Discrete Mathematics for Computer Science

Lecture 12: Counting

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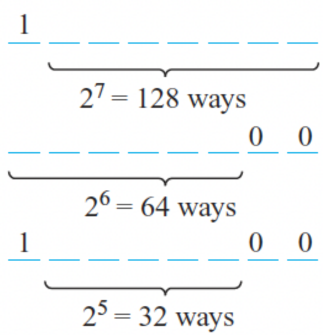
Inclusion-Exclusion Principle (Subtraction Rule)

Used in counts where the decomposition yields two independent counting tasks with overlapping elements:

- If we use the sum rule, some elements would be counted twice.

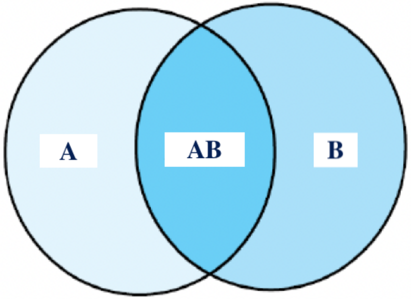
Inclusion-Exclusion Principle: uses a sum rule and then corrects the overlapping elements: $|A \cup B| = |A| + |B| - |A \cap B|$

Example: How many bit strings of length 8 either start with a '1' bit or end with the two bits '00'? $2^7 + 2^6 - 2^5$.



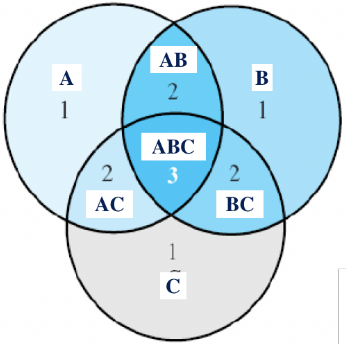
Inclusion-Exclusion Principle

Two sets A and B: $|A \cup B| = |A| + |B| - |A \cap B|$



Three sets A, B, and C:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

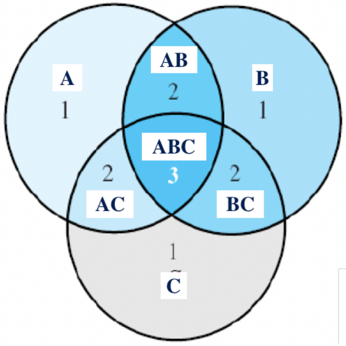


Inclusion-Exclusion Principle

Let $E_1, E_2, \ldots, E_n$ be finite sets:

$$|E_1 \cup E_2 \cup \cdots \cup E_n| = \sum_{1 \le i \le n} |E_i| - \sum_{1 \le i < j \le n} |E_i \cap E_j|$$
$$+ \sum_{1 \le i < j < k \le n} |E_i \cap E_j \cap E_k| - \ldots + (-1)^{n+1} |E_1 \cap E_2 \cap \ldots \cap E_n|.$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion Principle

Let $E_1, E_2, \ldots, E_n$ be finite sets:

$$|E_1 \cup E_2 \cup \cdots \cup E_n| = \sum_{1 \le i \le n} |E_i| - \sum_{1 \le i < j \le n} |E_i \cap E_j| + \sum_{1 \le i < j < k \le n} |E_i \cap E_j \cap E_k| - \ldots + (-1)^{n+1} |E_1 \cap E_2 \cap \ldots \cap E_n|.$$

Or equivalently,

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Inclusion-Exclusion Principle

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Proof by Induction:

Base step $(n = 2)$:

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

Inclusion-Exclusion Principle

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Proof by Induction:

Inductive Step: Consider inductive hypothesis

$$|\cup_{i=1}^{n-1} E_i| = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Set $E = E_1 \cup \ldots \cup E_{n-1}$, and $F = E_n$. By $|E \cup F| = |E| + |F| - |E \cap F|$.

$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right|$$

Inclusion-Exclusion Principle

Inductive Step: Consider inductive hypothesis

$$\left|\cup_{i=1}^{n-1} E_i\right| = \sum_{k=1}^{n-1}(-1)^{k+1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n-1} |E_{i_1}\cap E_{i_2}\cap\cdots\cap E_{i_k}|$$

Set $E = E_1 \cup \ldots \cup E_{n-1}$, and $F = E_n$. By $|E\cup F| = |E| + |F| - |E\cap F|$.

$$|\cup_{i=1}^{n} E_i| = \boxed{\left|\cup_{i=1}^{n-1} E_i\right|} + |E_n| - \boxed{\left|\left(\cup_{i=1}^{n-1} E_i\right)\cap E_n\right|}$$

- The first term is given by inductive hypothesis
- For the third term, by distributive law,

$$\left|\left(\cup_{i=1}^{n-1} E_i\right)\cap E_n\right| = \left|\cup_{i=1}^{n-1}(E_i\cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

where $G_i = E_i \cap E_n$.

Inclusion-Exclusion Principle

$$|\cup_{i=1}^{n} E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |\cup_{i=1}^{n-1} G_i|, \text{ where } G_i = E_i \cap E_n.$$

Using inductive hypothesis on both $|\bigcup_{i=1}^{n-1} E_i|$ and $|\bigcup_{i=1}^{n-1} G_i|$:

$$\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n-1} |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}| + |E_n|$$

$$- \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n-1} |G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_k}|$$

Note that

$$-(-1)^{k+1} |G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_k}| = (-1)^{k+2} |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k} \cap E_n|$$

Inclusion-Exclusion Principle

$$\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{k=1}^{n-1}(-1)^{k+1} \sum_{1\le i_1<i_2<\ldots<i_k\le n-1} |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}|$$

$$+ |E_n| + \sum_{k=1}^{n-1}(-1)^{(k+1)+1} \sum_{1\le i_1<i_2<\ldots<i_k\le n-1} |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k} \cap E_n|$$

- The first term sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
- The second term sums $|E_n|$ and $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that contain n

That is, we have the inductive conclusion

$$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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An Alternative Form of Inclusion–Exclusion Principle

This form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \ldots, P_n$.

- A_i: the subset containing the elements that have property P_i.
- $N(P_{i_1}, P_{i_2}, \ldots, P_{i_k})$: The number of elements with all the properties.

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = N(P_{i_1}, P_{i_2}, \ldots, P_{i_k}).$$

- $N(P'_{i_1}, P'_{i_2}, \ldots, P'_n)$: The number of elements with none of the properties $P_1, P_2, \ldots, P_n$.

$$N(P'_{i_1}, P'_{i_2}, \ldots, P'_n) = N - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

$$N(P'_1 P'_2 \ldots P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n).$$

Inclusion-Exclusion Principle: Example

How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3.

Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively.

Let A_1, A_2, A_3 be the corresponding subsets of functions.

$$N(P_1', P_2', ..., P_3') = N - |A_1 \cup A_2 \cup A_3|$$

$$N(P_1'P_2'P_3') = N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3),$$

Inclusion-Exclusion Principle: Example

How many onto functions are there from a set with six elements to a set with three elements?

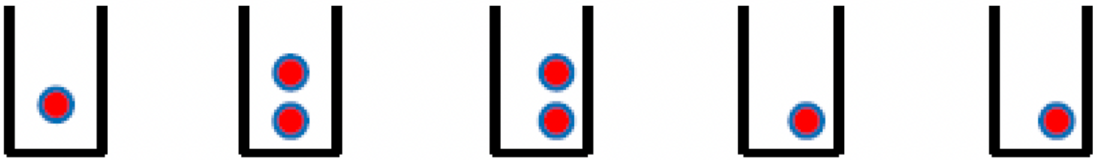
$$N(P_1'P_2'P_3') = N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3),$$

- N: the total number of functions. $N = 3^6$
- $N(P_i)$: the number of functions that do not have b_i in their range. $N(P_i) = 2^6$
- $N(P_i, P_j)$: The number of functions that do not have b_i and b_j in their range. $N(P_i, P_j) = 1^6$
- $N(P_1, P_2, P_3) = 0$

Pigeonhole Principle

Assume that there are a set of objects and a set of bins to store them.

The Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



Pigeonhole Principle: Example 1

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games.

Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the j th day of the month. Then,

$$a_1, a_2, ..., a_{30},$$

which is an increasing sequence of distinct integers, with $1 \leq a_j \leq 45$.

Moreover, $a_1 + 14, a_2 + 14, ..., a_{30} + 14$ is also an increasing sequence of distinct integers, with $15 \leq a_j + 14 \leq 59$.

Pigeonhole Principle: Example 1

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games.

Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: The 60 integers $a_1, a_2, ..., a_{30}, a_1 + 14, a_2 + 14, ..., a_{30} + 14$ are all less than or equal to 59. By the pigeonhole principle, two of these integers are equal.

Since the integers in each sequence are distinct, there must be indices i and j with $a_i = a_j + 14$.

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers:

- A subsequence of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where $1 \leq i_1 < i_2 < \ldots < i_m \leq N$.
- A sequence is called strictly increasing if each term is larger than the one that precedes it.

Pigeonhole Principle: Example 2

Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Example: The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$.

There are four strictly increasing subsequences of length four:

$$1, 4, 6, 12 \qquad 1, 4, 6, 7$$

$$1, 4, 6, 10 \qquad 1, 4, 5, 7$$

There is also a strictly decreasing subsequence of length four:

$$11, 9, 6, 5$$

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate (i_k, d_k) to the term a_k:

- i_k: the length of the longest increasing subsequence starting at a_k
- d_k: the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. I.e., $i_k \leq n$ and $d_k \leq n$ for $k = 1, 2, ..., n^2 + 1$.

By the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal.

That is, there exist terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$.

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof: There exist terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible.

The terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$:

- $a_s < a_t$: Since $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built, i.e., a_s, a_t, ... (followed by an increasing subsequence of length i_t beginning at a_t)
- $a_s > a_t$, Since $d_s = d_t$, an decreasing sequence of length $d_t + 1$ can be built, i.e., a_s, a_t, ...

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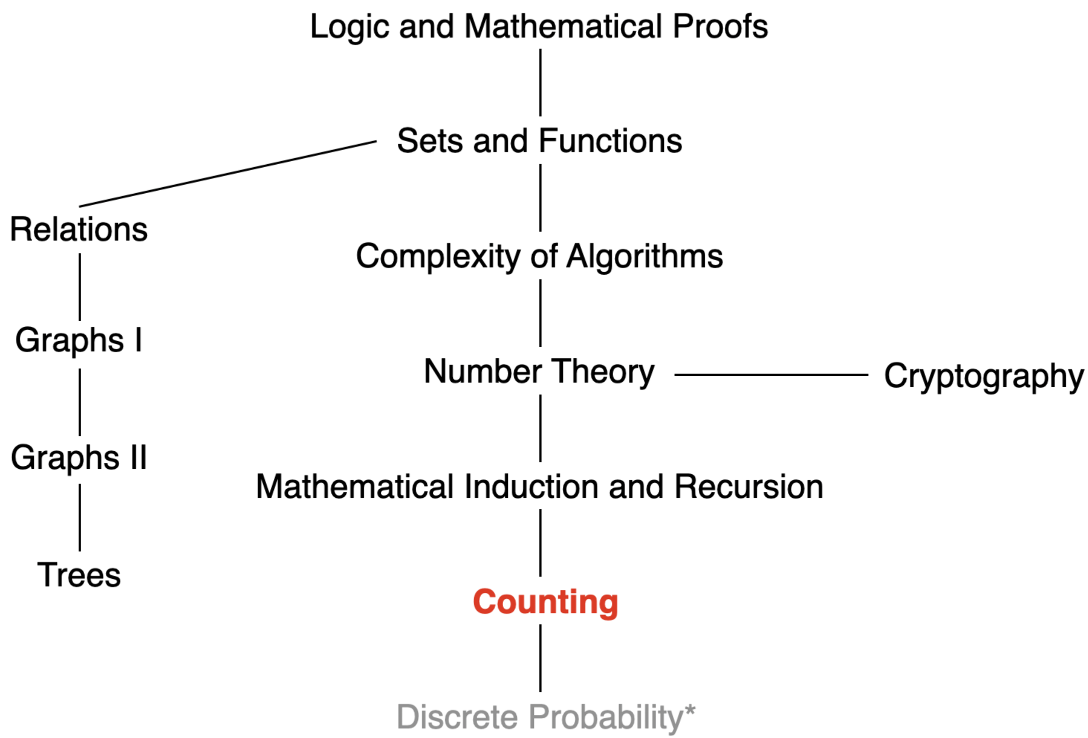


Counting basis, Permutations and Combinations, Binomial Coefficients, ...

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

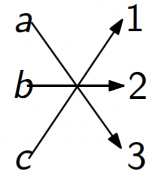
An n-element permutation of a set of size n is simply called a permutation.

Example: Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements (a, b), (a, c), (b, a) (b, c), (c, a). (c, b).

An Alternative Definition: Permutations and Bijection

A function that is both one-to-one and onto is called a bijection, or a one-to-one correspondence.

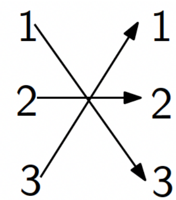
$$\boxed{f : \{a, b, c\} \to \{1, 2, 3\} \text{ defined by } f(a) = 3, f(b) = 2, f(c) = 1 \text{ is a bijection.}}$$



- In a bijection, exactly one arrow leaves each item on the left and exactly one arrow arrives at each item on the right.
- Thus, the left and right sides must have the same size.

A bijection from a set onto itself is called a permutation.

$$\boxed{f : \{1, 2, 3\} \to \{1, 2, 3\} \text{ defined by } f(1) = 3, f(2) = 2, f(3) = 1 \text{ is a bijection.}}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Permutations

How many 3-permutations of $\{1, 2, ..., n\}$ are there?

Based on product rule:

- n choices for first number.
- For each way of choosing first number, there are $n - 1$ choices for the second.
- For each way of choosing first two numbers, there are $n - 2$ choices for the third number.

By product rule, there are $n(n-1)(n-2)$ ways to choose the permutation.

Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof by the Product Rule: The first element of the permutation can be chosen in n ways, because there are n elements in the set.

There are $n-1$ ways to choose the second element of the permutation.

...

Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary: If n and r are integers with $0 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Permutations: Example

Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$P(100, 3) = 100 \times 99 \times 98 = 970,200.$

Example 2: How many permutations of the letters ABCDEFGH contain the string ABC?

The letters ABC must occur as a block. Thus, it is equivalent to finding the number of permutations of six objects:

$$\text{ABC, D, E, F, G, H.}$$

Thus, there are $P(6, 6) = 6! = 720$ permutations.

Combinations

An r-combination of elements of a set is an unordered selection of r elements from the set.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Example: The 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. Thus, $C(4, 2) = 6$.

Binomial Coefficient

Theorem: For integers n and r with $0 \leq r \leq n$, the number of r-element subsets of an n-element set is

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Proof: The $P(n,r)$ r-permutations of the set can be obtained by

- forming the $C(n,r)$ r-combinations of the set.
- ordering the elements in each r-combination, which can be done in $P(r,r)$ ways.

By the product rule,

$$P(n,r) = C(n,r)P(r,r).$$

Some Properties of Binomial Coefficients

- $C(n, 0) = 1$: one set of size 0.
- $C(n, n) = 1$: one set of size n.
- $C(n, r) = C(n, n - r)$

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

$$C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!}$$

Any other ideas to prove?

We will address this later.

Combinations: Example

Example 1: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

$C(10, 5) = \frac{10!}{5!5!} = 252.$

Example 2: There are 9 faculty members in the mathematics department and 11 in the computer science department.

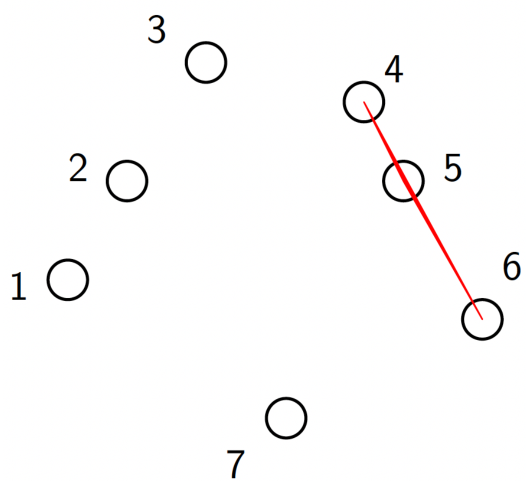
How many ways are there to form a committee with 3 faculty members from the mathematics department and 4 from the computer science department?

$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 27,720.$

Example: Counting Triangle

Design an algorithm to count the number of triangles formed by n points in the plane:

- 3 points form a triangle if and only if they are non-collinear



$1-2-7$: yes

$1-2-5$: yes

$1-2-3$: no

$1-5-6$: yes

$3-4-7$: yes

$4-5-6$: no

Example: Counting Triangle

The following loop is a part of program to determine the number of triangles formed by n points in the plane:

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

Question: Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

This corresponds to the total number of combinations. Why?

Example: Counting Triangle

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

- First loop begins with $i = 1$ and i increases up to n.
- Second loop begins with $j = i + 1$ and j increases up to n.
- Third loop begins with $k = j + 1$ and k increases up to n.

Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

Example: Counting Triangle

Want to compute the number of increasing triples (i, j, k) with $1 \leq i < j < k \leq n$.

Claim: The number of increasing triples is exactly the same as the number of 3-combinations from $\{1, 2, ..., n\}$.

- X: set of increasing triples
- Y: set of 3-combinations from $\{1, 2, ..., n\}$

Define: $f : X \to Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a bijection, so $|X| = |Y|$.

Example: Counting Triangle

- X: set of increasing triples
- Y: set of 3-combinations from $\{1, 2, ..., n\}$

Define: $f : X \to Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a bijection, so $|X| = |Y|$.

- One-to-one: if $f((i, j, k)) = f((i', j', k'))$, then $(i, j, k) = (i', j', k')$.
- Onto: for any $\{i, j, k\}$, there exists a (i, j, k) such that $f((i, j, k)) = \{i, j, k\}$.

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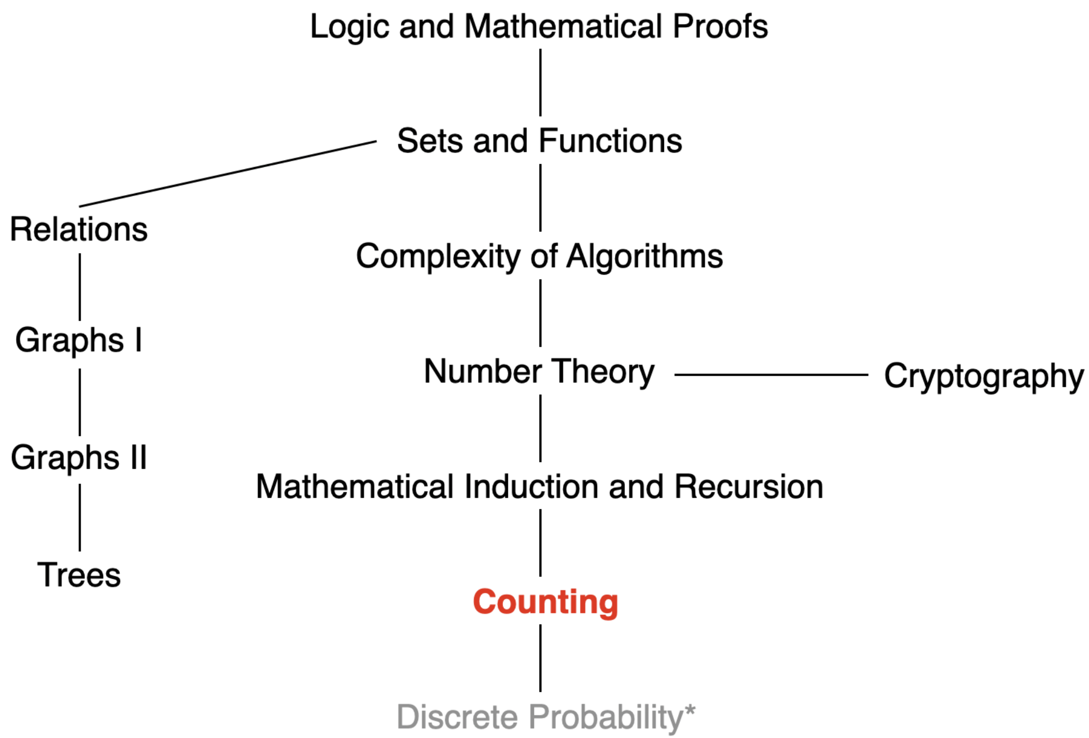


Counting basis, Permutations and Combinations, Binomial Coefficients, ...

Combinatorial Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Definition: A combinatorial proof of an identity is

- a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.

These two types of proofs are called double counting proofs and bijective proofs, respectively.

Combinatorial Proof: Bijective Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Bijective Proof: Suppose that S is a set with n elements.

The function that maps a subset A of S to $\bar{A}$ is a bijection between subsets of S with r elements and subsets with $n - r$ elements.

- X: the set of all possible A, where $|X| = C(n, r)$
- Y: the set of all possible $\bar{A}$, where $|Y| = C(n, n - r)$
- $f : X \to Y$ is defined as $f(A) = \bar{A}$
  - One-to-one: if $f(A_1) = f(A_2)$, then $A_1 = A_2$.
  - Onto: for any $\bar{A}$, there exists an A such that $f(A) = \bar{A}$.

Since there is a bijection between two finite sets X and Y, they must have the same number of elements. Thus, $C(n, r) = C(n, n - r)$.

Combinatorial Proof: Bijective Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$.

Double Counting Proof:

- Left-hand side $C(n,r)$: The number of subsets A of S with r elements.
- Right-hand side $C(n,n-r)$: The number of subsets $\bar{A}$ (i.e., the complement of A) of S with $n-r$ elements.

Each subset A of S is also determined by specifying which elements are not in A, so are in $\bar{A}$. Thus, both sides count the same thing

It follows that $C(n,r) = C(n,n-r)$.

The Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, ..., n$.

To count the number of terms of the form $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums (so that the other j terms in the product are ys).

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is $\binom{n}{j}$.

The Binomial Theorem: Example

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 1: What is the expansion of $(x+y)^4$?

$$\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\
&= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4.
\end{aligned}$$

The Binomial Theorem: Example

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 2: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

First, note that this expression equals $(2x + (-3y))^{25}$

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

The coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$:

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}.$$

The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

This is proven by substituting $x = 1$ and $y = 1$.
Any other ideas to prove?

The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Double Counting Proof: Let P denote the set of all subsets of $\{1, 2, ..., n\}$.

- Left-hand side: Let S_k be the set of all subsets of $\{1, 2, ..., n\}$ with k elements.

$$|P| = \sum_{i=0}^{n} |S_k| = \sum_{i=0}^{n} \binom{n}{k}$$

- Right-hand side: A set with n elements has a total of 2^n different subsets, i.e., $|P| = 2^n$.

The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Bijective Proof: Let P denote the set of all subsets of $\{1, 2, ..., n\}$. Let S_k be the set of all subsets of $\{1, 2, ..., n\}$ with k elements.

$$|P| = \sum_{i=0}^{n} |S_k| = \sum_{i=0}^{n} \binom{n}{k}$$

Consider $L = L_1 L_2 ... L_n$ be a list of size n from $\{0, 1\}$. Let $\mathcal{L}$ be the set of all such lists, we have $|\mathcal{L}| = 2^n$.

Objective: there is a bijection between $\mathcal{L}$ and P, so $|P| = |\mathcal{L}| = 2^n$.

The Binomial Theorem

Define the following function $f : \mathcal{L} \to P$

- If $L \in \mathcal{L}$, then $f(L)$ is the set $S \subset \{1, 2, ..., n\}$ defined by

$$i \in S, \text{for } L_i = 1$$

f is a bijection between $\mathcal{L}$ and P.

- one-to-one: If $f(L_1) = f(L_2)$, then $L_1 = L_2$
- onto: for any S, there exists an L such that $f(L) = S$

The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a positive integer.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

This is proven by substituting $x = -1$ and $y = 1$.

Corollary: Let n be a nonnegative integer.

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

This is proven by substituting $x = 1$ and $y = 2$.

Pascal's Identity

Theorem: Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: Suppose that T is a set containing $n+1$ elements.

- let a be an element in T
- let $S = T - a$.

Left-hand side counts the number of subsets of T containing k elements, i.e., $\binom{n+1}{k}$.

Note that a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a.

Right-hand side counts

- the subsets of $k-1$ elements of S, i.e., $\binom{n}{k-1}$
- the subsets of k elements of T, i.e., $\binom{n}{k}$.

Pascal's Identity

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$$

Consider $S = \{A, B, C, D, E\}$.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts:

- S_2: the 2-subsets that contain E
- S_3: the set of 2-subsets that do not contain E

$$S_1 = \{\color{red}{\{A, B\}, \{A, C\}, \{A, D\}}, \color{blue}{\{A, E\}}, \color{red}{\{B, C\}, \{B, D\}}, \color{blue}{\{B, E\}}, \color{red}{\{C, D\}}, \color{blue}{\{C, E\}, \{D, E\}}\}.$$

Pascal's Triangle

$$
\begin{array}{c}
\binom{0}{0} \\[6pt]
\binom{1}{0} \quad \binom{1}{1} \\[6pt]
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\[6pt]
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\[6pt]
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\[6pt]
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\[6pt]
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\[6pt]
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\[6pt]
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\[6pt]
\cdots
\end{array}
$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients.

Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: Consider bit strings of length $n+1$.

The left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.
We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

- This final one must occur at position $r+1, r+2, \ldots, $ or $n+1$.
- If the last one is the k-th bit there must be r ones among the first $k-1$ positions. There are $\binom{k-1}{r}$ such bit strings.

Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}.$$

Labelling and Trinomial Coefficients

Suppose we have k_1 labels of one kind (e.g., red) and $k_2 = n - k_1$ labels of another (e.g., blue). How many different ways to label n distinct objects?

$$C(n, k_1) = \frac{n!}{k_1!k_2}$$

If we have k_1 labels of one kind (e.g., red), k_2 labels of a second kind (e.g., blue), and $k_3 = n - k_1 - k_2$ labels of a third kind (e.g., green). How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

Labelling and Trinomial Coefficients

How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!}$$
$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$

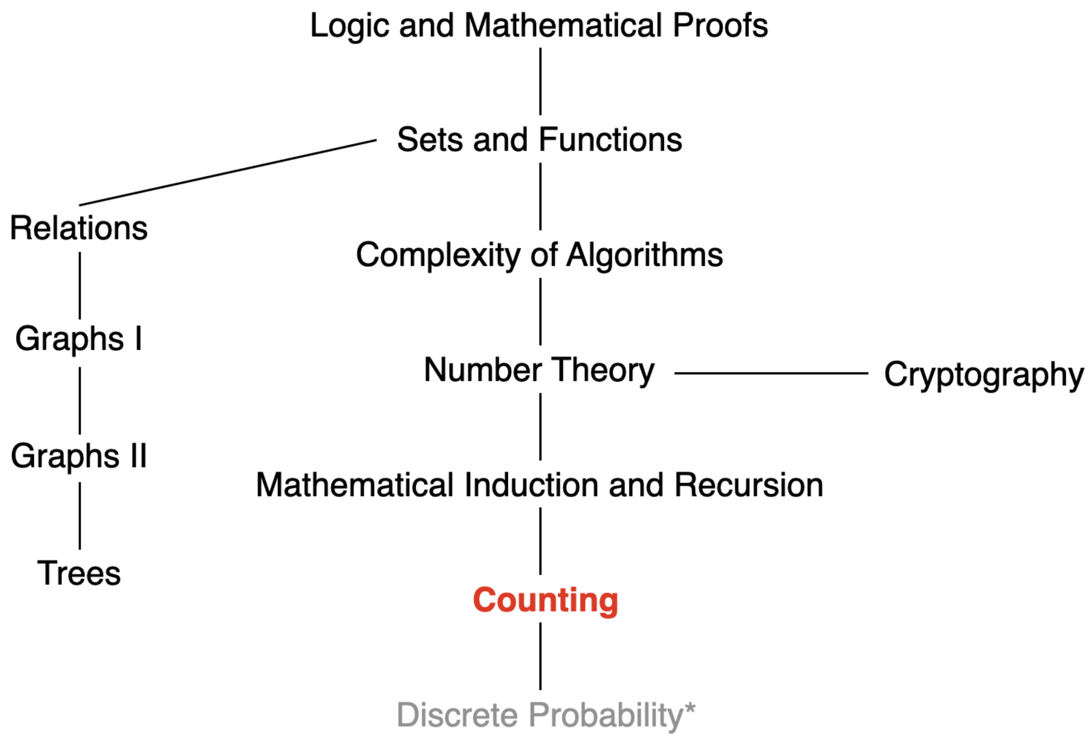
This is called a trinomial coefficient and denote it as

$$\binom{n}{k_1 \quad k_2 \quad k_3} = \frac{n!}{k_1!k_2!k_3!},$$

where $k1 + k2 + k3 = n$.

What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$?

Next Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, Generalized Permutations and Combinations, ...