CS201 · Discrete Mathematics

# Assignment 2

刘乐奇 / 12011327

March 24, 2022

## 1 Q1

1. False.
   Suppose $A = \{1\}, B = \{2\}$, then $A - B = A$ but $B$ is not a proper subset of $A$.

2. True
   Since $\forall x \in (A \cap B \cap C)$, i.e., $\forall x \in A \wedge B \wedge C$, we have $x \in A$. Since $A \subseteq (A \cup B)$, then $x \in (A \cup B)$.
   Therefore, $(A \cap B \cap C) \subseteq (A \cup B)$.

3. False
   Since
   $\overline{(A - B)} \cap (B - A) = \overline{(A \cap \overline{B})} \cap (B \cap \overline{A}) = (\overline{A} \cup B) \cap (B \cap \overline{A}) = ((\overline{A} \cap B) \cup (B \cap B)) \cap \overline{A} = ((\overline{A} \cap B) \cap \overline{A}) \cup (B \cap \overline{A}) = B \cap \overline{A} = B - A$.
   $B - A = B$ if and only if $A \cap B = \emptyset$, which does not always hold.

## 2 Q2

1. By definition, we have $A \oplus B = (A - B) \cup (B - A) = (A \cap \overline{B}) \cup (B \cap \overline{A})$, then

$$
\begin{aligned}
& A \oplus (B \oplus C) \\
&= A \oplus ((B - C) \cup (C - B)) = A \oplus (B \cap \overline{C}) \cup (C \cap \overline{B}) && \textit{By definition} \\
&= (A \cap \overline{(B \cap \overline{C}) \cup (C \cap \overline{B})}) \cup (\overline{A} \cap ((B \cap \overline{C}) \cup (C \cap \overline{B}))) && \textit{By definition} \\
&= (A \cap (\overline{B} \cup C) \cap (\overline{C} \cup B)) \cup (\overline{A} \cap ((B \cap \overline{C}) \cup (C \cap \overline{B}))) && \textit{De Morgen's law} \\
&= (((A \cap \overline{B}) \cup (A \cap C)) \cap (\overline{C} \cup B)) \cup ((\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)) && \textit{Distributive law} \\
&= (((A \cap \overline{B}) \cap (\overline{C} \cup B)) \cup ((A \cap C) \cap (\overline{C} \cup B))) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) && \textit{Distributive law} \\
&= (A \cap ((\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B))) \cup (\overline{A} \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) && \textit{Distributive law} \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) && \textit{Distributive law} \\
&= (A \cap B \cap C) \cup (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap \overline{C}) && \textit{Commutative law}
\end{aligned}
$$

   Similarly, we have

$$
\begin{aligned}
& (A \oplus B) \oplus C \\
&= ((A \cap \overline{B}) \cup (B \cap \overline{A})) \oplus C && \textit{By definition} \\
&= (\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})} \cap C) \cup (((A \cap \overline{B}) \cup (B \cap \overline{A})) \cap \overline{C}) && \textit{By definition} \\
&= ((\overline{A} \cup B) \cap (\overline{B} \cup A) \cap C) \cup (((A \cap \overline{B}) \cup (B \cap \overline{A})) \cap \overline{C}) && \textit{De Morgen's law} \\
&= ((\overline{A} \cup B) \cap ((\overline{B} \cap C) \cup (A \cap C))) \cup ((A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C})) && \textit{Distributive law} \\
&= (((\overline{A} \cup B) \cap (\overline{B} \cap C)) \cup ((\overline{A} \cup B) \cap (A \cap C))) \cup (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) && \textit{Distributive law} \\
&= ((\overline{A} \cap \overline{B} \cap C) \cup (B \cap \overline{B} \cap C) \cup ((\overline{A} \cap A \cap C) \cup (B \cap A \cap C))) \cup (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) && \textit{Distributive law} \\
&= (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C) \cup (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) && \textit{Complement law} \\
&= (A \cap B \cap C) \cup (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap \overline{C}) && \textit{Commutative law}
\end{aligned}
$$

   Therefore, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

2. It must be the case.
   First we need to prove that if $X \oplus Y = Z$, then $X \oplus Z = Y$
   $\because X \oplus Y = Z$
   $\therefore Z = (X \cup Y) - (X \cap Y)$
   Substitute in $X \oplus Z$, then

$$
\begin{aligned}
& X \oplus Z \\
&= X \oplus ((X \cup Y) - (X \cap Y)) && By\ definition \\
&= (X \cup ((X \cup Y) - (X \cap Y))) - (X \cap ((X \cup Y) - (X \cap Y))) && By\ definition \\
&= (X \cup ((X \cup Y) \cap \overline{(X \cap Y)})) - (X \cap ((X \cup Y) \cap \overline{(X \cap Y)})) && By\ definition \\
&= ((X \cup (X \cup Y)) \cap (X \cup \overline{(X \cap Y)})) - (X \cap (X \cup Y) \cap \overline{(X \cap Y)}) && By\ definition \\
&= ((X \cup Y) \cap (X \cup (\overline{X} \cup \overline{Y}))) - (X \cap \overline{(X \cap Y)}) && De\ Morgen's\ law \\
&= (X \cup Y) - (X \cap \overline{(X \cap Y)}) && Complement\ law \\
&= (X \cup Y) \cap \overline{(X \cap \overline{(X \cap Y)})} && By\ definition \\
&= (X \cup Y) \cap (\overline{X} \cup (X \cap Y)) && De\ Morgen's\ law \\
&= (X \cup Y) \cap (\overline{X} \cup Y) && Complement\ law \\
&= (X \cup \overline{X}) \cap Y && Distributive\ law \\
&= Y && Complement\ law
\end{aligned}
$$

   Second we need to prove the commutative law of symmetric difference.
   Since $A \oplus B = (A - B) \cup (B - A)$, according to the commutative law of conjunction, we have $A \oplus B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \oplus A$.
   Finally we can start to prove if $A \oplus C = B \oplus C$, then $B = C$.
   Suppose $A \oplus C = B \oplus C = D$, then by what we prove just now, we have

$$
\begin{aligned}
& A \oplus C = C \oplus A = D \implies C \oplus D = A \\
& B \oplus C = C \oplus B = D \implies C \oplus D = B \\
& \therefore A = C \oplus D = B
\end{aligned}
$$

# 3 Q3

1.

$$
\begin{aligned}
(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) && (By\ definition) \\
&= (B \cup C) \cap \overline{A} && (Distributive\ law) \\
&= (B \cup C) - A && (By\ definition)
\end{aligned}
$$

2.

$$
\begin{aligned}
(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) &= (A \cap B) \cap (\overline{B} \cup \overline{C}) \cap (A \cap C) && (De\ Morgen's\ law) \\
&= (B \cap A) \cap (A \cap C) \cap (\overline{B} \cup \overline{C}) && (Commutative\ law) \\
&= B \cap (A \cap A) \cap C \cap (\overline{B} \cup \overline{C}) && (Associative\ law) \\
&= B \cap A \cap C \cap (\overline{B} \cup \overline{C}) && (Idempotent\ law) \\
&= ((B \cap \overline{B}) \cup (B \cap \overline{C})) \cap A \cap C && (Commutative\ \&\ Distributive\ law) \\
&= B \cap \overline{C} \cap A \cap C && (Complement\ \&\ Identity\ law) \\
&= B \cap A \cap \emptyset && (Commutative\ \&\ Complement\ law) \\
&= \emptyset && (Domination\ law)
\end{aligned}
$$

# 4 Q4

Sufficiency: if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$
Suppose when $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $A \nsubseteq B$, then $\exists a \in A, a \notin B$, i.e., $\{a\} \not\subset B$.
However, $\{a\} \subset \mathcal{P}(A)$ and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., $\{a\} \subset \mathcal{P}(B)$. So $a$ is an element of set $B$, which contradicts the hypothesis. Thus, if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.
Necessity: if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$
Take a set $C \subset A$, then $C \subset B$. So we have $C \in \mathcal{P}(B)$. Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
After all, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$

# 5 Q5

1. ii. A function which is neither one-to-one nor onto.
   domain A: $\mathbb{Z}$; codomain B: $\mathbb{Z}$; map: $f(x) = |2x|$
   - Function
     Since $\forall x \in \mathbb{Z}^+, f(x) = 2x \in \mathbb{Z}^+, \forall x \in \mathbb{Z}^-, f(x) = -2x \in \mathbb{Z}^+$, and for an arbitrary $x$ the correspond $f(x)$ is unique.
   - Not one-to-one
     Take $x = -1$ and $y = 1$, then $f(x) = f(y)$. Thus, not one-to-one.
   - Not onto
     Take $y = -1 \in \mathbb{Z}$, but $\forall x \in \mathbb{Z}, f(x) = |2x| \in \mathbb{Z}^+$, which is impossible for $f(x) = y$.
2. i. Not a function.
   domain A: $\{1, 3\}$; codomain B: $\{2, 4\}$; map: $f(x) = 2x$
   - Not function
     For element 3 in domain $A$, there does not exists correspond element in codomain $B$
3. v. A function which is both one-to-one and onto.
   domain A: $\mathbb{R}$; codomain B: $\mathbb{R}$; map: $f(x) = 8 - 2x$
   - Function
     For every element in domain $A$, there is exactly one correspond element in codomain $B$.
   - One-to-one
     If $\forall x, y \in A, f(x) = f(y)$, we have $8 - 2x = 8 - 2y$, i.e., $x = y$.
   - Onto
     For all $y \in B$, there exists $x \in A$ where $x = \frac{8-y}{2}$ such that $f(x) = y$.
4. iii. A function which is onto but not one-to-one.
   domain A: $\mathbb{R}$; codomain B: $\mathbb{Z}$; map: $f(x) = \lfloor x + 1 \rfloor$
   - Function
     For every element in domain $A$, there is exactly one correspond element in codomain $B$.
   - Not one-to-one
     If $\forall x, y \in A, f(x) = f(y)$, we have $\lfloor x+1 \rfloor = \lfloor y+1 \rfloor$. However, if let $x = 0.5, y = 0.6$, the equation holds but it does not satisfy the condition of injunction.
   - Onto
     For all $y \in B$, there exists $x \in A$ where $x \in [y - 1, y) \subset A$ such that $f(x) = y$.
5. i. Not a function.
   domain A: $\mathbb{R}^+$; codomain B: $\mathbb{R}^+$; map: $f(x) = x - 1$
   - Not function
     For elements $x \in (0, 1] \subset A$, there is no correspond element in codomain $B$.

6. iv. A function which is one-to-one but not onto.
   domain A: $\mathbb{Z}^+$; codomain B: $\mathbb{Z}^+$; map: $f(x) = x + 1$

   - Function
     For every element in domain $A$, there is exactly one correspond element in codomain $B$.
   - One-to-one
     $\forall x, y \in A, f(x) = f(y)$, we have $x + 1 = y + 1$, which leads to $x = y$.
   - Not onto
     Let $y = 1 \in B$, there is no correspond $x \in A$ such that $f(x) = x + 1 = y$.

# 6 Q6

The mapping $f$ from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 8 - 2x$ has an inverse function $g(x) = \frac{8-x}{2}$. Because in all mappings in Q5, only this mapping is one-to-one correspondence.

# 7 Q7

Let $x = n + \varepsilon$, where $n$ is an integer and $0 \le \varepsilon < 1$.

- $0 \le \varepsilon < \frac{1}{3}$
  In this case, $3x = 3n + 3\varepsilon$. Since $0 \le 3\varepsilon < 1$, we have $\lfloor 3x \rfloor = 3n$. Similarly, $x + \frac{1}{3} = n + \varepsilon + \frac{1}{3}$, $x + \frac{2}{3} = n + \varepsilon + \frac{2}{3}$. Since $0 \le \varepsilon + \frac{1}{3} < \frac{2}{3}$, $0 \le \varepsilon + \frac{2}{3} < 1$, we have $\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \varepsilon + \frac{1}{3} \rfloor = n$, $\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \varepsilon + \frac{2}{3} \rfloor = n$. Thus, $\lfloor 3x \rfloor = 3n = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.
- $\frac{1}{3} \le \varepsilon < \frac{2}{3}$
  In this case, $3x = 3n + 3\varepsilon = (3n + 1) + (3\varepsilon - 1)$. Since $0 \le 3\varepsilon - 1 < 1$, we have $\lfloor 3x \rfloor = 3n + 1$. Similarly, $x + \frac{1}{3} = n + \varepsilon + \frac{1}{3}$, $x + \frac{2}{3} = n + \varepsilon + \frac{2}{3}$. Since $\frac{2}{3} \le \varepsilon + \frac{1}{3} < 1$, $1 \le \varepsilon + \frac{2}{3} < \frac{4}{3}$, we have $\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \varepsilon + \frac{1}{3} \rfloor = n$, $\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \varepsilon + \frac{2}{3} \rfloor = n + 1$. Thus, $\lfloor 3x \rfloor = 3n + 1 = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.
- $\frac{2}{3} \le \varepsilon < 1$
  In this case, $3x = 3n + 3\varepsilon$. Since $2 \le 3\varepsilon < 3$, we have $\lfloor 3x \rfloor = 3n + 2$. Similarly, $x + \frac{1}{3} = n + \varepsilon + \frac{1}{3}$, $x + \frac{2}{3} = n + \varepsilon + \frac{2}{3}$. Since $1 \le \varepsilon + \frac{1}{3} < 2$, $\frac{4}{3} \le \varepsilon + \frac{2}{3} < \frac{5}{3}$, we have $\lfloor x + \frac{1}{3} \rfloor = \lfloor n + \varepsilon + \frac{1}{3} \rfloor = n + 1$, $\lfloor x + \frac{2}{3} \rfloor = \lfloor n + \varepsilon + \frac{2}{3} \rfloor = n + 1$. Thus, $\lfloor 3x \rfloor = 3n + 2 = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

# 8 Q8

1. No.
   Giving a counterexample. Let $A = \{1, 2\}, B = \{1, 2, 3\}, C = \{1, 2\}$ and $g(1) = 1, g(2) = 2; f(1) = 1, f(2) = f(3) = 2$. It is obvious that $f \circ g$ is one-to-one and $g$ is one-to-one while $f$ is not one-to-one.
2. Yes.
   $\forall m, n \in A \wedge m \ne n$, assume that $g(m) = g(n)$. Since $f \circ g$ is one-to-one, we have $f \circ g(m) \ne f \circ g(n)$. However, $f \circ g(m) = f(g(m)) = f(g(n)) = f \circ g(n)$. This leads to a contradiction. Therefore, $g(m) \ne g(n)$, i.e., $g$ must be one-to-one.
3. Yes.
   $\forall m, n \in A \wedge m \ne n$, assume that $g(m) = g(n)$. Since $f \circ g$ is one-to-one, we have $f \circ g(m) \ne f \circ g(n)$. However, $f \circ g(m) = f(g(m)) = f(g(n)) = f \circ g(n)$. This leads to a contradiction. Therefore, $g(m) \ne g(n)$, i.e., $g$ must be one-to-one.
4. Yes.
   Since $f \circ g$ is onto, then $\forall m \in B, \exists n \in A(f \circ g(n) = m)$.
   Suppose $f$ is not onto, then $\exists y \in B, \forall x \in A(f(x) \ne y)$.
   Here we take $x = g(n)$, then we have $\exists y \in B, f(g(n)) \ne y$, which contradicts the hypothesis.

5. No
   Giving a counterexample. Let $A = \{1,2\}, B = \{1,2,3\}, C = \{1,2\}$ and $g(1) = 1, g(2) = 2; f(1) = 1, f(2) = f(3) = 2$. It is obvious that $f \circ g$ is onto while $g$ is not onto.

# 9 Q9

Since

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

we have

$$\begin{aligned}
1^3 - 0^3 &= 3 \times 1^2 - 3 \times 1 + 1 \\
2^3 - 1^3 &= 3 \times 2^2 - 3 \times 2 + 1 \\
3^3 - 2^3 &= 3 \times 3^2 - 3 \times 3 + 1 \\
&\ldots\ldots \\
n^3 - (n-1)^3 &= 3n^2 - 3n + 1
\end{aligned}$$

Add both sides, we have

$$\begin{aligned}
n^3 &= 3 \times (1^2 + 2^2 + 3^2 + \ldots + n^2) - 3 \times (1 + 2 + 3 + \ldots + n) + n \\
&= 3 \sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n \\
&= 3 \sum_{k=1}^{n} k^2 - \frac{n(3n+1)}{2}
\end{aligned}$$

Therefore, we have

$$\sum_{k=1}^{n} k^2 = \frac{n^3 + \frac{n(3n+1)}{2}}{3} = \frac{n(n+1)(2n+1)}{6}$$

# 10 Q10

1. $A = [0,1], B = (0,1)$ are uncountable, and $A - B = \{0,1\}$ is finite.
2. $A = \mathbb{R}, B = \{x | x \in \mathbb{R} \wedge x \notin \mathbb{Z}\}$ are uncountable, and $A - B = \mathbb{Z}$ is infinitely countable.
3. $A = \mathbb{R}, B = (0,1)$ are uncountable, and $A - b = \mathbb{R}/(0,1)$ is uncountable.

# 11 Q11

1. Countable.
   Since the set of students in CS201 is finite, its power set is definitely finite by definition, i.e., the set of all subsets of students in CS201 is countable.
2. Countable.
   The set is the same as $\mathbb{N} \times \mathbb{N}$. We know that there is a bijection between it and $\mathbb{Z}^+$ by listing:
   $$(0,0),(0,1),(1,0),(1,1),(1,2),\ldots,(i,i),(i,i+1),(i+1,i),(i+1,i+1),\ldots$$
3. Uncountable.
   Since $\mathbb{R}$ is uncountable, any sequence that includes an element from $\mathbb{R}$ must also be uncountable.

# 12 Q12

$A - B$ must be uncountable.
Suppose $A - B$ is countable. Since $A = (A - B) \cup (A \cap B)$, $A - B$ (by assumption) and $A \cap B$ (by definition) are countable, the elements of $A$ can also be listed in a sequence. It contradicts the fact that $A$ is uncountable.

# 13 Q13

Suppose $m+n=x$, we can get the function ranges from $(x-2)(x-1)/2+1$ to $(x-2)(x-1)/2+(x-1)$ since $m,n \in \mathbb{Z}^+$, i.e., $m$ can take value ranges as $1,2,3,\ldots,(x-1)$. When the value of $m+n$ is confirmed, the first term of the function is a fixed positive integer. Thus, to prove this function is bijective, we only need to prove that $f(1,x)=f(x-1,1)+1$. Since $f(x-1,1)+1=(x-2)(x-1)/2+(x-1)+1=(x^2-x+2)/2=(x-1)x/2+1=f(1,x)$.

# 14 Q14

Let $f:(0,1)\mapsto[0,1], f(x)=x$. It is a one-to-one function from (0,1) to [0,1], so $|(0,1)|\leq|[0,1]|$.
Let $g:[0,1]\mapsto(0,1), g(x)=\frac{x}{2}+\frac{1}{3}$. It is a one-to-one function from [0,1] to (0,1), so $|[0,1]|\leq|(0,1)|$.
By Schroder-Bernstein Theorem, we have $|[0,1]|=|(0,1)|$.

# 15 Q15

Since $|A|=|B|$, then there is a one-to-one correspondence $f$ between $A$ and $B$. Similarly, there is a one-to-one correspondence $g$ between $B$ and $C$.
Suppose $g\circ f(x)=g\circ f(y)$ for $x,y\in A$, i.e. $g(f(x))=g(f(y))$. Since $g$ is bijective, we have $f(x)=f(y)$. Similarly, $f$ is bijective, $x=y$. So $g\circ f:A\mapsto C$ is injective.
Since $g$ is bijective, $\forall c\in C,\exists b\in B, c=g(b)$. Similarly, $\forall b\in B.\exists a\in A, b=f(a)$. Then we have $c=g(f(a))=g\circ f(a)$, i.e., $g\circ f:A\mapsto C$ is surjective.
Therefore, there is a one-to-one correspondence between $A$ and $C$. Hence, $|A|=|C|$.

# 16 Q16

Since $f(x)=\Theta(g(x))$, then $\exists c_1>0,\exists c_2>0\ (\forall x\geq x_0\ (c_1g(x)\leq f(x)\leq c_2g(x)))$. Similarly, we have $\exists c_3>0,\exists c_4>0\ (\forall x\geq x_0\ (c_3h(x)\leq g(x)\leq c_4h(x)))$.
Therefore, $c_1c_3h(x)\leq c_1g(x)\leq f(x)\leq c_2g(x)\leq c_2c_4h(x)$, i.e., $f(x)=\Theta(h(x))$.

# 17 Q17

1. multiplications: 2n
   additions: n
2. $\Theta(n)$