CS201 Discrete Math for Computer Science
Assignment 1

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1 Question 1

p: You get 100 marks on the final.
q: You get an A in this course.

1. $\neg p$
2. $p \wedge \neg q$
3. $p \rightarrow q$
4. $\neg p \rightarrow \neg q$
5. $p \rightarrow q$
6. $\neg p \wedge q$
7. $q \rightarrow p$

2 Question 2

1.

p	q	p ⊕ q	p ∧ q	(p ⊕ q) → (p ∧ q)
F	F	F	F	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

2.

p	q	p ↔ q	¬ p ↔ q	(p ↔ q) ⊕ (¬ p ↔ q)
F	F	T	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

3.

p	q	p ⊕ q	p ⊕ ¬ q	(p ⊕ q) → (p ⊕ ¬ q)
F	F	F	T	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

3 Question 3

1. Since

p	q	p → q	¬ p ∨ q
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

then the two propositions are equivalent.

2. Since

p	q	p ⊕ q	¬ p ∨ ¬ q
F	F	F	T
F	T	T	T
T	F	T	F
T	T	F	F

then the two propositions are not equivalent.

3. Since

p	q	r	p → q	q → r	(p → q) → r	p → (q → r)
F	F	F	T	T	F	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

then the two propositions are not equivalent.

4. Since

p	q	¬ (p → q)	¬ q ∧¬ (p → q)	¬ p
F	F	F	F	T
F	T	F	F	T
T	F	T	T	F
T	T	F	F	F

then the two propositions are not equivalent.

5. Since

p	q	r	(p ∨ q) → r	(p → r) ∧ (q → r)
F	F	F	T	T
F	F	T	T	T
F	T	F	F	F
F	T	T	T	T
T	F	F	F	F
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

then the two propositions are equivalent.

4 Question 4

1. To prove $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent, we can firstly prove $\neg(p \oplus q) \rightarrow (p \leftrightarrow q)$ (sufficiency), then prove $(p \leftrightarrow q) \rightarrow \neg(p \oplus q)$ (necessity), since $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.
Sufficiency

Step	*Reason*
1. $\neg(p \oplus q)$	*Premise*
2. $\neg((\neg p \wedge q) \vee (p \wedge \neg q))$	$p \oplus q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$
3. $\neg(\neg p \wedge q) \wedge \neg(p \wedge \neg q)$	*De Morgan's law*
4. $(p \vee \neg q) \wedge (\neg p \vee q)$	*Double negation law* & *De Morgan's law*
5. $(q \rightarrow p) \wedge (p \rightarrow q)$	*Useful law*
6. $p \leftrightarrow q$	$p \leftrightarrow q \equiv (q \rightarrow p) \wedge (p \rightarrow q)$

Necessity

Step	*Reason*
1. $p \leftrightarrow q$	*Premise*
2. $(p \to q) \land (q \to p)$	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
3. $(p \lor \neg q) \land (\neg p \lor q)$	*Useful law*
4. $\neg(\neg p \land q) \land \neg(p \land \neg q)$	*Double negation law & De Morgan's law*
5. $\neg((\neg p \land q) \lor (p \land \neg q))$	*De Morgan's law*
6. $\neg(p \oplus q)$	$p \oplus q \equiv (\neg p \land q) \lor (p \land \neg q)$

2.

Step	*Reason*
1. $\neg(p \to q) \to \neg q$	*Premise*
2. $\neg(\neg p \lor q) \to \neg q$	*Useful law*
3. $(p \land \neg q) \to \neg q$	*De Morgan's law & Double negation law*
4. $\neg(p \land \neg q) \lor \neg q$	*Useful law*
5. $(\neg p \lor q) \lor \neg q$	*De Morgan's law & Double negation law*
6. $\neg p \lor (q \lor \neg q)$	*Associative law*
7. $\neg p \lor T$	*Negation law*
8. T	*Identity law*

3.

Step	*Reason*
1. $(p \to q) \to ((r \to p) \to (r \to p))$	*Premise*
2. $(\neg p \lor q) \to ((\neg r \lor p) \to (\neg r \lor p))$	*Useful Law*
3. $\neg(\neg p \lor q) \lor (\neg(\neg r \lor p) \lor (\neg r \lor p))$	*Useful law*
4. $(p \land \neg q) \lor ((r \land \neg p) \lor (\neg r \lor p))$	*De Morgan's law & Double negation law*
5. $(p \land \neg q) \lor (((r \lor \neg r) \land (\neg p \lor \neg r)) \lor p)$	*Associative law & Distributive law*
6. $(p \land \neg q) \lor ((\neg p \lor \neg r) \lor p)$	*Negation law & Identity law*
7. $(p \land \neg q) \lor (T \lor \neg r)$	*Communicative law & Associative law*
8. T	*Domination law*

5 Question 5

Since $p \to q \equiv \neg p \lor q$ and $p \lor q \equiv q \lor p$, then we have

$$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \equiv (q \to p) \land (r \to q) \land (p \to r)$$

When $p,\ q,\ r$ have the same truth value, T or F, then all the three implications hold to be true (since implication is false if and only if the premise is true and the conclusion is false). It follows that the whole proposition is to be true (since conjunction is true if and only if propositions on both sides to be true).
However, when $p,\ q,\ r$ do not have the same truth value, one of the three implications will be false (since there at least one to be true and another one to be false in $p,\ q,\ r$). It follows that the whole proposition is to be false (since conjunction is true if and only if propositions on both sides to be true).

6 Question 6

1. equivalent.
   Since

$$(p \to q) \lor (p \to r) \equiv (\neg p \lor q) \lor (\neg p \lor r) \equiv \neg p \lor (q \lor r) \equiv p \to (q \lor r)$$

2. equivalent.
   Since

$$\neg p \to (q \to r) \equiv p \lor (q \to r) \equiv p \lor (\neg q \lor r) \equiv \neg q \lor (p \lor r) \equiv q \to (p \lor r)$$

3. equivalent.
   Since

$$(p \to q) \land (p \to r) \equiv (\neg p \lor q) \land (\neg p \lor r) \equiv \neg p \lor (q \land r) \equiv p \to (q \land r)$$

4. not equivalent.
   If we take p as T, q and r as F, then the former will be F but the latter will be T.

7 Question 7

Domain: all students in your class.

1. $\exists x(C(x) \land D(x) \land F(x))$
2. $\forall x(C(x) \lor D(x) \lor F(x))$
3. $\exists x(C(x) \land \neg D(x) \land F(x))$
4. $\forall x(\neg C(x) \lor \neg D(x) \lor \neg F(x))$
5. $\exists x_1 \exists x_2 \exists x_3 (C(x_1) \lor D(x_2) \lor F(x_3))$

8 Question 8

Domain: all people in the world
Fr=Fred, Ev=Evelyn, Na=Nancy

1. $\forall x F(x, Fr)$
2. $\forall x F(Ev, x)$
3. $\forall x \exists y \neg F(x, y)$
4. $\exists x_1 \exists x_2 ((x_1 \neq x_2) \land (F(Na, x_1) \land F(Na, x_2)))$
5. $\forall x \exists y \forall z (F(x, z) \leftrightarrow (y = z))$
6. $\exists x \exists y \forall z (F(x, z) \leftrightarrow ((x \neq y) \land (y = z)))$

9 Question 9

1. $\exists y \forall x \forall z (\neg T(x, y, z) \land \neg Q(x, y))$
2. $\forall x \forall y \neg P(x, y) \lor \exists x \exists y \neg Q(x, y)$
3. $\forall x \forall y ((Q(x, y) \land \neg Q(y, x)) \lor (Q(y, x) \land \neg Q(x, y)))$
4. $\exists x \forall y \exists z \neg T(x, y, z)$

10 Question 10

Step	*Reason*
1. $p \land q$	*Premise*
2. $p \to \neg(q \land r)$	*Premise*
3. $s \to r$	*Premise*
4. p	*Simplification using* (1)
5. q	*Simplification using* (1)
6. $\neg(q \land r)$	*Modus ponens using* (2) *and* (4)
7. $\neg q \lor \neg r$	*De Morgen's law using* (6)
8. $\neg r$	*Disjunctive syllogism using* (5) *and* (7)
9. $\neg s$	*Modus tollens using* (3) *and* (8)

11 Question 11

1. W(x): x enjoys whale watching.
   O(x): x cares about ocean pollution.
   Domain: everyone in this class.
   Premises: $\exists x W(x)$, $\forall x(W(x) \to O(x))$.
   Conclusion: $\exists x O(x)$.

Step	*Reason*
1. $\exists x W(x)$	*Premise*
2. $W(a)$	*Existential instantiation from* (1)
3. $\forall x(W(x) \to O(x))$	*Premise*
4. $W(a) \to O(a)$	*Universal instantiation from* (3)
5. $O(a)$	*Modus ponens from* (3) *and* (4)
6. $\exists x O(x)$	*Existential generalization from* (5)

2. P(x): x owns a personal computer.
   W(x): x can use a word processing program.
   Domain: students in this class.
   Premises: $\forall x P(x)$, $\forall x(P(X) \to W(x))$.
   Conclusion: $\exists x W(x)$.

Step	*Reason*
1. $\forall x P(x)$	*Premise*
2. $P(a)$	*Universal instantiation from* (1)
3. $\forall x(P(x) \to W(x))$	*Premise*
4. $P(a) \to W(a)$	*Universal instantiation from* (3)
5. $W(a)$	*Modus ponens from* (3) *and* (4)
6. $\exists x W(x)$	*Existential generalization from* (5)

3. D(x): x has taken a course in discrete mathematics.
   A(x): x can take a course in algorithms.
   Domain: the five roommates.
   Premises: $\forall x D(x)$, $\forall x(D(x) \to A(x))$.
   Conclusion: $\forall x A(x)$.

Step	*Reason*
1. $\forall x D(x)$	*Premise*
2. $D(a)$ *for arbitrary a*	*Universal instantiation from* (1)
3. $\forall x(D(x) \to A(x))$	*Premise*
4. $D(a) \to A(a)$ *for arbitrary a*	*Universal instantiation from* (3)
5. $A(a)$ *for arbitrary a*	*Modus ponens from* (3) *and* (4)
6. $\forall x A(x)$	*Universal generalization from* (5)

12 Question 12

1. $\exists n \in \mathbb{N}(n^3 + 6n + 5 \text{ } is \text{ } odd \wedge n \text{ } is \text{ } odd)$.

2. The original statement in (a) is true.
   Suppose the negation statement is true, then $\exists k \in \mathbb{Z}, n = 2k + 1$. It follows that

$$\begin{aligned} n^3 + 6n + 5 &= (2k+1)^3 + 6(2k+1) + 5 \\ &= 8k^3 + 12k^2 + 18k + 12 \\ &= 2(4k^3 + 6k^2 + 9k + 6) \\ &= 2m \ (m = 4k^3 + 6k^2 + 9k + 6 \in \mathbb{Z}) \end{aligned}$$

   which means $n^3 + 6n + 5$ is even, contradicting to what the statement said $n^3 + 6n + 5$ is odd. So the original statement is true.

13 Question 13

Suppose $a^2 + b^2 = 2k(k \in \mathbb{Z})$, then we have $a^2 + 2ab + b^2 = 2k + 2ab$, i.e. $(a + b)^2 = 2(k + ab)$. Since $k, a, b \in \mathbb{Z}$, it is obvious that $(a + b)^2$ is even. Therefore, $(a + b)$ is also even.

14 Question 14

Suppose $a = 2, b = \frac{1}{2}$ that both are rational numbers. However $a^b = \sqrt{2}$ is irrational number.So the origin proposition is false.

15 Question 15

Suppose $\sqrt[3]{2}$ is rational. Then there exists integers a,b such that $\sqrt[3]{2} = \frac{a}{b}$, where $b \neq 0$ and a, b have no common factor.
Since $\sqrt[3]{2} = \frac{a}{b}$, it follows that $2b^3 = a^3$. So a^3 is even. Therefore, a is even.
Since a^3 is even, $\exists k \in \mathbb{Z}, a = 2k$. Thus, $b^3 = 4k^3$, which implies that b^3 is even, so b is even.
As a result, a and b have a common factor 2, which contradicts the assumption.

16 Question 16

As the given theorem said, we have an irrational number $\sqrt{6}$, then $5 + 2\sqrt{6}$ also irrational. Since $5 + 2\sqrt{6} = (\sqrt{2} + \sqrt{3})^2$, then $\sqrt{2} + \sqrt{3}$ is also irrational.
Here we will prove if x^2 is irrational then x is also irrational.
Suppose x^2 is irrational but x is rational. According to the product of rational numbers is also a rational number, x^2 must be rational, which contradicts to hypothesis. So x^2 is irrational then x is irrational.

17 Question 17

Let h as the highest bid of others, p as my payoff.
When $v_n < h$, then

$$\begin{gathered} p = 0,\ if\ b_n = v_n \\ p < 0,\ if\ b_n > v_n \\ p = 0,\ if\ b_n < v_n \\ \therefore p(b_n = v_n) \geq p(b_n \neq v_n) \end{gathered}$$

When $v_n = h$, then

$$\begin{gathered} p = 0,\ if\ b_n = v_n \\ p = 0,\ if\ b_n > v_n \\ p = 0,\ if\ b_n < v_n \\ \therefore p(b_n = v_n) = p(b_n \neq v_n) \end{gathered}$$

When $v_n > h$, then

$$\begin{gathered} p = v_n - h,\ if\ b_n = v_n \\ p = v_n - h,\ if\ b_n > v_n \\ p = 0,\ if\ b_n < v_n,\ b_n < h \\ p = v_n - h,\ if\ b_n < v_n,\ h < b_n < v_n \\ \therefore p(b_n = v_n) \geq p(b_n \neq v_n) \end{gathered}$$

After all, $b_n = v_n$ is the best choice.