Discrete Mathematics for Computer Science

Lecture 18: Graph

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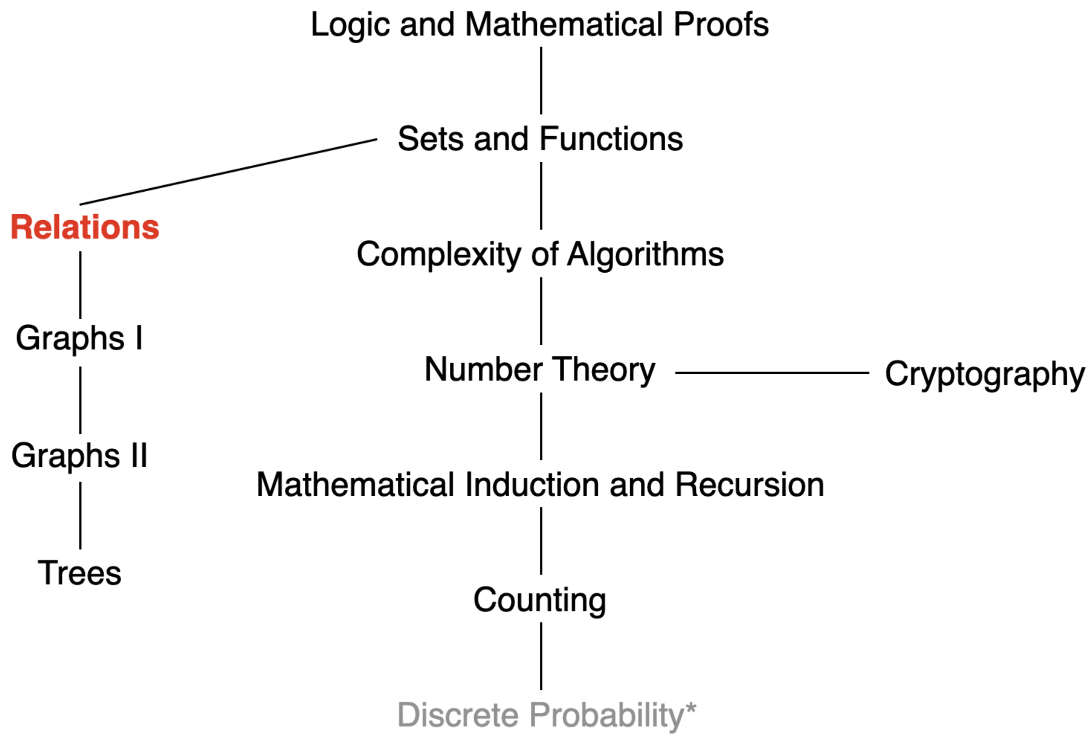
This Lecture



Relation, *n*-ary Relations, Representing Relations, Closures of Relations, Relation Equivalence, Partial Ordering, ...

Partial Ordering

- Partial Ordering
  - reflexive, antisymmetric, and transitive.
  - e.g., $\leq$, $|$
  - $a \preccurlyeq b$ denotes $(a, b) \in R$ in a poset (S, R); $(S, \preccurlyeq)$
- Comparable
  - if either $a \preccurlyeq b$ or $b \preccurlyeq a$
  - e.g., $S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$|$" relation: $2, 4$ are comparable, $3, 5$ are incomparable.
- Total Ordering
  - $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable
  - "$\leq$" is a total order, "$|$" is not a totally order
- Well-ordered set
  - total ordering; every nonempty subset of S has a least element
  - e.g., $a \preccurlyeq b \preccurlyeq c \ldots$; exists an a such that $(a, b) \in R$ for all $b \in S$

Well-Ordered Set

$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Example: The set of ordered pairs of positive integers, $\mathrm{Z}^+ \times \mathrm{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set.

The set Z, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of Z, has no least element.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Note: We do not need a basis step in a proof. Let x_0 is the least element of a well ordered set. This is because we need to prove

- if $P(x)$ is true for all $x \in S$ with $x \prec x_0$, then $P(x_0)$ is true.
- there are no elements $x \in S$ with $x \prec x_0$, so we know (using a vacuous proof) that $P(x)$ is true for all $x \in S$ with $x \prec x_0$.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false.

Consequently, the set $A = \{x \in S | P(x) \text{is false}\}$ is nonempty. Because S is well ordered, A has a least element a.

By the choice of a as a least element of A, we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. By the inductive step, $P(a)$ is true.

This contradiction shows that $P(x)$ must be true for all $x \in S$.

Questions from Section 5 (Induction)

The Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.

The principle of mathematical induction follows from the well-ordering property.

Question from students: Consider the set of negative integers. Although it does not has a least element, it has a greatest element. Can we solve it using mathematical induction?

Yes. We can solve it using the principle of well-ordered induction if we can find a relation $\preccurlyeq$ such that $(S, \preccurlyeq)$ is a well-ordered set.

Questions from Section 5 (Induction)

(i) The principle of mathematical induction, (ii) strong induction, and (iii) well-ordering property are all equivalent principles.

That is, the validity of each can be proved from either of the other two. (See Section 5.2 Exercise 41, 42, 43)

- (i) $\rightarrow$ (ii): The inductive hypothesis of a proof by mathematical induction is part of the inductive hypothesis in a proof by strong induction.
- (ii) $\rightarrow$ (iii) Use strong induction to show that the set of nonnegative integers has a least element.
- (iii) $\rightarrow$ (i) The principle of mathematical induction follows from the well-ordering property.

Questions from Section 5 (Induction)

(ii) $\to$ (iii): Use strong induction to show that the set of nonnegative integers has a least element.

- Suppose that the well-ordering property were false. Let S be a nonempty set of nonnegative integers that has no least element.
- $P(n)$: $i \notin S$ for $i = 0, 1, ..., n$.
- Strong induction:
  - $P(0)$ is true because if $0 \in S$ then S has a least element, namely, 0.
  - Suppose that $P(0)$, ... $P(n)$ is true, i.e., Thus, $0 \notin S$, $1 \notin S$, . . . , $n \notin S$. Clearly, $n + 1$ cannot be in S, for if it were, it would be its least element. Thus $P(n + 1)$ is true.
- As a result, $n \notin S$ for all nonnegative integers n. Thus, $S = \emptyset$, a contradiction.

Lexicographic Ordering

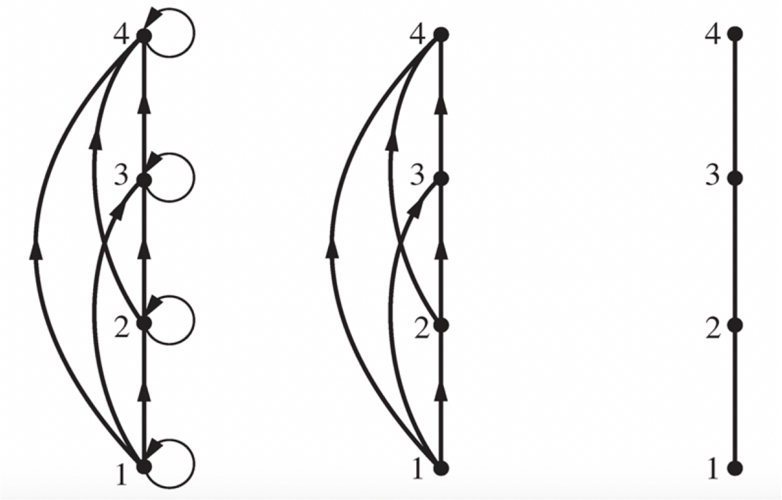
Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the lexicographic ordering on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- discreet $\prec$ discrete
- discreet $\prec$ discreetness

Hasse Diagram

A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

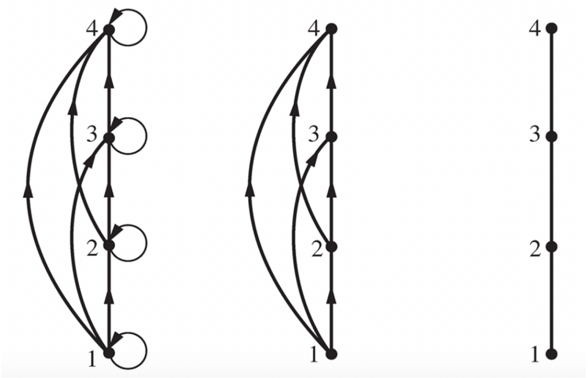


- A partial ordering. The loops are due to the reflexive property.
- The edges that must be present due to the transitive property are deleted.
- The Hasse diagram for the partial ordering (a).

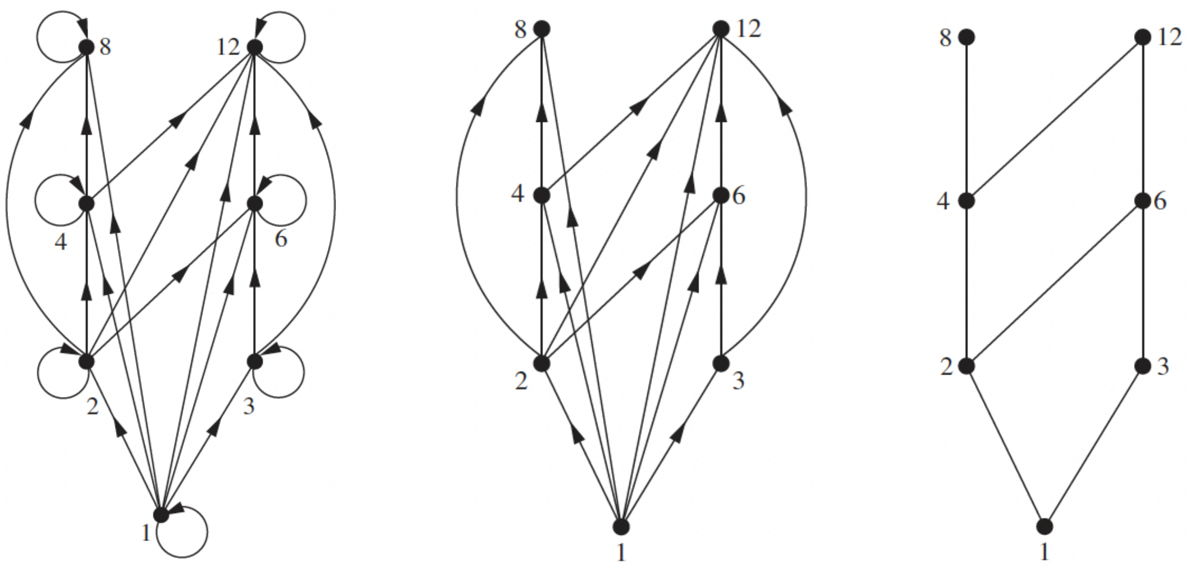
Procedure for Constructing Hasse Diagram

Start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ s.t. $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.



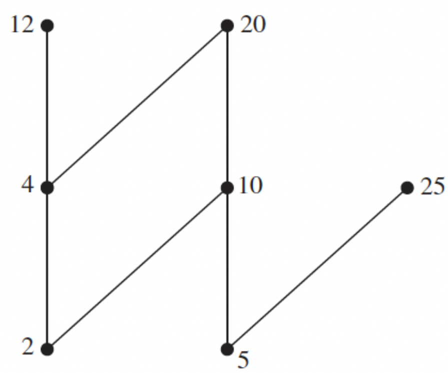
Hasse Diagram Example



Maximal and Minimal Elements

Definition: a is a maximal (resp. minimal) element in poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp. $b \prec a$).

Example: Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?



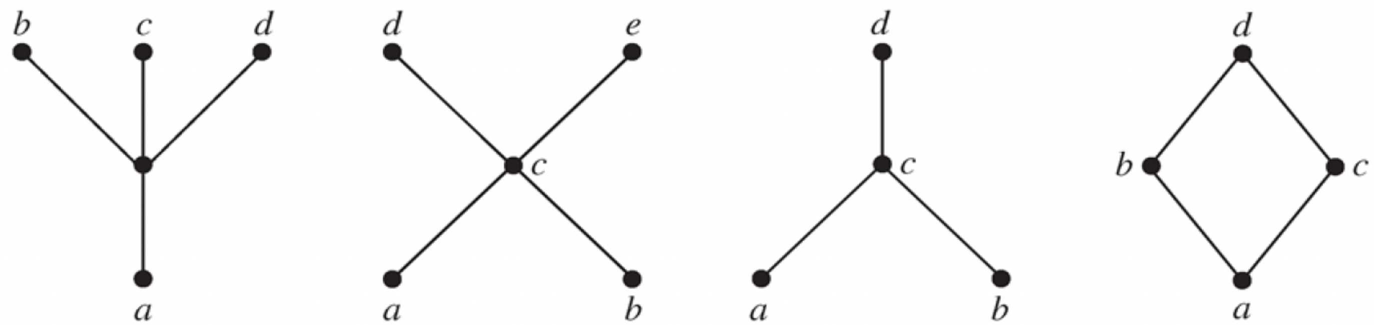
The maximal elements are 12, 20, and 25.

The minimal elements are 2 and 5.

A poset can have more than one maximal element and more than one minimal element.

Greatest and Least Elements

Definition: a is the greatest (resp. least) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp. $a \preccurlyeq b$) for all $b \in S$.



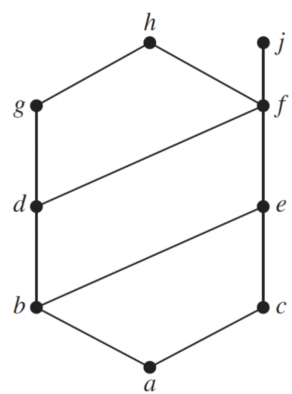
- (a): a least element a, no greatest element
- (b): neither a least nor a greatest element
- (c): no least element., a greatest element d
- (d): a least element a, a greatest element d

Upper and Lower Bound

Definition: Let A be a subset of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an upper bound (resp. lower bound) of A if $a \preccurlyeq u$ (resp. $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the least upper bound (resp. greatest lower bound) of A if x is an upper bound (resp. lower bound) that is less than any other upper bounds (resp. lower bounds) of A.

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist.



g is the least upper bound, b is the greatest lower bound.

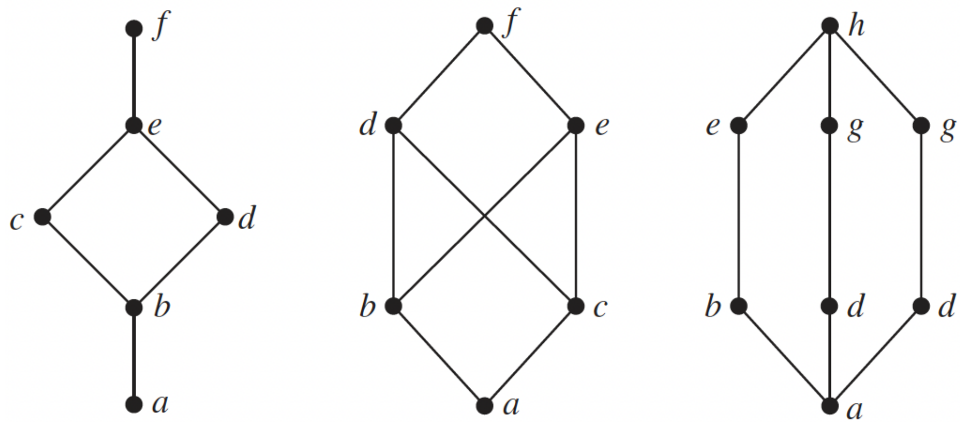
Upper and Lower Bound

Example: Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(Z^+, |)$.

- Lower bound of $\{3, 9, 12\}$: 1 and 3; the greatest lower bound: 3.
- Lower bound of $\{1, 2, 4, 5, 10\}$: 1; the greatest lower bound: 1.
- Upper bound of $\{3, 9, 12\}$: multiple of 36; the least upper bound: 36.
- Upper bound of $\{1, 2, 4, 5, 10\}$: multiple of 20; the least upper bound: 20.

Lattices

Definition: A partial ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.



- (a) and (c): lattices
- (b): not a lattice, because the elements b and c have no least upper bound.

Lattices: Example

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound.

- The least upper bound of two elements in this poset is the larger of the elements
- The greatest lower bound of two elements is the smaller of the elements

Hence, this second poset is a lattice.

Topological Sorting

Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever aRb. $\preccurlyeq$ is said compatible with R.

Topological Sorting for Finite Posets

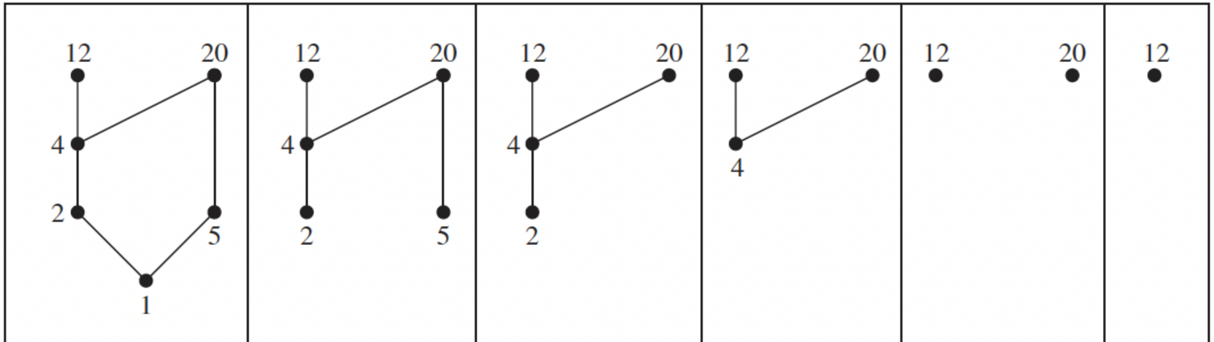
Lemma: Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

ALGORITHM 1 Topological Sorting.

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
  $S := S - \{a_k\}$
  $k := k + 1$
return $a_1, a_2, \ldots, a_n$ {$a_1, a_2, \ldots, a_n$ is a compatible total ordering of S}

Topological Sorting for Finite Posets

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.



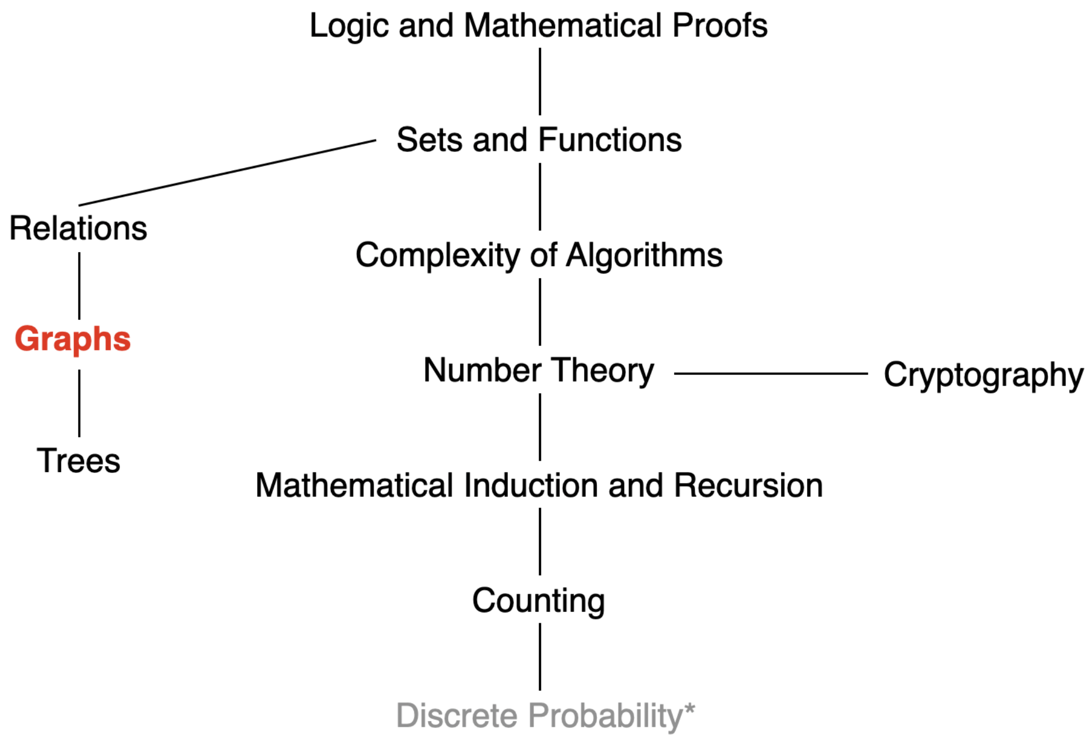
Minimal element chosen	1	5	2	4	20	12

This produces the total ordering

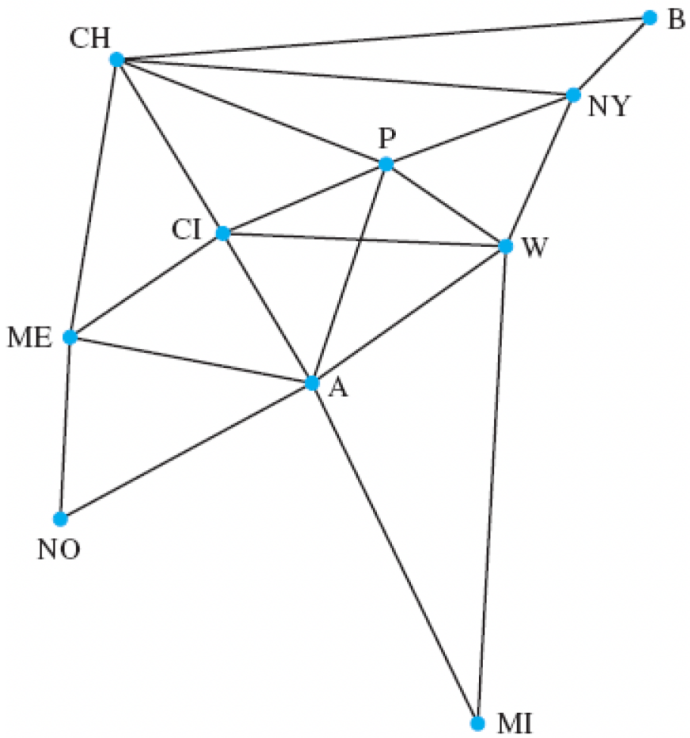
$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$$

Recall the Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

This Lecture



Example



- What is the minimum number of links to send a message from *B* to *NO*?

  3: B - CH - ME - NO

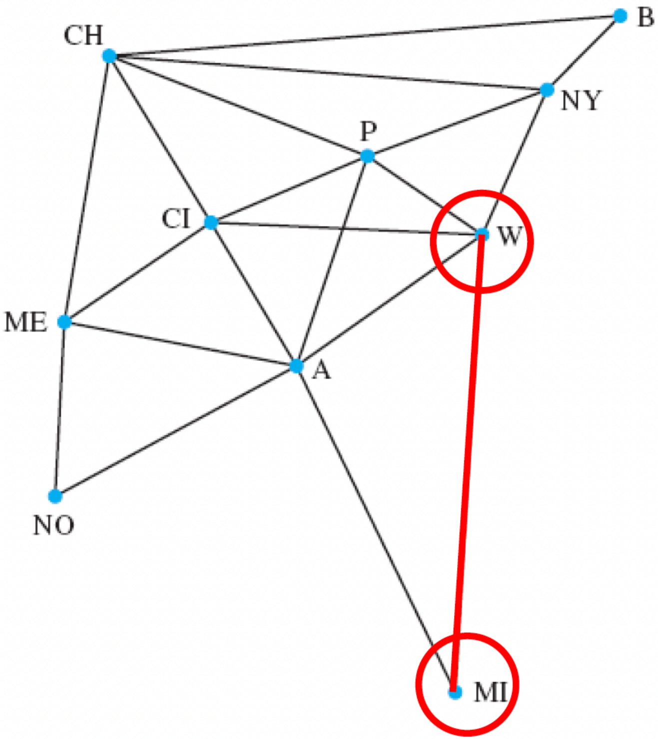
- Which city/cities has/have the most communication links emanating from it/them?

  A: 6 links

- What is the total number of communication links?

  20 links

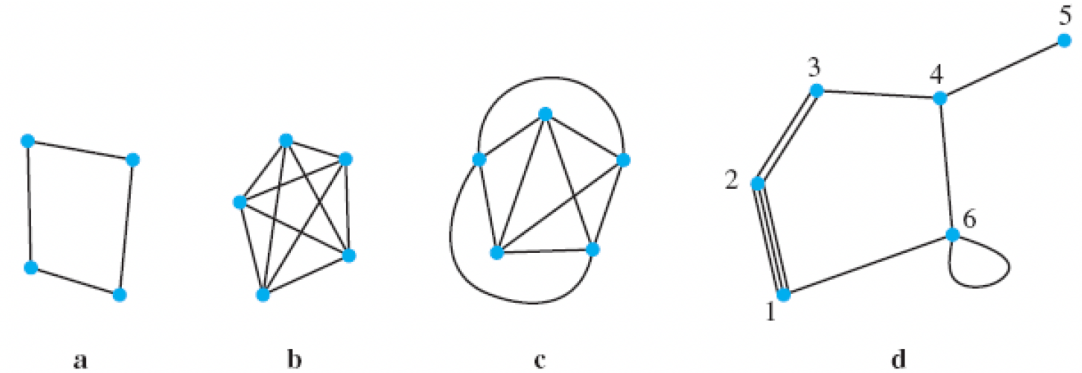
Graph G



- Consists of a set of vertices V, $|V| = n$
- and a set of edges E, $|E| = m$
- Each edge has two endpoints
- An edge joins its endpoints, two endpoints are adjacent if they are joined by an edge
- When a vertex is an endpoint of an edge, we say that the edge and the vertex are incident to each other.

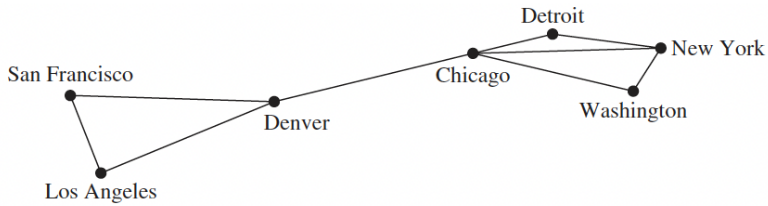
Definition of a Graph

Definition: A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to be incident to (or connect) its endpoints.

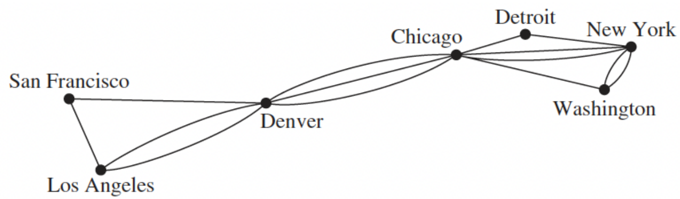


Simple Graph, Multigraph, Pseudograph

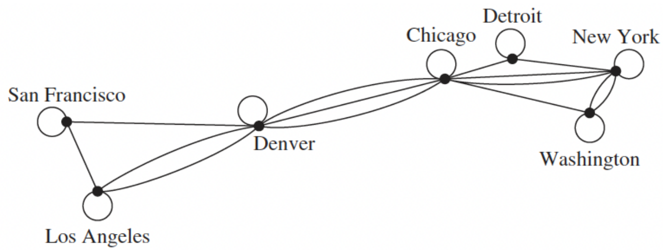
- simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.



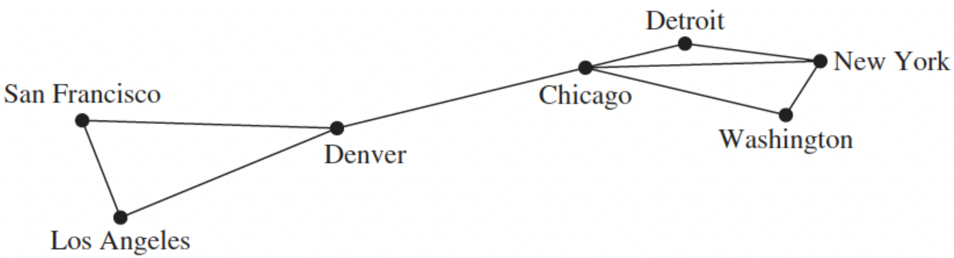
- Multigraph: Graphs that may have multiple edges connecting the same vertices.



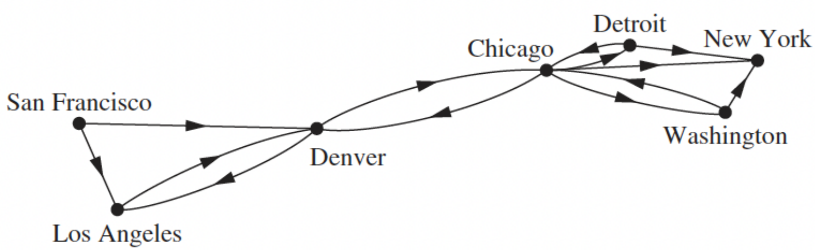
- Pseudograph: Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself.



Directed and Undirected Graph



A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

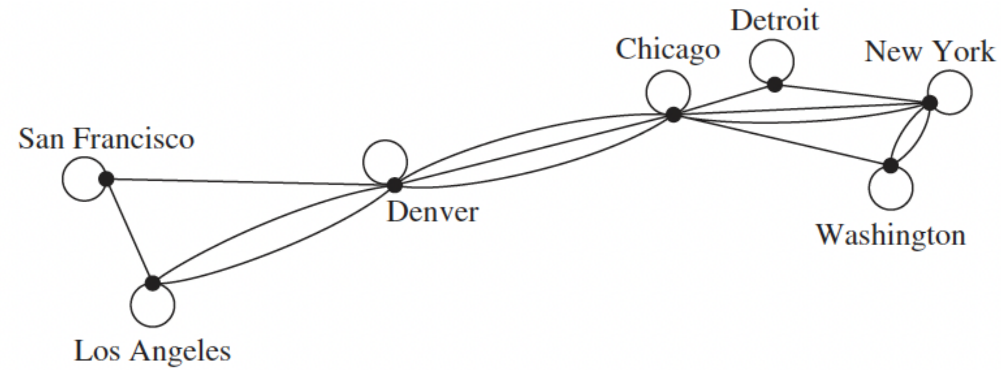


Graph: Example

- Computer networks
- Social networks
- Communication networks
- Information networks
- Software design
- Transportation networks
- Biological networks

Computer Networks

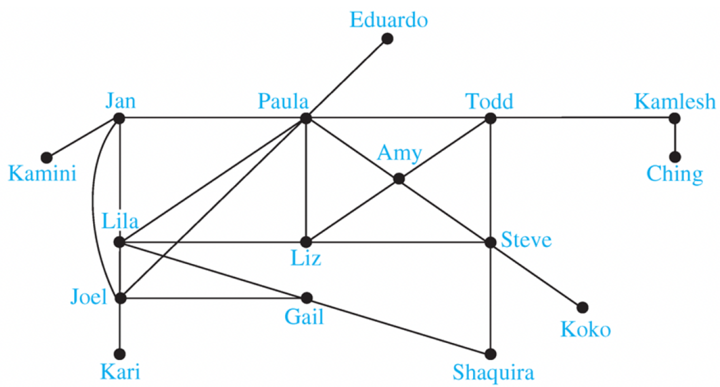
- Vertices: computers
- Edges: connections



Social Networks

- Vertices: individuals
- Edges: relationships

Friendship graphs: undirected graphs where two people are connected if they are friends (in the real world, wechat, or Facebook, etc.)



Social Networks

Influence graphs: directed graphs where there is an edge from one person to another if the first person can influence the second one.

Collaboration graphs: undirected graphs where two people are connected if they collaborate in some way.

- Hollywood graph
- Academic collaboration graph

Undirected Graphs

Definition: Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

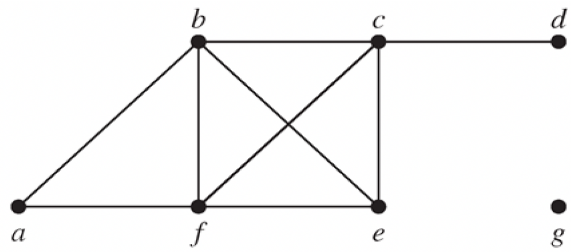
Definition: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

Definition: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Undirected Graphs: Example

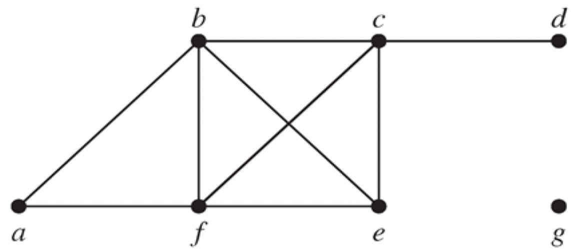
What are the degrees and neighborhoods of the vertices in the graph G?

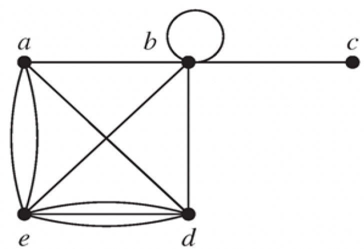


$deg(a) = 2$, $deg(b) = deg(c) = deg(f) = 4$, $deg(d) = 1$, $deg(e) = 3$, and $deg(g) = 0$.
$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$.

Undirected Graphs: Example

What are the degrees and neighborhoods of the vertices in the graph G?





$deg(a) = 4$, $deg(b) = deg(e) = 6$, $deg(c) = 1$, and $deg(d) = 5$.
$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

Undirected Graphs

Theorem (Handshaking Theorem): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Undirected Graphs

Theorem: An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 be the vertices of even degrees and V_2 be the vertices of odd degree.

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

Directed Graphs

Definition: An directed graph $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of directed edges.

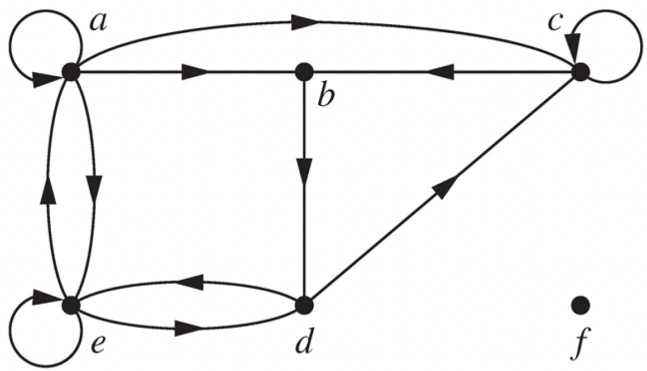
Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

Definition: Let (u, v) be an edge in G. Then u is the initial vertex of the edge and is adjacent to v and v is the terminal vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are the same.

Directed Graphs

Definition: The in-degree of a vertex v, denoted by $deg^-(v)$, is the number of edges which terminate at v. The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.



The in-degrees are $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, and $deg^-(f) = 0$.
The out-degrees are $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, and $deg^+(f) = 0$.

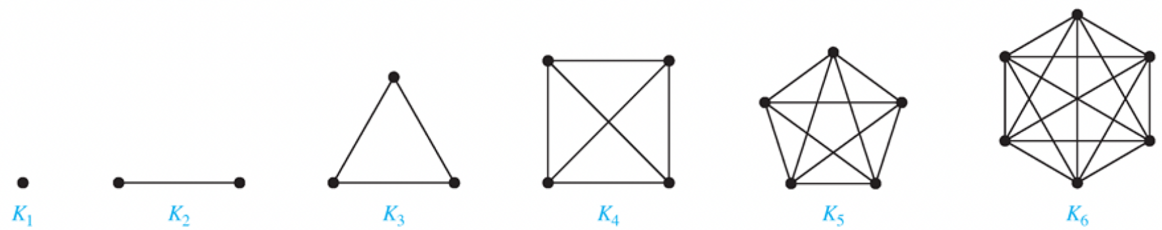
Directed Graphs

Theorem: Let $G = (V, E)$ be a graph with directed edges. Then,

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$$

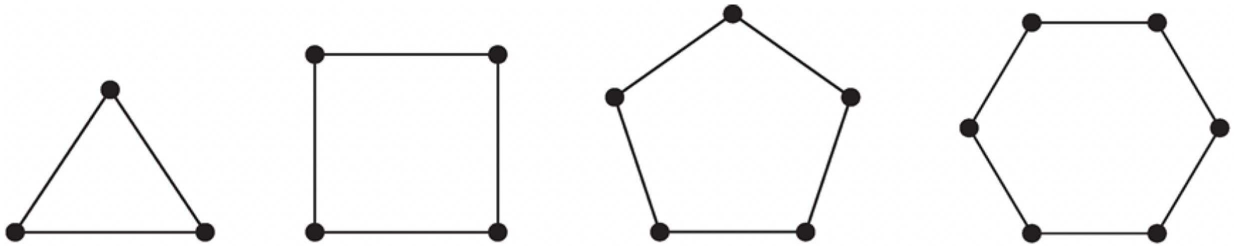
Complete Graphs

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Cycles

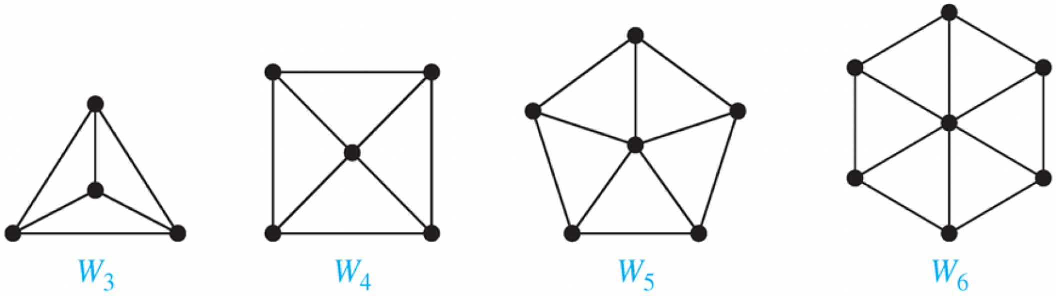
A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



C_3 C_4 C_5 C_6

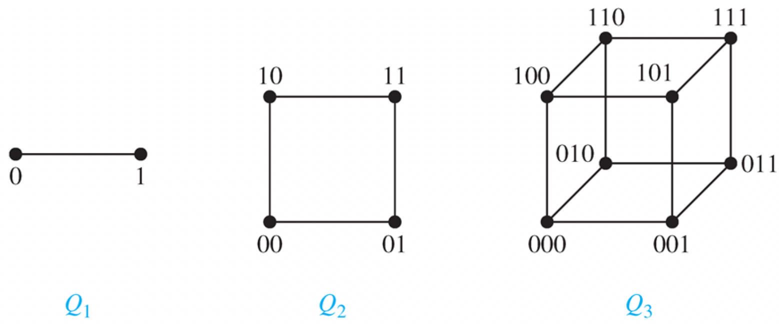
Wheels

A wheel W_n is obtained by adding an additional vertex to a cycle C_n.



N-dimensional Hypercube

An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



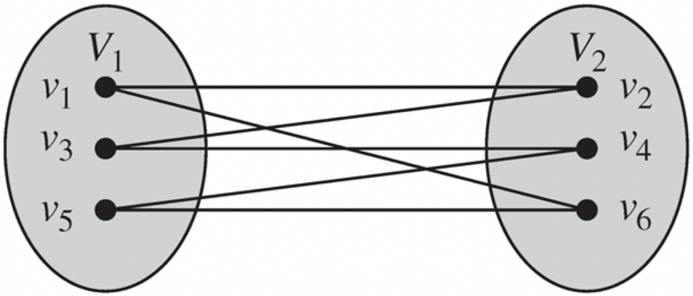
How many edges?

Construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit.

Bipartite Graphs

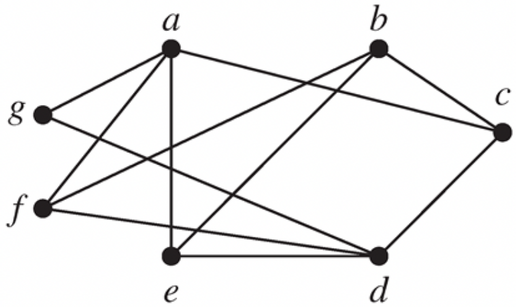
Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

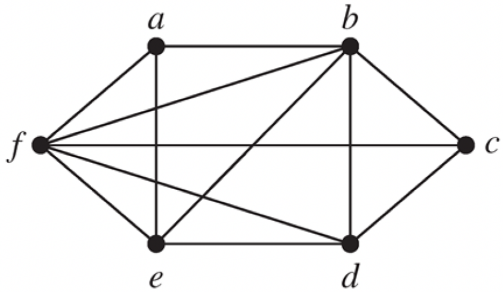
An equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are of the same color.



Bipartite Graphs

Are these graphs bipartite?





(a) Bipartite: Its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

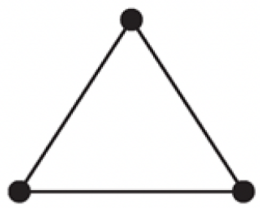
(b) Not bipartite: Its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset.

Bipartite Graphs: Examples

Show that C_6 is bipartite.

C_6

Show that C_3 is not bipartite.

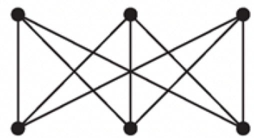


C_3

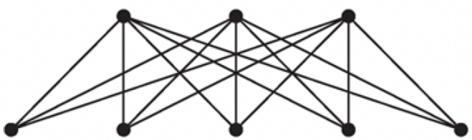
Complete Bipartite Graphs

Definition: A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

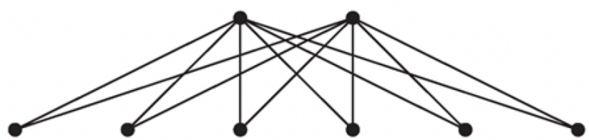
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



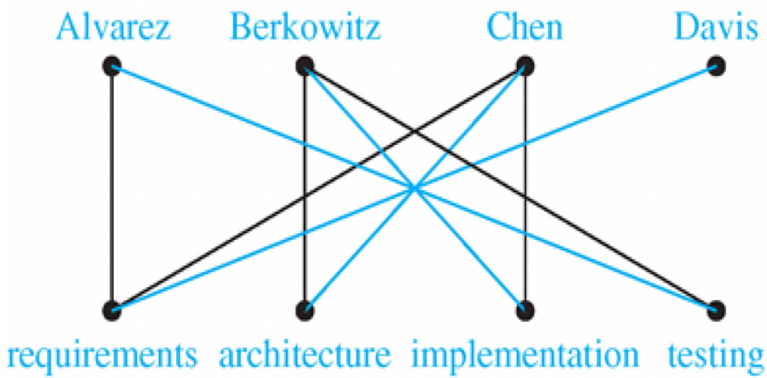
$K_{2,6}$

Bipartite Graphs and Matchings

Matching the elements of one set to elements in another. A matching is a subset of E s.t. no two edges are incident with the same vertex.

In other words, a matching is a subset of edges such that if $\{s, t\}$ and $\{u, v\}$ are distinct edges of the matching, then s, t, u, and v are distinct.

Job assignments: vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done.



Bipartite Graphs and Matchings

A maximum matching is a matching with the largest number of edges.

A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a complete matching from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Proof of Hall's Theorem

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Proof: "only if"

Suppose that there is a complete matching M from V_1 to V_2. Consider an arbitrary subset $A \subseteq V_1$.

Then, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2.

Thus, there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1.

Hence, $|N(A)| \geq |A|$.

Proof of Hall's Theorem

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Proof: "if", use strong induction to prove it.

Basic Step: $|V_1| = 1$

Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), and $|V_1| = j \leq k$, then there is a complete mathching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

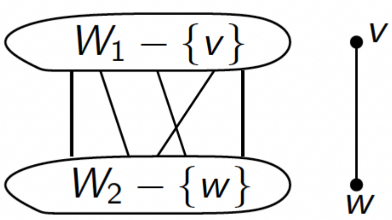
Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

Proof of Hall's Theorem

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

(i) For all integers j with $1 \le j \le k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2.

We select a vertex $v \in W_1$ and an element $w \in N(\{v\})$. The inductive hypothesis tells us there is a complete matching from $W_1 - \{v\}$ to $W_2 - \{w\}$.



(ii) For some integer j with $1 \le j \le k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2.

Proof of Hall's Theorem

Inductive step: Suppose that $H = (W, F)$ is a bipartite graph with bipartition (W_1, W_2) and $|W_1| = k + 1$.

(i) For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2.

(ii) For some integer j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2.

Let W_2' be the set of these neighbors. Then by i.h., there is a complete matching from W_1' to W_2'. Now consider the graph $K = (W_1 - W_1', W_2 - W_2')$.

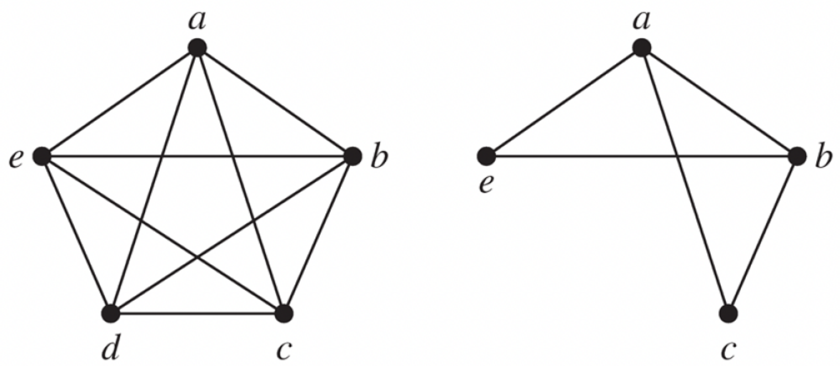
We will show that the condition $|N(A)| \geq |A|$ is met for all subsets A of $W_1 - W_1'$.

If not, there is a subset B of t vertices with $1 \leq t \leq k + 1 - j$ s.t. $|N(B)| < t$, contradicting the hypothesis that $|N(A)| \geq |A|$ for all $A \subseteq W_1$.

Subgraphs

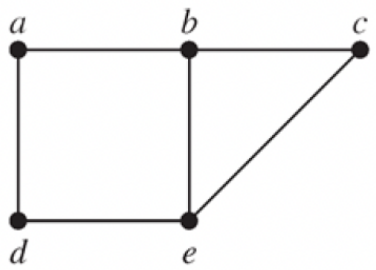
Definition: A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$.

A subgraph H of G is a proper subgraph of G if $H \neq G$.

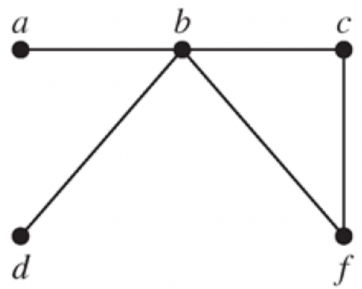


Union of Graphs

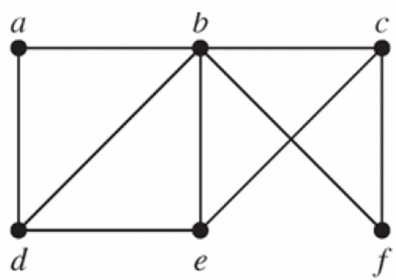
Definition: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, denoted by $G_1 \cup G_2$.



G_1



G_2



$G_1 \cup G_2$

This Lecture