CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 6
Due: June 2th, 2022, please submit one pdf file through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Plagiarism in an Assignment or a Quiz:

- **For the first time: the score of the assignment or quiz will be zero**
- **For the second time: the score of the course will be zero**
- **When two assignments are nearly identical, the policy will apply to BOTH students, unless one confesses having copied without the knowledge of the other.**

Any late submission will lead to a zero point with no exception.

Q. 1. (5 points) Let G be a *simple* graph with n vertices.

(a) What is the *maximum* number of edges G can have?

(b) If G is connected, what is the *minimum* number of edges G can have?

(c) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Q. 2. (5 points) Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Q. 3. (5 points) The complementary graph $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in G if and only if they are not adjacent in G. A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Q. 4. (10 points) Suppose that G is a graph on a finite set of n vertices. Prove the following:

(a) If every vertex of G has degree 2, then G contains a cycle.

(b) If G is disconnected, then its complement $\bar{G}$ is connected.

Q. 5. (5 points) Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Q. 6. (5 points) Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Q. 7. (10 points) Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\mathrm{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. That is, $\max_{u,v \in V} d_G(u, v)$. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\mathrm{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v \in V} \mathrm{ecc}(v)$. Prove the following.

(a) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

(b) For every positive integer n, there are connected graphs G_1 and G_2 with $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$ and $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$.

Q. 8. (5 points) Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

Q. 9. (10 points) Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

Q. 10. (5 points) Which of the these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph? a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

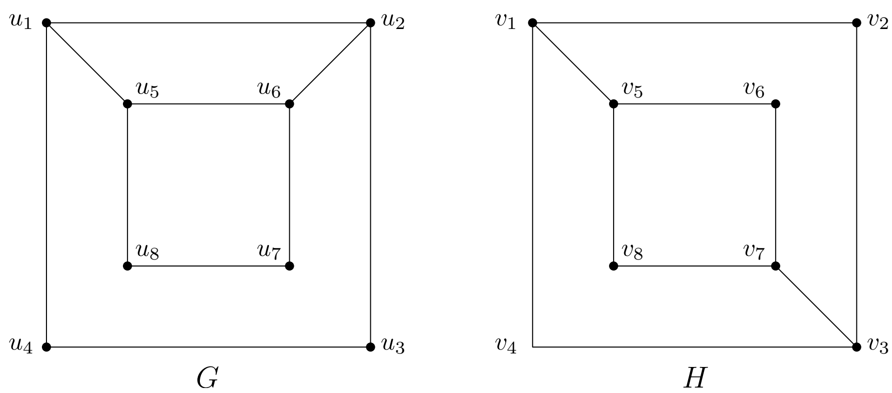

Figure 1: Q.19

Q. 11. (5 points) Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Q. 12. (10 points) An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Show that every n-cube has a Hamilton circuit.

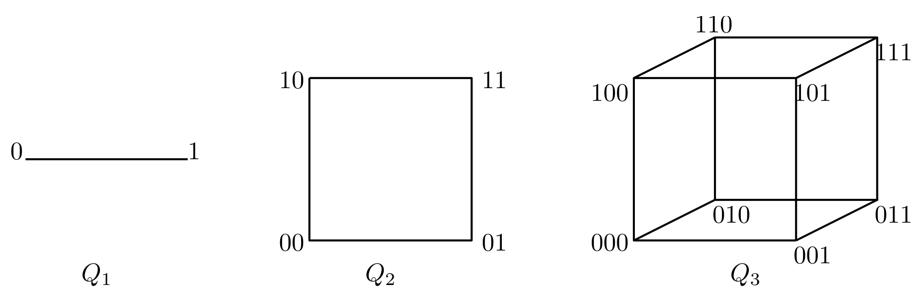

Figure 2: Q.36

Q. 13. (5 points) Consider the two graphs G and H. Answer the following three questions, and explain your answers.

(1) Which of the two graphs is/are *bipartite*?

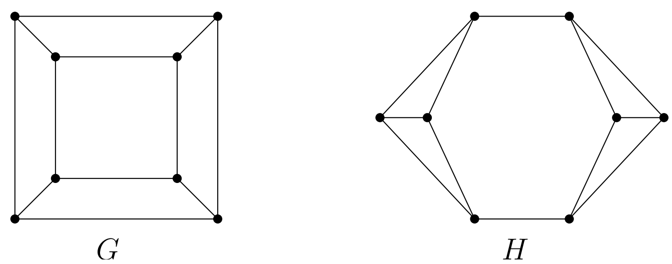

Figure 3: Q.37

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Q. 14. (5 points) There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Q. 15. (5 points) How many different spanning trees does each of these simple graphs have? a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Q. 16. (5 points) How many nonisomorphic spanning trees does each of these simple graphs have?
a) K_3 b) K_4 c) K_5