CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment #1
Due: 23:59 on Mar. 9th, 2021, please submit through Sakai

Q.1 Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You do not get 100 marks on the final.

(b) You get 100 marks on the final, but you do not get an A in this course.

(c) You will get an A in this course if you get 100 marks on the final.

(d) If you do not get 100 marks on the final, then you will not get an A in this course.

(e) Getting 100 marks on the final is sufficient for getting an A in this course.

(f) You get an A in this course, but you do not get 100 marks on the final.

(g) Whenever you get an A in this course, you got 100 marks on the final.

Solution:

(a) $\neg p$

(b) $p \wedge \neg q$

(c) $p \rightarrow q$

(d) $\neg p \rightarrow \neg q$

(e) $p \rightarrow q$

(f) $q \wedge \neg p$

(g) $q \rightarrow p$

$\square$

Q.2 Construct a truth table for each of these compound propositions.

(a) $(p \oplus q) \to (p \wedge q)$

(b) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(c) $(p \oplus q) \to (p \oplus \neg q)$

Solution:

(a)

p	q	$(p \oplus q) \to (p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

p	q	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

(c)

p	q	$(p \oplus q) \to (p \oplus \neg q)$
T	T	T
T	F	F
F	T	F
F	F	T

$\square$

Q.3 Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $p \to q$ and $\neg p \vee q$ (This is the Useful Law)

(b) $p \oplus q$ and $\neg p \vee \neg q$

(c) $(p \to q) \to r$ and $p \to (q \to r)$

(d) $(\neg q \wedge \neg(p \to q))$ and $\neg p$

(e) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$

Solution:

(a) The combined truth table is:

p	q	$p \to q$	$\neg p \lor q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

By comparing the last two columns, we have that they are equivalent.

(b) The combined truth table is:

p	q	$p \oplus q$	$\neg p \lor \neg q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(c) The combined truth table is:

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Since the fifth and last columns are not the same in both truth tables, we know that these two propositions are not equivalent.

(d) The combined truth table is:

p	q	$p \to q$	$\neg q$	$\neg(p \to q)$	$\neg q \wedge \neg(p \to q)$	$\neg p$
F	F	T	T	F	F	T
F	T	T	F	F	F	T
T	F	F	T	T	T	F
T	T	T	F	F	F	F

By comparing the last two columns, we have that they are not equivalent.

(e) The truth table for $(p \vee q) \to r$ is :

p	q	r	$p \vee q$	$(p \vee q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

The truth table for $(p \to r) \wedge (q \to r)$ is

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

$\square$

Q.4 Use logical equivalences to prove the following statements.

(a) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

(b) $\neg(p \to q) \to \neg q$ is a tautology.

(c) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
&\neg(p \oplus q) \\
&\quad \equiv \quad \neg((p \wedge \neg q) \vee (\neg p \wedge q)) \quad \text{Definition} \\
&\quad \equiv \quad \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \quad \text{De Morgan} \\
&\quad \equiv \quad (\neg p \vee q) \wedge (p \vee \neg q) \quad \text{De Morgan} \\
&\quad \equiv \quad (p \to q) \wedge (q \to p) \quad \text{Useful} \\
&\quad \equiv \quad p \leftrightarrow q \quad \text{Definition}
\end{aligned}$$

Thus, they are equivalent.

(b) We have

$$\begin{aligned}
&\neg(p \to q) \to \neg q \\
&\quad \equiv \quad \neg\neg(p \to q) \vee \neg q \quad \text{Useful} \\
&\quad \equiv \quad (p \to q) \vee \neg q \quad \text{Double negation} \\
&\quad \equiv \quad (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\quad \equiv \quad \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\quad \equiv \quad T \qquad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(c) We have

$$
\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv T \vee \neg r \quad \text{Complement} \\
&\equiv T \quad \text{Identity.}
\end{aligned}
$$

Thus, it is a tautology.

□

Q.5 Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true, when p, q, and r have the same truth value and it is false otherwise.
Solution: The statement is true if and only if all the three clauses, $p \vee \neg q$, $q \vee \neg r$, and $r \vee \neg p$ are true. Suppose that p, q and r are all true, or all false, it is checked that each clause is true, and the statement is true. On the other hand, if one of the variables is true, and the other two false, then the clause containing the negation of that variable will be false, making the entire conjunction false. Similarly, if one of the variable is false and the other two true, then the clause containing that variable unnegated will be false, again making the entire statement false.

□

Q.6 Determine whether or not the following two are logically equivalent, and explain your answer.

(a) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(b) $\neg p \to (q \to r)$ and $q \to (p \vee r)$

(c) $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$.

(d) $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$.

Solution:

(a) The second statement is false only when p is true and $q \vee r$ is false, which means both q and r are false.

The first statement if false only when both $p \to q$ and $p \to r$ are false. This only happens when p is true, and both q and r are false.

Thus, these two statements are logically equivalent.

(b) The second statement is false only when q is true and $p \vee r$ is false, which means both p and r are false.

The first statement if false only when $\neg p$ is true, and $q \to r$ is false. This only happens when p is false, and q is true, r is false.

Thus, these two statements are logically equivalent.

(c) We use logical equivalences as follows.

$$\begin{aligned}
&p \to (q \wedge r) \\
&\equiv \neg p \vee (q \wedge r) \text{ Useful} \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee r) \text{ Distributive} \\
&\equiv (p \to q) \wedge (p \to r)\text{Useful.}
\end{aligned}$$

These two statements are logically equivalent.

(d) These two statements are not logically equivalent. What we need only is to find an assignment of truth values such that one of these propositions is true and the other is false. Let p be false, q be true, and r be false. Then the first statement is false. However, the second statement is true.

□

Q.7 Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret." Express each of these sentences in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution:

(a) $\exists x(C(x) \wedge D(x) \wedge F(x))$

(b) $\forall x(C(x) \vee D(x) \vee F(x))$

(c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

(d) $\neg\exists x(C(x) \wedge D(x) \wedge F(x))$

(e) $(\exists x C(x)) \wedge (\exists x D(x)) \wedge (\exists x F(x))$

$\square$

Q.8 Let $F(x, y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express each of these statement.

(a) Everybody can fool Fred.

(b) Evelyn can fool everybody.

(c) There is no one who can fool everybody.

(d) Nancy can fool exactly two people.

(e) There is exactly one person whom everybody can fool.

(f) There is someone who can fool exactly one person besides himself or herself.

Solution:

(a) $\forall x F(x, \text{Fred})$

(b) $\forall y F(\text{Evelyn}, y)$

(c) $\neg\exists x \forall y F(x, y)$

(d) $\exists y_1 \exists y_2 (F(\text{Nancy}, y_1) \wedge F(\text{Nancy}, y_2) \wedge y_1 \neq y_2 \wedge \forall y (F(\text{Nancy}, y) \rightarrow (y = y_1 \vee y = y_2)))$

(e) $\exists y (\forall x F(x, y) \wedge \forall z (\forall x F(x, z) \rightarrow z = y))$

(f) $\exists x \exists y (x \neq y \wedge F(x, y) \wedge \forall z ((F(x, z) \wedge z \neq x) \rightarrow z = y) \wedge F(x, x))$
Note: for simplicity, we regard the following answer as correct as well: $\exists x \exists y (x \neq y \wedge F(x, y) \wedge \forall z ((F(x, z) \wedge z \neq x) \rightarrow z = y))$

$\square$

Q.9 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

(b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

(c) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

(d) $\forall x \exists y \forall z T(x, y, z)$

Solution:

(a)

$$\begin{aligned}
\neg\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y)) &\equiv \exists y \neg\exists x \exists z (T(x, y, z) \vee Q(x, y)) \\
&\equiv \exists y \forall x \neg\exists z (T(x, y, z) \vee Q(x, y)) \\
&\equiv \exists y \forall x \forall z \neg(T(x, y, z) \vee Q(x, y)) \\
&\equiv \exists y \forall x \forall z (\neg T(x, y, z) \wedge \neg Q(x, y))
\end{aligned}$$

(b)

$$\begin{aligned}
\neg(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)) &\equiv \neg\exists x \exists y P(x,y) \vee \neg\forall x \forall y Q(x,y) \\
&\equiv \forall x \neg\exists y P(x,y) \vee \exists x \neg\forall y Q(x,y) \\
&\equiv \forall x \forall y \neg P(x,y) \vee \exists x \exists y \neg Q(x,y)
\end{aligned}$$

(c)

$$\begin{aligned}
\neg\exists x \exists y (Q(x,y) \leftrightarrow Q(y,x)) &\equiv \forall x \neg\exists y (Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x \forall y \neg(Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x \forall y (\neg Q(x,y) \leftrightarrow Q(y,x))
\end{aligned}$$

(d)

$$\begin{aligned}
\neg\forall x \exists y \forall z T(x,y,z) &\equiv \exists x \neg\exists y \forall z T(x,y,z) \\
&\equiv \exists x \forall y \neg\forall y T(x,y,z) \\
&\equiv \exists x \forall y \exists y \neg T(x,y,z).
\end{aligned}$$

□

Q.10 Prove that if $p \wedge q$, $p \to \neg(q \wedge r)$, $s \to r$, then $\neg s$.
Solution:

(1)	$p \wedge q$	Premise
(2)	p	Simplication of (1)
(3)	$p \to \neg(q \wedge r)$	Premise
(4)	$\neg(q \wedge r)$	Modens ponens (2) (3)
(5)	$\neg q \vee \neg r$	De Morgan
(6)	q	Simplication of (1)
(7)	$\neg r$	Disjunctive syllogism (6) (7)
(8)	$s \to r$	Premise
(9)	$\neg s$	Modus tollens (7) (8)

□

Q.11 For each of these arguments, explain which rules of inference are used for each step.

(a) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

(b) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

(c) "Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Solution:

(a) Let $c(x)$ denote "x is in this class", $w(x)$ denote "x enjoys whale watching", and $p(x)$ denote "x cares about ocean pollution." The premises are $\exists x(c(x) \wedge w(x))$ and $\forall x(w(x) \rightarrow p(x))$. From the first premise, $c(y) \wedge w(y)$ for a particular person y. Using simplification, $w(y)$ follows. Using the second premise and universal instantiation, $w(y) \rightarrow p(y)$ follows. Using modus ponens, $p(y)$ follows, and by conjunction, $c(y) \wedge p(y)$ follows. Finally, by existential generalization, the desired conclusion, $\exists x(c(x) \wedge p(x))$ follows.

(b) Let $c(x)$ be "x is in this class," $p(x)$ be "x owns a PC", and $w(x)$ be "x can use a word-processing program". The premises are $c(Zeke)$, $\forall x(c(x) \rightarrow p(x))$, and $\forall x(p(x) \rightarrow w(x))$. Using the second premise and universal instantiation and modus ponens, $c(Zeke) \rightarrow p(Zeke)$ follows. Using the first premise and modus ponens, $p(Zeke)$ follows. Using the third premise and universal instantiation, $p(Zeke) \rightarrow w(Zeke)$ follows. Finally, using modus ponens, $w(Zeke)$, the desired conclusion follows.

(c) Let $r(x)$ be "x is one of the five roommates listed", let $d(x)$ be "x has taken a course in discrete mathematics", and let $a(x)$ be "x can take a course in algorithms". We are given premises $\forall x(r(x) \rightarrow d(x)$, $\forall x(d(x) \rightarrow a(x)$, and we want to conclude $\forall x(r(x) \wedge a(x))$.

Step	**Reason**
1. $\forall x(r(x) \to d(x))$	Hypothesis
2. $r(y) \to d(y)$	Universal Instantiation using 1.
3. $\forall x(d(x) \to a(x))$	Hypothesis
4. $d(y) \to a(y)$	Universal instantiation using 3.
5. $r(y) \to a(y)$	Hypothetical syllogism using 2. and 4.
6. $\forall x(r(x) \to a(x))$	Universal generalization using 5.

□

Q.12

(a) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Solution:

(a) The negation is

$$\exists x \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \wedge n \text{ is odd}).$$

(b) If n is odd then $n^3 + 6n + 5$ is even because n^3 is then odd and $6n$ is then even. Therefore, the original statement is true.

□

Q.13 Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.
Solution: Observe that $a^2 + b^2 = (a+b)^2 - 2ab$. Thus, $(a+b)^2$ has the same parity as $a^2 + b^2$. So $(a+b)^2$ is even. Then $a + b$ is also even.

□

Q.14 Prove or disprove that if a and b are rational numbers, then a^b is also rational.
Solution: Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and this number is not rational.

□

Q.15 Prove that $\sqrt[3]{2}$ is irrational.
Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

□

Q.16 Suppose that we have a theorem: "$\sqrt{n}$ is irrational whenever n is a positive integer that is *not* a perfect square." Use this theorem to prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Solution: We give a proof by contradiction. If $\sqrt{2}+\sqrt{3}$ is a rational number, then its square is also rational, which is $5+2\sqrt{6}$. Subtracting 5 and dividing by 2, we have $\sqrt{6}$ is also rational. However, this contradicts the theorem.

□

Q.17 (Second-Price Auction) Please read the following description carefully and answer questions. In an auction, an auctioneer is responsible for selling a product, and bidders bid for the product. The winner of the auction wins the product and pays for it. We consider a single object sealed second-price auction. The detailed settings are as follows:

- There is one product to be sold.
- There are N bidders, denoted by $\mathcal{N} = \{1, 2, ..., N\}$. Bidder $n \in \mathcal{N}$ has a valuation over the product of v_n.
- Every bidder submits his or her bid in a sealed envelope, so other bidders do not know his or her bid. Bidder $n \in \mathcal{N}$ submits a bid of b_n.
- After receiving the bids from all bidders, the auctioneer announces the winner and payment. The winner is the bidder who submits the highest bid. The payment of this winner is the second highest bid. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 4$, $b_3 = 5$. Then, the winner is bidder 3, and the payment is the second highest bid 4.

- If multiple bidders have the same bid, then they draw a lottery. Each of them has equally probability of winning. In this case, the payment is equal to their bids. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 5$, $b_3 = 5$. The winner is either 2 or 3 with equal probability. The payment is 5.

- After the auction, the payoffs of the bidders are as follows:

  - If bidder n loses, his or her payoff is zero.
  - If bidder n wins, his or her payoff is equal to its valuation v_n minus the payment.

  For bidder n, the higher payoff, the better.

Now, suppose you are a bidder in this auction, e.g., bidder n, and you do not know any other bidders' valuations and bids. You know your valuation v_n. You can choose your bid b_n to maximize your payoff. Prove that for an arbitrary bidder $n \in \mathcal{N}$, submitting a bid $b_n = v_n$ will always lead to a payoff that is no smaller than submitting a bid with $b_n \neq v_n$.

(Note: This second-price auction is commonly used, due to the property that bidders are willing to submit their valuation as their bid.)

(Hint: Use proof by cases; consider the highest bid of the others, and compare it with your valuation v_n; enumerate all possibilities.)

Solution: Let b^* be the highest bid of all bidders except bidder n, i.e., $b^* = \max_{\mathcal{N} \setminus \{n\}} b_n$. Note that if bidder n wins, then b^* is his or her payment.

Consider an arbitrary bidder $n \in \mathcal{N}$. There are three cases:

- $v_n < b^*$:

  - Submitting $b_n = v_n$: loses; payoff is zero
  - Submitting $b_n < v_n$: loses; payoff is zero
  - Submitting $b^* > b_n > v_n$: loses; payoff is zero
  - Submitting $b_n = b^*$: may win; if wins, payoff is negative, as $v_n - b^* < 0$; if loses, payoff is zero
  - Submitting $b_n > b^*$: wins; payoff is negative, as $v_n - b^* < 0$

- $v_n = b^*$

- Submitting $b_n = v_n$: may win; if wins, payoff is $v_n - b^* = 0$; if loses, payoff is zero
- Submitting $b_n > v_n$: wins; payoff is $v_n - b^* = 0$
- Submitting $b_n < v_n$: loses; payoff is zero

- $v_n > b^*$
  - Submitting $b_n = v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b_n > v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b^* < b_n < v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b_n = b^*$: may win; if wins, payoff is $v_n - b^* > 0$; if loses, payoff is zero
  - Submitting $b_n < b^*$: loses; payoff is zero

For all these three cases, submitting bid $b_n = v_n$ will always lead to a payoff that is no smaller than submitting bid $b_n \neq v_n$.