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Introduction

Without adopting one of the various views advocated in the foundations of mathematics, we may agree that mathematicians need and use a language, if only for the communication of their results and their problems. While mathematicians have been claiming the greatest possible exactness for their methods, they have been less sensitive as to their means of communication. It is well known that Leibniz proposed to put the practice of mathematical communication and mathematical reasoning on a firm base; it was, however, not before the nineteenth century that those enterprises were (more) successfully undertaken by G. Frege and G. Peano. No matter how ingeniously and rigorously Frege, Russell, Hilbert, Bernays and others developed mathematical logic, it was only in the second half of this century that logic and its language showed any features of interest to the general mathematician. The sophisticated results of Gödel were of course immediately appreciated, but they remained for a long time technical highlights without practical use. Even Tarski's result on the decidability of elementary algebra and geometry had to bide its time before any applications turned up.

Nowadays the application of logic to algebra, analysis, topology, etc. are numerous and well-recognised. It seems strange that quite a number of simple facts, within the grasp of any student, were overlooked for such a long time. It is not possible to give proper credit to all those who opened up this new territory, any list would inevitably show the preferences of the author, and neglect some fields and persons.

Let us note that mathematics has a fairly regular, canonical way of formulating its material, partly by its nature, partly under the influence of strong schools, like the one of Bourbaki. Furthermore the crisis at the beginning of this century has forced mathematicians to pay attention to the finer details of their language and to their assumptions concerning the nature and the extent of the mathematical universe. This attention started to pay off when it was discovered that there was in some cases a close connection between classes of mathematical structures and their syntactical description. Here is an example:

It is well known that a subset of a group G which is closed under multiplication and inverse, is a group; however, a subset of an algebraically closed field F which is closed under sum, product, minus and inverse, is in general not an algebraically closed field. This phenomenon is an instance of something quite general: an axiomatizable class of structures is axiomatised by a set of universal sentences (of the form $\forall x_1, \ldots, x_n \varphi$, with φ quantifier free) iff it is closed under substructures. If we check the axioms of group theory we see that indeed all axioms are universal, while not all the axioms of the theory of algebraically closed fields are universal. The latter fact could of course be accidental, it could be the case that we were not clever enough to discover a universal axiomatization of the class of algebraically closed fields. The above theorem of Tarski and Los tells us, however, that it is impossible to find such an axiomatization!

The point of interest is that for some properties of a class of structures we have simple syntactic criteria. We can, so to speak, read the behaviour of the real mathematical world (in some simple cases) off from its syntactic description.

There are numerous examples of the same kind, e.g. *Lyndon's Theorem*: an axiomatisable class of structures is closed under homomorphisms iff it can be axiomatised by a set of positive sentences (i.e. sentences which, in prenex normal form with the open part in disjunctive normal form, do not contain negations).

The most basic and at the same time monumental example of such a connection between syntactical notions and the mathematical universe is of course *Gödel's completeness theorem*, which tells us that provability in the familiar formal systems is extensionally identical with *truth* in all structures. That is to say, although provability and truth are totally different notions, (the first is combinatorial in nature, the latter set theoretical), they determine the same class of sentences: φ is provable iff φ is true in all structures.

Given the fact that the study of logic involves a great deal of syntactical toil, we will set out by presenting an efficient machinery for dealing with syntax. We use the technique of *inductive definitions* and as a consequence we are rather inclined to see trees wherever possible, in particular we prefer natural deduction in the tree form to the linear versions that are here and there in use.

One of the amazing phenomena in the development of the foundations of mathematics is the discovery that the language of mathematics itself can be studied by mathematical means. This is far from a futile play: Gödel's incompleteness theorems, for instance, lean heavily on a mathematical analysis of the language of arithmetic, and the work of Gödel and Cohen in the field of the independence proofs in set theory requires a thorough knowledge of the mathematics of mathematical language. Set theory remains beyond the scope of this book, but a simple approach to the incompleteness of arithmetic has been included. We will aim at a thorough treatment, in the hope that the reader will realise that all these things which he suspects to be trivial, but cannot see why, are perfectly amenable to proof. It may help the reader to think of himself as a computer with great mechanical capabilities, but with no creative insight, in those cases where he is puzzled because ‘why should we prove something so utterly evident’! On the other hand the reader should keep in mind that he is not a computer and that, certainly when he gets beyond chapter 2, certain details should be recognised as trivial.

For the actual practice of mathematics predicate logic is doubtlessly the perfect tool, since it allows us to handle individuals. All the same we start this book with an exposition of propositional logic. There are various reasons for this choice.

In the first place propositional logic offers in miniature the problems that we meet in predicate logic, but there the additional difficulties obscure some of the relevant features e.g. the completeness theorem for propositional logic already uses the concept of ‘maximal consistent set’, but without the complications of the Henkin axioms.

In the second place there are a number of truly propositional matters that would be difficult to treat in a chapter on predicate logic without creating a impression of discontinuity that borders on chaos. Finally it seems a matter of sound pedagogy to let propositional logic precede predicate logic. The beginner can in a simple context get used to the proof theoretical, algebraic and model theoretic skills that would be overbearing in a first encounter with predicate logic.

All that has been said about the role of logic in mathematics can be repeated for computer science; the importance of syntactical aspects is even more pronounced than in mathematics, but it does not stop there. The literature of theoretical computer science abounds with logical systems, completeness proofs and the like. In the context of type theory (typed lambda calculus) intuitionistic logic has gained an important role, whereas the technique of normalisation has become a staple diet for computer scientists.