CS327 · Theory of Computation

Homework 2

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1 What is Cantor's Theorem?

The set of all sets of positive integers (i.e., the power set of the set of positive integers) is not enumerable. [JD.Cheng_ToC1]

2 Mathematics notion

Suppose $\mathbb{N}^+$ is the set of positive integers, then its power set $\mathbf{P}(\mathbb{N}^+)$ is not enumerable.

3 Proof

3.1 Method 1: reductio ad absurdum

Suppose S_i ($i \in \mathbb{N}$) are the sets of positive integers, i.e., S_i are the elements of $\mathbf{P}(\mathbb{N}^+)$. Then we can construct a sequence

$$L : S_0, S_1, S_2, ...$$

Our goal is to find that if this sequence captures all S_i.
With the set

$$\Delta(L) = \{n | n \in \mathbb{N}^+, n \notin S_n\}$$

we can assume that $\exists m \in \mathbb{N}^+, S_m = \Delta(L)$. However, this contradicts the definition of $\Delta(L)$. Therefore, $\forall m \in \mathbb{N}^+, S_m \neq \Delta(L)$, i.e., some S_i are not in L. Thus, we cannot list S_i as a sequence, i.e., the set $\mathbf{P}(\mathbb{N}^+)$ is not enumerable.

3.2 Method 2: diagonalization method

The subsets of set can be represented by binary arrays, where 0 represents not existence, 1 represents existence. And suppose these subsets can be listed in order.
Suppose a power set $\mathbf{P}(\mathbb{N}^+)$ with elements $S_0, S_1, S_2, ...$, there are binary arrays $s_0, s_1, s_2, ...$ Then the power set can be written as a matrix. For example,

$$\begin{array}{rccccccc}
s_0 = & 1 & 1 & 0 & 1 & 0 & 1 & ... \\
s_1 = & 1 & 0 & 1 & 1 & 1 & 0 & ... \\
s_2 = & 0 & 1 & 1 & 1 & 0 & 1 & ... \\
s_3 = & 1 & 1 & 0 & 1 & 1 & 1 & ... \\
s_4 = & 1 & 0 & 1 & 1 & 1 & 1 & ... \\
... & & & & & & &
\end{array}$$

Extract the diagonal of the matrix and denote it as D:

$$D = 10111...$$

No matter how to permutate the binary arrays, D is determined as long as the mapping relation is determined. Thus, among all the elements in $\mathbf{P}(\mathbb{N}^+)$, the i-th number of the i-th element must equal to the

i-th number of D.
Then we have

$$\overline{D} = 01000...$$

Since $\overline{D}$ is also an binary array, then it must be an element in $\mathbf{P}(\mathbb{N}^+)$. However, $\overline{D}$ breaks the rule of the matrix. Because if we put $\overline{D}$ in the i-th line of the matrix, the i-th number of $\overline{D}$ must not equal to the i-th number of D. After all, the set $\mathbf{P}(\mathbb{N}^+)$ is not enumerable.