CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 4
Due: Apr. 30th, 2022, please submit one pdf file through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

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Q. 1. (5 points) Prove by induction that, for any sets $A_1, A_2, \cdots, A_n$, De Morgan's law can be generalized to

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Solution: The base case here is $n = 2$, that is De Morgan's law. (It is clear that $\overline{A_1} = \overline{A_1}$, so we could also use $n = 1$ as the base case.) It remains to show the inductive step. Suppose the statement holds for $n = k$, we now show it holds for $n = k + 1$. We have

$$\begin{aligned} \overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \\ &= \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} \quad \text{De Morgan} \\ &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}} \quad \text{by i.h.} \end{aligned}$$

Then by mathematical induction, we have proved the conclusion.

□

Q. 2. (5 points) Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a - b)$.

Solution: It turns out to be easier to think about the given statement as $na^{n-1}(a - b) \geq a^n - b^n$. The basic step ($n = 1$) is true since $a - b \geq a - b$. Assume that the inductive hypothesis, that $ka^{k-1}(a - b) \geq a^k - b^k$; we must show that $(k + 1)a^k(a - b) \geq a^{k+1} - b^{k+1}$. We have

$$\begin{aligned} (k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b) \\ &\geq a(a^k - b^k) + a^k(a-b) \\ &= a^{k+1} - ab^k + a^{k+1} - ba^k. \end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \geq a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \geq 0$, which factors into $(a^k - b^k)(a - b) \geq 0$, and this is true, because we are given that $a > b$.

$\square$

Q. 3. (10 points) A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Solution:

By checking the first few values $10, 20, 25, 30, 35, 40, 45, 50, \ldots$, we guess that we can make \$$n$ in amount of money, where

$$n \in \{10\} \cup \{5m : \ m \geq 4 \text{ and } m \in \mathbb{Z}^+\}.$$

Let $P(n)$ be the statement "we can make \$$5m$ in gift certificate in amount of \$10 and \$25."

- Basic step: $m = 4, 5$, we can make \$20 and \$25 in gift certificate.
- Inductive step: Suppose \$$5k$ for $4 \leq k < m$. We now prove $P(m)$ for $m \geq 6$. Note that $5m = 10 + 5(m-2)$. Since $4 \leq m-2 < m$, $P(m-2)$ is true. So we can make \$$5(m-2)$ in gift certificate. It then follows that we can \$$5m$ in gift certificate by adding an extra \$10 certificate.

$\square$

Q. 4. (5 points) Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Solution: The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction, it follows that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus, we have proved $\forall n Q(n)$ by ordinary induction.

$\square$

Q. 5. (5 points) Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. (use the equality $2^{2^{n+1}} = (a^{2^n})^2$)

Solution:

Algorithm 1 twopower (n: positive integer, a: real number)

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if n = 1 then
    return a^2
else
    return twopower(n − 1, a)^2
end if
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$\square$

Q. 6. (5 points) Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Solution: $f(n) = 25n^{\log_4 5} - 24n$.

$\square$

Q. 7. (5 points) How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Solution:

(a) 2 if $n = 1$, 2 if $n = 2$, and 0 if $n \geq 3$.

(b) 2^{n-2} for $n > 1$; 1 if $n = 1$.

(c) $2(n - 1)$.

$\square$

Q. 8. (5 points) How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks.

You should leave your answer as an equation.

Solution: First, we choose the ranks of the 2 pairs, noting that the order we pick these two ranks does not matter, so there are $\binom{13}{2}$ options here. Next we pick the 2 suits for the first pair, $\binom{4}{2}$ and the suits for the second pair $\binom{4}{2}$. Then we decide which 2 ranks of the remaining 11 to use for the other cards, $\binom{11}{2}$, and finally choose each of their suits $\binom{4}{1}\binom{4}{1}$. Altogether, by the product rule, this gives $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ hands.

$\square$

Q. 9. (10 points) Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Solution:

Since $\gcd(2, 121) = 1$, it suffices to prove that $2 \Big| \binom{240}{120}$ and $121 \Big| \binom{240}{120}$. We prove these two divisibilities in general, i.e.,

$$2 \Big| \binom{2n}{n}, \text{ and } (n+1) \Big| \binom{2n}{n}.$$

Since

$$\begin{aligned} \binom{2n}{n} &= \frac{(2n)!}{n!n!} \\ &= \frac{2n \cdot (2n-1)!}{n!n!} \\ &= \frac{2 \cdot (2n-1)!}{(n-1)!n!} \\ &= 2 \cdot \binom{2n-1}{n}, \end{aligned}$$

we have 2 divides $\binom{2n}{n}$. Since

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(2n)!}{(n+1)!n!} \\ &= \frac{1}{n+1}\binom{2n}{n},\end{aligned}$$

which is an integer, we have $n+1$ divides $\binom{2n}{n}$. This completes the proof.

$\square$

Q. 10. (5 points) How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution: Working modulo 5 there are 25 pairs: $(0,0), (0,1), \ldots, (4,4)$. Thus, we could have 25 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

$\square$

Q. 11. (10 points) Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Solution: The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. There are four possible pairs of parities: (odd, odd), $(odd, even)$, $(even, odd)$, $(even, even)$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

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Q. 12. (5 points) Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution:

Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore, the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

□

Q. 13. (5 points) Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Solution: The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are $1, -1$, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6$, $\alpha_2 = -1$ and $\alpha_3 = -1$. Therefore, the answer is $a_n = 6 - 2(-1)^n - 2^n$.

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Q. 14. (5 points) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 0$.

Solution: The *characteristic equation* is $r^2 - 5r + 6 = 0$, and the solutions are $r = 2, 3$. So the closed-form is $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$. In particular, we have $1 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0 = \alpha_1 + \alpha_2$, and $0 = 2\alpha_1 + 3\alpha_2$. Solving for 2 unknowns with 2 equations is $\alpha_1 = 3$ and $\alpha_2 = -2$. Therefore, we have

$$a_n = 3 \cdot 2^n - 2 \cdot 3^n.$$

□

Q. 15. (5 points) Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Solution: We may split S_n into 3 cases :

> Case (1): item 1 is not in the subset. We must now choose a non-empty subset of $\{2, \ldots, n\}$. There are a_{n-1} ways to do this.
>
> Case (2): item 1 is in the subset, and there are more elements. We must now choose a non-empty subset of $\{3, \ldots, n\}$. There are a_{n-2} ways to do this.
>
> Case (3): item 1 is in the subset, and no other elements are. There is 1 way to do this.

Thus, we have the recurrence relation as: $a_n = a_{n-1} + a_{n-2} + 1$.

$\square$

Q. 16. (10 points) Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned} \sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r. \end{aligned}$$

Thus,

$$1 + \left(\sum_{r=1}^{n-1} C(n,r)x^r \right) + x^n = 1 + \left(\sum_{r=1}^{n-1} (C(n-1,r) + C(n-1,r-1))x^r \right) + x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n, r)$ must equal $C(n-1, r) + C(n-1, r-1)$ for $1 \le r \le n-1$, as desired.

$\square$