CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 4
Due: Apr. 30th, 2022, please submit one pdf file through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. (5 points) Prove by induction that, for any sets $A_1, A_2, \cdots, A_n$, De Morgan's law can be generalized to

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Q. 2. (5 points) Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

Q. 3. (10 points) A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Q. 4. (5 points) Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Q. 5. (5 points) Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. (use the equality $2^{2^{n+1}} = (a^{2^n})^2$)

Q. 6. (5 points) Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Q. 7. (5 points) How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Q. 8. (5 points) How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks.

You should leave your answer as an equation.

Q. 9. (10 points) Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Q. 10. (5 points) How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Q. 11. (10 points) Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Q. 12. (5 points) Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Q. 13. (5 points) Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Q. 14. (5 points) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 0$.

Q. 15. (5 points) Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Q. 16. (10 points) Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]