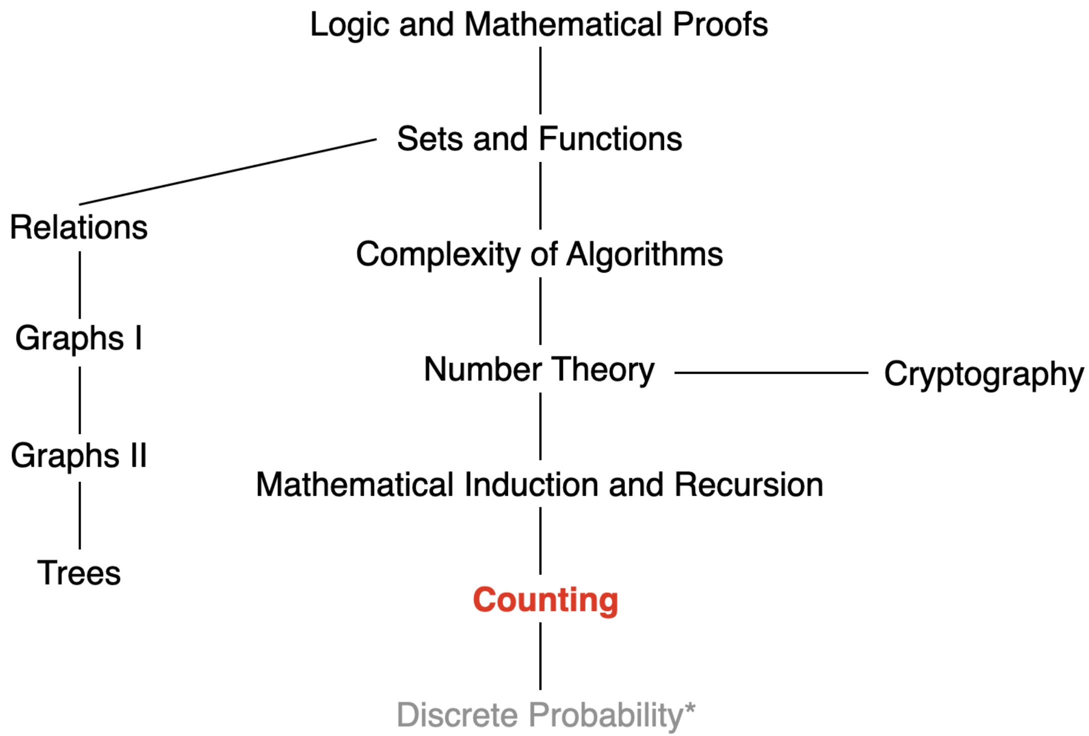
Discrete Mathematics for Computer Science

Lecture 14: Counting

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Solving Linear Recurrence Relations, ...

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Solving Linear Recurrence Relations

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

By induction, such a recurrence relation is uniquely determined by this recurrence relation and k initial conditions $a_0, a_1, \ldots, a_{k-1}$.

Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, ...,$$

where α_1 and α_2 are constants.

Solve Linear Recurrence Relations:

- Solve r_1 and r_2 with $r^2 - c_1 r - c2 = 0$.
- Solve α_1 and α_2 with the initial conditions.

Solving Linear Recurrence Relations of Degree k

Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

$$a_n = \sum_{i-1}^{k} c_i a_{n-i}.$$

The characteristic equation (CE) is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0.$$

Theorem: If this CE has k distinct roots r_i, then the solutions to the recurrence are of the form

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i's are constants.

The Case of Degenerate Roots: Degree Two

Theorem: If the $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then

$$a_n = (\alpha_1 + \alpha_2 n) r_0^n,$$

for all $n \geq 0$ and two constants α_1 and α_2.

The Case of Degenerate Roots: Degree k

Theorem: Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then,

$$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$ and constants $\alpha_{i,j}$.

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

In particular, write $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$ as

$$(r - r_1)^{m_1} (r - r_2)^{m_2} \ldots (r - r_t)^{m_t} = 0.$$

What if There Is No Real Root?

If our recurrence relation has two complex conjugate roots, we could write our solution the way we did in the case where we had two real roots: $a_n = c_1(a+bi)^n + c_2(a-bi)^n$. However, there is a more compact way to write our solution in terms of real numbers. We can write $a+bi = re^{i\theta}$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$. Then $a - bi = re^{-i\theta}$. Using DeMoivre's Theorem $(re^{i\theta})^n = r^n(\cos n\theta + i\sin n\theta)$. Thus

$$c_1(re^{i\theta})^n + c_2(re^{-i\theta})^n = r^n[(c_1+c_2)\cos n\theta + i(c_1-c_2)\sin n\theta].$$

If we set $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$, then our solution is

$$a_n = r^n(C_1\cos n\theta + C_2\sin n\theta).$$

Again we can use k_1 and k_2 to solve a system of equations in C_1 and C_2.

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Linear Nonhomogeneous Recurrence Relations

Definition: A linear nonhomogeneous relation with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Example:

- $a_n = a_{n-1} + 2^n$ $\quad a_n = a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ $\quad a_n = a_{n-1} + a_{n-2}$
- $a_n = 3a_{n-1} + n3^n$ $\quad a_n = 3a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ $\quad a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Linear Nonhomogeneous Recurrence Relations

Every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.

Linear Nonhomogeneous Recurrence Relations

Proof: Suppose $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation,

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + \ldots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.
Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Linear Nonhomogeneous Recurrence Relations

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.

The key is to find the particular solution to the linear nonhomogeneous relation. However, there is no general method for finding such a solution.

Example 1

There are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants.

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- Compute $a_n^{(h)}$
- Compute $a_n^{(p)}$
- Initial condition

Example 1

To compute $a_n^{(h)}$:
The characteristic equation is

$$r^2 - 3r = 0.$$

The roots are $r_1 = 3$ and $r_2 = 0$. By So, assume that

$$a_n^{(h)} = \alpha 3^n.$$

To compute $a_n^{(p)}$: Try $a_n^{(p)} = cn + d$. Thus,

$$cn + d = 3(c(n-1) + d) + 2n.$$

We get $c = -1$ and $d = -3/2$. Thus, $a_n^{(p)} = -n - 3/2$.

Example 1

To compute $a_n^{(h)}$: $a_n^{(h)} = \alpha 3^n$.

To compute $a_n^{(p)}$: $a_n^{(p)} = -n - 3/2$.

Initial condition:

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - n - 3/2.$$

Base on the initial condition $a_1 = 3$. We have $3 = -1 - 3/2 + 3\alpha$, which implies $\alpha = 11/6$. Thus, $a_n = -n - 3/2 + (11/6)3^n$.

Example 2

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$. (Since we do not provide the initial conditions, obtain the general form would be sufficient.)

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- Try $a_n^{(p)} = C \cdot 7^n$:

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$$

Thus, $C = 49/20$, and $a_n^{(p)} = (49/20)7^n$.

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Linear Nonhomogeneous Recurrence Relations

For previous two examples, we made a guess that there are solutions of a particular form. This was not an accident.

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

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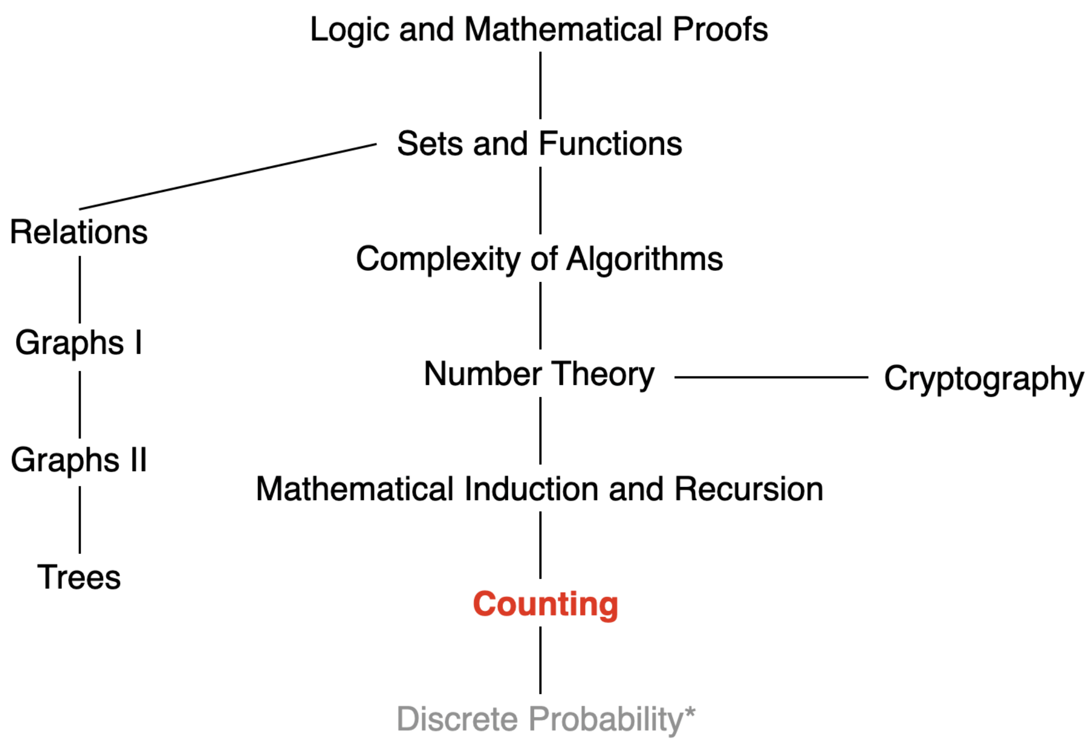


Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Solving Linear Recurrence Relations, Generalized Permutations and Combinations, Generating Function, ...

Generalized Permutations and Combinations

- Permutations with repetition
- Permutations with indistinguishable objects
- Combinations with repetition

Repetition: Distinct objects; each object can be selected multiple times

Indistinguishable objects: E.g., "SUCCESS"

Permutations with Repetition

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet? 26^r

Theorem: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word SUCCESS?

Solution:

- The three S's can be placed among the seven positions in $C(7,3)$ different ways.
- The two C's can be placed in $C(4,2)$ ways.
- The U can be placed in $C(2,1)$ ways.
- The E can be placed in $C(1,1)$ way.

From the product rule,

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

Theorem: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, . . . , and n_k indistinguishable objects of type k, is

$$C(n,n_1)C(n-n_1,n_2)\cdots C(n-n_1-\cdots n_{k-1},n_k)=\frac{n!}{n_1!n_2!\cdots n_k!}.$$

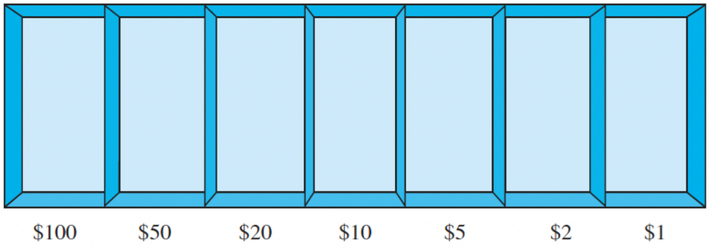
Combinations with Repetition

Example: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?

Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

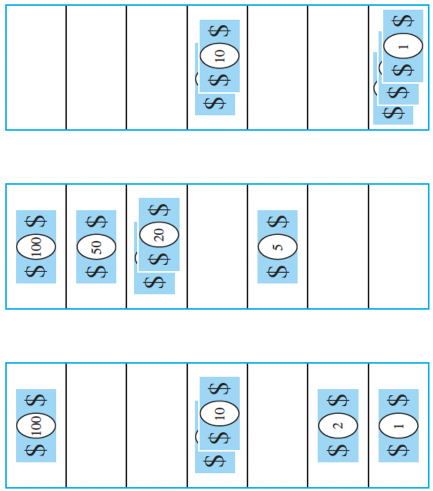
Combinations with Repetition

Solution: Suppose that a cash box has seven compartments, one to hold each type of bill.



- These compartments are separated by six dividers
- The choice of five bills corresponds to placing five markers in the compartments holding different types of bills.

Combinations with Repetition



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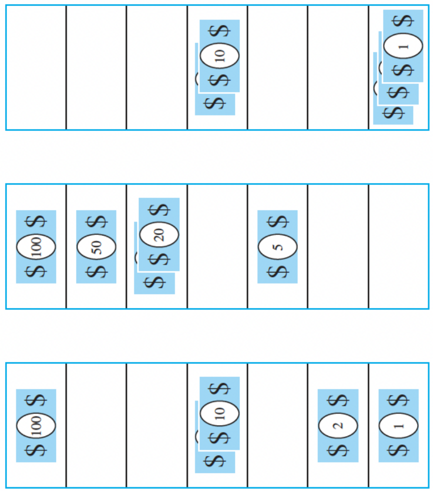
The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row with a total of 11 positions.

Combinations with Repetition



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Consequently, the number of ways to select the five bills is the number of ways to select the positions of the five stars from the 11 positions.

$$C(11,5) = \frac{11!}{5!6!} = 462$$

Combinations with Repetition

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

In the previous example:

- Selecting five bills: $r = 5$
- Seven types of bills: $n = 7$

Combinations with Repetition: Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: This is equivalent to finding the number of ways of selecting 11 items from three types of items, so that x_1 items of type one, x_2 items of type two, and x_2 items of type three:

$$C(3 + 11 - 1, 11) = 78$$

Combinations with Repetition: Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers?

Solution: A solution corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three:

- At least one item of type one, two items of type two, and three items of type three.

A solution corresponds to a choice of one item of type one, two of type two, and three of type three, together with a choice of five additional items of any type.

$$C(3+5-1, 5) = 21$$

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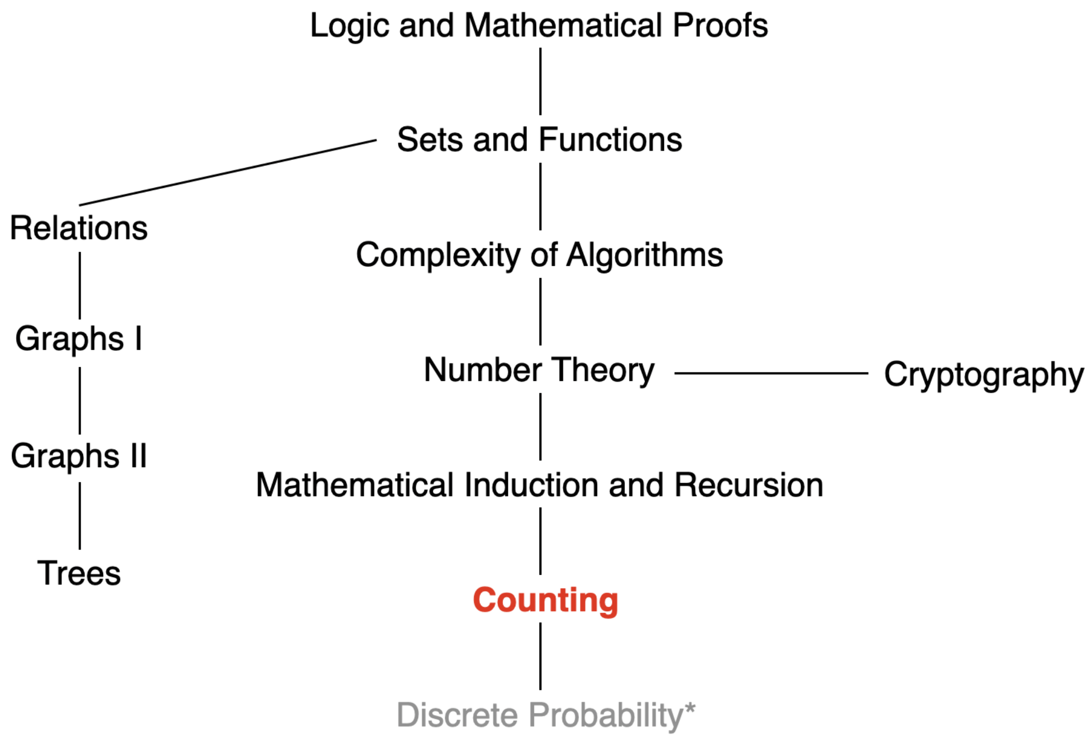

Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Solving Linear Recurrence Relations, Generalized Permutations and Combinations, Generating Function, ...

Generating Function

- Definition of generating function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Generating Function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

Example:

- The sequence $\{a_k\}$ with $a_k = 3$

$$\sum_{k=0}^{\infty} 3x^k$$

- The sequence $\{a_k\}$ with $a_k = 2^k$

$$\sum_{k=0}^{\infty} 2^k x^k$$

Generating Function: Finite Series

A finite sequence a_0, a_1, . . . , a_n can be easily extended by setting $a_{n+1} = a_{n+2} = ... = 0$.

The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n, i.e.,

$$G(x) = a_0 + a_1 x + ... + a_n x^n.$$

Example: What is the generating function for the sequence a_0, a_1, ... , a_m, with $a_k = C(m, k)$?

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + ... + C(m, m)x^m.$$

Based on binomial theorem, $G(x) = (1 + x)^m$.

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Generating Function

- Definition of generation function
- Useful facts
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Useful Facts

- For $|x| < 1$, function $G(x) = 1/(1-x)$ is the generating function of the sequence 1, 1, 1, 1, . . . ,

$$1/(1-x) = 1 + x + x^2 + \dots$$

- For $|ax| < 1$, function $G(x) = 1/(1-ax)$ is the generating function of the sequence 1, a, a^2, a^3, . . . ,

$$1/(1-ax) = 1 + ax + a^2x^2 + \dots$$

- For $|x| < 1$, $G(x) = 1/(1-x)^2$ is the generating function of the sequence 1, 2, 3, 4, 5, . . .

$$1/(1-x)^2 = 1 + 2x + 3x^2 + \dots$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x_k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Example 1: To obtain the corresponding sequence of $G(x) = 1/(1-x)^2$: Consider $f(x) = 1/(1-x)$ and $g(x) = 1/(1-x)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, 1, 1, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) x^k.$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x_k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Example 2: To obtain the corresponding sequence of $G(x) = 1/(1-ax)^2$ for $|ax| < 1$:

Consider $f(x) = 1/(1-ax)$ and $g(x) = 1/(1-ax)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, a, a^2, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) a^k x^k.$$

Useful Generating Functions

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$$

$$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$$

$$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$$

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$

$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Useful Generating Functions

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$
$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$
$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Extended Binomial Coefficient

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Here, u can be any real number, e.g., negative integers, non-integers, ...

Extended Binomial Coefficient

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Example: Find the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Taking $u = -2$ and $k = 3$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Taking $u = 1/2$ and $k = 3$

$$\begin{aligned} \binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16. \end{aligned}$$

Extended Binomial Coefficient

When u is a negative integer:

$$
\begin{aligned}
\binom{-n}{r} &= \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \\
&= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} \\
&= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} \\
&= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} \\
&= (-1)^r \binom{n+r-1}{r} \\
&= (-1)^r C(n+r-1, r).
\end{aligned}
$$

Extended Binomial Theorem

Theorem: Let x be a real number with $|x| < 1$ and let u be a real number. Then,

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Example:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

Generating Function

- Definition of generation function
- Useful facts
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Generating Function and Combinations with Repetitions

Recall the following example:

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers?

This type of counting problem can be solved with generating function.

Generating Function and Combinations with Repetitions

Formally, generating functions can also be used to solve counting problems of the following type:

$$e_1 + e_2 + \cdots + e_n = C,$$

where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint.

Example 1

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

By enumerating all possibilities, we have that the coefficient of x^{17} in this product is 3.

Example 2

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution: This corresponds to the coefficient of x^8 of expansion

$$(x^2 + x^3 + x^4)^3$$

This coefficient equals 6.

Example 3

Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in the cases

- Case 1: when the order does not matter

  E.g., three \$1 tokens; one \$1 token and a \$2 token

- Case 2: when the order does matter

  E.g., three \$1 tokens; a \$1 token and then a \$2 token; a \$2 token and then a \$1 token

Example 3

Case 1: when the order does not matter

The answer is the coefficient of x^r in the generating function

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots).$$

Case 2: when the order does matter

The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x+x^2+x^5)^n$$

Because any number of tokens may be inserted,

$$1+(x+x^2+x^5)+(x+x^2+x^5)^2+\cdots=\frac{1}{1-(x+x^2+x^5)}$$

Example 4

Use generating functions to find the number of r-combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1+x)^n$$

But by the binomial theorem, we have

$$f(x) = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Thus, $\binom{n}{r}$ is the answer.

Example 5

Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (1 + x + x^2 + \cdots)^n.$$

As long as $|x| < 1$, we have $1 + x + x^2 + \cdots = 1/(1-x)$, so

$$G(x) = 1/(1-x)^n = (1-x)^{-n}.$$

Applying the extended binomial theorem

$$(1-x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

Hence, the coefficient of x^r equals $\binom{-n}{r}(-1)^r = C(n+r-1, r)$.

Example 6

Use generating functions to find the number of ways to select r objects of n different kinds if we must select at least one object of each kind.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (x + x^2 + x^3 + \cdots)^n = x^n(1 + x + x^2 + \cdots)^n = x^n/(1-x)^n.$$

$$\begin{aligned}
G(x) &= x^n/(1-x)^n \\
&= x^n \cdot (1-x)^{-n} \\
&= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\
&= x^n \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)(-1)^r x^r
\end{aligned}
\qquad
\begin{aligned}
&= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r} \\
&= \sum_{t=n}^{\infty} C(t-1, t-n) x^t \\
&= \sum_{r=n}^{\infty} C(r-1, r-n) x^r.
\end{aligned}$$

Hence, there are $C(r-1, r-n)$ ways to select r objects of n different kinds if we must select at least one object of each kind.

Generating Function and Combinations with Repetitions

- Based on the combination problem, transfer the problem as finding the coefficient of x^r of a generating function, e.g.,

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n$$

- Find the coefficient of x^r
  - Enumerate all possibilities or
  - Use useful generating functions

Generating Function

- Definition of generation function
- Useful facts
- Generating function and combinations with repetition
- Generating function to solve recurrence relations

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. We aim to first derive the formulation of $G(x)$.

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2, \end{aligned}$$

Thus, $G(x) - 3xG(x) = (1 - 3x)G(x) = 2$:

$$G(x) = \frac{2}{(1-3x)}.$$

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: We aim to first derive the formulation of $G(x)$.

$$G(x) = \frac{2}{(1 - 3x)}.$$

Then, derive a_k using the identity $1/(1 - ax) = \sum_{k=0}^{\infty} a_k x^k$. That is,

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^k$.

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: We extend this sequence by setting $a_0 = 1$. We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1-10x),
\end{aligned}$$

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$.

Solution: Thus,

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Thus, $a_n = \frac{1}{2}(8^n + 10^n)$.

Generating function to solve recurrence relations

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Based on the recurrence relations, derive the formulation of $G(x)$.
- Using identities (or the useful facts of generating functions), derive sequence $\{a_k\}$.

Next Lecture