CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 2
Due: Mar. 24th, 2022, please submit through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Q. 1 *(5 points) Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.*

(a) $(A - B = A) \to (B \subset A)$

(b) $(A \cap B \cap C) \subseteq (A \cup B)$

(c) $\overline{(A - B)} \cap (B - A) = B$

Solution:

(a) False. As an counterexample, let $A = \{1\}$, and $B = \{2\}$. Then $A - B = A$, but B is not a subset of A.

(b) True. $A \cap B \cap C \subseteq A \cap B \subseteq A \cup B$.

(c) False. Let $A = B = \{1\}$. Then, $\overline{A - B} \cap (B - A) = U \cap \emptyset \neq B = \{1\}$.

□

Q. 2 *(5 points) The* symmetric difference *of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.*

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Solution:

(a) Using membership table, one can show that each side consists of the elements that are in an odd number of the sets A, B and C. Thus, it follows.

(b) Yes. We prove that for every element $x \in A$, we have $x \in B$ and vice versa. We use proof by cases.

First, for elements $x \in A$ and $x \notin C$, since $A \oplus C = B \oplus C$, we know that $x \in A \oplus C$ and thus $x \in B \oplus C$. Since $x \notin C$, we must have $x \in B$. For elements $x \in A$ and $x \in C$, we have $x \notin A \oplus C$. Thus, $x \notin B \oplus C$. Since $x \in C$, we must have $x \in B$.

The proof of the other way around is similar.

□

Q. 3 *(5 points) Let A, B and C be sets. Prove the following using set identities.*

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Solution:

(1) We have

$$\begin{aligned}(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) && \text{by definition}\\ &= \overline{A} \cap (B \cup C) && \text{ditributive law}\\ &= (B \cup C) - A && \text{by definition}\end{aligned}$$

(2) We have

$$\begin{aligned}&(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C)\\ &= (A \cap B) \cap (A \cap C) \cap \overline{(B \cap C)} && \text{commutative law}\\ &= (A \cap B \cap C) \cap \overline{(B \cap C)} && \text{associative law}\\ &= (A \cap B \cap C) \cap (\overline{B} \cup \overline{C}) && \text{De Morgan}\\ &= ((A \cap B \cap C) \cap \overline{B}) \cup ((A \cap B \cap C) \cap \overline{C}) && \text{distributive law}\\ &= \emptyset \cup \emptyset && \text{Complement}\\ &= \emptyset.\end{aligned}$$

□

Q. 4 *(5 points) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.*

Solution: For the "if" part, given $A \subseteq B$, we want to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., if $C \subseteq A$, then $C \subseteq B$. Since $A \subseteq B$, $A \subseteq C$ directly follows.

For the "only if" part, given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show that $A \subseteq B$. Suppose that $a \in A$. Then $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$, which means that $\{a\} \subseteq B$. This implies that $a \in B$, and completes the proof.

$\square$

Q. 5 *(10 points) For each of the following mappings, use the following options to describe them, and explain your answers.*
i. Not a function.
ii. A function which is neither one-to-one nor onto.
iii. A function which is onto but not one-to-one.
iv. A function which is one-to-one but not onto.
v. A function which is both one-to-one and onto.

(a) The mapping f from $\mathbf{Z}$ to $\mathbf{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbf{R}$ to $\mathbf{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbf{R}^+$ to $\mathbf{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ defined by $f(x) = x + 1$.

Solution:

(a) ii. It is a function, because it assigns an element in the codomain $\mathbf{Z}$ to every element in the domain $\mathbf{Z}$. It is not one-to-one, because $f(-2) = f(2)$. It is not onto, because there does no exists any x such that $f(x) = -10$.

(b) i. It is not a function, because when $x = 3$, $f(x) = 2 \times 3 = 6$ is not in the codomain $\{2, 4\}$.

(c) v. It is one-to-one. Specifically, consider $x, x' \in \mathbf{R}$, if $f(x) = f(x')$, i.e., $8 - 2x = 8 - 2x'$. Thus, $x = x'$. It is onto, because for any $y \in \mathbf{R}$, we can find an $x \in \mathbf{R}$ such that $y = 8 - 2x$.

(d) iii. ...

(e) i. ...

(f) iv. ...

$\square$

Q. 6 *(5 points) Which of the mappings in Q. 5 have an inverse function? What is the inverse function? Please list all such mappings and explain your answer.*

Solution: The mapping in (c) has an inverse function, because it is a function which is one-to-one and onto. The inverse function is $f^{-1}(y) = (8-y)/2$. All the other mappings do not have inverse function.

$\square$

Q. 7 *(5 points) Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.*

Solution:

Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed $n = \lfloor x \rfloor$.

- if $x \in [n, n+\frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n+1)$, so $\lfloor 3x \rfloor = 3n$. Moreover in this case $x + \frac{1}{3}$ is still less than $n+1$, and $x + \frac{2}{3}$ is still less than $n+1$. So, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + n = 3n$ as well.

- if $x \in [n+\frac{1}{3}, n+\frac{2}{3})$, then $3x \in [3n+1, 3n+2)$, so $\lfloor 3x \rfloor = 3n+1$. Moreover in this case $x+\frac{1}{3}$ is in $[n+\frac{2}{3}, n+1)$, and $x+\frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + (n+1) = 3n+1$ as well.

- if $x \in [n+\frac{2}{3}, n+1)$, similar and both sides equal $3n+2$.

$\square$

Q. 8 *(10 points) Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the* composition *function.*

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) *If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.*

(c) *If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.*

(d) *If $f \circ g$ is onto, must f be onto? Explain your answer.*

(e) *If $f \circ g$ is onto, must g be onto? Explain your answer.*

Solution:

(a) No. We prove this by giving a counterexample. Let $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = A$. Define the function g by $g(1) = a$ and $g(2) = b$, and define the function f by $f(a) = 1$, and $f(b) = f(c) = 2$. Then it is easily verified that $f \circ g$ is one-to-one and g is one-to-one. But f is not one-to-one.

(b) Yes. For any two elements $x, y \in A$ with $x \neq y$, assume to the contrary that $g(x) = g(y)$. On one hand, since $f \circ g$ is one-to-one, we have $f \circ g(x) \neq f \circ g(y)$. One the other hand, $f \circ g(x) = f(g(x)) = f(g(y)) = f \circ g(y)$. This leads to a contradiction. Thus, $g(x) \neq g(y)$, which means that g must be one-to-one.

(c) Yes. Similar to (b), the condition that f is one-to-one is in fact not used.

(d) Yes. Since $f \circ g$ is onto, we know that $f \circ g(A) = C$, which means that $f(g(A)) = C$. Note that $g(A)$ is a subset of B, thus, $f(B)$ must also be C. This means that f is also onto.

(e) No. A counterexample is the same as that in (a).

$\square$

Q. 9 *(5 points) Derive the formula for $\sum_{k=1}^{n} k^2$.*

Solution: First, we note that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. Then, we sum this equation for all values of k from 1 to n. On the left, because of telescoping, we have just n^3; on the right we have

$$3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3 \sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n.$$

Equating the two sides and solving for $\sum_{k=1}^{n} k^2$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left(n^3 + \frac{3n(n+1)}{2} - n\right) \\
&= \frac{n}{3}\left(\frac{2n^2 + 3n + 3 - 2}{2}\right) \\
&= \frac{n}{3}\left(\frac{2n^2 + 3n + 1}{2}\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

□

Q. 10 *(10 points) Give an example of two uncountable sets A and B such that the difference $A - B$ is*

(a) finite,

(b) countably infinite,

(c) uncountable.

Solution: In each case, let A be the set of real numbers.

(a) Let B be the set of real numbers as well, then $A - B = \emptyset$, which is finite.

(b) Let B be the set of real numbers that are not positive integers, then $A - B = \mathbf{Z}^+$, which is countably infinite.

(c) Let B be the set of positive real numbers. Then $A - B$ is the set of negative real numbers, which is uncountable.

□

Q. 11 *(10 points) For each set defined below, determine whether the set is* countable *or* uncountable*. Explain your answers. Recall that $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ denotes the set of real numbers.*

(a) The set of all subsets of students in CS201

(b) $\{(a, b)|a,\ b \in \mathbf{N}\}$

(c) $\{(a, b)|a \in \mathbf{N},\ b \in \mathbf{R}\}$

Solution:

(a) Countable. The number of students in CS201 is finite, so the size of its power set is also finite. All finite sets are countable.

(b) Countable. The set is the same as $\mathbf{N} \times \mathbf{N}$. We now show that these elements can be listed in a sequence:

$$(0, 0), (1, 0), (1, 1), (0, 1), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), \ldots.$$

That is, we start with $a = 0$, list $(0, 0)$. Then, we work on $a = 1$, list $(1, 0)$, $(1, 1)$, $(0, 1)$. Subsequently, for any $a = i$, we list $(i, 0)$, $(i, 1)$, ..., (i, i), $(i-1, i)$, ...$(0, i)$. Then, we set $a = i+1$ and continue the process. It can be easily checked that all elements in set $\{(a, b)|a,\ b \in \mathbf{N}\}$ are in this sequence. (Note: as long as students can show there is a sequence that can list all the elements or there is a one-to-one corresponds from the set of positive integers to this set, then it is correct.)

(c) Uncountable. We will prove by contradiction. Suppose $\{(a, b) \mid a \in \mathbf{N},\ b \in \mathbf{R}\}$ is countable. Then, it's subset $\{(a, b) \mid a = 1,\ b \in \mathbf{R}, 0 < b < 1\}$ is also countable. Thus, we can list all the elements in this set in a sequence. Let $(1, r_1)$, $(1, r_2)$, $(1, r_3)$ ... be the elements in the sequence, where

- $r_1 = 0.d_{11}d_{12}d_{13}...$
- $r_2 = 0.d_{21}d_{22}d_{23}...$
- $r_3 = 0.d_{31}d_{32}d_{33}...$
- ...

Now, we aim to construct a tuple $(1, r)$ that is not in this sequence. Let $r = d_1d_2d_3....$ Set $d_i = 3$ if $d_{ii} \neq 3$, and $d_i = 2$ if $d_{ii} = 3$. It can be seen that r is different from any element in the sequence. Thus, this leads to a contradiction.

$\square$

Q. 12 *(5 points) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?*

Solution: Since $A = (A-B) \cup (A \cap B)$, if $A-B$ is countable, the elements of A can be listed in a sequence by alternating elements of $A - B$ and elements of $A \cap B$. This contradicts the uncountability of A.

$\square$

Q. 13 *(5 points) Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m,n) = (m+n-2)(m+n-1)/2 + m$ is one-to-one and onto.*

Solution: It is clear from the formula that the range of values the function takes on for a fixed value of $m+n$, say $m+n = x$, is $(x-2)(x-1)/2+1$ through $(x-2)(x-1)/2+(x-1)$, because m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m+n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x+1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1,1)+1 = f(1,x)$. We have $f(x-1,1)+1 = (x-2)(x-1)/2 + (x-1) + 1 = (x^2 - x + 2)/2 = (x-1)x/2 + 1 = f(1,x)$.

$\square$

Q. 14 *(5 points) By the Schröder-Bernstein theorem, prove that $(0,1)$ and $[0,1]$ have the same cardinality.*

Solution: By the Schröder-Bernstein theorem, it suffices to find one-to-one functions $f : (0,1) \to [0,1]$ and $g : [0,1] \to (0,1)$. Let $f(x) = x$ and $g(x) = (x+1)/3$. It is then straightforward to prove that f and g are both one-to-one.

$\square$

Q. 15 *(5 points) Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.*

By definition, we have one-to-one and onto functions $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a one-to-one and onto function from A to C, so we have $|A| = |C|$.

$\square$

Q. 16 *(5 points) Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.*

Solution: The definition of "$f(x)$ is $\Theta(g(x))$" is that $f(x)$ if both $O(g(x))$ and $\Omega(g(x))$. This means that there are positive constants C_1, k_1, C_2, and k_2 such that $|f(x)| \leq C_2|g(x)|$ for all $x > k_2$ and $|f(x) \geq C_1|g(x)|$ for all $x > k_1$. Similarly, we have that there are positive constants C_1', k_1', C_2', and k_2' such that $|g(x)| \leq C_2'|h(x)|$ for all $x > k_2'$ and $|g(x)| \geq C_1'|h(x)|$ for all $x > k_1'$. We can combine these inequalities to obtain $|f(x)| \leq C_2C_2'|h(x)|$ for all $x > \max(k_2, k_2')$ and $|f(x)| \geq C_1C_1'|h(x)|$ for all $x > \max(k_1, k_1')$. This means that $f(x)$ is $\Theta(h(x))$.

$\square$

Q. 17 *(5 points) Consider the following algorithm for evaluating the value of a polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots a_1x + a_0$ at $x = c$.*

Algorithm 1 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
power := 1
y := a_0
for i := 1 to n do
    power := power * c
    y := y + a_i * power
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0}
```

(a) How many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).

(b) Under the operations considered in (a), what is the time complexity with respect to n (in Big-Theta Notation)?

Solution: (a) $2n$ multiplications and n additions. (b) Given the results in (a), there are $3n$ operations. Thus, the time complexity is $\Theta(n)$.

$\square$