Discrete Mathematics for Computer Science

Lecture 1b: Propositional Logic

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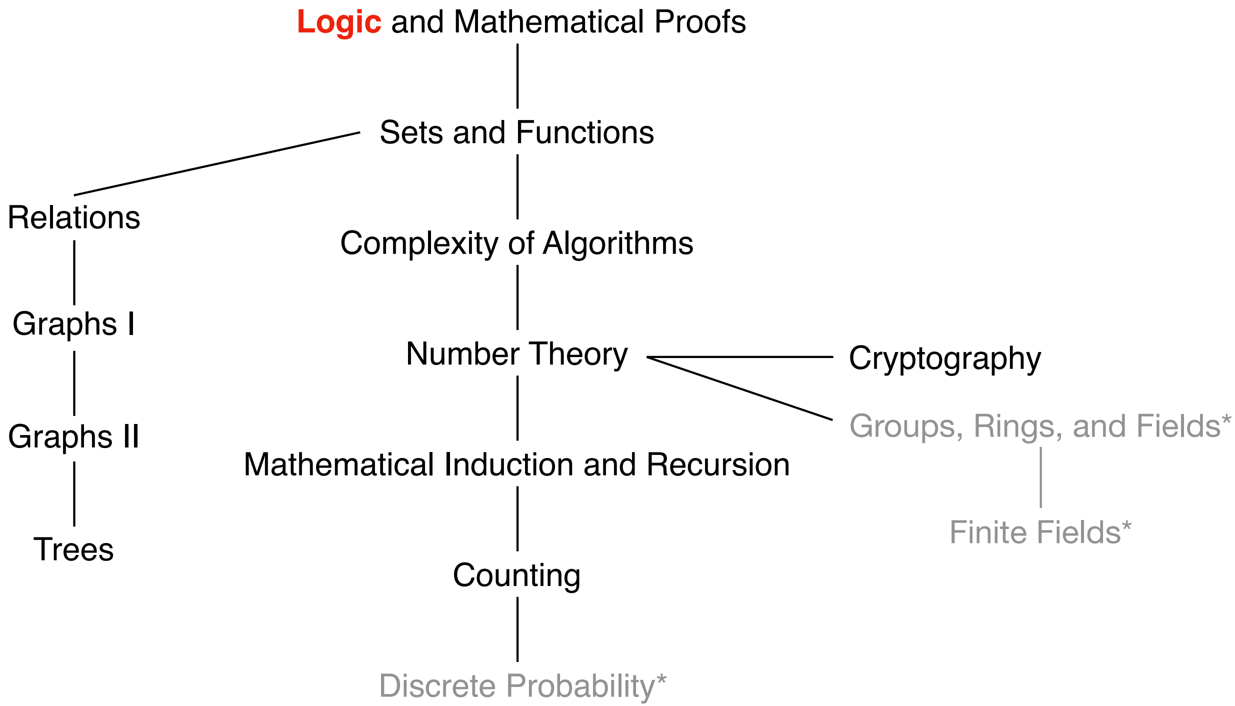
This Lecture

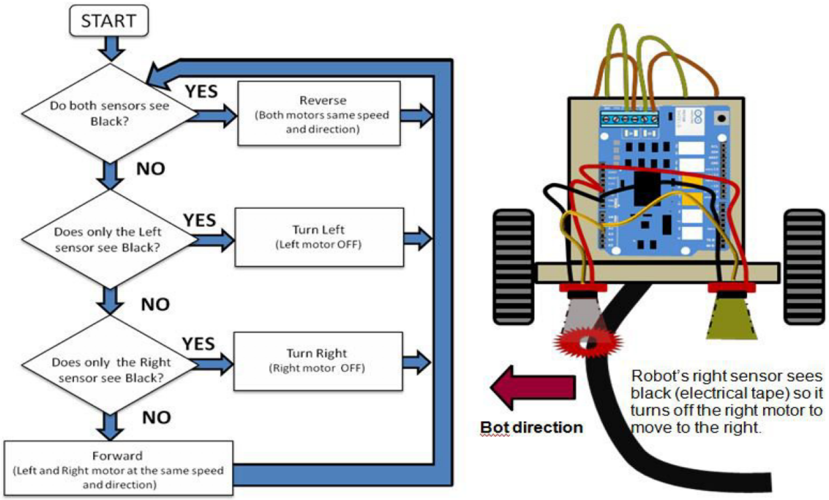


Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

What is Logic?

Logic is the basis of all mathematical reasoning:

- Syntax of statements
- The meaning of statements
- The rules of logical inference



What is Propositional Logic?

Proposition: a declarative sentence that is either true or false (not both).

- Declarative sentence: a sentence that makes a statement, while it does not ask a question or give an order
- Either true or false: fixed; no variable involved

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Propositional variables: variables that represent propositions

- Conventional letters used for propositional variables are p, q, r, s, ...

Examples

Examples of propositions:

- SUSTech is located in Shenzhen. (T)
- $2 + 2 = 3$ (F)
- It is raining today. (either T or F)

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- No parking.
- How old are you?
- $x + 2 = 5$
- Computer x is functioning properly.

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Examples of propositions:

- SUSTech is located in Shenzhen.
- $2 + 2 = 3$
- It is raining today. (The date is specified)

Examples which are not propositions:

- No parking. $\rightarrow$ Not a declarative sentence
- How old are you? $\rightarrow$ Not a declarative sentence
- $x + 2 = 5$ $\rightarrow$ Neither true nor false
- Computer x is functioning properly.
  (Computer "x" is not specified) $\rightarrow$ Neither true nor false

Examples

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- SUSTech is located in Shenzhen.
- $2 + 2 = 3$
- It is raining today.

Examples which are not propositions:

- No parking.
- How old are you?
- $x + 2 = 5$ (Related to predicate logic!)
- Computer x is functioning properly.
  (Related to predicate logic!)

How about the following?

- Do not pass go.
- What time is it?
- There is no pollution in New Jersey.
- $2^n \geq 100$
- 13 is a prime number.

How about the following?

- Do not pass go. Not a proposition
- What time is it? Not a proposition
- There is no pollution in New Jersey. A proposition; either T or F
- $2^n \geq 100$ Not a proposition
- 13 is a prime number. A proposition; T

Compound Propositions

Many mathematical statements are constructed by combining one or more propositions $\rightarrow$ compound propositions.

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- p: It rains outside.
- q: We will watch a movie.
- A new proposition r: If it rains outside, then we will watch a movie.

(Recall that p, q, r are propositional variables that represent propositions.)

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(Recall that p, q, r are propositional variables that represent propositions.)

Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Negation

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> "It is not the case that p."

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Negation of the following propositions?

- $5 + 2 \neq 8$
- 10 is not a prime number.
- Class does not begin at 8:30am.

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- It is not the case that 10 is not a prime number. That is, 10 is a prime number.
- It is not the case that class does not begin at 8:00am. That is, class begins at 8:00am.

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Negation: Truth Table

A **truth table** displays the relationships between truth values (T or F) of different propositions.

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The truth table for the negation of a proposition:

p	$\neg p$
T	F
F	T

A row for each of the two possible truth values of a proposition p

Conjunction (And)

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q".

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

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- q: $5 + 2 = 8$
- $p \wedge q$: SUSTech is located in Shenzhen, and $5 + 2 = 8$

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Example:

- p: SUSTech is located in Shenzhen. (T)
- q: $5 + 2 = 8$ (F)
- $p \wedge q$: SUSTech is located in Shenzhen, and $5 + 2 = 8$ (F)

Conjunction (And)

Conjunction of the following?

- p: Rebecca's PC has more than 16 GB free hard disk space.
- q: The processor in Rebecca's PC runs faster than 1 GHz.

Conjunction (And)

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- p: Rebecca's PC has more than 16 GB free hard disk space.
- q: The processor in Rebecca's PC runs faster than 1 GHz.

Conjunction:

- $p \wedge q$: Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz.

Disjunction (Or)

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q" (inclusive or).

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Disjunction (Or)

Disjunction of the following proposition?

- p: Students who have taken calculus can take this class.
- q: Students who have taken computer science can take this class.

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Disjunction:

- $p \vee q$: Students who have taken calculus or computer science can take this class.

  Note: This is an inclusive or. We mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.

Conjunction and Disjunction: Truth Table

p	q	$p \land q$	$p \lor q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Rows: all possible values of elementary propositions.

If there are n propositional variables, there are 2^n rows.

Exclusive Or

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive Or

Exclusive or of the following proposition?

- p: Students who have taken calculus can take this class.
- q: Students who have taken computer science can take this class.

Exclusive Or

Exclusive or of the following proposition?

- p: Students who have taken calculus can take this class.
- q: Students who have taken computer science can take this class.

Exclusive or:

- $p \oplus q$: Students who have taken calculus or computer science, but not both, can enroll in this class.

Conditional Statement (Implication)

Let p and q be propositions. The conditional statement (a.k.a. implication) $p \rightarrow q$, is the proposition "if p, then q".

Proposition $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

In $p \rightarrow q$, p is called the hypothesis and q is called the conclusion.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statement (Implication)

$p \rightarrow q$ is read in a variety of equivalent ways:

- if p then q
- p implies q
- p is sufficient for q
- q is necessary for p
- q follows from p
- q unless $\neg p$
- p only if q

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- p only if q (Or equivalently, only if you get an A, you may get 100 on the final.)

Example:

- p: you get 100 on the final
- q: you will get an A
- If you get 100 on the final, then you will get an A.

Conditional Statement (Implication)

$p \to q$ is read in a variety of equivalent ways:

- if p then q
- p implies q
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- q unless $\neg p$ (Or equivalently, if you does not get an A, it cannot be the case that you get 100 on the final.)
- p only if q (Or equivalently, only if you get an A, you may get 100 on the final.)

  ("If" indicates sufficient condition; "only if" indicates necessary condition)

Example:

- p: you get 100 on the final
- q: you will get an A
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Conditional Statement (Implication)

- The converse of $p \to q$ is $q \to p$.
- The contrapositive of $p \to q$ is $\neg q \to \neg p$.
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Examples:

- If you get 100 on the final, then you will get an A. $(p \to q)$
- If you get an A, then you get 100 on the final. $(q \to p)$
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Which is equivalent to $p \to q$?

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Which is equivalent to $p \to q$?

$\neg q \to \neg p$ is equivalent to $p \to q$

- **Equivalent** means that two compound propositions always have the same truth value.
- Try to write the truth table of $p \to q$ and $\neg q \to \neg p$?

Equivalent

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Biconditional

Let p and q be propositions. The biconditional statement (a.k.a. bi-implications), denoted by $p \leftrightarrow q$, is the proposition "p if and only if q", is true when p and q have the same truth values, and false otherwise.

- p is necessary and sufficient for q
- if p then q, and conversely
- p iff q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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A Quick Summary of Compound Proposition

A proposition is a declarative statement that is either true or false.

Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Determining the Truth Value

- p: 2 is a prime (T)
- q: 6 is a prime (F)

Determine the truth value of the following:

- $\neg p$
- $p \wedge q$
- $p \wedge \neg q$
- $p \vee q$
- $p \oplus q$
- $p \rightarrow q$
- $q \rightarrow p$

Determining the Truth Value

- p: 2 is a prime (T)
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Determine the truth value of the following:

- $\neg p$ F
- $p \wedge q$ F
- $p \wedge \neg q$ T
- $p \vee q$ T
- $p \oplus q$ T
- $p \rightarrow q$ F
- $q \rightarrow p$ T

Constructing the Truth Table

Construct a truth table for $p \vee q \rightarrow \neg r$

Constructing the Truth Table

Construct a truth table for $p \vee q \to \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \to \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Computer Representation of True and False

- A **bit** is a symbol with two possible values: 0 (false) or 1 (true)
- A variable that takes on values 0 and 1 is called a Boolean variable.
- A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

Computer Representation of True and False

Bitwise operations:

- Each bit is represented by 1 (True) and 0 (False)
- Compute OR ($\vee$), AND ($\wedge$), XOR ($\oplus$) in a bitwise fashion

$$\begin{array}{ll} 01\ 1011\ 0110 & \\ \underline{11\ 0001\ 1101} & \\ & \text{bitwise } \mathit{OR} \\ & \text{bitwise } \mathit{AND} \\ & \text{bitwise } \mathit{XOR} \end{array}$$

Computer Representation of True and False

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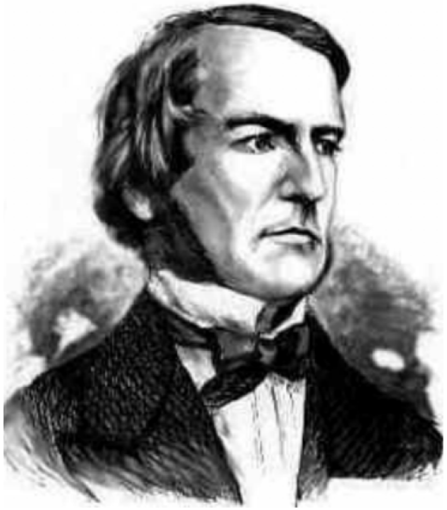
- Each bit is represented by 1 (True) and 0 (False)
- Compute OR ($\vee$), AND ($\wedge$), XOR ($\oplus$) in a bitwise fashion

01 1011 0110	
11 0001 1101	
11 1011 1111	bitwise *OR*
01 0001 0100	bitwise *AND*
10 1010 1011	bitwise *XOR*

George Boole

- British mathematician (b. 1815, d. 1864)
- The inventor of Boolean algebra (Chapter 12 in our textbook)
- Truth tables are an example of Boolean algebra. Other examples contain gates, minimization of circuits, ...



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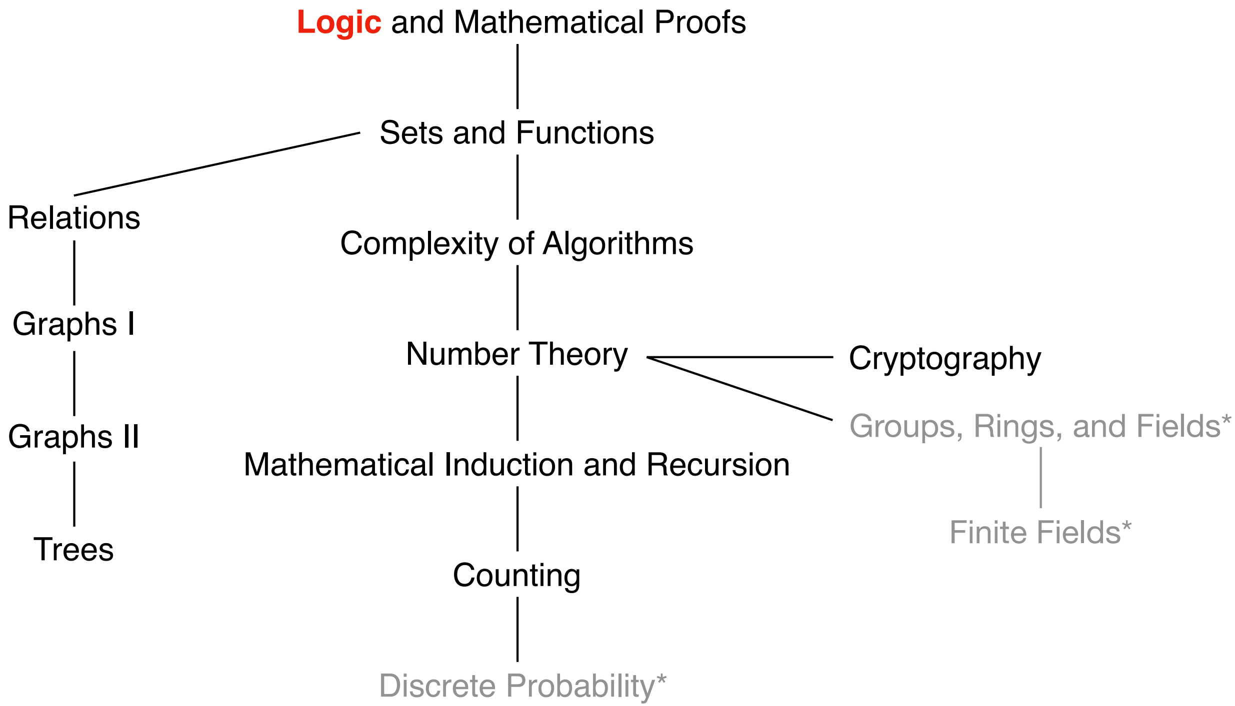

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