Discrete Mathematics for Computer Science

Lecture 15: Relation

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Overview of Last Lecture

Linear Recurrence Relations

- Linear homogeneous recurrence relations
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$
- Linear nonhomogeneous recurrence relations
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$
  - $a_n = a_n^{(p)} + a_n^{(h)}$
  - $a_n^{(h)}$: the associated linear homogeneous recurrence relation
  - $a_n^{(p)}$ for certain functions $F(n)$: polynomials and powers of constants.

Generalized Permutations and Combinations

- Permutations with repetition
- Permutations with indistinguishable objects
- Combinations with repetition

Linear Nonhomogeneous Recurrence Relations

Particular solution $a_n^{(p)}$ with certain $F(n)$:

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2 + 1)3^n$.

To compute $a_n^{(h)}$:
The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

This characteristic equation has a single root $r = 3$ of multiplicity $m = 2$.

$$a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n.$$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = n^2 2^n$:

Since $s = 2$ is not a root of the characteristic equation, we have

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0)2^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

$$\begin{aligned}(p_2 n^2 + p_1 n + p_0)2^n = {} & 6(p_2(n-1)^2 + p_1(n-1) + p_0)2^{n-1} \\ & - 9(p_2(n-2)^2 + p_1(n-2) + p_0)2^{n-2} + n^2 2^n.\end{aligned}$$

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + (p_2 n^2 + p_1 n + p_0)2^n.$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = (n^2+1)3^n$:

Since $s = 3$ is a root of the characteristic equation with multiplicity $m = 2$, we have

$$a_n^{(p)} = n^2(p_2 n^2 + p_1 n + p_0)3^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

...

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + n^2(p_2 n^2 + p_1 n + p_0)3^n.$

Example 2: The Term n^m

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- $a_n^{(p)}$ should be in the form of $np_0 2^n$.
- Try $a_n^{(p)} = p_0 \cdot 2^n$:

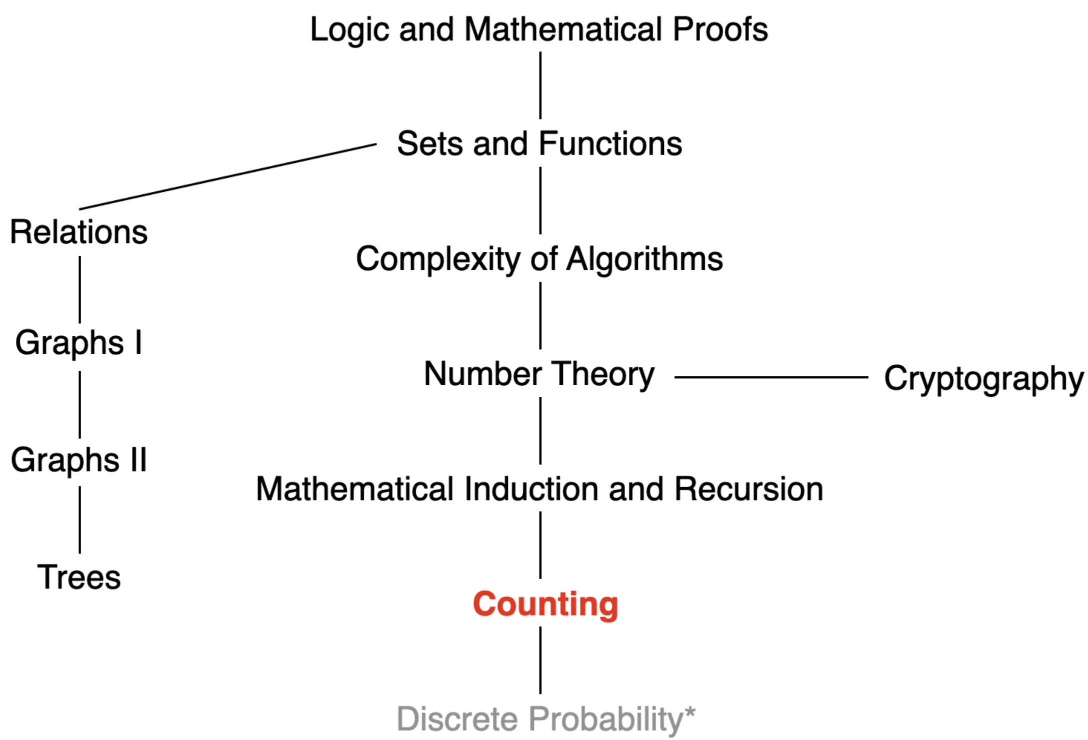
$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2} + 2^n.$$

Since $s = 2$ is a root of the characteristic equation,

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2}$$

always holds. Thus, we obtain $0 = 4$.

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Solving Linear Recurrence Relations, Generalized Permutations and Combinations, Generating Function, ...

Generating Function

- Definition of generating function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Generating Function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \ldots + a_kx^k + \ldots = \sum_{k=0}^{\infty} a_kx^k.$$

Example:

- The sequence $\{a_k\}$ with $a_k = 3$

$$\sum_{k=0}^{\infty} 3x^k$$

- The sequence $\{a_k\}$ with $a_k = 2^k$

$$\sum_{k=0}^{\infty} 2^kx^k$$

Generating Function: Finite Series

A finite sequence $a_0, a_1, \ldots, a_n$ can be easily extended by setting $a_{n+1} = a_{n+2} = \ldots = 0$.

The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n, i.e.,

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Example: What is the generating function for the sequence $a_0, a_1, \ldots, a_m$, with $a_k = C(m, k)$?

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \ldots + C(m, m)x^m.$$

Based on binomial theorem, $G(x) = (1 + x)^m$.

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Generating Function

- Definition of generation function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Useful Facts

- For $|x| < 1$, function $G(x) = 1/(1-x)$ is the generating function of the sequence 1, 1, 1, 1, . . . ,

$$1/(1-x) = 1 + x + x^2 + \dots$$

- For $|ax| < 1$, function $G(x) = 1/(1-ax)$ is the generating function of the sequence $1, a, a^2, a^3, \dots,$

$$1/(1-ax) = 1 + ax + a^2x^2 + \dots$$

- For $|x| < 1$, $G(x) = 1/(1-x)^2$ is the generating function of the sequence 1, 2, 3, 4, 5, . . .

$$1/(1-x)^2 = 1 + 2x + 3x^2 + \dots$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Example 1: To obtain the corresponding sequence of $G(x) = 1/(1-x)^2$: Consider $f(x) = 1/(1-x)$ and $g(x) = 1/(1-x)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, 1, 1, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) x^k.$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x_k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x) g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Example 2: To obtain the corresponding sequence of $G(x) = 1/(1 - ax)^2$ for $|ax| < 1$:

Consider $f(x) = 1/(1 - ax)$ and $g(x) = 1/(1 - ax)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, a, a^2, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) a^k x^k.$$

Useful Generating Functions

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$$
$$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$$
$$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$$
$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$
$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$
$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$
$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Useful Generating Functions

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$
$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$
$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Extended Binomial Coefficient

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Here, u can be any real number, e.g., negative integers, non-integers, ...

Extended Binomial Coefficient

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Example: Find the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Taking $u = -2$ and $k = 3$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Taking $u = 1/2$ and $k = 3$

$$\begin{aligned} \binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16. \end{aligned}$$

Extended Binomial Coefficient

When u is a negative integer:

$$\begin{aligned}
\binom{-n}{r} &= \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \\
&= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} \\
&= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} \\
&= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} \\
&= (-1)^r \binom{n+r-1}{r} \\
&= (-1)^r C(n+r-1, r).
\end{aligned}$$

Extended Binomial Theorem

Theorem: Let x be a real number with $|x| < 1$ and let u be a real number. Then,

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Example:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

Generating Function

- Definition of generation function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Generating Function and Combinations with Repetitions

Recall the following example:

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers?

This type of counting problem can be solved with generating function.

Generating Function and Combinations with Repetitions

Formally, generating functions can also be used to solve counting problems of the following type:

$$e_1 + e_2 + \cdots + e_n = C,$$

where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint.

Example 1

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \leq e_1 \leq 5$, $3 \leq e_2 \leq 6$, and $4 \leq e_3 \leq 7$.

Solution: The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

By enumerating all possibilities, we have that the coefficient of x^{17} in this product is 3.

Example 2

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution: This corresponds to the coefficient of x^8 of expansion

$$(x^2 + x^3 + x^4)^3$$

This coefficient equals 6.

Example 3

Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in the cases

- Case 1: when the order does not matter

  E.g., three \$1 tokens; one \$1 token and a \$2 token

- Case 2: when the order does matter

  E.g., three \$1 tokens; a \$1 token and then a \$2 token; a \$2 token and then a \$1 token

Example 3

Case 1: when the order does not matter
The answer is the coefficient of x^r in the generating function

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots).$$

Case 2: when the order does matter
The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x+x^2+x^5)^n$$

Because any number of tokens may be inserted,

$$1+(x+x^2+x^5)+(x+x^2+x^5)^2+\cdots=\frac{1}{1-(x+x^2+x^5)}$$

Example 4

Use generating functions to find the number of r-combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1+x)^n$$

But by the binomial theorem, we have

$$f(x) = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Thus, $\binom{n}{r}$ is the answer.

Example 5

Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (1 + x + x^2 + \cdots)^n.$$

As long as $|x| < 1$, we have $1 + x + x^2 + \cdots = 1/(1-x)$, so

$$G(x) = 1/(1-x)^n = (1-x)^{-n}.$$

Applying the extended binomial theorem

$$(1-x)^{-n} = (1+(-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

Hence, the coefficient of x^r equals $\binom{-n}{r}(-1)^r = C(n+r-1, r)$.

Example 6

Use generating functions to find the number of ways to select r objects of n different kinds if we must select at least one object of each kind.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (x + x^2 + x^3 + \cdots)^n = x^n(1 + x + x^2 + \cdots)^n = x^n/(1-x)^n.$$

$$\begin{aligned}
G(x) &= x^n/(1-x)^n \\
&= x^n \cdot (1-x)^{-n} \\
&= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\
&= x^n \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)(-1)^r x^r
\end{aligned}
\qquad
\begin{aligned}
&= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r} \\
&= \sum_{t=n}^{\infty} C(t-1, t-n) x^t \\
&= \sum_{r=n}^{\infty} C(r-1, r-n) x^r.
\end{aligned}$$

Hence, there are $C(r-1, r-n)$ ways to select r objects of n different kinds if we must select at least one object of each kind.

Generating Function and Combinations with Repetitions

- Based on the combination problem, transfer the problem as finding the coefficient of x^r of a generating function, e.g.,

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n$$

- Find the coefficient of x^r
  - Enumerate all possibilities or
  - Use useful generating functions

Generating Function

- Definition of generation function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. We aim to first derive the formulation of $G(x)$.

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})x^k \\ &= 2, \end{aligned}$$

Thus, $G(x) - 3xG(x) = (1 - 3x)G(x) = 2$:

$$G(x) = \frac{2}{(1-3x)}.$$

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Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \dots$ and initial condition $a_0 = 2$.

Solution: We aim to first derive the formulation of $G(x)$.

$$G(x) = \frac{2}{(1-3x)}.$$

Then, derive a_k using the identity $1/(1-ax) = \sum_{k=0}^{\infty} a_k x^k$. That is,

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^k$.

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: We extend this sequence by setting $a_0 = 1$. We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1 - 10x),
\end{aligned}$$

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$.

Solution: Thus,

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Thus, $a_n = \frac{1}{2}(8^n + 10^n)$.

Generating function to solve recurrence relations

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Based on the recurrence relations, derive the formulation of $G(x)$.
- Using identities (or the useful facts of generating functions), derive sequence $\{a_k\}$.

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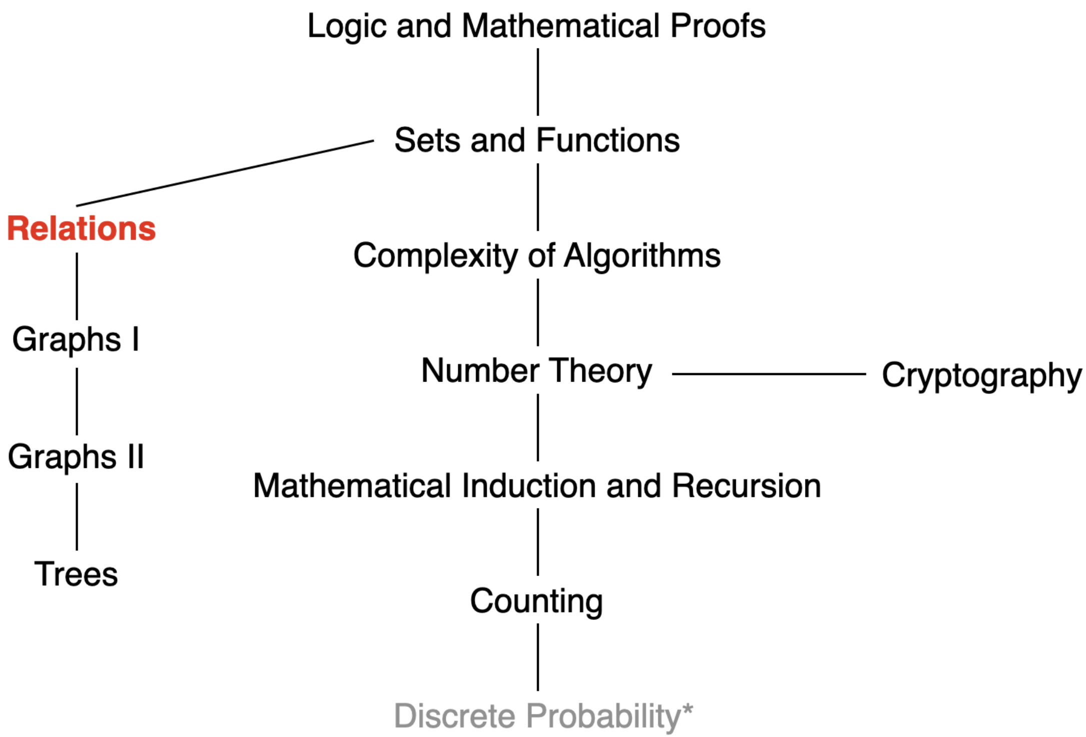


Cartesian Product

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, the Cartesian product $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), ..., (a_1, b_n), ..., (a_m, b_n)\}.$$

Cartesian product defines a set of all ordered arrangements of elements in the two sets.

A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.

Binary Relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

We use the notation aRb to denote $(a, b) \in R$, and $a \not{R} b$ to denote $(a, b) \notin R$.

Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

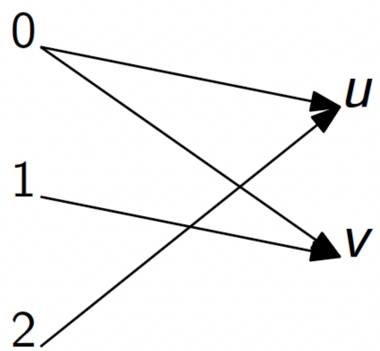
- Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?
- Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?
- Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

Representing Binary Relations

We can graphically represent a binary relation R as:

if aRb, then we draw an arrow from a to b: $a \to b$

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$



Representing Binary Relations

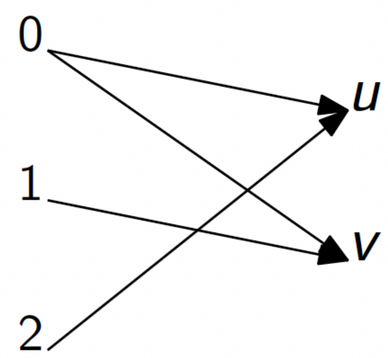
We can also represent a binary relation R by a table showing the ordered pairs of R.

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$

R	u	v
0	×	×
1	×	
2		×

Representing Binary Relations

Relations represent one to many relationships between elements in A and B.



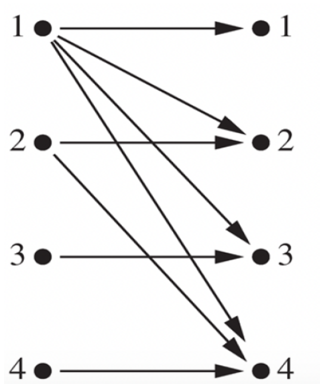
What is the difference between a relation and a function from A to B?

Relation on the Set

Definition: A relation on the set A is a relation from A to itself.

Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a \text{ divides } b\}$. What does R_{div} consist of?

$$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Number of Binary Relations

Theorem: The number of binary relations on a set A, where $|A| = n$, is 2^{n^2}.

Proof: If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

R is a binary relation on A if $R \subseteq A \times A$ (R is subset).

The number of subsets of a set with k elements is 2^k.

Properties of Relations: Reflexive Relation

Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Is R_{div} reflexive?

Yes. $(1,1),(2,2),(3,3),(4,4) \in R_{div}$.

Reflexive Relation

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Is R_{div} reflexive?

Yes. $(1,1), (2,2), (3,3), (4,4) \in R_{div}$.

Relation Matrix (binary matrix):

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Examples

Consider the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations reflexive?

R_1, R_3, and R_4.

Properties of Relations: Irreflexive Relation

Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

Is $R_{\neq}$ irreflexive?

$$R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4),\\ (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}.$$

Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \notin R_{\neq}$.

Irreflexive Relation

Example: Assume that $R_{\neq} = \{(a,b) : a \neq b\}$ on $A = \{1,2,3,4\}$.

Is $R_{\neq}$ irreflexive?

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is irreflexive if and only if MR has 0 in every position on its main diagonal.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations irreflexive?

R_2 and R_5.

Properties of Relations: Symmetric Relation

Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Is R_{div} symmetric?

No. $(1, 2) \in R_{div}$ but $(2, 1) \notin R$.

Symmetric Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),$$
$$(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ symmetric?

Yes. If $(a, b) \in R_{\neq}$ then $(b, a) \in R_{\neq}$.

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is symmetric if and only if MR is symmetric.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a,b) \mid a \le b\},\\
R_2 &= \{(a,b) \mid a > b\},\\
R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a,b) \mid a = b\},\\
R_5 &= \{(a,b) \mid a = b + 1\},\\
R_6 &= \{(a,b) \mid a + b \le 3\}.
\end{aligned}
$$

Which of these relations symmetric?

R_3, R_4, and R_6.

Properties of Relations: Antisymmetric Relation

Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R antisymmetric? Yes.

$$\mathrm{MR} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A relation R is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

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Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations antisymmetric?

R_1, R_2, R_4 and R_5.

Properties of Relations: Transitive Relation

Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Is R_{div} transitive?

Yes. If $a|b$ and $b|c$, then $a|c$.

Transitive Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),$$
$$(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ transitive?

No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

Transitive Relation

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R transitive?

Yes.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a,b) \mid a \le b\},\\
R_2 &= \{(a,b) \mid a > b\},\\
R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a,b) \mid a = b\},\\
R_5 &= \{(a,b) \mid a = b + 1\},\\
R_6 &= \{(a,b) \mid a + b \le 3\}.
\end{aligned}
$$

Which of these relations transitive?

R_1, R_2, R_3 and R_4.

Combining Relations

Since relations are sets, we can combine relations via set operations.

Set operations: union, intersection, difference, etc.

Example: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
$R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$,
$R_2 = \{(1, v), (3, u), (3, v)\}$

What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

Combining Relations

Example: $R_1 = \{(x, y)|x < y\}$ and $R_2 = \{(x, y)|x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R1 \oplus R_2$?

- $R_1 \cup R_2 = \{(x, y)|x \neq y\}$
- $R_1 \cap R_2 = \emptyset$
- $R_1 - R_2 = R_1$
- $R_2 - R_1 = R_2$
- $R_1 \oplus R_2 = \{(x, y)|x \neq y\}$

Composite of Relations

Definition: Let R be a relation from a set A to a set B and S be a relation from B to C. The composite of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We denote the composite of R and S by $S \circ R$.

Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$:

- $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$
- $S = \{(0, b), (1, a), (2, b)\}$
- $S \circ R = \{(1, b), (3, a), (3, b)\}$

Next Lecture