CS327 · Theory of Computation

Homework 5

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December 10, 2022

1 Function

Square of the sum of two natural numbers.

$$Di(x, y) = (x + y)^2$$

2 Prerequisite

2.1 Summation Function

By definition, we have

$$\begin{aligned} sum(x, 0) &= x \\ sum(x, y') &= sum(x, y)' \end{aligned}$$

Turn it to the pattern as Pr, we have

$$\begin{aligned} sum(x, 0) &= id_1^1(x) \\ sum(x, s(y)) &= Cn[s, id_3^3](x, y, sum(x, y)) \end{aligned}$$

Or in simplification

$$sum = Pr[id_1^1, Cn[s, id_3^3]]$$

2.2 Product Function

By definition, we have

$$\begin{aligned} prod(x, 0) &= 0 \\ prod(x, y') &= x + prod(x, y) \end{aligned}$$

Turn it to the pattern as Pr, we have

$$\begin{aligned} prod(x, 0) &= z(x) \\ prod(x, s(y)) &= g(x, y, prod(x, y)) \end{aligned}$$

By setting $f = sum, g_1 = id_3^3, g_2 = id_3^3$ in Cn, we have

$$\begin{aligned} g(x, y, prod(x, y)) &= Cn[sum, id_1^3, id_3^3](x, y, prod(x, y)) \\ &= sum(id_1^3(x, y, prod(x, y)), id_3^3(x, y, prod(x, y))) \\ &= x + prod(x, y) \end{aligned}$$

which satisfies out request. If we simplify it, we have

$$prod = Pr[z, Cn[sum, id_1^3, id_3^3]]$$

3 Recursive Function

We can rewrite the function as below:

$$\begin{aligned}
Di(x,0) &= x^2 \\
&= prod(x,x) \\
Di(x,y') &= (x+y')^2 \\
&= prod(x+y', x+y') \\
&= prod(x+y', (x+y)') \\
&= (x+y') + prod(x+y', x+y) \\
&= (x+y') + prod((x+y)', x+y) \\
&= (x+y') + (x+y) + (x+y)^2 \\
&= sum(x,y') + sum(x,y) + Di(x,y) \\
&= sum(sum(x,y'), sum(x,y), Di(x,y))
\end{aligned}$$

To put these equations in the format of Pr, we should find functions f and g such that

$$\begin{aligned}
f(x) &= prod(x,x) \\
g(x,y,-) &= sum(Di(x,y), sum(x,y'), sum(x,y))
\end{aligned}$$

for all natural numbers x, y and $-$.
It is easy to get $f = Pr[z, Cn[sum, id_1^3, id_3^3]]$ by the prerequisite. But for g, we need to break it into parts as below.

$$\begin{aligned}
g(x,y,-) = sum(A,B,Di(x,y)) &= Cn[sum, id_1^3, id_2^3, id_3^3](A,B,Di(x,y)) \\
A = sum(x,y') &= Cn[sum, id_1^3, Cn[s, id_2^3]](x,y) \\
B = sum(x,y) &= Cn[sum, id_1^3, id_2^3](x,y)
\end{aligned}$$

By backsubstitution, we have

$$g(x,y,-) = Cn[sum, Cn[sum, id_1^3, Cn[s, id_2^3]], Cn[sum, id_1^3, id_2^3], id_3^3](x,y,Di(x,y))$$

Above all, we have

$$Di = Pr[Pr[z, Cn[sum, id_1^3, id_3^3]], Cn[sum, Cn[sum, id_1^3, Cn[s, id_2^3]], Cn[sum, id_1^3, id_2^3], id_3^3]]$$