Discrete Mathematics for Computer Science

Lecture 6: Complexity of Algorithms

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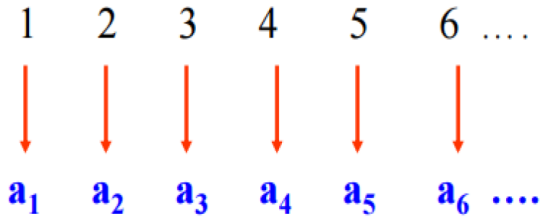
Countable Sets

Theorem: An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers):

- Each element appears once
  - We can relax this. Why?
- All elements must be listed

A sequence is a function from a subset of the set of integers to a set S.



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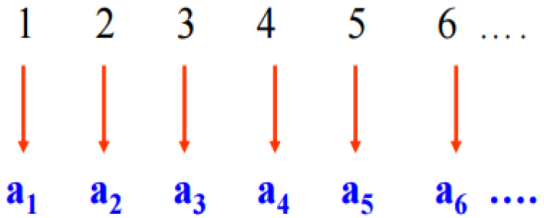
Theorem: An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers):

- Each element appears once
  - We can relax this. Why?
  - Because we can remove the repeated elements, and the remaining is still a sequence.
- All elements must be listed

A sequence is a function from a subset of the set of integers to a set S.



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Countable Sets

Theorem: The set of finite strings S over a finite alphabet A is countably infinite. (Assume an alphabetical ordering of symbols in A)

For example, let $A = \{\text{'a', 'b', 'c'}\}$. Then, set
$S = \{\text{'', 'a', 'b', 'c', 'ab' ..., 'aaaaa', ...}\}$

Theorem: The set of real numbers $\mathbf{R}$ is uncountable.

Proof by Contradiction: Suppose $\mathbf{R}$ is countable. Then, the interval from 0 to 1 is countable. This implies that the elements of this set can be listed as $r_1, r_2, r_3, ...$, where

- $r_1 = 0.d_{11}d_{12}d_{13}d_{14}$
- $r_2 = 0.d_{21}d_{22}d_{23}d_{24}$
- $r_3 = 0.d_{31}d_{32}d_{33}d_{34}$
- ...

where all $d_{ij} \in \{0, 1, 2, ..., 9\}$.

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Uncountable Sets

Theorem: The set $\mathcal{P}(\mathbf{N})$ is uncountable.

Proof by contradiction:

Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

- $S_0 = b_{00}b_{01}b_{02}b_{03}\cdots$
- $S_1 = b_{10}b_{11}b_{12}b_{13}\cdots$
- $S_2 = b_{20}b_{21}b_{22}b_{23}\cdots$

$\vdots$

all $b_{ij} \in \{0, 1\}$.

Form a new set called $R = b_0b_1b_2b_3...$, where $b_i = 0$ if $b_{ii} = 1$, and $b_i = 1$ if $b_{ii} = 0$. R is different from each set in the list. Each bit string is unique, and R and S_i differ in the i-th bit for all i.

Uncountable Sets

Theorem: The set $\mathcal{P}(\mathbf{N})$ is uncountable.

Theorem: The power set of any countably infinite set is uncountable.

Proof by contradiction:

Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

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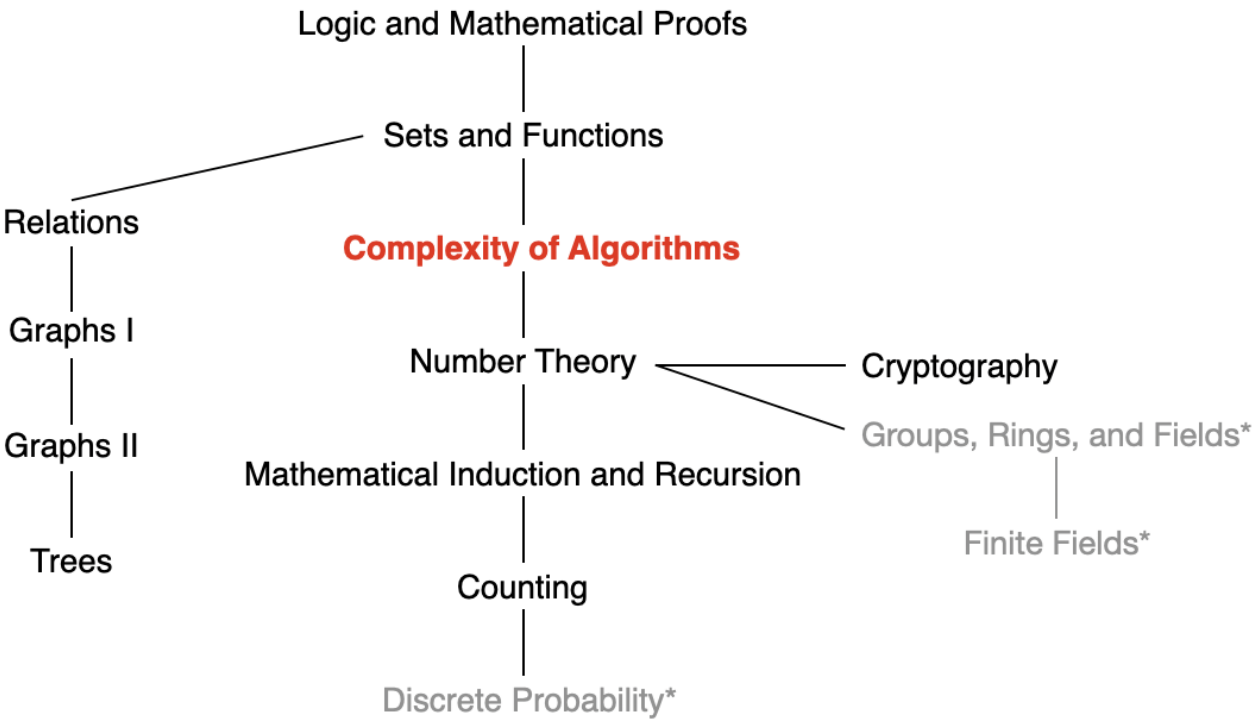
This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

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Algorithm

An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.

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ALGORITHM 4 The Bubble Sort.

procedure *bubblesort*($a_1, \ldots, a_n$: real numbers with $n \geq 2$)
for $i := 1$ **to** $n - 1$
    for $j := 1$ **to** $n - i$
        if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}
{$a_1, \ldots, a_n$ is in increasing order}

ALGORITHM 5 The Insertion Sort.

procedure *insertion sort*($a_1, a_2, \ldots, a_n$: real numbers with $n \geq 2$)
for $j := 2$ **to** n
    $i := 1$
    while $a_j > a_i$
        $i := i + 1$
    $m := a_j$
    for $k := 0$ **to** $j - i - 1$
        $a_{j-k} := a_{j-k-1}$
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Before we get into details, the growth of functions ...

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The Growth of Functions

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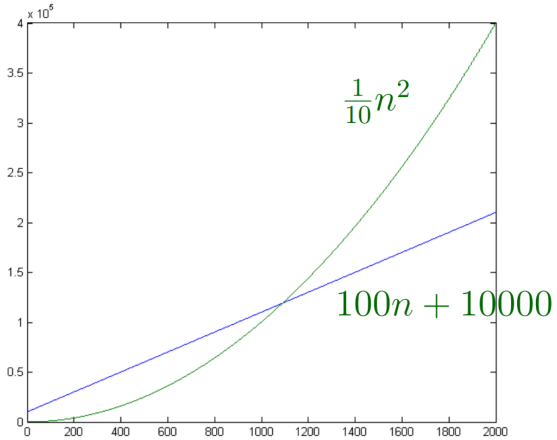

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The Growth of Functions

- Big-O notation, e.g., $O(n^2)$
- Big-Omega notation, e.g., $\Omega(n^2)$
- Big-Theta notation, e.g., $\Theta(n^2)$

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Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|,$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]

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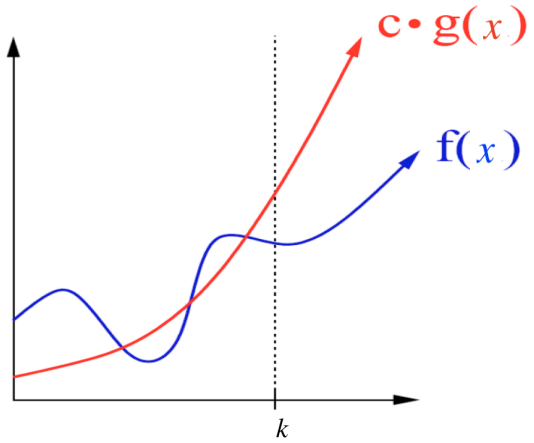

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Big-O Notation: Example

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Proof: We can readily estimate the size of $f(x)$ when $x > 1$:

$$0 \leq x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2.$$

This is because when $x > 1$, $x < x^2$ and $1 < x^2$. Thus, let $C = 4$, $k = 1$:

$$|f(x)| \leq C|x^2|, \text{ whenever } x > k.$$

Hence, $f(x) = O(x^2)$.

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Note that there are multiple ways for proving this. Alternatively, we can estimate the size of $f(x)$ when $x > 2$:

$$0 \leq x^2 + 2x + 1 \leq x^2 + x^2 + x^2 = 3x^2.$$

It follows that $C = 3$, $k = 2$. ...

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Big-O Notation: Example

Examples: The following formulas are all $O(x^2)$:

- $4x^2$
- $8x^2 + 2x - 3$
- $x^2/5 + \sqrt{x} - \log(x)$

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Observe that in the relationship "$f(x)$ is $O(x^2)$," x^2 can be replaced by any function with larger values than x^2. For example,

- $f(x)$ is $O(x^3)$
- $f(x)$ is $O(x^2 + x + 7)$, ...

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When $f(x)$ is $O(g(x))$, and $h(x)$ is a function that has larger absolute values than $g(x)$ does for sufficiently large values of x, it follows that

$$f(x) \text{ is } O(h(x)).$$

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Big-O Estimates for Polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_n$ are real numbers. Then, $f(x) = O(x^n)$.

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Proof:

Assuming $x > 1$, we have

$$\begin{aligned} |f(x)| &= |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\ &\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\ &= x^n (|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\ &\leq x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|). \end{aligned}$$

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The leading term $a_n x^n$ of a polynomial dominates its growth.

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Big-O Estimates for Some Functions

$1 + 2 + \cdots + n = O(n^2)$

$n! = O(n^n)$

$\log n! = O(n \log n)$

$\log_a n = O(n)$ for an integer $a \geq 2$

$n^a = O(n^b)$ for integers $a \leq b$

$n^a = O(2^n)$ for an integer a

In our textbook, p211-212

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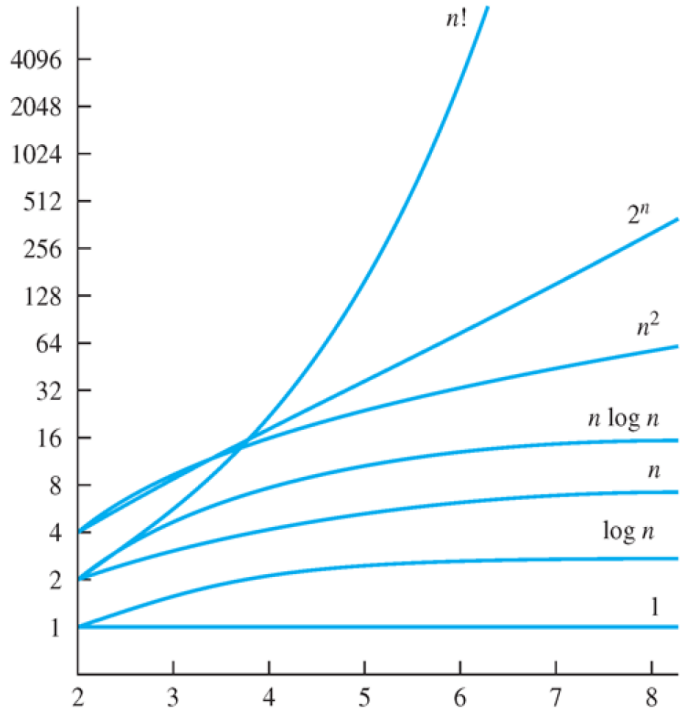
Big-O Estimates for Some Functions



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Combinations of Functions

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$.

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Proof:

By definition, there exist constants C_1, C_2, k_1, k_2 such that $|f_1(x)| \leq C_1|g_1(x)|$ when $x > k_1$ and $|f_2(x)| \leq C_2|g_2(x)|$ when $x > k_2$. Then

$$\begin{aligned} |(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \\ &\leq |f_1(x)| + |f_2(x)| \\ &\leq C_1|g_1(x)| + C_2|g_2(x)| \\ &\leq C_1|g(x)| + C_2|g(x)| \\ &= (C_1 + C_2)|g(x)| \\ &= C|g(x)|, \end{aligned}$$

where $g(x) = \max(|g_1(x)|, |g_2(x)|)$ and $C = C_1 + C_2$.

$k = \max\{k_1, k_2\}$.

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Proof:

When k $> \max(k_1, k_2)$,

$$\begin{aligned} |(f_1 f_2)(x)| &= |f_1(x)||f_2(x)| \\ &\leq C_1|g_1(x)|C_2|g_2(x)| \\ &\leq C_1 C_2|(g_1 g_2)(x)| \\ &\leq C|(g_1 g_2)(x)|, \end{aligned}$$

where $C = C_1 C_2$.

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Big-Omega Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-Omega of $g(x)$."]

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Big-Ω tells us that a function grows at least as fast as another.

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Note: $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$.

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Big-Theta Notation (Big-O & Big-Omega)

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Theta(g(x))$ if

- $f(x)$ is $O(g(x))$ and
- $f(x)$ is $\Omega(g(x))$.

When $f(x)$ is $\Theta(g(x))$, we say that $f(x)$ is big-Theta of $g(x)$, that $f(x)$ is of order $g(x)$, and that $f(x)$ and $g(x)$ are of the same order.

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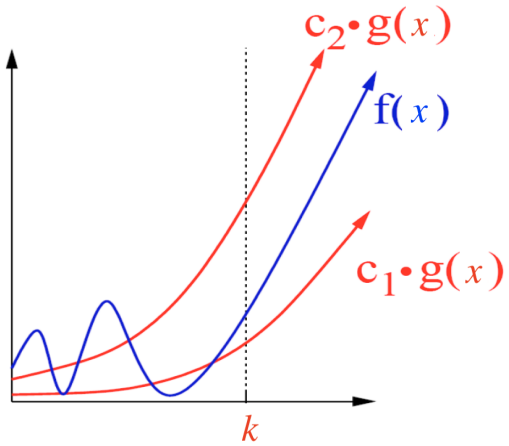

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Big-Theta Notation

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where a_0, a_1, . . . , a_n are real numbers with $a_n \neq 0$. Then $f(x)$ is of order x^n.

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- $f(x) = O(x^n)$
- $f(x) = \Omega(x^n)$

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Big-Theta Notation: Examples

$3n^2 + 4n = \Theta(n)$?

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$n/5 + 10n \log n = \Theta(n^2)$?

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Big-Theta Notation: Examples

$3n^2 + 4n = \Theta(n)$?	No
$3n^2 + 4n = \Theta(n^2)$?	Yes
$3n^2 + 4n = \Theta(n^3)$?	No, but $O(n^3)$
$n/5 + 10n \log n = \Theta(n^2)$?	No, but $O(n^2)$
$n^2/5 + 10n \log n = \Theta(n \log n)$?	No
$n^2/5 + 10n \log n = \Theta(n^2)$?	Yes

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The Growth of Functions

- Big-O notation, e.g., $O(n^2)$
  - Upper bound
- Big-Omega notation, e.g., $\Omega(n^2)$
  - Lower bound
- Big-Theta notation, e.g., $\Theta(n^2)$
  - Of the same order

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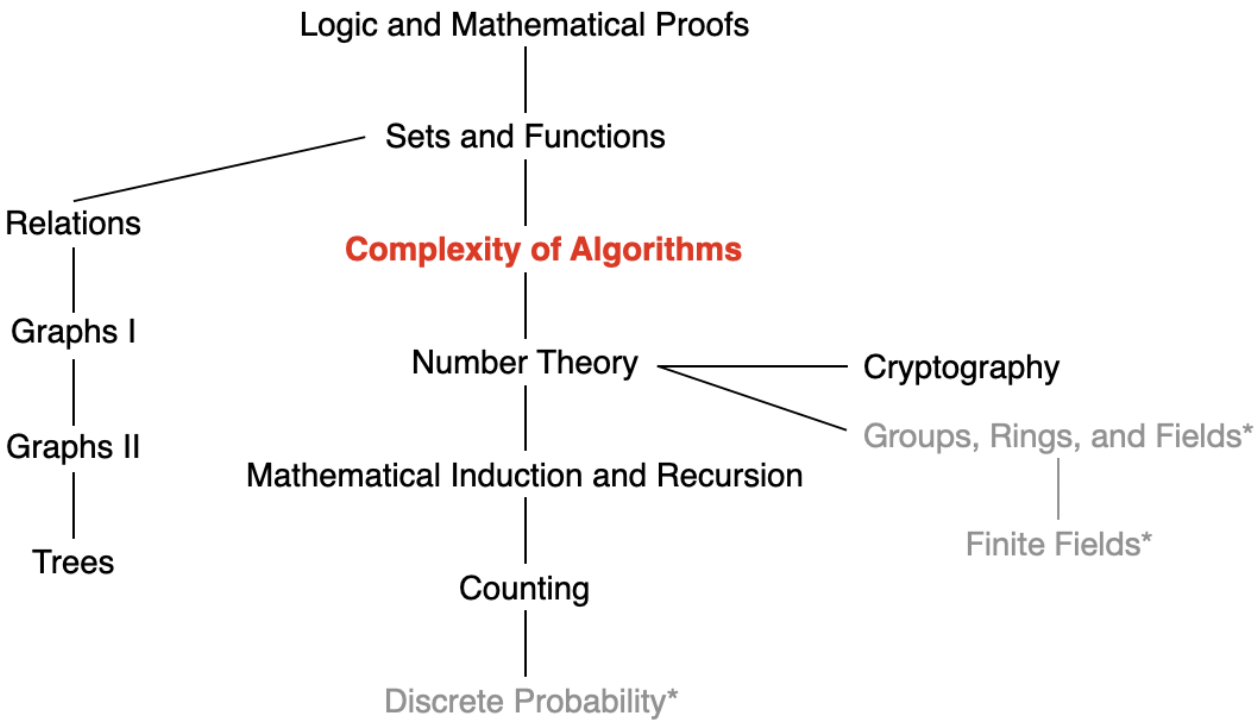
This Lecture



The growth of functions, complexity of algorithm, P and NP, ...

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Algorithms

An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.

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Example (Computational Problem and Algorithm):

- Computational Problem: Input n numbers $a_1, a_2, ..., a_n$; Output the sum of the n numbers.

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Example (Computational Problem and Algorithm):

- Computational Problem: Input n numbers $a_1, a_2, ..., a_n$; Output the sum of the n numbers.
- Algorithm: the following procedures

  Step 1: set $S = 0$

  Step 2: for $i = 1$ to n, replace S by $S + a_i$

  Step 3: output S

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Instance

An instance of a problem is a realization of all the inputs needed to compute a solution to the problem.

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Example: $8, 3, 6, 7, 1, 2, 9$

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An instance of a problem is a realization of all the inputs needed to compute a solution to the problem.

Example: $8, 3, 6, 7, 1, 2, 9$

A correct algorithm halts with the correct output for every input instance. We can then say that the algorithm solves the problem.

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Time and Space Complexity

- Time complexity: The number of machine operations (addition, multiplication, comparison, replacement, etc) needed in an algorithm.

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Example (Algorithm)

Step 1: set $S = 0$

Step 2: for $i = 1$ to n, replace S by $S + a_i$

Step 3: output S

Time Complexity:

- Steps 1 and 3 take one operation.
- Step 2 takes $2n$ operations.

Therefore, altogether this algorithm takes $1 + 2n + 1$ operations. The time complexity is $O(n)$.

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Horner's Algorithm and Its Complexity

Example: Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$. Direct computation takes 3 additions and 6 multiplications.

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Example: Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$. Direct computation takes 3 additions and 6 multiplications.

Can we do better?

Another way is $f(x) = 1 + x(2 + x(3 + 4x))$, which takes 3 additions and 3 multiplications.

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Horner's algorithm for computing
$f(x) = a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n = a_0 + x(a_1 + ... + x(a_{n-1} + a_nx))$
at a particular x:

Step 1: set $S = a_n$

Step 2: for $i = 1$ to n, replace S by $a_{n-i} + Sx$

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Horner's algorithm for computing
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at a particular x:

Step 1: set $S = a_n$

Step 2: for $i = 1$ to n, replace S by $a_{n-i} + Sx$

Step 3: output S

The number of operations needed in this algorithm is $1 + 3n + 1 = 3n + 2$. So the time complexity of this algorithm is $O(n)$.

Note: Operations: addition, multiplication, comparison, replacement, etc.

Time Complexity: Example

Determine the time complexity of the following algorithm:

```
for i := 1 to n
    for j := 1 to n
        a := 2 * n + i * j;
    end for
end for
```

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- Computing $a := 2 \times n + i \times j$ takes 4 operations (two multiplications, one addition, and one replacement).

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- For each i, it takes $4n$ operations to complete the second loop.
- Thus, this algorithm takes $n \times 4n = 4n^2$ operations to complete the two loops. The time complexity of this algorithm is $O(n^2)$.

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Time Complexity: Example

Determine the time complexity of the following algorithm:

$S := 0$
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- Computing $S := S + i \times j$ takes 3 operations.

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end for

- Computing $S := S + i \times j$ takes 3 operations.
- For each i , completing the second loop takes $3i$ operations.
- Thus, this algorithm takes

$$1 + \sum_{i=1}^{n} 3i = 1 + 3\frac{n(n+1)}{2}$$

So the complexity of this algorithm is $O(n^2)$.

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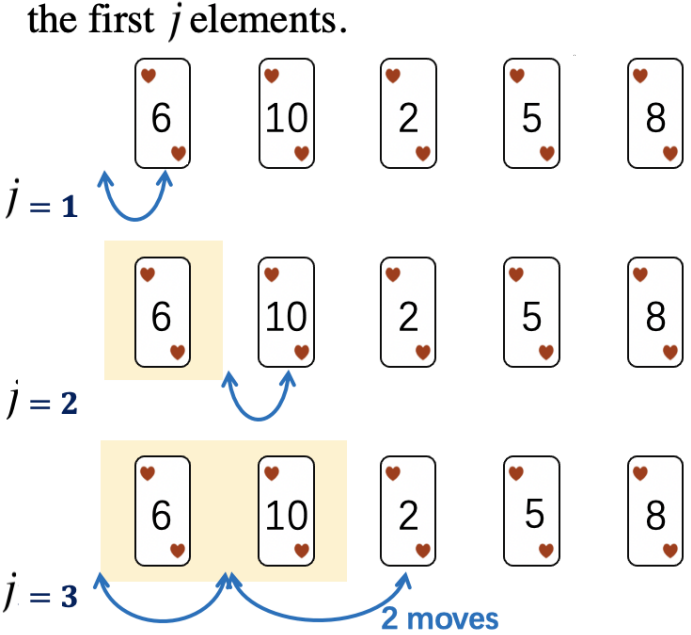
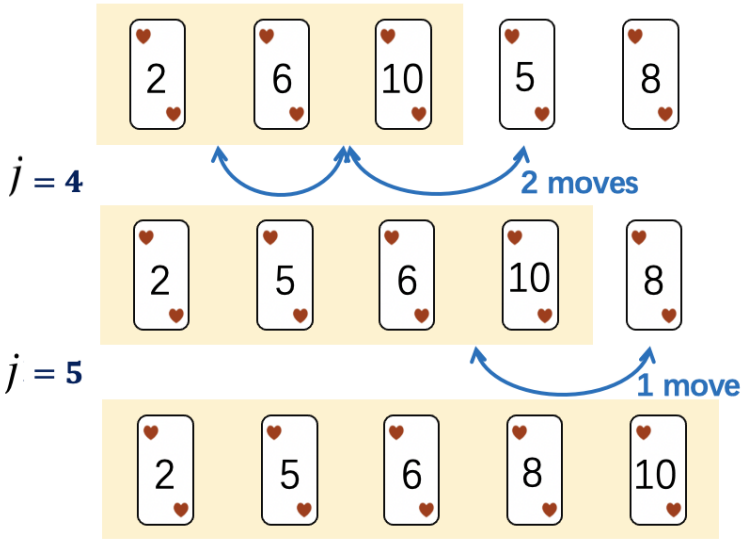
Time Complexity: Example - Insertion Sort

In iteration j, we move the j-th element left until its correct place is found among the first j elements.





Code?

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Time Complexity: Example - Insertion Sort

Input: $A[1 \ldots n]$ is an array of numbers

for $j := 2$ to n

  $key = A[j]$;

  $i = j - 1$;

  while $i \geq 1$ and $A[i] > key$ do

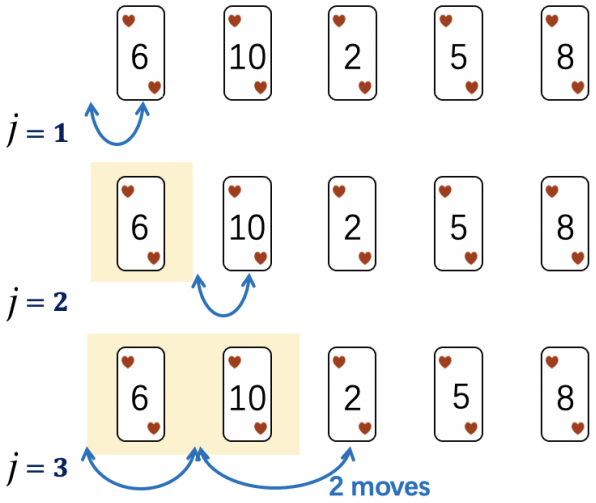
    $A[i + 1] = A[i]$;

    $i - -$;

  end while

  $A[i + 1] = key$;

end for



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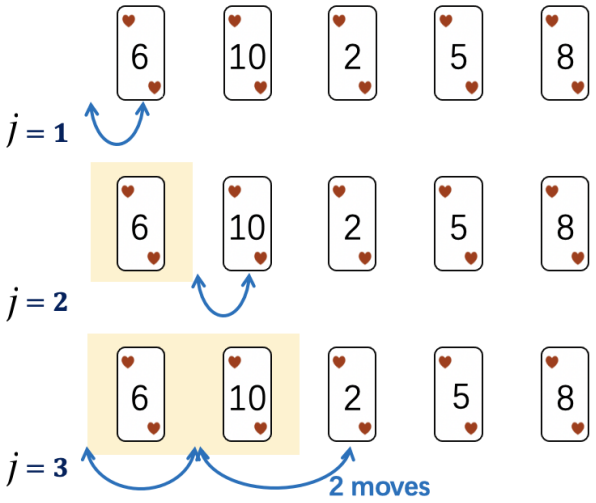

The time complexity depends on the input array $A[1, ..., n]$.

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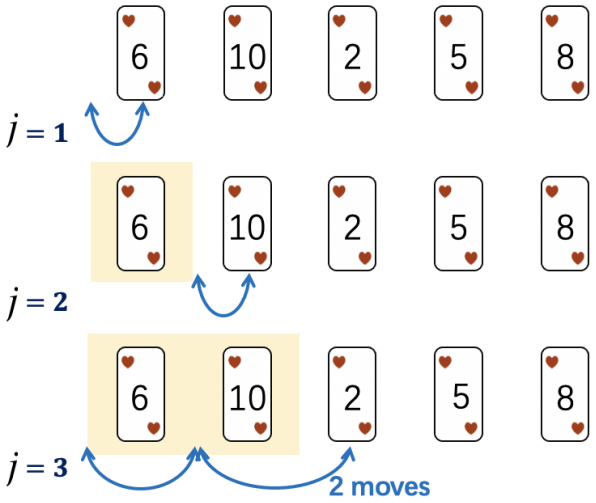

The time complexity depends on the input array $A[1, ..., n]$.

Consider only the number of comparisons

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Three Cases of Analysis: Best-Case

Best-Case Complexity: The smallest number of operations needed to solve the given problem using this algorithm on input of specified size.

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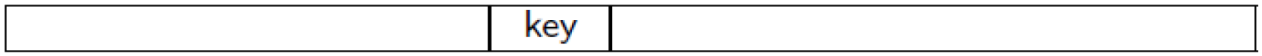
Example: (Insertion Sort)

$$A[1] \le A[2] \le A[3] \le \cdots \le A[n]$$

The number of comparisons needed is

$$\underbrace{1+1+1+\cdots+1}_{n-1} = n-1 = \Theta(n)$$



Sorted Unsorted

"key" is compared to only the element right before it.

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Three Cases of Analysis: Worst-Case

Worst-Case Complexity: The largest number of operations needed to solve the given problem using this algorithm on input of specified size.

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Example: (Insertion Sort)

$$A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$$

The number of comparisons needed is

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \Theta(n^2)$$

Sorted Unsorted

"key" is compared to everything element before it.

Three Cases of Analysis: Average-Case

Average-Case Complexity: The average number of operations used to solve the problem over all possible inputs of a given size is found in this type of analysis.

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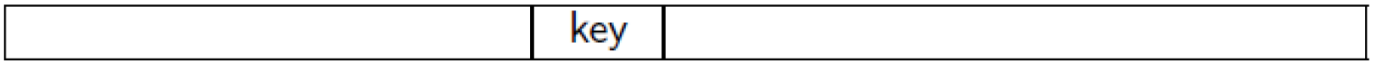
Example: (Insertion Sort)

$\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely



Sorted Unsorted

On average, "key" is compared to half of the elements before it.

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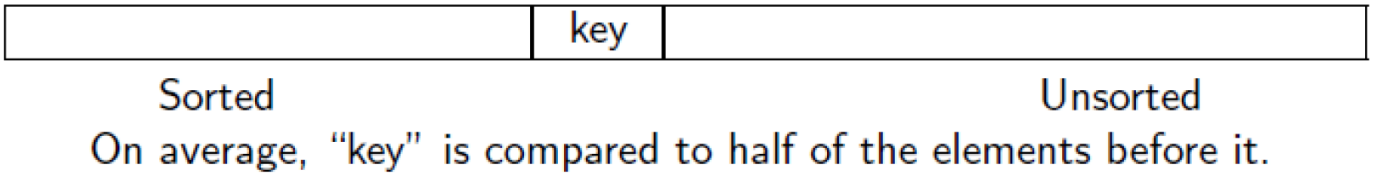

- For a particular instance, compute the number of comparisons
- Since we assume equal probability, take the average

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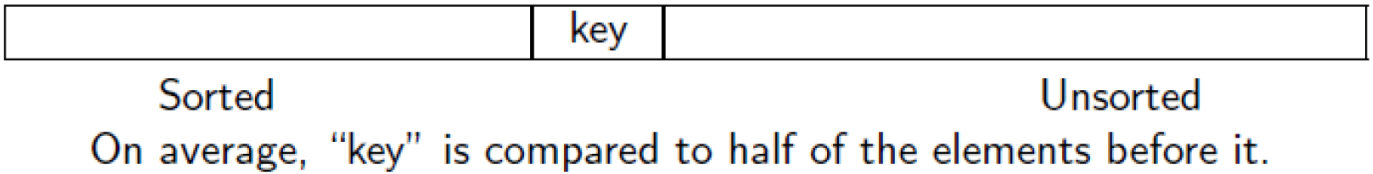

- For a particular instance, compute the number of comparisons
- Since we assume equal probability, take the average

Average-case complexity is usually difficult to compute.

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Some Thoughts on Algorithm Design

Algorithm Design is mainly about designing algorithms that have small Big-O running time.

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Being able to do good algorithm design lets you identify the hard parts of your problem and deal with them effectively.

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Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code!

- The most straightforward manner based on the statement of the problem and the definitions of terms

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- The most straightforward manner based on the statement of the problem and the definitions of terms

A few hours of abstract thought devoted to algorithm design could speed up the solution substantially and simplified it!

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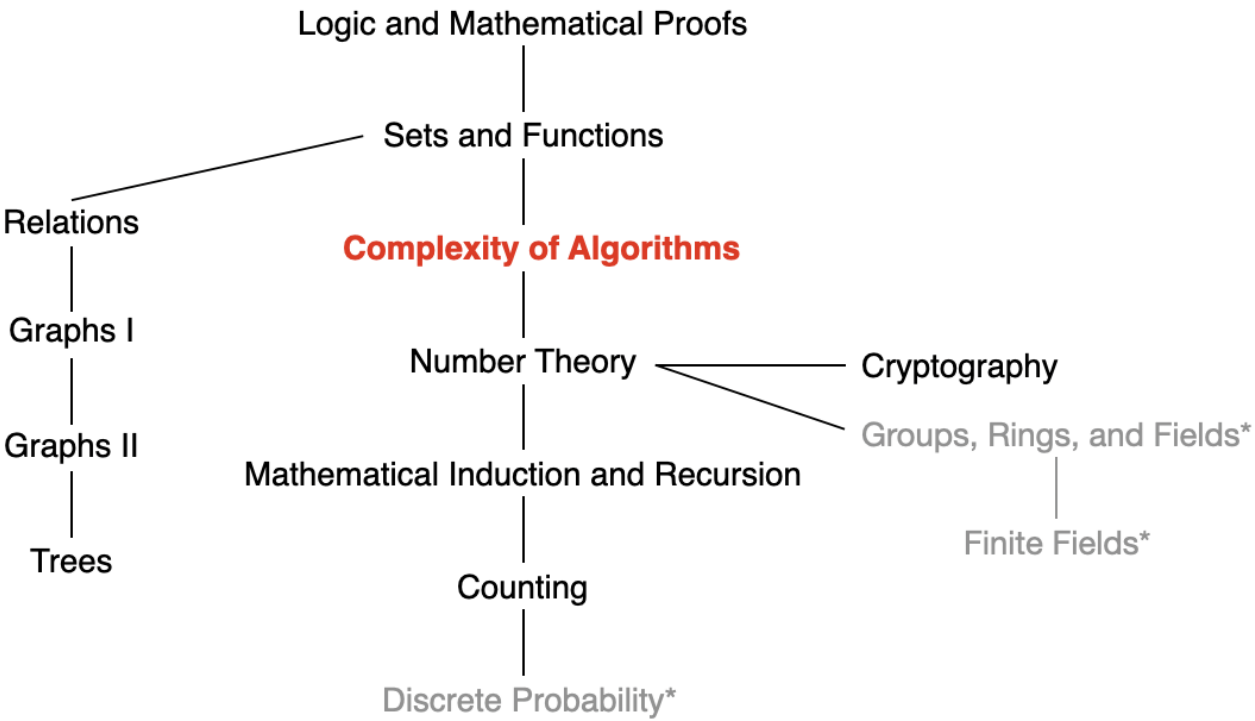
This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

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Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

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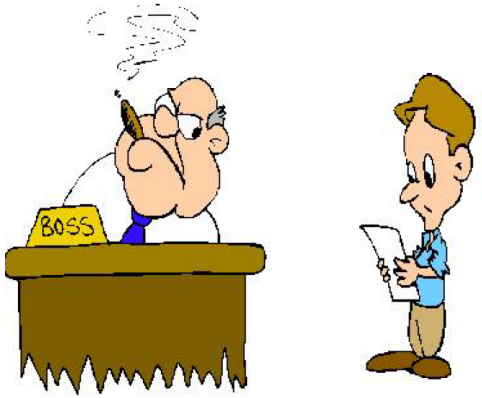
Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Blame yourself.



I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

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Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Show that no-efficient algorithm exists.

I couldn't find a polynomial-time algorithm, because no such algorithm exists.

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Dealing with Hard Problems

Showing that a problem has an efficient algorithm is, relatively easy:

- Design such an algorithm.

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Proving that no efficient algorithm exists for a particular problem is difficult:

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Showing that a problem has an efficient algorithm is, relatively easy:

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How can we prove the non-existence of something?

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Dealing with Hard Problems

Showing that a problem has an efficient algorithm is, relatively easy:

- Design such an algorithm.

Proving that no efficient algorithm exists for a particular problem is difficult:

How can we prove the non-existence of something?

We will now learn about NP-Complete problems, which provides us with a way to approach this question.

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NP-Complete

P: Problems that are solvable using an algorithm with polynomial worst-case complexity

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NP: Problems for which a solution can be checked in polynomial time.

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NP: Problems for which a solution can be checked in polynomial time.

NP-Complete: If any of these problems can be solved by a polynomial worst-case time algorithm, then all problems in the class NP can be solved by polynomial worst-case time algorithms.

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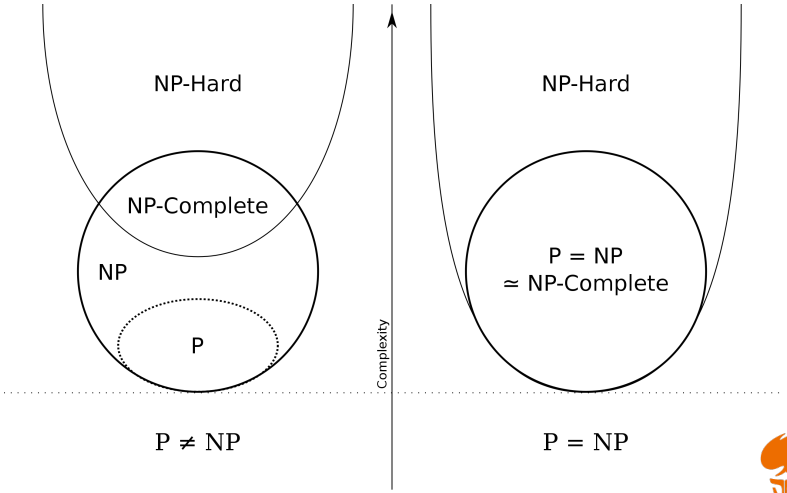

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NP-Complete

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NP-Complete and NP-Hard problems are very likely to be hard.

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Thus, to proving that no efficient algorithm exists for a particular problem?

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NP-Complete

Researchers have spent many years trying to find efficient solutions to these problems but failed.

NP-Complete and NP-Hard problems are very likely to be hard.

Thus, to proving that no efficient algorithm exists for a particular problem?

Prove that your problem is NP-Complete or even NP-Hard:

- Show that your problem can be reduced to a typical (well-known) NP-Complete or NP-Hard problem.

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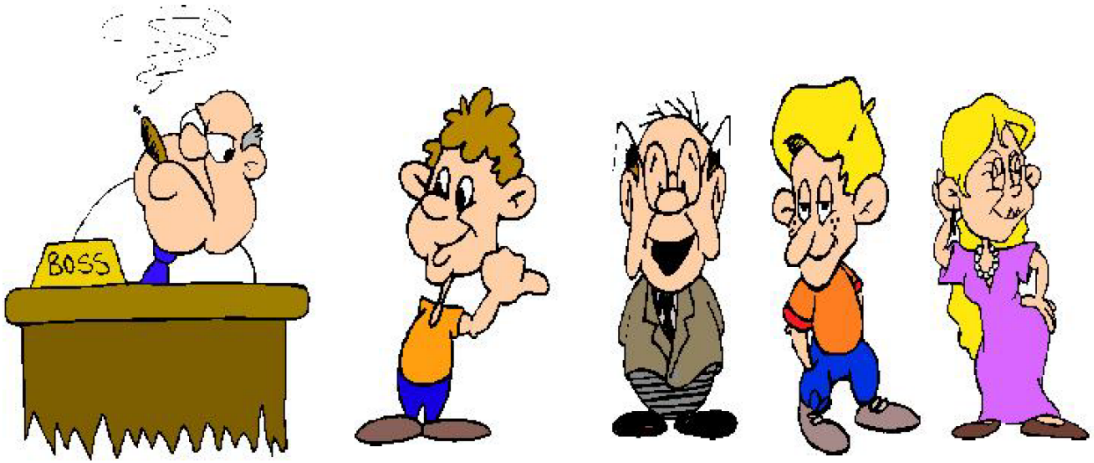
NP-Complete

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

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Dealing with Hard Problems

- Input size of a problem
- Different types of problems
- Polynomial-time algorithms
- Details for P and NP

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Encoding the Inputs of Problems

Complexity of a problem is measure with respect to the size of input:

- E.g., for insertion sort, $\Theta(n^2)$ is the average-case complexity, where n is the length of the array.

In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

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The Input Size of Problems

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For most problems, it is sufficient to choose some natural and (usually) simple encoding and use the size s of this encoding.

- E.g., 5 can be encoded as 101.

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Input Size Example: Composite

Example: Input a positive integer n; output if there are integers j, $k > 1$ such that $n = jk$? (i.e., is n a composite number?)

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Any integer $n > 0$ can be represented in the binary number system as a string $a_0 a_1 ... a_k$ of length $\lceil \log_2(n+1) \rceil$.

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Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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Note: Back to our earlier discussions for complexity, when we use fixed length encoding regardless of a_i for $i = 1, 2, ..., n$, the value of m becomes a constant. Thus, we can omit the constant m.

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Complexity in terms of Input Size

Example (Composite): The naive algorithm for determining whether n is composite compares n with the first $n-1$ numbers to see if any of them divides n.

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This makes $\Theta(n)$ comparisons, so it might seem linear and very efficient.

But, the input size of this problem is $\log_2 n$ instead of n. The number of comparisons performed is actually $\Theta(n)$, which can be represented as $\Theta(2^{(\log_2 n)})$. It is exponential with respect to the input size.

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Functions of the Same Type

Definition: Two positive functions $f(n)$ and $g(n)$ are of the same type if

$$c_1 g(n^{a_1})^{b_1} \leq f(n) \leq c_2 g(n^{a_2})^{b_2}$$

for all large n, where a_1, b_1, c_1, a_2, b_2, c_2 are some positive constants.

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Example:

- All polynomials are of the same type
- Polynomials and exponentials are of different types.

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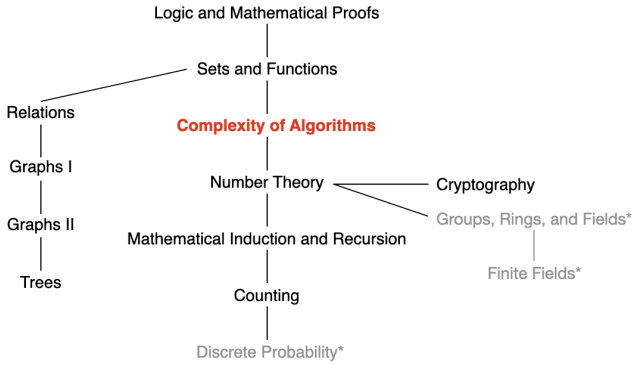
This Lecture



The growth of functions, complexity of algorithm, P and NP:

- Input size of a problem
- Different types of problems
- Polynomial-time algorithms
- Details for P and NP

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