CS201 · Discrete Mathematics

Assignment 6

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1 Q1

1. $\frac{n(n-1)}{2}$
2. $n-1$
3.
   - For $n=1$, it is always true.
   - For $n=2$, we have $\min_{v\in V} deg(v) \geq \frac{n-1}{2} = \frac{1}{2}$, i.e., $\min_{v\in V} deg(v) \geq 1$. It means the two vertices must be connected.
   - For $n=3$, we have $\min_{v\in V} deg(v) \geq \frac{n-1}{2} = 1$. It means the two vertices must be connected.
   - For $n \geq 4$, suppose G has m edges. Then we have

$$\begin{aligned}2m &= \sum_{v\in V} deg(v)\\ &\geq \sum_{v\in V} \min_{v\in V} deg(v)\\ &\geq \sum_{v\in V} \frac{n-1}{2}\\ &= \frac{n(n-1)}{2}\end{aligned}$$

   i.e., $m \geq \frac{n(n-1)}{4}$.
   In connected graph, $m \geq n-1$. Since $n \geq 4$, i.e., $\frac{n(n-1)}{4} \geq n-1$, graph G satisfies $m \geq \frac{n(n-1)}{4} \geq n-1$. Therefore, G is connected.

After all, G must be connected.

2 Q2

1. Figure 1 is G_5.

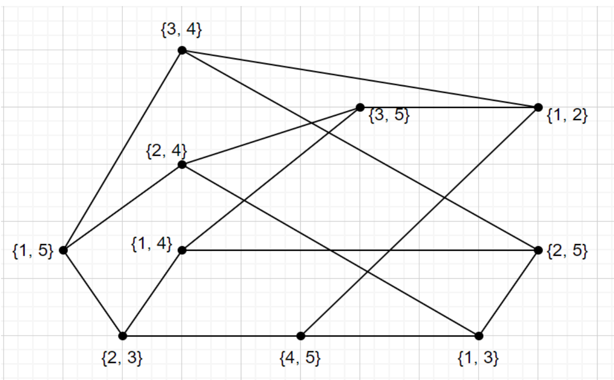

Figure 1: G_5

2. The degree of each vertex in G_n is $C(n-2, 2)$. Reason is as following:
   Suppose we randomly choose two numbers a, b from $\{1, ..., n\}$, then for disjointness, a' and b' must be chosen from another $n-2$ numbers of $\{1, ..., n\}$, i.e., from $\{1, ..., n\} - \{a, b\}$. Thus, for the vertex $\{a, b\}$, there are $C(n-2, 2)$ edges connected on it.

3 Q3

First we need to prove that the complete undirected graph with n vertices has $\frac{n(n-1)}{2}$ edges.

- For $n = 1$, there is no edge, which is true. For $n = 2$, there is one edge in the complete graph, which is also true.
- Suppose it is true for $n = k$ ($k \geq 3$), i.e., the graph has $\frac{k(k-1)}{2}$ edges. Then for $n = k+1$, the $k+1$ vertex connects another k vertices producing k additional edges. So the total amount of edges is $\frac{k(k-1)}{2} + k = \frac{k^2-k+2k}{2} = \frac{(k+1)k}{2}$.

After all, the complete undirected graph with n vertices has $\frac{n(n-1)}{2}$ edges.
By definition, the amount of edges of the graph G and its complementary graph $\bar{G}$ is equal to the amount of edges of the complete graph having the same vertices as G. Suppose G has m edges and $\bar{G}$ has m' edges, for v vertices, then we have $m + m' = \frac{v(v-1)}{2}$. Since G is self-complementary, then $m = m'$, i.e., $m = m' = \frac{v(v-1)}{4}$. Since v and $v-1$ are two continuous integers, they are coprime. Since m is also an integer, then either v or $v-1$ is divided by 4. Therefore, $v \equiv 0 \ or \ 1 \pmod 4$

4 Q4

1. If there is at least a vertex in G that is not adjacent with other vertices, then it must form a cycle, i.e., G contains a cycle.
   If the vertices in G are adjacent, then we can randomly choose two adjacent vertices u, v to construct a path P. And then we try to choose a vertex from G that is adjacent with the lately added vertex in P (at the first time the lately added vertex is v) but not in P, then we have
   - if we cannot choose this vertex, then P has already contained cycle. Since $deg(w) \geq 2$ for every vertex w, the lately added vertex must be in a cycle.
   - if we can choose this vertex, then add it in P and repeat the operations.

   Since the set of vertices is finite, when the vertices in G are all chosen, it must be the first situation. Therefore, G contains a cycle.

2. - For $n = 1$, obviously true.
   - For $n = 2$, disconnected G has only two isolated vertices, then $\bar{G}$ has two vertices and an edge connected them, i.e., $\bar{G}$ is connected.
   - For $n = 3$, disconnected G can be either three isolated vertices or one isolated vertex and two vertices with an edge connected them. But no matter which situation, $\bar{G}$ is still connected.
   - For $n \geq 4$, suppose G has m edges and $\bar{G}$ has m' edges. As proved in Q3, or as pasted below, the complete undirected graph with n vertices has $\frac{n(n-1)}{2}$ edges.
     - For $n = 1$, there is no edge, which is true. For $n = 2$, there is one edge in the complete graph, which is also true.
     - Suppose it is true for $n = k$ ($k \geq 3$), i.e., the graph has $\frac{k(k-1)}{2}$ edges. Then for $n = k+1$, the $k+1$ vertex connects another k vertices producing k additional edges. So the total amount of edges is $\frac{k(k-1)}{2} + k = \frac{k^2-k+2k}{2} = \frac{(k+1)k}{2}$.

     After all, the complete undirected graph with n vertices has $\frac{n(n-1)}{2}$ edges.
     Then $m + m' = \frac{n(n-1)}{2}$. So we have $m' = \frac{n(n-1)}{2} - m$. For disconnected G, we have $m < n - 1$.

Thus we have

$$\begin{aligned} m' &= \frac{n(n-1)}{2} - m \\ &> \frac{n(n-1)}{2} + 1 - n \\ &= \frac{n^2 - 3n + 2}{2} \\ &= \frac{(n-1)(n-2)}{2} \end{aligned}$$

Since $n \geq 4$, we have $m' = \frac{(n-1)(n-2)}{2} \geq n - 1$. Therefore, $\bar{G}$ is connected.

After all, if G is disconnected, then its complement $\bar{G}$ is connected.

5 Q5

1. Suppose a vertex $x \in N(A \cup B)$, and a vertex $y \in A \cup B$ is adjacent to x. Then we have $y \in A$ or $y \in B$. If $y \in A$, then $x \in N(A)$; if $y \in B$, then $x \in N(B)$. So $x \in N(A) \cup N(B)$, i.e., $N(A \cup B) \subseteq N(A) \cup N(B)$.
   Suppose a vertex $x \in N(A) \cup N(B)$, and a vertex $y \in A$ or $y \in B$ is adjacent to x. So $y \in A \cup B$. Therefore, $x \in N(A \cup B)$, i.e., $N(A) \cup N(B) \subseteq N(A \cup B)$.
   After all, $N(A) \cup N(B) = N(A \cup B)$.

2. Suppose a vertex $x \in N(A \cap B)$, and a vertex $y \in A \cap B$ is adjacent to x. Then we have $y \in A$ and $y \in B$. So it must be $x \in N(A)$ and $x \in N(B)$, i.e., $x \in N(A) \cap N(B)$. Therefore, $N(A \cap B) \subseteq N(A) \cap N(B)$.
   Let $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, c), (d, b), (b, c)\}$. And take $A = \{a\}, B = \{d\}$, i.e., $N(A) = \{b, c\}, N(B) = \{b, c\}, N(A) \cap N(B) = \{b, c\}$. However, $A \cap B = \emptyset$, i.e., $N(A \cap B) = \emptyset$. Therefore, $N(A \cap B) \neq N(A) \cap N(B)$.

6 Q6

For a bipartite simple graph, we can divide the vertices into two sets. Suppose one set has v_1 vertices and the other has v_2, i.e., $v = v_1 + v_2$. Then the maximum number of edges is $e_{max} = v_1 v_2$ when one vertex in one set is adjacent to all vertices in the other set. By AM-GM inequality, we have $v = v_1 + v_2 \geq 2\sqrt{v_1 v_2}$, i.e., $v_1 v_2 \leq \frac{v^2}{4}$. Therefore, $e \leq v_1 v_2 \leq \frac{v^2}{4}$.

7 Q7

1. Since $diam(G) = \max_{u,v \in V} d_G(u, v)$ and $rad(G) = \min_{v \in V} ecc(v) = \min_{v \in V} \max_{u \in V} d_G(u, v)$, it is obvious that $rad(G) \leq diam(G)$.
   Suppose the two endings of $diam(G)$ are $e_1, e_2 \in V$ and $rad(G) = ecc(r)$ for $r \in V$. Then by triangle inequality, we have $diam(G) = d_G(e_1, e_2) \leq d_G(e_1, r) + d_G(r, e_2)$. Since $rad(G) = ecc(r) = \max_{u \in V} d_G(u, r)$, then $rad(G) \geq d_G(w, r)$ for $w \in V$. Thus, we have $diam(G) \leq rad(G) + rad(G) = 2rad(G)$.
   After all, we have proved that

$$rad(G) \leq diam(G) \leq 2rad(G)$$

2. Let G_1 be a cycle with n vertices, then it will always be $diam(G_1) = rad(G_1) = n$.
   Let G_2 be a linked list with $2n + 1$ vertices, then it will always be $diam(G_2) = 2rad(G_2) = 2n$.
   Figure 2 is an example.

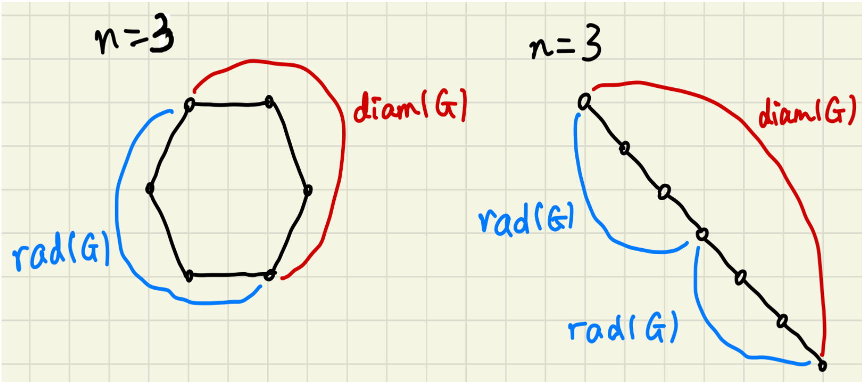

Figure 2: $n = 3$, G_1 and G_2

8 Q8

We can construct a Hamilton circuit in graph G as $u_1 \to u_5 \to u_8 \to u_7 \to u_6 \to u_2 \to u_3 \to u_4 \to u_1$. But we cannot do that for graph H but Hamilton path as $v_4 \to v_1 \to v_2 \to v_3 \to v_7 \to v_8 \to v_5 \to v_6$. We cannot find a function f that can map v_i to u_j $(i, j = 1, 2, 3, 4, 5, 6, 7, 8)$ such that $f(v_i)$ can construct a Hamilton circuit. Therefore, G and H are not isomorphic.

9 Q9

The graph we talk below is a directed multigraph having no isolated vertices.

- Sufficiency:
  If the graph $G = (V, E)$ is weakly connected, then $\forall v \in V$, there must exist a vertex that is incident to it.
  If the in-degree and out-degree of each vertex are equal, then for every vertex, the number of edges that points to it is equal to the number of edges that starts from it. Suppose we start from an initial vertex u, and it has an edge that points its adjacent vertex named v. Similarly, the same for v, for v's adjacent vertex w that there is an edge from v to w, etc.. Then finally, the last vertex would have an edge that starts from it and ends at v. In this way, we construct a Euler circuit.
- Necessity:
  If the graph $G = (V, E)$ has an Euler circuit, then it is strongly connected, i.e., it must be weakly connected.
  Since Euler circuit contains all edges of G, and if we start from any vertex in G and follow the Euler circuit, we will end at the initial vertex. Therefore, the in-degree and out-degree of each vertex are equal.

After all, we have proved what the problem asked.

10 Q10

ac

11 Q11

For $\min\{m, n\} = 1$ and $\max\{m, n\} \geq 1$, the complete bipartite graphs $K_{m,n}$ are trees.

12 Q12

- For $n = 2$, it is trivial that the Hamilton circuit is $00 \to 01 \to 11 \to 10 \to 00$

- Suppose it is true for $n = k$ ($k \geq 3$), and the Hamilton circuit is $g_1 \to g_2 \to ... \to g_m \to g_1$, where $m = 2^k$. Then for $n = k+1$, we can construct the Hamilton circuit as $0g_1 \to 0g_2 \to ... \to 0g_m \to 1g_m \to ... \to 1g_2 \to 1g_1 \to 0g_1$.

After all, every n-cube has a Hamilton circuit. (It is equivalent to prove there exists n-bit Gray code for positive integer n)

13 Q13

1. The graph G is bipartite while H is not. The reason is as following.
   For convenience, we can label every vertex of graph with alphabets as below (Figure 3).

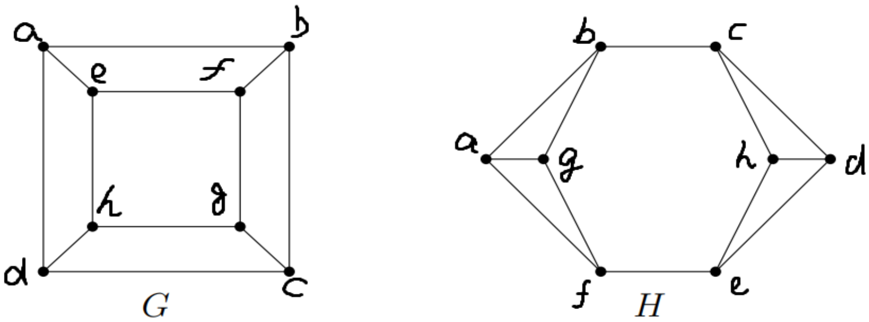

Figure 3: G and H

   - First let us focus on graph G. Without loss of generality, we can color vertex a as red, then vertex b, d, e must be blue in case that the two endings of an edge have the same color. Similarly, vertex c, f, h must be red and thus vertex g must be blue. After all, all the vertices are colored with either red or blue and no edge connects two vertices with same color. Therefore, G is bipartite.
   - Then let us focus on graph H. Similarly, we can start from vertex a with color red. Then vertex b, g, f are blue and c, e are red. It necessitates two adjacent vertices d, h to be blue at the same time, which makes H no possibility to be bipartite.

2. As proved in (1), graph G is bipartite while H is not. The vertex set of G can be divided into two set $\{a, c, f, h\}$ and $\{b, d, e, g\}$ and edges exist between the two set. However, if we divide the vertex set of H according to vertex color in (1) as $\{a, c, e\}$ and $\{b, d, f, g, h\}$, then the edges exist not only between the two set, but also between g and h. It makes that there is no function that can make $f(g)$ and $f(h)$ are adjacent in a bipartite graph. Therefore, G and H are not isomorphic to each other.

3. Neither graphs have an Euler circuit. Since every vertex in each graph has 3 edges connected it, i.e., the degree is 3, which is not even.

14 Q14

This problem can be simplified as:
For a complete graph K_{17}, we need to use 3 colors to color every edge. And we should prove that there at least exists a triangle with 3 edges that are in the same color.
Then we can start to prove it.

- In a complete graph K_{17}, for any vertex t, there are 16 edges incident to it. By pigeonhole principle, there at least 6 edges are in the same color c_1. Suppose the other end of these edges are u, v, w, x, y, z. And we delete vertex t and the edges incident to t.

- If there exists an edge that connects any two vertices in u, v, w, x, y, z and has the same color of c_1, then it is proved.
- If not, then we can take u (or any other vertex in u, v, w, x, y, z, without loss of generality), and consider the 15 edges incident to it. For the 5 edges that respectively connects u and one of v, w, x, y, z, we need to use 2 colors to color them. Then among the 5 edges, there at least 3 of them are in the same color c_2. Suppose the other end of these 3 edges are v, w, x.
  * If there exists an edge that connects any two vertices in v, w, x and has the same color of c_2, then it is proved.
  * If not, then we need to color $(v, w), (w, x), (v, x)$ with only one remaining color, i.e., they are in the triangle with edges in the same color.

After all, it is proved.

15 Q15

1. 3
2. 16
3. 4
4. 5

16 Q16

1. 1
2. 2
3. 2