Discrete Mathematics for Computer Science

Lecture 16: Relation

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Example 3

Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in the cases. **Case 2: when the order does matter:**

The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x + x^2 + x^5)^n$$

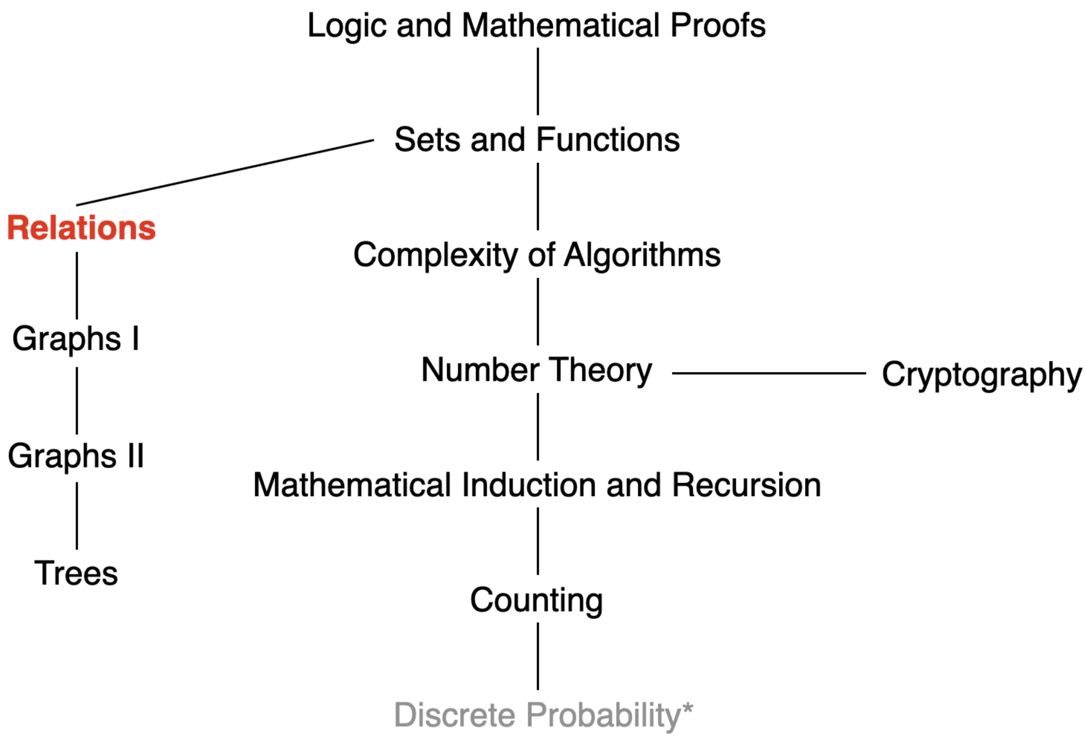
Because any number of tokens may be inserted,

$$G(x) = 1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots = \frac{1}{1 - (x + x^2 + x^5)}$$

Since $1/(1-x) = \sum_{k=0}^{\infty} x^k$, we have

$$G(x) = \sum_{k=0}^{\infty} (x + x^2 + x^5)^k.$$

This Lecture



Relation, *n*-ary Relations, Representing Relations, Closures of Relations, ...

Binary Relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations: Symmetric Relation

Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Is R_{div} symmetric?

No. $(1, 2) \in R_{div}$ but $(2, 1) \notin R$.

Symmetric Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}.$$

Is $R_{\neq}$ symmetric?

Yes. If $(a, b) \in R_{\neq}$ then $(b, a) \in R_{\neq}$.

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is symmetric if and only if MR is symmetric.

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Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations symmetric?

R_3, R_4, and R_6.

Properties of Relations: Antisymmetric Relation

Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R antisymmetric? Yes.

$$\mathrm{MR} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A relation R is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a,b) \mid a \le b\},\\
R_2 &= \{(a,b) \mid a > b\},\\
R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a,b) \mid a = b\},\\
R_5 &= \{(a,b) \mid a = b+1\},\\
R_6 &= \{(a,b) \mid a + b \le 3\}.
\end{aligned}
$$

Which of these relations antisymmetric?

R_1, R_2, R_4 and R_5.

Properties of Relations: Transitive Relation

Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Is R_{div} transitive?

Yes. If $a|b$ and $b|c$, then $a|c$.

Transitive Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ transitive?

No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

Transitive Relation

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R transitive?

Yes.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a,b) \mid a \le b\},\\
R_2 &= \{(a,b) \mid a > b\},\\
R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a,b) \mid a = b\},\\
R_5 &= \{(a,b) \mid a = b+1\},\\
R_6 &= \{(a,b) \mid a+b \le 3\}.
\end{aligned}
$$

Which of these relations transitive?

R_1, R_2, R_3 and R_4.

Combining Relations

Since relations are sets, we can combine relations via set operations.

Set operations: union, intersection, difference, etc.

Example: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
$R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$,
$R_2 = \{(1, v), (3, u), (3, v)\}$

What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

Combining Relations

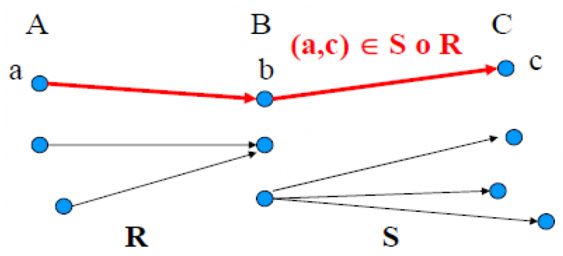
Example: $R_1 = \{(x, y)|x < y\}$ and $R_2 = \{(x, y)|x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R1 \oplus R_2$?

- $R_1 \cup R_2 = \{(x, y)|x \neq y\}$
- $R_1 \cap R_2 = \emptyset$
- $R_1 - R_2 = R_1$
- $R_2 - R_1 = R_2$
- $R_1 \oplus R_2 = \{(x, y)|x \neq y\}$

Composite of Relations

Definition: Let R be a relation from a set A to a set B and S be a relation from B to C. The composite of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We denote the composite of R and S by $S \circ R$.



Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$:

- $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$
- $S = \{(0, b), (1, a), (2, b)\}$
- $S \circ R = \{(1, b), (3, a), (3, b)\}$

Power of a Relation

Definition: Let R be a relation on A. The powers R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

- $R^1 = R$
- $R^2 = R \circ R = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$
- $R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$
- $R^4 = R^3 \circ R = \{(1, 3), (2, 3), (3, 3)\}$
- $R^k =?$ for $k > 3$

Transitive Relation and R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof:

- "if" part: In particular, $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, we have $(a, c) \in R^2 \subseteq R$.
- "only if: part: by induction.
  - $n = 1$: $R^1 \subseteq R$
  - Suppose $R^n \subseteq R$:
    - ★ $(a, c) \in R^{n+1} \triangleq R^n \circ R$: there is a $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R^n \subseteq R$
    - ★ Since R is transitive, $(a, b) \in R$ and $(b, c) \in R^n \subseteq R$ implies that $(a, c) \in R$

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Number of Binary Relations

Theorem: The number of binary relations on a set A, where $|A| = n$, is 2^{n^2}.

Proof: If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

R is a binary relation on A if $R \subseteq A \times A$ (R is subset).

The number of subsets of a set with k elements is 2^k.

Number of Reflexive Relations

Theorem: The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

Proof: A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

All other pairs in R are of the form (a, b) with $a \neq b$, s.t. $a, b \in A$.

How many of these pairs are there?

How many subsets on $n(n-1)$ elements are there?

Summary on Properties of Relations

- Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
- Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.
- Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.
- Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

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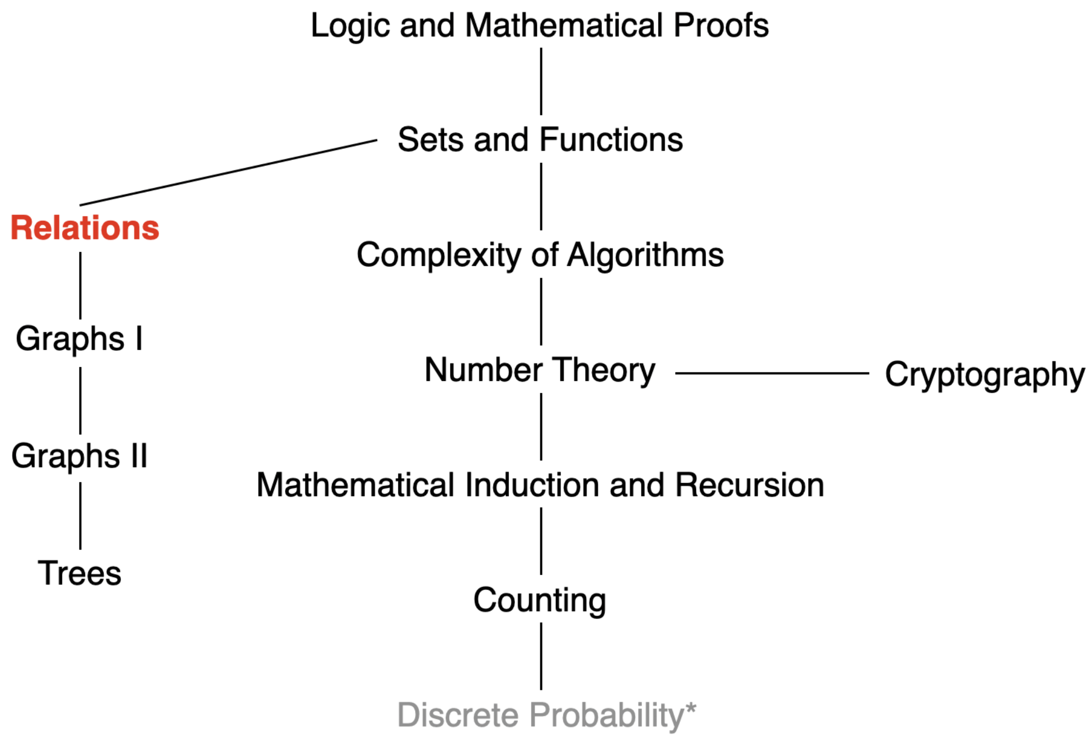

Relation, *n*-ary Relations, Representing Relations, Closures of Relations, ...

n-ary Relations

Definition: An n-ary relation R on sets $A_1, ..., A_n$, written as $R : A_1, ..., A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

- The sets $A_1, ..., A_n$ are called the domains of R.
- The degree of R is n.
- R is functional in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

A relational database is essentially an n-ary relation R.

A domain A_i is a primary key for the database if the relation R is functional in A_i.

Which domains are primary keys for the n-ary relation, assuming that no n-tuples will be added in the future?

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Student name, student ID.

Relational Databases

A composite key for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Example: Is the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation, assuming that no n-tuples are ever added?

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Yes.

Selection Operator

Let A be any n-ary domain $A = A_1 \times \cdots \times A_n$, and let $C : A \rightarrow \{T, F\}$ be any condition (predicate) on elements (n-tuples) of A.

The selection operator s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.

$$\forall R \subseteq A, s_C(R) = R \cap \{a \in A | s_C(a) = T\} = \{a \in R | s_C(a) = T\}.$$

Selection Operator: Example

Suppose that we have a domain

$$A = \text{StudentName} \times \text{Standing} \times \text{SocSecNos}$$

Suppose that we have a condition

$$UpperLevel(name, standing, ssn) \\ :\equiv [(standing = junior) \vee (standing = senior)]$$

Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the juniors and seniors.

Projection Operator

Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, ..., i_m)$ be a sequence of indices all falling in the range 1 to n. That is, where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.

Then the projection operator on n-tuples

$$P_{i_k} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

is defined by

$$P_{i_k}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

Projection Operator: Example

Suppose that we have a domain

$$Cars = Model \times Year \times Color (n = 3)$$

Consider the index sequence $\{i_k\} = (1, 3)$ $(m = 2)$.

Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1, a_2, a_3) = (model, year, color)$ to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of model/color combinations available.

Projection Operator: Example

What is the table obtained when the projection $P_{1,2}$ is applied to the relation as follows?

Student	***Major***	***Course***
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

Student	***Major***
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

Join Operator

Puts two relations together to form a sort of combined relation.

If the tuple (A, B) appears in R_1, and the tuple (B, C) appears in R_2, then the tuple (A, B, C) appears in the join $J(R_1, R_2)$.

A, B, C can also be sequences of elements rather that single elements.

Join Operator: Example

Suppose that R_1 is a teaching assignment table, relating Professors to Courses.

Suppose that R2 is a room assignment table relating Courses to Rooms and Times.

Then $J(R1, R2)$ is like your class schedule, listing $(professor, course, room, time)$.

Join Operator: Example

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

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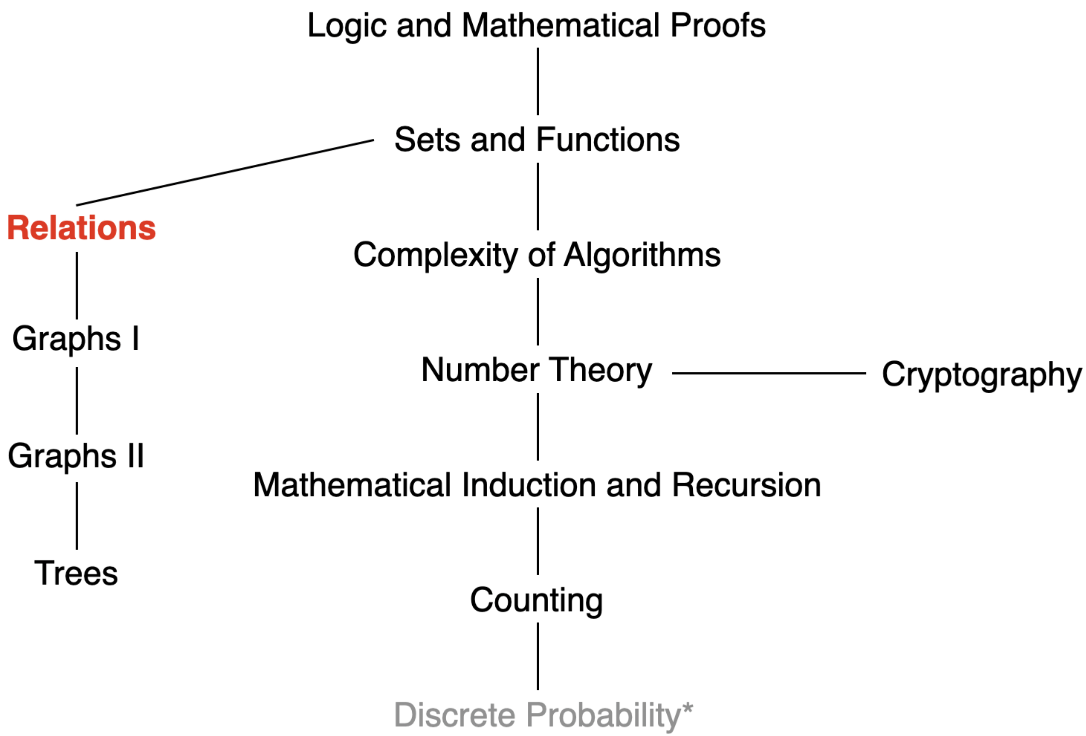

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Representing Relations

Some ways to represent n-ary relations:

- with an explicit list or table of its tuples
- with a function from the domain to $\{T, F\}$

Some special ways to represent binary relations:

- with a zero-one matrix
- with a directed graph

Zero-One Matrix

$$m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases} \tag{1}$$

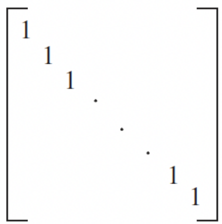
Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

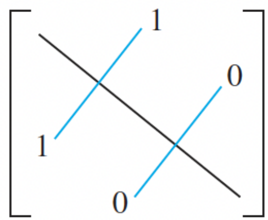
Solution: $R = \{(2, 1), (3, 1), (3, 2)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

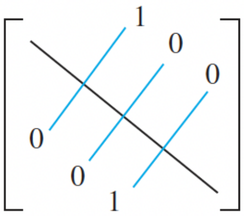
Zero-One Matrix



Reflexive



Symmetric



Antisymmetric

Example: Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?
Reflexive, symmetric. Not antisymmetric.

Zero-One Matrix: Join and Meet

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero–one matrices.

The join of A and B is the zero–one matrix with (i, j)-th entry $a_{ij} \vee b_{ij}$.
The join of A and B is denoted by $A \vee B$.

The meet of A and B is the zero–one matrix with (i, j)-th entry $a_{ij} \wedge b_{ij}$.
The meet of A and B is denoted by $A \wedge B$.

Zero-One Matrix: Join and Meet

Consider relations R_1 and R_2 on a set A:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Example: Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

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Zero-One Matrix: Composite of Relations

Let $A = [a_{ij}]$ be an $m \times k$ zero–one matrix and $B = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then, the Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with (i, j)-th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Zero-One Matrix: Composite of Relations

Suppose that R is a relation from A to B and S is a relation from B to C:

$$M_{S\circ R} = M_R \odot M_S.$$

The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S.

Example:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{M}_{S\circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

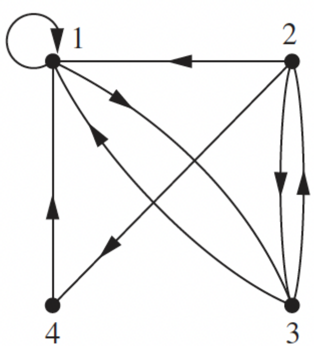
Directed Graph

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

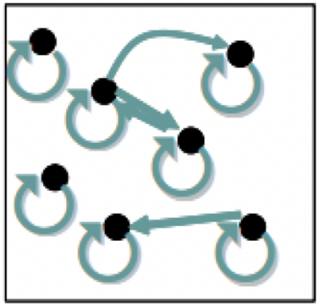
Example: Relation R is defined on $\{1, 2, 3, 4\}$:

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

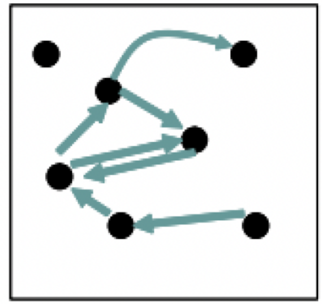


Directed Graph

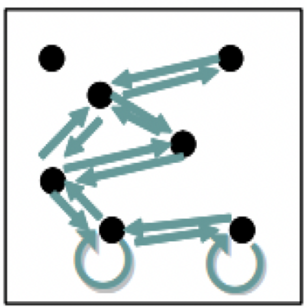
Reflexive, irreflexive, symmetric, antisymmetric?



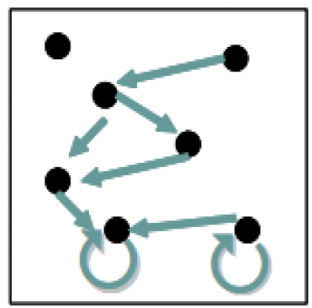
reflexive



irreflexive

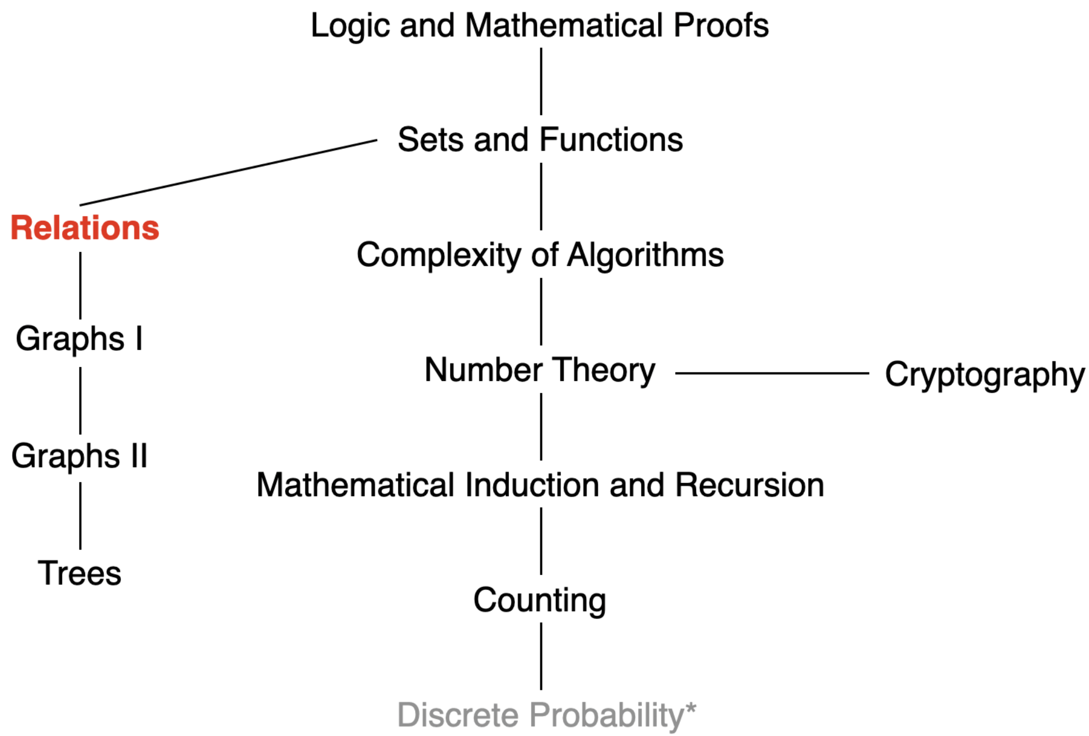


symmetric



antisymmetric

This Lecture



Relation, *n*-ary Relations, Representing Relations, Closures of Relations, ...

Closures of Relations

Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

Is this relation R reflexive?

No. $(2,2)$ and $(3,3)$ are not in R.

The question is what is the minimal relation $S \supseteq R$ that is reflexive?

How to make R reflexive by minimum number of additions?

Add $(2,2)$ and $(3,3)$

Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$.

The minimal set $S \supseteq R$ is called the reflexive closure of R.

Closures of Relations

The set S is called the reflexive closure of R if it:

- contains R
- is reflexive
- is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

Relations can have different properties:

- reflexive
- symmetric
- transitive

We define:

- reflexive closures
- symmetric closures
- transitive closures

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

S is the minimal set containing R satisfying the property P.

Example: $R = \{(1,2),(2,3),(2,2)\}$ on $A = \{1,2,3\}$. What is the symmetric closure S of R?

$S = \{(1,2),(2,3),(2,2),(2,1),(3,2)\}$.

What is the transitive closure S of R?

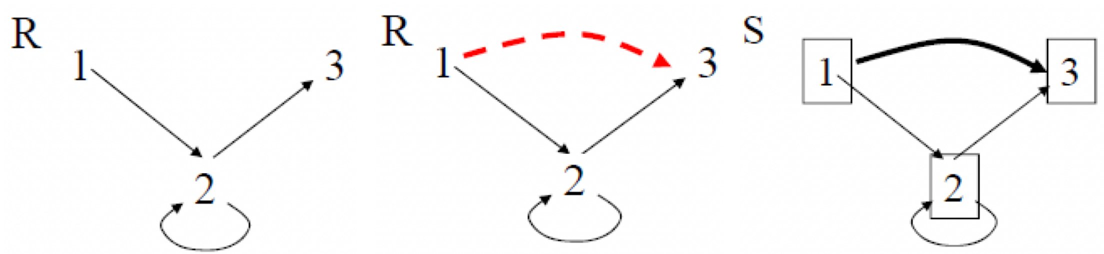
$S = \{(1,2),(2,2),(2,3),(1,3)\}$.

Transitive Closure

We can represent the relation on the graph.

Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Example: $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$. Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



Paths in Directed Graphs

Definition: A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$.

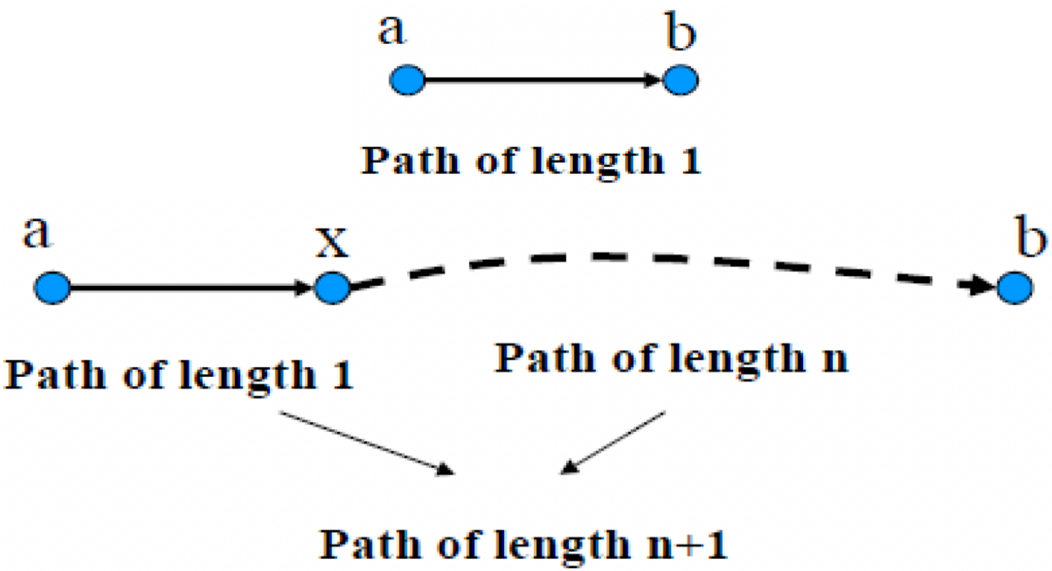
A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Path Length

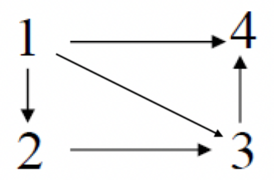
Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Proof (by induction):



Recall that $R^{n+1} = R^n \circ R$

Path Length: Example



$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$

$R^2 = \{(1, 3), (2, 4), (1, 4)\}$

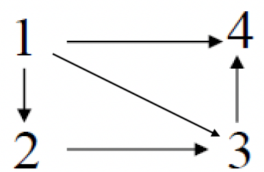
$R^3 = \{(1, 4)\}$

$R^4 = \emptyset$

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$



$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$, $R^2 = \{(1, 3), (2, 4), (1, 4)\}$

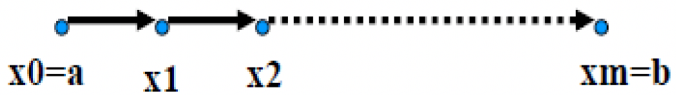
$R^3 = \{(1, 4)\}$, $R^4 = \emptyset$

$R^* = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

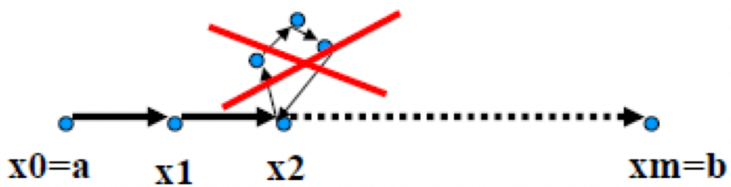
Connectivity

Lemma: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Proof (by intuition): There are at most n different elements we can visit on a path if the path does not have loops:



Loops may increase the length but the same node is visited more than once

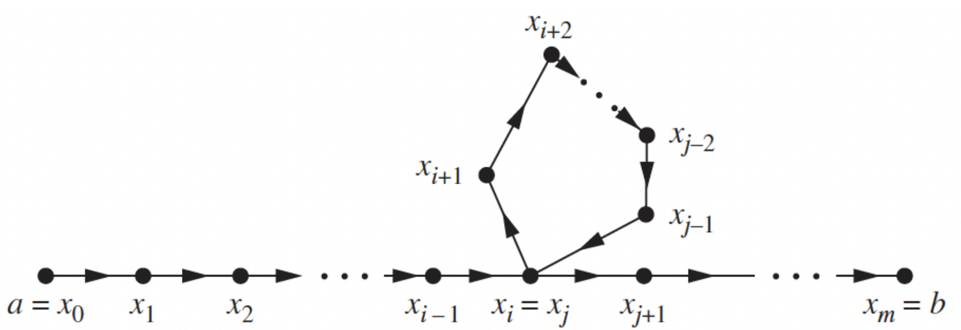


Connectivity

Lemma: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Proof: Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x1, x2, ..., x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a \neq b$ and that $m \geq n$. The $m+1$ vertices are from n elements. According to the pigeonhole principle and $a \neq b$, at least two of the vertices $x_0, x_1, ..., x_{m-1}$ are equal.



There is a circuit that can be deleted until the length is $< n$.

Connectivity

Lemma: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Lemma: If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Connectivity

Theorem: The transitive closure of a relation R equals the connectivity relation R^*:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Recall: Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Recall: The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

Connectivity

Theorem: The transitive closure of a relation R equals the connectivity relation R^*:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

- R^* is transitive

  If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.
- $R^* \subseteq S$ whenever S is a transitive relation containing R
  - Suppose that S is a transitive relation containing R.
  - $S^n \subseteq S$ for integer $n \geq 1$. (Recall S is transitive iff $S^n \subseteq S$).
  - We have $S^* \subseteq S$.
  - If $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S.
  - Thus, $R^* \subseteq S^* \subseteq S$.

Find Transitive Closure

Recall that if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Thus,

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]},$$

where $M_R^{[n]} = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{n\ M_R's}$

Find Transitive Closure: Example

nd the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

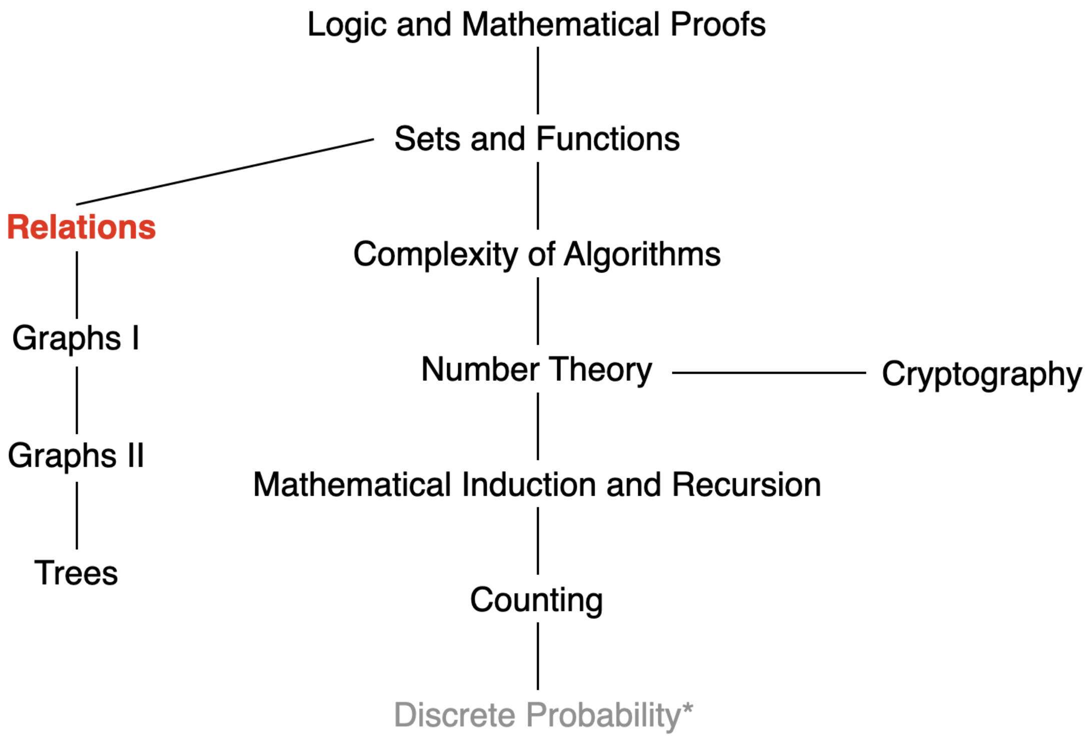
Find Transitive Closure: Algorithm

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
  $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
  $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return B{**B** is the zero–one matrix for R^*}

- $n - 1$ Boolean products
- Each of these Boolean products use $n^2(2n-1)$ bit operations.
- $O(n^4)$ bit operations.

This Lecture



Relation, *n*-ary Relations, Representing Relations, Closures of Relations, Relation Equivalence, ...