Discrete Mathematics for Computer Science

Lecture 4: Set and Function

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A Question from Students Last Lecture

Are $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$ and $\forall x \exists y((x \neq 0) \rightarrow (xy = 1))$ equivalent?

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Quick Answer: Yes. This is because $p \rightarrow \exists y Q(y)$ and $\exists y(p \rightarrow Q(y))$ are equivalent. To prove this, see page 45 in the textbook:

- If $p \rightarrow \exists y Q(y)$ is true, then $\exists y(p \rightarrow Q(y))$ is true;
- If $\exists y(p \rightarrow Q(y))$ is true, then $p \rightarrow \exists y Q(y)$ is true.

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Complicated Answer: (out of the scope of this course)

- Free variable: $\sum_{k=1}^{10} f(k, n)$, n is a free variable

Where φ is any formula and where x is *not* a free variable in ψ:

$\forall x\, \varphi \rightarrow \psi \not\Leftrightarrow \forall x(\varphi \rightarrow \psi)$ (No!)

$\psi \rightarrow \forall x\, \varphi \Leftrightarrow \forall x(\psi \rightarrow \varphi)$ (Yes!)

$\exists x\, \varphi \rightarrow \psi \not\Leftrightarrow \exists x(\varphi \rightarrow \psi)$ (No!)

$\psi \rightarrow \exists x\, \varphi \Leftrightarrow \exists x(\psi \rightarrow \varphi)$ (Yes!)

Review of Last Lecture

- Nested Quantifiers: $\forall x \exists y P(x, y)$, $\exists x \forall y P(x, y)$
  - The order matters if quantifiers are of different type.
    - ★ $\forall x \exists y P(x, y)$, $\exists y \forall x P(x, y)$
  - The order does no matter if quantifiers are of the same type.
    - ★ $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
    - ★ $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- Argument and Inference
  - Argument form is valid, if

$$(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q \text{ is a tautology.}$$

  - The validity of an argument follows from the validity of argument form.
  - Rules of inference
- Proofs: direct proof, proof by contrapositive, proof by contradiction, proof by cases, ...

Review: Using Rules of Inference to Build Arguments

- "It is not sunny this afternoon and it is colder than yesterday."
- "We will go swimming only if it is sunny."
- "If we do not go swimming then we will take a canoe trip."
- "If we take a canoe trip, then we will be home by sunset."
- Show the conclusion that "we will be home by sunset."

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- p: It is sunny this afternoon.
- q: It is colder than yesterday.
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- "It is not sunny this afternoon and it is colder than yesterday."

  $\neg p \land q$
- "We will go swimming only if it is sunny."

  $r \to p$
- "If we do not go swimming then we will take a canoe trip."

  $\neg r \to s$
- "If we take a canoe trip, then we will be home by sunset."

  $s \to t$
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Premises: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$

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Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

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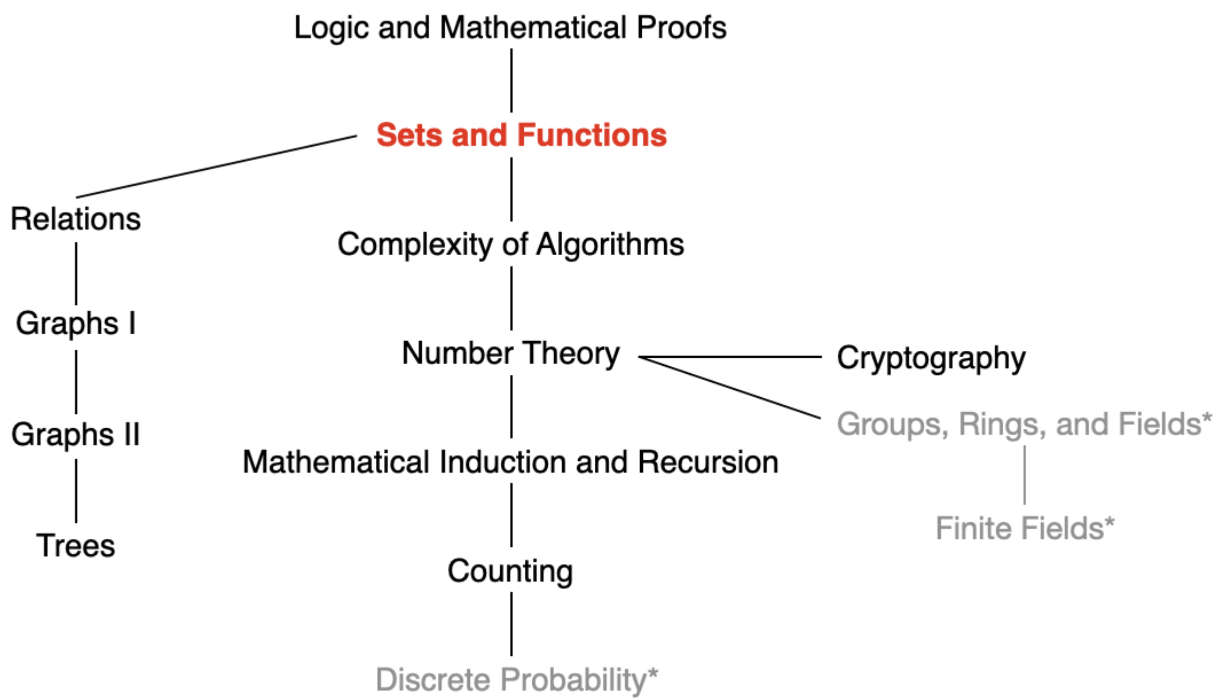
Summary of Logic and Proof

- **Logic**:
  - Proposition, applications, equivalence $\equiv$, ...
  - Predicates $P(x)$
  - Quantifiers $\forall x P(x)$, $\exists x P(x)$
- **Mathematical Proofs**:
  - Argument: premises, conclusion
  - Rules of inference
  - Proofs

This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Sets

A set is an unordered collection of objects. These objects are called elements or members.

- $A = \{1, 2, 3, 4\}$
- $B = \{a, b, c, d\}$
- $C = \{a, 2, 1, \text{Mary}\}$

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Many discrete structures are built with sets:

- combinations
- relations
- graphs

Set Representation

Examples:

- $A = \{2, 3, 5, 7\}$
- $B = \{1, 2, 3, ..., 100\}$
- $C = \{a \geq 2 \mid a \text{ is a prime}\}$
- $D = \{2n \mid n = 0, 1, 2, ..., \}$

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Representing a set by:

- listing (enumerating) the elements
- if enumeration is hard, use ellipses (...)
- definition by property, using the set builder

$$\{x \mid x \text{ has property } P \text{ or property } P(x))\}$$

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Notation:

- $a \in A$: a is an element of set A
- $a \notin A$: a is not an element of set A

Important sets

- Natural numbers:
  - $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
- Integers:
  - $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Positive integers:
  - $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$
- Rational numbers:
  - $\mathbf{Q} = \{\frac{p}{q} \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$
- Real numbers:
  - $\mathbf{R}$
- Complex numbers:
  - $\mathbf{C}$

Important sets

- $[a, b] = \{x \mid a \leq x \leq b\}$
  $[a, b) = \{x \mid a \leq x < b\}$
  $(a, b] = \{x \mid a < x \leq b\}$
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- Two sets A, B are *equal* if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$.

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Universal and Empty Set

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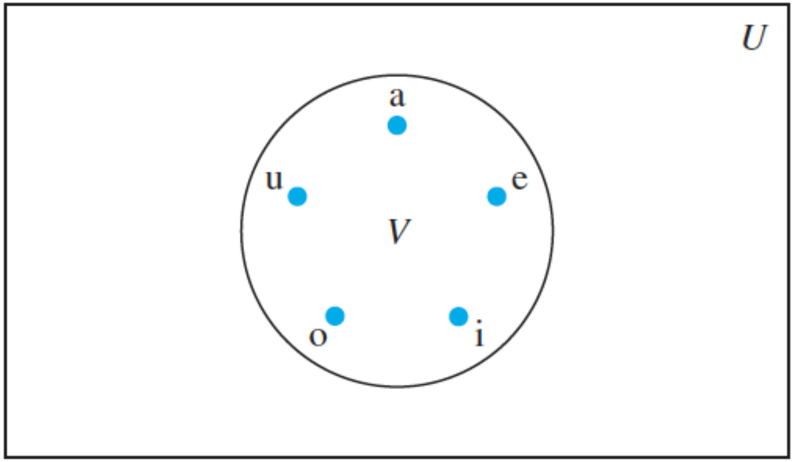
- Are $\emptyset$ and $\{\emptyset\}$ equal? No

Venn Diagrams

A set can be visualized using Venn diagrams



Subset

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If $A \subseteq B$, but $A \neq B$, then we say A is a proper subset of B, i.e., $\forall x(x \in A \to x \in B) \land \exists x(x \in B \land x \notin A)$, denoted by $A \subset B$.

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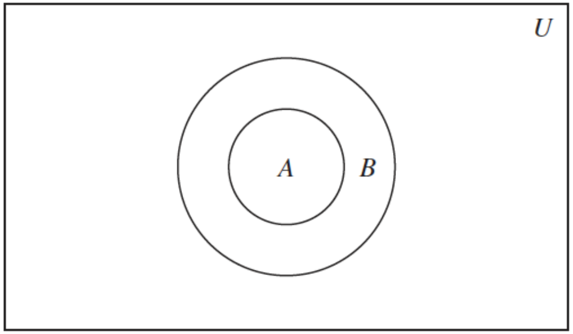

Proof of Subset

Proof:

- Showing $A \subseteq B$: if x belongs to A, then x also belongs to B.
- Showing $A \nsubseteq B$: find a single $x \in A$ such that $x \notin B$.

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Prove that $\emptyset \subseteq S$.

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Note: two sets are equal if and only if each is a subset of the other:

$$\forall x(x \in A \leftrightarrow x \in B)$$

The Size of a Set – Cardinality

Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and n is the cardinality of S, denoted by $|S|$.

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Examples:

- $A = \{1, 2, 3, ..., 20\}$, where $|A| = 20$
- $B = \{1, 2, 3, ...\}$, which is infinite
- $|\emptyset| = 0$
- $|\{\emptyset\}| = 1$

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If S is a set with $|S| = n$, then $|\mathcal{P}(S)| = 2^n$. Why?

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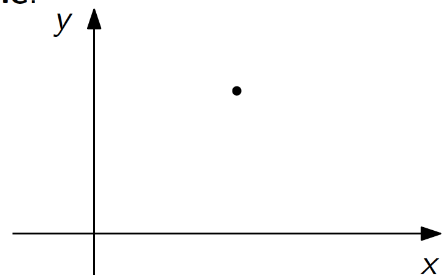
Tuples

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on.

Example:



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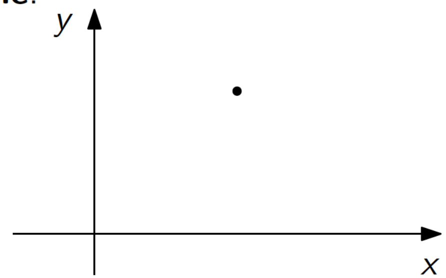

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Two ordered n-tuples are equal if and only if each corresponding pair of their elem-ents is equal. That is, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

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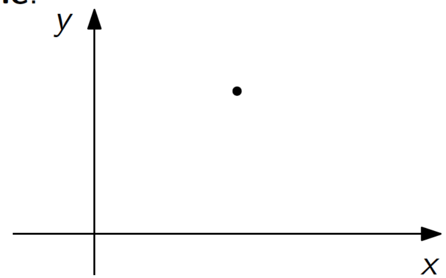

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Ordered 2-tuples are called ordered pairs

Cartesian Product

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$:

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- $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

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The Cartesian product of the sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times ... \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$ where $a_i \in A_i$ for $i = 1, ..., n$:

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Example:

$A = \{0, 1\},\ B = \{1, 2\},\ C = \{0, 1, 2\}$

$A \times B \times C =$

$\{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),$
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A^n denotes $A \times A \times ... \times A$ with n sets:

$$A^n = \{(a_1, a_2, ..., a_n) \mid a_i \in A \text{ for } i = 1, 2, ..., n\}$$

Relation

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Example: What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently,

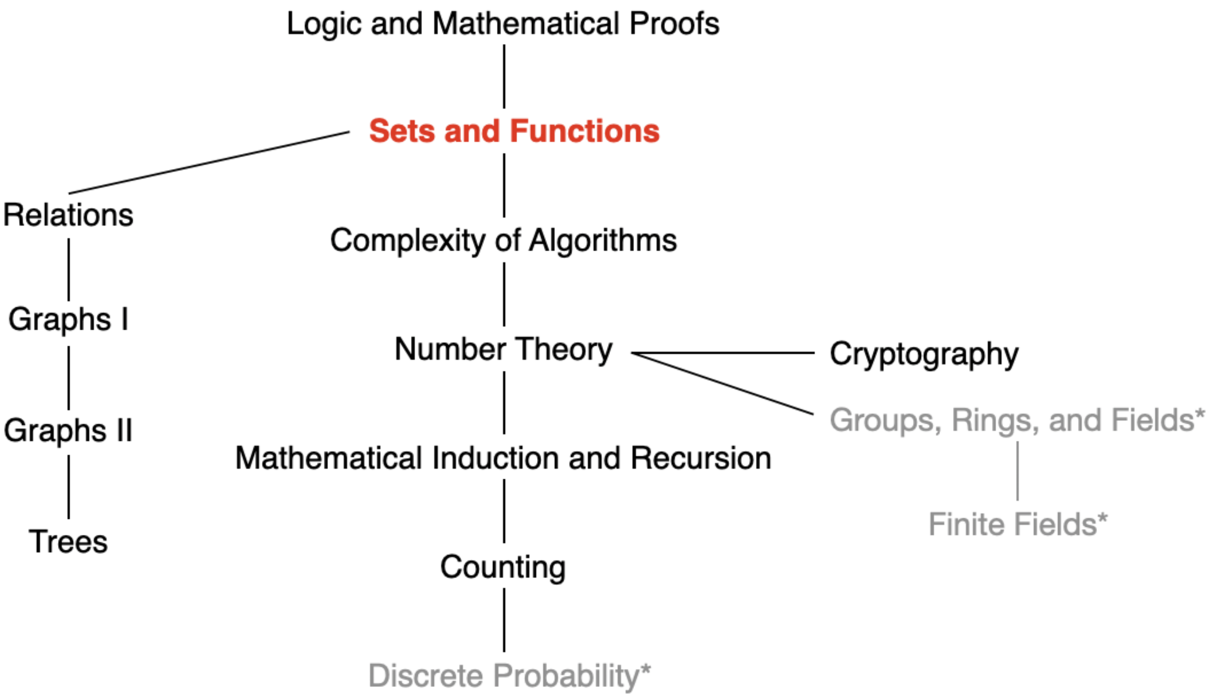
$$R = \{(0,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

.

Summary of Set

- Set: unordered collection of objects
- Subset $A \subseteq B$
- Cardinality: size of set
- Power of set $\mathcal{P}(A)$
- Tuple: (a, b)
- Cartesian Product $A \times B$
- Relation: a subset of $A \times B$

This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Set Operations

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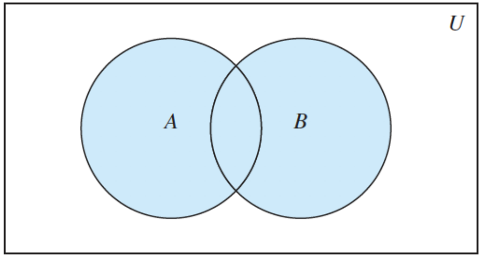

$A \cup B$ is shaded.

Set Operations

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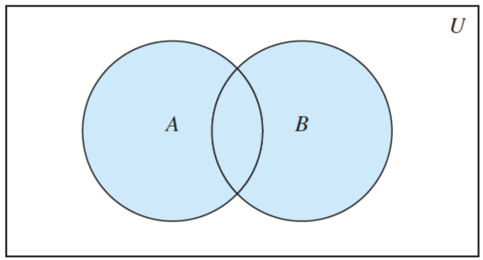
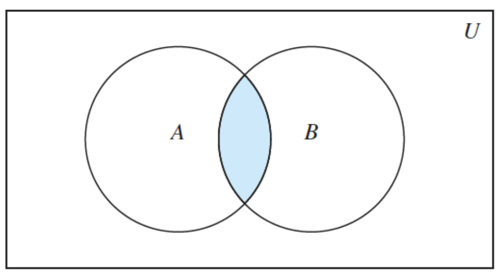


$A \cup B$ is shaded.

Intersection: The intersection of the sets A and B, denoted by $A \cap B$, is the set $\{x \mid x \in A \land x \in B\}$.



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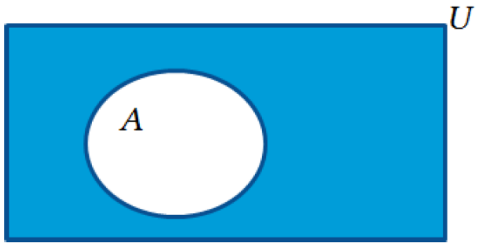
Set Operations

Complement: If A is a set, then the complement of the set A (with respect to U), denoted by $\bar{A}$ is the set $U - A$, $\bar{A} = \{x \in U \mid x \notin A\}$



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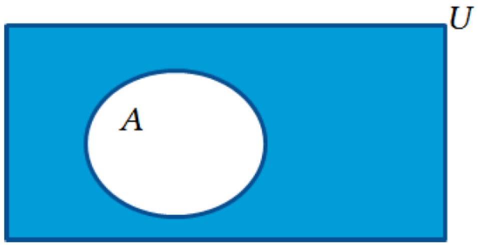
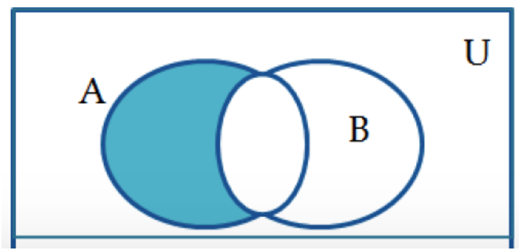


Difference: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.
$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$.



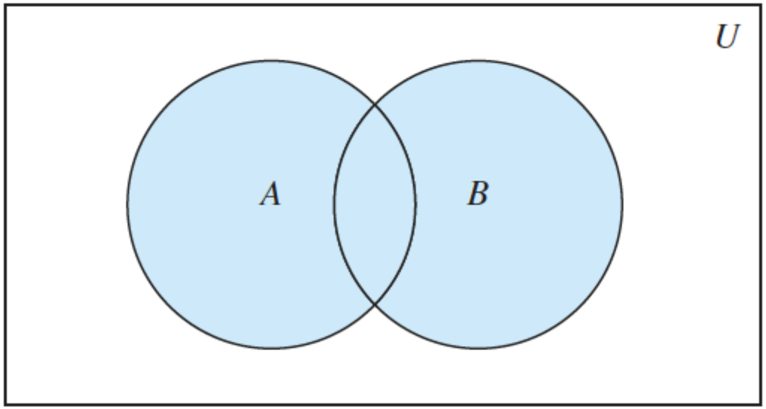
Disjoint Sets

Two sets A and B are called disjoint if their intersection is empty, i.e., $A \cap B = \emptyset$.

Example: $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ are disjoint, because $A \cap B = \emptyset$.

Cardinality of the Union

What is the cardinality of $A \cup B$?



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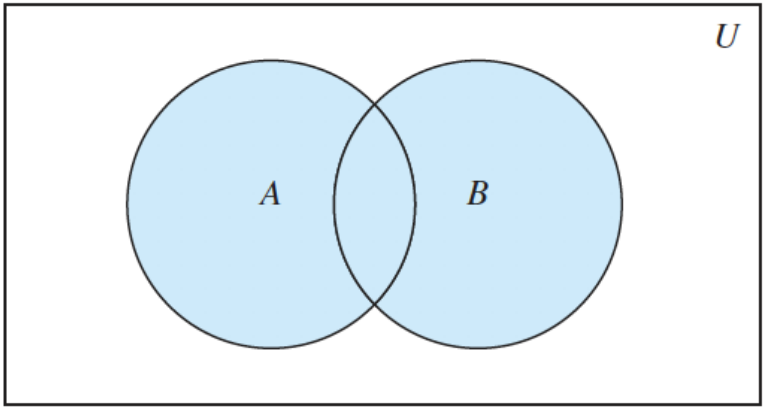

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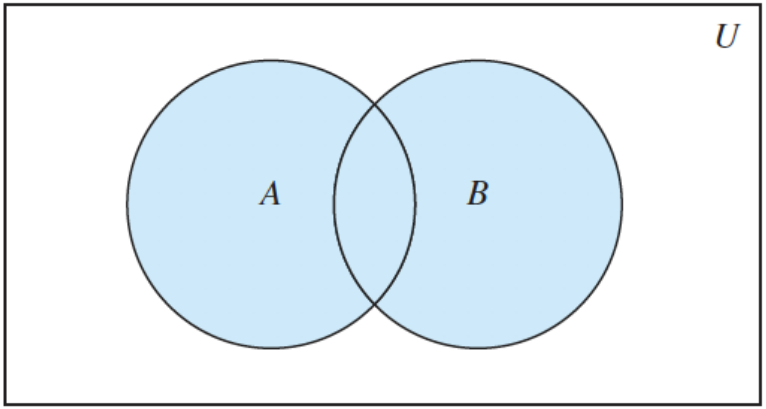

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$$|A \cup B| = |A| + |B| - |A \cap B|$$

The generalization of this result to unions of an arbitrary number of sets is called the principle of inclusion–exclusion.

Exercises

- $U = \{0, 1, 2, \ldots, 10\}, A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$
2. $A \cap B$
3. $\bar{A}$
4. $\bar{B}$
5. $A - B$
6. $B - A$

Set Identities

The properties and laws of sets that help us demonstrate and prove set operations, subsets and equivalence.

- Identity laws
  - $A \cup \emptyset = A$
  - $A \cap U = A$
- Domination laws
  - $A \cup U = U$
  - $A \cap \emptyset = \emptyset$
- Idempotent laws
  - $A \cup A = A$
  - $A \cap A = A$
- Complementation laws
  - $\bar{\bar{A}} = A$

Set Identities

- Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative laws
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's laws
  - $\overline{A \cap B} = \bar{A} \cup \bar{B}$
  - $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Set Identities

- Absorbtion laws
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- Complement laws
  - $A \cup \bar{A} = U$
  - $A \cap \bar{A} = \emptyset$

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof of Set Identities

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Proof 1: Using membership tables. Consider an arbitrary element x: 1, x is in A; 0, x is not in A.

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	1	1

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Proof 1: Using membership tables. Consider an arbitrary element x: 1, x is in A; 0, x is not in A.

Proof 2: by showing that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

- $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$:
  -
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$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Generalized Unions and Intersections

- The *union of a collection of sets* is the set that contains those elements that are members of at least one set in the collection $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$.

- The *intersection of a collection of sets* is the set that contains those elements that are members of all sets in the collection $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$.

Computer Representation of Sets

Question: How to represent sets in a computer?

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- A better solution: assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is in the set.
  - Universal set U is finite and with n elements
  - Represent a subset A of U with n bits, where the i-th bit is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Computer Representation of Sets

Example: $U = \{1, 2, 3, 4, 5\}$
$A = \{2, 5\}$. Thus, A is represented by 01001
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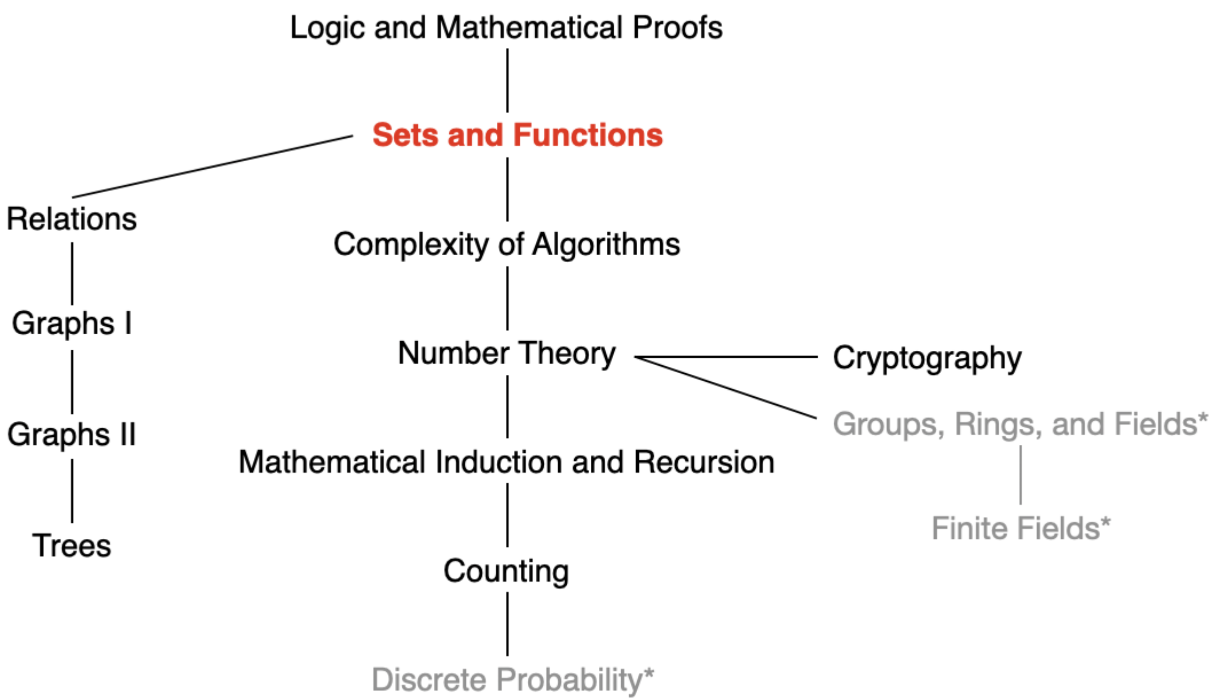
- Union: $A \vee B = 11001$, i.e., $\{1, 2, 5\}$
- Intersection: $A \wedge B = 00001$, i.e., $\{5\}$
- Complement: $\bar{A} = 10110$, i.e., $\{1, 3, 4\}$

Summary of Set Operations

- Union $A \cup B$, cardinality (principle of inclusion-exclusion)
- Intersection $A \cap B$
- Complement $\bar{A}$
- Difference $A - B$
- Disjoint set
- Set identities
- Proof of set identities
  - membership table, subset, set build and logical equivalences
- Computer representations

This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Function

Let A and B be two sets. A function from A to B, denoted by $f : A \to B$, is an assignment of exactly one element of B to each element of A.

- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

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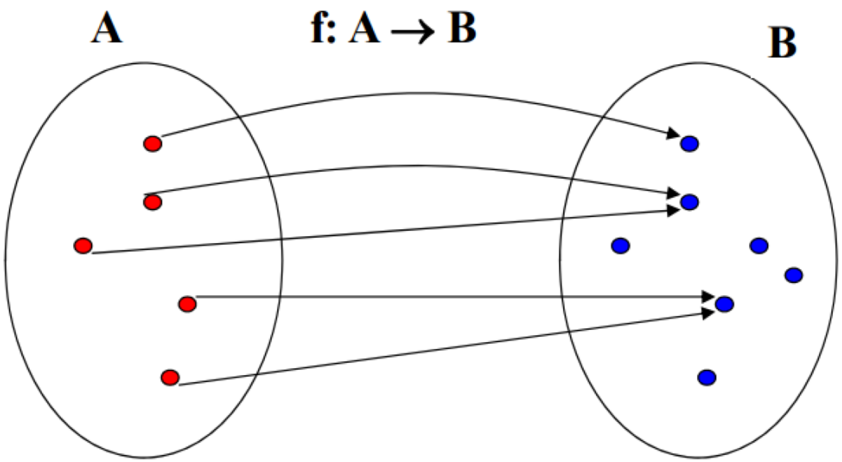

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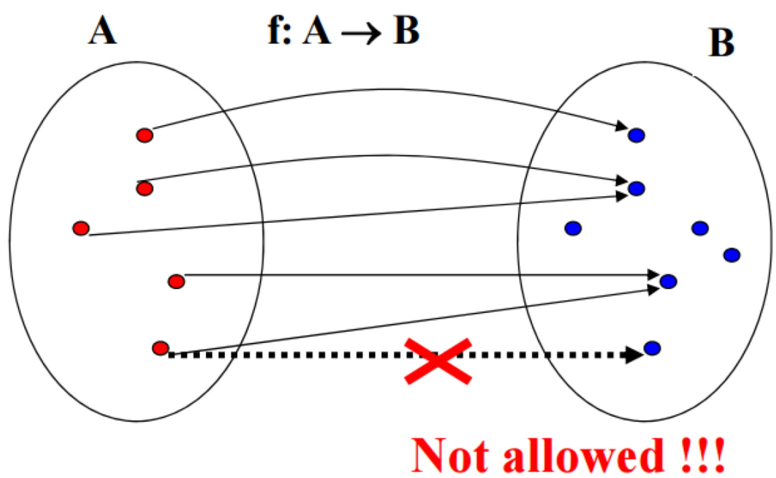
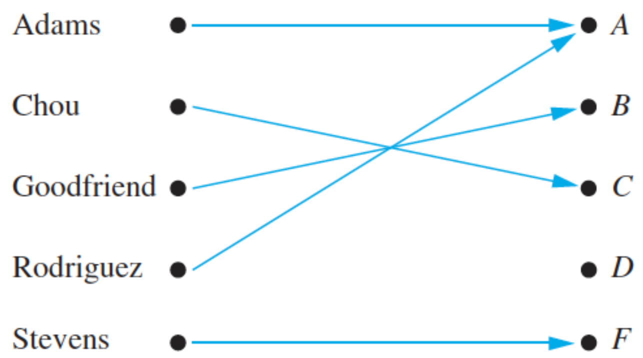


Representing Functions

1 Explicitly state the assignments between elements of the two sets



Note: Admas $\mapsto A$, Chou $\mapsto C$, ...

2 By a formula

3 By a relation from A to B

Representing Functions

1. Explicitly state the assignments between elements of the two sets
2. By a formula: $f(x) = x + 1$
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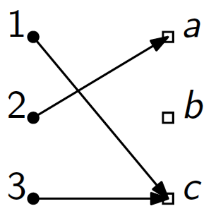
$A = \{1, 2, 3\}$, $B = \{a, b, c\}$



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Example:

$A = \{1, 2, 3\}$, $B = \{a, b, c\}$

– c is the image of 1
– 2 is a preimage of a
– the domain of f is $\{1, 2, 3\}$
– the codomain of f is $\{a, b, c\}$
– the range of f is $\{a, c\}$

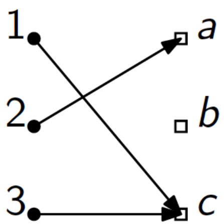

Image of a Subset

For a function $f : A \to B$ and $S \subseteq A$, the image of S is a subset of B that consists of the images of the elements of S, denoted by $f(S)$, where
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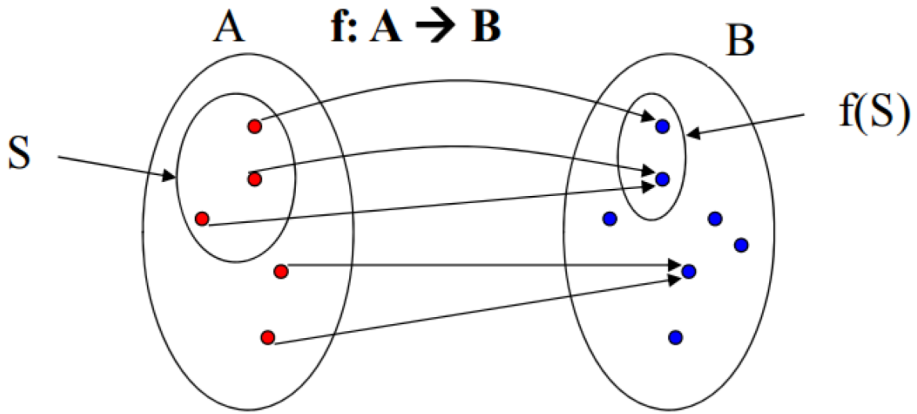

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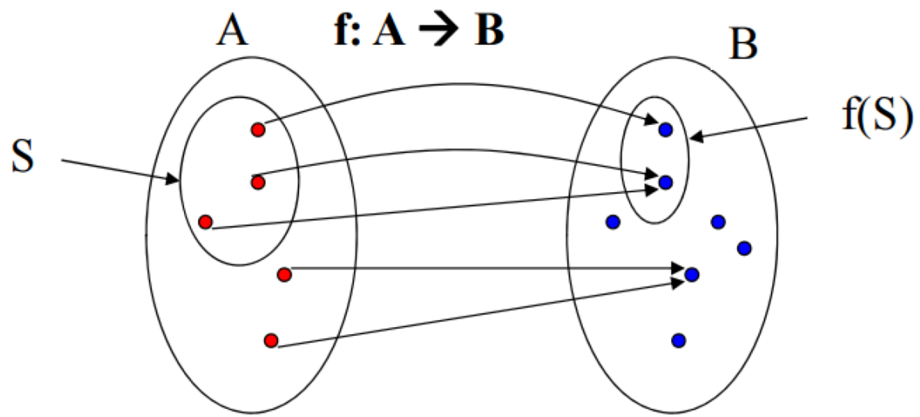

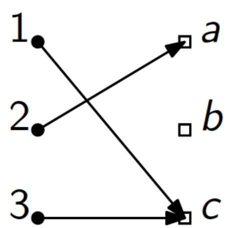

Let $S = \{1, 3\}$, what is $f(S)$?

One-to-One and Onto Functions

- **One-to-one function**
  - never assign the same value to two different domain elements.
- **Onto function**
  - every member of the codomain is the image of some element of the domain.
- **One-to-one correspondence**
  - One-to-one and onto

One-to-One (Injective) Function

A function f is called one-to-one or injective if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. Also called an injunction.

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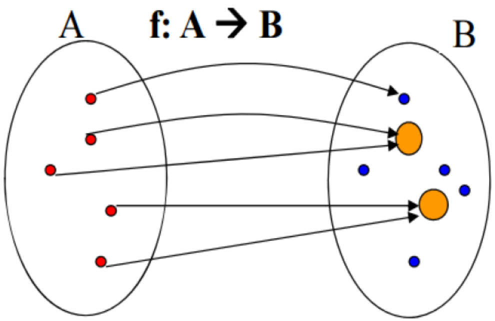

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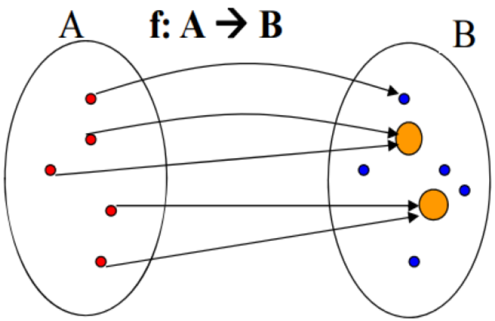

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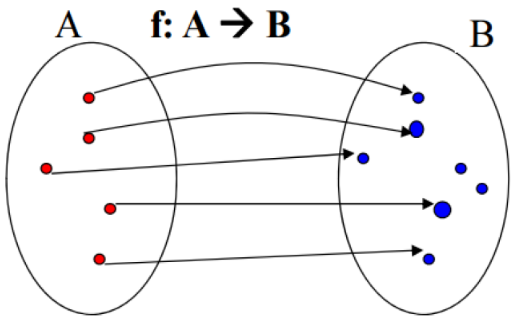

Injective function

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Example 1:

Whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one?

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Whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one? Yes.

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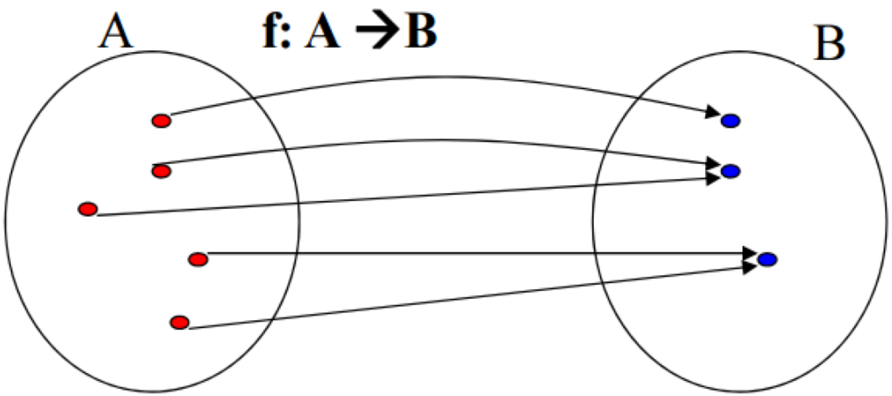

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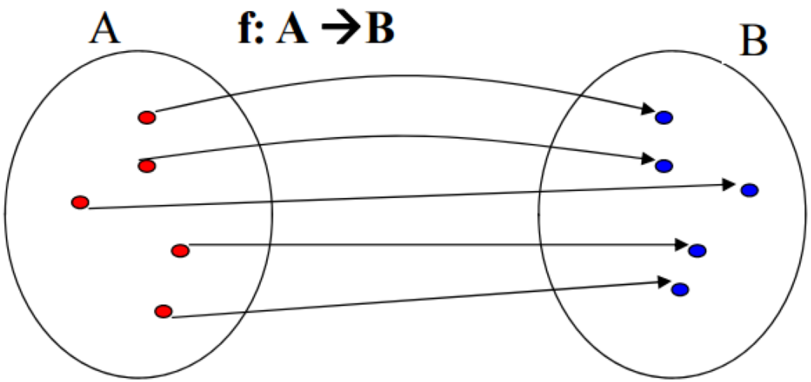

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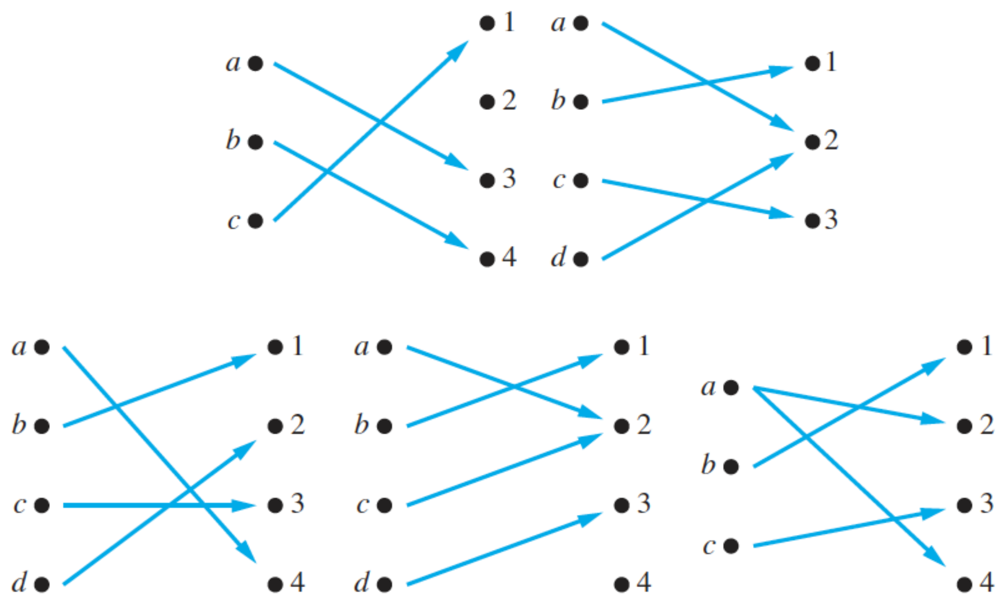

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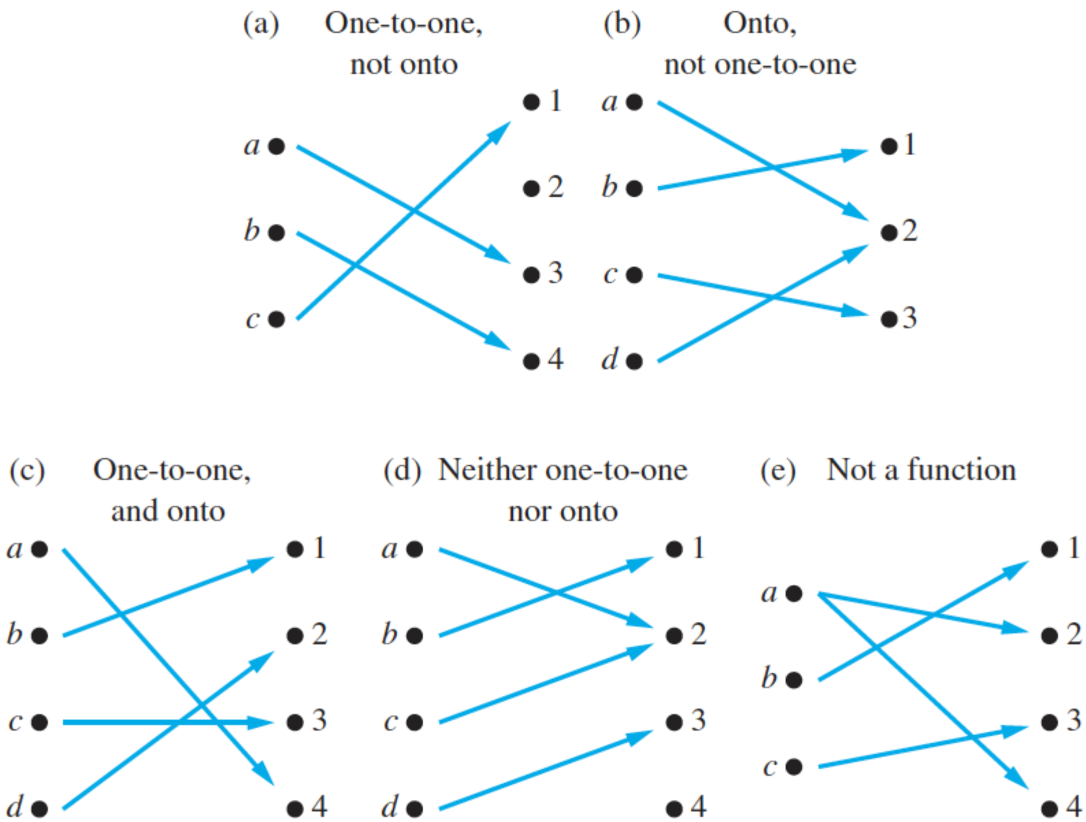

Proof for One-to-One and Onto

Suppose that $f : A \to B$.

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
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- Bijective (one-to-one correspondence): injective and surjective

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- If f is onto, then f is one-to-one (contradiction): Suppose that f is onto. Suppose that f is not one-to-one. Thus, $f(x_i) = f(x_j)$ for some $i \neq j$. Then, $|\{f(x_1), ..., f(x_n)\}| \leq n - 1$. Note that $|f(A)| = |B| = n$, which leads to a contradiction.

One-to-One and Onto

Consider an infinite set A and a function from A to A. Consider the statement "For any arbitrary $f : A \to A$, f is one-to-one if and only if f is onto". Is this statement true?

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Proof (Counterexample): Consider the following $f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = 2x$. f is one-to-one but not onto:

- $f(1) = 2$
- $f(2) = 4$
- $f(3) = 6$
- ...

We can prove that 3 has no preimage.

Two Functions on Real Numbers

Let f_1 and f_2 be functions from A to $\boldsymbol{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\boldsymbol{R}$ defined for all $x \in A$ by

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Example:

$f_1 = x - 1$ and $f_2 = x^3 + 1$

Then

$$(f_1 + f_2)(x) = x^3 + x$$
$$(f_1 f_2)(x) = x^4 - x^3 + x - 1$$

Inverse Functions

Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

The inverse function of f is denoted by f^{-1}.
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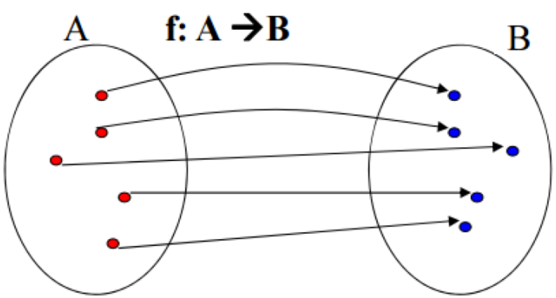

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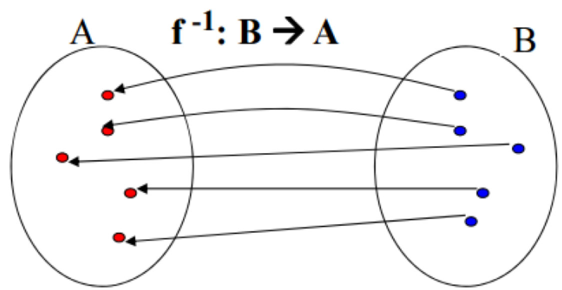

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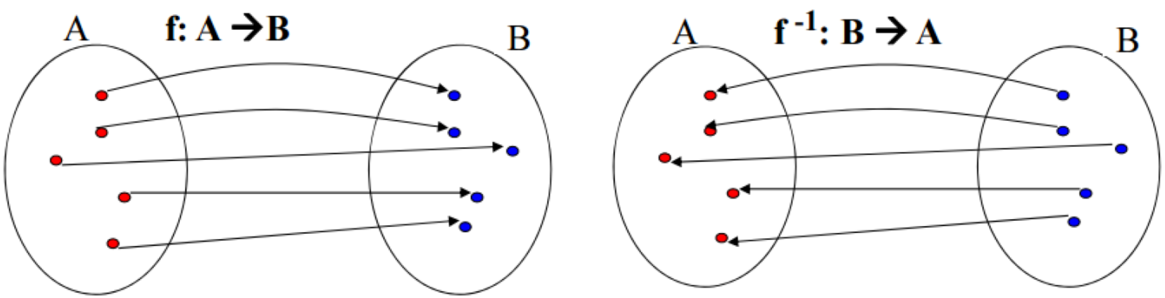

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A bijection is called invertible.

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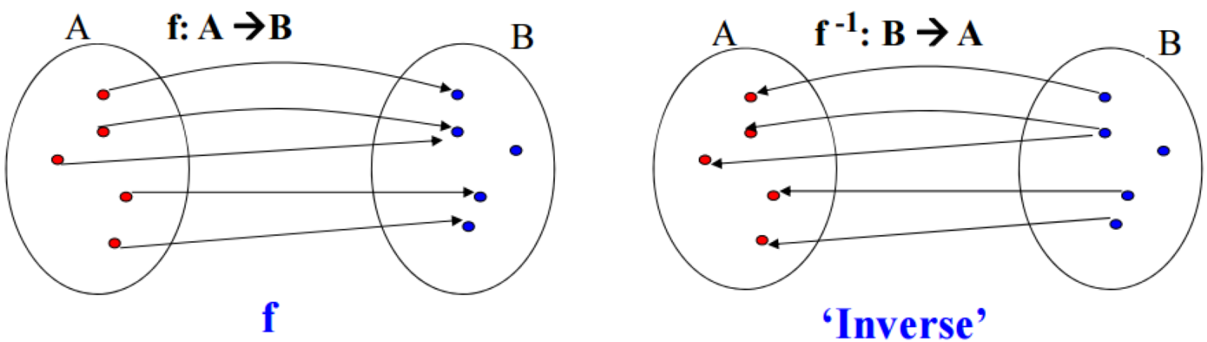

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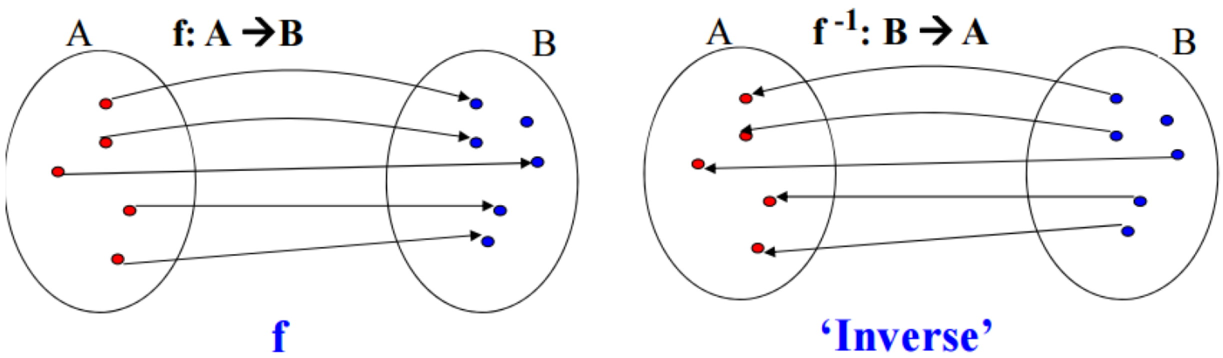

The inverse is not a function: one element of B is not assigned an element of A.

Proof for Inverse Function

1 Prove function f is a bijection: injective, surjective

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2 If f is a bijection, then it is invertible

3 Determine the inverse function

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To reverse the function, suppose that y is the image of x, so that $y = x + 1$. Then, $x = y - 1$. This means that $y - 1$ is the unique element of $\boldsymbol{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

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Proof: No, f is not invertible. This is because f is not injective, as $f(-2) = f(2)$.

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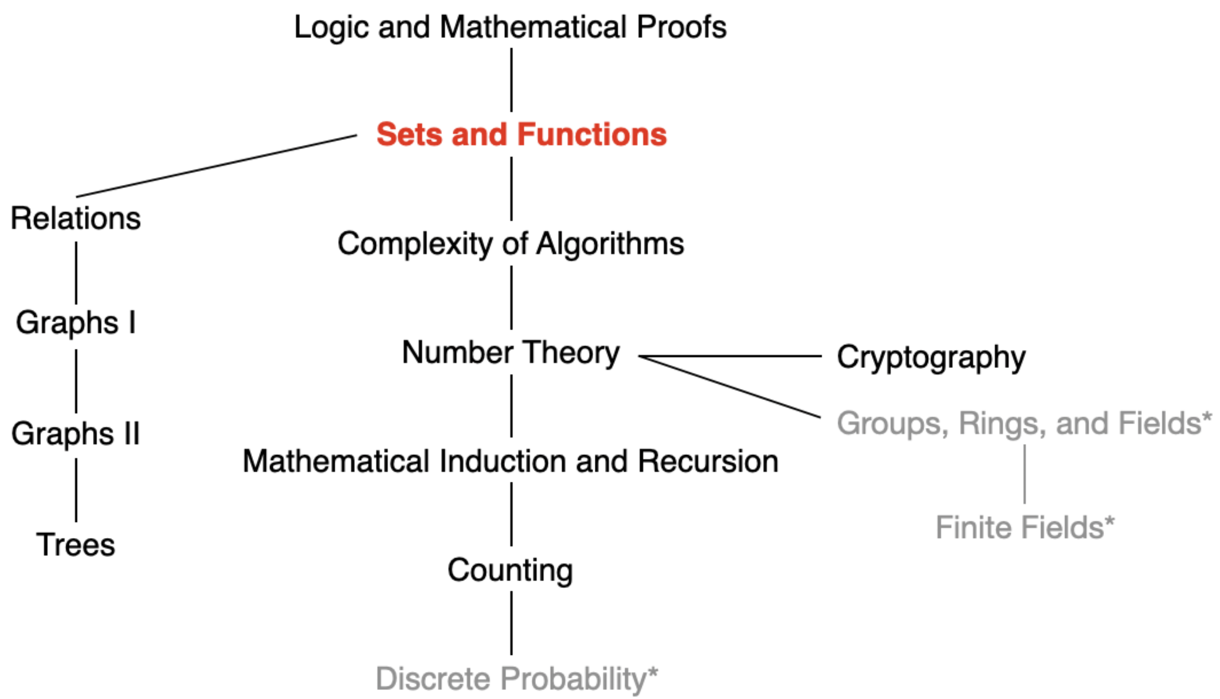
Summary of Function

- Function $f : A \to B$: an assignment of exactly one element of B to each element of A
- Domain, codedomain, image, preimage, range
- One-to-one function
  - also called an injunction or injective function
- Onto function
  - also called a surjection or surjective function
- One-to-one correspondence
  - one-to-one and onto
  - also called a bijection or bijective function
- Inverse function
  - One-to-one correspondence

Next Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets