

Discrete Mathematics for Computer Science

Lecture 20: Graph

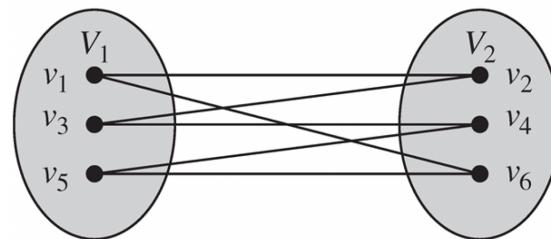
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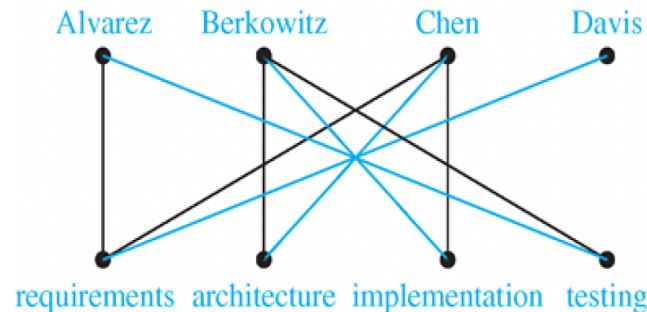


Bipartite Graphs

Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 .



Matching the elements of one set to elements in another. A matching is a subset of E s.t. no two edges are incident with the same vertex.



Bipartite Graphs and Matchings

A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching** from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1 .

Proof of Hall's Theorem

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1 .

Proof: “only if”

Suppose that there is a **complete matching** M from V_1 to V_2 . Consider an arbitrary subset $A \subseteq V_1$.

Then, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2 .

Thus, there are **at least** as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1 .

Hence, $|N(A)| \geq |A|$.

Proof of Hall's Theorem

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1 .

Proof: “if”, use **strong induction** to prove it.

Basic Step: $|V_1| = 1$

Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) , and $|V_1| = j \leq k$, then there is a complete matching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

Proof of Hall's Theorem

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

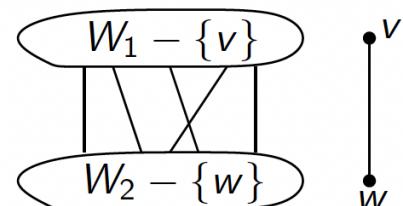
- (i) For **all** integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to **at least $j + 1$ elements** of W_2 .
- (ii) For **some** integer j with $1 \leq j \leq k$, there is a subset W'_1 of j vertices such that there are **exactly j neighbors** of these vertices in W_2 .

Proof of Hall's Theorem

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

- (i) For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2 .

We select a vertex $v \in W_1$ and an element $w \in N(\{v\})$. The inductive hypothesis tells us there is a complete matching from $W_1 - \{v\}$ to $W_2 - \{w\}$.



- (ii) For some integer j with $1 \leq j \leq k$, there is a subset W'_1 of j vertices such that there are exactly j neighbors of these vertices in W_2 .

Proof of Hall's Theorem

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

- (i) For **all** integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to **at least $j + 1$ elements** of W_2 .
- (ii) For **some** integer j with $1 \leq j \leq k$, there is a subset W'_1 of j vertices such that there are **exactly j neighbors** of these vertices in W_2 .

Let W'_2 be the set of these neighbors. Then by i.h., there is a complete matching from W'_1 to W'_2 . Now consider the graph $K = (W_1 - W'_1, W_2 - W'_2)$.

We will show that the condition $|N(A)| \geq |A|$ is met for all subsets A of $W_1 - W'_1$.

If not, there would be a subset of t vertices of $W_1 - W'$ such that have fewer than t vertices of $W_2 - W'_2$ as neighbors.

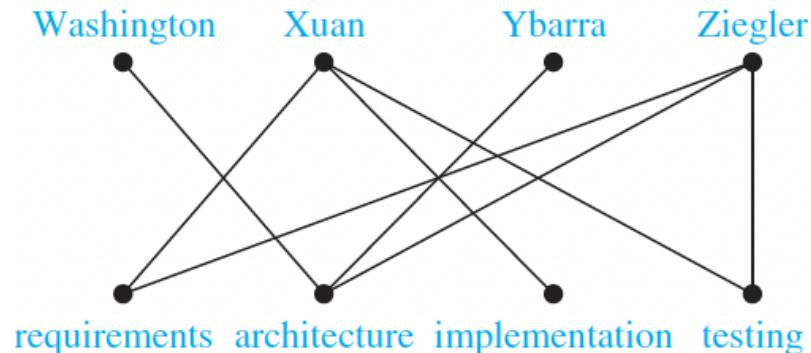
Then, the set of $j + t$ vertices of W_1 (i.e., these t vertices and the j vertices removed) has fewer than $j + t$ neighbors in W_2 , contradicting the hypothesis that $|N(A)| \geq |A|$ for all $A \subseteq W_1$.



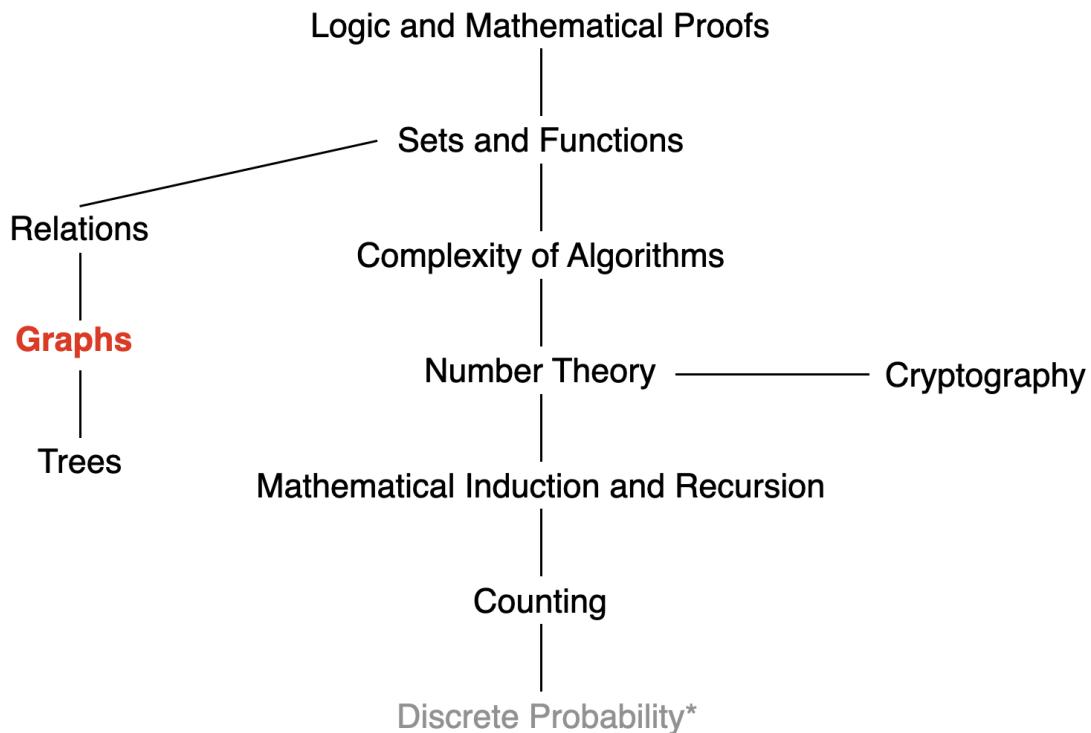
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Hall's Theorem: Example



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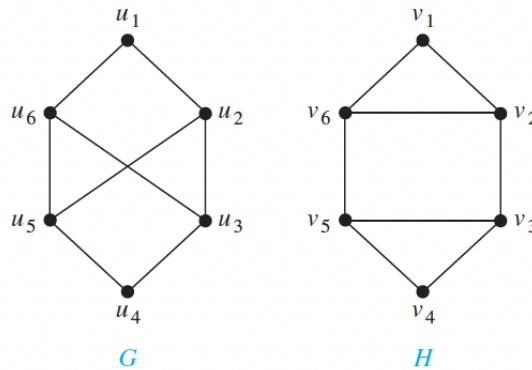


Graph and terminologies, representing graphs and graph isomorphism,
connectivity, Euler and Hamilton path, ...



Paths and Isomorphism

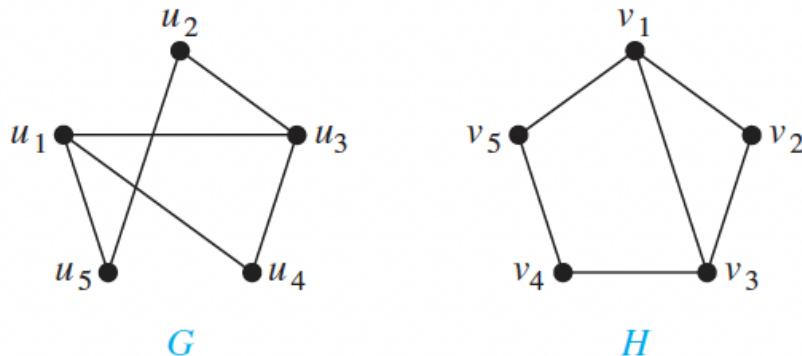
The existence of a simple circuit of length k is **isomorphic invariant**. This can be used to **construct mappings** that may be isomorphisms.



Not isomorphic. H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1 , whereas G has no simple circuit of length three.

Paths and Isomorphism

The existence of a simple circuit of length k is **isomorphic invariant**. This can be used to **construct mappings** that may be isomorphisms.



Because many isomorphic invariants (e.g., number of vertices/edges, degree, circuit) agree, G and H may be isomorphic. Let $f(u_1) = v_3$, $f(u_4) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_5$, and $f(u_5) = v_4$. We can show that f is an isomorphism, so G and H are isomorphic.

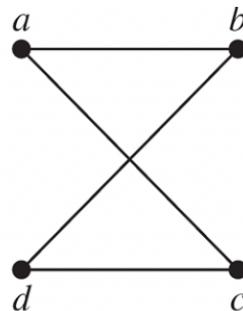
Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix A with respect to the ordering v_1, v_2, \dots, v_n of vertices. The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the (i, j) -th entry of A^r .

Note: with directed or undirected edges, multiple edges and loops allowed

Counting Paths between Vertices:

How many paths of length 4 are there from a to d in the graph G ?



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

$a, b, a, b, d;$
 $a, c, a, b, d;$

$a, b, a, c, d;$
 $a, c, a, c, d;$

$a, b, d, b, d;$
 $a, c, d, b, d;$

$a, b, d, c, d;$
 $a, c, d, SUSTech$

Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix A with respect to the ordering v_1, v_2, \dots, v_n of vertices. The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the (i, j) -th entry of A^r .

Proof (by induction):

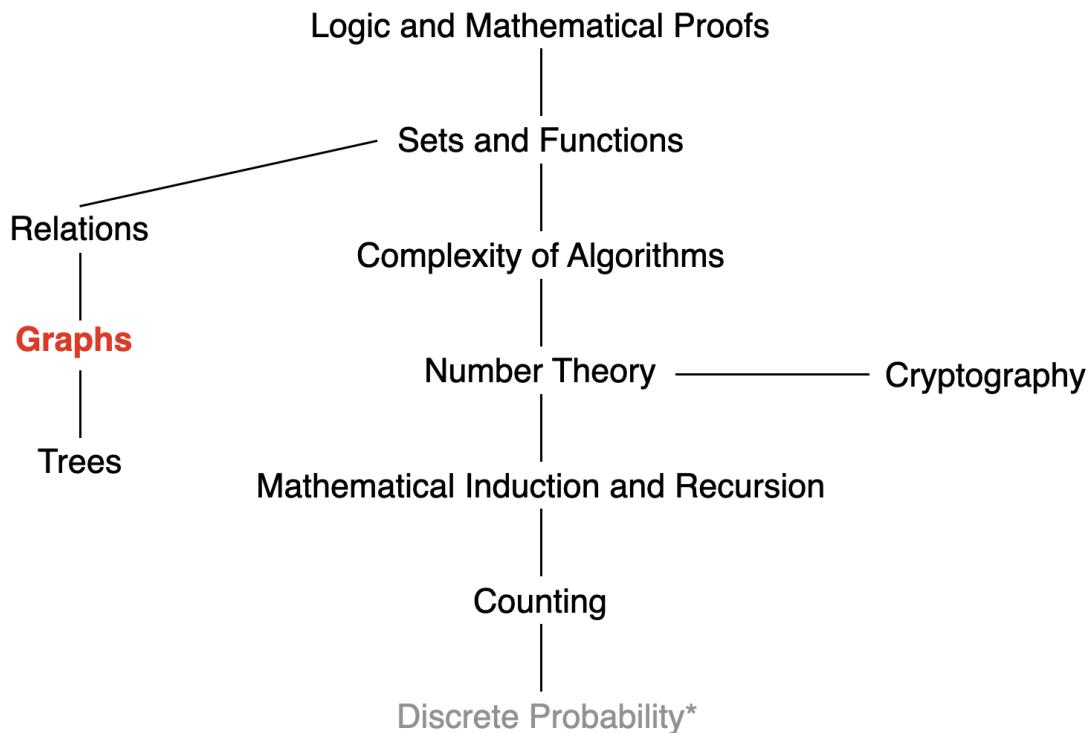
- **Basic Step:** The number of paths from v_i to v_j of length 1 is the (i, j) -th entry of A .
- **Inductive hypothesis:** Assume that the (i, j) -th entry of A^r is the number of different paths of length r from v_i to v_j .
- **Inductive Step:** $A^{r+1} = A^r A$. The (i, j) -th entry of A^{r+1} equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

where b_{ik} is the (i, k) -th entry of A^r . By the inductive hypothesis, b_{ik} is the number of paths of length r from v_i to v_k .

- **Inductive Conclusion:** (i, j) -th entry of A^{r+1} counts all paths of length $r + 1$ for all possible intermediate vertices v_k .

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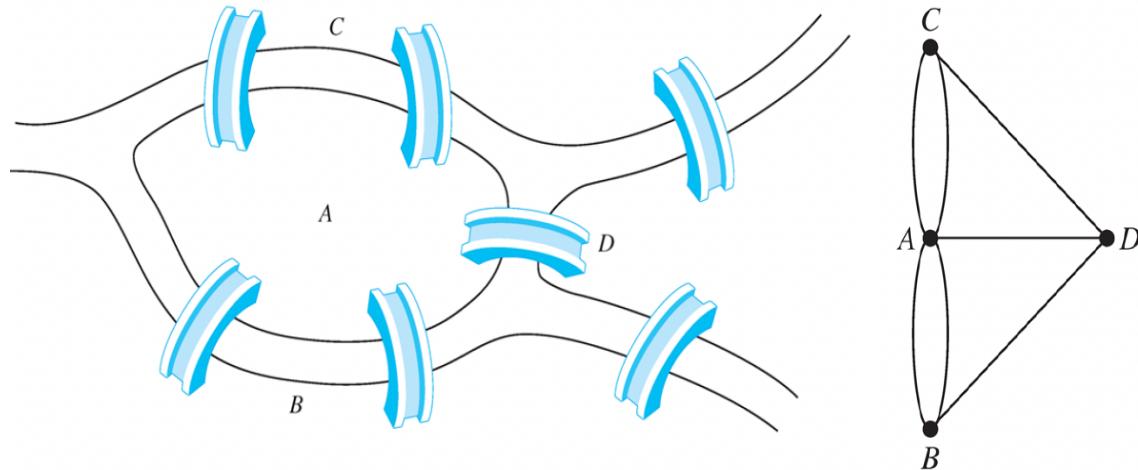


Graph and terminologies, representing graphs and graph isomorphism,
connectivity, Euler and Hamilton path, ...



Euler Paths

Königsberg seven-bridge problem: People wondered whether it was possible to start at some location in the town, travel across **all the bridges once** without crossing any bridge twice, and **return to the starting point**.

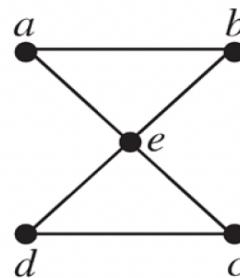


Euler Paths and Circuits

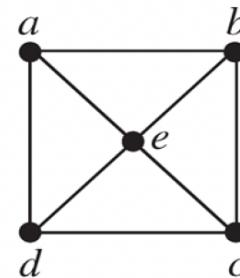
Definition: An **Euler circuit** in a graph G is a **simple circuit** containing every edge of G . An Euler path in G is a simple path containing every edge of G .

Recall that a path or circuit is **simple** if it does not contain the same edge more than once.

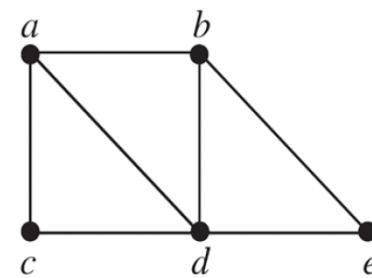
Example: Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



G_1



G_2



G_3

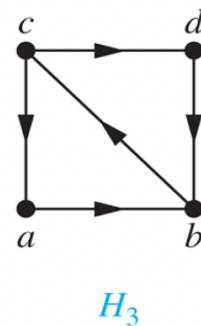
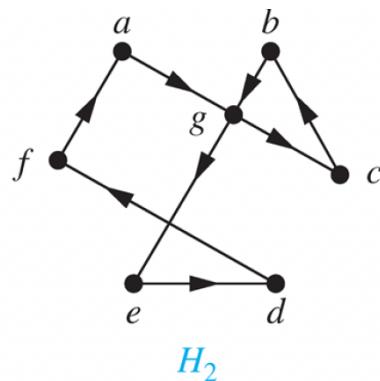
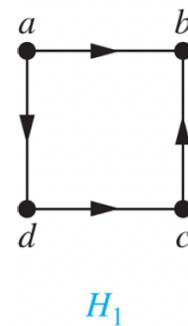
G_1 : an Euler circuit, e.g., a, e, c, d, e, b, a ;

G_2 : neither; G_3 : an Euler path, e.g., a, c, d, e, b, d, a, b

Euler Paths and Circuits

Definition: An **Euler circuit** in a graph G is a **simple circuit** containing every edge of G . An Euler path in G is a simple path containing every edge of G .

Example: Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



H_1 : neither; H_2 : an Euler circuit, e.g., a, g, c, b, g, e, d, f, a; H_3 : an Euler path, e.g., c, a, b, c, d, b

Necessary Conditions for Euler Circuits and Paths

Consider undirected graph:

Euler Circuit \Rightarrow The degree of every vertex must be even

- Each time the circuit passes through a vertex, it contributes two to the vertex's degree.
- The circuit starts with a vertex a and ends at a , then contributes two to $\deg(a)$.

Euler Path \Rightarrow The graph has exactly two vertices of odd degree

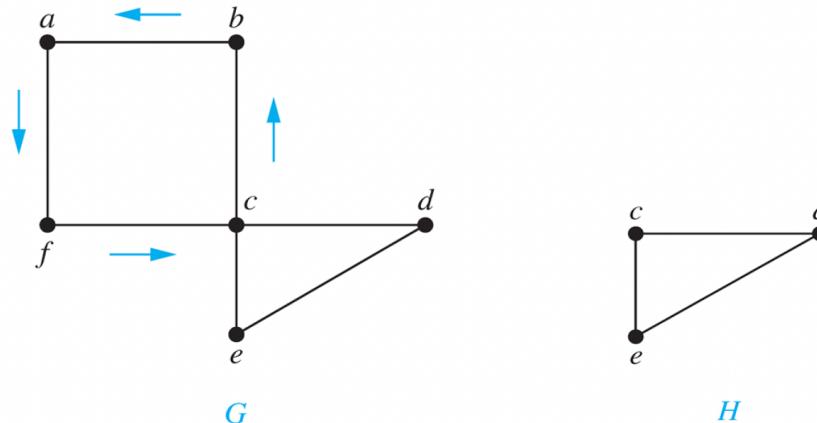
- The initial vertex and the final vertex of an Euler path have odd degree.

Are these conditions also sufficient?



Sufficient Conditions for Euler Circuits and Paths

G is a connected multigraph with ≥ 2 vertices, all of even degree.



We will form a simple circuit that begins at an arbitrary vertex a of G , building it edge by edge.

The path **begins** at a , and it must **terminate** at a . This is because every time we enter a vertex other than a , we can leave it.

An Euler circuit has been constructed if all the edges have been used. Otherwise, consider the subgraph H obtained from G by **deleting the edges** already used. Every vertex in H has even degree ...

Algorithm for Constructing an Euler Circuit

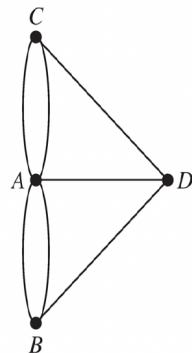
ALGORITHM 1 Constructing Euler Circuits.

```
procedure Euler(G: connected multigraph with all vertices of even degree)
  circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
  H := G with the edges of this circuit removed
  while H has edges
    subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of circuit
    H := H with edges of subcircuit and all isolated vertices removed
    circuit := circuit with subcircuit inserted at the appropriate vertex
  return circuit {circuit is an Euler circuit}
```

Euler Circuits and Paths

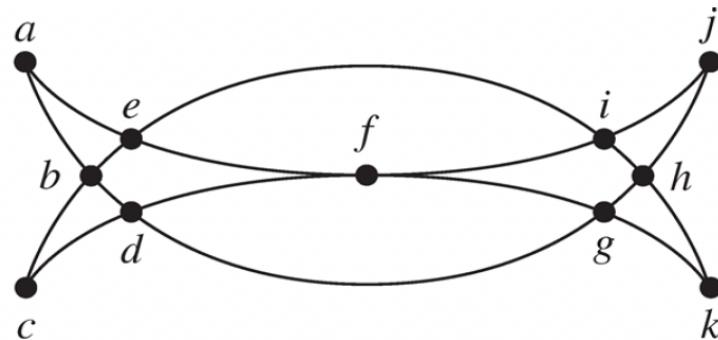
Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.



No Euler circuit, no Euler path

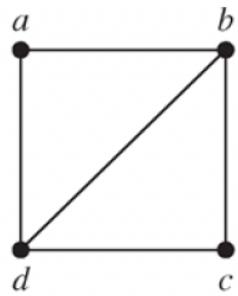
Euler Circuits and Paths: Example



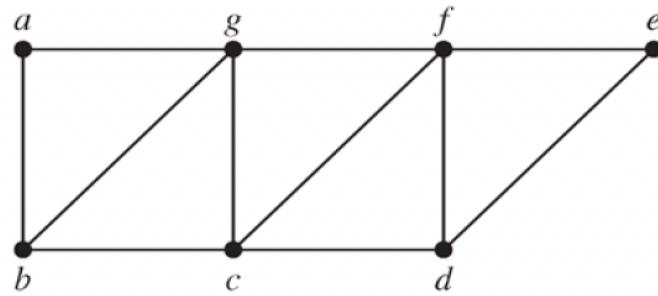
It has such a circuit because all its vertices have even degree.
We will use the algorithm to construct an Euler circuit:

- Form the circuit $a, b, d, c, b, e, i, f, e, a$;
- Obtain the subgraph H by deleting the edges in this circuit and all vertices that become isolated;
- Form the circuit $d, g, h, j, i, h, k, g, f, d$ in H ;
- Splice this new circuit into the first circuit at the appropriate place produces the Euler circuit
 $a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a$.

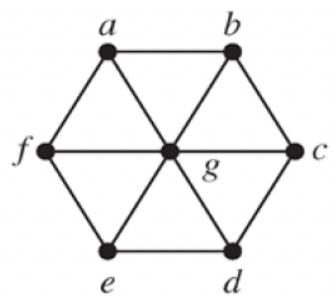
Euler Circuits and Paths: Example



G_1



G_2



G_3

- G_1 contains exactly two vertices of odd degree, namely, b and d . Hence, it has an Euler path that must have b and d as its endpoints.
- G_2 has exactly two vertices of odd degree, namely, b and d . So it has an Euler path that must have b and d as endpoints.
- G_3 has no Euler path because it has six vertices of odd degree.

Applications of Euler Paths and Circuits

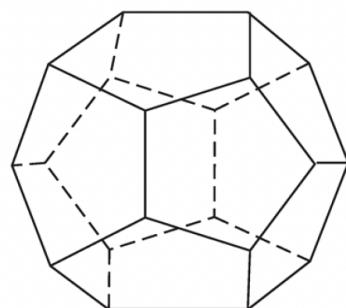
Finding a path or circuit that traverses each

- street in a neighborhood
- road in a transportation network
- link in a communication network
- ...

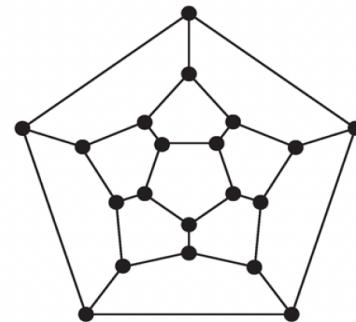
Hamilton Paths and Circuits

Euler paths and circuits contained every edge only once.

What about containing **every vertex** exactly once?



(a)

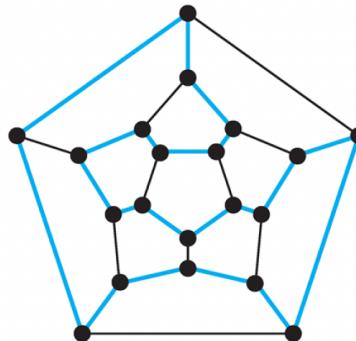


(b)

Hamilton Paths and Circuits

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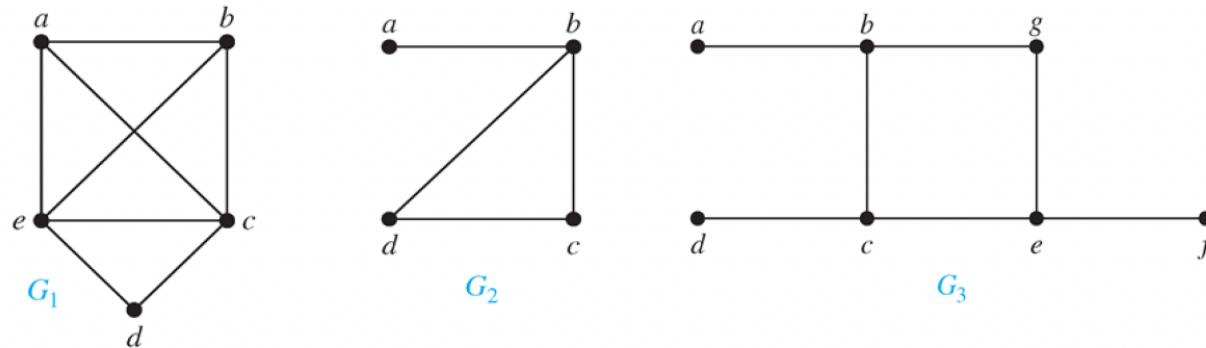
What about containing **every vertex** exactly once?



Hamilton Paths and Circuits

Definition: A simple path in a graph G that passes through every vertex exactly once is called a **Hamilton path**, and a simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**.

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



- G_1 has a Hamilton circuit: a, b, c, d, e, a ;
- G_2 has no Hamilton circuit (because containing every vertex must contain the edge a, b twice), but it has a Hamilton path;
- G_3 has neither, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Sufficient Conditions for Hamilton Circuits

No known simple necessary and sufficient conditions are known for the existence of a Hamilton circuit.

But, there are some useful **sufficient conditions**.

Dirac's Theorem: If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is $\geq n/2$, then G has a Hamilton circuit.

Ore's Theorem: If G is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices, then G has a Hamilton circuit.

Example: Show that K_n has a Hamilton circuit whenever $n \geq 3$.

Hamilton path problem \in NP-Complete

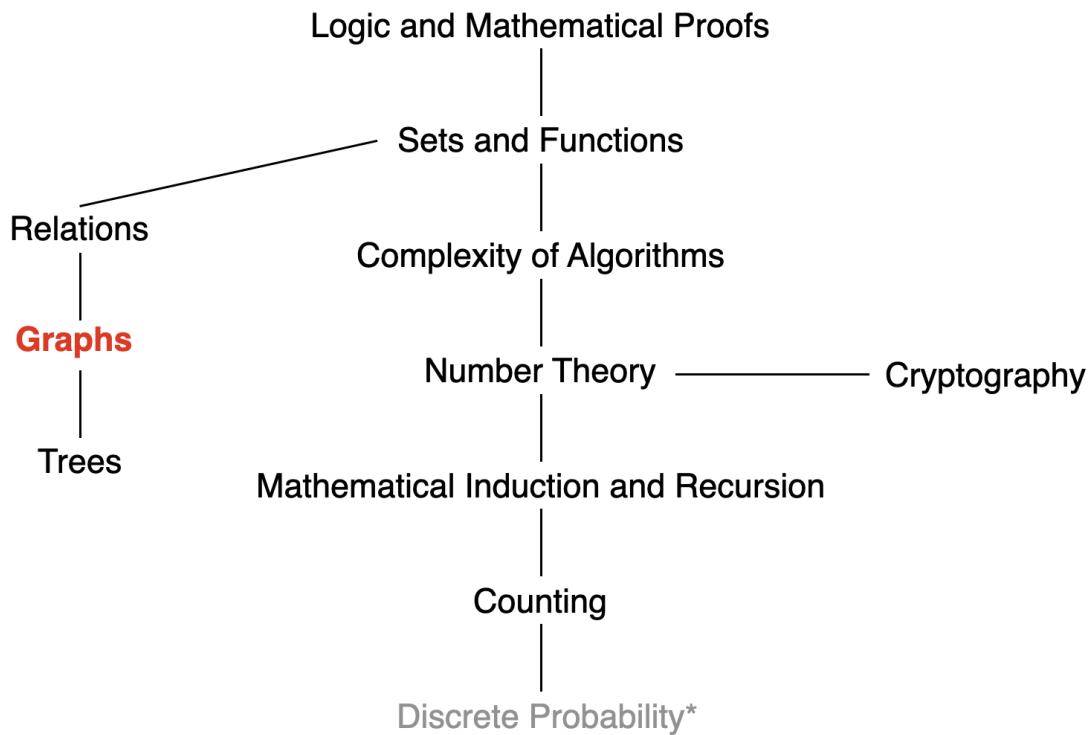
Applications of Hamilton Paths and Circuits

A path or a circuit that visits each city, or each node in a communication network **exactly once**, can be solved by finding a **Hamilton path**.

Traveling Salesperson Problem (TSP) asks for the **shortest route** a traveling salesperson should take to visit a set of cities.

the decision version of the TSP \in NP-Complete

This Lecture



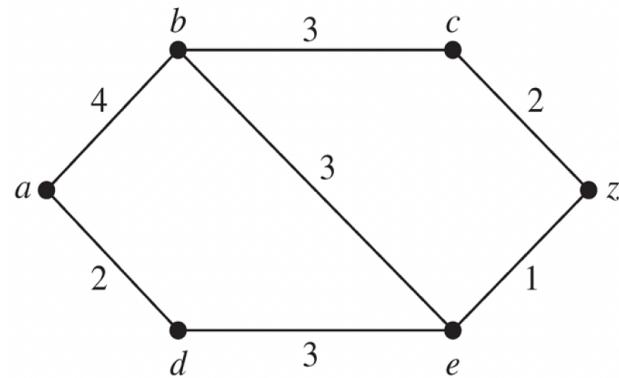
Graph and terminologies, representing graphs and graph isomorphism,
connectivity, Euler and Hamilton path, shortest-path problem ...



Shortest Path Problems

Using graphs with **weights** assigned to their **edges**

Such graphs are called weighted graphs and can model lots of questions involving distance, time consuming, fares, etc.



What is the length of a shortest path between a and z ?

Dijkstra's Algorithm

S : a distinguished set of vertices

$L(v)$: the length of a shortest path from a to v that contains vertices only in S

(i) Set $L(a) = 0$ and $L(v) = \infty$ for all v , $S = \emptyset$

(ii) While $z \notin S$

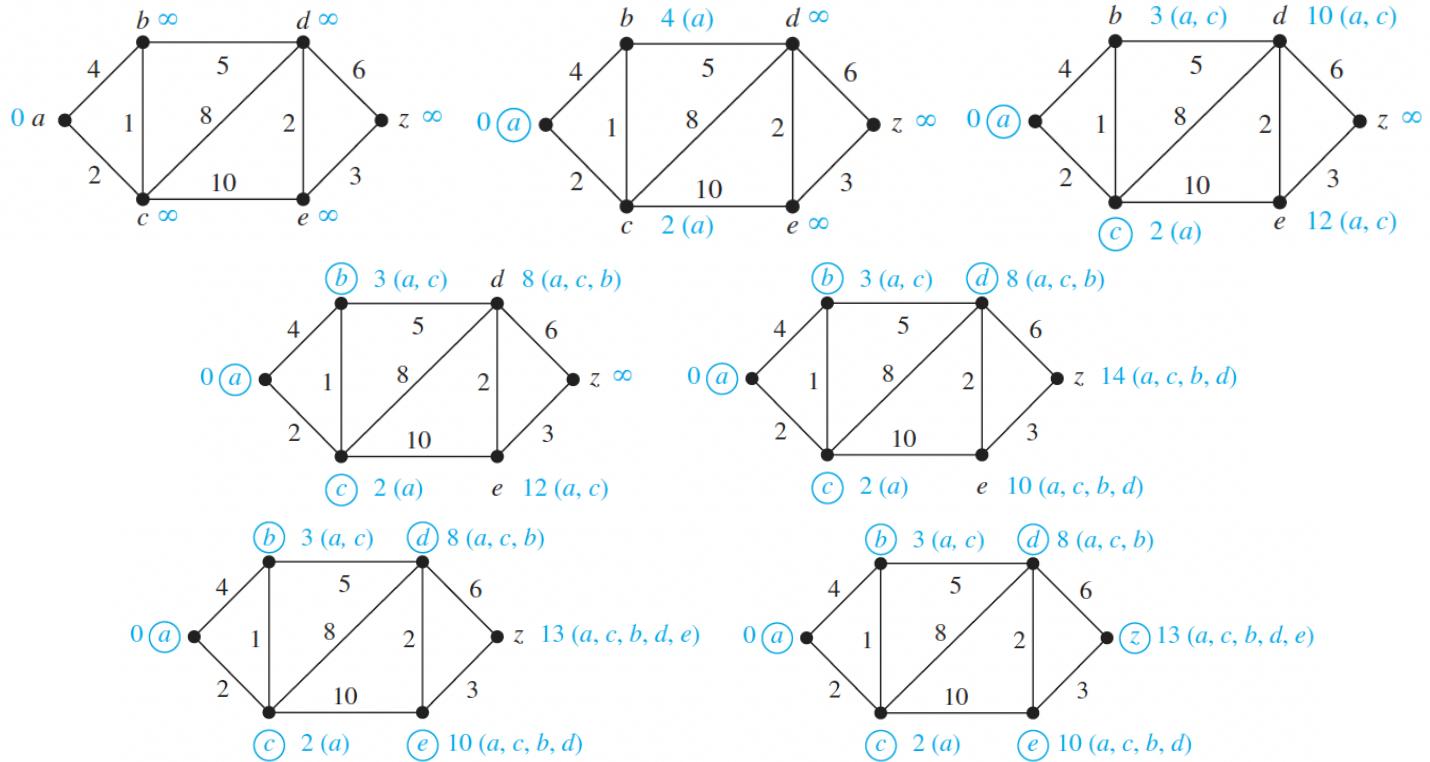
$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

For all vertices v not in S

$L(v) := \min\{L(u) + w(u, v), L(v)\}$

Dijkstra's Algorithm



$$S = \{a, c, b, d, e, z\}$$

$$L(a) = 0, L(b) = 3, L(c) = 2, L(d) = 8, L(e) = 10, L(z) = 13$$

Dijkstra's Algorithm

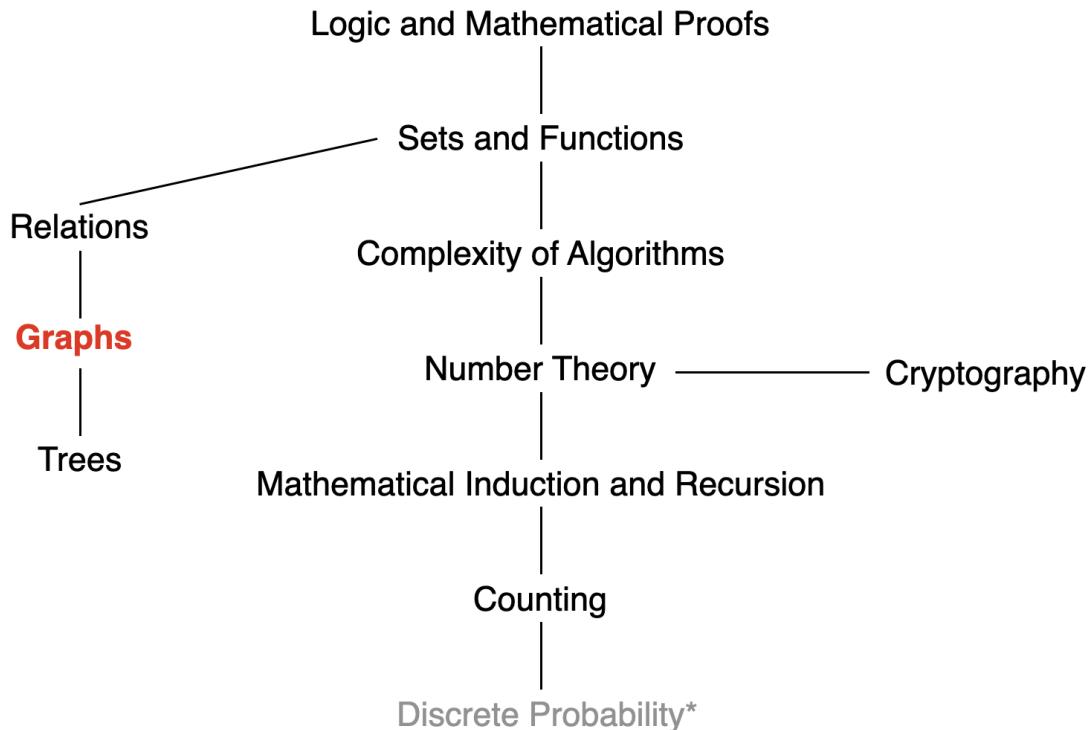
Theorem: Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

Proof by induction ... (P713 on textbook)

Theorem: Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) in a connected simple undirected weighted graph with n vertices.

- $n - 1$ iterations
- no more than $2(n - 1)$ operations are used at each iteration

This Lecture



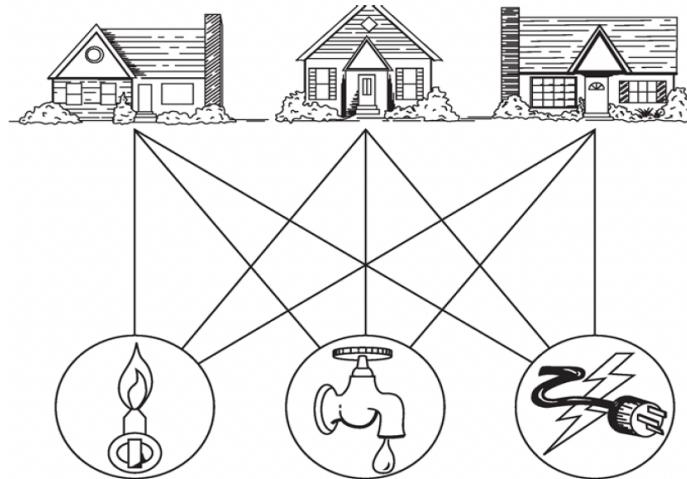
..., Euler and Hamilton path, shortest-path problem, Planar Graphs



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Planar Graphs

Join three houses to each of three separate utilities.

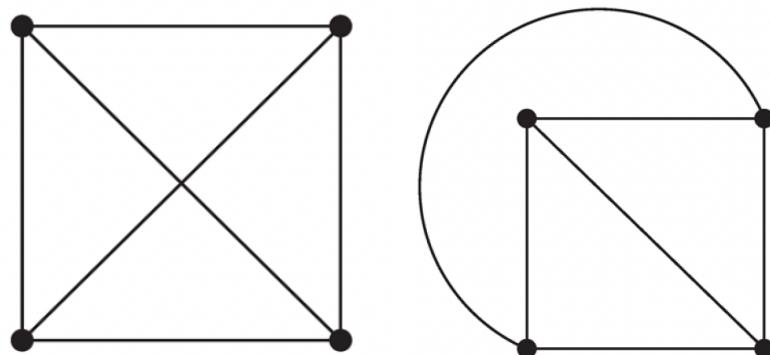


Can this graph be drawn in the plane such that no two of its edges cross?
Complete bipartite graph $K_{3,3}$

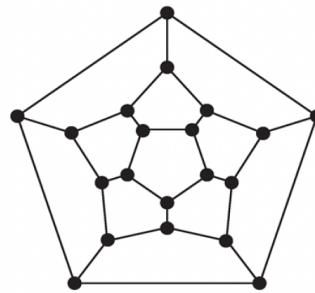
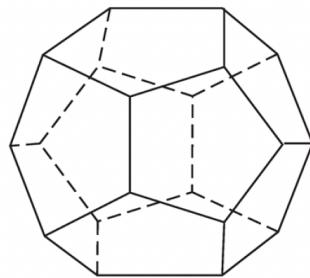
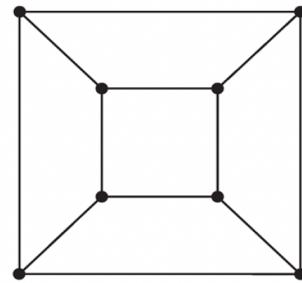
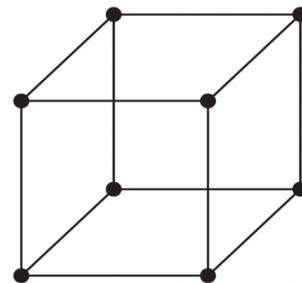
Planar Graphs

Definition: A graph is called **planar** if it can be drawn in the **plane without any edges crossing**. Such a drawing is called a **planar representation** of the graph.

Example: Is K_4 planar?



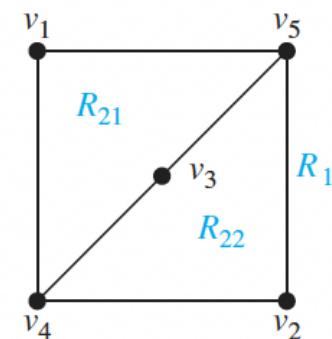
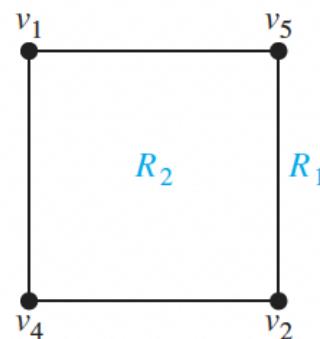
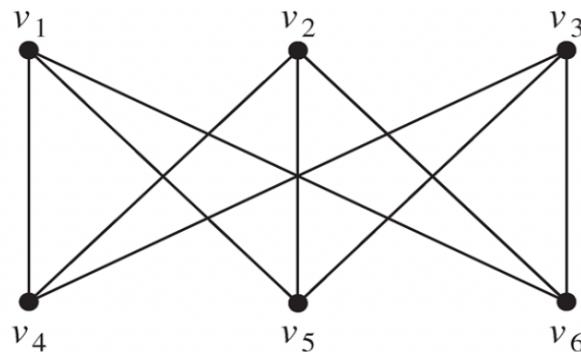
Planar Graphs: Example



- We can show that a graph is planar by displaying a planar representation.
- It is harder to show that a graph is nonplanar.

Planar Graphs: Example

Is $K_{3,3}$ planar?

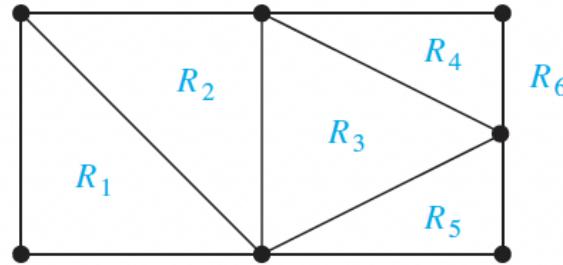


Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed.

- In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5 .
- These four edges form a closed curve that splits the plane into two regions, R_1 and R_2 .
- The vertex v_3 is in either R_1 or R_2 . Suppose v_3 is in R_2 , there is no way to place the final vertex v_6 without forcing a crossing.

Euler's Formula

A planar representation of a graph splits the plane into **regions**, including an unbounded region.



Theorem (Euler's Formula): Let G be a **connected planar simple graph** with e edges and v vertices. Let r be the number of regions in a planar representation of G . Then, $r = e - v + 2$.

Euler's Formula

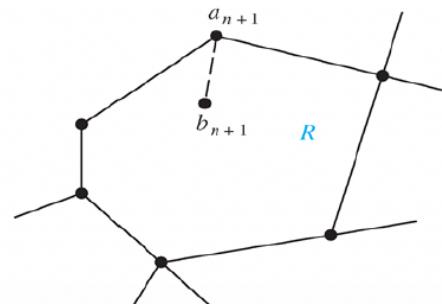
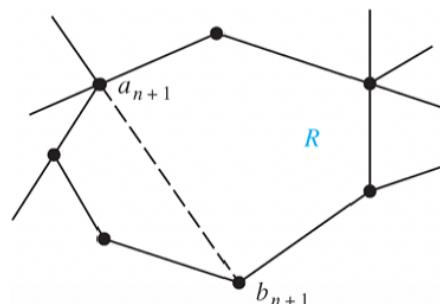
Theorem (Euler's Formula): Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G . Then, $r = e - v + 2$.

Proof (by induction): We will prove the theorem by [successively adding an edge](#) at each stage.

- Basic Step: $r_1 = e_1 - v_1 + 2$

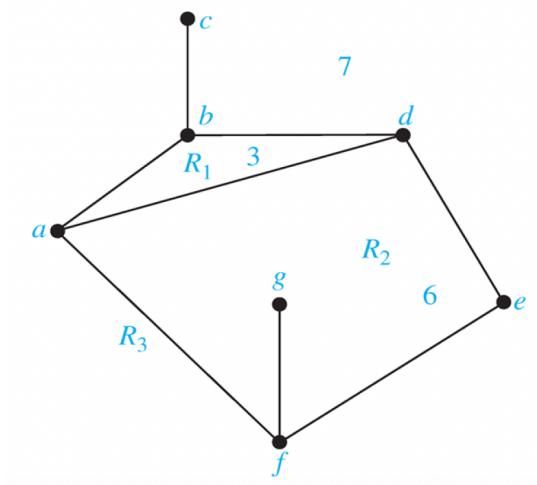


- Inductive Hypothesis: $r_k = e_k - v_k + 2$
- Inductive step: Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1} .



The Degree of Regions

Definition: The degree of a region is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary, it contributes two to the degree.



Corollaries

Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Proof: The degree of every region is at least 3.

- G is simple
- $v \geq 3$

The sum of the degrees of the regions is exactly twice the number of edges in the graph:

$$2e = \sum_{\text{all regions } R} \deg(R) \geq 3r$$

Hence, $(2/3)e \geq r$. By Euler's formula (i.e., $r = e - v + 2$), $e \leq 3v - 6$.

Corollaries

Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

Proof (by Contradiction):

If G has one or two vertices, the result is true.

If G has at least three vertices, by Corollary 1, $e \leq 3v - 6$, so $2e \leq 6v - 12$.

- If the degree of every vertex were at least six, then we would have $2e = \sum_{v \in V} \deg(v) \geq 6v$ (by handshaking theorem).
- This contradicts the inequality $2e \leq 6v - 12$.

It follows that there must be a vertex with degree no greater than five.

Corollary 3: In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e < 2v - 4$.



Examples

Show that K_5 is nonplanar.

$v = 5$ and $e = 10$.

Using Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Show that $K_{3,3}$ is nonplanar.

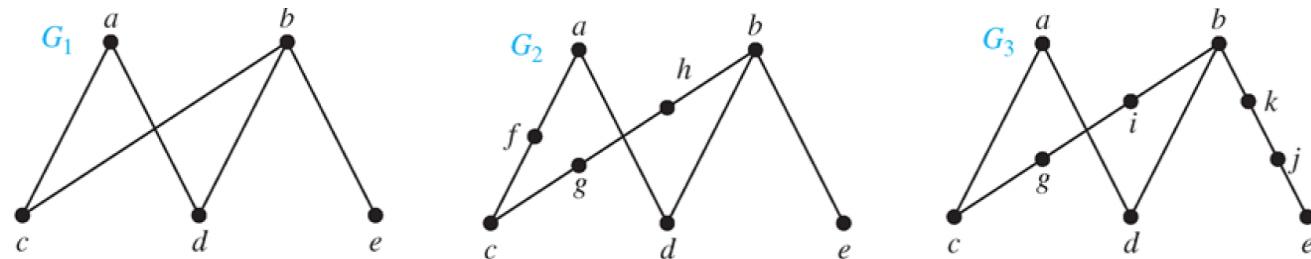
$v = 6$ and $e = 9$.

Using Corollary 3: In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Kuratowski's Theorem

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

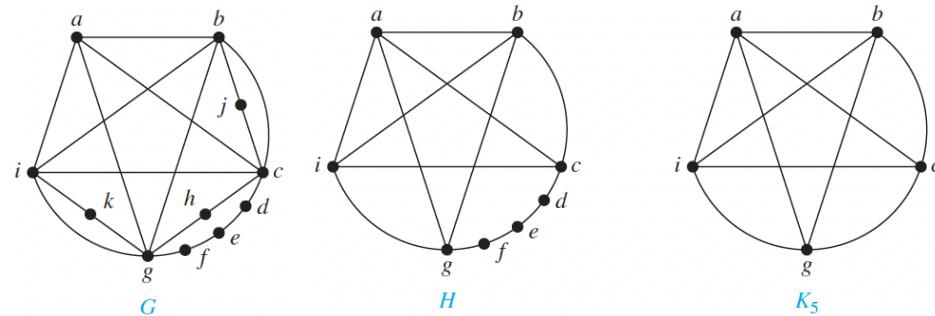
The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



Kuratowski's Theorem

Theorem: A graph is **nonplanar** if and only if it contains a **subgraph homomorphic** to $K_{3,3}$ or K_5 .

Example:

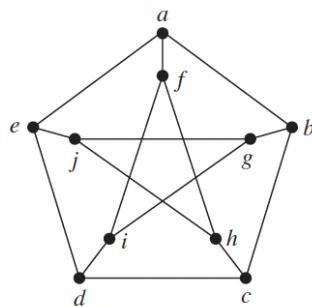


G has a subgraph H homeomorphic to K_5 .

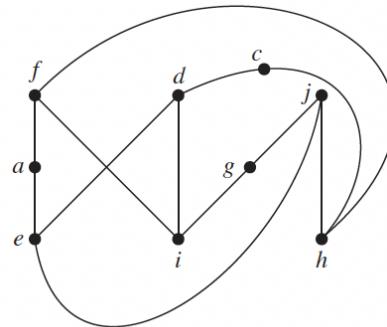
- H is obtained by deleting h, j , and k and all edges incident with these vertices.
- H is homeomorphic to K_5 because it can be obtained from K_5 by a sequence of elementary subdivisions.

Hence, G is nonplanar.

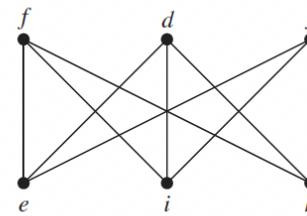
Kuratowski's Theorem: Example



(a)



(b) H



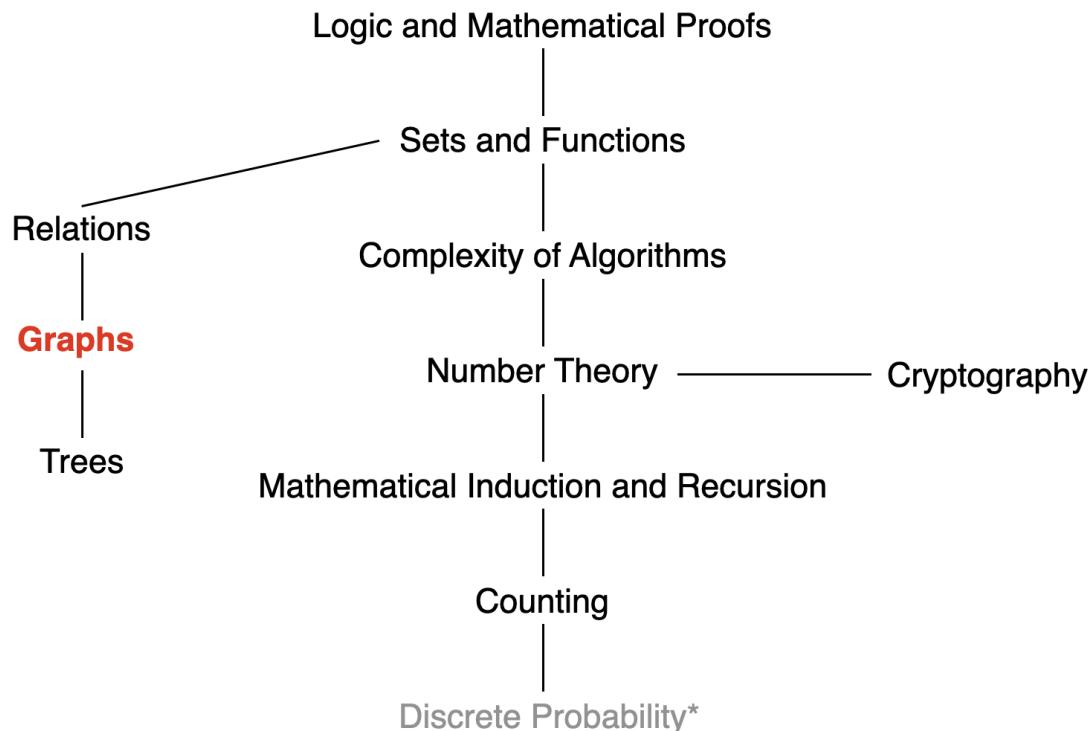
(c) $K_{3,3}$

G has a subgraph H homeomorphic to $K_{3,3}$.

- The subgraph H of the Petersen graph obtained by deleting b and the three edges that have b as an endpoint,
- H is homeomorphic to $K_{3,3}$, with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, because it can be obtained by a sequence of elementary subdivisions.

Hence, G is nonplanar.

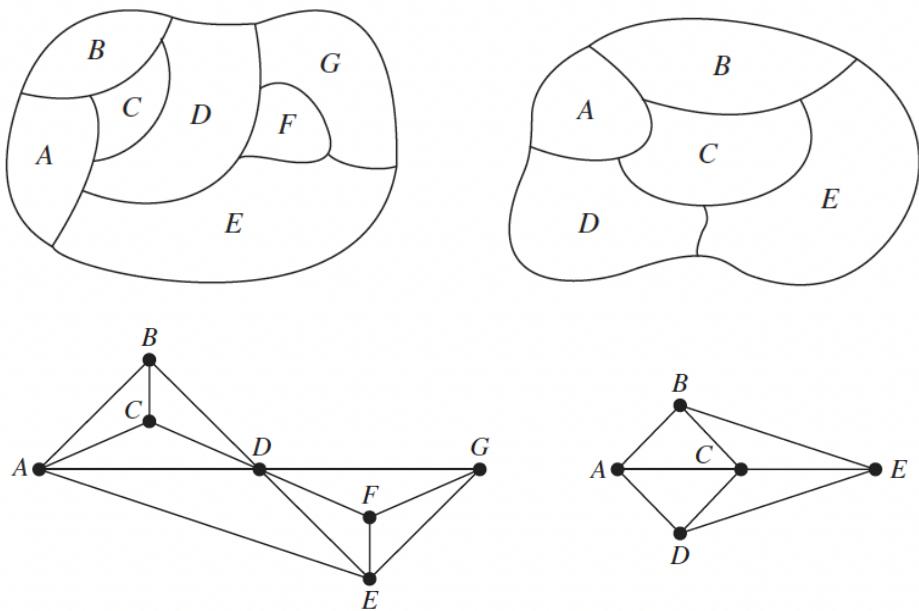
This Lecture



..., Euler and Hamilton path, shortest-path problem, planar graphs, graph coloring, ...



Graph Coloring



A **coloring** of a simple graph is the assignment of a color to each vertex of the graph so that **no two adjacent vertices** are assigned the same color.

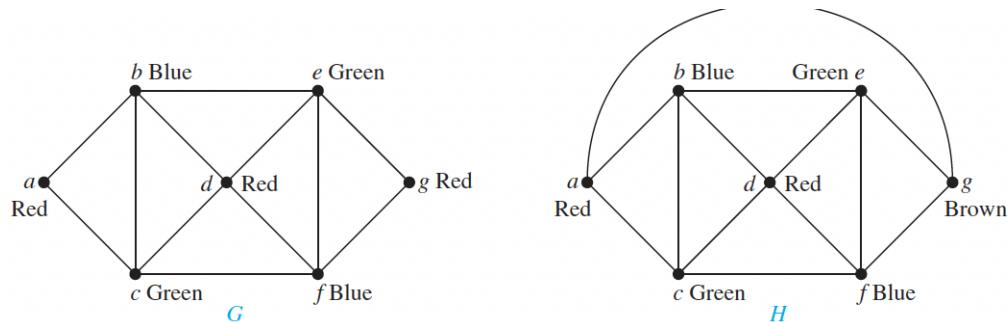
Graph Coloring

The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.

Theorem (Four Color Theorem): The **chromatic number** of a planar graph is **no greater than four**.



Graph Coloring: Example



The chromatic number of G is at least three, because the vertices a , b , and c must be assigned different colors.

- Assign red to a , blue to b , and green to c ;
- d can (and must) be colored red because it is adjacent to b and c ;
- e can (and must) be colored green because it is adjacent only to vertices colored red and blue;
- ...

This produces a coloring of G using exactly three colors.

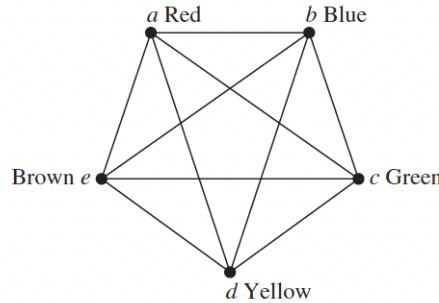


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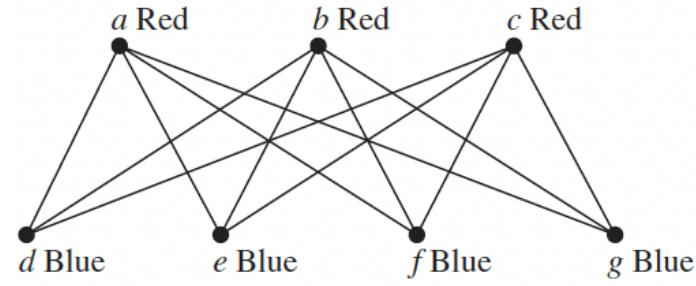
Southern University
of Science and
Technology

Graph Coloring: Example

What is the chromatic number of K_n ? $\chi(K_n) = n$

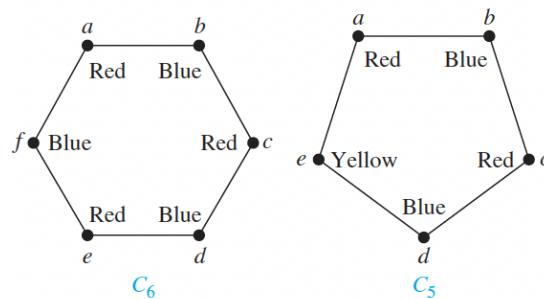


What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers? $\chi(K_{m,n}) = 2$



Graph Coloring: Example

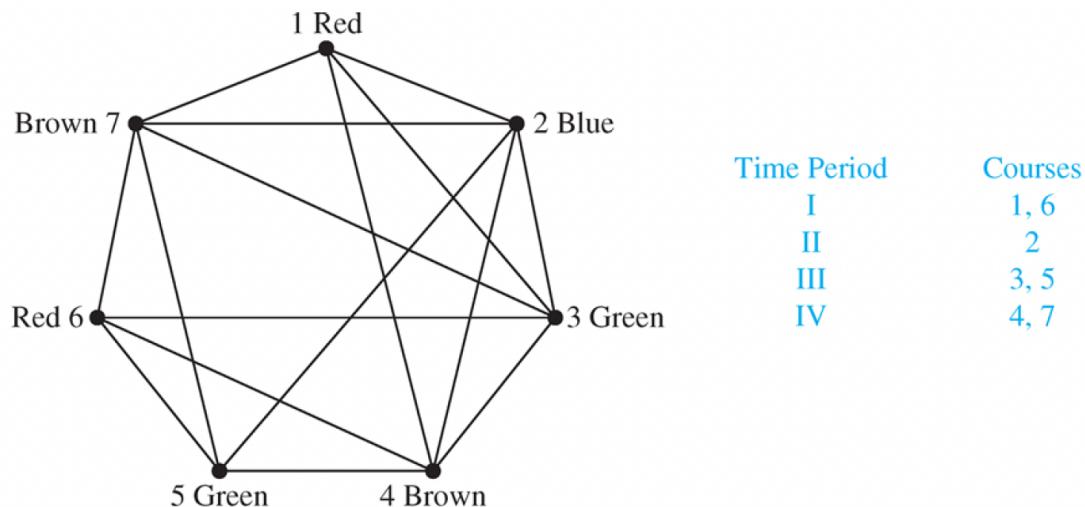
What is the chromatic number of the graph C_n , where $n \geq 3$? (Recall that C_n is the cycle with n vertices.)



- When n is even, $\chi(C_n) = 2$
- When n is odd and $n > 1$, $\chi(C_n) = 3$

Applications of Graph Coloring

Scheduling Final Exams: Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Applications of Graph Coloring

Channel Assignments: Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Graph Coloring \in NP-Complete