CS201 · Discrete Mathematics

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Assignment 4

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1 Q1

- For $n = 1$, $\overline{A_1} = \overline{A}_1$ is established.
- For $n = 2$, $\overline{A_1 \cup A_2} = \overline{A}_1 \cap \overline{A}_2$ is always true by De Morgan's law.
- For $n = k \geq 3$, suppose $\overline{A_1 \cup A_2 \cup \ldots \cup A_k} = \overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_k$ is true. Then for $n = k+1$, we have

$$\begin{aligned}\overline{A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}} &= \overline{A_1 \cup A_2 \cup \ldots \cup A_k} \cap \overline{A}_{k+1} \\ &= \overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_k \cap \overline{A}_{k+1}\end{aligned}$$

Thus, De Morgen's law can be generalized to

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n$$

2 Q2

- For $n = 1$, left $= a - b \leq a - b =$ right is established.
- For $n = k \geq 2$, where k is a positive integer, suppose $a^k - b^k \leq ka^{k-1}(a-b)$ holds. Then for $n = k+1$, where $k+1$ is a positive integer, since $0 < b < a$, we have

$$\begin{aligned}a^{k+1} - b^{k+1} &= (a-b)(a^k + a^{k-1}b + \ldots + ab^{k-1} + b^k) \\ &< (a-b)\underbrace{(a^k + a^k + \ldots + a^k + a^k)}_{k+1} \\ &= (a-b)(k+1)a^k = (k+1)a^k(a-b)\end{aligned}$$

Thus, for n a positive integer, $a^n - b^n \leq na^{n-1}(a-b)$ is true.

3 Q3

First we can try listing the amounts of money:

j	1	2	3	4	5	6	7	8	9	...
p	0	10	20	25	35	45	50	60	70	...
m	0	1	2	0	1	2	0	1	2	...
n	0	0	0	1	1	1	2	2	2	...

where $p = 10m + 25n$, m is the amounts of \$10 certificates, n is the amounts of \$25 certificates and p is the amounts of money.
Then we can guess that $p(n) = 10 \cdot ((n-1) \mod 3) + 25 \cdot ((n-1) \text{ div } 3)$, and we can prove it by using strong induction.
∘ $p(1) = 0, p(2) = 10$ is obvious from the list. Then suppose for $3 \leq j \leq k$, we have $p(j) = 10 \cdot ((j-1) \mod 3) + 25 \cdot ((n-1) \text{ div } 3)$.
∘ For $k+1$, we have two situations:

- If $(k-1) \mod 3 = 0$ or 1, then $k \mod 3 = 1$ or 2 and $(k-1)$ div $3 = k$ div 3. Thus, we have

$$\begin{aligned}p(k+1) = p(k) + 10 &= 10 \cdot ((k-1) \mod 3) + 25 \cdot ((k-1)\ div\ 3) + 10 \\ &= 10 \cdot (k \mod 3) + 25 \cdot (k\ div\ 3)\end{aligned}$$

- If $(k-1) \mod 3 = 2$, then $k \mod 3 = 0$ and k div $3 = (k-1)$ div $3 + 1$. Thus, we have

$$\begin{aligned} p(k+1) &= p(k) - 20 + 25 = 10 \cdot ((k-1) \mod 3) + 25 \cdot ((k-1)\ div\ 3) - 20 + 25 \\ &= 10 \cdot (k \mod 3) + 25 \cdot (k\ div\ 3) \end{aligned}$$

Above all, we have $p(k+1) = 10 \cdot (k \mod 3) + 25 \cdot (k \text{ div } 3)$, which completes the strong induction. Therefore, $p(n) = 10 \cdot ((n-1) \mod 3) + 25 \cdot ((n-1) \text{ div } 3)$.

4 Q4

Principle of mathematical induction $\implies$ Strong induction:
We can construct a proposition $Q(n) = P(1) \wedge P(2) \wedge ... \wedge P(n)$. From the premise of strong induction, we have $\forall k(Q(k) \rightarrow P(k+1))$. And it is obvious that $Q(k) \rightarrow Q(k)$. Then we have $Q(k) \rightarrow (Q(k) \wedge P(k+1))$, where the latter is equivalent to $Q(k+1)$. So we have $Q(1) \equiv P(1)$ is true, and $Q(k) \rightarrow Q(k+1)$, which satisfies the condition of the principle of mathematical induction. Thus, we can conclude by the principle of mathematical induction that $\forall n Q(n)$. Since $Q(n) \rightarrow P(n)$, we have $\forall n P(n)$, i.e., strong induction is proved.
Strong induction $\implies$ Principle of mathematical induction:
From the premise of the principle of mathematical induction, we have $\forall k(P(k) \rightarrow P(k+1))$. Then $P(1) \wedge P(2) \wedge ... \wedge P(k) \rightarrow P(k+1)$ is true, which satisfies the condition of strong induction. Thus, we have $\forall n P(n)$, i.e., the principle of mathematical induction is proved.

5 Q5

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fn recur(a: real, n: uint) -> real {
    if n==1 {
        return a*a;
    }
    return recur(a, n-1)*recur(a, n-1);
}
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6 Q6

$$\begin{aligned} f(n) &= 5f(\frac{n}{4}) + 6n \\ &= 5(5f(\frac{n}{4^2}) + 6 \cdot \frac{n}{4}) + 6n = 5^2 f(\frac{n}{4^2}) + \frac{5}{4} \cdot 6n + 6n \\ &= 5^2(5f(\frac{n}{4^3}) + 6 \cdot \frac{n}{4^2}) + \frac{5}{4} \cdot 6n + 6n = 5^3 f(\frac{n}{4^3}) + \frac{5^2}{4^2} \cdot 6n + \frac{5}{4} \cdot 6n + 6n \\ &\vdots \\ &= 5^{\log_4 n} f(1) + \frac{5^{\log_4 n - 1}}{4^{\log_4 n - 1}} \cdot 6n + ... + \frac{5}{4} \cdot 6n + 6n \\ &= 5^{\log_4 n} + 6n \sum_{i=0}^{\log_4 n - 1} (\frac{5}{4})^i \\ &= 5^{\log_4 n} + 24 \cdot (\frac{5}{4})^{\log_4 n} \cdot n - 24n \\ &= 5^k + 24 \cdot (\frac{5}{4})^k \cdot 4^k - 24 \cdot 4^k \\ &= 25 \cdot 5^k - 24 \cdot 4^k \end{aligned}$$

7 Q7

1.
$$\binom{n}{2} \cdot 2$$

2.
$$\sum_{j=0}^{n-2} \binom{n-2}{j} \cdot 2^j = 3^{n-2}$$

3.
$$n - 1$$

8 Q8

$$C(13,1)C(4,2)C(12,1)C(4,2)C(11,1)C(4,1)C(10,1)C(4,1)$$

9 Q9

By definition, we have

$$\begin{aligned}\binom{240}{120} &= \frac{240!}{120!\,120!} \\ &= \frac{239!}{119!\,120!} \cdot \frac{240}{120} \\ &= \binom{239}{120} \cdot 2\end{aligned}$$

Since $\binom{239}{120}$ is an integer, $\binom{240}{120} = \binom{239}{120} \cdot 2$ can be divided by 2.

In the meanwhile, since $\gcd(121, 241) = 1$, we just need to prove that $\binom{240}{120} \cdot 241$ can be divided by 121.

$$\begin{aligned}\binom{240}{120} \cdot 241 &= \frac{240!}{120!\,120!} \cdot 241 \\ &= \frac{241!}{121!\,120!} \cdot 121 \\ &= \binom{241}{121} \cdot 121\end{aligned}$$

Since $\binom{241}{121}$ is an integer, $\binom{240}{120} \cdot 241 = \binom{241}{121} \cdot 121$ can be divided by 121.

Above all, $\binom{240}{120}$ can be divided by 2 and 121, and $\gcd(2, 121) = 1$, then we can conclude that $\binom{240}{120}$ can be divided by $242 = 2 \cdot 121$.

10 Q10

The possible values for $n \mod 5$ are $0, 1, 2, 3, 4$. Totally, there are $5 \times 5 = 25$ kinds of combinations for ($a \mod 5, b \mod 5$). By pigeonhole principle, it needs at least 26 ordered pairs.

11 Q11

The 4 possible value pair types for (x_i, y_i) are (even, even), (odd, odd), (even, odd), (odd, even). By the pigeonhole principle, there must be two points that have the same value pair type. Since sum of two even or two odd can be divided by 2, there at least one pair of points in it has integer coordinates midpoint.

12 Q12

Suppose there are n ($n \geq 2$) people in the party, then for any one of the n people, he or she may know 0, 1, ..., $n-1$ other people.
Suppose there is no situation for two people who know the same number of other people. We can denote the people know 0 other people as A and the people know $n-1$ other people as B. However, it leads to a contradiction that A is not acquaint with B but B knows A. Therefore, there are two people who know the same number of other people.

13 Q13

The characteristic equation of $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ is

$$\begin{aligned} & r^3 - 2r^2 - r + 2 = 0 \\ \implies & (r-2)(r^2-1) = 0 \end{aligned}$$

its roots are $r_1 = 2, r_2 = 1, r_3 = -1$. Thus, the solution is

$$a_n = \alpha_1 2^n + \alpha_2 + \alpha_3(-1)^n$$

from the initial condition, we have

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 + \alpha_3 = 3 \\ a_1 &= 2\alpha_1 + \alpha_2 - \alpha_3 = 6 \\ a_2 &= 4\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{aligned}$$

by solving it, we have $\alpha_1 = -1, \alpha_2 = 6, \alpha_3 = -2$. Therefore, the solution is

$$a_n = -2^n + 6 + (-2) \cdot (-1)^n$$

14 Q14

The characteristic equation of $a_n = 5a_{n-1} - 6a_{n-2}$ is

$$\begin{aligned} & r^2 - 5r + 6 = 0 \\ \implies & (r-2)(r-3) = 0 \end{aligned}$$

its roots are $r_1 = 2, r_2 = 3$. Thus, the solution is

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

from the initial condition, we have

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 = 1 \\ a_1 &= 2\alpha_1 + 3\alpha_2 = 0 \end{aligned}$$

by solving it, we have $\alpha_1 = 3, \alpha_2 = -2$. Therefore, the solution is

$$a_n = 3 \cdot 2^n + (-2) \cdot 3^n$$

15 Q15

Let $S_n = \{1, 2, ..., n\}$ and let a_n denote the number of non-empty subsets of S_n that contain no two consecutive integers.
Then for $S_{n-1} = \{1, 2, ..., n-1\}$, comparing to S_n, it lacks an integer n, and as a result, the subsets containing n and one integer ranging from 1 to $n-2$ are not counted in a_{n-1}.
And then, we need to consider if we take two non-consecutive integers ranging from 1 to $n-2$ and combine them with n to form a subset, which is obviously counted in a_n but not in a_{n-1}. And for three non-consecutive integers ranging from 1 to $n-2$, for four, five, etc.. However, all the situations are included in a_{n-2}.
After all, the recurrence relation for a_n is

$$a_n = a_{n-1} + a_{n-2}$$

where $a_0 = 0, a_1 = 1$.

16 Q16

By binomial theorem, we have $C(n, r)$ to be the coefficient of x^r in $(1 + x)^n$. At the same time, we have

$$\begin{aligned}(1 + x)^n &= (1 + x)^{n-1} + x(1 + x)^{n-1} \\ &= \sum_{j=0}^{n-1} C(n-1, j)x^{n-1-j} + x \sum_{j=0}^{n-1} C(n-1, j)x^{n-1-j}\end{aligned}$$

in this expansion, the coefficient of x^r is

$$C(n-1, n-r-1) + C(n-1, n-r)$$

where the previous one is from $\sum_{j=0}^{n-1} C(n-1, j)x^{n-1-j}$ taking $j = n-r-1$ and the latter one is from $x \sum_{j=0}^{n-1} C(n-1, j)x^{n-1-j}$ taking $j = n-r$.
By the property of combination, we have $C(n-1, n-r-1) = C(n-1, r)$ and $C(n-1, n-r) = C(n-1, r-1)$.
Thus, the coefficient of x^r is $C(n-1, r) + C(n-1, r-1)$. Therefore, $C(n, r) = C(n-1, r) + C(n-1, r-1)$.