2006.
- H. van Ditmarsch, W. van der Hoek, and B. Kooi, "Dynamic Epistemic Logic," Springer, 2007.
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- G. Priest, "An Introduction to Non-Classical Logic: From If to Is," Cambridge University Press, 2001, 2008 (2nd Edition).
- R. T. Cook, "A Dictionary of Philosophical Logic," Edinburgh University Press, 2009.
- D. M. Gabbay and F. Guenthner (Eds.), "Handbook of Philosophical Logic, 2nd Edition," Vol.1-18, 2001-2018.

An Elementary Introduction to Modal Logics

- **Modal Logic: What Is It and Why Study It ?**
- **Relational Structures**
- **Modal Languages**
- **Model (Semantic) Theory of Modal Logics**
- **Normal Modal Logics**
- **Temporal Logics**
- **Spatio-Temporal Logics**
- **Deontic Logics**
- **Epistemic Logics**

Modal Logic: What Is It and Why Study It ?

Modal Logic: What Is It?

Modal Logic: What Is It ?

- **"Modal logic is the logic of necessity and possibility, of 'must be' and 'may be'. By this is meant that it considers not only truth and falsity applied to what is or is not so as things actually stand, but considers what would be so if things were different. If we think of how things are as the actual world then we may think of how things might have been as how things are in an alternative, non-actual but possible, state of affairs – or possible world. Logic is concerned with truth and falsity. In modal logic we are concerned with truth or falsity in other possible worlds as well as the real one. In this sense a proposition will be necessary in a world if is true in all worlds which are possible relative to that world, and possible in a world if it is true in at least one world possible relative to that world."**
  -- G. E. Hughes and M. J. Cresswell, "A New Introduction to Modal Logic," Routledge, 1996.

Modal Logic: What Is It ?

- **"Modal logic is a branch of mathematical logic studying mathematical models of correct reasoning which involves various kinds of necessity-like and possibility-like operators."**
  -- A. Chagrov and M. Zakharyaschev, "Modal Logic," Oxford University Press, 1997.

Modal Logic: What Is It ?

- **"Ask three modal logicians what modal logic is, and you are likely to get at least three different answers."**
  "a number of general ideas guide our thinking about the subject, and we'll present the most important right away as a series of three slogans. "
  "Slogan 1: Modal languages are simple yet expressive languages for talking about relational structures.
  Slogan 2: Modal languages provide an internal, local perspective on relational structures.
  Slogan 3: Modal languages are not isolated formal systems."
  -- P. Blackburn, M. de Rijke, and Y. Venema, "Modal Logic," Cambridge University Press, 2001.

Modal Logic: What Is It ?

- **"Modal logic is the logic of necessity and possibility, of 'must be' and 'may be'. These may be interpreted in various ways. If necessity is necessary truth, there is alethic modal logic, if it is moral or normative necessity, there is deontic logic. It may refer to what is known or believed to be true, in which case, there is an epistemic logic, or to what always has been or to what henceforth always will be true, which gives an aspect of temporal logic. Another interpretation is to read 'necessarily p' as 'it is provable that p'."**
  – M. J. Cresswell, "Modal Logic", in Blackwell Guide to Philosophical Logic, 2001.

Modal Logic: What Is It ?

- "Modal logic, as the logic of necessity and possibility, takes into account not only truth and falsity of the way things actually are, but also what would be true or false if things were different. If one thinks of the way things are as the actual world, one may then think of how things might have been different as how they are in alternative, non-actual but possible possible worlds. As logic is concerned with truth and falsity, modal logic is concerned with truth or falsity in other possible worlds as well as the real one. A proposition is then necessary in a world just in case it is true in all the worlds that are possible alternatives to that world, and possible just in case it is true in some alternative possible world."
  -- M. J. Cresswell, "Modal Logic", in Blackwell Guide to Philosophical Logic, 2001.

Modal Logic: What Is It ?

- "Modal logic concerns itself with the modes in which things may be true or false, particularly their possibility, necessity and impossibility."
  -- G. Priest, "An Introduction to Non-Classical Logic: From If to Is," Cambridge University Press, 2001, 2008 (2nd Edition).
- "Modal logics are used to formalize statements where finer distinctions need to be made than just 'true' or 'false'. Classically, modal logic distinguished between statements that are necessarily true and those that are possibly true." "a form of modal logic (is) called temporal logic, where 'necessarily' is interpreted as always and 'possibly' is interpreted as eventually."
  -- M. Ben-Ari, "Mathematical Logic for Computer Science," Springer, 1993, 2001, 2012 (3rd Edition).

Modal Logic: What Is It ?

- "A modal is an expression (like 'necessarily' or 'possibly') that is used to qualify the truth of a judgement. Modal logic is, strictly speaking, the study of the deductive behavior of the expressions 'it is necessary that' and 'it is possible that'. However, the term 'modal logic' may be used more broadly for a family of related systems. These include logics for belief, for tense and other temporal expressions, for the deontic (moral) expressions such as 'it is obligatory that' and 'it is permitted that', and many others."
  "Narrowly construed, modal logic studies reasoning that involves the use of the expressions 'necessarily' and 'possibly'. However, the term 'modal logic' is used more broadly to cover a family of logics with similar rules and a variety of different symbols."
  -- J. Garson, "Modal Logic", in "Stanford Encyclopedia of Philosophy," 2000, 2018.

Various Modal Notions

- *Alethic* "it is necessary", "it is possible"
- *Cognitive* "it is believed", "it is consistent with the current knowledge base"
- *Deontic* "it is obligatory", "it is permitted"
- *Dynamic* "after every execution of the program", "after some execution of the program"
- *Epistemic* "it is known", "it does not contradict to what is known"
- *Provable* "it is provable in a given formal theory", "it is consistent with the theory"
- *Temporal* "henceforth", "sometimes"
- *Tense* "at all future times", "eventually"

Various Modal Notions/Logics [Garson-2018]

Logic	Symbols	Expressions Symbolized
Modal Logic	$\Box$	It is necessary that …
	$\Diamond$	It is possible that …
Deontic Logic	O	It is obligatory that …
	P	It is permitted that …
	F	It is forbidden that …
Temporal Logic	G	It will always be the case that …
	F	It will be the case that …
	H	It has always been the case that …
	P	It was the case that …
Doxastic Logic	Bx	x believes that …

Modal Logic: Why Study It ?

Modal Logic: Why Study It?

Modal Logic: Why Study It?

- The limitation of classical logics
  - In various arguments and/or judgements, there are some modal expressions like ‘it is necessary that’ and ‘it is possible that’.
  - The formal language of classical logics have no modal operators and therefore is difficult to represent various modal notions/concepts.
- The needs of introducing modal operators into object languages
  - Modal notions/concepts, as abstract/general but related to the evidence relationship in arguments, should be objects studied by logic, and therefore, should be introduced into the object languages of logics.

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The History of Modal Logic: Aristotle

- Aristotle’s the definitions of the modalities
  - Modern modal logic treats necessity and possibility as interdefinable: “necessarily *P*” is equivalent to “not possibly not *P*”, and “possibly *P*” to “not necessarily not *P*”: $\Box P \leftrightarrow \neg\Diamond\neg P$, $\Diamond P \leftrightarrow \neg\Box\neg P$. Aristotle gives these same equivalences in “On Interpretation.” However, in “Prior Analytics”, he makes a distinction between two notions of possibility.
  - On the first, which he takes as his preferred notion, “possibly P” is equivalent to “not necessarily P and not necessarily not P”. He then acknowledges an alternative definition of possibility according to the modern equivalence, but this plays only a secondary role in his system.

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The History of Modal Logic: Aristotle

- Aristotle’s syllogisms with modalities
  - Aristotle builds his treatment of modal syllogisms on his account of non-modal (assertoric) syllogisms: he works his way through the syllogisms he has already proved and considers the consequences of adding a modal qualification to one or both premises.

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The History of Modal Logic: G. W. Leibniz

- G. W. Leibniz’s modal metaphysics (the metaphysics of necessity, contingency, and possibility)
  - G. W. Leibniz (1646-1716), developed an approach to questions of modality — necessity, possibility, contingency — that not only served an important function within his general metaphysics, epistemology, and philosophical theology but also has continuing interest today.
  - Indeed, it has been suggested that 20th-century developments in modal logic were either based on Leibnizian insights or at least had a Leibnizian spirit.

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The History of Modal Logic: G. W. Leibniz

- G. W. Leibniz’s modal metaphysics (the metaphysics of necessity, contingency, and possibility)
  - Leibniz in “Discourse on Metaphysics”, presents his classic picture: “The nature of an individual substance or of a complete being is to have a notion so complete that it is sufficient to contain and to allow us to deduce from it all the predicates of the subject to which this notion is attributed.”
  - “I think there is an infinity of possible ways in which to create the world, according to the different designs which God could form, and that each possible world depends on certain principal designs or purposes of God which are distinctive of it, that is, certain primary free decrees (conceived sub ratione possibilitatis) or certain laws of the general order of this possible universe with which they are in accord and whose concept they determine, as they do also the concepts of all the individual substances which must enter into this same universe.” – in a letter (14 July 1686) from Leibniz to Arnauld

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The History of Modal Logic: G. W. Leibniz

- G. W. Leibniz’s modal metaphysics (the metaphysics of necessity, contingency, and possibility)
  - Leibniz distinguishes necessary and contingent truths in §13 of the Discourse on Metaphysics:
    “The one whose contrary implies a contradiction is absolutely necessary; this deduction occurs in the eternal truths, for example, the truths of geometry. The other is necessary only ex hypothesi and, so to speak, accidentally, but it is contingent in itself, since its contrary does not imply a contradiction. And this connection is based not purely on ideas and God’s simple understanding, but on his free decrees and on the sequence of the universe.”

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The History of Modal Logic: G. W. Leibniz

♣ G. W. Leibniz's modal metaphysics (the metaphysics of necessity, contingency, and possibility)

- In §33 of the Monadology, Leibniz makes a clear distinction between necessary truths and contingent truths:
"There are two kinds of truths, those of reasoning and those of fact. The truths of reason are necessary and their opposite is impossible; the truths of fact are contingent and their opposite is possible. When a truth is necessary, its reason can be found by analysis, resolving it into simpler ideas and simpler truths until we reach the primitives."

The History of Modal Logic: H. MacColl

♣ H. MacColl's work

- In 1880, MacColl claimed that ($p \rightarrow q$) and ($\neg p \vee q$) are not equivalent: ($\neg p \vee q$) follows from ($p \rightarrow q$), but not vice versa. This is the case because MacColl interprets $\vee$ as regular extensional disjunction, and $\rightarrow$ as intensional implication, but then from the falsity of p or the truth of q it does not follow that p without q is logically impossible.
- In 1897, MacColl distinguished between certainties, possibilities and variable statements, and introduces Greek letters as indices to classify propositions. So α^{ε} expresses that α is a certainty (operator ε means 'it is certain that'), α^{η} that α is an impossibility (operator η means 'it is impossible that'), and α^{0} that α is a variable, i.e., neither a certainty nor an impossibility. Using this threefold classification of statements, MacColl proceeds to distinguish between causal implication and general implication.

The History of Modal Logic: C.I. Lewis

♣ C.I. Lewis's work

- Modern modal logic began in 1912 when Lewis filed a complaint to the effect that classical logic fails to provide a satisfactory analysis of implication, "the ordinary 'implies' of ordinary valid inference".
- Lewis 1912 published his paper "Implication and the Algebra of Logic", started to voice his concerns on the so-called "paradoxes of material implication".

The History of Modal Logic: C.I. Lewis

♣ C.I. Lewis's work

- Lewis points out that in Russell and Whitehead's "Principia Mathematica" we find two "startling theorems: (1) a false proposition implies any proposition ($\neg p \rightarrow (p \rightarrow q)$), and (2) a true proposition is implied by any proposition ($p \rightarrow (q \rightarrow p)$)."
- Lewis has in mind an intended meaning for the conditional connective $\rightarrow$, and that is the meaning of the English word "implies".
- The meaning of "implies" is that "of ordinary inference and proof" according to which a proposition implies another proposition if the second can be logically deduced from the first.
- Given such an interpretation, (1) and (2) ought not to be theorems, and propositional logic may be regarded as unsound vis-à-vis the reading of $\rightarrow$ as logical implication.

The History of Modal Logic: C.I. Lewis

♣ C.I. Lewis's work

- In "The Calculus of Strict Implication" (1914) Lewis suggests two possible alternatives to the extensional system of Whitehead and Russell's "Principia Mathematica."
- One way of introducing a system of strict implication consists in eliminating from the system those theorems that, like (1) and (2) above, are true only for material implication but not for strict implication, thereby obtaining a sound system for both material and strict implication, but in neither case complete.
- The second, more fruitful alternative consists in introducing a new system of strict implication, still modeled on the Whitehead and Russell system of material implication, that will contain (all or a part of) extensional propositional logic as a proper part, but aspiring to completeness for at least strict implication.

The History of Modal Logic: C.I. Lewis

♣ C.I. Lewis's work

- Lewis 1918 published his book "Survey of Symbolic Logic", as the birth of modal logic as a mathematical discipline; there Lewis introduced the first modal logic system (named as "Survey System") of strict implication meant to capture the ordinary, strict sense of implication.
- Lewis extended propositional calculus with a unary modality I ('it is impossible that') and defined the binary modality $p \rightarrow\!3\ q$ (p strictly implies q) to be I($p \wedge \neg q$).
- Strict implication was meant to capture the notion of logical entailment, and Lewis presented an axiom system based on '$\rightarrow\!3$' (fish-hook).
- Fact: The system Lewis proposed is equivalent to S3.

The History of Modal Logic: Lewis and Langford

♣ Lewis and Langford' work

- Lewis and Langford's joint book "Symbolic Logic", published in 1932, contains a more detailed development of Lewis' ideas.
- Here ◇('it is possible that') is primitive and $p \rightarrow 3\ q$ is defined to be $\neg\Diamond(p \wedge \neg q)$.
- Five axiom systems of ascending strength, S1–S5, are discussed; S3 is equivalent to Lewis' system of 1918, and only S4 and S5 are normal modal logics.

The History of Modal Logic: G. H. Von Wright

♣ G. H. Von Wright's work

- In his book "A Essay in Modal Logic" published 1951, Von Wright remarks that, strictly speaking, modal logic is the logic of the modes of being.
- In the book "A Essay in Modal Logic" and the related paper "Deontic Logic" also published 1951, Von Wright sets out to explore modal logic in a wider sense, the logic of the modes of knowledge, belief, norms, and similar concepts; this wider sense of the term has since gained currency.
- These two works marked the beginning of much work in epistemic, doxastic, and deontic logic.

The History of Modal Logic: K. Gödel

♣ K. Gödel's work

- Lewis's axiomatic systems are not modular. Instead of extending a base system of propositional logic with specifically modal axioms (e.g., K and others), Lewis defines his axioms directly in terms of strict implication '→3'.
- The modular approach to modal Hilbert-style systems is due to Gödel. Gödel 1933 showed that propositional intuitionistic logic could be translated into S4 in a theorem-preserving way.
- Instead of using the Lewis's axiomatization, Gödel took □ as primitive and formulated S4 in the way that has become standard: he enriched a standard system for classical propositional logic with the rule of generalization, the K axiom, and the additional axioms ($\Box p \rightarrow p$ and $\Box p \rightarrow \Box\Box p$).

The History of Modal Logic: S. A. Kripke

♣ S. A. Kripke's work

- The relational semantics of modal logics is often called Kripke semantics. Kripke laid down the foundations of modern propositional and predicade modal logic in several influential papers published from 1959 to 1965.
- Kripke 1959 proved that S5-based modal predicate logic is complete with respect to models with an implicit global relation.
- Kripke 1963 introduced an explicit accessibility relation R and gives semantic characterization of some propositional modal logics in terms of this relation; he also defined relational semantics for first-order modal languages.

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- Deontic Logics
- Epistemic Logics

Relational Structures: General Definition

♣ Relational structure: What is it?

- A *relational structure* is a non-empty set on which a number of relations have been defined; they are widespread in mathematics, computer science, artificial intelligence and linguistics, and are also used to interpret first-order languages.

♣ General definition of relational structure

- A relational structure is a n-tuple F whose first component is a non-empty set W called the *universe* (or *domain*) of F, and whose remaining components are relations on W.
- We assume that every relational structure contains at least one relation. The elements of W have a variety of names, including: points, states, nodes, worlds, times, instants and situations.

Strict Partial Order Structures

♣ **Strict partial orders (SPOs)**

- **A *strict partial order structure* is an ordered pair (W, R) such that R is irreflexive and transitive.**
- **A strict partial order R is a *linear order* (or a *total order*) if it also satisfies the trichotomy condition/law.**

♣ **Examples**

- **Suppose that W = (1, 2, 3, 4, 6, 8, 12, 24) and Rxy means 'x and y are different, and y can be divided by x.' R is a strict partial order but not a linear order.**
- **If we define Rxy by 'x is numerically smaller than y,' then we obtain a linear order over the same universe W.**
- **Important examples of linear order structures are (N, <) (or (ω, <)), (Z <), (Q, <) and (R, <), the natural numbers, integers, rational numbers and real numbers in their usual order.**

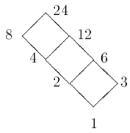


Fig. 1.1. A strict partial order.

Partial Order Structures

♣ **Partial orders (POs)**

- **We can think of a *partial order* as the reflexive closure of a strict partial order; that is, if R is a strict partial order on W, then $R \cup \{(u, u) \mid u \in W\}$ is a partial order.**
- **If a partial order is connected, then it is called a *reflexive linear order* (or a *reflexive total order*).**

♣ **Examples**

- **If we interpret the relation in Fig. 1.1 reflexively (that is, if we take Rxy to mean 'x and y are equal, or y can be divided by x'), then we have a simple example of a partial order, but it is not a reflexive linear order.**
- **Important examples of reflexive linear orders include (N, ≤) (or (ω, ≤)), (Z, ≤), (Q, ≤) and (R, ≤), the natural numbers, integers, rational numbers and real numbers under their respective 'less-than-or-equal-to' orderings.**

Labeled Transition Systems

♣ **Labeled Transition Systems (LTSs)**

- **A *Labeled Transition System (LTS)*, or more simply, *transition system*, is an ordered pair (W, $\{Ra \mid a \in A\}$) where W is a non-empty set of *states*, A is a non-empty set of *labels*, and for each label $a \in A$, *transition relation* $Ra \subseteq W \times W$.**

♣ **Example: A deterministic transition system**

- **In Fig. 1.2 a *deterministic transition system* with states $w1$, $w2$, $w3$, $w4$ and labels a, b, c is shown.**
- **Formally, $Ra = \{(w_1, w_2), (w_4, w_4)\}$, $Rb = \{(w_2, w_3)\}$, and $Rc = \{(w_4, w_3)\}$, and the transition relations are all partial functions from W to W.**

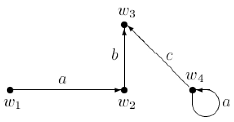


Fig. 1.2. A deterministic transition system.

Labeled Transition Systems

♣ **Example: A non-deterministic transition system**

- **A *non-deterministic transition system* is one in which the state we reach by making a particular kind of transition from a given state need not be fixed. That is, the transition relations do not have to be partial functions, but can be arbitrary relations.**

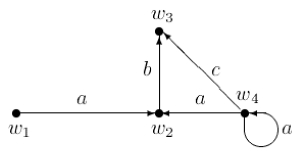


Fig. 1.3. A non-deterministic transition system.

Labeled Transition Systems

♣ **Transition systems as an abstract model of computation**

- **Transition systems can be viewed as an abstract model of computation: the states are the possible states of a computer, the labels stand for programs, and $(u, v) \in Ra$ means that there is an execution of the program a that starts in state u and terminates in state v.**
- **It is natural to depict states as nodes and transitions Ra as directed arrows.**

♣ **Example: A non-deterministic transition system**

- **In Fig. 1.3, label a is now a non-deterministic program, for if we execute it in state w_4 there are two possibilities: either we loop back into w_4, or we move to w_2.**

Trees

♣ **Trees**

- **A *tree* is a relational structure (T, S) where:**
  (1) T is the set of *nodes* and S is a binary relation (called *successor* or *daughter-of* relation) defined on T.
  (2) T contains a unique node $r \in T$ (called the *root*) such that for $\forall t \in T$, $(r, t) \in S^*$.
  (3) For every $t \neq r$ there is a unique $t' \in T$ such that $(t', t) \in S$. (t' is called the *S-predecessor* of t)
  (4) S is *acyclic*; that is, for $\forall t \in T$, $(t, t) \notin S^+$. (It follows that S is irreflexive.)

♣ **Homework**

- **Comparing the above definition of tree with those definitions of tree you learned in Discrete Mathematics / Data Structures, and showing the equivalence among them.**

Trees

♣Example: A tree representing phrase-structure

- The tree depicted in Fig. 1.5 represents some simple facts about phrase-structure, namely that a sentence (S) can consist of a noun phrase (NP) and a verb phrase (VP); an NP can consist of a proper noun (PN); and VPs can consist of a transitive verb (TV) and an NP.
- Fig. 1.5 contains enough information to give us a tree (T, S) in the sense just defined: the nodes in T are the displayed points, and the relation S is indicated by means of a straight line segment drawn from a node to a node immediately below. The root of the tree is the topmost node (the one labeled S).

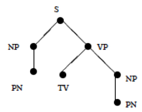


Fig. 1.5. A finite decorated tree.

Transitive Trees

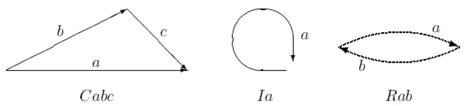
♣Transitive trees

- A *transitive tree* is an SPO (T, <) such that (1) there is a root $r \in T$ satisfying $r < t$ for all $t \in T$, and (2) for each $t \in T$, the set $\{s \in T \mid s < t\}$ of predecessors of t is finite and linearly ordered by <.
- (a) If (T, S) is a tree then (T, S^+) is a transitive tree. (b) (T, <) is a transitive tree IFF (T, $S_<$) is a tree, where $S_<$ is the immediate successor relation given by $(s, t) \in S_<$ IFF $s < t$ and $s < v < t$ for no $v \in T$.

Arrow Structures

♣Arrow structures

- An *arrow structure (frame)* is a quadruple $F = (W, C, R, I)$ such that C, R, and I are a ternary, a binary, and a unary relation on W, respectively, and pictorially as follows:

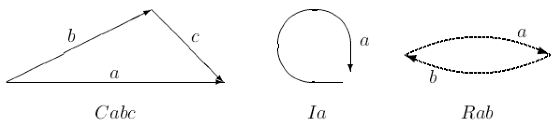


- The *objects* of an *arrow structure* are things that can be pictured as *arrows*.
- Note: Although arrows are the prime citizens of arrow structures, this does not mean that they should always be thought of as primitive entities. For example, in a two-dimensional arrow structure, an arrow a is thought of as an ordered pair (a_0, a_1) of which a_0 represents the starting point of a, and a_1 its endpoint.

Arrow Structures

♣Arrow structures: the composition relation

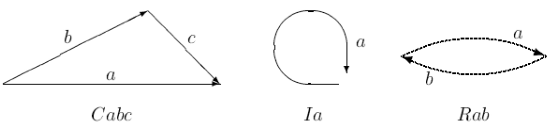
- The central relation on arrows is a ternary *composition* relation C, where *Cabc* says that arrow a is the outcome of composing arrow b with arrow c (or conversely, that a can be decomposed into b and c):
  Cabc IFF $a_0 = b_0$, $a_1 = c_1$, and $b_1 = c_0$



Arrow Structures

♣Arrow structures: the identity relation

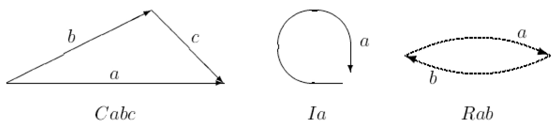
- Arrow structures contain degenerate arrows, transitions that do not lead to a different state.
- Formally, this means that arrow structures will contain a designated subset I of *identity* arrows; in the ordered-pair-representation:
  Ia IFF $a_0 = a_1$



Arrow Structures

♣Arrow structures: the reverse relation

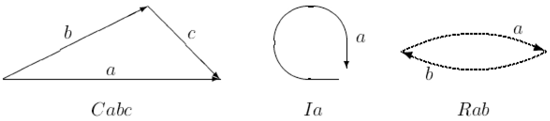
- Another relation on arrows a binary *reverse* relation R, in many cases this relation will be a partial function:
  Rab IFF $a_0 = b_1$, $a_1 = b_0$



Arrow Structures

♣ **Squares**

- The ***two-dimensional arrow structure***, in which the universe consists of all ordered pairs over the set U (and the relations C, R and I) is called the ***square*** over U, notated as S_U.
- The square arrow frame over U can be pictorially represented as a full graph over U: each arrow object (a_0, a_1) in S_U can be represented as a 'real' arrow from a_0 to a_1.



An Elementary Introduction to Modal Logics

- Modal Logic: What Is It and Why Study It ?
- Relational Structures
- **Modal Languages**
- Model (Semantic) Theory of Modal Logics
- Normal Modal Logics
- Temporal Logics
- Spatio-Temporal Logics
- Deontic Logics
- Epistemic Logics

The Basic Propositional Modal Language: Definition

♣ **The basic propositional modal language**

- The ***basic propositional modal language*** is defined using a set of countably infinite proposition letters (or proposition symbols or propositional variables) Φ whose elements are usually denoted p, q, r, and so on, usual classical logical connectives, and a unary ***modal operator*** ◇ ('***diamond***').
- The well-formed formulas A of the basic propositional modal language are defined as $A(B) =_{df} p \mid \bot \mid \neg A \mid A \vee B \mid \Diamond A$, where p ranges over elements of Φ.
- This definition means that a formula is either a proposition letter, the propositional constant falsum ('bottom'), a negated formula, a disjunction of formulas, or a formula prefixed by a diamond.
- A dual modal operator □ ('***box***') for the diamond can be defined as $\Box A =_{df} \neg\Diamond\neg A$ (Note: $\Diamond A \leftrightarrow \neg\Box\neg A$).

The Basic Propositional Modal Language: Examples

♣ **Examples: The first readings of diamond and box**

- In traditional modal logic, the diamond and box are read as follows.
- $\Diamond A$ can be read as 'it is ***possibly*** the case that A.' Under this reading, $\Box A$ means 'it is not possible that not A,' that is, '***necessarily*** A.'

♣ **Examples of formulas we would probably regard as correct principles**

- All instances of $\Box A \rightarrow \Diamond A$ ('whatever is necessary is possible')
- All instances of $A \rightarrow \Diamond A$ ('whatever is, is possible')

The Basic Propositional Modal Language: Examples

♣ **Important questions**

- Should $A \rightarrow \Box\Diamond A$ ('whatever is, is necessarily possible') be regarded as a general truth about necessity and possibility?
- Should $\Diamond A \rightarrow \Box\Diamond A$ ('whatever is possible, is necessarily possible') be regarded as a general truth about necessity and possibility?
- Are any of these formulas linked by a modal notion of logical consequence, or are they independent claims about necessity and possibility?
- The relational semantics of the basic propositional modal language offers a simple and intuitively compelling framework in which to discuss them.

The Basic Propositional Modal Language: Examples

♣ **Examples: The second readings of diamond and box**

- In epistemic logic, the basic propositional modal language is used to reason about knowledge.
- Instead of writing $\Box A$ for '**the agent *knows* that A**' it is usual to write KA.
- Given that we are talking about knowledge (as opposed to, say, belief or rumor), it seems natural to view all instances of $KA \rightarrow A$ as true: if the agent really knows that A, then A must hold. On the other hand (assuming that the agent is not omniscient) we would regard $A \rightarrow KA$ as false.

♣ **Important questions**

- If an agent knows that A, does he/she know that he/she knows it?

The Basic Propositional Modal Language: Examples

♣ Examples: The third readings of diamond and box

- In provability logic, the basic propositional modal language is used to reason about proofs.
- In provability logic, $\Box A$ is read as 'it is *provable* (in some arithmetical theory) that A.'
- A central theme in provability logic is the search for a complete axiomatization of the provability principles that are valid for various arithmetical theories (such as Peano Arithmetic).
- The *Lob formula* $\Box(\Box p \rightarrow p) \rightarrow \Box p$ plays a key role here.

Propositional Modal Languages: Modal Similarity Types

♣ Modal similarity types

- A *modal similarity type* is an ordered pair $\tau = (O, \rho)$ where where O is a non-empty set, and ρ is a function from O to N (the set of natural numbers).
- The elements of O are called *modal operators*; we use Δ ('*triangle*'), Δ_0, Δ_1, . . . to denote elements of O. The function ρ assigns to each operator $\Delta \in O$ a finite arity, indicating the number of arguments Δ can be applied to.
- The definition permits null-ary modalities (or *modal constants*), triangles that take no arguments at all.
- We often refer to unary triangles as diamonds, and denote them by $\Diamond_a$ or $<a>$, where a is taken from some index set.
- We often assume that the arity of operators is known, and do not distinguish between τ and O.

Propositional Modal Languages: Definition

♣ Propositional modal languages

- A *propositional modal language* $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ.
- The set $Form(\tau, \Phi)$ of propositional modal formulas over τ and Φ is defined as $A(B) =_{df} p \mid \bot \mid \neg A \mid A \vee B \mid \Delta(A_1, \ldots, A_{\rho(\Delta)})$, where p ranges over elements of Φ.
- The similarity type of the basic propositional modal language is called τ_0.

Propositional Modal Languages: Definition

♣ Dual operators for non-nullary triangles

- For each $\Delta \in O$ the dual ∇ of Δ is defined as $\nabla(A_1, \ldots, A_n) =_{df} \neg\Delta(\neg A_1, \ldots, \neg A_n)$.
- The dual of a triangle of arity at least 2 is called a *nabla*.
- As in the basic propositional modal language, the dual of a diamond is called a box, and is written $\Box_a$ or $[a]$.

♣ Examples of propositional modal languages

- The basic propositional temporal language
- The propositional dynamic logic
- An arrow language
- Feature logic and Description logic

The Basic Propositional Temporal Language

♣ The basic propositional temporal language

- The *basic propositional temporal language* is built using a set of unary operators $O = \{<F>, <P>\}$.
- The intended interpretation of a formula $<F>A$ is 'A will be true at some Future time,' and the intended interpretation of $<P>A$ is 'A was true at some Past time.'
- The basic propositional temporal language is the core language underlying a branch of modal logic called temporal logic.
- It is traditional to write <F> as F and <P> as P, and their duals are written as G and H, respectively. (The mnemonics here are: 'it is always Going to be the case' and 'it always Has been the case.') .

The Basic Propositional Temporal Language

♣ Representation examples

- $PA \rightarrow PGA$: This formula says 'whatever has happened will always have happened', and this seems a plausible candidate for a general truth about time.
- $FA \rightarrow FFA$: If we insist this formula 'must always be true', it shows that we are thinking of time as dense: between any two instants there is always a third.
- $GFp \rightarrow FGp$ (the *McKinsey formula*) : If we insist this formula is true, for all propositional symbols p, we are insisting that atomic information true somewhere in the future eventually settles down to being always true. (We might think of this as reflecting a 'thermodynamic' view of information distribution.)

The Propositional Dynamic Logic (PDL)

♣ The propositional dynamic logic

- The language of ***propositional dynamic logic***, has an infinite collection of diamonds. Each of these diamonds has the form $<\pi>$, where π denotes a (non-deterministic) program. The intended interpretation of $<\pi>A$ is ‘some terminating execution of π from the present state leads to a state bearing the information A.’
- The dual assertion $[\pi]A$ states that ‘every execution of π from the present state leads to a state bearing the information A.’

The Propositional Dynamic Logic (PDL)

♣ The expressive power of PDL

- PDL is highly expressive: we will make the inductive structure of the programs explicit in PDL’s syntax.
- Complex programs are built out of basic programs using some repertoire of program constructors.
- By using diamonds which reflect this structure, we obtain a powerful and flexible language.

The Propositional Dynamic Logic (PDL)

♣ The core language of PDL (*regular PDL*)

- Suppose we have fixed some set of basic (deterministic or non-deterministic) programs a, b, c, and so on (thus we have basic modalities $<a>$, $<b>$, $<c>$, … at our disposal). Then we are allowed to define complex programs π (and hence, modal operators $<\pi>$) over this base as follows:
  Choice: If π_1 and π_2 are programs, then so is $\pi_1 \cup \pi_2$. The program $\pi_1 \cup \pi_2$ (non-deterministically) executes π_1 or π_2.
  Composition: If π_1 and π_2 are programs, then so is $\pi_1; \pi_2$. This program first executes π_1 and then π_2.
  Iteration: If π is a program, then so is π^*. π^* is a program that executes π a finite (possibly zero) number of times.
- For the collection of diamonds this means that if $<\pi_1>$ and $<\pi_2>$ are modal operators, then so are $<\pi_1 \cup \pi_2>$, $<\pi_1; \pi_2>$ and $<\pi_1{}^*>$.

The Propositional Dynamic Logic (PDL)

♣ A straightforward example

- The formula $<\pi^*>A \leftrightarrow A \vee <\pi; \pi^*>A$ says that a state bearing the information A can be reached by executing π a finite number of times if and only if either we already have the information A in the current state, or we can execute π once and then find a state bearing the information A after finitely many more iterations of π.

♣ Another example (*Segerberg’s axiom* or the *induction axiom*)

- $[\pi^*](A \rightarrow [\pi]A) \rightarrow (A \rightarrow [\pi^*]A)$
- Try working out what exactly it is that this formula says.

The Propositional Dynamic Logic (PDL)

♣ Other constructors of PDL

- ***Intersection***: If π_1 and π_2 are programs, then so is $\pi_1 \cap \pi_2$. The intended meaning of $\pi_1 \cap \pi_2$ is: execute both π_1 and π_2, in parallel.
- The intended reading of $<\pi_1 \cap \pi_2>A$ is that if we execute both π_1 and π_2 in the present state, then there is at least one state reachable by both programs which bears the information A.
- ***Test***: If A is a formula, then $A?$ is a program. This program tests whether A holds, and if so, continues; if not, it fails.
- The key point to note about the test constructor is its unusual syntax: it allows us to make a modality out of a formula. Intuitively, this modality accesses the current state if the current state satisfies A.
- It can be used to build interesting programs. For example, (p?; a)∪(¬p?; b) is ‘if p then a else b.’

An Arrow Language

♣ An arrow language

- A similarity type with modal operators other than diamonds, is the type $\tau_{\rightarrow}$ of arrow logic.
- The language of arrow logic is designed to talk about the objects in arrow structures (entities which can be pictured as arrows).
- The well-formed formulas ϕ of the arrow language are defined as $A(B) =_{df} p \mid \bot \mid \neg A \mid A \vee B \mid A \bullet B \mid \otimes A \mid 1'$, where p ranges over elements of Φ.
- 1’ (‘identity’) is a null-ary modality (a modal constant), the ‘converse’ operator ⊗ is a diamond, and the ‘composition’ operator • is a dyadic operator.
- Possible readings of these operators are: 1’ identity ‘skip’, $\otimes A$ converse ‘A conversely’, $A \bullet B$ composition ‘first A, then B’.

Substitution

♣ Substitution

- Suppose we are working a modal similarity type τ and a set Φ of proposition letters. A substitution is a map $\sigma: \Phi \to \mathrm{Form}(\tau, \Phi)$.
- Now such a substitution σ induces a map $(\bullet)^\sigma : \mathrm{Form}(\tau, \Phi) \to \mathrm{Form}(\tau, \Phi)$ which we can recursively define as follows:

$$\begin{aligned} \bot^\sigma &= \bot \\ p^\sigma &= \sigma(p) \\ (\neg\psi)^\sigma &= \neg\psi^\sigma \\ (\psi \vee \theta)^\sigma &= \psi^\sigma \vee \theta^\sigma \\ (\Delta(\psi_1, \ldots, \psi_n))^\sigma &= \Delta(\psi_1^\sigma, \ldots, \psi_n^\sigma). \end{aligned}$$

- This definition spells out exactly what is meant by carrying out uniform substitution.
- Finally, we say that χ is a substitution instance of B if there is some substitution τ such that $B^\tau = \chi$.

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Frames (Kripke-frames): Definition

♣ Frames (Kripke-frames)

- A *frame (Kripke-frame)* for the basic modal language is an ordered pair $F = (W, R)$ where W is a non-empty set of objects, and R is a binary relation defined on W, $R \subseteq W \times W$. W is often called '*world*' and R is often called a relation of '*alternativeness*' or '*accessibility*'.
- That is, a frame for the basic modal language is simply a relational structure bearing a single binary relation.
- The semantics of the basic modal language based on frames is usually called *Kripke-style relational semantics*.

♣ Note

- We may refer to the elements of W by many different names.

Models: Definition

♣ Models

- A *model* for the basic modal language is an ordered pair $M = (F, V)$ ($M = (W, R, V)$) where F is a frame for the basic modal language, and V is function (called a *valuation*) from Φ to the power set of W that assigns to each propositional variable p in Φ a subset $V(p)$ of W, $V: \Phi \to P(W)$.
- Given a model $M = (F, V)$, we say that M is *based on* the frame F, or that F is the frame *underlying* M.

♣ Notes

- Informally we think of $V(p)$ as the set of points in the model where p is TRUE.
- Models for the basic modal language can be viewed as relational structures in a natural way, namely as structures of the form: $(W, R, V(p), V(q), V(r), \ldots)$.
- That is, a model is a relational structure consisting of a domain, a single binary relation R, and the unary relations given to us by V.

Frames and Models

♣ Frames and models

- Thus, viewed from a purely structural perspective, a frame F and a model M based on F, are simply two relational models based on the same universe; indeed, a model is simply a frame enriched by a collection of unary relations.
- But in spite of their mathematical kinship, frames and models are used very differently.

♣ Frames as mathematical pictures of ontologies

- Frames are essentially mathematical pictures of ontologies that we find interesting. We use the level of frames to make our fundamental assumptions mathematically precise.
- For example, we may view time as a collection of points ordered by a strict partial order, or feel that a correct analysis of knowledge requires that we postulate the existence of situations linked by a relation of 'being an epistemic alternative to.'

Frames and Models

♣ Models dressing frames with contingent information

- The unary relations provided by valuations are there to dress our frames with contingent information.
- Is it raining on Tuesday or not? Is the system write-enabled at time t? Is a situation where Janet does not love him an epistemic alternative for John?
- Such information is important, and we certainly need to be able to work with it — nonetheless, statements only deserve the description 'logical' if they are invariant under changes of contingent information.

♣ To define a modally reasonable notion of validity

- Because we have drawn a distinction between the fundamental information given by frames, and the additional descriptive content provided by models, it will be straightforward to define a modally reasonable notion of validity.

Satisfaction: Definitions

♣ Satisfaction of a formula

- Suppose w is a state in a model $M = (W, R, V)$. Then we inductively define the notion of a formula A being *satisfied* (or *true*) in M at state w as follows:
  $M, w \models p$ IFF $w \in V(p)$ where $p \in \Phi$,
  $M, w \models \perp$ never,
  $M, w \models \neg A$ IFF not $M, w \models A$ (i.e., $M, w \not\models A$)
  $M, w \models A \vee B$ IFF $M, w \models A$ or $M, w \models B$,
  $M, w \models \Diamond A$ IFF for some $v \in W$ with Rwv, $M, v \models A$.
  It follows from this definition that
  $M, w \models \Box A$ IFF for all $v \in W$ such that Rwv, $M, v \models A$.

♣ Satisfaction of a set of formulas

- A set Γ of formulas is true at a state w of a model M, notated as $M, w \models \Gamma$, if all members of Γ are true at w.

Satisfaction: Definitions

♣ Satisfaction of a formula

- If M does not satisfy A at w we often write $M, w \not\models A$, and say that A is *false* or *refuted* at w.
- A formula A is *satisfiable* in a model M if there is some state in M at which A is true; a formula is *falsifiable* or *refutable* in a model if its negation is satisfiable.
- When M is clear from the context, we write $w \models A$ for $M, w \models A$ and $w \not\models A$ for $M, w \not\models A$.
- It is convenient to extend the valuation V from proposition letters to arbitrary formulas so that $V(A)$ always denotes the set of states at which A is true: $V(A) =_{df} \{ w \mid M, w \models A \}$.

Satisfaction: Definitions

♣ Notes

- The notion of satisfaction is intrinsically *internal* and *local*.
- We evaluate formulas inside models, at some particular state w (the *current state*).
- Moreover, $\Diamond$ works locally: the final clause in the definition "$M, w \models \Diamond A$ IFF for some $v \in W$ with Rwv, $M, v \models A$" treats $\Diamond A$ as an instruction to scan states in search of one where A is satisfied.
- Crucially, only states *R-accessible* from the current one can be scanned by our operators.
- Much of the characteristic flavor of modal logic springs from the perspective on relational structures embodied in the satisfaction definition.

Satisfaction: Possible Worlds

♣ Possible worlds

- The use of the word '*world*' (or '*possible world*') for the entities in W derives from the reading of the basic modal language in which $\Diamond A$ is taken to mean 'possibly A,' and $\Box A$ to mean 'necessarily A.'

♣ Leibniz's view about necessity and possibility

- Given this reading, the machinery of frames, models, and satisfaction which we have defined is essentially an attempt to capture mathematically the view (often attributed to Leibniz) that necessity means truth in all possible worlds, and that possibility means truth in some possible world.

Satisfaction: Possible Worlds

♣ Notes

- The satisfaction definition stipulates that $\Diamond$ and $\Box$ check for truth not at all possible worlds (that is, at all elements of W) but only at R-accessible possible worlds.
- At first sight this may seem a weakness of the satisfaction definition — but in fact, it is its greatest source of strength.
- The point is this: varying R is a mechanism which gives us a firm mathematical grip on the pre-theoretical notion of access between possible worlds.
- For example, by stipulating that $R = W \times W$ we can allow all worlds access to each other; this corresponds to the Leibnizian idea in its purest form.

Satisfaction: Possible Worlds

♣ Notes

- Going to the other extreme, we might stipulate that no world has access to any other.
- Between these extremes there is a wide range of options to explore.
- Should inter-world access be reflexive?
  Should it be transitive?
  What impact do these choices have on the notions of necessity and possibility?
  For example, if we demand symmetry, does this justify certain principles, or rule others out?

Universal Satisfaction: Definitions

♣ **Universal satisfaction of a formula**

- A formula A is ***globally/universally true*** in a model M, notated as $M \models A$) if it is satisfied at all points in M (that is, if $M, w \models A$, for all $w \in W$).

♣ **Universal satisfaction of a set of formulas**

- A set Γ of formulas is globally true (satisfiable, respectively) at a model M, notated as $M \models \Gamma$, if $M, w \models \Gamma$ for all states w in M (some state w in M, respectively).

Satisfaction: Examples

♣ **A linear order**

- Consider the frame $F = (\{w_1, w_2, w_3, w_4, w_5\}, R)$, where Rw_iw_j IFF $j = i + 1$: $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_4 \rightarrow w_5$.
- If we choose a valuation V on F such that $V(p) = \{w_2, w_3\}$, $V(q) = \{w_1, w_2, w_3, w_4, w_5\}$, and $V(r) = \varnothing$, then in the model $M = (F, V)$ we have that $M, w_1 \models \Diamond\Box p$, $M, w_1 \not\models \Diamond\Box p \rightarrow p$, $M, w_2 \models \Diamond(p \wedge \neg r)$, and $M, w_1 \models q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$.
- Furthermore, $M \models \Box q$.
- Now, it is clear that $\Box q$ is true at w_1, w_2, w_3, w_4, but why is it true at w_5? Well, as w_5 has no successors at all (we often call such points '***dead ends***' or '***blind states***') it is vacuously true that q is true at all R-successors of w_5. Indeed, any 'boxed' formula $\Box q$ is true at any dead end in any model.

Satisfaction: Examples

♣ **A SPO**

- Let F be the SPO given in Figure 1.1, where $W = (1, 2, 3, 4, 6, 8, 12, 24)$ and Rxy means 'x and y are different, and y can be divided by x.' R is a strict partial order but not a linear order.
- Choose a valuation V on this frame such that $V(p) = \{4, 8, 12, 24\}$, and $V(q) = \{6\}$, and let $M = (F,$
- Then
  $M, 4 \models \Box p$,
  $M, 6 \models \Box p$,
  $M, 2 \not\models \Box p$, and
  $M, 2 \models \Diamond(q \wedge \Box p) \wedge \Diamond(\neg q \wedge \Box p)$.

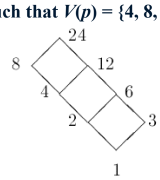


Fig. 1.1. A strict partial order.

Satisfaction: Examples

♣ **n occurrences of modal operators**

- Whereas a diamond $\Diamond$ corresponds to making a single R-step in a model, stacking diamonds one in front of the other corresponds to making a sequence of R-steps through the model.
- The following defined operators will sometimes be useful: we write $\Diamond^n A$ for A preceded by n occurrences of $\Diamond$, and $\Box^n A$ for A preceded by n occurrences of A.
- If we like, we can associate each of these defined operators with its own accessibility relation.
- We do so inductively: R^0xy is defined to hold if $x = y$, and $R^{n+1}xy$ is defined to hold if $(\exists z)(Rxz \wedge R^n zy)$.
- Then, for any model M and state w in M, $M, w \models \Diamond^n A$ IFF there exists a v such that $R^n wv$ and $M, v \models A$.

Satisfaction: Examples

♣ **Epistemic logic**

- In epistemic logic $\Box$ is written as K and KA is interpreted as 'the agent knows that A.'
- Under this interpretation, the intuitive reading for the semantic clause governing K is: the agent knows A in a situation w (that is, $w \models \mathrm{K}A$) IFF ϕ is true in all situations v that are compatible with her knowledge (that is, if $v \models A$ for all v such that Rwv).
- Thus, under this interpretation, W is to be thought of as a collection of situations, R is a relation which models the idea of one situation being epistemically accessible from another, and V governs the distribution of primitive information across situations.

τ-Frame and τ-Modal for Modal Languages of Arbitrary Similarity Type

♣ **τ-frame**

- Let τ be a modal similarity type. A ***τ-frame*** is a tuple F consisting of the following ingredients:
  (i) a non-empty set W,
  (ii) for each $n \geq 0$, and each n-ary modal operator Δ in the similarity type τ, an $(n+1)$-ary relation R_Δ.
- If τ contains just a finite number of modal operators $\Delta_1, \ldots, \Delta_n$, we write $F = (W, R_{\Delta 1}, \ldots, R_{\Delta n})$; otherwise we write $F = (W, R_\Delta)_{\Delta \in \tau}$ or $F = (W, \{R_\Delta \mid \Delta \in \tau\})$.

♣ **τ-model**

- A ***τ-model*** is an ordered pair $M = (F, V)$ where F is a τ-frame, and V is a valuation from Φ to the power set of W, where W is the universe of F.

Satisfaction for Modal Languages of Arbitrary Similarity Type

♣ Satisfaction

- The notion of a formula ϕ being satisfied (or true) at a state w in a model $M = ((W, \{R_\Delta \mid \Delta \in \tau\}), V)$, notated as $M, w \models \phi$, is defined inductively.
- The clauses for the atomic and Boolean cases are the same as for the basic modal language.
- As for the modal case, when $\rho(\Delta) > 0$ we define $M, w \models \Delta(A_1, \ldots, A_n)$ IFF for some $v_1, \ldots, v_n \in W$ with $R_\Delta w v_1 \ldots v_n$, we have, for each i, $M, v_i \models A_i$.
- When $\rho(\Delta) = 0$ (that is, when Δ is a null-ary modality) then R_Δ is a unary relation and we define $M, w \models \Delta$ IFF $w \in R_\Delta$.
- That is, unlike other modalities, null-ary modalities do not access other states.

Satisfaction for Modal Languages of Arbitrary Similarity Type

♣ Notes

- In fact, the semantics of null-ary modalities is identical to that of the propositional variables, save that the unary relations used to interpret them are not given by the valuation — rather, they are part of the underlying frame.
- As before, we often write $w \models A$ for $M, w \models A$ where M is clear from the context.
- The concept of global truth (or universal truth) in a model is defined as for the basic modal language: it simply means truth at all states in the model.
- And, as before, we sometimes extend the valuation V supplied by M to arbitrary formulas.

Satisfaction: Examples of Arbitrary Similarity Type

♣ A deterministic transition system

- Let τ be a similarity type with three unary operators $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$. Then a τ-frame has three binary relations Ra, Rb, and Rc (i.e., it is a LTS with three labels, as an example, W, Ra, Rb and Rc are shown in Fig. 1.2).
- Consider the formula $\langle a\rangle p \to \langle b\rangle p$. Informally, this formula is true at a state, if it has an Ra-successor satisfying p only if it has an Rb-successor satisfying p.
- Let V be a valuation with $V(p) = \{w_2\}$. Then the model $M = (W, Ra, Rb, Rc, V)$ has $M, w_1 \not\models \langle a\rangle p \to \langle b\rangle p$.

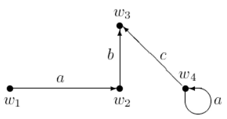


Fig. 1.2. A deterministic transition system.

Satisfaction: Examples of Arbitrary Similarity Type

♣ A frame containing a ternary relation and a 4-ary relation

- Let τ be a similarity type with a binary modal operator $\triangle$ and a ternary operator $\bigcirc$. Frames for this τ contain a ternary relation $R_\triangle$ and a 4-ary relation S.
- As an example, let $W = \{u, v, w, s\}$, $R_\triangle = \{(u, v, w)\}$, and $S = \{(u, v, w, s)\}$ as in Fig. 1.6, and consider a valuation V on this frame with $V(p_0) = \{v\}$, $V(p_1) = \{w\}$, and $V(p_2) = \{s\}$.

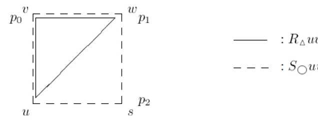


Fig. 1.6. A simple frame

Satisfaction: Examples of Arbitrary Similarity Type

♣ A frame containing a ternary relation and a 4-ary relation

- Now, let ϕ be the formula $\triangle(p_0, p_1) \to \bigcirc(p_0, p_1, p_2)$.
- An informal reading of A is 'any triangle of which the evaluation point is a vertex, and which has p_0 and p_1 true at the other two vertices, can be expanded to a rectangle with a fourth point at which p_2 is true.'
- We can verify that A is true at u, and indeed at all other points, and hence that it is globally true in the model.

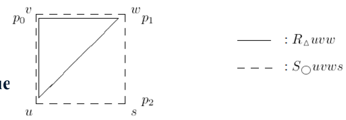


Fig. 1.6. A simple frame

Bidirectional Frames and Models

♣ The into-the-future and into-the-past relations

- Consider the basic temporal language has two unary operators F and P.
- Models for this language consist of a set bearing two binary relations, R_F *(the into-the-future relation)* and R_P *(the into-the-past relation)*, which are used to interpret F and P respectively.
- However, given the intended reading of the operators, most such models are inappropriate: clearly we ought to insist on working with models based on frames in which R_P is the *converse* of R_F (that is, frames in which $(\forall x)(\forall y)(R_F xy \leftrightarrow R_P yx)$).

Bidirectional Frames and Models

- **Bidirectional frames and models**
  - Let us denote the converse of a relation R by $R^{\smile}$.
  - We will call a frame of the form $(T, R, R^{\smile})$ a *bidirectional frame*, and a model built over such a frame a *bidirectional model*.
  - From now on, we will only interpret the basic temporal language in bidirectional models.
  - That is, if M = $(T, R, R^{\smile}, V)$ is a bidirectional model then:
    $M, t \models \mathrm{F}A$ IFF $(\exists s)(Rts \wedge M, s \models A)$,
    $M, t \models \mathrm{P}A$ IFF $(\exists s)(R^{\smile}ts \wedge M, s \models A)$.

Bidirectional Frames and Models

- **Bidirectional frames and models**
  - Once we have made this restriction, we do not need to mention $R^{\smile}$ explicitly any more: once R has been fixed, its converse is fixed too.
  - That is, we are free to interpret the basic temporal languages on frames (T, R) for the basic modal language using the clauses:
    $M, t \models \mathrm{F}A$ IFF $(\exists s)(Rts \wedge M, s \models A)$,
    $M, t \models \mathrm{P}A$ IFF $(\exists s)(Rst \wedge M, s \models A)$.
  - These clauses clearly capture a crucial part of the intended semantics: F looks forward along R, and P looks backwards along R.

Dense and Unbounded Bidirectional Frames

- **Dense bidirectional frames**
  - A bidirectional frame (T, R) is *dense* if there is a point between any two related points, i.e., $(\forall x)(\forall y)(Rxy \rightarrow (\exists z)(Rxz \wedge Rzy))$.
- **Unbounded bidirectional frames**
  - A bidirectional frame (T, R) is *right-unbounded* if every point has a successor, i.e., $(\forall x)(\exists y)(Rxy)$.
  - A bidirectional frame (T, R) is *left-unbounded* if every point has a predecessor, i.e., $(\exists x)(\forall y)(Rxy)$.
  - A bidirectional frame (T, R) is *unbounded* if it is both right and left unbounded.

Trichotomous Bidirectional Frames

- **Trichotomous bidirectional frames**
  - A bidirectional frame (T, R) is *trichotomous* if any two points are equal or are related one way or the other, i.e., $(\forall x)(\forall y)(Rxy \vee x = y \vee Ryx)$, and *weak total order* (or *weakly linear*) if it is both transitive and trichotomous.
- **DUWTO-frames**
  - We call a bidirectional frame with all these properties a *DUWTO*-frame.

Regular Frames and Models

- **Regular frames and models**
  - The language of PDL has an infinite collection of diamonds, each indexed by a program π built from basic programs using the constructors $\cup$, ;, and *. A model for this language has the form $(W, \{R_\pi \mid \pi \text{ is a program}\}, V)$. That is, a model is a labeled transition system together with a valuation.
  - Given our readings of $\cup$, ;, and *, as choice, composition, and iteration, it is clear that we are only interested in relations constructed using the following inductive clauses:
    $R_{\pi1\cup\pi2} = R_{\pi1} \cup R_{\pi2}$,
    $R_{\pi1;\pi2} = R_{\pi1} \bullet R_{\pi2}$ $(= \{(x, y) \mid (\exists z)(R_{\pi1}xz \wedge R_{\pi2}zy)\})$,
    $R_{\pi1^*} = (R_{\pi1})^*$, the reflexive transitive closure of $R_{\pi1}$.
  - These inductive clauses completely determine how each modality should be interpreted. Once the interpretation of the basic programs has been fixed, the relation corresponding to each complex program is fixed too.

Regular Frames and Models

- **Regular frames and models**
  - Suppose we have fixed a set of basic programs.
  - Let Π be the smallest set of programs containing the basic programs and all programs constructed over them using the regular constructors $\cup$, ;, and *.
  - Then a *regular frame* for Π is a labeled transition system $(W, \{R_\pi \mid \pi \in \Pi\})$ such that R_a is an arbitrary binary relation for each basic program a, and for all complex programs π, R_π is the binary relation inductively constructed in accordance with the previous clauses.
  - A *regular model* for Π is a model built over a regular frame; that is, a regular model is regular frame together with a valuation.

Regular Frames and Models

Regular frames and models

- When working with the language of PDL over the programs in Π, we will only be interested in regular models for Π, for these are the models that capture the intended interpretation.
- The intended reading of ∩ demands that $R_{\pi 1 \cap \pi 2} = R_{\pi 1} \cap R_{\pi 2}$.
- As for ?, it is clear that we want the following definition: $R_A? = \{(x, y) \mid x = y \text{ and } y \models A\}$.
- This is indeed the clause we want, but note that it is rather different from the others: it is not a frame condition. Rather, in order to determine the relation $R_A?$, we need information about the truth of the formula A, and this can only be provided at the level of models.

Arrow Models and Square Models

Arrow models

- An *arrow model* is a structure $M = (F, V)$ such that $F = (W, C, R, I)$ is an arrow frame and V is a valuation. Then:
  $M, a \models 1'$ IFF Ia,
  $M, a \models \otimes A$ IFF $M, b \models A$ for some b with Rab,
  $M, a \models A \bullet B$ IFF $M, b \models A$ and $M, c \models B$ for some b and c with $Cabc$.

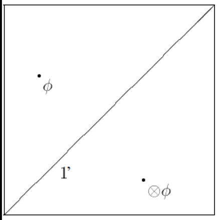
Square models

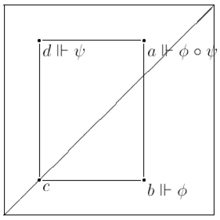
- When F is a *square frame* S_U, V now maps propositional variables to sets of ordered pairs over U; that is, to binary relations. The truth definition can be rephrased as follows:
  $M, (a_0, a_1) \models 1'$ IFF $a_0 = a_1$,
  $M, (a_0, a_1) \models \otimes A$ IFF $M, (a_1, a_0) \models A$,
  $M, (a_0, a_1) \models A \bullet B$ IFF $M, (a_0, u) \models A$ and $M, (u, a_1) \models B$ for some $u \in U$.

Square Models

Square models

- One could draw a square model two-dimensionally.
- It will be obvious that the modal constant 1' holds precisely at the diagonal points and that $\otimes A$ is true at a point IFF A holds at its mirror image with respect to the diagonal.

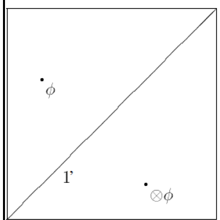


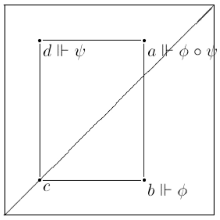


Square Models

Square models

- The formula $A \bullet B$ holds at a point a iff we can draw a rectangle abcd such that: b lies on the vertical line through a, d lies on the vertical line through a; and c lies on the diagonal.





Frames and Validity

Frames and validity

- So far we have been viewing modal languages as tools for talking about models. But models are composite entities consisting of a frame (our underlying ontology) and contingent information (the valuation).
- We often want to ignore the effects of the valuation and get a grip on the more fundamental level of frames. The concept of validity lets us do this.
- A formula is *valid* on a frame if it is true at every state in every model that can be built over the frame.
- In effect, this concept interprets modal formulas on frames by abstracting away from the effects of particular valuations.

Validity of a Formula: Definitions

Validity at a state in a frame

- A formula A is *valid* at a state w in a frame F, notated as $F, w \models A$, if A is true at w in every model (F, V) based on F.

Validity in a frame

- A formula A is valid in a frame F, notated as $F \models A$, if it is valid at every state in F.

Validity on a class of frames

- A formula A is valid on a class of frames FF, notated as $FF \models A$, if it is valid in every frame F in FF.
- The set of all formulas that are valid in a class of frames FF is called *the logic of FF*, notated as L_{FF}.

Validity

- A formula A is valid, notated as $\models A$, if it is valid on the class of all frames, i.e., it is valid in any frame.

Validity of a Set of Formulas: Definitions

♣ **Validity at a state in a frame**

- A set Γ of formulas is *valid* at a state w in a frame F, notated as $F, w \models \Gamma$, if for every $A \in \Gamma$, $F, w \models A$.

♣ **Validity in a frame**

- A set Γ of formulas is valid in a frame F, notated as $F \models \Gamma$, if it is valid at every state in F, i.e., for every $A \in \Gamma$, $F \models A$.

♣ **Validity on a class of frames**

- A set Γ of formulas is valid on a class of frames FF, notated as $FF \models \Gamma$, if it is valid in every frame F in FF, i.e., for every $A \in \Gamma$, $FF \models A$.

♣ **Validity**

- A set Γ of formulas is valid, notated as $\models \Gamma$, if it is valid on the class of all frames, i.e., it is valid in any frame.

Validity of a Formula: Examples

♣ **The validity of formula '$\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$'**

- The formula $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ is valid on all frames.
- To see this, take any frame F and state w in F, and let V be a valuation on F.
- We have to show that "if $(F, V), w \models \Diamond(p \vee q)$, then $(F, V), w \models \Diamond p \vee \Diamond q$." (Note: if … then … !)
- So assume that $(F, V), w \models \Diamond(p \vee q)$.
- Then, by definition there is a state v such that Rwv and $(F, V), v \models p \vee q$.
- But, if $v \models p \vee q$ then either $v \models p$ or $v \models q$.
- Hence either $w \models \Diamond p$ or $w \models \Diamond q$.
- Either way, $w \models \Diamond p \vee \Diamond q$.

Validity of a Formula: Examples

♣ **The validity of formula '$\Diamond\Diamond p \to \Diamond p$'**

- The formula $\Diamond\Diamond p \to \Diamond p$ is NOT valid on all frames.
- To see this, we need to find a frame F, a state w in F, and a valuation on F that falsifies the formula at w.
- So let F be a three-point frame with universe {0, 1, 2} and relation {(0, 1), (1, 2)}.
- Let V be any valuation on F such that $V(p) = \{2\}$.
- Then $(F, V), 0 \models \Diamond\Diamond p$, but $(F, V), 0 \not\models \Diamond p$ since 0 is not related to 2.

Validity of a Formula: Examples

♣ **The validity of formula '$\Diamond\Diamond p \to \Diamond p$'**

- But, there is a class of frames on which $\Diamond\Diamond p \to \Diamond p$ is valid: the class of transitive frames.
- To see this, take any transitive frame F and state w in F, and let V be a valuation on F.
- We have to show that if $(F, V), w \models \Diamond\Diamond p$, then $(F, V), w \models \Diamond p$. So assume that $(F, V), w \models \Diamond\Diamond p$.
- Then by definition there are states u and v such that Rwu and Ruv and $(F, V), v \models p$.
- But as R is transitive, it follows that Rwv, hence $(F, V), w \models \Diamond p$.
- As this example suggests, when additional constraints are imposed on frames, more formulas may become valid.

Validity of a Formula: Examples

♣ **Example: A deterministic transition system**

- Consider the frame depicted in Fig. 1.2. On this frame the formula $\langle a\rangle p \to \langle b\rangle p$ is not valid; a counter-model is obtained by putting $V(p) = \{w_2\}$.
- Now consider a frame satisfying the condition $R_a \subseteq R_b$; an example is depicted in Fig. 1.7. On this frame it is impossible to refute the formula $\langle a\rangle p \to \langle b\rangle p$ at w, because a refutation would require the existence of a point u with $R_a wu$ and p true at u, but not $R_b wu$; but such points are forbidden when we insist that $R_a \subseteq R_b$.

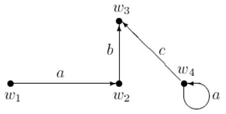


Fig. 1.2. A deterministic transition system.

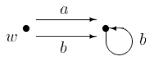


Fig. 1.7. A frame satisfying $R_a \subseteq R_b$.

Validity of a Formula: A General Point

♣ **The formula $\langle a\rangle p \to \langle b\rangle p$ defines the property that $R_a \subseteq R_b$**

- In fact, the previous example is a completely general point.
- In every frame F of the appropriate similarity type, if F satisfies the condition $R_a \subseteq R_b$, then $\langle a\rangle p \to \langle b\rangle p$ is valid in F.
- More over, the converse to this statement also holds: whenever $\langle a\rangle p \to \langle b\rangle p$ is valid on a given frame F, then the frame must satisfy the condition $R_a \subseteq R_b$.
- Therefore, the formula $\langle a\rangle p \to \langle b\rangle p$ defines the property that $R_a \subseteq R_b$.

Validity of a Formula: Examples of the Basic Temporal Language

♣ Example

- When interpreting the basic temporal language (Bidirectional frames and models), we observed that arbitrary frames of the form (W, R_P, R_F) were uninteresting given the intended interpretation of F and P, and we insisted on interpreting them using a relation R and its converse.
- Interestingly, there is a sense in which the basic temporal language itself is strong enough to enforce the condition that the relation R_P is the converse of the relation R_F: such frames are precisely the ones which validate both the formulas $p \rightarrow \mathrm{GP}p$ and $p \rightarrow \mathrm{HF}p$.

Validity of a Formula: Examples of the Basic Temporal Language

♣ Example

- The formula $\mathrm{F}q \rightarrow \mathrm{FF}q$ is not valid on all frames.
- To see this we need to find a frame $TT = (T, R)$, a state t in TT, and a valuation on TT that falsifies this formula at t.
- So let $T = \{0, 1\}$, and let R be the relation $\{(0, 1)\}$.
- Let V be a valuation such that $V(p) = (1)$.
- Then $(TT, V), 0 \models \mathrm{F}p$, but obviously $(TT, V), 0 \not\models \mathrm{FF}p$.

Validity of a Formula: Examples of the Basic Temporal Language

♣ Example

- But there is a frame on which $\mathrm{F}p \rightarrow \mathrm{FF}p$ is valid.
- As the universe of the frame take the set of all rational numbers Q, and let the frame relation be the usual $<$-ordering on Q.
- To show that $\mathrm{F}p \rightarrow \mathrm{FF}p$ is valid on this frame, take any point t in it, and any valuation V such that $(Q, <, V), t \models \mathrm{F}p$; we have to show that $t \models \mathrm{FF}p$.
- But this is easy: as $t \models \mathrm{F}p$, there exists a t' such that $t < t'$ and $t' \models p$.
- Because we are working on the rationals, there must be an s with $t < s$ and $s < t'$ (for example, $(t + t')/2$).
- As $s \models \mathrm{F}p$, it follows that $t \models \mathrm{FF}p$.

General Frames: Why?

♣ Why general frames ?

- At the level of models the fundamental concept is satisfaction that is a relatively simple concept involving only a frame and a single valuation.
- By ascending to the level of frames we get a deeper grip on relational structures, but there is a price to pay, i.e., validity lacks the concrete character of satisfaction, for it is defined in terms of all valuations on a frame.
- The level of general frames is an intermediate level between models and frames.

♣ General frames

- A general frame (F, A) is a frame F together with a restricted, but suitably well-behaved collection A of admissible valuations.

General Frames: Why?

♣ The most important reason to work with general frames

- General frames support a notion of validity that is mathematically simpler than the frame-based one, without losing too many of the concrete properties that make models so easy to work with.
- This ‘simpler behavior’ will only really become apparent when we discuss the algebraic perspective on completeness theory.
- It will turn out that there is a fundamental and universal completeness result for general frame validity, something that the frame semantics lacks.
- Moreover, we will discover that general frames are essentially a set-theoretic representation of boolean algebras with operators.

General Frames: Definitions

♣ General frames

- Let τ be a modal similarity type, and $F = (W, R_\Delta)_{\Delta\in\tau}$ a τ-frame. For $\Delta \in \tau$ we define the following function m_Δ on the power set of W:
  $m_\Delta(X_1, \ldots, X_n) = \{w \in W \mid$ there are $w_1, \ldots, w_n \in W$ such that $R_\Delta w w_1 \ldots w_n$ and $w_i \in X_i$; for all $i = 1, \ldots, n.\}$
- Let τ be a modal similarity type. A *general τ-frame* is an ordered pair (F, S) where $F = (W, R_\Delta)_{\Delta\in\tau}$ is a τ-frame, and S is a non-empty collection of subsets of W closed under the following operations:
  (i) union: if $X, Y \in S$ then $X \cup Y \in S$.
  (ii) relative complement: if $X \in A$, then $(W - X) \in S$.
  (iii) modal operations: if $X_1, \ldots, X_n \in S$, then $m_\Delta(X_1, \ldots, X_n) \in A$ for all $\Delta \in \tau$.

General Frames: Definitions

♣ Models based on general frames

- A *model based on a general frame* is a triple (F, S, V) where (F, S) is a general frame and V is a valuation satisfying the constraint that for each proposition letter p, $V(p)$ is an element of A.
- Valuations satisfying this constraint are called *admissible* for (F, S).

General Frames: Definitions

♣ Notes

- It follows immediately from the first two clauses of the definition of general frame that both the empty set and the universe of a general frame are always admissible.
- An ordinary frame $F = (W, R_\Delta)_{\Delta\in\tau}$ can be regarded as a general frame where $S = P(W)$ (that is, a general frame in which all valuations are admissible).
- If a valuation V is admissible for a general frame (F, S), then the closure conditions listed in the definition of general frame guarantee that $V(A)\in S$, for all formulas A. In short, a set of admissible valuations S is a 'logically closed' collection of information assignments.

Validity of a Formula: Definitions

♣ Validity at a state in a general frame

- A formula A is valid at a state w in a general frame (F, S), notated as (F, S), w $\models A$, if A is true at w in every admissible model (F, S, V) based on (F, S).

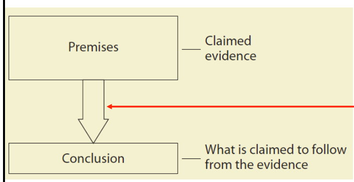
♣ Validity in a general frame

- A formula A is valid in a general frame (F, S), notated as (F, S) $\models A$, if A is true at every state in every admissible model (F, S, V) based on (F, S).

♣ Validity on a class of general frame

- A formula A is valid on a class of general frames G, notated as G $\models A$, if it is valid on every general frame (F, S) in G.
- If A is valid on the class of all general frames we say that it is *g-valid* and write $\models_g A$.

Logic: What Is It?



[From P. J. Hurley, "A Concise Introduction to Logic"]

What **entails** what?

What **follows from** what?

Why? What are the evaluation **criteria**?

How to **establish/define** the evaluation criteria?

How to evaluate **arguments/reasoning**?

It is **LOGIC** to answer these fundamental questions.

Local Semantic Consequence Relation

♣ Local semantic consequence relation

- Let τ be a similarity type, and let S be a class of structures of type τ (that is a class of models, a class of frames, or a class of general frames of this type).
- Let Γ and A be a set of formulas and a single formula from a language of type τ.
- We say that A is a *local semantic consequence* of Γ over S, notated as $\Gamma \models_S A$, if for all models M from S, and all points w in M, if M, w $\models \Gamma$ then M, w $\models A$.

♣ Note

- Local semantic consequence relation is the notion of logical entailment, i.e., premises guarantee conclusions, but here the guarantee covers local notions of correctness.

Global Semantic Consequence Relation

♣ Global semantic consequence relation

- Let τ be a similarity type, and let S be a class of structures of type τ (that is a class of models, a class of frames, or a class of general frames of this type).
- Let Γ and A be a set of formulas and a single formula from a language of type τ.
- We say that A is a *global semantic consequence* of Γ over S, notated as $\Gamma \models_S^g A$, if for all structures s in S, if s $\models \Gamma$ then s $\models A$.
- Here, depending on the kind of structures s contains, denotes either validity in a frame, validity in a general frame, or global truth in a model.

Semantic Consequence Relation: Examples

♣ **Local semantic consequence relation**

- Suppose that we are working with Tran, the class of transitive frames. Then ◇p is a local semantic consequence of {◇◇p} over the class Tran: {◇◇p} $\models_{Tran}$ ◇p.
- On the other hand, ◇p is not a local semantic consequence of {◇◇p} over the class of all frames.

♣ **Difference between local and global consequence relations**

- Consider the formulas *p* and □*p*. It is easy to see that *p* does not locally imply □*p* — indeed, that this entailment should not hold is pretty much the essence of locality.
- On the other hand, suppose that we consider a model *M* where p is globally true. Then *p* certainly holds at all successors of all states, so M $\models$ □p, and so p $\models^{g}$ □p.

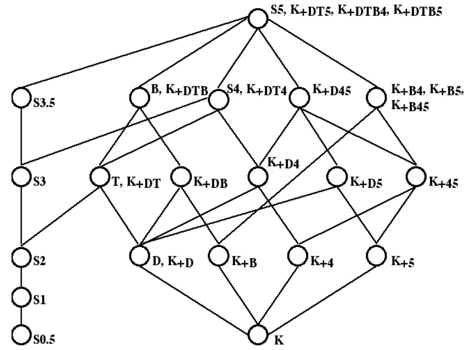
An Elementary Introduction to Modal Logics

- Modal Logic: What Is It and Why Study It ?
- Relational Structures
- Modal Languages
- Model (Semantic) Theory of Modal Logics
- Normal Modal Logics
- Temporal Logics
- Spatio-Temporal Logics
- Deontic Logics
- Epistemic Logics

Normal Modal Logics: What Are They ?

Normal Modal Logics: What are They?

The Relationships Between Various Modal Logics



The Minimal/Weakest Normal Modal Logic System K

♣ **The Hilbert-style axiomatic system of modal logic K**

- The language: The basic propositional modal language.
- The axiom schemata: All axiom schemata of CPC and the following axiom scheme (K):
  (K, Kripke) □(*A* → *B*) → (□*A* → □*B*)
- The inference rules: MP and the following (N) rule:
  N (Necessitation) From theorem *A*, to infer □*A*

♣ **Facts**

- K is a conservative extension of CPC,
  K = CPC + (K) + N, Th(K) ⊃ Th(CPC).
- All formulas of the following schemata are not theorems of K: □*A*→◇*A*, □*A*→*A*, *A*→◇*A*, ◇*A*∨◇¬*A*, ¬(□*A*∧□¬*A*), *A*→□◇*A*, □*A*→□□*A*, ◇*A*→□◇*A*.

The Minimal/Weakest Normal Modal Logic System K

♣ **The importance of axiom scheme (K)**

- The axiom scheme (K) is sometimes called the ***distribution axiom***, and is important because it lets us transform □(*A* → *B*) (a boxed formula) into □*A* → □*B* (an implication).
- This box-over-arrow distribution enables further purely propositional reasoning to take place.

♣ **Example**

- Suppose we are trying to prove □*B*, and have constructed a proof sequence containing both □(*A* → *B*) and □*A*. If we could apply MP under the scope of the box, we would have proved □*B*.
- And this is what distribution lets us do: as K contains the axiom scheme (K), by MP we can prove □(*A* → *B*) → (□*A*→ □*B*).
- But then a first application of modus ponens proves □*A* → □*B*, and a second proves □*B* as desired.

The Minimal/Weakest Normal Modal Logic System K

♣ The importance of inference rule N

- The necessitation rule N ‘modalizes’ provable formulas by stacking boxes in front.
- Roughly speaking, while the (K) axiom scheme lets us apply classical reasoning inside modal contexts, necessitation rule N creates new modal contexts for us to work with; modal proofs arise from the interplay of these two mechanisms.

♣ Notes

- Same as MP, the necessitation rule N preserves validity: if it is impossible to falsify A, then obviously we will never be able to falsify A at any accessible state!
- Similarly, the necessitation rule N preserves global truth.
- But the necessitation rule N does not preserve satisfaction: just because p is true in some state, we cannot conclude that p is true at all accessible states.

The Minimal/Weakest Normal Modal Logic System K

♣ The importance of modal logic K

- Modal logic K is the minimal/weakest modal logic system.
- K generates precisely the all formulas (*K-theorems*) that are valid on the class of all frames.

♣ The soundness and completeness of K

- The axioms of K are all valid on the class of all frames, and all inference rules preserve validity, hence its all theorems are valid on the class of all frames: K is sound with respect to the class of all frames.
- Moreover, the converse is also true: if a basic modal formula is valid on the class of all frames, then it is provable in K: K is complete with respect to the class of all frames.

Proving Theorems in K: An Example

♣ Proving the formula (theorem) $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ in K

1. $p \rightarrow (q \rightarrow (p \wedge q))$ { a theorem of CPC }
2. $\Box(p \rightarrow (q \rightarrow (p \wedge q)))$ { applying N to 1 }
3. $\Box(p \rightarrow (q \rightarrow (p \wedge q))) \rightarrow (\Box p \rightarrow \Box(q \rightarrow (p \wedge q)))$ { (K) axiom }
4. $\Box p \rightarrow \Box(q \rightarrow (p \wedge q))$ { applying MP to 2 and 3 }
5. $\Box(q \rightarrow (p \wedge q)) \rightarrow (\Box q \rightarrow \Box(p \wedge q))$ { (K) axiom }
6. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ { an axiom scheme of CPC }
7. $\Box p \rightarrow (\Box q \rightarrow \Box(p \wedge q))$ { applying 6 and MP to 4 and 5 }
8. $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ { applying the deduction theorem of CPC to 7 }

Normal Modal Logic Systems: Extensions of K

♣ K is too weak

- All formulas of the following schemata are not theorems of K: $\Box A \rightarrow \Diamond A$, $\Box A \rightarrow A$, $A \rightarrow \Diamond A$, $\Diamond A \vee \Diamond \neg A$, $\neg(\Box A \wedge \Box \neg A)$, $A \rightarrow \Box \Diamond A$, $\Box A \rightarrow \Box\Box A$, $\Diamond A \rightarrow \Box \Diamond A$.
- We know that $\Diamond\Diamond p \rightarrow \Diamond p$ is valid on all transitive frames, so we would want a proof system that generates this formula; K does not do this, for $\Diamond\Diamond p \rightarrow \Diamond p$ is not valid on all frames.

♣ Extending K

- We can extend K to cope with many restrictions by adding extra axioms, and therefore, we can obtain various modal logic systems, called *normal modal logic systems*, that are stronger than K.

Normal Modal Logic Systems

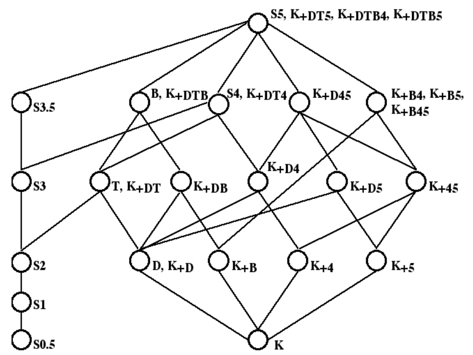
♣ Normal modal logic systems

- A *normal modal logic* (with the basic propositional modal language) is an extension of K such that it includes all axiom schemata of CPC, the axiom scheme (K) and some other axiom scheme(schemata), and the inference rules MP and N.
- Any normal modal logic is an extension of K and therefore a conservative extension of CPC:
  NML = CPC + (K) + N + .. , Th(NML) ⊃ Th(K) ⊃ Th(CPC).

♣ Modal logic systems

- A *modal logic* (with the basic propositional modal language) is a conservative extension of CPC such that it includes all axiom schemata of CPC, and the inference rules MP.
- Any modal logic is a conservative extension of CPC.

The Relationships Between Various Modal Logics



The Normal Modal Logic System D (K_{+D})

♣ The Hilbert-style axiomatic system of normal modal logic D

- The axiom schemata: All axiom schemata of K and the following axiom scheme (D):
  (D, Deontic) $\Box A \rightarrow \Diamond A$

♣ Notes

- D is a conservative extension of K.
- D (K_{+D}) = K + (D), Th(D) ⊃ Th(K).

♣ The frame/model of D

- D is sound and complete with respect to the class of right-unbounded frames.
- Note: R: $W \rightarrow W$ is said to be ***right-unbounded*** if every point has a successor, i.e., $(\forall x)(\exists y)(Rxy)$.

The Normal Modal Logic System T (K_{+T}, K_{+DT})

♣ The Hilbert-style axiomatic system of normal modal logic T

- The axiom schemata: All axiom schemata of K and the following axiom scheme (T):
  (T) $\Box A \rightarrow A$ or $A \rightarrow \Diamond A$

♣ Notes

- T is a conservative extension of K and D.
- T (K_{+T}, K_{+DT}) = K + (T), Th(T) ⊃ Th(K),
  T (K_{+T}, K_{+DT}) = D + (T), Th(T) ⊃ Th(D) ⊃ Th(K).

♣ The frame/model of T

- T is sound and complete with respect to the class of reflexive frames.
- Fact: If R: $W \rightarrow W$ is reflexive, then it is right-unbounded.

The Normal Modal Logic System B (K_{+DTB})

♣ The Hilbert-style axiomatic system of normal modal logic B

- The axiom schemata: All axiom schemata of T and the following axiom scheme (B):
  (B, Brouwer) $A \rightarrow \Box\Diamond A$

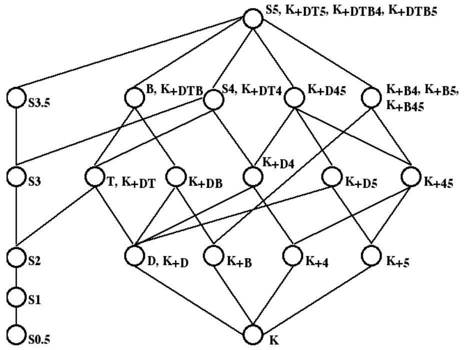
♣ Notes

- B is a conservative extension of T.
- B (K_{+DTB}) = T + B = K + (T) + (B),
  Th(B) ⊃ Th(T) ⊃ Th(D) ⊃ Th(K).

♣ The frame/model of B

- B is sound and complete with respect to the class of reflexive and symmetric frames.

The Relationships Between Various Modal Logics



The Normal Modal Logic System K_{+B}

♣ The Hilbert-style axiomatic system of normal modal logic K_{+B}

- The axiom schemata: All axiom schemata of K and the following axiom scheme (B):
  (B) $A \rightarrow \Box\Diamond A$

♣ Notes

- K_{+B} is a conservative extension of K.
- K_{+B} = K + (B), Th(K_{+B}) ⊃ Th(K).

♣ The frame/model of K_{+B}

- K_{+B} is sound and complete with respect to the class of symmetric frames.

The Normal Modal Logic System K_{+4}

♣ The Hilbert-style axiomatic system of normal modal logic K_{+4}

- The axiom schemata: All axiom schemata of K and the following axiom scheme (4):
  (4) $\Box A \rightarrow \Box\Box A$ or $\Diamond\Diamond A \rightarrow \Diamond A$

♣ Notes

- K_{+4} is a conservative extension of K.
- K_{+4} = K + (4), Th(K_{+4}) ⊃ Th(K).

♣ The frame/model of K_{+4}

- K_{+4} is sound and complete with respect to the class of transitive frames.

The Normal Modal Logic System K_{+5}

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+5}**

- **The axiom schemata: All axiom schemata of K and the following axiom scheme (5):**
  (5, E, Euclidean) $\Diamond A \rightarrow \Box\Diamond A$

♣ **Notes**

- **K_{+5} is a conservative extension of K.**
- **K_{+5} = K + (5), Th(K_{+5}) ⊃ Th(K).**

♣ **The frame/model of K_{+5}**

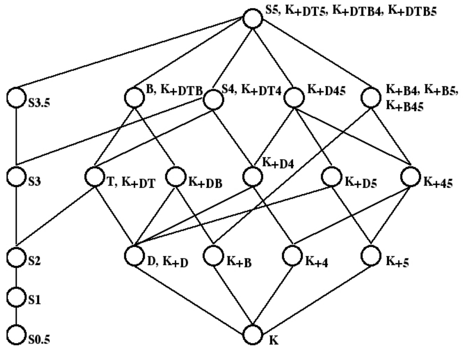
- **K_{+5} is sound and complete with respect to the class of Euclidean frames.**
- **Note: R: $W \rightarrow W$ is said to be *Euclidean* if $(\forall x)(\forall y)(\forall z)((Rxy \wedge Rxz) \rightarrow Ryz)$.**

The Characteristic Axioms of Normal Modal Logics

♣ **The corresponding relationship between modal axioms and properties of relations**

(K)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	
(D)	$\Box A \rightarrow \Diamond A$	Right-unbounded
(T)	$\Box A \rightarrow A$ or $A \rightarrow \Diamond A$	Reflexive
(B)	$A \rightarrow \Box\Diamond A$	Symmetric
(4)	$\Box A \rightarrow \Box\Box A$ or $\Diamond\Diamond A \rightarrow \Diamond A$	Transitive
(5, E)	$\Diamond A \rightarrow \Box\Diamond A$	Euclidean

The Relationships Between Various Modal Logics



Normal Modal Logic Systems K_{+DB} and K_{+D4}

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+DB}**

- **K_{+DB} = D + (B) = K + (D) + (B)**
- **Th(K_{+DB}) ⊃ Th(D) ⊃ Th(K), Th(K_{+DB}) ⊃ Th(B) ⊃ Th(K).**
- **K_{+DB} is sound and complete with respect to the class of right-unbounded and symmetric frames.**

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+D4}**

- **K_{+D4} = D + (4) = K + (D) + (4).**
- **Th(K_{+D4}) ⊃ Th(D) ⊃ Th(K), Th(K_{+D4}) ⊃ Th(K_{+4}) ⊃ Th(K).**
- **K_{+D4} is sound and complete with respect to the class of right-unbounded and transitive frames.**

Normal Modal Logic Systems K_{+D5} and K_{+45}

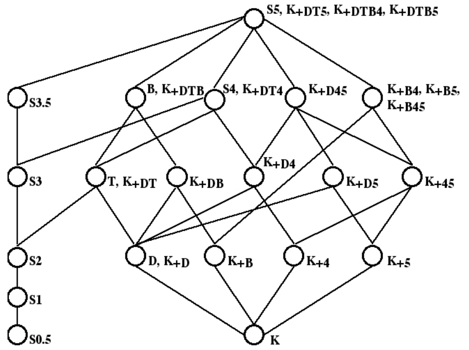
♣ **The Hilbert-style axiomatic system of normal modal logic K_{+D5}**

- **K_{+D5} = D + (5) = K + (D) + (5)**
- **Th(K_{+D5}) ⊃ Th(D) ⊃ Th(K), Th(K_{+D5}) ⊃ Th(K_{+5}) ⊃ Th(K).**
- **K_{+D5} is sound and complete with respect to the class of right-unbounded and Euclidean frames.**

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+45}**

- **K_{+45} = K_{+4} + (5) = K_{+5} + (4) = K + (4) + (5).**
- **Th(K_{+45}) ⊃ Th(K_{+4}) ⊃ Th(K), Th(K_{+45}) ⊃ Th(K_{+5}) ⊃ Th(K).**
- **K_{+45} is sound and complete with respect to the class of transitive and Euclidean frames.**

The Relationships Between Various Modal Logics



The Normal Modal Logic System S4 (K_{+DT4})

♣ **The Hilbert-style axiomatic system of normal modal logic S4 (K_{+DT4})**

- S4 (K_{+DT4}) = T + (4) = K + (T) + (4)
- Th(K_{+DT4}) ⊃ Th(T) ⊃ Th(D) ⊃ Th(K),
  Th(K_{+DT4}) ⊃ Th(K_{+D4}) ⊃ Th(D) ⊃ Th(K),
  Th(K_{+DT4}) ⊃ Th(K_{+D4}) ⊃ Th(K_{+4}) ⊃ Th(K).
- S4 (K_{+DT4}) is sound and complete with respect to the class of **reflexive and transitive frames**.

The Normal Modal Logic System K_{+D45}

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+D45}**

- K_{+D45} = D + (4) + (5) = K_{+D4} + (5) = K_{+D5} + (4)
  = K + (D) + (4) + (5)
- Th(K_{+D45}) ⊃ Th(K_{+D4}) ⊃ Th(D) ⊃ Th(K),
  Th(K_{+D45}) ⊃ Th(K_{+D4}) ⊃ Th(K_{+4}) ⊃ Th(K),
  Th(K_{+D45}) ⊃ Th(K_{+D5}) ⊃ Th(D) ⊃ Th(K),
  Th(K_{+D45}) ⊃ Th(K_{+D5}) ⊃ Th(K_{+5}) ⊃ Th(K).
- K_{+D45} is sound and complete with respect to the class of **right-unbounded, transitive, and Euclidean frames**.

The Normal Modal Logic System K_{+B45} (K_{+B4}, K_{+B5})

♣ **The Hilbert-style axiomatic system of normal modal logic K_{+B45} (K_{+B4}, K_{+B5})**

- K_{+B45} (K_{+B4}, K_{+B5}) = K + (B) + (4) + (5)
  = K + (B) + (4) = K + (B) + (5)
- Th(K_{+B45}) ⊃ Th(K_{+B}) ⊃ Th(K),
  Th(K_{+B45}) ⊃ Th(K_{+45}) ⊃ Th(K_{+4}) ⊃ Th(K),
  Th(K_{+B45}) ⊃ Th(K_{+45}) ⊃ Th(K_{+5}) ⊃ Th(K).
- K_{+B45} is sound and complete with respect to the class of **symmetric, transitive, and Euclidean frames**.

♣ **Facts**

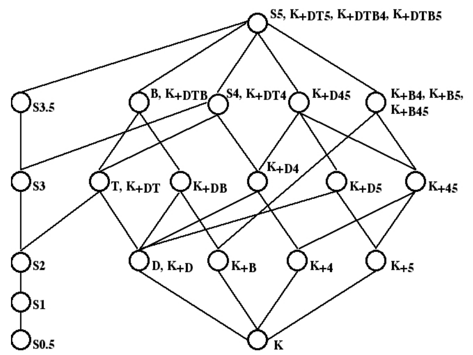
- If R: $W \to W$ is symmetric and transitive, then it is Euclidean; if R: $W \to W$ is symmetric and Euclidean, then it is transitive.

The Normal Modal Logic System S5 (K_{+DT5})

♣ **The Hilbert-style axiomatic system of normal modal logic S5 (K_{+DT5}, K_{+DTB4}, K_{+DTB5})**

- S5 (K_{+DT5}, K_{+DT54}, K_{+DTB4}, K_{+DTB5})
  = T + (5) = K + (T) + (5)
  = K + (T) + (5) + (4) ((4) is provable in S5)
  = S4 + (B) = K + (T) + (4) + (B) = K + (T) + (4) + (B) + (5)
  ((B) is provable in S5)
- Th(K_{+B4}) ⊃ Th(K_{+B}) ⊃ Th(K),
  Th(K_{+D5}) ⊃ Th(5) ⊃ Th(K).
- S5 is sound and complete with respect to the class of **reflexive, symmetric, and transitive (equivalence) frames**.
- Note: A reflexive, symmetric, and transitive relation is an equivalence relation.

The Relationships Between Various Modal Logics



Modal Logic System S0.5

♣ **The Hilbert-style axiomatic system of modal logic S0.5**

- The language: The basic propositional modal language.
- The axiom schemata: All axiom schemata of K and the following axiom scheme (T):
  (T) $\Box A \to A$ or $A \to \Diamond A$
- The inference rules: MP and the following (PCL) rule:
  PCL From CPC theorem A, to infer $\Box A$

♣ **Facts**

- S0.5 is a conservative extension of CPC,
  S0.5 = CPC + (T) + PCL, Th(S0.5) ⊃ Th(CPC).
- Th(S0.5) ⊄ Th(K), $\Box A \to A$ ∉ Th(K), $\Box A \to A$ ∈ Th(S0.5)
- Th(K) ⊄ Th(S0.5), $\Box((\Box(A \to B)) \to (\Box A \to \Box B))$ ∈ Th(K),
  $\Box((\Box(A \to B)) \to (\Box A \to \Box B))$ ∉ Th(S0.5)

Modal Logic System S2

♣ **The Hilbert-style axiomatic system of modal logic S2**

- The axiom schemata: All axiom schemata of S1 and the following axiom scheme (T).
- The inference rules: MP, PCL, and the following rules:
  R1 From axiom A, to infer $\Box A$
  R2 From theorem $\Box(A \rightarrow B)$, to infer $\Box(\Box A \rightarrow \Box B)$
- S2 = S0.5 + R1 + R2.
- Th(S2) ⊃ Th(S0.5),
  Th(T) ⊃ Th(S2).

Modal Logic System S3 and S3.5

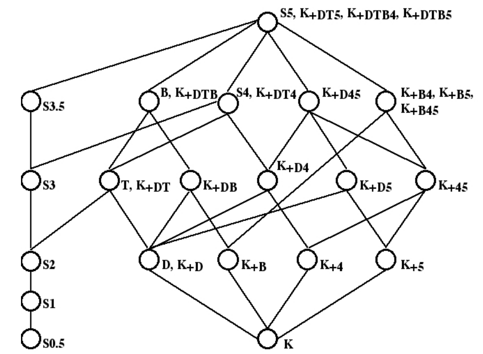
♣ **The Hilbert-style axiomatic system of modal logic S3**

- The axiom schemata: All axiom schemata of CPC, axiom scheme (T), and the following axiom scheme (3):
  (3) $\Box(A \rightarrow B) \rightarrow \Box(\Box A \rightarrow \Box B)$.
- The inference rules: MP, PCL, R1.
- Th(S3) ⊃ Th(S2), Th(S4) ⊃ Th(S3).

♣ **The Hilbert-style axiomatic system of modal logic S3.5**

- The axiom schemata: All axiom schemata of S3 and the axiom scheme (5).
- The inference rules: MP, PCL, and the following rule R1':
  R1' From axiom A that is not a instance of (5), to infer $\Box A$
- Th(S3.5) ⊃ Th(S3), Th(S5) ⊃ Th(S3.5).

The Relationships Between Various Modal Logics



Lewis's Modal Logic Systems

♣ **Lewis's modal logics**

- The main aim of Lewis's work beginning in 1912 on the establishment of modern modal logic was to find a satisfactory theory of implication which is better than CML in that it can avoid those implicational paradoxes.
- Lewis 1918 developed the first modern modal logic system.

♣ **The notion of strict implication**

- The symbol Lewis used for strict implication was "$\rightarrow 3$"
- *Strict implication*: $(A \rightarrow 3\ B) =_{df} \Box(A \rightarrow B)$ $(\neg\Diamond(A \land \neg B))$

Lewis's Modal Logic System S1

♣ **The axiom schemata of Lewis's modal logic S1**

- AS1.1 $(A \land B) \rightarrow 3\ (B \land A)$
  AS1.2 $(A \land B) \rightarrow 3\ A$
  AS1.3 $A \rightarrow 3\ (A \land A)$
  AS1.4 $((A \land B) \land C) \rightarrow 3\ (A \land (B \land C))$
  AS1.5 $((A \rightarrow 3\ B) \land (B \rightarrow 3\ C)) \rightarrow 3\ (A \rightarrow 3\ C)$
  AS1.6 $(A \land (A \rightarrow 3\ B)) \rightarrow 3\ B$

♣ **The inference rules of Lewis's modal logic S1**

- *MP with strict implication (Strict Detachment)*: From A and $A \rightarrow 3\ B$, to infer B
- *Adjunction*: From A and B, to infer $A \land B$

♣ **Facts**

- S1 is a conservative extension of CPC, Th(S1) ⊃ Th(CPC).
- S0.5 = S1 − AS1.6, Th(S1) ⊃ Th(S0.5).

Lewis's Modal Logic Systems S2 and S3

♣ **Lewis's modal logic S2 (The system Lewis preferred)**

- Axiom schemata: All axiom schemata of S1 and the following axiom scheme:
  AS2.1 $\Diamond(A \land B) \rightarrow 3\ (\Diamond A \land \Diamond B)$ (Lewis call this the Consistent Postulate)
- S2 = S1 + AS2.1
- Th(S2) ⊃ Th(S1), Th(T) ⊃ Th(S2).

♣ **Lewis's modal logic S3 (the first system Lewis developed 1918)**

- Axiom schemata: All axiom schemata of S1 and the following axiom scheme:
  AS3.1 $(A \rightarrow 3\ B) \rightarrow 3\ ((\neg\Diamond B) \rightarrow 3\ (\neg\Diamond A))$
- S3 = S1 + AS3.1
- Th(S3) ⊃ Th(S2).

Lewis's Modal Logic Systems S4 and S5

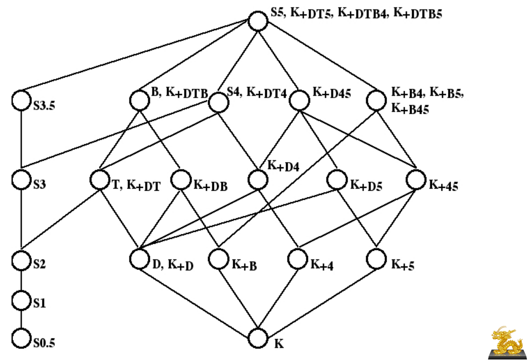
♣ **Lewis's modal logic S4**

- **Axiom schemata: All axiom schemata of S1 and the following axiom scheme:**
  AS4.1 $\Box A \to_3 \Box\Box A$
- **S4 = S1 + AS4.1**
- **Th(S4) ⊃ Th(S3).**

♣ **Lewis's modal logic S5**

- **Axiom schemata: All axiom schemata of S1 and the following axiom scheme:**
  AS5.1 $\Diamond A \to_3 \Box\Diamond A$
- **S5 = S1 + AS5.1**
- **Th(S5) ⊃ Th(S4).**

The Relationships Between Various Modal Logics



Various Modal Axioms [Garson-2018]

Name	Axiom	Condition on Frames	R is…
(D)	$\Box A \to \Diamond A$	$\exists u wRu$	Serial
(M)	$\Box A \to A$	wRw	Reflexive
(4)	$\Box A \to \Box\Box A$	$(wRv \;\&\; vRu) \Rightarrow wRu$	Transitive
(B)	$A \to \Box\Diamond A$	$wRv \Rightarrow vRw$	Symmetric
(5)	$\Diamond A \to \Box\Diamond A$	$(wRv \;\&\; wRu) \Rightarrow vRu$	Euclidean
(CD)	$\Diamond A \to \Box A$	$(wRv \;\&\; wRu) \Rightarrow v = u$	Functional
$(\Box M)$	$\Box(\Box A \to A)$	$wRv \Rightarrow vRv$	Shift Reflexive
$(C4)$	$\Box\Box A \to \Box A$	$wRv \Rightarrow \exists u(wRu \;\&\; uRv)$	Dense
(C)	$\Diamond\Box A \to \Box\Diamond A$	$wRv \;\&\; wRx \Rightarrow \exists u(vRu \;\&\; xRu)$	Convergent

Map of the Relationships Between Modal Logics [Garson-2018]

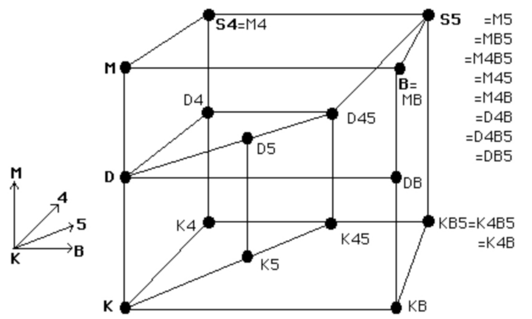
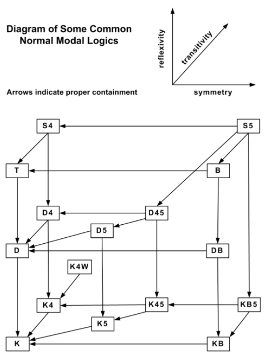


Diagram of Modal Logics

Map of the Relationships Between Modal Logics [Wikipedia]



Various Modal Axioms [B&S-2001]

D.	$\Box P \to \Diamond P$,
T.	$\Box P \to P$,
4.	$\Box P \to \Box\Box P$,
E.	$\Diamond P \to \Box\Diamond P$,
B.	$P \to \Box\Diamond P$,
Tr.	$\Box P \leftrightarrow P$,
V.	$\Box P$,
M.	$\Box\Diamond P \to \Diamond\Box P$,
G.	$\Diamond\Box P \to \Box\Diamond P$,
H.	$(\Diamond P \wedge \Diamond Q) \to (\Diamond(P \wedge Q) \vee \Diamond(P \wedge \Diamond Q) \vee \Diamond(Q \wedge \Diamond P))$,
Grz.	$\Box(\Box(P \to \Box P) \to P) \to P$,
Dum.	$\Box(\Box(P \to \Box P) \to P) \to (\Diamond\Box P \to P)$,
W.	$\Box(\Box P \to P) \to \Box P$.

Various Modal Logics [B&S-2001]

KT = **T** = the Gödel/Feys/Von Wright system,
KT4 = **S4**
KT4B = **KT4E** = **S5**
KD = deontic **T**,
KD4 = deontic **S4**,
KD4E = deontic **S5**,
KTB = the Brouwer system (‘the em Brouwersche system’),
KT4M = **S4.1**,
KT4G = **S4.2**,
KT4H = **S4.3**,
KT4Dum = **D** = Prior’s Diodorean logic,
KT4Grz = **KGrz** = Grzegoczyk’s system,
K4W = **KW** = Löb’s system,
KTr = **KT4BM** = the trivial system,
KV = the *verum* system.

Various Modal Axioms [B&S-2001]

Table 5.

Label	Formula	Equation	Condition on R
T	$\Box P \to P$	$\mathbf{l}a \leq a$	$\forall x(xRx)$
B	$\Diamond\Box P \to P$	$\mathbf{ml}a \leq a$	$\forall x \forall y(xRy \to yRx)$
4	$\Box P \to \Box\Box P$	$\mathbf{l}a \leq \mathbf{ll}a$	$\forall x \forall y \forall z((xRy \wedge yRz) \to xRz)$

Table 6.

Label	Formula	Condition on R
.3	$\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$	$\forall x \forall y \forall z((xRy \wedge xRz) \to (yRz \vee zRy))$
M	$\Box\Diamond P \to \Diamond\Box P$	$\forall x \exists y(xRy \wedge \forall z \forall w((yRz \wedge yRw) \to z = -w))$
Grz	$\Box(\Box(P \to \Box P) \to P) \to P$	There is no infinite chain $x_0, x_1, x_2, \ldots$ with $x_i R x_{i+1}$ and $x_i \neq x_{i+1}$, for all i.

Various Normal Modal Logics [Z&W&C-2001]

Table 1. A list of standard normal modal logics.

D	=	$\mathbf{K} \oplus \Box p \to \Diamond p$
T	=	$\mathbf{K} \oplus \Box p \to p$
KB	=	$\mathbf{K} \oplus p \to \Box\Diamond p$
K4	=	$\mathbf{K} \oplus \Box p \to \Box\Box p$
K5	=	$\mathbf{K} \oplus \Diamond\Box p \to \Box p$
$\mathbf{Alt}_n$	=	$\mathbf{K} \oplus \Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1})$
D4	=	$\mathbf{K4} \oplus \Diamond\top$
S4	=	$\mathbf{K4} \oplus \Box p \to p$
GL	=	$\mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p$
Grz	=	$\mathbf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p$
K4.1	=	$\mathbf{K4} \oplus \Box\Diamond p \to \Diamond\Box p$
K4.2	=	$\mathbf{K4} \oplus \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
K4.3	=	$\mathbf{K4} \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
S4.1	=	$\mathbf{S4} \oplus \Box\Diamond p \to \Diamond\Box p$
S4.2	=	$\mathbf{S4} \oplus \Diamond\Box p \to \Box\Diamond p$
S4.3	=	$\mathbf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$
Triv	=	$\mathbf{K4} \oplus \Box p \leftrightarrow p$
Verum	=	$\mathbf{K4} \oplus \Box p$
S5	=	$\mathbf{S4} \oplus p \to \Box\Diamond p$
K4B	=	$\mathbf{K4} \oplus p \to \Box\Diamond p$
$\mathbf{A}^*$	=	$\mathbf{GL} \oplus \Box\Box p \to \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
Dum	=	$\mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$
$\mathbf{K4BW}_n$	=	$\mathbf{K4} \oplus \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$
$\mathbf{K4BD}_n$	=	$\mathbf{K4} \oplus B_n$
$\mathbf{K4}_{n,m}$	=	$\mathbf{K4} \oplus \Box^n p \to \Box^m p$, for $1 \leq m < n$

An Elementary Introduction to Modal Logics

- ◆Modal Logic: What Is It and Why Study It ?
- ◆Relational Structures
- ◆Modal Languages
- ◆Model (Semantic) Theory of Modal Logics
- ◆Normal Modal Logics
- ◆Temporal Logics
- ◆Spatio-Temporal Logics
- ◆Deontic Logics
- ◆Epistemic Logics