CS201 · Discrete Mathematics

Assignment 5

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1 Q1

Suppose a relation R on a set A is antisymmetric, then

$$\forall a, b \in A, ((a, b) \in R \land (b, a) \in R) \implies (a = b)$$

Let $R' \subseteq R$, then for $(x, y) \in R'$ and $(y, x) \in R'$, we have also $(x, y) \in R$ and $(y, x) \in R$, by the definition of antisymmetric relation R, we have $x = y$. Therefore, the subset R' is also antisymmetric.

2 Q2

Since R is symmetric, we have if $(a, b) \in R$, then $(b, a) \in R$. Since $R^* = \bigcup_{k=1}^{\infty} R^k = R \cup R^2 \cup R^3 \cup \ldots$. Thus, $R \subseteq R^*$, i.e., if $(a, b) \in R \subseteq R^*$, then $(b, a) \in R \subseteq R^*$. Therefore, R^* is also symmetric.

3 Q3

Since R is reflexive, then $\forall a \in A$, we have $(a, a) \in R$. Thus, $(a, a) \in R^2 = R \circ R$. Suppose $(a, b) \in R$, then for $R^2 = R \circ R$, $(a, a) \circ (a, b) = (a, b)$ is also in R^2. Therefore, $R \subseteq R^2$.

4 Q4

Suppose R is symmetric while $\overline{R}$ is not. Then for an arbitrary element $(a, b) \in \overline{R}$, it must exist at least an element (b, a) such that $(b, a) \notin \overline{R}$. Since $(b, a) \notin \overline{R}$, then it must be in R. Since R is symmetric, then $(a, b) \in R$. However, (a, b) is already in $\overline{R}$, which contradicts the assumption. Therefore, $\overline{R}$ is also symmetric.

5 Q5

1. It is reflexive.
   First we need to prove that the sum of the same positive integers is no larger than their product.

   - For only a value a, $a \leq a$ is always true.
   - For two values a, b, we can assume that $a \leq b$ (if not, just exchange their role). Then $a \cdot b = \underbrace{b + b + \ldots + b}_{a} \geq \underbrace{a + b + \ldots + b}_{a} \geq a + b$
   - Suppose it is true for k values $a, b, c, \ldots, n$, i.e., $\underbrace{a \cdot b \cdot c \cdot \ldots \cdot n}_{k} \geq \underbrace{a + b + c + \ldots + n}_{k}$. Then for $k + 1$ values, we have

     $$\underbrace{a \cdot b \cdot c \cdot \ldots \cdot n}_{k} \cdot m \geq (\underbrace{a + b + c + \ldots + n}_{k}) \cdot m$$

     Let $p = (\underbrace{a + b + c + \ldots + n}_{k})$, then it turns to prove $p \cdot m \geq p + m$. Since p and m are still arbitrary, it is still true.

Therefore, the sum of the same positive integers is no larger than their product.
Since the prime factors of a positive integer m are larger than 1, then the sum of them is less or equal than the product of them. So $m \preceq m$.

2. It is not.
$21 \preceq 17$ since $3+7=10 \leq 17$, and $17 \preceq 21$ since $17 \leq 3 \cdot 7 = 21$. However, $21 \neq 17$. Therefore, it is not antisymmetric.

3. It is not.
$17 \preceq 35$ since $17 \leq 5 \cdot 7 = 35$, and $35 \preceq 13$ since $5+7=12 \leq 13$. However, $17 \npreceq 13$ since $17 > 13$. Therefore, it is not transitive.

6 Q6

Suppose relation $R = \{(A,C),(B,A),(C,D),(C,A)\}$ on the set $S = \{A,B,C,D\}$. Then we have

$$R^2 = \{(A,C),(B,A),(C,D),(C,A)\} \qquad R^3 = \{(B,D),(B,C)\}$$

And thus,

$$\begin{aligned} R \cup R^2 \cup R^3 &= \{(A,C),(B,A),(C,D),(C,A),(A,C),(C,D),(C,A),(B,D)\} \\ R \cup R^2 &= \{(A,C),(B,A),(C,D),(C,A),(A,C),(C,D),(C,A)\} \end{aligned}$$

Therefore, $R^* = R \cup R^2 \cup R^3 \neq R \cup R^2$.

7 Q7

Suppose we have a set S, whose elements are all people. All the discussions below are on the set S.

1. It is equivalence relation.
   - Reflexive:
     Since one and himself have the same sign of the zodiac, then $\forall x \in S, (x,x) \in S$. So it is reflexive.
   - Symmetric:
     Suppose x and y have the same sign of the zodiac, then obviously, y and x also have the same sign of the zodiac, due to commutative law of and.
   - Transitive:
     Suppose x and y have the same sign of the zodiac, and so do y and z. Since every sign of zodiac is distinct, x and z must have the same sign of the zodiac.

2. It is equivalence relation.
   - Reflexive:
     Since one and himself were born in the same year, then $\forall x \in S, (x,x) \in S$. So it is reflexive.
   - Symmetric:
     Suppose x and y were born in the same year, then obviously, y and x were also born in the same year, due to commutative law of and.
   - Transitive:
     Suppose x and y were born in the same year, and so do y and z. Since every year is distinct, x and z must born in the same year.

3. It is not equivalence relation.
   - Not transitive:
     Suppose x and y have been in the same city, and so do y and z. Since it can be that x and y have been in the city A while y and z have been in the city B, with different city, x and z can have not been in the same city.

8 Q8

Suppose $x, y, z \in \mathbb{Q}$.

- Reflexive:
  Since $x - x = 0 \in \mathbb{Q}$, then $(x, x) \in \{(x, y)|x - y \in \mathbb{Q}\}$. So the relation is reflexive.
- Symmetric:
  Suppose $x - y = k \in \mathbb{Q}$, then we have $y - x = -k \in \mathbb{Q}$. So the relation is symmetric.
- Transitive:
  Suppose $x - y = m \in \mathbb{Q}$ and $y - z = n \in \mathbb{Q}$, then we have $x - z = (x - y) - (y - z) = m - n \in \mathbb{Q}$. So the relation is transitive.

Therefore, the relation $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation.
Since the difference of two rational numbers is also rational but the difference of a rational number and an irrational number is irrational. Therefore,

$$[1] = \mathbb{Q}, \quad \left[\frac{1}{2}\right] = \mathbb{Q}, \quad [\pi] = \{\pi + q|q \in \mathbb{Q}\}$$

9 Q9

First we have

- Reflexive:
  Since for all functions f from $\mathbf{N}^+$ to $\mathbf{R}$, we always have $f \leq Cf$ for some number C, i.e., $f = O(f)$. So $\propto$ is reflexive.
- Transitive:
  Suppose we have $f \propto g$ and $g \propto h$, i.e., $\exists C_1, C_2, f \leq C_1 g, g \leq C_2 h$. Then we have $f \leq C_1 C_2 h$. Let $C' = C_1 C_2$, then $f \leq C' h$, i.e., $f \propto h$. So $\propto$ is transitive.

So we just need to focus on symmetry or antisymmetry.

1. $\propto$ is not an equivalence relation.
   Suppose $\propto$ is symmetric, then we have $f \propto g$ and $g \propto f$, i.e., $f = O(g)$ and $g = O(f)$. So f and g must satisfy that $f = \Theta(g)$. This contradicts that f is randomly chosen from the functions from $\mathbf{N}^+$ to $\mathbf{R}$.
2. $\propto$ is not a partial ordering.
   Suppose $f : x \to x$ and $g : x \to 2x$, then $f = O(g)$ and $g = O(f)$, but $f \neq g$.
3. $\propto$ is not a total ordering.
   Since it is even not a partial ordering, it must not be a total ordering.

10 Q10

- Reflexive:
  For $R \in \mathrm{R}(S)$, it always has $R \subseteq R$. So it is reflexive.
- Anti-symmetric:
  Suppose we have $R_1, R_2 \in \mathrm{R}(S)$ such that $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$, then by the property of subset, we have $R_1 = R_2$. So it is anti-symmetric.
- Transitive:
  Suppose we have $R_1, R_2, R_3 \in \mathrm{R}(S)$ such that $R_1 \subseteq R_2$ and $R_2 \subseteq R_3$, then by the transitivity of subset, we have $R_1 \subseteq R_2 \subseteq R_3$. So it is transitive.

Therefore, $\propto$ is a partial ordering. Thus, $(\mathrm{R}(S), \propto)$ is a poset.

11 Q11

1. We can construct $R = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, ...\}$ with infinite but countable set elements. The latter set element has one more element than the former set element and the augmenting element is 1 greater than the largest element in former set element. Since the former set element is always the subset of the latter one, this R has no maximal element.
2. We can construct R just as the reverse of R in (a). Let $R = \{..., \{0, 1, 2\}, \{0, 1\}, \{0\}\}$. The former set element has one more element than the latter set element and the augmenting element is 1 greater than the largest element in latter set element. Since the latter set element is always the subset of the former one, this R has no minimal element.
3. We can construct $R = \{..., S_{k-1}, S_k, S_{k+1}, ...\}$. In each set element S_k, there are infinite elements in it. And $S_{k+1} = \{elementsinS_kandanelementnotinS_k\}$, $S_{k-1} = \{elementsinS_kexceptanelement\}$. Since $S_{k-1} \subseteq S_k \subseteq S_{k+1}$, this R has neither minimal nor maximal elements.

12 Q12

1. maximal element: n
2. minimal elements: a, b, c
3. greatest element: n
4. no least element.
5. upper bounds of $\{a, b, c\}$: l, n
6. least upper bound of $\{a, b, c\}$: l
7. no lower bounds of $\{f, g, h\}$.
8. no greatest lower bound of $\{f, g, h\}$.