Discrete Mathematics for Computer Science

Lecture 5: Set and Function

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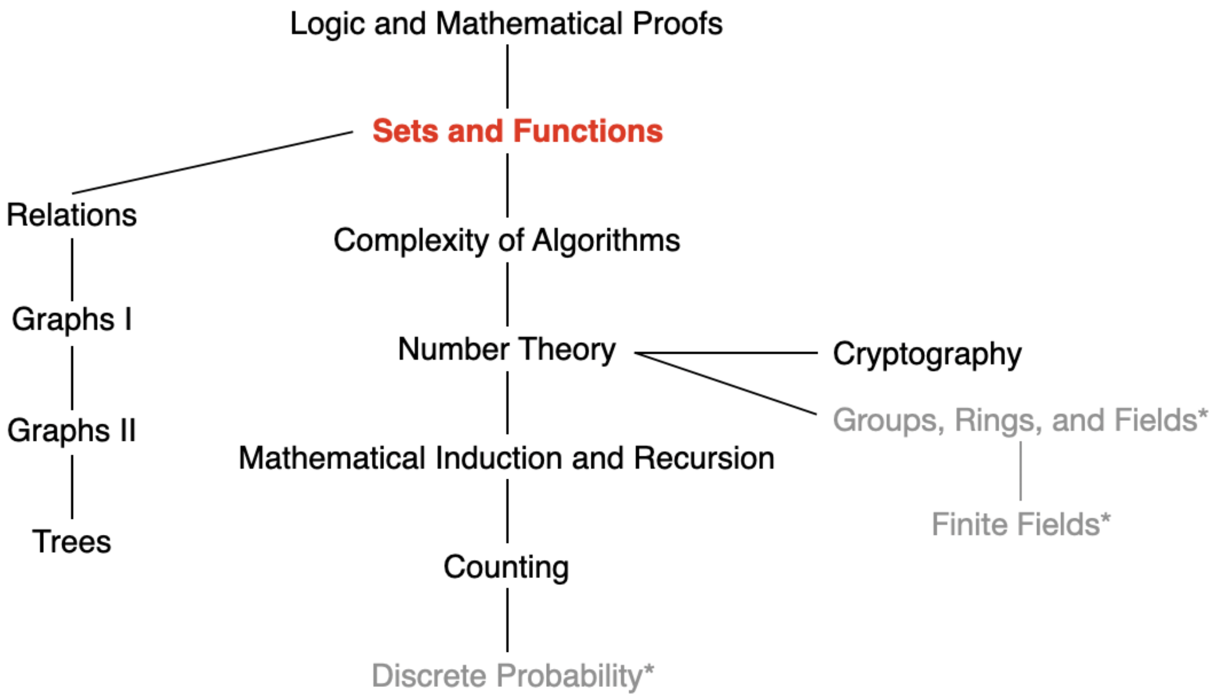
This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Inverse Functions

Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns an element b belonging to B to the unique element a in A such that $f(a) = b$.

The inverse function of f is denoted by f^{-1}.
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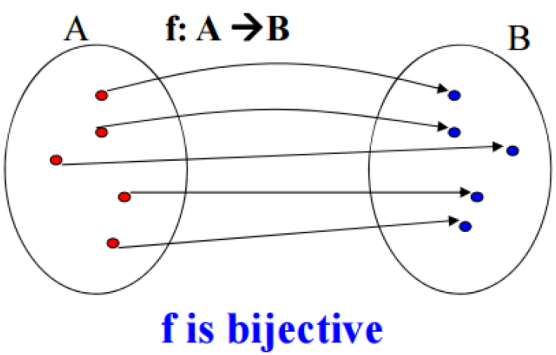

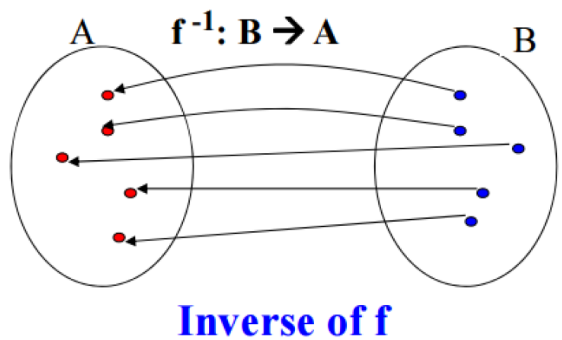

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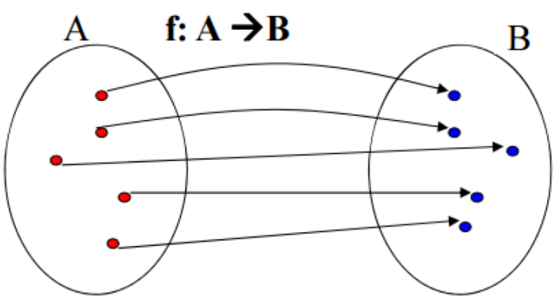
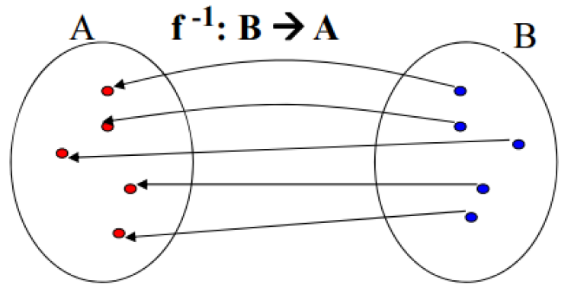


f is bijective



Inverse of f

A bijection is called invertible.

Inverse Functions

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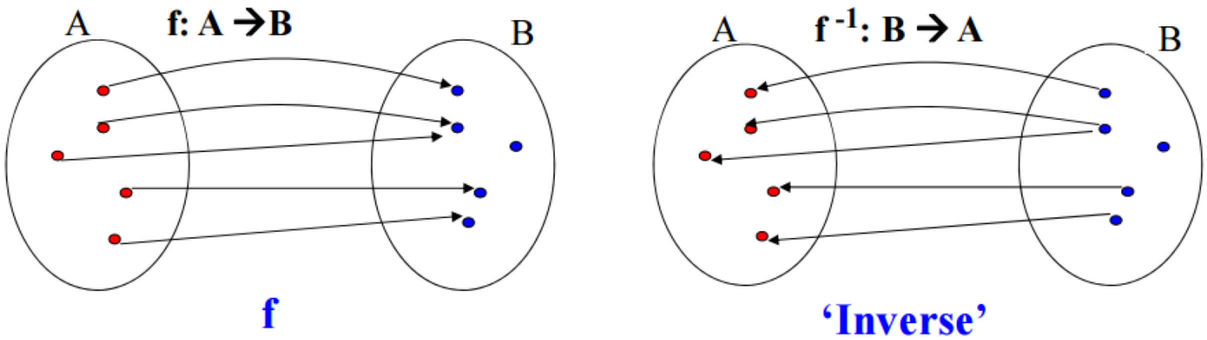

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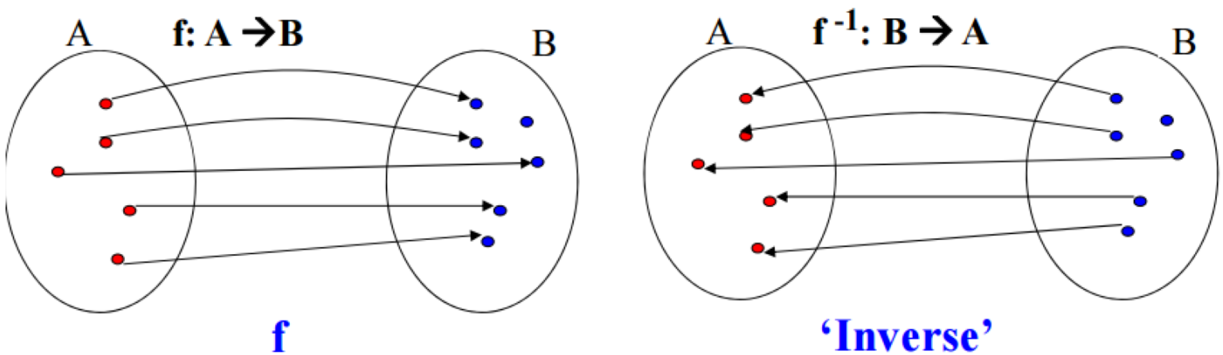

The inverse is not a function: one element of B is not assigned an element of A.

Proof for Inverse Function

1. Prove function f is a bijection: injective, surjective

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
To show that f is *surjective*	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$
To show that f is not *surjective*	Find a specific element $y \in B$ such that $f(x) \neq y$ for all $x \in A$

2. If f is a bijection, then it is invertible
3. Determine the inverse function

Inverse Functions: Example 1

$f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = x + 1$. Is f invertible? If yes, then what is the inverse function f^{-1}?

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To determine the inverse function, suppose that y is the image of x, so that $y = x + 1$. Then, $x = y - 1$. This means that $y - 1$ is the unique element of $\boldsymbol{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Inverse Functions: Example 2

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Proof: It is invertible, as it is a bijection:

- Injective: Consider x and x'. If $f(x) = f(x')$ (i.e., $x^2 = (x')^2$), then we have $x^2 - (x')^2 = (x + x')(x - x') = 0$. Since we consider the set of all nonnegative real numbers, we must have $x = x'$.
- Surjective: Consider an arbitrary nonnegative real number y. There exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$.

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To reverse the function, suppose that y is the image of x, so that $y = x^2$. Then, $x = \sqrt{y}$. Consequently, $f^{-1}(y) = \sqrt{y}$.

Composition of Functions

Let f be a function from B to C and let g be a function from A to B. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.

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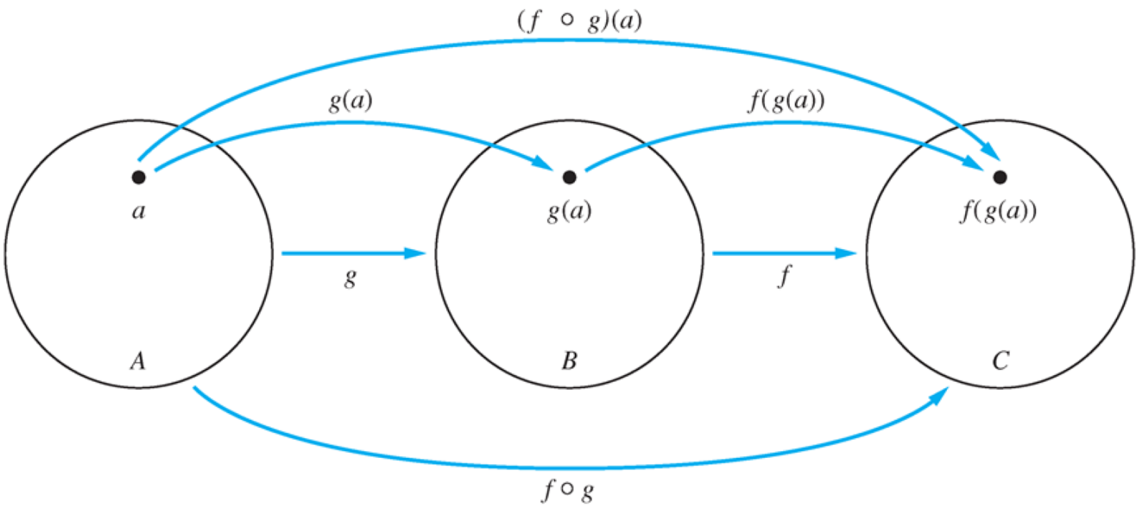

Composition of Functions

- **Example 1**:

  Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$.

  $g : A \to A$ | $f : A \to B$

  $1 \mapsto 3$ | $1 \mapsto b$

  $2 \mapsto 1$ | $2 \mapsto a$

  $3 \mapsto 2$ | $3 \mapsto d$

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  $f \circ g : A \to B$
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  $3 \mapsto a$

Composition of Functions

- **Example 2**:

  Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g : \mathbf{Z} \rightarrow \mathbf{Z}$, where $f(x) = 2x$ and $g(x) = x^2$.

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Note: In general, the order of composition matters.

Composition of Functions

- Suppose that f is a bijection from A to B. Then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$, Since
$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$
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where I_A, I_B denote the *identity functions* on the sets A and B, respectively.

Note: Identity function is sometimes denoted by $\iota_A(\cdot)$:

$$\iota_A(x) = x$$

Floor and Ceiling Functions

- The floor function assigns a real number x the largest integer that is $\leq x$, denoted by $\lfloor x \rfloor$. E.g., $\lfloor 3.5 \rfloor = 3$.
- The ceiling function assigns a real number x the smallest integer that is $\geq x$, denoted by $\lceil x \rceil$. E.g., $\lceil 3.5 \rceil = 4$.

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(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Note: n is an integer, x is a real number.

Floor and Ceiling Functions: Example 1

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- $0 \leq \epsilon < \frac{1}{2}$: In this case, $2x = 2n + 2\epsilon$. Since $0 \leq 2\epsilon < 1$, we have $\lfloor 2x \rfloor = 2n$. Similarly, $x + \frac{1}{2} = n + \frac{1}{2} + \epsilon$. Since $0 \leq \frac{1}{2} + \epsilon < 1$, we have $\lfloor x + \frac{1}{2} \rfloor = n$. Thus, $\lfloor 2x \rfloor = 2n$, and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = 2n$.

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- $\frac{1}{2} \leq \epsilon < 1$: In this case, $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$. Since $0 \leq 2\epsilon - 1 < 1$, we have $\lfloor 2x \rfloor = 2n + 1$. ....

Floor and Ceiling Functions: Example 2

Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Floor and Ceiling Functions: Example 2

Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Proof: This statement is false. Consider a counterexample $x = \frac{1}{2}$ and $\frac{1}{2}$. We can find that $\lceil x + y \rceil = 1$, but $\lceil x \rceil + \lceil y \rceil = 2$.

Factorial Function

The factorial function $f : \mathbf{N} \rightarrow \mathbf{Z}^{+}$ is the product of the first n positive integers when n is a nonnegative integer, denoted by $f(n) = n!$.

Summary of Function

- Function $f : A \to B$: an assignment of exactly one element of B to each element of A
- One-to-one function
- Onto function
- One-to-one correspondence: one-to-one function and onto
- Inverse function
- Floor function, ceiling function, factorial function

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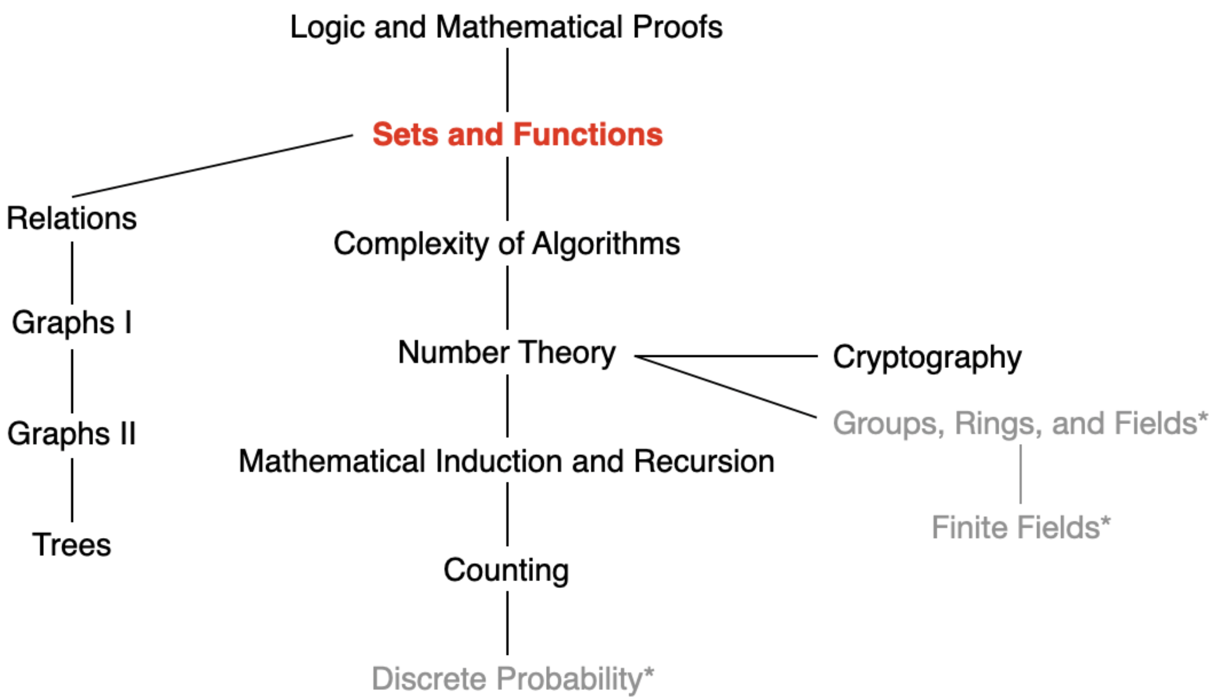

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Sequences

A sequence is a function from a subset of the set of integers (typically the set $\{0, 1, 2, ...\}$ or $\{1, 2, 3, ...\}$) to a set S.

We use the notation a_n to denote the image of the integer n. $\{a_n\}$ represents the ordered list $\{a_1, a_2, a_3, ...\}$

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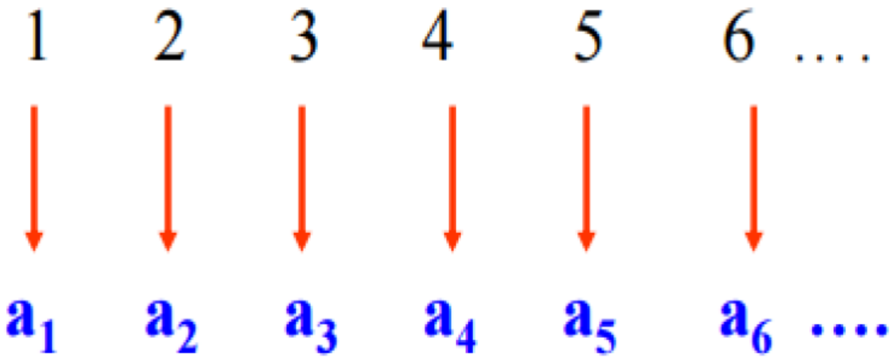

Sequences

Examples:

- $a_n = n^2$, where $n = 1, 2, 3, \ldots$
- $a_n = (-1)^n$, where $n = 1, 2, 3, \ldots$
- $a_n = 2^n$, where $n = 1, 2, 3, \ldots$

Geometric Progression

A geometric progression is a sequence of the form

$$a, ar, ar^2, ..., ar^n, ...$$

where the initial term a and the common ratio r are real numbers.

Example: $a_n = 3 \times (\frac{1}{2})^n$, where $n = 0, 1, 2, 3, ...$

Arithmetic Progression

An arithmetic progression is a sequence of the form

$$a, a + d, a + 2d, a + 3d, ..., a + nd, ...$$

where the initial term a and common difference d are real numbers.

Example: $a_n = -1 + 4n$, where $n = 0, 1, 2, 3, ...$

Recursively Defined Sequences

1 Providing explicit formulas, e.g., $a_n = -1 + 4n$, where $n = 0, 1, 2, 3, \ldots$

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2 **Recursively Defined Sequences**: provide

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- a rule for determining subsequent terms from those that precede them.

The n-th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

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Examples:

- $a_0 = 1$, $a_n = a_{n-1} + 2$ for $n = 1, 2, 3, \ldots.$
- $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$ (Fibonacci sequence)

Summations

The summation of the terms of a sequence is

$$\sum_{j=m}^{n} = a_m + a_{m+1} + ... + a_n$$

- j: the index of summation; the choice of the letter is arbitrary
- m: the lower limit of the summation
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$$\sum_{j=1}^{n}(ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j$$

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Summations

The sum of the first n terms of the arithmetic progression:

$$S_n = \sum_{j=0}^{n} (a + jd) = (n+1)a + d \sum_{j=0}^{n} j = (n+1)a + d\frac{n(n+1)}{2}$$

The sum of the first n terms of the geometric progression:

- $r \neq 1$

$$S_n = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = \frac{ar^{n+1} - a}{r - 1}$$

- $r = 1$

$$S_n = \sum_{j=0}^{n} (ar^j) = (n+1)a$$

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Summations: Example

- **Examples**:
  - $S = \sum_{j=1}^{5}(2+3j)$
  - $S = \sum_{j=3}^{5}(2+3j)$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)$
  - $S = \sum_{j=0}^{3} 2(5)^j$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{3} ij$

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Summations: Example

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  - $S = \sum_{j=0}^{3} 2(5)^j$ 312
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{3} ij$ 60

Infinite Series

Infinite geometric series can be computed in the closed form for $|x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{x^{n+1} - 1}{x - 1} = \frac{1}{1 - x}$$

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$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Some Useful Summation Formulas

$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

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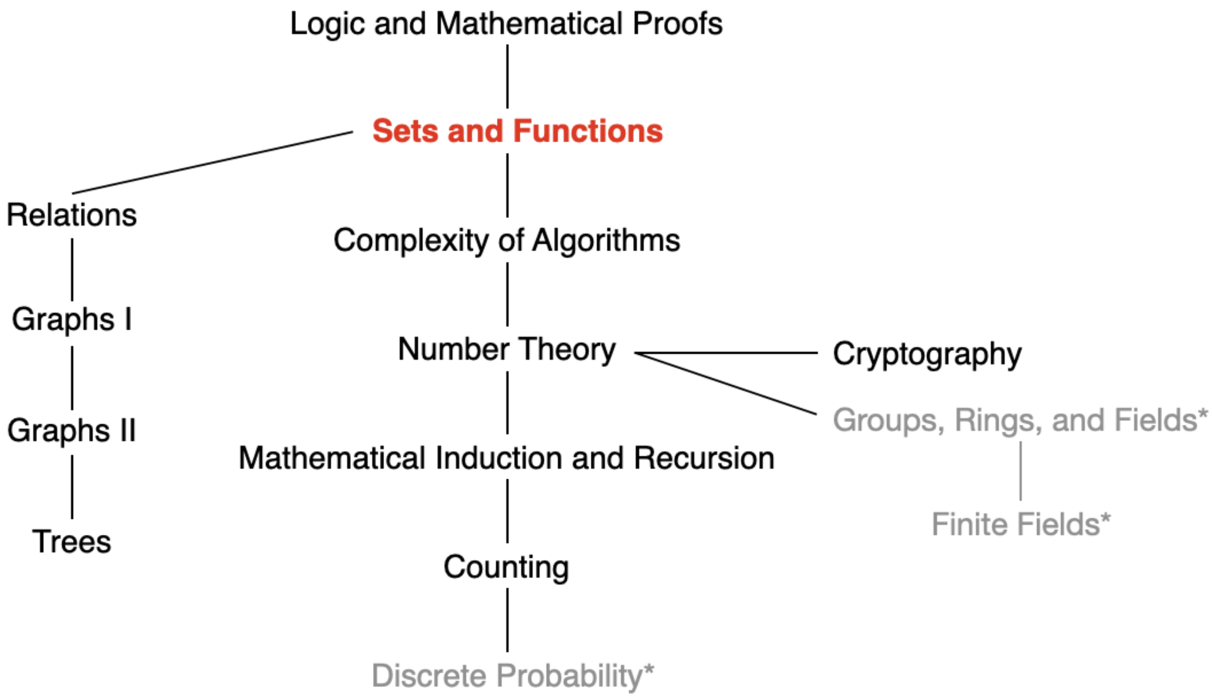

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The sets A and B have the same cardinality if there is a one-to-one correspondence between elements in A and B.

If there is a one-to-one function from A to B, the cardinality of A is less than or equal to the cardinality of B, denoted by $|A| \leq |B|$.

Cardinality of Sets

Recall: the cardinality of a finite set is defined by the number of the elements in the set.

The sets A and B have the same cardinality if there is a one-to-one correspondence between elements in A and B.

If there is a one-to-one function from A to B, the cardinality of A is less than or equal to the cardinality of B, denoted by $|A| \leq |B|$.

Moreover, when $|A| \leq |B|$ and A and B have different cardinalities, we say that the cardinality of A is less than the cardinality of B, denoted by $|A| < |B|$.

Countable Sets

A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called countable. A set that is not countable is called uncountable.

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The elements of the set can be enumerated and listed.

Hilbert's Paradox: Grand Hotel

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel.

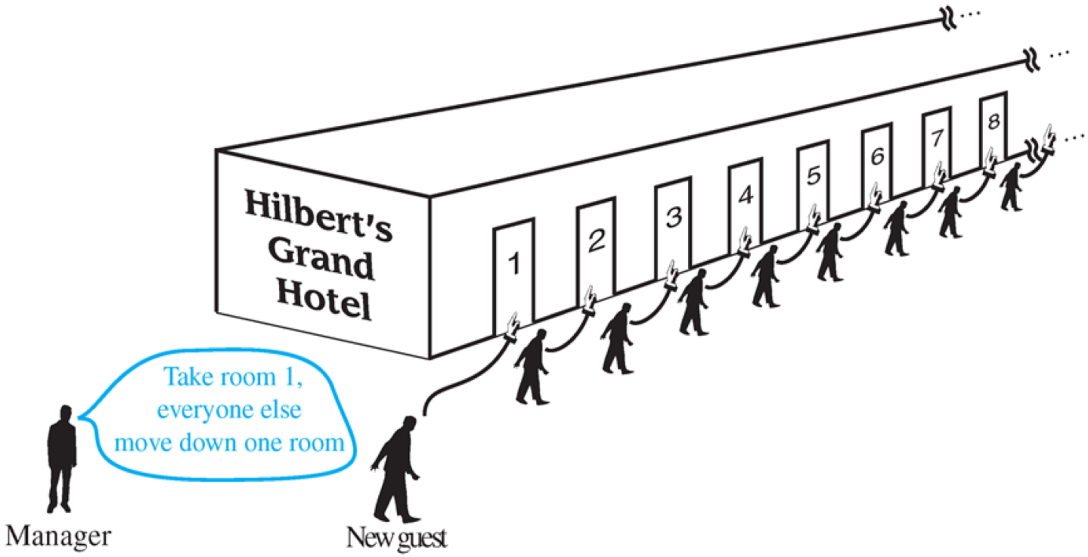

FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

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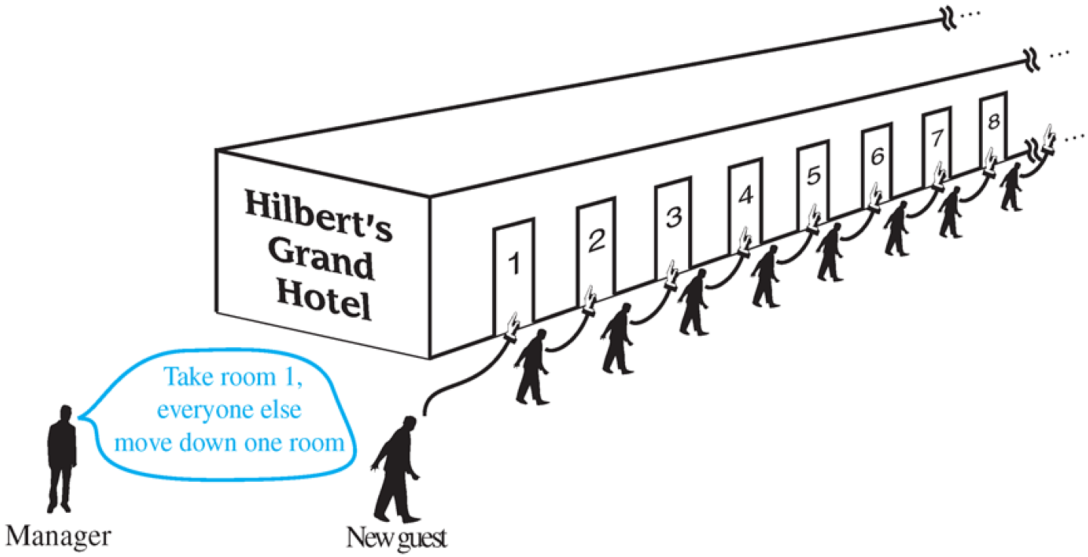

FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

Finitely many room: "All rooms are occupied" is equivalent to "no new guests can be accommodated".
Infinitely many room: This equivalence no longer holds.

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- Onto: For any arbitrary element in $t \in A$, we have an $n = (t + 1)/2 \in \mathbf{Z}^+$ such that $f(n) = t$.

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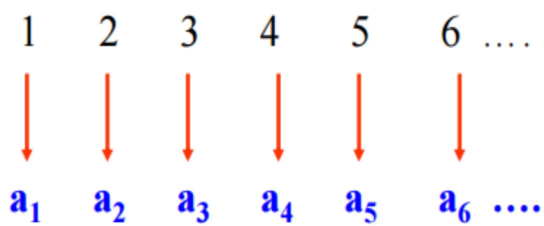
A sequence is a function from <u>a subset of the set of integers</u> to <u>a set S</u>.



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Alternatively, show there is a one-to-one correspondence from $\mathbf{Z}^+$ to $\mathbf{Z}$:

- when n is even: $f(n) = n/2$
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Do $\mathbf{Z}^+$ and $\mathbf{Z}$ have the same cardinality? Yes, because there is a one-to-one correspondence between $\mathbf{Z}^+$ and $\mathbf{Z}$.

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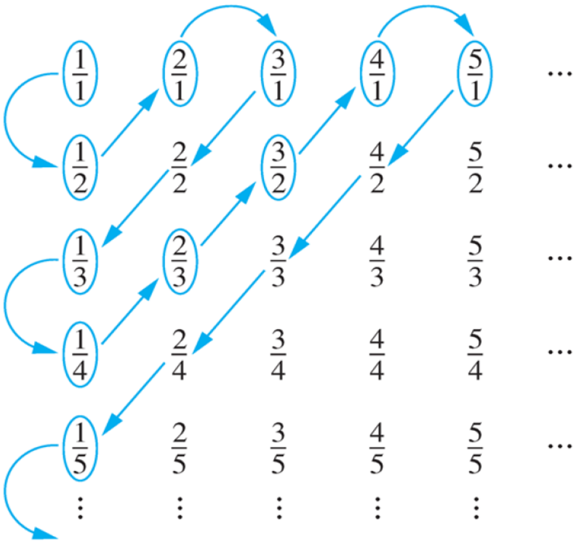
Solution:

Constructing the list: first list p/q with $p+q=2$, next list p/q with $p+q=3$, and so on.

$1, 1/2, 2, 3, 1/3, 1/4, 2/3, \ldots$



Countable Sets: Example 4

Theorem: The set of finite strings S over a finite alphabet A is countably infinite. (Assume an alphabetical ordering of symbols in A)

For example, let $A = \{$'a', 'b', 'c'$\}$. Then, set
$S = \{$'', 'a', 'b', 'c', 'ab' ..., 'aaaaa', ...$\}$

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Solution:

We show that the strings can be listed in a sequence. First list

(i) all the strings of length 0 in alphabetical order.

(ii) then all the strings of length 1 in lexicographic order.

(iii) and so on.

This implies a bijection from $\mathbf{Z}^+$ to S.

Countable Sets: Example 5

The set of all Java programs is countable.

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Solution:

Let S be the set of strings constructed from the characters which may appear in a Java program. Use the ordering from the previous example. Take each string in turn

– feed the string into a Java compiler

– if the complier says YES, this is a syntactically correct Java program, we add this program to the list

– we move on to the next string

In this way, we construct a bijection from $\mathbf{Z}^+$ to the set of Java programs.

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Theorem: If A and B are countable sets, then $A \cup B$ is also countable.

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A set that is not countable is called uncountable.

Theorem: The set of real numbers $\mathbf{R}$ is uncountable.

Proof by Contradiction:

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Proof by Contradiction: Suppose $\mathbf{R}$ is countable. Then, the interval from 0 to 1 is countable. This implies that the elements of this set can be listed as $r_1, r_2, r_3, ...,$ where

- $r_1 = 0.d_{11}d_{12}d_{13}d_{14}$
- $r_2 = 0.d_{21}d_{22}d_{23}d_{24}$
- $r_3 = 0.d_{31}d_{32}d_{33}d_{34}$

where all $d_{ij} \in \{0, 1, 2, ..., 9\}$.

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We want to show that not all real numbers in the interval between 0 and 1 are in this list. Form a new number called $r = 0.d_1 d_2 d_3 d_4$, where $d_i = 2$ if $d_{ii} \neq 2$, and $d_i = 3$ if $d_{ii} = 2$.

Example: suppose	r1 = 0.75243...	d1 = 2
	r2 = 0.524310...	d2 = 3
	r3 = 0.131257...	d3 = 2
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Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

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Schroder-Bernstein Theorem

Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B, and hence $|A| = |B|$.

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Example: Show that $|(0,1)| = |(0,1]|$

$f(x) = x$, $g(x) = x/2$

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Definition: We say that a function is computable if there is a computer program in some programming language that finds the values of this function. If a function is not computable, we say it is uncomputable.

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Cantor's theorem: If S is a set, then $|S| < |P(S)|$.

This Lecture