An Elementary Introduction to Set Theory

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Text and Reference Books on Naïve Set Theory

- M. L. O'Leary, "A First Course in Mathematical Logic and Set Theory," Wiley, 2016. [O'Leary]
- E. Mendelson, "Introduction to Mathematical Logic," Chapman & Hall, 1964, 1979, 1987, 1997, 2010, 2015 (6th Edition). [Mendelson]
- P. J. Cameron, "Sets, Logic and Categories," Springer, 1998. [Cameron]
- S. Lipschutz, "Set Theory and Related Topics," Schaums's Outline Series, McGraw-Hill, 1964, 1998 (2nd Edition). [Lipschutz]
- K. Devlin, "The Joy of Sets – Fundamentals of Contemporary Set Theory," Springer, 1979, 1993 (2nd Edition). [Devlin]
- R. R. Stoll, "Set Theory and Logic," Dover, 1963. [Stoll]
- P. R. Halmos, "Naïve Set Theory," Litton Educational Publishing, 1960, Springer, 1974. [Halmos]

Text and Reference Books on Axiomatic Set Theory

- M. L. O'Leary, "A First Course in Mathematical Logic and Set Theory," Wiley, 2016. [O'Leary]
- E. Mendelson, "Introduction to Mathematical Logic," Chapman & Hall, 1964, 1979, 1987, 1997, 2010, 2015 (6th Edition). [Mendelson]
- C. C. Pinter, "A Book of Set Theory," Addison-Wesley, 1971, Dover, 2014 (Revised and corrected republication). [Pinter]
- T. Jech, "Set Theory," Springer, 1978, 1997, 2003, 2006 (Corrected 4th printing). [Jech]
- P. G. Hinman, "Fundamentals of Mathematical Logic, A K Peters, 2005. [Hinman]
- G. Tourlakis, "Lectures in Logic and Set Theory, Vol. 2: Set Theory," 2003. [Tourlakis]

Text and Reference Books on Axiomatic Set Theory

- K. Hrbacek and T. Jech, "Introduction to Set Theory," Marcel Dekker, 1978, 1984, 1999 (3rd Edition, Revised and Expanded). [H&J]
- P. J. Cameron, "Sets, Logic and Categories," Springer, 1998. [Cameron]
- K. Devlin, "The Joy of Sets – Fundamentals of Contemporary Set Theory," Springer, 1979, 1993 (2nd Edition). [Devlin]
- K. Kunen, "Set Theory – An Introduction to Independence Proofs," Elsevier, 1980. [Kunen]
- H. B. Enderton, "Elements of Set Theory," Academic Press, 1977. [Enderton]
- D. W. Barnes and J. M. Mack, "An Algebraic Introduction to Mathematical Logic," Springer, 1975. [B&M]
- J. R. Shoenfield, "Mathematical Logic," Association for Symbolic Logic / Addison-Wesley, 1967. [Shoenfield]
- R. R. Stoll, "Set Theory and Logic," Dover, 1963. [Stoll]

An Elementary Introduction to Set Theory

- Naïve Set Theory
- Axiomatic Set Theory
- Ordinal Numbers
- Categories

Set: What is It? (From the Viewpoint of Naïve Set Theory)

♣ Cantor's concept of a set [Cantor, 1870s]

- The only fundamental idea of set theory is to regard and represent a collection of objects as a *single entity*, i.e., a *set*.
- A *set* S is a collection of definite and distinguishable objects (called *elements* or *members* of S), to be conceived as a whole.
- Examples of sets: {1, 2, 3}, {1, 2, 3, a, b, c}, {1, 2, 3, …}
- Examples of collections that are not sets: {1, 2, 2, 3}, {1, 2, 3, a, b, c, c}

♣ The membership relation

- $x \in (\notin) A$: x *is (not) an element of* A; x *(does not) belongs* A.
- Note: The symbol "∈" is fashioned after the Greek letter epsilon.

The Empty Set

♣ **The empty set**

- **The *empty set* ∅ (slashed 0, slashed zero, 0 with stroke) includes no element.**
- **$\varnothing =_{df} (\forall x)(x \notin \varnothing)$ (for all x, x is not an element of ∅)**
- **Sometime, the empty set ∅ is represented as { }.**

♣ **Notes**

- **The notion of the empty set is fundamentally important in Set Theory.**
- **The role of the empty set in Set Theory is similar to that of zero in Number Theory.**
- **$\varnothing \neq \{\varnothing\}$! ($|\varnothing| = 0$, $|\{\varnothing\}| = 1$)**

Finite Sets and Infinite Sets

♣ **Finite sets**

- ***Finite sets*: Sets including finite elements.**
- **The *size of a finite set* A (represented as $|A|$): the number of elements of A.**
- **$|\varnothing| =_{df} 0$ ($|\varnothing| = 0$, $|\{\varnothing\}| = 1$)**
- **Ex: {1, 2, 3, 4, 5, 6, 7, 8, 9}, |{1, 2, 3, 4, 5, 6, 7, 8, 9}| = 9**

♣ **Infinite sets**

- ***Infinite sets*: Sets including infinite elements.**
- **Note: There is no concept of "size" about infinite sets.**
- **Ex: {1, 2, 3, 4, 5, 6, 7, 8, 9, …, n, … }**

Definition and Representation of Finite/Infinite Sets

♣ **Extensional definition and representation of finite/infinite sets**

- **Enumerate/list all elements of a set explicitly and usually surround them with braces: $A =_{df} \{a_1, a_2, \ldots, a_k, \ldots\}$.**
- **Note: "…" must be "definite and distinguishable objects".**
- **This way to define/represent a set is called the *roster method* and the list is called a *roster*.**
- **It is often difficult, sometime impossible, to define an infinite set by the roster method.**

♣ **Examples**

- **$N_{100} =_{df} \{0, 1, 2, 3, \ldots, 99\}$**
- **$N =_{df} \{0, 1, 2, 3, \ldots, n, \ldots\}$**
- **$Alphabet =_{df} \{a, b, c, \ldots, z\}$**

Definition and Representation of Finite/Infinite Sets

♣ **Intensional definition and representation of finite/infinite sets**

- **Define a property that all elements of a set have and usually show a typical element a that satisfies the property P: $A =_{df} \{ a \mid P(a) \}$**
- **This way to define/represent a set is called the *abstraction method*.**
- **It is this abstraction method that arises some paradox problems in the naïve Set Theory.**

♣ **Examples**

- **$N_{100} =_{df} \{ x \mid x \in N \land x < 100 \}$**
- **$N =_{df} \{ x \mid x \in N \}$ (Is this definition OK?)**
- **$Z =_{df} \{ n \mid n \in N \lor -n \in N \}$**
- **$Q =_{df} \{ x/y \mid x \in Z \land y \in Z \land y \neq 0 \}$**

Famous Sets [O'Leary]

Famous Sets

Although sets can contain many different types of elements, numbers are probably the most common for mathematics. For this reason particular important sets of numbers have been given their own symbols.

Symbol	Name
N	*The set of natural numbers*
Z	*The set of integers*
Q	*The set of rational numbers*
R	*The set of real numbers*
C	*The set of complex numbers*

Let $a, b \in \mathbf{R}$ such that $a < b$.

closed interval	$[a, b]$	*closed ray*	$[a, \infty)$
open interval	(a, b)	*closed ray*	$(-\infty, a]$
half-open interval	$[a, b)$	*open ray*	(a, ∞)
half-open interval	$(a, b]$	*open ray*	$(-\infty, a)$

As rosters,

$$\mathbf{N} = \{0, 1, 2, \ldots\}$$

and

$$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

Notice that we define the set of natural numbers to include zero and do not make a distinction between counting numbers and whole numbers. Instead, write

$$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$$

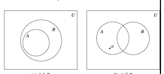
and

$$\mathbf{Z}^- = \{\ldots, -3, -2, -1\}.$$

The Subsets and Power Set of a Set

♣ **Subsets and inclusion relation**

- **Set A is called a *subset* of set B, if all elements of A are also elements of B.**
- ***Inclusion relation*: $A \subseteq B$ IFF $(\forall x)(x \in A \Rightarrow x \in B)$.**
- **For any set A, $A \subseteq A$, $\varnothing \subseteq A$.**
- ***Proper subset*: $A \subset B$ IFF $(A \subseteq B) \land (A \neq B)$.**



(a) $A \subseteq B$ (b) $A \nsubseteq B$

Figure 3.6 Venn diagrams for a subset and a nonsubset.

♣ **The power set of a set**

- **The *power set* of set A is the set that includes all subsets of A.**
- **$P(A) =_{df} \{x \mid x \subseteq A\}$, $2^A =_{df} \{x \mid x \subseteq A\}$ ($|P(A)| = 2^{|A|}$ if A is finite)**
- **$A \subseteq B$ IFF $P(A) \subseteq P(B)$, $A = B$ IFF $P(A) = P(B)$**
- **Ex: $A = \{1, 2, 3\}$, $P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$**

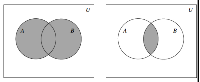
Set Operations

♣ ***Equivalence*** **of two sets**

- $A = B$ IFF $(A \subseteq B) \wedge (B \subseteq A)$
- $A \neq B$ IFF $\neg(A = B) = \neg(A \subseteq B) \vee \neg(B \subseteq A)$

♣ ***Join (Union, Sum)*** **of sets**

- $A \cup B =_{df} \{x \mid (x \in A) \vee (x \in B)\}$



(a) $A \cup B$ (b) $A \cap B$

Figure 3.2 Venn diagrams for union and intersection.

♣ ***Disjoint join (union, sum)*** **of sets**

- $A + B =_{df} \{x_A \text{ or } y_B \mid (x \in A) \wedge (y \in B)\}$
- $x_A =_{df} (x, A)$, $y_B =_{df} (y, B)$

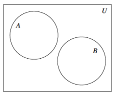


Figure 3.3 A Venn diagram for disjoint sets.

♣ ***Meet (Intersection)*** **of sets**

- $A \cap B =_{df} \{x \mid (x \in A) \wedge (x \in B)\}$
- If $A \cap B = \varnothing$, then A and B are said to be ***mutually disjoint***.

Set Operations

♣ ***Difference*** **of sets**

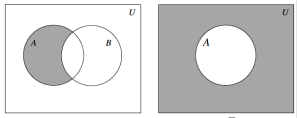
- $A - B =_{df} \{x \mid (x \in A) \wedge (x \notin B)\}$
- $A - B = A$ if $A \cap B = \varnothing$, $A - A = \varnothing$

♣ ***Symmetric difference*** **of sets**

- $A \oplus B =_{df} \{x \mid ((x \in A) \wedge (x \notin B)) \vee ((x \notin A) \wedge (x \in B))\}$
- $A \oplus B = A \cup B$ if $A \cap B = \varnothing$, $A \oplus A = \varnothing$

♣ ***Complement*** **of sets**

- For $A \subseteq U$, $A^C =_{df} U - A$.
- $A \cup A^C = U$, $A \cap A^C = \varnothing$
- $(A^C)^C = A$.



(a) $A \setminus B$ (b) $\overline{A}$

Figure 3.4 Venn diagrams for set difference and complement.

Set Operation Laws

♣ ***Idempotent laws***

- $A \cup A = A$, $A \cap A = A$

♣ ***Commutative laws***

- $A \cup B = B \cup A$, $A \cap B = B \cap A$

♣ ***Associative laws***

- $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

♣ ***Distributive laws***

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

♣ ***Absorption law***

- $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

♣ ***De Morgan's laws***

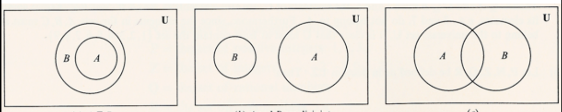
- $(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C$

Set Operation Laws [Lipschutz]

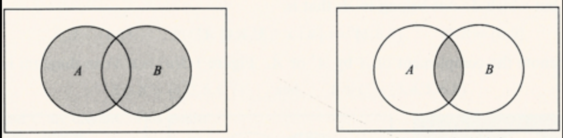
Table 1-1 Laws of the Algebra of Sets

Idempotent laws	
(1*a*) $A \cup A = A$	(1*b*) $A \cap A = A$
Associative laws	
(2*a*) $(A \cup B) \cup C = A \cup (B \cup C)$	(2*b*) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3*a*) $A \cup B = B \cup A$	(3*b*) $A \cap B = B \cap A$
Distributive laws	
(4*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5*a*) $A \cup \varnothing = A$	(5*b*) $A \cap \mathbf{U} = A$
(6*a*) $A \cup \mathbf{U} = \mathbf{U}$	(6*b*) $A \cap \varnothing = \varnothing$
Involution law	
(7) $(A^c)^c = A$	
Complement laws	
(8*a*) $A \cup A^c = \mathbf{U}$	(8*b*) $A \cap A^c = \varnothing$
(9*a*) $\mathbf{U}^c = \varnothing$	(9*b*) $\varnothing^c = \mathbf{U}$
DeMorgan's laws	
(10*a*) $(A \cup B)^c = A^c \cap B^c$	(10*b*) $(A \cap B)^c = A^c \cup B^c$

Venn Diagrams [Lipschutz]

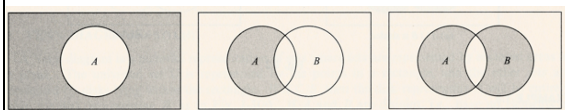


(*a*) $A \subseteq B$ (*b*) A and B are disjoint (*c*)

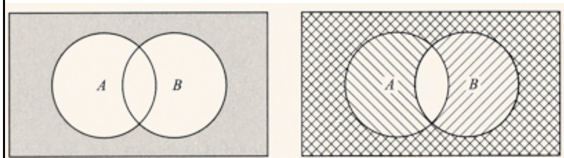


(*a*) $A \cup B$ is shaded (*b*) $A \cap B$ is shaded

Venn Diagrams [Lipschutz]

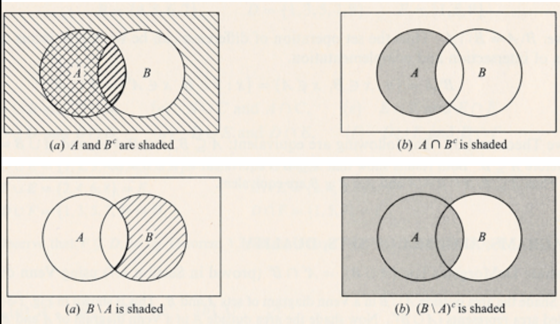


(*a*) A^c is shaded (*b*) $A \setminus B$ is shaded (*b*) $A \oplus B$ is shaded

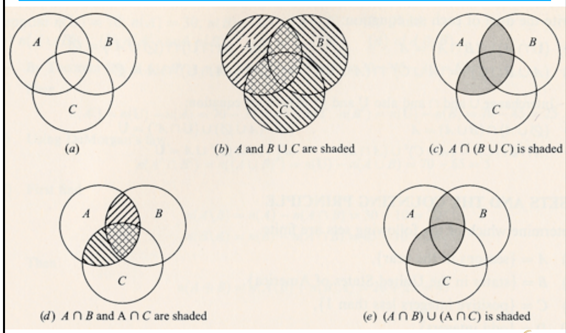


(*a*) Shaded area: $(A \cup B)^c$ (*b*) Cross-hatched area: $A^c \cap B^c$

Venn Diagrams [Lipschutz]



Venn Diagrams [Lipschutz]



How to Define Natural Numbers Mathematically?

- **$0 =_{df} \varnothing$ (define 0 as the empty set)**
- **$1 =_{df} \{0\} = \{\varnothing\} = \varnothing \cup \{\varnothing\} = 0 \cup \{0\}$**
- **$2 =_{df} \{0, 1\} = \{\varnothing, \{\varnothing\}\} = \varnothing \cup \{\varnothing, \{\varnothing\}\} = \varnothing \cup (\{\varnothing\} \cup \{\{\varnothing\}\})$**
  $= \varnothing \cup (\{\varnothing\} \cup \{\varnothing \cup \{\varnothing\}\}) = \varnothing \cup \{\varnothing\} \cup \{\varnothing \cup \{\varnothing\}\}$
  $= 0 \cup \{0\} \cup \{1\} = 1 \cup \{1\}$
- **$3 =_{df} \{0, 1, 2\} = 0 \cup \{0\} \cup \{1\} \cup \{2\} = 2 \cup \{2\}$**
  :
- **$S(n)(n+1) =_{df} \{0, 1, \ldots, n\} = 0 \cup \{0\} \cup \{1\} \cup \ldots \{n\} = n \cup \{n\}$**
  (define $S(n)(n + 1)$, as the set of from 0 to n, or the union of 0, $\{0\}$, $\{1\}$, …, $\{n\}$, or the union of n and $\{n\}$)
- **$N =_{df} \{ 0, 1, 2, 3, \ldots, n, S(n), \ldots \}$; $N =_{df} \{ n \mid (\exists_1 x)(x=S(n)) \}$**
- **Fact: For any natural number n, there must be only one natural number $S(n)$ $(n+1)$, as its only successor.**

How to Define Natural Numbers Mathematically?

- **$n + 1 = S(n) =_{df} \{0, 1, \ldots, n\} = \varnothing(0) \cup \{0\} \cup \ldots \cup \{n\} = n \cup \{n\}$**
  (define $n + 1$ as the set of from 0 to n, or the union of $\varnothing(0)$, $\{0\}$, $\{1\}$, …, $\{n\}$, or the union of the set n and $\{n\}$)
- **$n + 2 = S(S(n)) =_{df} \{0, 1, 2, \ldots, n, n+1\} = n+1 \cup \{n+1\}$**
  $(n+2 = (n+1)+1)$
- **$n + 3 = S(S(S(n))) =_{df} \ldots\ldots$ $(n+3 = ((n+1)+1)+1)$**
  :
- **Questions: $n + m =_{df}$?**
- **$n + m = S(S(\ldots S(n)\ldots)) =_{df} \ldots\ldots$ $(n+m = ((n+1)+1) + \ldots +1)$**
  (m times) (m times)

Ordered Pairs and Direct (Cartesian) Products

♣ ***Ordered pair***

- **$(a, b) =_{df} \{\{a\}, \{a, b\}\}$ [K. Kuratowski, 1921]**
- **Special case: (a, a)**

♣ ***Direct (Cartesian) product of two sets***

- **$A \times B =_{df} \{(a, b) \mid a \in A, b \in B\}$**
- **Ex: $A = \{1, 2, 3\}$, $B = \{x, y, z\}$**
  $A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\}$

♣ **Notes**

- **R. Descartes realized that, by taking two perpendicular axes and setting up coordinates, the points of the Euclidean plane can be labelled in a unique way by ordered pairs of real numbers.**
- **A point is an ordered pair of real numbers, so that the set of points of the Euclidean plane is the Cartesian product $R \times R$.**

Direct (Cartesian) Products and Set Operations [O'Leary]

♣ ***Direct (Cartesian) product and set operations***

- **$A \times (B \cap C) = (A \times B) \cap (A \times C)$**
- **$A \times (B \cup C) = (A \times B) \cup (A \times C)$**
- **$A \times (B - C) = (A \times B) - (A \times C)$**
- **$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$**

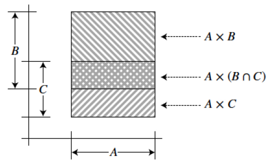


Figure 3.7 $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

N-tuples and Direct Products

♣ *3-tuples (triples)*

- $(a, b, c) =_{df} \{\{a\}, \{a, b\}, \{a, b, c\}\}$
- $(a, b, c) =_{df} ((a, b), c) = \{\{\{a\}, \{a, b\}\}, \{\{\{a\}, \{a, b\}\}, c\}\}$

♣ *n-tuples*

- $(a_1, a_2, \ldots, a_{n-1}, a_n) =_{df} \{\{a_1\}, \{a_1, a_2\}, \ldots, \{a_1, a_2, \ldots, a_{n-1}, a_n\}\}$
- $(a_1, a_2, \ldots, a_{n-1}, a_n) =_{df} ((a_1, a_2, \ldots, a_{n-1}), a_n)$

♣ *Direct (Cartesian) products of sets*

- $A_1 \times A_2 \times \ldots \times A_n =_{df} \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$

♣ *Cartesian power (exponentiation)*

- $A^n =_{df} A \times A \times \ldots \times A$

The Notion of Relation: Binary Relations

♣ **Binary relations**

- **A** ***binary relation*** R **from set** A **(*source*, *from-set*) to set** B **(*target*, *to-set*) is defined as** $R: A \to B =_{df} R \subseteq A \times B$. **We write** aRb **if** $(a, b) \in R$.
- **Any binary relation is a set of ordered pairs.**
- $R: A \to B$ **defines an *abstract binary relation* (related two sets** A **and** B**) that may have many instances.**
- **Any concrete (explicitly enumerates all elements) subset of** $A \times B$ **defines a *concrete binary relation* from** A **to** B.

♣ **Domain and range of a binary relation** $R: A \to B$

- ***Domain***: $\mathrm{dom}(R) =_{df} \{a \mid (\exists b)((a, b) \in R)\}$, $\mathrm{dom}(R) \subseteq A$
- ***Range***: $\mathrm{ran}(R) =_{df} \{b \mid (\exists a)((a, b) \in R)\}$, $\mathrm{ran}(R) \subseteq B$

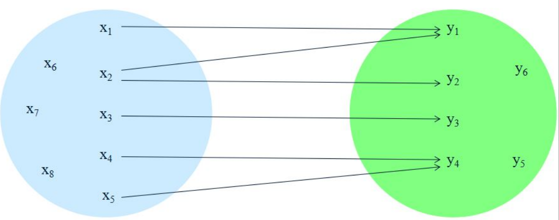
The Source, From-set, Target, and To-set of a Relation

♣ **Binary relations**

- $R: X \to Y =_{df} R \subseteq X \times Y$.

♣ **Source (From-set) and target (To-set) of relation** $R: X \to Y$

Source, From-set **R** **Target, To-set**

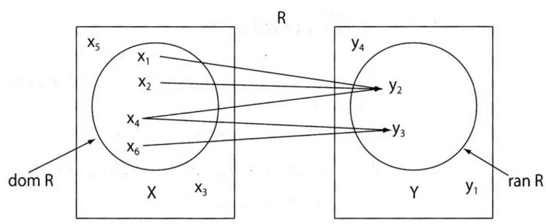


The Domain and Range of a Relation

♣ **Binary relations**

- $R: X \to Y =_{df} R \subseteq X \times Y$.

♣ **Domain and range of relation** $R: X \to Y$



Various Binary Relations

♣ **Universal relation**

- $R: A \to B$ **is said to be a *universal relation* if** $R = A \times B$.
- **Ex.:** $A = \{1, 2, 3\}$, $B = \{x, y, z\}$
  $R = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z)\} = A \times B$
  $R' = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y)\} \neq A \times B$
- **Note: There are many (infinite) universal relations, because there are many different source** A **and target** B.

♣ **The empty relation**

- $R = \varnothing$.
- **Note: There is only one empty relation.**

Various Binary Relations

♣ **Identity relation**

- $id_A =_{df} \{(a, a) \mid (a \in A)\}$.
- **Ex.:** $A = \{1, 2, 3\}$. $id_A = \{(1, 1), (2, 2), (3, 3)\}$ **is an identity relation;** $R' = \{(1, 1), (2, 2)\}$ **and** $R'' = \{(1, 1), (2, 2), (3, 3), (3, 1)\}$ **are not identity relations.**

♣ **The inverse relation of a relation**

- $R^{-1}: B \to A =_{df} \{(b, a) \mid (a, b) \in R\}$ **where** $R \subseteq A \times B$.
- $(R^{-1})^{-1} = R$; $(A \times A)^{-1} = A \times A$; $(\varnothing)^{-1} = \varnothing$; $(id_A)^{-1} = id_A$
- **Ex.:** $A = \{1, 2, 3\}$, $B = \{x, y, z\}$.
  $R = \{(1, x), (2, y), (3, z)\}$
  $R^{-1} = \{(x, 1), (y, 2), (z, 3)\}$
  $R' = \{(x, 1), (y, 2), (z, 3), (z, 2)\} \neq R^{-1}$

Various Binary Relations

♣ **Reflexive relation**

- $R: A \to A$ is said to be *reflexive* if $(\forall a)(a \in A \Rightarrow (a, a) \in R)$.
- Note: Any identity relation must be a reflexive relation, but the reverse does not necessarily hold.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3)\}$, $R' = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ are reflexive relations; $R'' = \{(1, 1), (2, 2)\}$, $R''' = \{(1, 1), (3, 3), (1, 2)\}$ are not reflexive relations.

♣ **Irreflexive relation**

- $R: A \to A$ is said to be *irreflexive* if $(\forall a)(a \in A \Rightarrow (a, a) \notin R)$.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 2), (2, 3), (1, 3)\}$ is an irreflexive relation; $R' = \{(1, 2), (2, 3), (2, 2)\}$ is not an irreflexive relation.
- Note: "Irreflexive" is not the same as "not reflexive".

Various Binary Relations

♣ **Symmetric relation**

- $R: A \to A$ is said to be *symmetric* if $(\forall a)(\forall b)[(a \in A \land b \in A) \Rightarrow ((a, b) \in R \Rightarrow (b, a) \in R)]$.
- Note: Any identity relation must be a symmetric relation, but the reverse does not necessarily hold.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ is a symmetric relation; $R' = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3)\}$ is not a symmetric relation.

♣ **Asymmetric relation**

- $R: A \to A$ is said to be *asymmetric* if $(\forall a)(\forall b)[(a \in A \land b \in A) \Rightarrow ((a, b) \in R \Rightarrow (b, a) \notin R)]$.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 2), (2, 3), (3, 1)\}$ is an asymmetric relation; $R' = \{(1, 2), (2, 3), (3, 1), (2, 1)\}$ and $R'' = \{(1, 2), (2, 3), (3, 1), (1, 1)\}$ are not asymmetric relations.
- Note: "Asymmetric" is not the same as "not symmetric".

Various Binary Relations

♣ **Antisymmetric relation**

- $R: A \to A$ is said to be *antisymmetric* if $(\forall a)(\forall b)((a \in A \land b \in A) \Rightarrow (((a, b) \in R \land (b, a) \in R) \Rightarrow a = b))$.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3)\}$, $R' = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$ are antisymmetric relations; $R'' = \{(1, 2), (2, 1)\}$, $R''' = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ are not antisymmetric relations.
- Note: "Antisymmetric" is not the same as "not symmetric".

♣ **Connected relation**

- $R: A \to A$ is said to be *connected* if $(\forall a)(\forall b)((a \in A \land b \in A) \Rightarrow (a \neq b \Rightarrow ((a, b) \in R \lor (b, a) \in R)))$.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 1)\}$ and $R' = \{(1, 1), (2, 2), (1, 2), (2, 3), (3, 1)\}$ are connected relations; $R'' = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$ is not a connected relation.

Various Binary Relations

♣ **Transitive relation**

- $R: A \to A$ is said to be *transitive* if $(\forall a)(\forall b)(\forall c)((a \in A \land b \in A \land c \in A) \Rightarrow (((a, b) \in R \land (b, c) \in R) \Rightarrow (a, c) \in R))$.
- Note: Any identity relation must be a transitive relation, but the reverse does not necessarily hold.
- Ex.: $A = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ and $R' = \{(1, 1), (2, 2), (1, 2), (2, 3), (1, 3)\}$ are transitive relations; $R'' = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$ and $R''' = \{(1, 1), (2, 2), (1, 2), (2, 3), (3, 1)\}$ are not transitive relations.

Various Binary Relations [Devlin]

R	is *reflexive*	if	$(\forall a \in x)(aRa)$;
R	is *symmetric*	if	$(\forall a, b \in x)(aRb \to bRa)$;
R	is *antisymmetric*	if	$(\forall a, b \in x)[(aRb \land a \neq b) \to \neg(bRa)]$;
R	is *connected*	if	$(\forall a, b \in x)[(a \neq b) \to (aRb \lor bRa)]$;
R	is *transitive*	if	$(\forall a, b, c \in x)[(aRb \land bRc) \to (aRc)]$.

Various Binary Relations [Lipschutz]

Reflexive Relations: A relation R on a set A is *reflexive* if $a\,R\,a$ for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

Symmetric Relations: A relation R on a set A is *symmetric* if whenever $a\,R\,b$ then $b\,R\,a$, that is, if whenever $(a, b) \in R$, then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Antisymmetric Relations: A relation R on a set A is *antisymmetric* if whenever $a\,R\,b$ and $b\,R\,a$ then $a = b$, that is, if whenever (a, b) and (b, a) belong to R then $a = b$. Thus R is not antisymmetric if there exist $a, b \in A$ such that (a, b) and (b, a) belong to R, but $a \neq b$.

Transitive Relations: A relation R on a set A is *transitive* if whenever $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in A$ such that $(a, b), (b, c) \in R$, but $(a, c) \notin R$.

Example of Various Binary Relations [Lipschutz]

EXAMPLE 3.6 Consider the following five relations on the set $A = \{1,2,3,4\}$:

$$R_1 = \{(1,1),\ (1,2),\ (2,3),\ (1,3),\ (4,4)\}$$
$$R_2 = \{(1,1),\ (1,2),\ (2,1)\ (2,2),\ (3,3), (4,4)\}$$
$$R_3 = \{(1,3),\ (2,1)\}$$
$$R_4 = \varnothing, \text{ the empty relation}$$
$$R_5 = A \times A, \text{ the universal relation}$$

Determine which of the relations are: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

(*a*) Since A contains the four elements 1, 2, 3, 4, a relation R on A is reflexive if it contains the four pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, $(2,2)$ does not belong to any of them.

(*b*) R_1 is not symmetric since $(1,2) \in R_1$ but $(2,1) \notin R_1$. R_3 is not symmetric since $(1,3) \in R_3$ but $(3,1) \notin R_3$. The other relations are symmetric.

(*c*) R_2 is not antisymmetric since $(1,2)$ and $(2,1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_5 is not antisymmetric. All the other relations are antisymmetric.

(*d*) The relation R_3 is not transitive since $(2,1), (1,3) \in R_3$ but $(2,3) \notin R_3$. All the other relations are transitive.

Example of Various Binary Relations [Lipschutz]

EXAMPLE 3.7 Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.
(2) Set inclusion $\subseteq$ on a collection $\mathscr{C}$ of sets.
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.
(5) Relation $|$ of divisibility on the set **P** of positive integers. (Recall that $x|y$ if there exists z such that $xz = y$.)

Determine which of the relations are: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

(*a*) The relation (3) is not reflexive since no line is perpendicular to itself. Also, (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every integer x in **Z**, $A \subseteq A$ for any set A in $\mathscr{C}$, and $n|n$ for every positive integer n in **P**.

(*b*) The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to a. The other relations are not symmetric. For example, $3 \leq 4$ but $4 \leq 3$; $\{1,2\} \subseteq \{1,2,3\}$ but $\{1,2,3\} \not\subseteq \{1,2\}$; and $2|6$ but $6|2$.

(*c*) The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on **P** is antisymmetric since whenever $m|n$ and $n|m$ then $m = n$. (Note that divisibility on **Z** is not antisymmetric since $3|-3$ and $-3|3$ but $3 \neq -3$.) The relation $\perp$ is not antisymmetric since we can have distinct lines a and b such that $a \perp b$ and $b \perp a$. Similarly, $\parallel$ is not antisymmetric.

(*d*) The relations $\leq$, $\subseteq$ and $|$ are transitive. That is:

(i) If $a \leq b$ and $b \leq c$, then $a \leq c$.
(ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iii) If $a|b$ and $b|c$, then $a|c$.

On the other hand, the relation $\perp$ is not transitive. If $a \perp b$ and $b \perp c$, then it is not true that $a \perp c$. Since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \nparallel a$. Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

Equivalence Relations and Partial Order Relations

♣ **Equivalence relations**

- **$R: A \to A$ is an *equivalence relation* IFF it is reflexive, symmetric, and transitive.**
- **Ex: $=_N =_{df} \{(x, y) \mid x \in N \wedge y \in N \wedge y = x\}$**
- **Ex: $=_Z =_{df} \{(x, y) \mid x \in Z \wedge y \in Z \wedge y = x\}$**

♣ **Partial order relations**

- **$R: A \to A$ is a *partial order relation* IFF it is reflexive, antisymmetric, and transitive.**
- **Ex: $\leq_N =_{df} \{(x, y) \mid x \in N \wedge y \in N \wedge x \leq y\}$**
- **Ex: $\leq_Z =_{df} \{(x, y) \mid x \in Z \wedge y \in Z \wedge x \leq y\}$**

♣ **Note**

- **The equivalence relation and partial order relation are two types of the most important and useful relations.**

The Notion of Relation: Composition of Relations

♣ **Composite relation (composition of relations)**

- **Let $R: A \to B$ and $S: B \to C$.**
- **$R \bullet S =_{df} \{(a, c) \mid (\exists b)((b \in B) \wedge (a, b) \in R \wedge (b, c) \in S)\}$ ($R \bullet S \subseteq A \times C$).**
- **The target (to-set) of R must be the source (from-set) of S.**
- **$R \bullet (S \bullet T) = (R \bullet S) \bullet T$**

♣ **Example**

- **$A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{7, 8, 9\}$.**
- **$R = \{(1, x), (2, y), (3, z)\}$,**
  $S = \{(x, 7), (y, 8), (z, 9)\}$,
  $R \bullet S = \{(1, 7), (2, 8), (3, 9)\}$

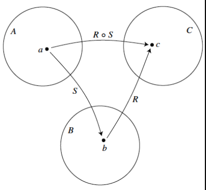


Figure 4.2 A composition of relations.

The Notion of Relation: Composition of Relations

♣ **Power (exponentiation) of relations ($R: A \to A$)**

- **$R^0 =_{df} \{(a, a) \mid a \in A\}$,**
  $R^1 =_{df} R$,
  $R^2 =_{df} R \bullet R$,
  … ,
  $R^{n+1} =_{df} R^n \bullet R$.

♣ **Transitive closure**

- **$R^+ =_{df} R^1 \cup R^2 \cup \ldots \cup R^n \cup \ldots$**
- **Ex: Let R be the parent-child relationship, then R^+ is the ancestor-descendant relationship.**

♣ **Reflexive transitive closure**

- **$R^* =_{df} R^0 \cup R^+ = R^0 \cup R^1 \cup R^2 \cup \ldots \cup R^n \cup \ldots$**
- **Ex: Let R be the blood relationship, then R^* is the consanguinity relation.**

The Notion of Relation: Composition of Relations

♣ **Composite relation examples**

- **Let $Successor =_{df} \{(x, y) \mid x \in N \wedge y \in N \wedge y = x + 1\}$.**
- **$Successor^0 = \{(x, x) \mid x \in N\}$ is the relation "=".**
- **$Successor^1 = Successor$ is the relation "successor".**
- **$Successor^2 = Successor \bullet Successor$ is the relation "x+2 = y".**
- **$Successor^n = Successor^{n-1} \bullet Successor$ is the relation "x+n = y".**

♣ **Transitive closure example**

- **$Successor^+$ is the relation "$x < y$".**

♣ **Reflexive transitive closure example**

- **$Successor^*$ is the relation "$x \leq y$".**

Properties of Various Binary Relations

- $R: A \rightarrow A$ is a reflexive relation IFF $R = R \cup id_A$.
- $R: A \rightarrow A$ is a symmetric relation IFF $R = R \cup R^{-1}$.
- $R: A \rightarrow A$ is a transitive relation IFF $R = R \cup R^{+}$.
- $R: A \rightarrow A$ is a reflexive and symmetric relation IFF $R = R \cup id_A \cup R^{-1}$.
- $R: A \rightarrow A$ is a reflexive and transitive relation IFF $R = R^{*}$.
- $R: A \rightarrow A$ is a symmetric and transitive relation IFF $R = (R \cup R^{-1})^{+}$.

Equivalence Classes and Quotient Set

♣ Equivalence classes

- Let $\equiv : A \rightarrow A$ be an equivalence relation on A, i.e., it is reflexive, symmetric, and transitive: for any $a, b, c \in A$,
  $(a, a) \in \equiv$,
  $(a, b) \in \equiv \Rightarrow (b, a) \in \equiv$, and
  $((a, b) \in \equiv \wedge (b, c) \in \equiv) \Rightarrow (a, c) \in \equiv$.
  The *equivalence classes* of A defined by (with respect to) $\equiv$ are defined as follows: $[a]_{\equiv} =_{df} \{b \mid (a, b) \in \equiv\}$.

♣ Quotient set

- The *quotient set* of A defined by (with respect to) $\equiv$ (*A modulo $\equiv$*) is defined as follows:
  $A/\equiv \; =_{df} \{[a]_{\equiv} \mid a \in A\}$.
- The quotient set of A is a *partition* of A.

Equivalence Classes and Quotient Set: Properties

♣ Properties of equivalence classes

- Let $\equiv : A \rightarrow A$ be an equivalence relation on A.
- (1) $a \in [a]_{\equiv}$
- (2) $b \in [a]_{\equiv} \Leftrightarrow a \in [b]_{\equiv}$
- (3) $a \equiv b \Leftrightarrow [a]_{\equiv} = [b]_{\equiv}$
- (4) $b \in [a]_{\equiv} \wedge c \in [a]_{\equiv} \Rightarrow b \equiv c$
- (5) $([a]_{\equiv} \neq [b]_{\equiv} \Rightarrow [a]_{\equiv} \cap [b]_{\equiv} = \varnothing) \wedge ([a]_{\equiv} \cap [b]_{\equiv} \neq \varnothing \Rightarrow [a]_{\equiv} = [b]_{\equiv})$
- (6) $\bigcup_{a \in A} [a]_{\equiv} = A$

♣ Quotient set and partition

- The quotient set of A, $A/\equiv \; =_{df} \{[a]_{\equiv} \mid a \in A\}$, as a partition of A, satisfies (5) and (6).

Equivalence Classes and Quotient Set: Examples

♣ A mathematical puzzle

- 1 = 4,
  2 = 8,
  3 = 24,
  4 = ?
- Note: Regard "=" as just a symbol!

♣ A solution form the viewpoint of equivalence relation

- 1 = 4,
  2 = 8,
  3 = 24,
  4 = 1,
  8 = 2,
  24 = 3.

♣ A further question: Which equivalence relation?

- 5 = ? ? = 5

Equivalence Classes and Quotient Set: Examples

♣ Congruent (Congruential) relations on N

- $a \equiv r \pmod{m}$ $[a \equiv r \bmod m]$ IFF $a = qm + r$ $(0 \leq r \leq m - 1)$
- Congruent (Congruential) relations are equivalence relations.

♣ A partition of N by a congruent relation ($m = 5$)

- 1 ≡ 1 (mod 5), 2 ≡ 2 (mod 5), 3 ≡ 3 (mod 5), 4 ≡ 4 (mod 5), 5 ≡ 5(0) (mod 5),
  6 ≡ 1 (mod 5), 7 ≡ 2 (mod 5), 8 ≡ 3 (mod 5), 9 ≡ 4 (mod 5), 10 ≡ 5(0) (mod 5),
  11 ≡ 1 (mod 5), 12 ≡ 2 (mod 5), 13 ≡ 3 (mod 5), 14 ≡ 4 (mod 5), 15 ≡ 5(0) (mod 5),
  16 ≡ 1 (mod 5), 17 ≡ 2 (mod 5), 18 ≡ 3 (mod 5), ……

♣ A partition of N by a congruent relation ($m = 3$)

- 1 ≡ 1 (mod 3), 2 ≡ 2 (mod 3), 3 ≡ 3(0) (mod 3),
  4 ≡ 1 (mod 3), 5 ≡ 2 (mod 3), 6 ≡ 3(0) (mod 3),
  7 ≡ 1 (mod 3), 8 ≡ 2 (mod 3), 9 ≡ 3(0) (mod 3), …

Partial Order Relations

♣ Partial order relations

- Let $\leq : A \rightarrow A$ be a partial order relation on A, i.e., it is reflexive, antisymmetric, and transitive: for any $a, b, c \in A$,
  $(a, a) \in \leq$,
  $((a, b) \in \leq \wedge (b, a) \in \leq) \Rightarrow a = b$, and
  $((a, b) \in \leq \wedge (b, c) \in \leq) \Rightarrow (a, c) \in \leq$.
- The notation "$a<b$" means $a \leq b$ and $a \neq b$.

♣ An example: Inclusion relation $\subseteq$ on power set $P(A)$

- For any $a, b, c \in P(A)$,
  $(a, a) \in \subseteq$,
  $((a, b) \in \subseteq \wedge (b, a) \in \subseteq) \Rightarrow a = b$, and
  $((a, b) \in \subseteq \wedge (b, c) \in \subseteq) \Rightarrow (a, c) \in \subseteq$.
- Note: There are always some subsets of A which cannot be compared with each other.

Partial Order Relations: An Example [O'Leary]

- Let S be a set of symbols and let S^* denote the set of all strings over S.
- Use the symbol □ to denote the empty string, the string of length zero. The empty string is always an element of S^*.
- Take $\sigma, \tau \in S^*$. The concatenation of σ and τ is denoted by $\sigma\hat{}\tau$ and is the string consisting of the elements of σ followed by those of τ.
- Finally, for all $\sigma, \tau \in S^*$, define $\sigma \le \tau$ IFF there exists $v \in S^*$ such that $\tau = \sigma\hat{}v$, then $\le$ is a partial order on S^*.

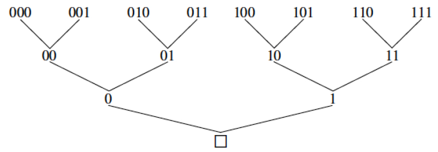


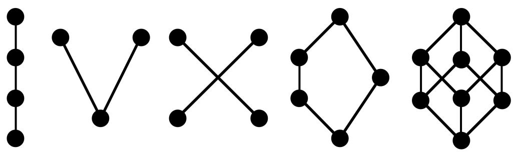
Figure 4.5 A partial order defined on $\{0, 1\}^*$.

Partially Ordered Sets (Posets)

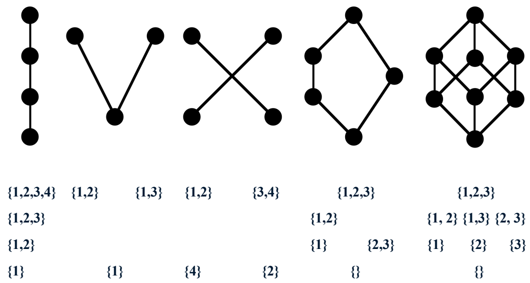
- **Partially ordered sets (posets)**
  - Let $\le : P \rightarrow P$ be a partial order relation on P. $(P, \le)$ is called a *partially ordered set (poset)*.
  - Note: When we say a poset, we have to represent both the set P and the partial order relation $\le$ defined on it.
- **Comparable elements in partially ordered sets**
  - Let $(P, \le)$ be a poset. For any $a, b \in P$, a and b are said to be *comparable* if either $a \le b$ or $b \le a$ holds.
  - Any two elements of a poset are not necessarily comparable.
- **Examples**
  - $(N, \le_N)$, $(Z, \le_Z)$, $(P(A), \subseteq)$, $(Z, |)$ (exact division relation "$a \mid b$" means that $b = q \bullet a$ for $a, b, q \in Z$).

Hasse Diagrams

- **Hasse diagrams [H. Hasse]**
  - Let $(P, \le)$ be a poset. For any $a, b \in P$, if $a \le b$ and $a \ne b$, then put a point representing b higher than a point representing a and draw down a line from b to a.
- **Examples**



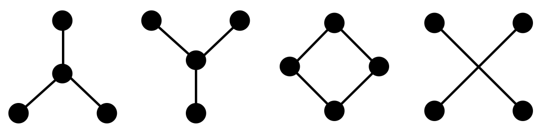
Hasse Diagram Examples: Inclusion Relation among Sets



Maximal/Minimal and Greatest/Least Elements of Posets

- **Maximal elements and Minimal elements**
  - Let $(P, \le)$ be a poset. For $a \in (P, \le)$, if $\neg(\exists b)(a < b)$, then a is called a *maximal element* of $(P, \le)$; For $a \in (P, \le)$, if $\neg(\exists b)(b < a)$, then a is called a *minimal element* of $(P, \le)$.
  - A poset may have multiple maximal/minimal elements.
  - "maximal/minimal" means "nothing else is larger/smaller".
- **Greatest (Maximum) elements and Least (Minimum) elements**
  - Let $(P, \le)$ be a poset. For $\top \in (P, \le)$, if $(\forall b)(b \le \top)$, then $\top$ is called the *greatest (maximmu) element* of $(P, \le)$; For $\bot \in (P, \le)$, if $(\forall b)(\bot \le b)$, then $\bot$ is called the *least (minimum) element* of $(P, \le)$.
  - A poset may have at most one greatest (maximum) / least (minimum) element. (greatest vs. maximal ?)
  - "greatest/least" means "larger/smaller than everything else".

Maximal/Minimal and Greatest/Least Elements of Posets



Bounds of Partially Ordered Sets (Posets)

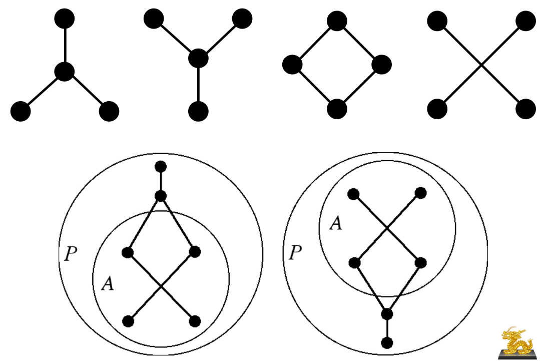
♣ Upper bounds and lower bounds

- Let $(P, \leq)$ be a poset and $A \subseteq P$. For $a \in A$, $b \in P$, if $(\forall a)(a \leq b)$, then b is called an *upper bound (u.b.)* of A; For $a \in A$, $b \in P$, if $(\forall a)(b \leq a)$, then b is called a *lower bound (l.b.)* of A.
- A subset of a poset may have multiple upper/lower bounds.

♣ Least upper bounds and greatest lower bounds

- Let $(P, \leq)$ be a poset and $A \subseteq P$. For $B \in P$, if $B \leq b$ for all u.b. b, then B is called the *least upper bound (l.u.b.)* of A; For $B \in P$, if $b \leq B$ for all l.b. b, then B is called the *greatest lower bound (g.l.b.)* of A.
- A subset of a poset may have at most one least upper/ greatest lower bound .

Bounds of Partially Ordered Sets: Examples



Bounds of Partially Ordered Sets [Jech]

Definition 2.2. If $(P, <)$ is a partially ordered set, X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X$ and $(\forall x \in X)\, a \not< x$;
a is a *minimal* element of X if $a \in X$ and $(\forall x \in X)\, x \not< a$;
a is the *greatest* element of X if $a \in X$ and $(\forall x \in X)\, x \leq a$;
a is the *least* element of X if $a \in X$ and $(\forall x \in X)\, a \leq x$;
a is an *upper bound* of X if $(\forall x \in X)\, x \leq a$;
a is a *lower bound* of X if $(\forall x \in X)\, a \leq x$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted $\sup X$ $(\inf X)$. Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

Well-Founded Sets and Well-Founded Orders

♣ Well-founded sets and well-founded orders

- Let $(P, \leq)$ be a poset. If every non-empty subset of P has a minimal element, then $(P, \leq)$ is called a *well-founded set* and $\leq$ is called a *well-founded order*.

♣ Examples

- $(N, \leq_N)$, $(P(A), \subseteq)$, and $(N, |)$ are well-founded sets (orders).
- $(Z, \leq_Z)$ and $(Z, |)$ are not well-founded sets (orders).

Total (Linear) Order Relations

♣ Total (Linear) order relations

- A partial order relation is called a *total (linear) order relation* if it is connected, i.e., a relation is called a total (linear) order relation if it is reflexive, antisymmetric, transitive, and connected.

♣ Examples

- Ex: $\leq_N =_{\mathrm{df}} \{(x, y) \mid x \in N \wedge y \in N \wedge x \leq y\}$
- Ex: $\leq_Z =_{\mathrm{df}} \{(x, y) \mid x \in Z \wedge y \in Z \wedge x \leq y\}$

Totally Ordered Sets (Tosets, Chains)

♣ Totally ordered sets (Toset, Chains)

- Let $\leq : T \rightarrow T$ be a totally ordered relation on T, i.e., it is reflexive, antisymmetric, transitive, and connected:
  for any $a, b, c \in T$,
  $(a, a) \in \leq$,
  $((a, b) \in \leq \wedge (b, a) \in \leq) \Rightarrow a = b$,
  $((a, b) \in \leq \wedge (b, c) \in \leq) \Rightarrow (a, c) \in \leq$, and
  $(a \neq b) \Rightarrow ((a, b) \in \leq \vee (b, a) \in \leq)$.
- $(T, \leq)$ is called a *totally ordered set (toset, chain)*.
- *Trichotomy law*: For any $a, b \in (T, \leq)$, exactly one of the following holds: $a < b$, $b < a$, or $a = b$ (Any two elements of a totally ordered set are necessarily comparable).

♣ Examples

- $(N, \leq_N)$, $(Z, \leq_Z)$

Inductive Partially Ordered Sets

- **Inductive partially ordered sets**
  - A poset $(P, \leq)$ is said to be ***inductive*** if every chain in P has an upper bound in P.
- **Zorn's Lemma**
  - Every inductive set has at least one maximal element.
  - Note: While Zorn's lemma and the Axiom of Choice are set-theoretically equivalent, it is much more difficult to derive the former from the latter than vice-versa. [Bell]

Well-Ordered Sets (Wosets) and Well-Orders

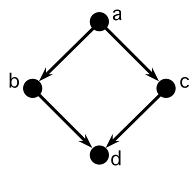
- **Well-ordered sets (Wosets) and well-orders**
  - Let $(T, \leq)$ be a toset. If every non-empty subset of T has a least (minimum) element, then $(T, \leq)$ is called a ***well-ordered set (woset)*** and $\leq$ is called a ***well-order***.
- **Examples**
  - $(N, \leq_N)$ is a well-ordered set (order).
  - $(N, |)$ is not a well-ordered set (order).
- **The axiom of choice and well-ordering theorem**
  - The ***axiom of choice*** [E. Zermelo, 1904]: For any set (family) F of non-empty pairwise disjoint sets, there is a set C (called a ***choice set***) that contains exactly one element in common with each set in F.
  - The ***well-ordering theorem***: Under the axiom of choice, any set can be well-ordered.

Compatible Relations and Pseudo Order Relations

- **Compatible relations**
  - $R : A \to A$ is called a ***compatible relation*** if it is reflexive and symmetric, i.e.,
    for any $a, b \in R$, $(a, a) \in R$, $(a, b) \in R \Rightarrow (b, a) \in R$.
- **Pseudo order (preorder) relations**
  - $\leq : A \to A$ is called a ***pseudo order (preorder) relation*** if it is reflexive and transitive, i.e.,
    for any $a, b, c \in \leq$,
    $(a, a) \in \leq$, $((a, b) \in \leq \wedge (b, c) \in \leq) \Rightarrow (a, c) \in \leq$.

Church-Rosser Property (Diamond Property)

- **Church-Rosser property (Diamond property)**
  - $(\forall a)(\forall b)(\forall c)(((a, b)\in R \wedge (a, c)\in R) \Rightarrow$
    $(\exists d)((b, d)\in R \wedge (c, d)\in R))$
  - Church-Rosser property is a vary useful property in theoretical computer science.



The Notion of Relation: Ternary Relations and N-ary Relations

- **Ternary relations**
  - $R =_{df} R \subseteq A_1 \times A_2 \times A_3$.
  - Example: Parents-Child relationship
    Parents-Child $=_{df} \{(f, m, c) \mid (\exists f)((f \in F) \wedge (\exists m)((m \in M) \wedge (\exists c)((c \in C) \wedge (f, c) \in FC \wedge (m, c) \in MC)\}$
    where F is the set of fathers, M is the set of mothers, C is the set of children, FC is the relation of father-child, and MC is the relation of mother-child.
- **N-ary relations**
  - $R =_{df} R \subseteq A_1 \times A_2 \times \ldots \times A_n$.
  - Example: student record
    SR $=_{df} \{(i, n, d, c1, c2, \ldots,) \mid (\exists i)((i \in ID) \wedge (\exists n)((n \in NAME) \wedge (\exists d)((d \in DEPT) \wedge (\exists c)((c \in COURSE) \wedge \ldots \}$

How to Define the Addition Relation Mathematically?

- **Addition relation on natural numbers**
  - Let N be the set of natural numbers.
  - $AddR$: $N \to N =_{df} AddR \subseteq N \times N$.
- **Addition relation on natural numbers less than 5**
  - $N_5 = \{0, 1, 2, 3, 4\}$
  - $AddR_5$: $N_5 \to N_5$
    $=_{df}$ { (0,0), (0,1), (0,2), (0,3), (0,4),
    (1,0), (1,1), (1,2), (1,3), (0,4),
    (2,0), (2,1), (2,2), (2,3), (2,4),
    (3,0), (3,1), (3,2), (3,3), (3,4),
    (4,0), (4,1), (4,2), (4,3), (4,4) }

The Notion of Function: Binary Functions

♣ **Binary functions (2-ary functions, 1-ary functions (sometime))**

- A *binary function* f from set A (*source*, *from-set*) to set B (*target*, *to-set*) is defined as
  $f: A \to B =_{df} f \subseteq (A \times B) \wedge$
  $(\forall x)(\forall y)(\forall z)((x \in A \wedge y \in B \wedge z \in B) \Rightarrow (((x, y) \in f \wedge (x, z) \in f) \Rightarrow y = z))$.
- For $(x, y) \in f$, $y = f(x)$ is called the *image of x under f*.
- Any binary function is a binary relation (the contrary is NOT necessarily true) and therefore a set of ordered pairs.
- Binary function $f: A \to B$ defines an *abstract binary function* that may have many instances.
- Some (NOT all !) subsets of $A \times B$ define *concrete binary functions* from A to B.

The Notion of Function: Binary Functions

♣ **Domain and range of a binary function $f: A \to B$**

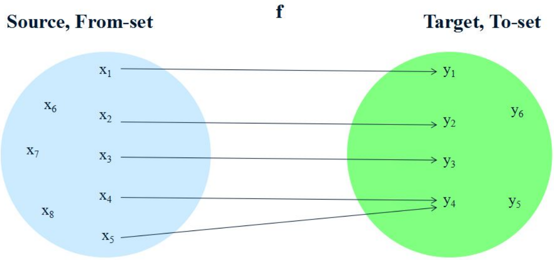
- *Domain*: $\mathrm{dom}(f) =_{df} \{a \mid (\exists b)((a, b) \in f)\}$, $\mathrm{dom}(f) \subseteq A$
- *Range*: $\mathrm{ran}(f) =_{df} \{b \mid (\exists a)((a, b) \in f)\}$, $\mathrm{ran}(f) \subseteq B$
- *Image*: For $A' \subseteq A$, $f(A') =_{df} \{b \mid (\exists a)((a \in A') \wedge ((a, b) \in f))\}$, $f(A') \subseteq \mathrm{ran}(f) \subseteq B$

The Source, From-set, Target, and To-set of a Function

♣ **Binary functions**

- $f: X \to Y =_{df} f \subseteq X \times Y$.

♣ **Source (From-set) and target (To-set) of function $f: X \to Y$**

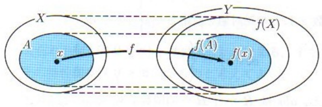


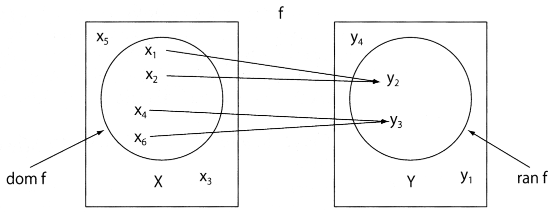
The Source, From-set, Target, and To-set of a Function

♣ **Binary functions**

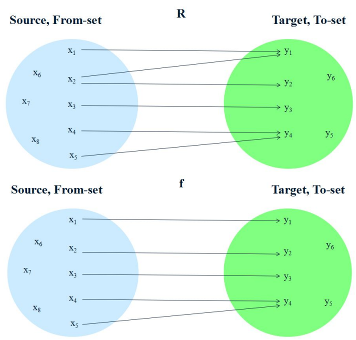
- $f: X \to Y =_{df} f \subseteq X \times Y$.

♣ **Domain and range of function $f: X \to Y$**





Difference between Relations and Functions



How to Define the Addition Function Mathematically?

♣ **Addition function on natural numbers**

- Let N be the set of natural numbers.
- $AddF: (N \to N) \to N =_{df} AddF \subseteq N \times N \times N$.

♣ **Addition function on natural numbers less than 5**

- $N_5 = \{0, 1, 2, 3, 4\}$, $N_9 = \{0, 1, 2, 3, 4, ..., 8\}$
- $AddF_5: (N_5 \to N_5) \to N_9$
  $=_{df}$ { (0,0,0), (0,1,1), (0,2,2), (0,3,3), (0,4,4),
  (1,0,1), (1,1,2), (1,2,3), (1,3,4), (1,4,5),
  (2,0,2), (2,1,3), (2,2,4), (2,3,5), (2,4,6),
  (3,0,3), (3,1,4), (3,2,5), (3,3,6), (3,4,7),
  (4,0,4), (4,1,5), (4,2,6), (4,3,7), (4,4,8) }

Injections (Injective Functions)

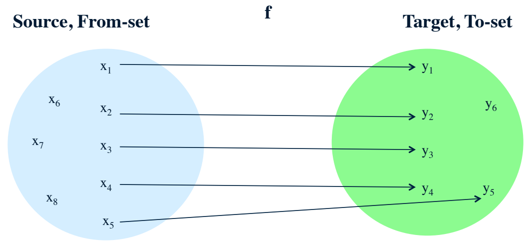
♣ Injections (Injective functions)

- An *injection*, or *injective function*, or "*one-to-one*", is a function f: $A \rightarrow B$ which maps different values of the source to different values of the target, i.e., it must satisfy the following condition:
  $(\forall x)(\forall y)(\forall u)(\forall v)((x,y \in A \wedge u,v \in B) \Rightarrow$
  $(((x, u) \in f \wedge (y, v) \in f \wedge x \neq y) \Rightarrow u \neq v))$.

♣ Examples

- $A = \{1, 2, 3, 4, 5\}$, $B = \{r, s, t, x, y, z\}$
  f: $A \rightarrow B$ ($f \subseteq A \times B$) = $\{(1, x), (2, y), (3, z), (4, r), (5, s)\}$ is an injection.
- $A = \{1, 2, 3, 4, 5\}$, $B = \{x, y, z\}$
  f: $A \rightarrow B$ ($f \subseteq A \times B$) = $\{(1, x), (2, y), (3, x), (4, x), (5, y)\}$ is not an injection.

Injections (Injective Functions)



Surjections (Surjective Functions)

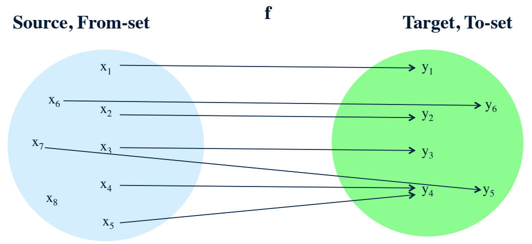
♣ Surjections (Surjective functions)

- A *surjection*, or *surjective function*, or "*on-to*", is a function f: $A \rightarrow B$ for which its range is the whole of its target, i.e., it must satisfy the following condition: ran(f) = B.

♣ Examples

- $A = \{1, 2, 3, 4, 5\}$, $B = \{x, y, z\}$
  f: $A \rightarrow B$ ($f \subseteq A \times B$) = $\{(1, x), (2, y), (3, z), (4, x), (5, y)\}$ is a surjection.
- $A = \{1, 2, 3, 4, 5\}$, $B = \{x, y, z\}$
  f: $A \rightarrow B$ ($f \subseteq A \times B$) = $\{(1, x), (2, y), (3, x), (4, x), (5, y)\}$ is not a surjection.

Surjections (Surjective Functions)



Partial Functions and Total Functions

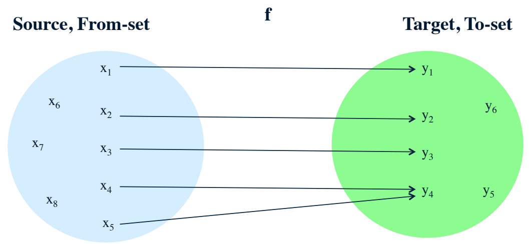
♣ Partial functions

- A *partial function* is a function f: $A \rightarrow B$ for which its domain is the proper subset of its source, i.e., it must satisfy the following condition: dom(f) $\subset A$.

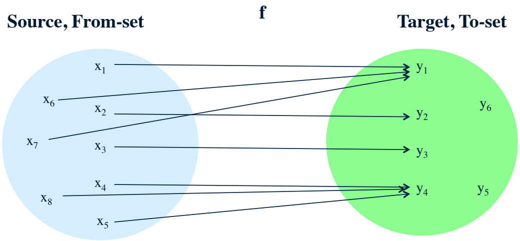
♣ Total functions

- A *total function* is a function f: $A \rightarrow B$ for which its domain is the whole of its sourse, i.e., it must satisfy the following condition: dom(f) = A.

Partial Functions



Total Functions



Bijections (Bijective Functions)

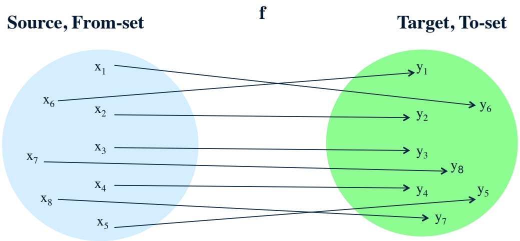
♣ **Bijections**

- **A *bijection*, or *bijective function*, or "*one-to-one correspondence*" is a function which maps every element of the source on to every element of the target in a one-to-one relationship.**
- **A *bijection* is injective, surjective, and total.**

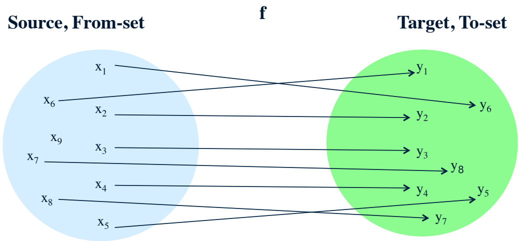
♣ **Invertible functions**

- **A function f: $A \rightarrow B$ is called to be *invertible*, if f^{-1} : $B \rightarrow A$ is a function.**
- **A function is invertible IFF it is a bijection.**

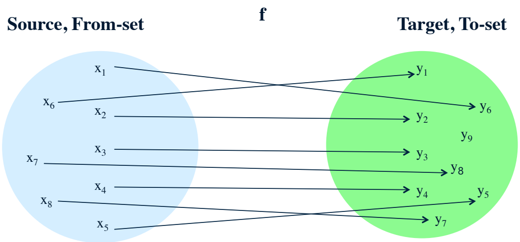
Bijections (Injective, Surjective, and Total Functions)



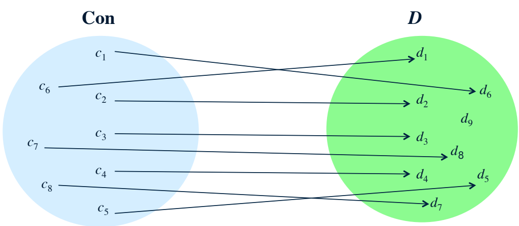
Injective and Surjective Functions



Injective and Total Functions



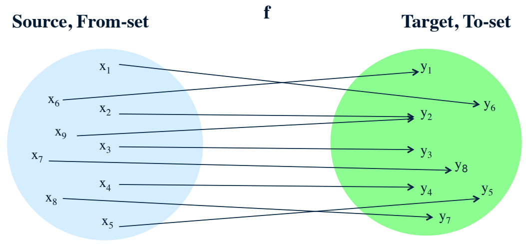
An Injective and Total Function *I*



injective and total function ***I***

$c_1{}^{I} = d_6$, ……

Surjective and Total Functions



Order-Preserving Functions

♣ **Order-preserving functions**

◆ Let R be a relation on A and S be a relation on B. A function f: $A \to B$ is called an ***order-preserving function*** if for all a_1, $a_2 \in A$, $(a_1, a_2) \in R$ IFF $(f(a_1), f(a_2)) \in S$, and we say that f ***preserves*** R with S.

♣ **Order isomorphism**

◆ An order-preserving bijection is also called an ***order isomorphism***.

◆ We say that (A, R) is ***order isomorphic*** to (B, S) and write $(A, R) \cong (B, S)$, if there exists an order isomorphism preserving R with S. Sometimes (A, R) and (B, S) are said to have the same order type when they are order isomorphic.

◆ An isomorphism pairs elements from two sets in such a way that the orders on the two sets appear to be the same.

Equipotent Relations and Cardinalities of Infinite Sets

♣ **Equipotent (equipollent) relations**

◆ Two (infinite) sets A and B are said to be ***equipotent (equipollent)***, denoted by $A \cong B$, IFF there is a bijection between A and B.

◆ Theorem: Equipotent relations are equivalence relations.

♣ **The cardinality (power, cardinal number) of an infinite set**

◆ Two infinite sets A and B are said to have the same cardinality (power), written as $c(A) = c(B)$, if $A \cong B$.

♣ **The first-property of infinite sets**

◆ There is a bijection between an infinite set and one of its own proper subsets, i.e, an infinite set may equipotent to (have the same cardinality of) one of its own proper subsets.

◆ Note: This first-property can be used to define infinite sets.

Enumerable (Countable) Sets

♣ **Enumerable (Countable) sets**

◆ An infinite set A is said to be ***enumerable (countable)*** if $A \cong N$.

◆ Cantor first used symbol "$\aleph_0$" (aleph-zero, aleph-nought) to represent the cardinality of the set N of natural numbers (and any enumerable (countable) set).

♣ Examples of enumerable (countable) sets

◆ Odd number set N_{odd}, Even number set N_{even},
any infinite subset of N,
Integer set Z,
the set of the nth power of integers,
Rational number set Q,
Algebraic number set.

Properties of Enumerable (Countable) Sets

◆ Any infinite subset of an enumerable (countable) set is also a an enumerable (countable) set.

◆ The sum (difference) of an enumerable (countable) set and a finite set is also an enumerable (countable) set.

◆ The sum of finite enumerable (countable) sets is also enumerable (countable).

◆ The direct product of finite enumerable (countable) sets is also enumerable (countable).

◆ The direct product of enumerable (countable) number of enumerable (countable) sets is not an enumerable (countable) set!

Cardinality Operations

♣ **Sum of cardinality**

◆ The cardinality of the disjoint join (union, sum) of two (infinite) sets A and B, $A + B$, is written as $c(A) + c(B)$, and called the ***sum*** of $c(A)$ and $c(B)$.

◆ Theorem: $\aleph_0 + \aleph_0 = \aleph_0$, $n + \aleph_0 = \aleph_0$.

♣ **Product of cardinality**

◆ The cardinality of the direct (Cartesian) product of two (infinite) sets A and B, $A \times B$, is written as $c(A) \bullet c(B)$, and called the ***product*** of $c(A)$ and $c(B)$.

◆ Theorem: $\aleph_0 \bullet \aleph_0 = \aleph_0$, $n \bullet \aleph_0 = \aleph_0$.

Properties of Cardinality Operations [Lipschutz]

Cardinal numbers	Sets
(1) $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$	(1) $(A\cup B)\cup C=A\cup(B\cup C)$
(2) $\alpha+\beta=\beta+\alpha$	(2) $A\cup B=B\cup A$
(3) $(\alpha\beta)\gamma=\alpha(\beta\gamma)$	(3) $(A\times B)\times C\approx A\times(B\times C)$
(4) $\alpha\beta=\beta\alpha$	(4) $A\times B\approx B\times A$
(5) $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$	(5) $A\times(B\cup C)=(A\times B)\cup(A\times C)$
(6) If $\alpha\leq\beta$, then $\alpha+\gamma\leq\beta+\gamma$	(6) If $A\subseteq B$, then $(A\cup C)\subseteq(B\cup C)$
(7) If $\alpha\leq\beta$, then $\alpha\gamma\leq\beta\gamma$	(7) If $A\subseteq B$, then $(A\times C)\subseteq(B\times C)$

Ordering of Cardinalities

♣ **Cardinality comparison (smaller cardinality)**

◆ For any two (infinite) sets A and B, if there is an injection from A to B, then we say that the set A *has smaller cardinality than* the set B and denote as $c(A) \leq c(B)$.

♣ **Cardinality comparison (strictly smaller cardinality)**

◆ For any two (infinite) sets A and B, if $c(A) \leq c(B)$ and $c(A) \neq c(B)$ (i.e., there is no bijection between A and B), then say that the set A *has strictly smaller cardinality than* the set B and denote as $c(A) < c(B)$.

Cantor's Theorem

♣ **Schröder-Bernshtein theorem**

◆ If there is an injection from A to B and an injection from A to B, then there is a bijection from A to B.

◆ If $c(A) \leq c(B)$ and $c(B) \leq c(A)$, then $c(A) = c(B)$.

◆ The smaller cardinality relation $\leq$ is a partial order relation.

♣ **Theorem (Law of cardinality trichotomy)**

◆ For any two infinite sets A and B, exactly one of the following holds: $c(A) < c(B)$, $c(B) < c(A)$, or $c(A) = c(B)$.

♣ **Cantor's theorem**

◆ For any set A, there is an injection from A to its power set $P(A)$ but no bijection between these sets, $c(A) < c(P(A))$.

◆ Corollary: $\aleph_0 < c(P(N))$.

The Cardinality of Real Number Set R

♣ **The cardinality of continuum**

◆ The cardinality of real number set R (also called *continuum*) is represented by $\aleph$.

◆ Theorem (by Cantor): $R[0,1]$ is not enumerable (countable).

◆ Theorem (by Cantor): R is not enumerable (countable).

◆ Theorem (by Cantor): $\aleph = c(P(N))$.

♣ Examples of sets with the cardinality $\aleph$

◆ $c(R - Q) = \aleph$.

◆ $c(R\ [0, 1]) = \aleph$.

◆ c(the set of all points on the plane) $= \aleph$.

The Continuum Hypothesis (CH)

♣ **The continuum hypothesis (CH)**

◆ $\aleph_0 < c(P(N))$ (Cantor's theorem; Recall $\aleph = c(P(N))$)

◆ Let us list all cardinalities according to the cardinality comparison relation $\leq$ as $\aleph_0, \aleph_1, \ldots, \aleph_\omega, \ldots$, then $\aleph_1 \leq c(P(N))$.

◆ $\aleph_1 = \aleph = c(P(N))$? (There is no $\aleph'$ such that $\aleph_0 < \aleph' < \aleph$)

♣ **The general continuum hypothesis (GCH)**

◆ If $c(A) = \aleph_\omega$, then $\aleph_{\omega+1} = c(P(A))$?

◆ The CH/GCH is the biggest open problem in modern Set Theory.

Russell's Paradox [Cameron]

♣ **Frege's work**

◆ The logician Gottlob Frege was the first to develop mathematics on the foundation of set theory. He learned of Russell's Paradox while his work was in press, and wrote as the follows:
"A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr Bertrand Russell as the work was nearly through the press."

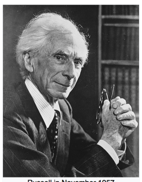
♣ **Russell's question**

◆ Russell asked: Let S be the set of all sets which are not members of themselves. Is S a member of itself?

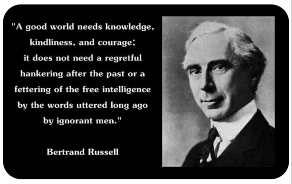
Russell's Paradox

♣ *Russell's Paradox* [Russell, 1901] [SEP]

- Russell's paradox is the most famous of the logical or set-theoretical paradoxes. Also known as the Russell-Zermelo paradox, the paradox arises within naïve set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself. Hence the paradox.



Russell in November 1957



Russell's Paradox

♣ A classification of all sets in Naïve Set Theory

- The first type: All sets that do not include itself as an element.
- The second type: All sets that include itself as an element.

♣ Russell's question

- Question: Let $M =_{df} \{x \mid x \notin x\}$. M is a set of the first type, or the second type?
- If M is a set of the first type, then because it does not include itself as an element, i.e., $x \notin x$, therefore, $M \in M$, it should be a set of the second type.
- If M is a set of the second type, then because it includes itself as an element, i.e., $x \in x$, therefore, $M \notin M$, it should be a set of the first type.

An Elementary Introduction to Set Theory

- Naïve Set Theory
- Axiomatic Set Theory
- Ordinal Numbers
- Categories