Cryptography

Cryptography = kryptos + graphos

- kryptos: secret
- graphos: writing

One-sentence definition: "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called adversaries." – Ronald L. Rivest

Shift Ciphers

Shift Ciphers: Make messages secret by shifting each letter several letters forward in the alphabet.

Encryption:

- Assign each letter an integer $p \in Z_{26} = \{0, 1, ..., 25\}$ based on the location of the letter in the alphabet.
- Replace p with $f(p)$:

$$f(p) = (p + k) \bmod 26.$$

- Maps $f(p)$ back to the alphabet.

Decryption:

- Assign each letter an integer $p \in Z_{26} = \{0, 1, ..., 25\}$ based on the location of the letter in the alphabet.
- Replace p with $f^{-1}(p)$:

$$f^{-1}(p) = (p - k) \bmod 26.$$

- Maps $f^{-1}(p)$ back to the alphabet.

We can generalize shift ciphers further to slightly enhance security by using a function of the form

$$f(p) = (ap + b) \bmod 26.$$

仿射变换.

How about the decryption? Suppose $\gcd(a, 26) = 1$.

Suppose that $c = (ap + b) \bmod 26$ with $\gcd(a, 26) = 1$. To decrypt we need to show how to express p in terms of c. That is, we solve the congruence for p:

$$c \equiv ap + b \pmod{26}.$$

Subtract b from both sides, we have $c - b \equiv ap \pmod{26}$. Since $\gcd(a, 26) = 1$, we know that there is an inverse $\bar{a}$ of a modulo 26:

$$p \equiv \bar{a}(c - b) \pmod{26}.$$

Crytanalysis

The process of recovering plaintext from ciphertext without knowledge of both the encryption method and the key is known as crytanalysis or breaking codes.

How to break messages that were encrypted using a shift cipher?

Solution 1: Try each 26 possible shifts.

Solution 2: Try different values of k based on the frequency of letters in the ciphertext. The nine most common letters in English text: E: 13%, T: 9%, A: 8%, O: 8%, I: 7%, N: 7%, S: 7%, H: 6%, and R: 6%.

Private Key Cryptosystem

In a **private key cryptosystem**, once you know an encryption key, you can quickly find the decryption key.

When a private key cryptosystem is used, two parties who wish to communicate in secret must share a secret key.

Any problems?

- Two people who want to communicate securely need to securely exchange this key.
- New key is used for each communication session between two parties.

私钥密码系统中，密钥一旦被发现，解钥便很容易解出.

Public Key Cryptography

In **public key cryptosystems**, knowing how to send an encrypted message does not help decrypt messages.

- Public key is known to the public.
- Private key is kept secret: only the intended recipient of a message can decrypt it.

公钥密码系统中不需要交换密钥.

RSA Encryption

1 Translate a plaintext message into integers, each with two digits, e.g, A is translated into 00, B into 01, . . . , and Z into 25.

2 Divide this string into equally sized blocks of $2N$ digits
   - $2N$ is the largest even number such that the number 2525...25 with $2N$ digits does not exceed n.

3 For each block, transform it into a ciphertext block:

$$C = M^e \bmod n$$

Encrypt the message "STOP" with key $(n = 2537, e = 13)$. Note that $2537 = 43 \cdot 59$, where $p = 43$ and $q = 59$ are primes, and $\gcd(e, (p-1)(q-1)) = 1$.

Solution:

1 Translate into integers: 18191415

2 Divide this into blocks of 4 digits (because $2525 < 2537 < 252525$): 1819 1415

3 Encrypt each block using the mapping

$$C = M^{13} \bmod 2537.$$

We have $1819^{13} \bmod 2537 = 2081$ and $1415^{13} \bmod 2537 = 2182$. The encrypted message is 2081 2182.

RSA Decryption

For each block, transform the ciphertext into plaintext message:

$$M = C^d \bmod n$$

Example: What is the decrypted message of 0981 0461 with $e = 13$, $p = 43$, $q = 59$?

Solution: Recall that $ed \equiv 1 \pmod{(p-1)(q-1)}$. Thus, $d = 937$ is an inverse of 13 modulo $42 \cdot 58 = 2436$.

For each block, transform it into plaintext message:

$$M = C^{937} \bmod 2537.$$

Since $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$, the plaintext message is 0704 1115, which is "HELP".

RAS Cryptosystem

Pick two large primes p and q. Let $n = pq$. **Encryption key** (n, e) and **decryption key** (n, d) are selected such that

(1) $\gcd(e, (p-1)(q-1)) = 1$

(2) $ed \equiv 1 \pmod{(p-1)(q-1)}$

RSA encryption: $C = M^e \bmod n$;
RSA decryption: $M = C^d \bmod n$. Why?

According to (1), the inverse d exists. According to (2), there exists an integer k such that

$$de = 1 + k(p-1)(q-1).$$

It follows that $C^d \equiv (M^e)^d = M^{de} = M^{1+k(p-1)(q-1)} \pmod{n}$.

Assuming that $\gcd(M, p) = \gcd(M, q) = 1$, we have $M^{p-1} \equiv 1 \pmod{p}$ and $M^{q-1} \equiv 1 \pmod{q}$. (see Theorem 3 in Section 4.4)

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$$C^d \equiv M \cdot (M^{p-1})^{k(q-1)} \equiv M \cdot 1 = M \pmod{p}$$
$$C^d \equiv M \cdot (M^{q-1})^{k(p-1)} \equiv M \cdot 1 = M \pmod{q}.$$

Because $\gcd(p, q) = 1$, we have

$$C^d \equiv M \pmod{pq}.$$

This basically implies that

$$M = C^d \bmod n$$

RSA as Public Key System

RSA as a Public Key System

- Public key: (n, e); Private key: d
- p, q must be kept secret!

Why is the RSA cryptosystem suitable for public key cryptography?

- It is possible to rapidly construct a public key by finding two large primes p and q, each with more than 200 digits.
- When we know p and q, we can quickly find an inverse d.
- However, no method is known to decrypt messages that is not based on finding a factorization of n.
  - Factorization is believed to be a difficult problem.
  - The most efficient factorization methods known (as of 2010) require billions of years to factor 400-digit integers.

RSA as Digital Signature

Alice's RSA public key is (n, e) and her private key is d.

Alice splits the plaintext message into blocks and applies her decryption function:

$$S = M^d \bmod n \quad (\textbf{RSA signature})$$

When a recipient receives her message, they apply Alice's encryption function:

$$M = S^e \bmod n \quad (\textbf{RSA verification})$$

Alice can send her message to as many people as she wants and by signing it in this way, every recipient can be sure it came from Alice.

Diffie-Hellman Key Exchange Protocol

Two parties exchange a secret key over an insecure communications channel without having shared any information in the past.

Diffie-Hellman Key Exchange Protocol

Before introducing the protocol:

Definition: A primitive root modulo a prime p is an integer r in Z_p such that every nonzero element of Z_p is a power of r.

Example: Whether 2 is a primitive root modulo 11?

When we compute the powers of 2 in Z_{11}, we obtain $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 5$, $2^5 = 10$, $2^6 = 9$, $2^7 = 7$, $2^8 = 3$, $2^9 = 6$, $2^{10} = 1$.
Because every element of Z_{11} is a power of 2, 2 is a primitive root of 11.

Suppose that Alice and Bob want to share a common key. Consider Z_p.

(1) Alice and Bob agree to use a prime p and a primitive root a of p.
(2) Alice chooses a secret integer k_1 and sends a^{k_1} **mod** p to Bob.
(3) Bob chooses a secret integer k_2 and sends a^{k_2} **mod** p to Alice.
(4) Alice computes $(a^{k_2})^{k_1}$ **mod** p.
(5) Bob computes $(a^{k_1})^{k_2}$ **mod** p.

Alice and Bob have computed their shared key:

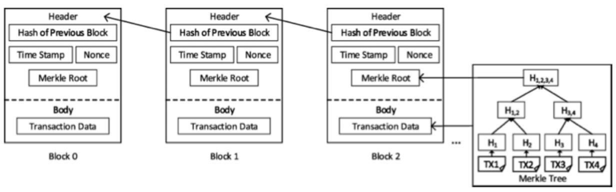
$$(a^{k_2})^{k_1} \ \mathbf{mod}\ p = (a^{k_1})^{k_2} \ \mathbf{mod}\ p.$$

- Public information: p, a, a^{k_1} **mod** p, and a^{k_2} **mod** p
- Secret: k_1, k_2, $(a^{k_2})^{k_1}$ **mod** $p = (a^{k_1})^{k_2}$ **mod** p

Note that it is very hard to determine k_1 with a, p, and a^{k_1} **mod** p.

Blockchain

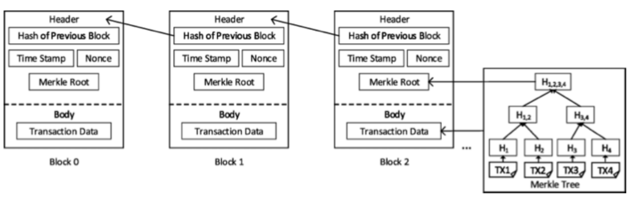
A blockchain is a decentralized, distributed, and oftentimes public, digital ledger consisting of records called blocks that are used to record transactions (or data) across many computers.



Any involved block cannot be altered retroactively, without the alteration of all subsequent blocks.

Mining

Given a set of transactions, generate a new block:



Proof of Work: Take the current block's header, guess the nonce such that the hash of the header (SHA-256) smaller than a target value.

The target can be changed to adjust the difficulty of mining.
Transaction data cannot be tampered:

- If transaction data is tampered, then the Merkle root needs to be changed;
- Then nonce needs to be changed. (very difficult!)
- Even if a nonce is found, the hash of the header is changed
- The header of the subsequent block also needs to be changed.
- ...