Discrete Mathematics for Computer Science

Lecture 2: Propositional and Predicate Logic

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

南方科技大学 SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY | 计算机科学与工程系 Department of Computer Science and Engineering

Undergraduate Students Declaration Form

This is______________(student ID: ______________, who has enrolled in _________ course of the Department of Computer Science and Engineering. I have read and understood the regulations on courses according to "Regulations on Academic Misconduct in courses for Undergraduate Students in the SUSTech Department of Computer Science and Engineering". I promise that I will follow these regulations during the study of this course.

SUSTech Southern University of Science and Technology

Last Lecture

A proposition is a declarative statement that is either true or false.

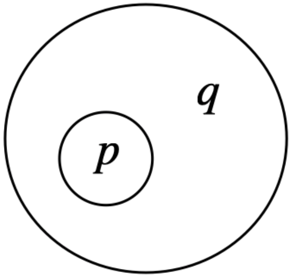
Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

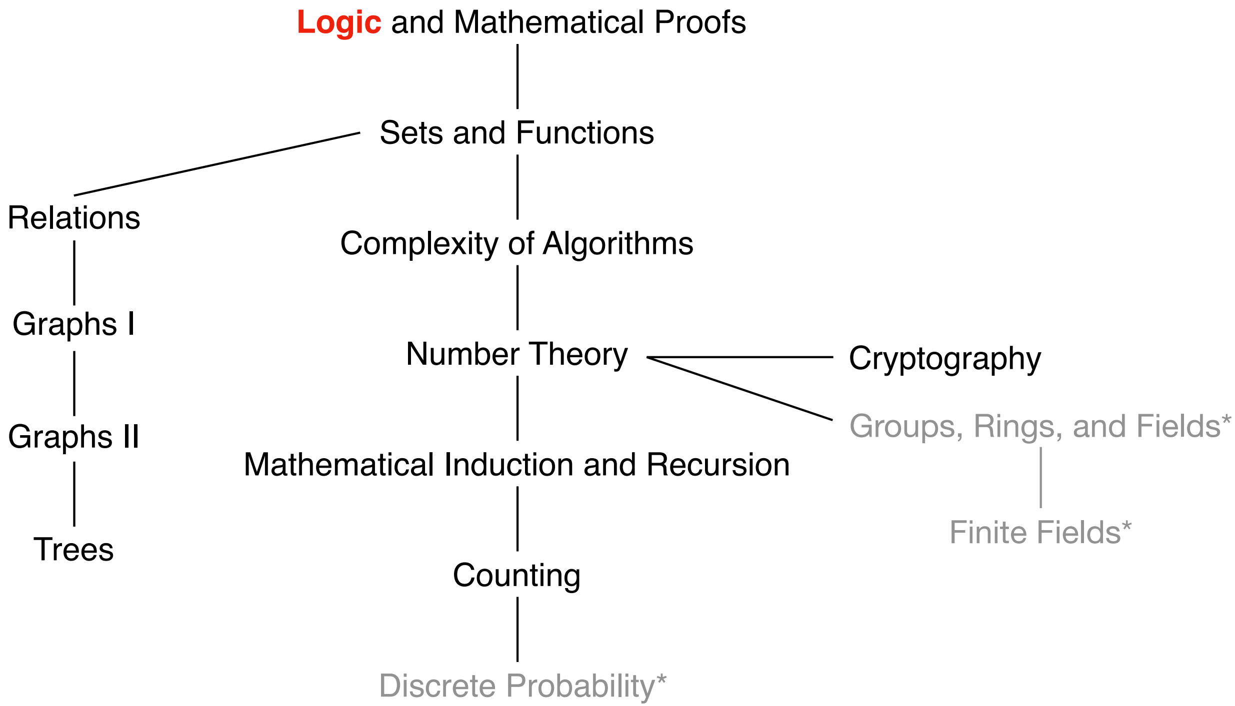
Review: Implication

$p \to q$ is read in a variety of equivalent ways:

- if p then q
- p implies q
- p is sufficient for q
- q is necessary for p
- q follows from p
- q unless $\neg p$
- p only if q



This Lecture



Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

Applications of Propositional Logic

- Translation of English sentences to remove ambiguous
  - Use combinations of atomic (elementary) propositions
  - Sentence to logical expression: determine the true value
- Inference and reasoning
  - New true propositions are inferred from existing ones
  - Used in Artificial Intelligence
- Design of logic circuit

Translation of English

If you are older than 13 or you are with your parents, then you can watch this movie.

SUSTech Southern University of Science and Technology

Translation of English

If you are older than 13 or you are with your parents, then you can watch this movie.

$\Rightarrow$ If (you are older than 13) or (you are with your parents), then (you can watch this movie).

Translation of English

If you are older than 13 or you are with your parents, then you can watch this movie.

$\Rightarrow$ If (you are older than 13) or (you are with your parents), then (you can watch this movie).

Atomic (elementary) propositions:

- p: you are older than 13
- q: you are with your parents
- r: you can watch this movie

Translation of English

If you are older than 13 or you are with your parents, then you can watch this movie.

$\Rightarrow$ If (you are older than 13) or (you are with your parents), then (you can watch this movie).

Atomic (elementary) propositions:

- p: you are older than 13
- q: you are with your parents
- r: you can watch this movie

Translation: $p \vee q \to r$

Try to Translate This Sentence

You can access the Internet from campus only if you are a computer science major or you are not a freshman.

Try to Translate This Sentence

You can access the Internet from campus only if you are a computer science major or you are not a freshman.

Atomic (elementary) propositions:

- p: You can access the Internet from campus
- q: You are a computer science major
- r: You are a freshman

Try to Translate This Sentence

You can access the Internet from campus only if you are a computer science major or you are not a freshman.

Atomic (elementary) propositions:

- p: You can access the Internet from campus
- q: You are a computer science major
- r: You are a freshman

Translation: $p \to (q \vee \neg r)$

(Recall that "p only if q" means "if p, then q".)

Inference and Reasoning

If (you are older than 13) or (you are with your parents), then (you can watch this movie).

Translation: $p \vee q \rightarrow r$

Given that p is true.

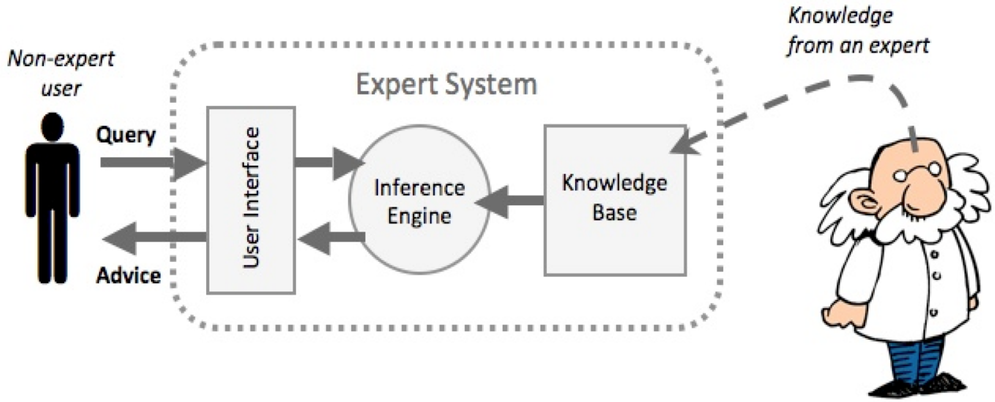
With the help of the logic, we can infer the following statement:

You can watch this movie.

Inference and Reasoning: Artificial intelligence

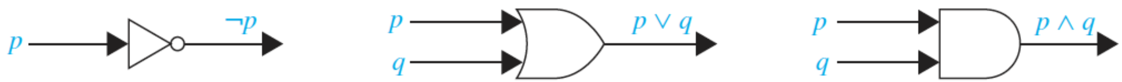
Artificial intelligence (AI): builds programs that act intelligently

- Expert System

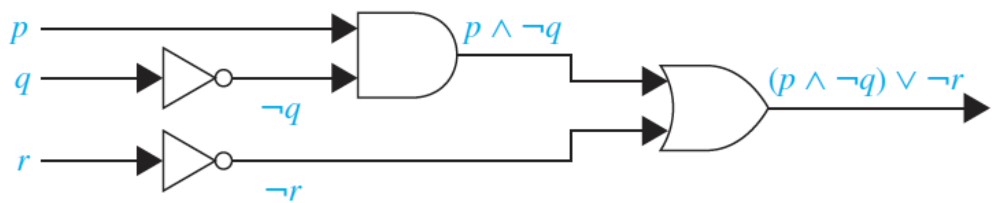


- Automated Theorem Proving
  - Automated reasoning dealing with proving mathematical theorems by computer programs

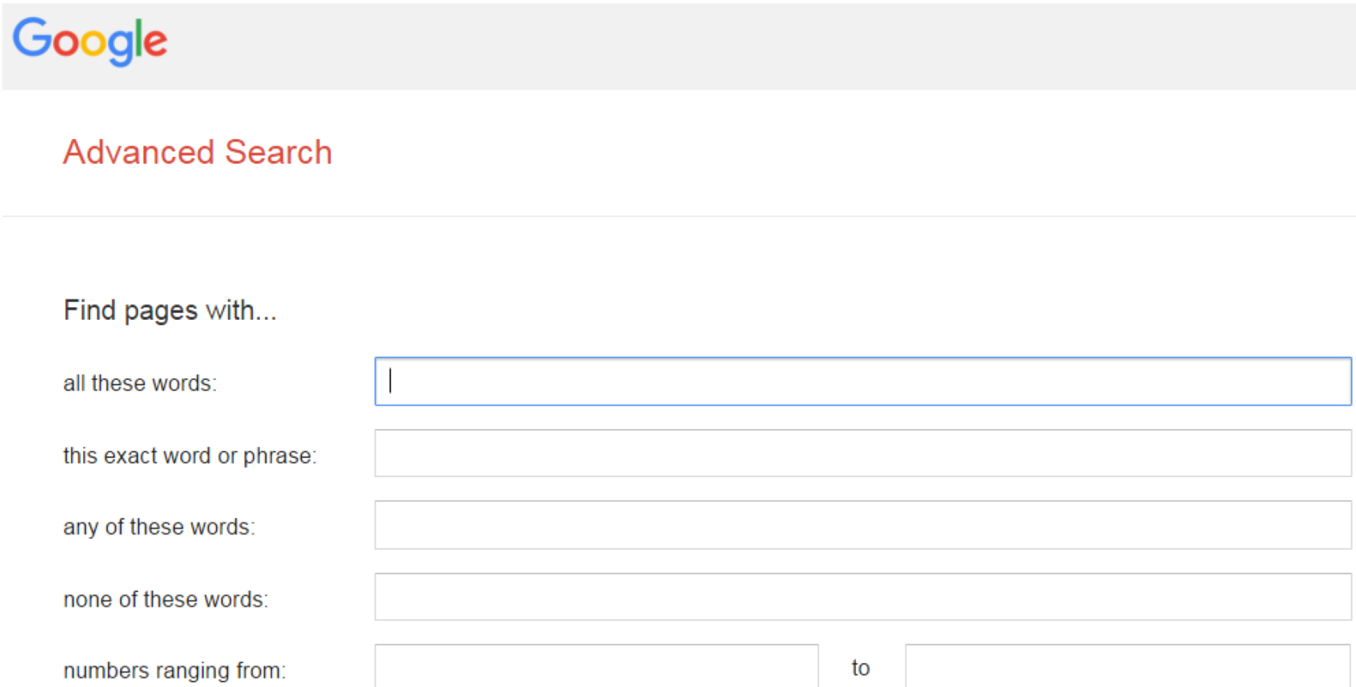
Design of Logic Circuits



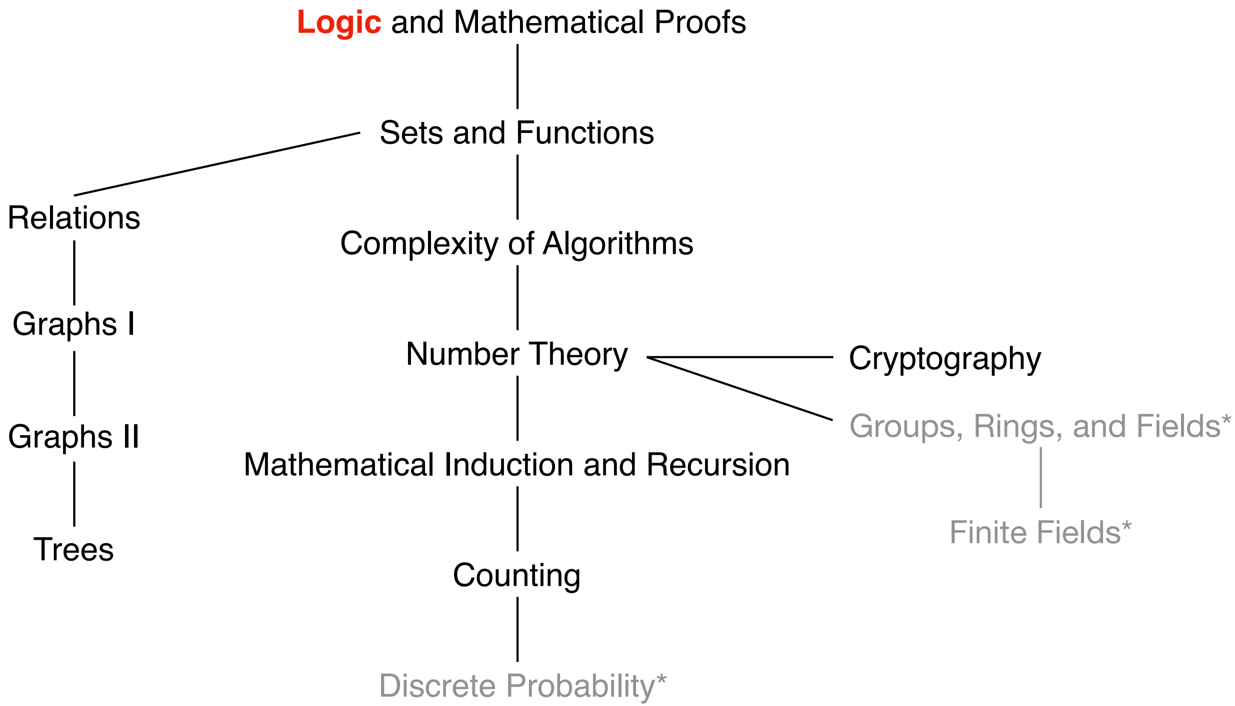
Inverter | OR gate | AND gate



Other Applications



This Lecture



Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

Tautology and Contradiction

- Tautology: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.
- Contradiction: A compound proposition that is always false.
- Contingency: A compound proposition that is neither a tautology nor a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

The compound propositions p and q are called logically equivalent, denoted by $p \equiv q$ or $p \Leftrightarrow q$, if $p \leftrightarrow q$ is a tautology.

That is, two compound propositions are equivalent if they always have the same truth value.

Logical Equivalences

The compound propositions p and q are called logically equivalent, denoted by $p \equiv q$ or $p \Leftrightarrow q$, if $p \leftrightarrow q$ is a tautology.

That is, two compound propositions are equivalent if they always have the same truth value.

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Logical Equivalences

The compound propositions p and q are called logically equivalent, denoted by $p \equiv q$ or $p \Leftrightarrow q$, if $p \leftrightarrow q$ is a tautology.

That is, two compound propositions are equivalent if they always have the same truth value.

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical Equivalences: Example

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
    ((j > q) || (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

```
(1) if (((i+j ≤ p+q) && (i ≤ p) && (j > q))
    || ((i+j ≤ p+q) && (i ≤ p)
        && (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

Consider the two pieces of codes taken from two different versions of Mergesort. Do they do the same thing?

SUSTech Southern University of Science and Technology

Logical Equivalences: Example

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
    ((j > q) || (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

```
(1) if (((i+j ≤ p+q) && (i ≤ p) && (j > q))
    || ((i+j ≤ p+q) && (i ≤ p)
       && (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

Logical Equivalences: Example

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
    ((j > q) || (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

```
(1) if (((i+j ≤ p+q) && (i ≤ p) && (j > q))
      || ((i+j ≤ p+q) && (i ≤ p)
         && (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

- $s \sim (i + j \leq p + q)$
- $t \sim (i \leq p)$
- $u \sim (j > q)$
- $v \sim (List1[i] \leq List2[j])$

Logical Equivalences: Example

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
       ((j > q) || (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

```
(1) if (((i+j ≤ p+q) && (i ≤ p) && (j > q))
      || ((i+j ≤ p+q) && (i ≤ p)
          && (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
(5)   List3[k] = List2[j]
(6)   j = j+1
(7) k = k+1
```

- $s \sim (i + j \leq p + q)$
- $t \sim (i \leq p)$
- $u \sim (j > q)$
- $v \sim (List1[i] \leq List2[j])$

Left

- $s \wedge t \wedge (u \vee v)$

Right

- $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

Let $w \sim (s \wedge v)$.

Logical Equivalences: Example

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
       ((j > q) || (List1[i] ≤ List2[j])))
(2)    List3[k] = List1[i]
(3)    i = i+1
(4) else
(5)    List3[k] = List2[j]
(6)    j = j+1
(7) k = k+1
```

```
(1) if (((i+j ≤ p+q) && (i ≤ p) && (j > q))
      || ((i+j ≤ p+q) && (i ≤ p)
          && (List1[i] ≤ List2[j])))
(2)    List3[k] = List1[i]
(3)    i = i+1
(4) else
(5)    List3[k] = List2[j]
(6)    j = j+1
(7) k = k+1
```

- $s \sim (i + j \leq p + q)$
- $t \sim (i \leq p)$
- $u \sim (j > q)$
- $v \sim (List1[i] \leq List2[j])$

Let $w \sim (s \wedge v)$.

Left

- $w \wedge (u \vee v)$

Right

- $(w \wedge u) \vee (w \wedge v)$

Logical Equivalences: Example

(1) $w \wedge (u \vee v)$

w	u	v	$u \vee v$	$w \wedge (u \vee v)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(1') $(w \wedge u) \vee (w \wedge v)$

w	u	v	$w \wedge u$	$w \wedge v$	$(w \wedge u) \vee (w \wedge v)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

17

Distributive Laws

- $w \wedge (u \vee v)$ is equivalent to $(w \wedge u) \vee (w \wedge v)$
- $w \vee (u \wedge v)$ is equivalent to $(w \vee u) \wedge (w \vee v)$

Distributive Laws

- $w \wedge (u \vee v)$ is equivalent to $(w \wedge u) \vee (w \wedge v)$
- $w \vee (u \wedge v)$ is equivalent to $(w \vee u) \wedge (w \vee v)$

Equivalent statements are important for logical reasoning since they can be substituted and can help us to:

- make a logical argument
- infer new propositions

Example: $p \to q \equiv \neg q \to \neg p$

De Morgan's Laws

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

- *Identity laws*
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- *Domination laws*
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- *Idempotent laws*
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$

Important Logical Equivalences

- *Double negation laws*
  - $\neg(\neg p) \equiv p$
- *Commutative laws*
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- *Associative laws*
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Important Logical Equivalences

- *Distributive laws*
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's laws*
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- *Others*
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

    Absorption laws
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

    Negation laws
  - $p \rightarrow q \equiv \neg p \vee q$ *Useful law*

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $(p \wedge q) \rightarrow p$ is a tautology.

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $(p \land q) \rightarrow p$ is a tautology.

Proof:
$$
\begin{aligned}
(p \land q) \rightarrow p &\equiv \neg(p \land q) \lor p && \text{Useful} \\
&\equiv (\neg p \lor \neg q) \lor p && \text{De Morgan's} \\
&\equiv (\neg q \lor \neg p) \lor p && \text{Commutative} \\
&\equiv \neg q \lor (\neg p \lor p) && \text{Associative} \\
&\equiv \neg q \lor T && \text{Negation} \\
&\equiv T && \text{Domination}
\end{aligned}
$$

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $(p \wedge q) \to p$ is a tautology.

Proof (alternatively):

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $p \to q \equiv \neg q \to \neg p$

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $p \to q \equiv \neg q \to \neg p$

Proof:

$\neg q \to \neg p \equiv \neg(\neg q) \vee (\neg p)$	Useful
$\equiv q \vee (\neg p)$	Double negation
$\equiv (\neg p) \vee q$	Communtative
$\equiv p \to q$	Useful

Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Example 1: $1^2 \geq 0$

Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Example 1: $1^2 \geq 0$

However, we also have

- $2^2 \geq 0$, $3^2 \geq 0$, ...
- $(-1)^2 \geq 0$, $(-2)^2 \geq 0$, ...

Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Example 1: $1^2 \geq 0$

However, we also have

- $2^2 \geq 0$, $3^2 \geq 0$, ...
- $(-1)^2 \geq 0$, $(-2)^2 \geq 0$, ...

What is a more natural solution to express the knowledge?

Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Example 1: $1^2 \geq 0$

However, we also have

- $2^2 \geq 0$, $3^2 \geq 0$, ...
- $(-1)^2 \geq 0$, $(-2)^2 \geq 0$, ...

What is a more natural solution to express the knowledge?

Include variables!

- Predicates: $P(x)$: $x^2 \geq 0$
- Quantifiers: For all integer x, we have $x^2 \geq 0$.

.

Limitations of Propositional Logic

Example 2:

- Every computer in Room 101 is functioning properly.
- MATH3 is a computer in Room 101.

Can we conclude "MATH3 is functioning properly" using the rules of propositional logic?

Limitations of Propositional Logic

Example 2:

- Every computer in Room 101 is functioning properly.
- MATH3 is a computer in Room 101.

Can we conclude "MATH3 is functioning properly" using the rules of propositional logic?

NO!

Limitations of Propositional Logic

Example 2:

- Every computer in Room 101 is functioning properly.
- MATH3 is a computer in Room 101.

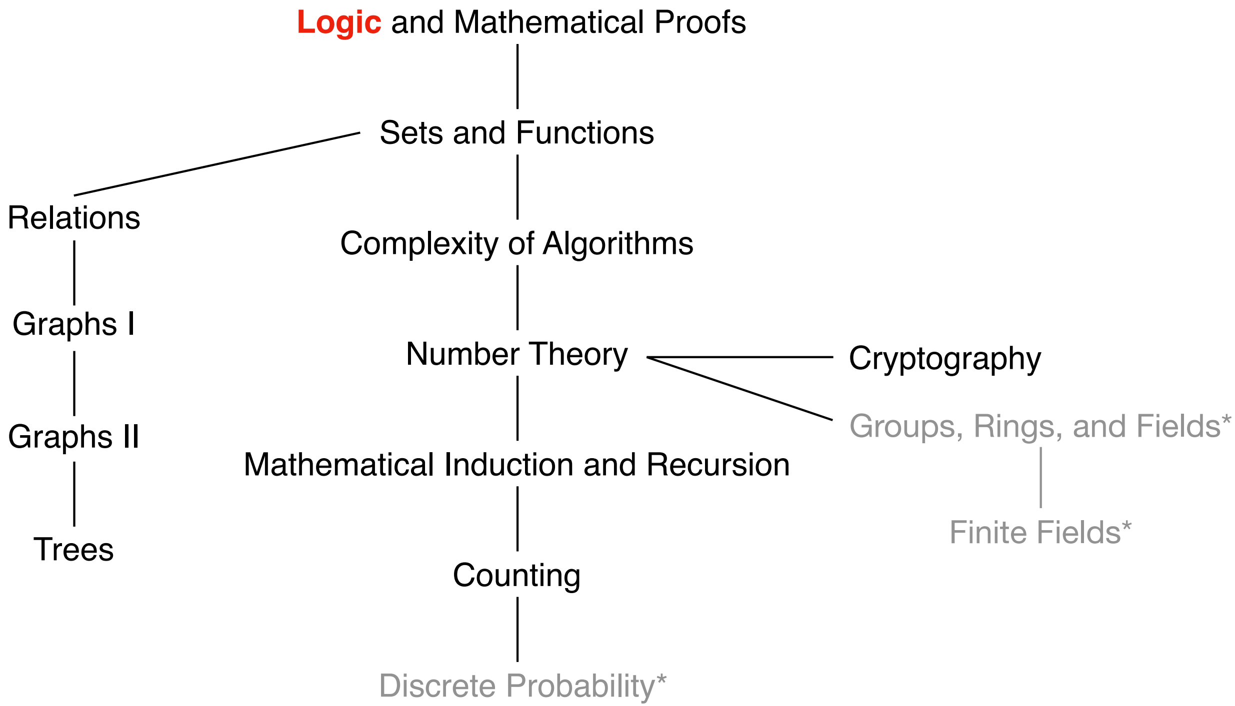
Can we conclude "MATH3 is functioning properly" using the rules of propositional logic?

NO!

Solution: Predicates and Quantifiers

- $P(x)$: Computer x is functioning properly.
- $\forall x P(x)$: $P(x)$ holds for all computer x in Room 101.
- Universal quantifier, existential quantifier

This Lecture



Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

Predicate Logic

Predicate Logic: make statements with variables

Example: x is greater than 3

- Variable x
- Predicate P: "is greater than 3"
- Propositional function $P(x)$: the truth value of P at x

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Example: $P(x)$ denote the statement "$x > 3$":

- $P(2)$ is F
- $P(4)$ is T

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Example: $P(x)$ denote the statement "$x > 3$":

- $P(2)$ is F
- $P(4)$ is T

Is $P(x)$ a proposition?

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Example: $P(x)$ denote the statement "$x > 3$":

- $P(2)$ is F
- $P(4)$ is T

Is $P(x)$ a proposition? No!

Is $P(2)$ a proposition?

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Example: $P(x)$ denote the statement "$x > 3$":

- $P(2)$ is F
- $P(4)$ is T

Is $P(x)$ a proposition? No!

Is $P(2)$ a proposition? Yes!

Predicates

- A predicate is a statement $P(x_1, x_2, ..., x_n)$ that contains n variables x_1, x_2, ... x_n. It becomes a proposition when specific values are substituted for the variables x_1, x_2, ... x_n.

Predicates

- A predicate is a statement $P(x_1, x_2, ..., x_n)$ that contains n variables x_1, x_2, ... x_n. It becomes a proposition when specific values are substituted for the variables x_1, x_2, ... x_n.
- The domain (universe) D of the predicate variables x_1, x_2, ... x_n is the set of all values that may be substituted in place of the variables.

Predicates

- A predicate is a statement $P(x_1, x_2, ..., x_n)$ that contains n variables x_1, x_2, ... x_n. It becomes a proposition when specific values are substituted for the variables x_1, x_2, ... x_n.
- The domain (universe) D of the predicate variables x_1, x_2, ... x_n is the set of all values that may be substituted in place of the variables.
- The truth set of $P(x_1, x_2, ..., x_n)$ is the set of all values of the predicate variables $(x_1, x_2, ..., x_n)$ such that the proposition $P(x_1, x_2, ..., x_n)$ is true.

Predicates: Example 1

Let $P(x)$ be the predicate "$x^2 > x$" with domain of the real numbers.

1. What are the truth values of $P(2)$ and $P(1)$?

2. What is the truth set of $P(x)$?

Predicates: Example 1

Let $P(x)$ be the predicate "$x^2 > x$" with domain of the real numbers.

1. What are the truth values of $P(2)$ and $P(1)$?

   $P(2) = \text{T}$, $P(1) = \text{F}$

2. What is the truth set of $P(x)$?

Predicates: Example 1

Let $P(x)$ be the predicate "$x^2 > x$" with domain of the real numbers.

1. What are the truth values of $P(2)$ and $P(1)$?

   $P(2) = \mathrm{T}$, $P(1) = \mathrm{F}$

2. What is the truth set of $P(x)$?

   $x > 1$ or $x < 0$

Predicates: Example 2

Let $Q(x, y)$ be the predicate "$x = y + 3$" with domain of the real numbers.

1. What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?

2. What is the truth set of $Q(x, y)$?

Predicates: Example 2

Let $Q(x, y)$ be the predicate "$x = y + 3$" with domain of the real numbers.

1. What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?

   $Q(1, 2) = \text{F}$, $Q(3, 0) = \text{T}$

2. What is the truth set of $Q(x, y)$?

Predicates: Example 2

Let $Q(x, y)$ be the predicate "$x = y + 3$" with domain of the real numbers.

1. What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?

   $Q(1, 2) = \text{F}$, $Q(3, 0) = \text{T}$
2. What is the truth set of $Q(x, y)$?

   $(a, a - 3)$ for all real numbers a

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$:
- $P(2) \wedge Q(2)$:
- $Q(x) \rightarrow P(x)$:

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$: Both 2 and 3 are primes.
- $P(2) \wedge Q(2)$: 2 is a prime or an integer.
- $Q(x) \to P(x)$: If x is an integer, then x is a prime.

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$: Both 2 and 3 are primes. (T)
- $P(2) \wedge Q(2)$: 2 is a prime or an integer. (T)
- $Q(x) \rightarrow P(x)$: If x is an integer, then x is a prime.

SUSTech Southern University of Science and Technology

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$: Both 2 and 3 are primes. (T)
- $P(2) \wedge Q(2)$: 2 is a prime or an integer. (T)
- $Q(x) \rightarrow P(x)$: If x is an integer, then x is a prime. (Not a proposition!)

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$: Both 2 and 3 are primes. (T)
- $P(2) \wedge Q(2)$: 2 is a prime or an integer. (T)
- $Q(x) \rightarrow P(x)$: If x is an integer, then x is a prime. (Not a proposition!)

Note: Researchers may use Prime(x) to refer to "x is a prime", Integer(x) to refer to "x is an integer", and others. It is only a way of notation. If you use such notations, please define it clearly beforehand.

Quantified Statements

Propositional function $P(x) \overset{\text{specify } x}{\Longrightarrow}$ Proposition

Quantified Statements

Propositional function $P(x) \overset{\text{specify } x}{\Longrightarrow}$ Proposition

An alternative way to obtain proposition:

Propositional function $P(x) \overset{\text{for all/some } x \text{ in domain}}{\Longrightarrow}$ Proposition

Quantified Statements

Propositional function $P(x) \overset{\text{specify } x}{\Longrightarrow}$ Proposition

An alternative way to obtain proposition:

Propositional function $P(x) \overset{\text{for all/some } x \text{ in domain}}{\Longrightarrow}$ Proposition

Predicate logic permits quantified statement where variables are substituted for statements about the group of objects.

Quantified Statements

Two types of quantified statements:

- Universal quantifier $\forall x P(x)$

- Existential quantifier $\exists x P(x)$

Quantified Statements

Two types of quantified statements:

- Universal quantifier $\forall x P(x)$
  - All CS-major graduates have to pass CS201.
  - (This is true for all CS-major graduates.)
- Existential quantifier $\exists x P(x)$
  - Some CS-major students graduate with honor.
  - (This is true for some students.)

Universal Quantifier

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$."

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers?

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers? True

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers? True
- All real numbers?

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers? True
- All real numbers? False

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers? True
- All real numbers? False

The domain must always be specified!

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

Question 1: Is $\forall x P(x)$ a proposition?

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$P(x)$ for all values of x in the domain.

Question 1: Is $\forall x P(x)$ a proposition?

Yes. Its truth value?

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

Question 1: Is $\forall x P(x)$ a proposition?

Yes. Its truth value?

- True if $P(x)$ is true for all x in the domain.
- False if there is an x in the domain such that $P(x)$ is false. (counterexample)

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

Question 1: Is $\forall x P(x)$ a proposition?

Yes. Its truth value?

- True if $P(x)$ is true for all x in the domain.
- False if there is an x in the domain such that $P(x)$ is false. (counterexample)

Question 2: What is the truth value of $\forall x P(x)$ when the domain is empty?

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

Question 1: Is $\forall x P(x)$ a proposition?

Yes. Its truth value?

- True if $P(x)$ is true for all x in the domain.
- False if there is an x in the domain such that $P(x)$ is false. (counterexample)

Question 2: What is the truth value of $\forall x P(x)$ when the domain is empty?

Proposition $\forall x P(x)$ is true for every propositional function $P(x)$.

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Example: $P(x)$: $x > 0$

What is the truth value of $\exists x P(x)$?

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Example: $P(x)$: $x > 0$

What is the truth value of $\exists x P(x)$?

- What if assuming the domain to be all real numbers? True

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Example: $P(x)$: $x > 0$

What is the truth value of $\exists x P(x)$?

- What if assuming the domain to be all real numbers? True
- What if all negative real numbers? False

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Example: $P(x)$: $x > 0$

What is the truth value of $\exists x P(x)$?

- What if assuming the domain to be all real numbers? True
- What if all negative real numbers? False

The domain must always be specified!

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Yes. Its truth value?

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Yes. Its truth value?

- True if there is an x in the domain such that $P(x)$ is true. (an example)
- False if $P(x)$ is false for all x in the domain.

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Yes. Its truth value?

- True if there is an x in the domain such that $P(x)$ is true. (an example)
- False if $P(x)$ is false for all x in the domain.

Question 2: What is the truth value of $\exists x P(x)$ when the domain is empty?

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Yes. Its truth value?

- True if there is an x in the domain such that $P(x)$ is true. (an example)
- False if $P(x)$ is false for all x in the domain.

Question 2: What is the truth value of $\exists x P(x)$ when the domain is empty?

Proposition $\exists x P(x)$ is false for every propositional function $P(x)$.

Summary of Quantified Statements

Statement	When true?	When false?
$\forall x\ P(x)$	P(x) true for all x	There is an x where P(x) is false.
$\exists x\ P(x)$	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose that the elements in the domain can be enumerated as $x_1, x_2, \ldots, x_n$ then:

- $\forall x P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ is true.
- $\exists x P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$ is true.

Properties of Quantifiers

The truth values of $\forall x P(x)$ and $\exists x P(x)$ depend on both the propositional function $P(x)$ and the domain.

Example: $P(x)$: $x < 2$

- domain: the positive integers

  $\forall x P(x)$: , $\exists x P(x)$:

- domain: the negative integers

  $\forall x P(x)$: , $\exists x P(x)$:

- domain: $\{3, 4, 5\}$

  $\forall x P(x)$: , $\exists x P(x)$:

Properties of Quantifiers

The truth values of $\forall x P(x)$ and $\exists x P(x)$ depend on both the propositional function $P(x)$ and the domain.

Example: $P(x)$: $x < 2$

- domain: the positive integers

  $\forall x P(x)$: F, $\exists x P(x)$: T
- domain: the negative integers

  $\forall x P(x)$: T, $\exists x P(x)$: T
- domain: $\{3, 4, 5\}$

  $\forall x P(x)$: F, $\exists x P(x)$: F

Precedence of Proposition and Quantifiers

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

- $\neg p \wedge q$ means $(\neg p) \wedge q$ rather than $\neg(p \wedge q)$
- $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$

Translation with Quantifiers

Every student in this class has studied algebra.

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 1:

- $A(x)$: "x has studied algebra".
- Domain: the students in the class
- $\forall x A(x)$

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
-

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
- $\forall x(C(x) \rightarrow A(x))$

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
- $\forall x(C(x) \rightarrow A(x))$

Note: Implication $p \rightarrow q$.

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
- $\forall x(C(x) \rightarrow A(x))$

How about $\forall x(C(x) \wedge A(x))$?

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
- $\forall x(C(x) \rightarrow A(x))$

How about $\forall x(C(x) \wedge A(x))$? All students are in this class and has studied algebra.

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 3:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- $S(x)$: "x is a student"
- Domain: all people
- $\forall x(S(x) \land C(x) \to A(x))$

Translation with Quantifiers

Some student in this class has visited Mexico.

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 1:

- $M(x)$: "x has visited Mexico".
- Domain: the students in the class
- $\exists x M(x)$

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 2:

- $M(x)$: "x has visited Mexico".
- $C(x)$: "x is a student in this class."
- Domain: all people
-

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 2:

- $M(x)$: "x has visited Mexico".
- $C(x)$: "x is a student in this class."
- Domain: all people
- $\exists x(C(x) \wedge M(x))$

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 2:

- $M(x)$: "x has visited Mexico".
- $C(x)$: "x is a student in this class."
- Domain: all people
- $\exists x(C(x) \land M(x))$

How about $\exists x(C(x) \rightarrow A(x))$?

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 2:

- $M(x)$: "x has visited Mexico".
- $C(x)$: "x is a student in this class."
- Domain: all people
- $\exists x(C(x) \land M(x))$

How about $\exists x(C(x) \rightarrow A(x))$? No! This is even true when there is some people not in the class.

Negation of Quantifiers

Every student in this class has taken a course in calculus.

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\forall x P(x)$

Negation of Quantifiers

Every student in this class has taken a course in calculus.

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\forall x P(x)$

The negation of this statement: It is not the case that every student in this class has taken a course in calculus.

- $\neg(\forall x P(x))$
- $\exists x(\neg P(x))$

Negation of Quantifiers

Every student in this class has taken a course in calculus.

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\forall x P(x)$

The negation of this statement: It is not the case that every student in this class has taken a course in calculus.

- $\neg(\forall x P(x))$
- $\exists x(\neg P(x))$

$$\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$$

Negation of Quantifiers

There is a student in this class who has taken a course in calculus."

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\exists x P(x)$

Negation of Quantifiers

There is a student in this class who has taken a course in calculus."

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\exists x P(x)$

The negation of this statement: It is not the case that there is a student in this class has taken a course in calculus.

- $\neg(\exists x P(x))$
- $\forall x(\neg P(x))$

Negation of Quantifiers

There is a student in this class who has taken a course in calculus."

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\exists x P(x)$

The negation of this statement: It is not the case that there is a student in this class has taken a course in calculus.

- $\neg(\exists x P(x))$
- $\forall x(\neg P(x))$

$$\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$$

Negation of Quantified Statements

A.k.a, De Morgan laws for quantifiers

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x\ P(x)$	$\forall x\ \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x\ P(x)$	$\exists x\ \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Next Lecture