Discrete Mathematics for Computer Science

Lecture 23: Review Part 2

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There are a group of people. If we count them by 2's, we have 1 left over; by 3's, we have nothing left; by 4, we have 1 left over; by 5, we have 4 left over; by 6, we have 3 left over; by 7, we have nothing left; by 8, we have 1 left over; by 9, nothing is left. How many people are there? Give the details of your calculation.

$$x \equiv 1 \pmod 2 \qquad x = 2k+1 \quad 2$$
$$x \equiv 0 \pmod 3 \qquad x = 3k \quad 1$$
$$x \equiv 1 \pmod 4 \qquad x = 4k+1 \quad 2$$
$$x \equiv 4 \pmod 5 \qquad x = 5k+4 \quad 3$$
$$x \equiv 3 \pmod 6 \qquad x = 6k+3 \quad 1$$
$$x \equiv 0 \pmod 7 \qquad x = 7k \quad 4$$
$$x \equiv 1 \pmod 8 \qquad x = 8k+1 \quad 2$$
$$x \equiv 0 \pmod 9. \qquad x = 9k \quad 1$$

$$x \equiv 4 \pmod 5$$
$$x \equiv 0 \pmod 7$$
$$x \equiv 1 \pmod 8$$
$$x \equiv 0 \pmod 9.$$

Lecture Schedule

1 Logic
2 Mathematical Proofs
3 Sets and Functions
4 Complexity of Algorithms
5 Number Theory
6 Cryptography
7 Mathematical Induction
8 Recursion
9 Counting

   How to use generating function to solve counting problem?

   No need to remember those equations.

10 Relations
11 Graph
12 Trees

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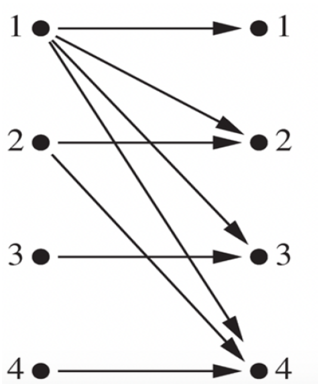
Cartesian Product

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, the Cartesian product $A \times B$ is the set of pairs $\{(a_1, b_1), (a_2, b_2), ..., (a_1, b_n), ..., (a_m, b_n)\}$.

Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

A relation on the set A is a relation from A to itself.

We use the notation aRb to denote $(a, b) \in R$, and $a \not R b$ to denote $(a, b) \notin R$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Summary on Properties of Relations

- Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
- Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.
- Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.
- Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Combining Relations

Definition: Let R be a relation from a set A to a set B and S be a relation from B to C. The composite of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$:

- $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$
- $S = \{(0, b), (1, a), (2, b)\}$
- $S \circ R = \{(1, b), (3, a), (3, b)\}$

Power of a Relation

Definition: Let R be a relation on A. The powers R^n, for $n = 1, 2, 3, ...$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

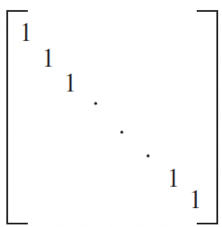
Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, ...$

Physical meaning of R^n?

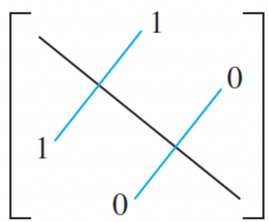
Representing Relations

Some special ways to represent binary relations:

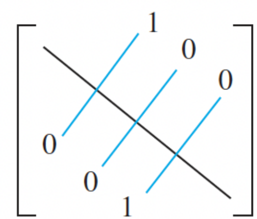
- with a zero-one matrix
- with a directed graph



Reflexive



Symmetric



Antisymmetric

Zero-One Matrix

Consider relations R_1 and R_2 on a set A:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Suppose that R is a relation from A to B and S is a relation from B to C:

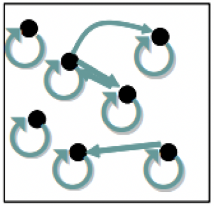
$$M_{S \circ R} = M_R \odot M_S.$$

The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S.

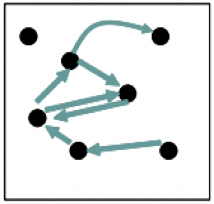
Directed Graph

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

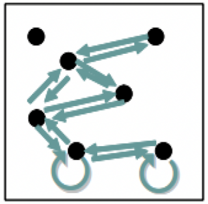
The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.



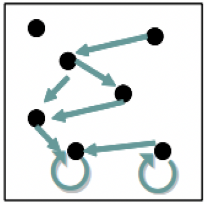
reflexive



irreflexive



symmetric



antisymmetric

Closures of Relations

The set S is called the reflexive closure of R if it:

- contains R
- is reflexive
- is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Relations can have different properties:

- Reflexive closures
- Symmetric closures
- Transitive closures: Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Paths in Directed Graphs

Definition: A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$.

A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Proof (by induction)

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Find Transitive Closure

Lemma: If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Thus,

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]},$$

where $M_R^{[n]} = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{n\ M_R's}$

Roy-Warshall Algorithm

Consider a list of vertices $v_1, v_2, ..., v_k, ..., v_n$. Define a zero-one matrix

$$\mathbf{W}_k = [w_{ij}^{(k)}],$$

where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, ..., v_k\}$ and is 0 otherwise.

Warshall's algorithm computes M_{R^*} by efficiently computing $\mathbf{W}_0 = M_R, W_1, W_2, ..., \mathbf{W}_n = M_{R^*}$.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

Equivalence Relation

Definition: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Partition of a Set S

Definition: Let S be a set. A collection of nonempty subsets of S, i.e A_1, A_2, . . . , A_k, is called a partition of S if:

$$A_i \cap A_j = \emptyset, i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Theorem: The equivalence classes form a partition of A.

Theorem: Let $\{A_1, A_2, ..., A_i, ...\}$ be a partition of S. Then, there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Partial Ordering

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set, or poset, denoted by (S, R).

The notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R).

The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$.

Comparability

Definition: The elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are called incomparable.

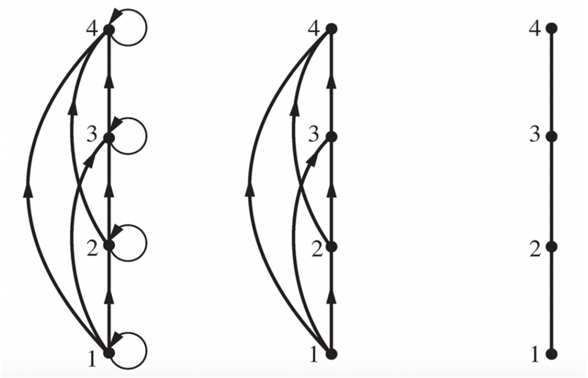
Definition: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order.

$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Hasse Diagram

Start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ s.t. $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.



Maximal and Minimal Elements

Definition: a is a maximal (resp. minimal) element in poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp. $b \prec a$).

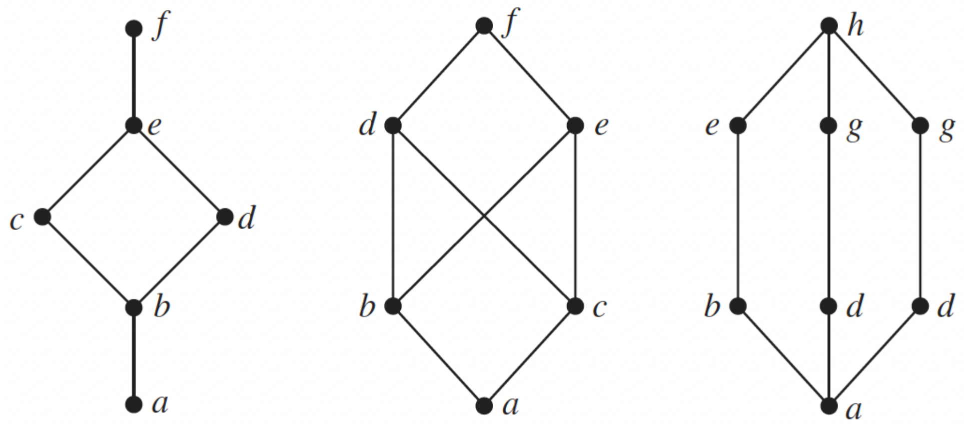
Definition: a is the greatest (resp. least) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp. $a \preccurlyeq b$) for all $b \in S$.

Definition: Let A be a subset of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an upper bound (resp. lower bound) of A if $a \preccurlyeq u$ (resp. $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the least upper bound (resp. greatest lower bound) of A if x is an upper bound (resp. lower bound) that is less than any other upper bounds (resp. lower bounds) of A.

Upper and Lower Bound

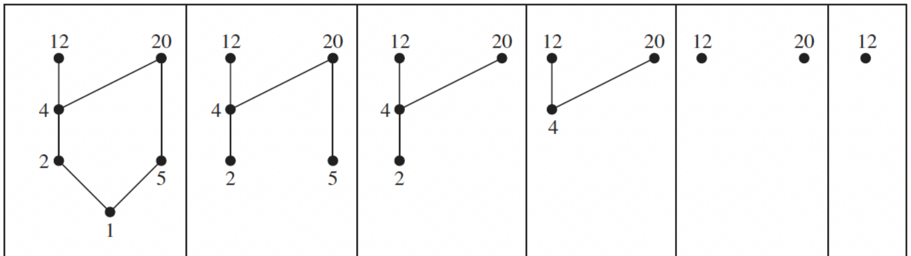
Definition: A partial ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.



- (a) and (c): lattices
- (b): not a lattice, because the elements b and c have no least upper bound.

Topological Sorting for Finite Posets

Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever aRb. $\preccurlyeq$ is said compatible with R.
Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.



Minimal element chosen 1	5	2	4	20	12

This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$$

Lecture Schedule

Definition of a Graph

A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to be incident to (or connect) its endpoints.

Simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.

Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Undirected Graphs

Definition: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

Definition: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Theorem (Handshaking Theorem): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} deg(v)$$

Directed Graphs

Definition: Let (u, v) be an edge in G. Then u is the initial vertex of the edge and is adjacent to v and v is the terminal vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are the same.

Definition: The in-degree of a vertex v, denoted by $deg^{-}(v)$, is the number of edges which terminate at v. The out-degree of v, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

Theorem: Let $G = (V, E)$ be a graph with directed edges. Then,

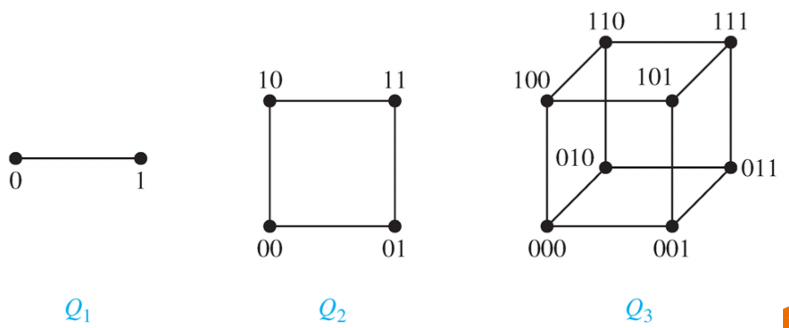
$$|E| = \sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v)$$

Special Graphs

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



Bipartite Graphs

Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

Definition: A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Bipartite Graphs and Matchings

Matching the elements of one set to elements in another. A matching is a subset of edges such that no two edges are incident with the same vertex.

A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a complete matching from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Subgraphs

Definition: A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$.

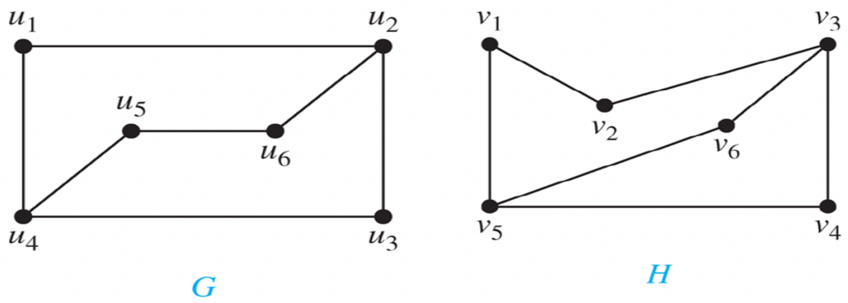
Definition: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, denoted by $G_1 \cup G_2$.

To represent a graph, we may use adjacency lists, adjacency matrices, and incidence matrices.

Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an isomorphism.

Useful graph invariants include the number of vertices, number of edges, degree sequence, existence circuit with certain length, etc.



Path: Undirected Graph

Definition: Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, e_2, \ldots,$ e_n of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \ldots, n$.

The path is a circuit if it begins and ends at the same vertex, i.e., if $u = v$, and has length greater than zero.

A path or circuit is simple if it does not contain the same edge more than once.

Length of a path = the number of edges on path

Path and circuit in directed graph?

Connectivity

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

Lemma: If there is a path between two distinct vertices x and y of a graph G, then there is a simple path between x and y in G.

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connectivity

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Definition: A directed graph is strongly connected if there is a path from a to b and a path from b to a whenever a and b are vertices in the graph.

Definition: A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Cut Vertices and Cut Edges

Sometimes the removal from a graph of a vertex and all incident edges disconnect the graph. Such vertices are called cut vertices.

Similarly we define cut edges.

A set of edges E' is called an edge cut of G if the subgraph $G - E'$ is disconnected. The edge connectivity $\lambda(G)$ is the minimum number of edges in an edge cut of G.

Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of A^r.

Proof (by induction)

Note: with directed or undirected edges, multiple edges and loops allowed

Euler Paths

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

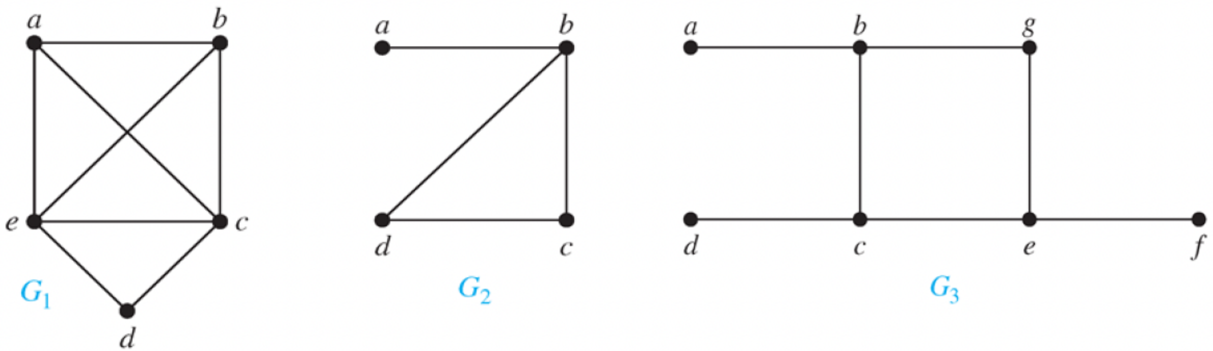
Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Hamilton Paths and Circuits

Definition: A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?

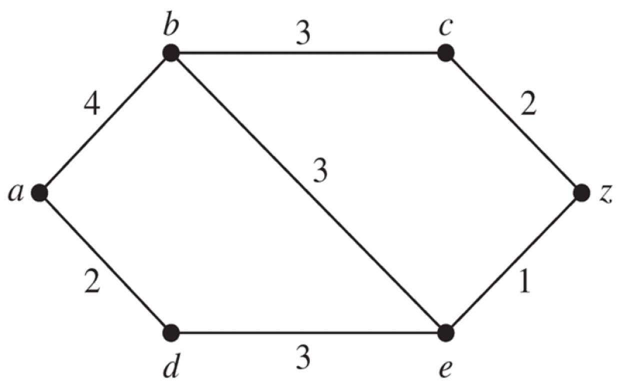


- G_1 has a Hamilton circuit: a, b, c, d, e, a;
- G_2 has no Hamilton circuit (because containing every vertex must contain the edge a, b twice), but it has a Hamilton path;
- G_3 has neither, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Shortest Path Problems

Using graphs with weights assigned to their edges

Such graphs are called weighted graphs and can model lots of questions involving distance, time consuming, fares, etc.



What is the length of a shortest path between a and z?

Dijkstra's Algorithm

Planar Graphs

Definition: A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

A planar representation of a graph splits the plane into regions, including an unbounded region.

Theorem (Euler's Formula): Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then, $r = e - v + 2$.

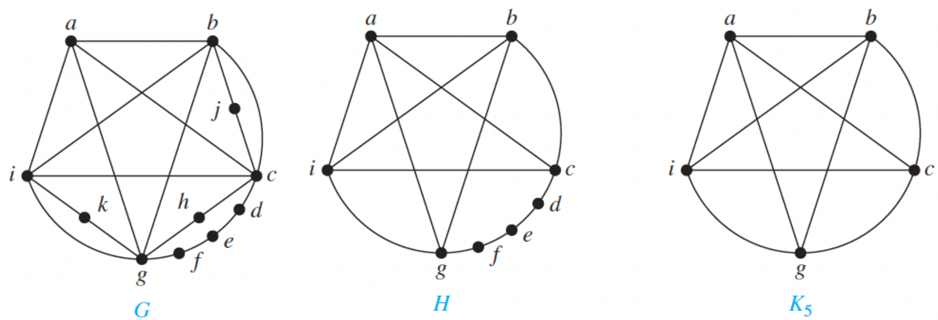
Kuratowski's Theorem

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

Theorem: A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Example:



Graph Coloring

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The chromatic number of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.

Theorem (Four Color Theorem): The chromatic number of a planar graph is no greater than four.

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Trees

Definition: A tree is a connected undirected graph with no simple circuits.

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Definition: A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

m-Ary Trees

Definition: A rooted tree is called an m-ary tree if every internal vertex has no more than m children.

The tree is called a full m-ary tree if every internal vertex has exactly m children.

In particular, an m-ary tree with $m = 2$ is called a binary tree.

Definition: An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right.

Counting Vertices in a Full m-Ary Trees

Theorem: A tree with n vertices has $n-1$ edges.

Theorem: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices.

Level and Height

The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

The height of a rooted tree is the maximum of the levels of the vertices.

A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h - 1$.

Tree Traversal

The procedures for systematically visiting every vertex of an ordered tree are called traversals.

Definition Let T be an ordered rooted tree with root r. If T consists only of r, then r is the preorder traversal of T.

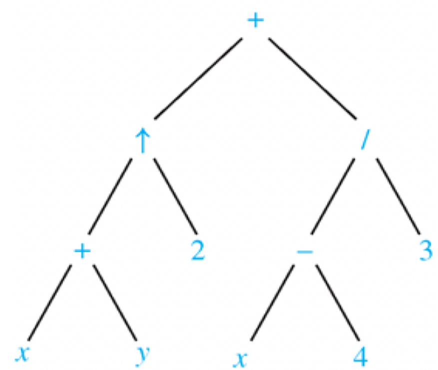
Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the inorder traversal of T.

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the postorder traversal of T.

Expression Trees

Complex expressions can be represented using ordered rooted trees.

- the internal vertices represent operations
- the leaves represent the variables or numbers



Prefix Notation

Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Example: $+ \; - \; * \; 2 \; 3 \; 5 \; / \; \uparrow 2 \; 3 \; 4$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8/4=2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

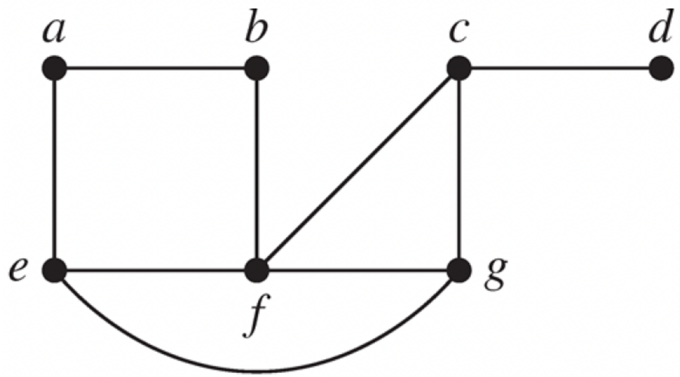
$$+ \quad \underbrace{- \quad 6 \quad 5}_{6-5=1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1+2=3}$$

The postorder traversal of expression trees leads to the postfix form of the expression (reverse Polish notation).

Spanning Trees

Definition: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

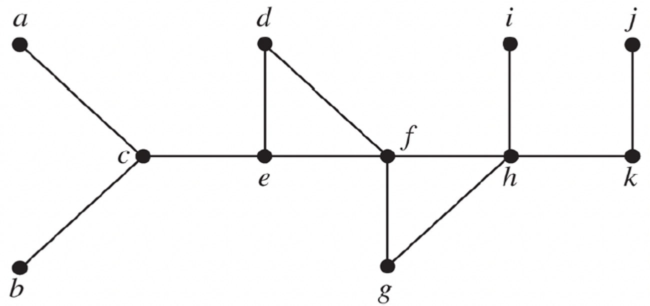


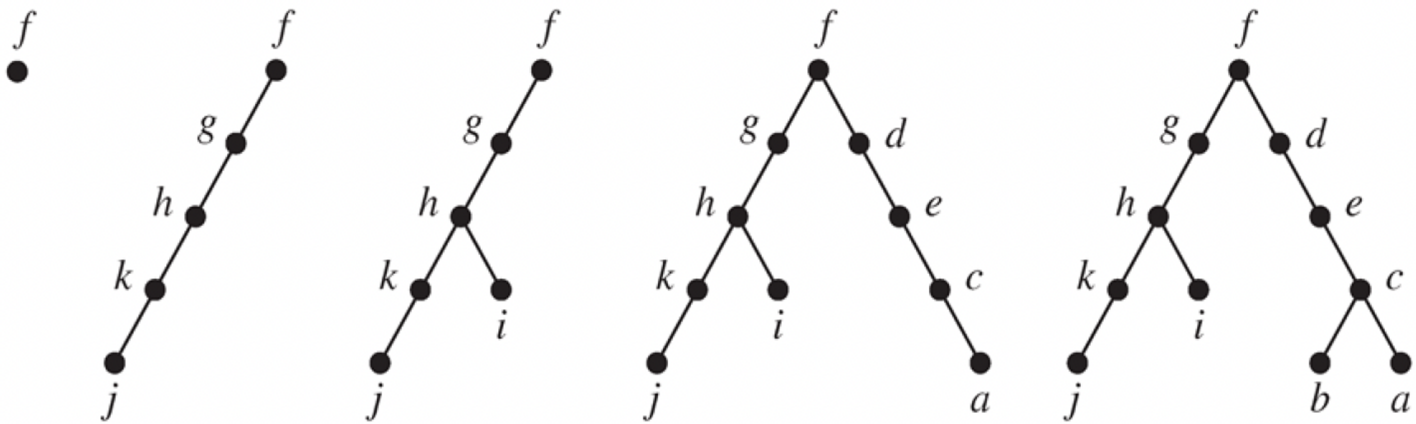
Remove edges to avoid circuits.

Depth-First Search

- First, arbitrarily choose a vertex of the graph as the root.
- Form a path by successively adding vertices and edges. Continue adding to this path as long as possible.
- If the path goes through all vertices of the graph, the tree is a spanning tree.
- Otherwise, move back to some vertex to repeat this procedure (backtracking).

Depth-First Search: Example





Breadth-First Search

This is the second algorithm that we build up spanning trees by successively adding edges.

- First arbitrarily choose a vertex of the graph as the root.
- Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges
- For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.
- Continue in this manner until all vertices have been added.

Breadth-First Search: Example