Discrete Mathematics for Computer Science

Lecture 9: Cryptography

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Modular Arithmetic in CS

Modular arithmetic and congruencies are used in CS:

- Pseudorandom number generators
- Hash functions
- Cryptography

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Pseudorandom Number Generators

Linear congruential method

We choose four numbers:

- the modulus m
- multiplier a
- increment c
- seed x_0

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Pseudorandom Number Generators

Linear congruential method

We choose four numbers:

- the modulus m
- multiplier a
- increment c
- seed x_0

We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \leq x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

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Pseudorandom Number Generators

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Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

- $x_1 = 7*3+4 \bmod 9 = 25 \bmod 9 = 7$
- $x_2 = 53 \bmod 9 = 8$
- $x_3 = 60 \bmod 9 = 6$
- $x_4 = 46 \bmod 9 = 1$
- $x_5 = 11 \bmod 9 = 2$
- $x_6 = 18 \bmod 9 = 0$
- ....

This sequence contains nine different numbers before repeating.

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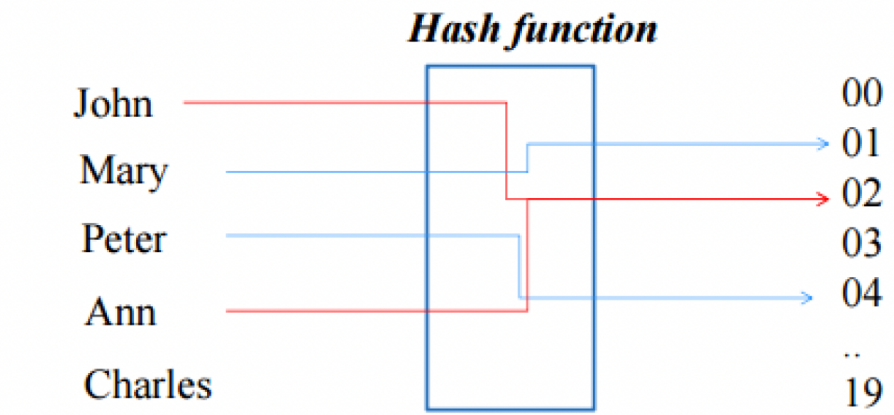
Hash Functions

A hash function is an algorithm that maps data of arbitrary length to data of a fixed length. The values returned by a hash function are called hash values or hash codes.

Example:



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Hash Functions

Problem: Given a large collection of records, how can we store and find a record quickly?

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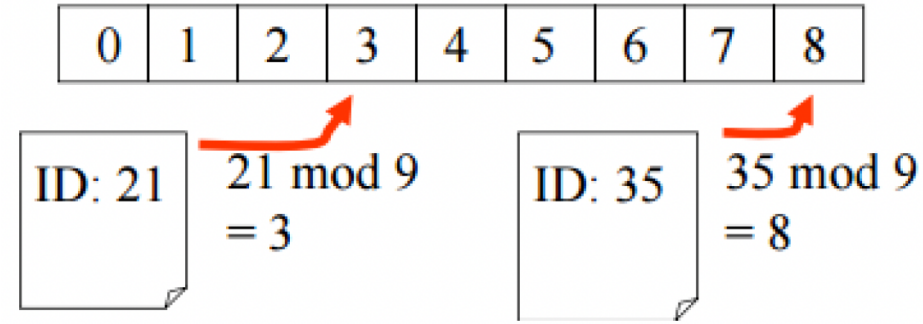

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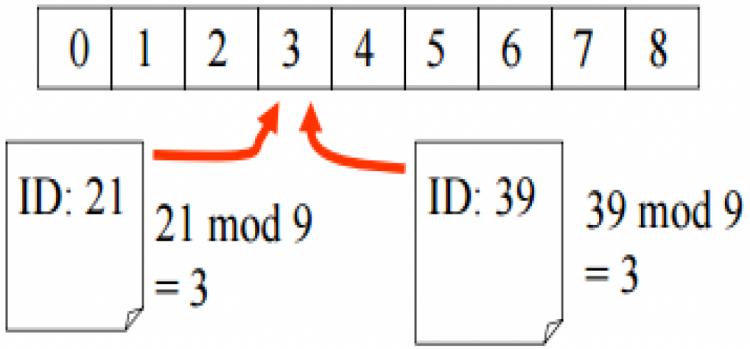
Hash Functions

Two records mapped to the same location



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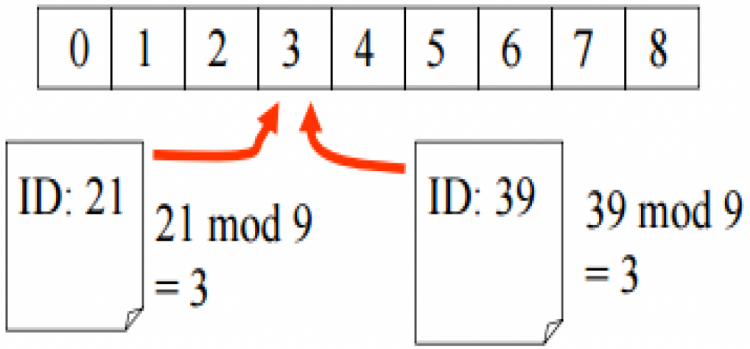

How to address this?

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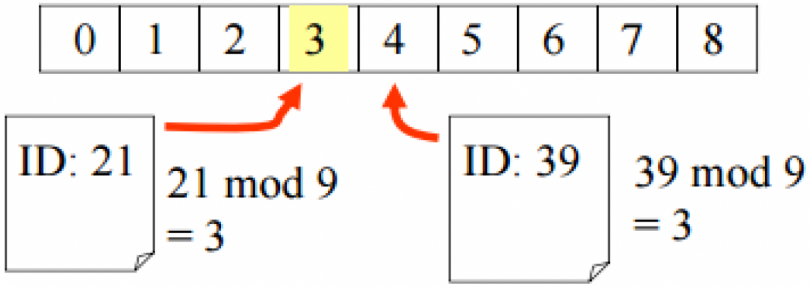
Hash Functions

One way is to assign the first free location following the occupied memory location assigned by the hashing function.

try

$$h_0(k) = k \bmod n$$
$$h_1(k) = (k+1) \bmod n$$
$$\ldots$$
$$h_m(k) = (k+m) \bmod n$$



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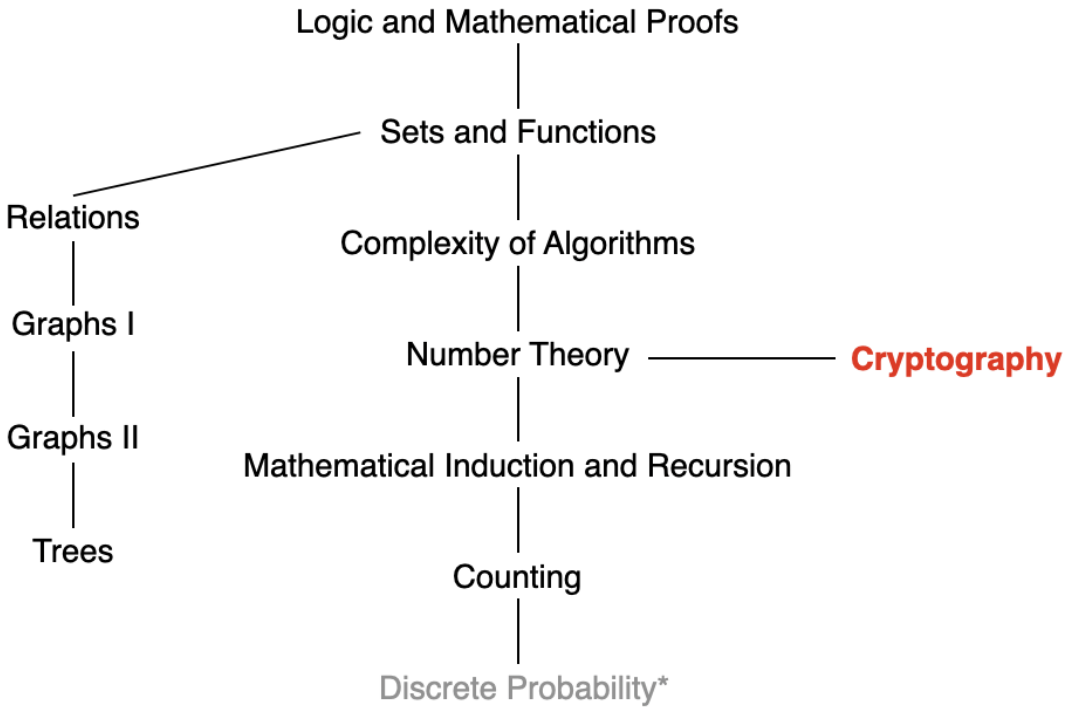

Cryptograhy: classical cryptography, RAS cryptosystem, ...

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Cryptography

History of almost 4000 years (from 1900 B.C.)

Cryptography = kryptos + graphos

- kryptos: secret
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One-sentence definition: "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called adversaries." – Ronald L. Rivest

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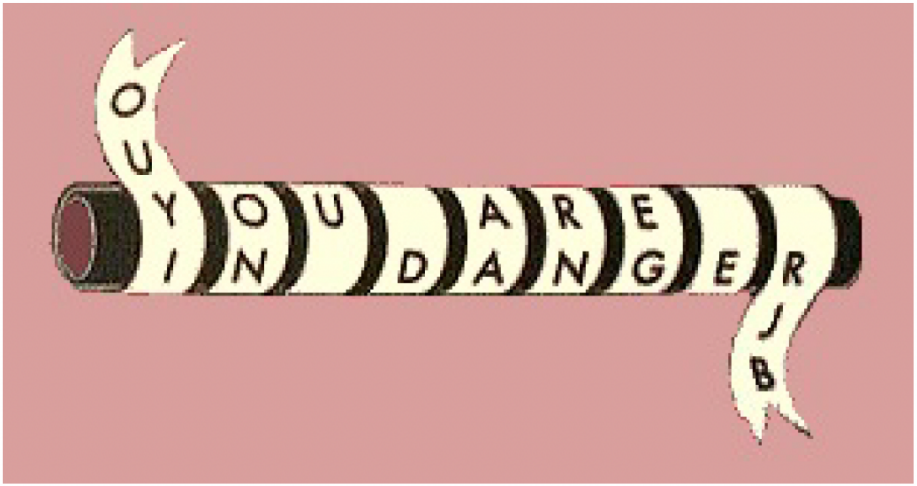
Some Examples

In 405 B.C., the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



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Some Examples

The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during World War I.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

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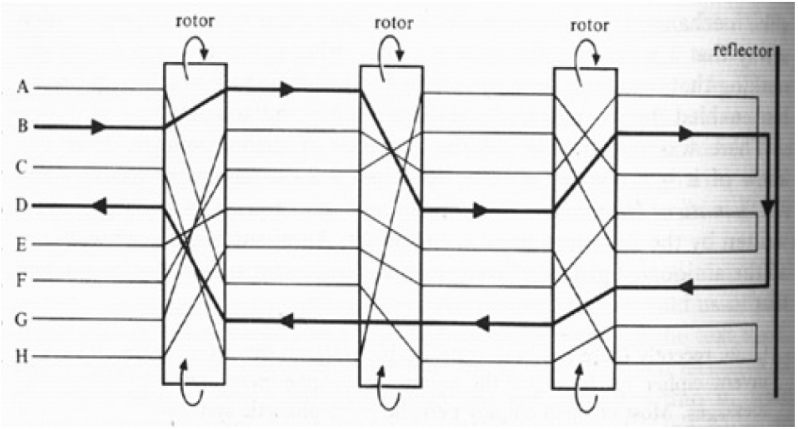
Some Examples

Enigma, Germany coding machine in World War II.



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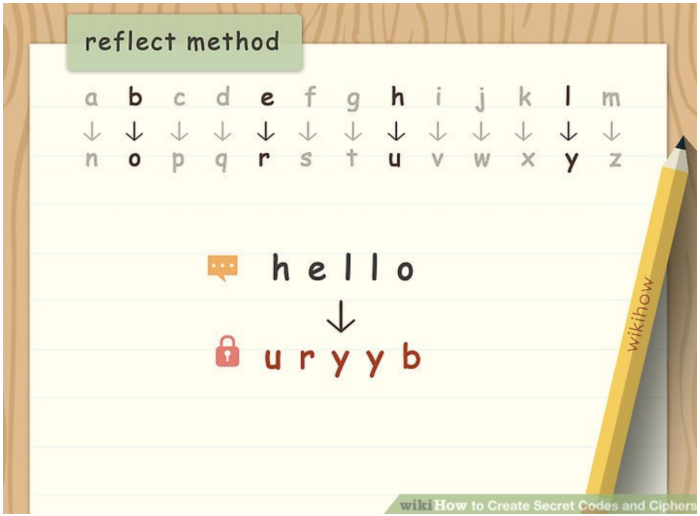

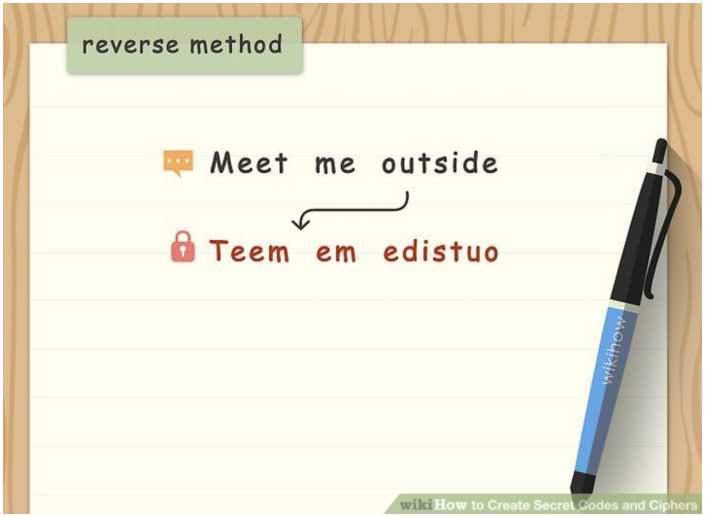

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Shift Ciphers

Shift Ciphers: Make messages secret by shifting each letter several letters forward in the alphabet.

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Encryption:

- Assign each letter an integer $p \in Z_{26} = \{0, 1, ..., 25\}$ based on the location of the letter in the alphabet.
- Replace p with $f(p)$:

$$f(p) = (p + k) \bmod 26.$$

- Maps $f(p)$ back to the alphabet.

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Shift Ciphers

Example: What is the secret message produced from the message "MEET YOU IN THE PARK" using the Shift cipher with $k = 3$?

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Solution: First replace the letters in the message with numbers. This produces

12 4 4 19 24 14 20 8 13 19 7 4 15 0 17 10.

Now replace each of these numbers p by $f(p) = (p + 3) \bmod 26$. This gives

15 7 7 22 1 17 23 11 16 22 10 7 18 3 20 13.

Translating this back to letters produces the encrypted message "PHHW BRX LQ WKH SDUN."

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- Replace p with $f^{-1}(p)$:

$$f^{-1}(p) = (p-k) \bmod 26.$$

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Suppose that $c = (ap + b) \bmod 26$ with $\gcd(a, 26) = 1$. To decrypt we need to show how to express p in terms of c. That is, we solve the congruence for p:

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$$p \equiv \bar{a}(c - b) \pmod{26}.$$

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Crytanalysis

The process of recovering plaintext from ciphertext without knowledge of both the encryption method and the key is known as crytanalysis or breaking codes.

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Solution 1: Try each 26 possible shifts.

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How to break messages that were encrypted using a shift cipher?

Solution 1: Try each 26 possible shifts.

Solution 2: Try different values of k based on the frequency of letters in the ciphertext. The nine most common letters in English text: E: 13%, T: 9%, A: 8%, O: 8%, I: 7%, N: 7%, S: 7%, H: 6%, and R: 6%.

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Private Key Cryptosystem

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Any problems?

- Two people who want to communicate securely need to securely exchange this key.
- New key is used for each communication session between two parties.

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Public Key Cryptography

In **public key cryptosystems**, knowing how to send an encrypted message does not help decrypt messages.

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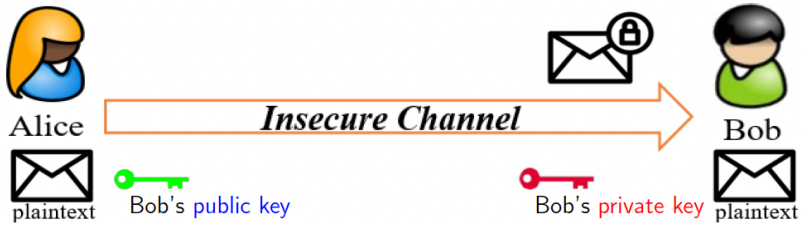

- Public key is known to the public.
- Private key is kept secret: only the intended recipient of a message can decrypt it.

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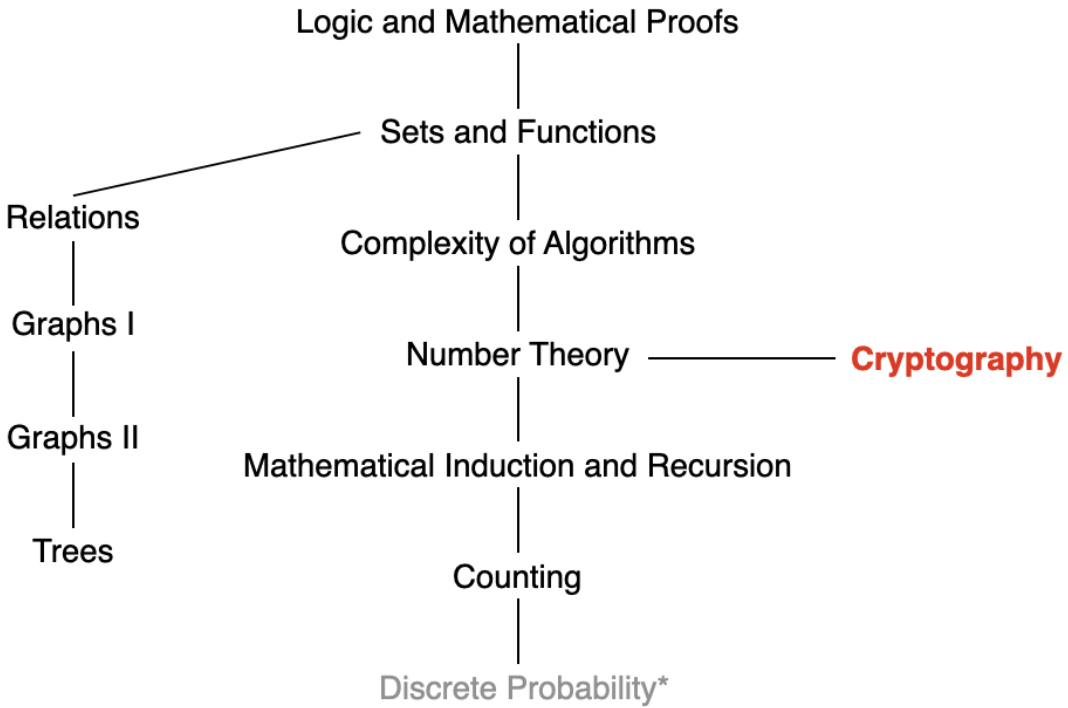

Cryptograhy: classical cryptography, RAS cryptosystem, ...

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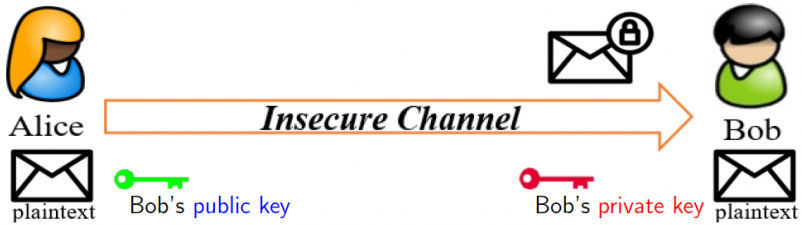
Overview

- RSA as Public Key System
  - Only target recipient can decrypt the message:



- RSA as Digital Signature
- Diffie-Hellman Key Exchange Protocol

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RAS Cryptosystem

Rivest-Shamir-Adleman **2002 Turing Award**

2002	Ronald L. Rivest, Adi Shamir and Leonard M. Adleman	For their ingenious contribution for making **public-key cryptography** useful in practice.

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RSA encryption: $C = M^e \bmod n$

RSA decryption: $M = C^d \bmod n$

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RSA Encryption

1. Translate a plaintext message into integers, each with two digits, e.g, A is translated into 00, B into 01, . . . , and Z into 25.

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3. For each block, transform it into a ciphertext block:

$$C = M^e \bmod n$$

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RSA Encryption: Example

Encrypt the message "STOP" with key $(n = 2537, e = 13)$. Note that $2537 = 43 \cdot 59$, where $p = 43$ and $q = 59$ are primes, and $\gcd(e, (p-1)(q-1)) = 1$.

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Solution:

1. Translate into integers: 18191415
2. Divide this into blocks of 4 digits (because $2525 < 2537 < 252525$): 1819 1415
3. Encrypt each block using the mapping

$$C = M^{13} \bmod 2537.$$

We have $1819^{13} \bmod 2537 = 2081$ and $1415^{13} \bmod 2537 = 2182$. The encrypted message is 2081 2182.

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RSA Decryption

For each block, transform the ciphertext into plaintext message:

$$M = C^d \bmod n$$

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Example: What is the decrypted message of 0981 0461 with $e = 13$, $p = 43$, $q = 59$?

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Solution: Recall that $ed \equiv 1 \pmod{(p-1)(q-1)}$. Thus, $d = 937$ is an inverse of 13 modulo $42 \cdot 58 = 2436$.

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For each block, transform it into plaintext message:

$$M = C^{937} \bmod 2537.$$

Since $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$, the plaintext message is 0704 1115, which is "HELP".

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$$de = 1 + k(p-1)(q-1).$$

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Assuming that $\gcd(M, p) = \gcd(M, q) = 1$, we have $M^{p-1} \equiv 1 \pmod{p}$ and $M^{q-1} \equiv 1 \pmod{q}$. (see Theorem 3 in Section 4.4)

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This basically implies that

$$M = C^d \bmod n$$

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RSA as Public Key System

Pick two large primes p and q. Let $n = pq$. **Encryption key** (n, e) and **decryption key** (n, d) are selected such that

(1) $\gcd(e, (p-1)(q-1)) = 1$

(2) $ed \equiv 1 \pmod{(p-1)(q-1)}$

RSA encryption: $C = M^e \bmod n$;
RSA decryption: $M = C^d \bmod n$.

RSA as a Public Key System

- Public key: (n, e)
- Private key: d
- p, q must be kept secret!

Why is the RSA cryptosystem suitable for public key cryptography?

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- When we know p and q, we can quickly find an inverse d.
- However, no method is known to decrypt messages that is not based on finding a factorization of n.
  - Factorization is believed to be a difficult problem.
  - The most efficient factorization methods known (as of 2010) require billions of years to factor 400-digit integers.

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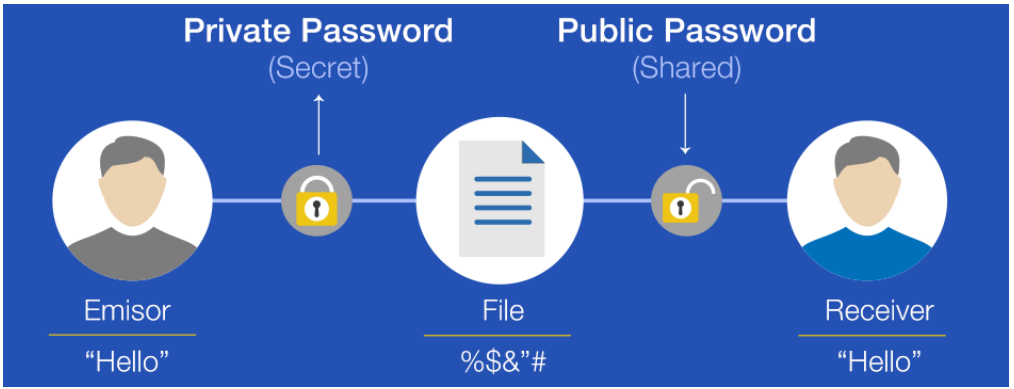
Overview

- RSA as Public Key System
- RSA as Digital Signature
  - The recipient of the message knows that it came from the person they think it came from.



- Diffie-Hellman Key Exchange Protocol

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RSA as Digital Signature

Alice's RSA public key is (n, e) and her private key is d.

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Alice can send her message to as many people as she wants and by signing it in this way, every recipient can be sure it came from Alice.

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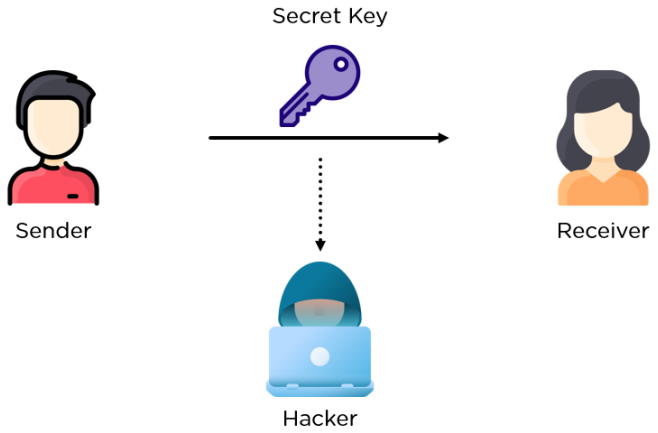
Overview

- RSA as a Public Key System
- RSA as Digital Signature
- Diffie-Hellman Key Exchange Protocol
  - Exchange a secret key over an insecure communications channel



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Diffie-Hellman Key Exchange Protocol

Two parties exchange a secret key over an insecure communications channel without having shared any information in the past.

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When we compute the powers of 2 in Z_{11}, we obtain $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 5$, $2^5 = 10$, $2^6 = 9$, $2^7 = 7$, $2^8 = 3$, $2^9 = 6$, $2^{10} = 1$.
Because every element of Z_{11} is a power of 2, 2 is a primitive root of 11.

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Diffie-Hellman Key Exchange Protocol

Suppose that Alice and Bob want to share a common key. Consider Z_p.

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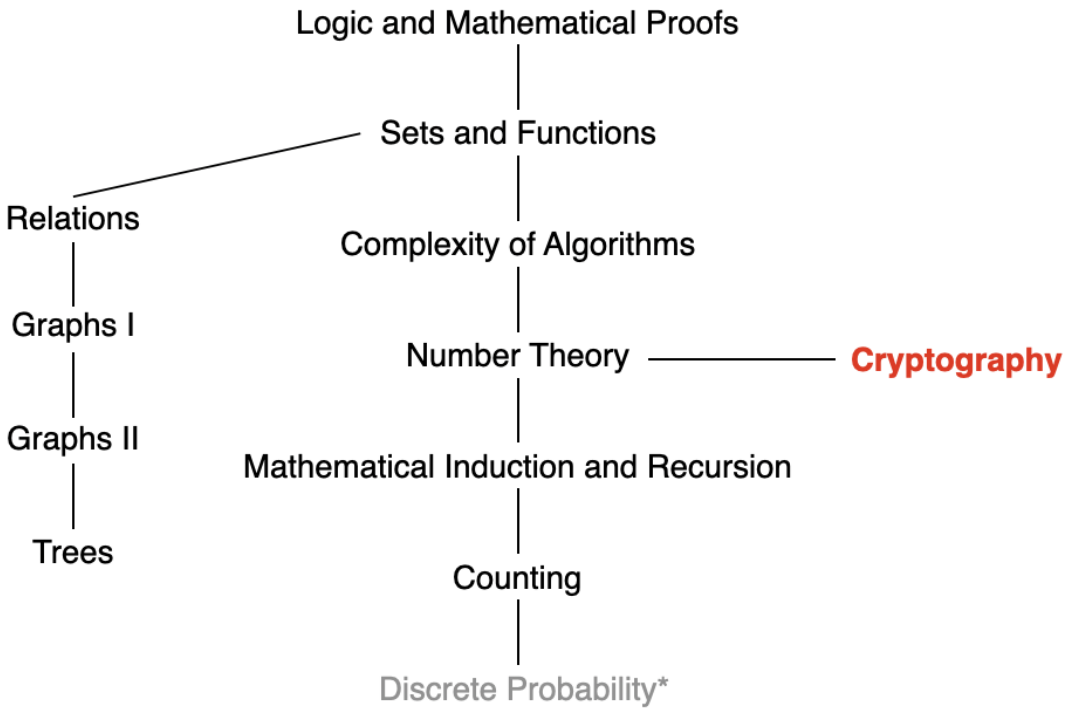
- Public information: p, a, a^{k_1} **mod** p, and a^{k_2} **mod** p
- Secret: k_1, k_2, $(a^{k_2})^{k_1}$ **mod** $p = (a^{k_1})^{k_2}$ **mod** p

Note that it is very hard to determine k_1 with a, p, and a^{k_1} **mod** p.

This Lecture



Cryptograhy: classical cryptography, RAS cryptosystem, blockchain, ...

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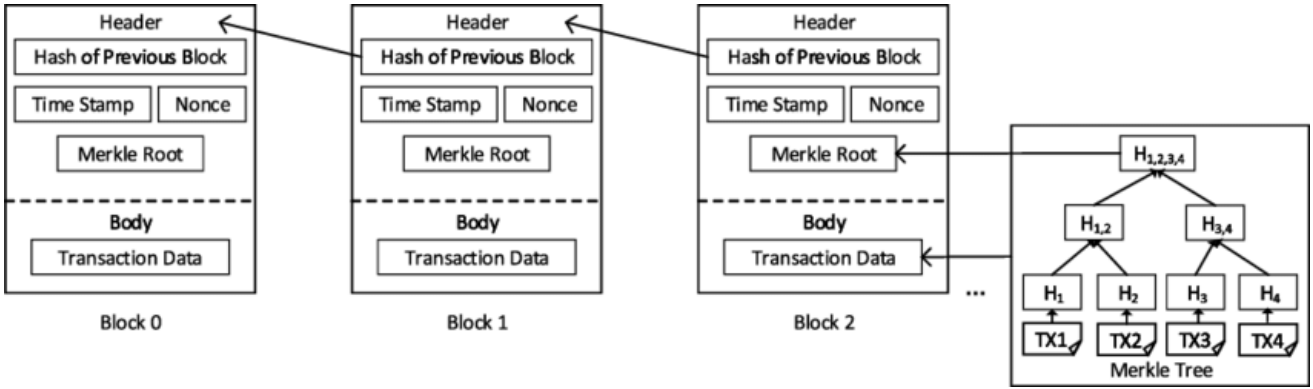
Blockchain

A blockchain is a decentralized, distributed, and oftentimes public, digital ledger consisting of records called blocks that are used to record transactions (or data) across many computers.



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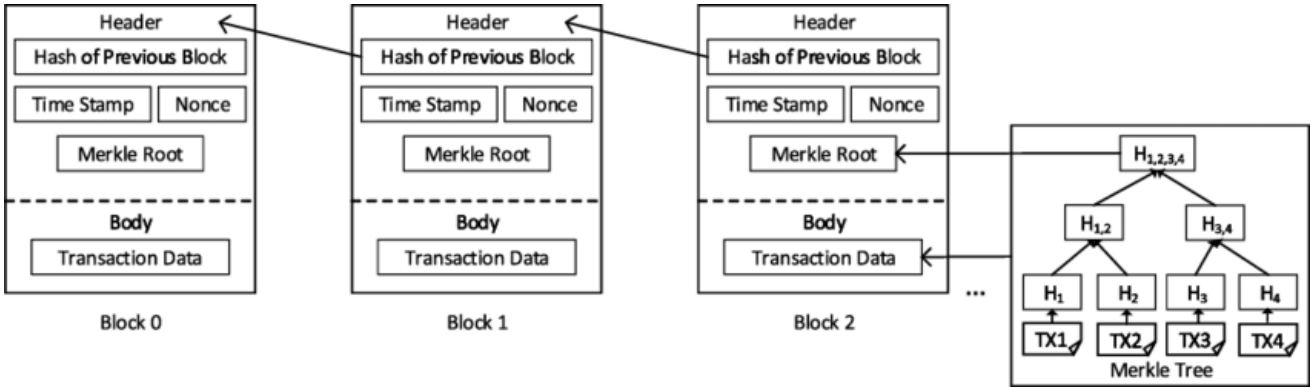

Any involved block cannot be altered retroactively, without the alteration of all subsequent blocks.

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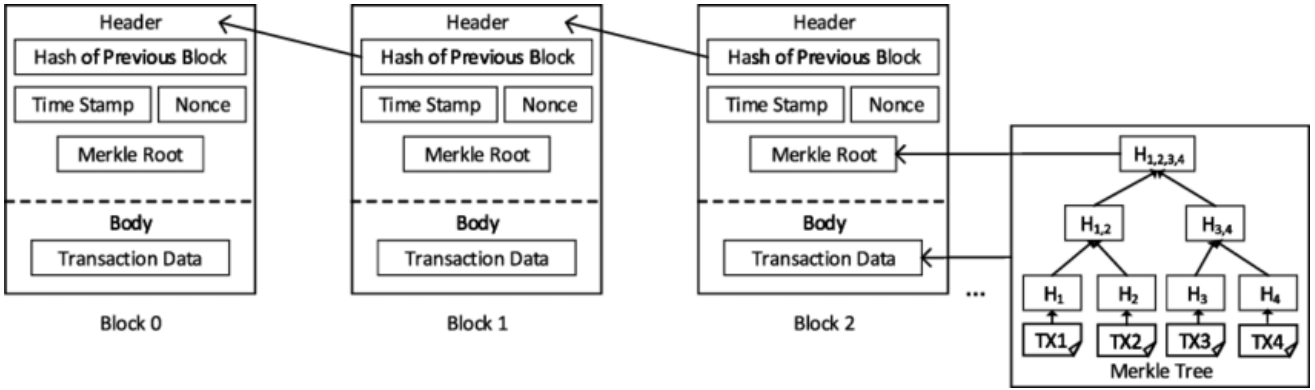
What is Mining?

Given a set of transactions, generate a new block:



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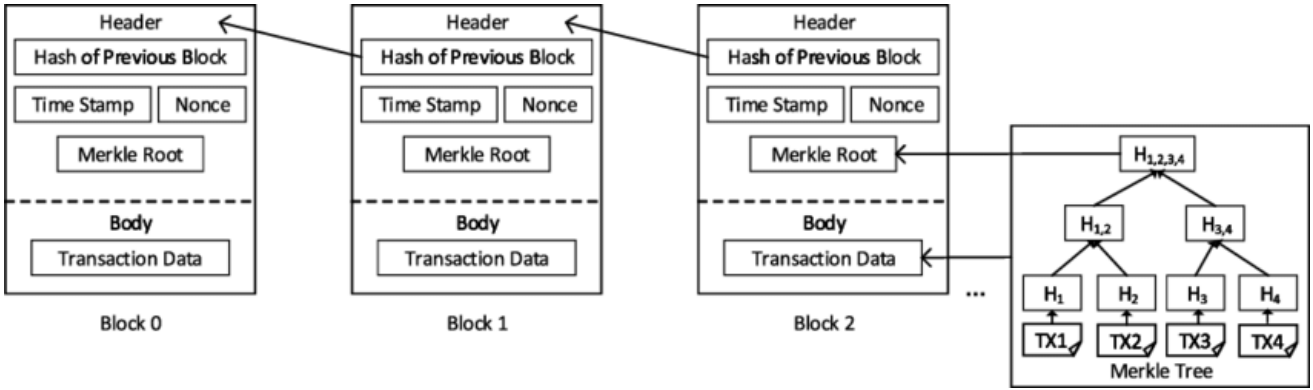

Proof of Work: Take the current block's header, guess the nonce such that the hash of the header (SHA-256) smaller than a target value.

The target can be changed to adjust the difficulty of mining.

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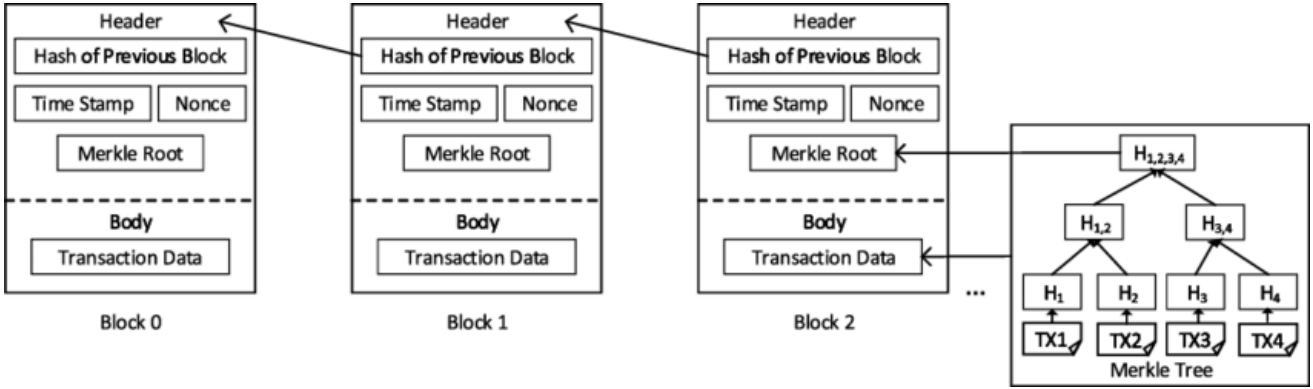
How to Make Sure the Data are not Tampered?



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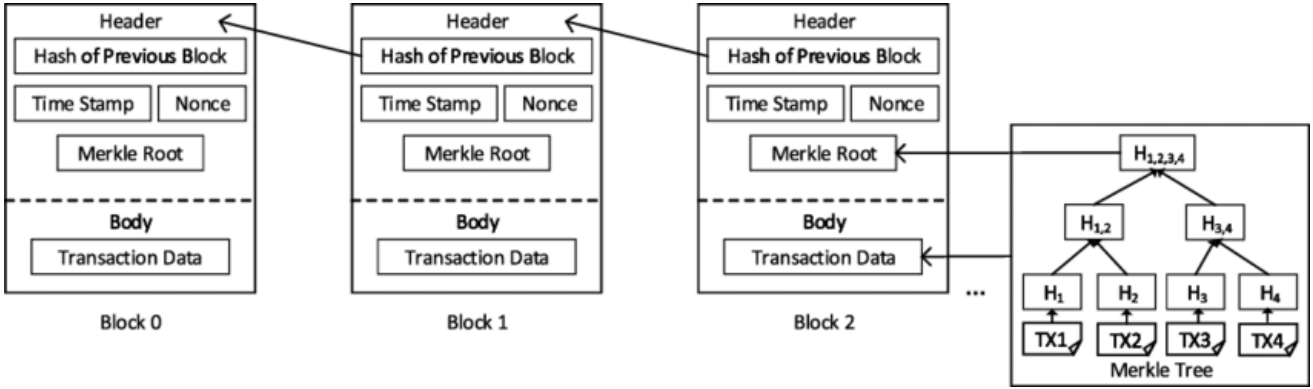

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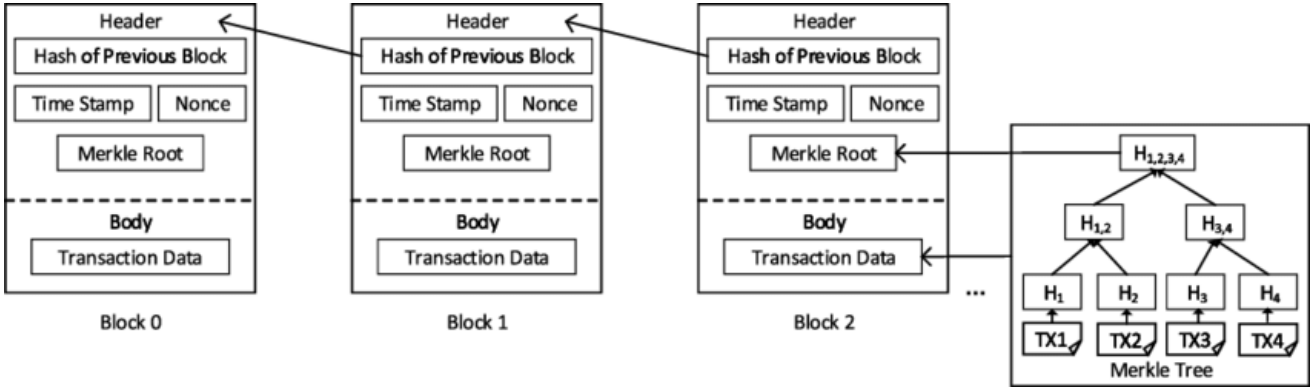

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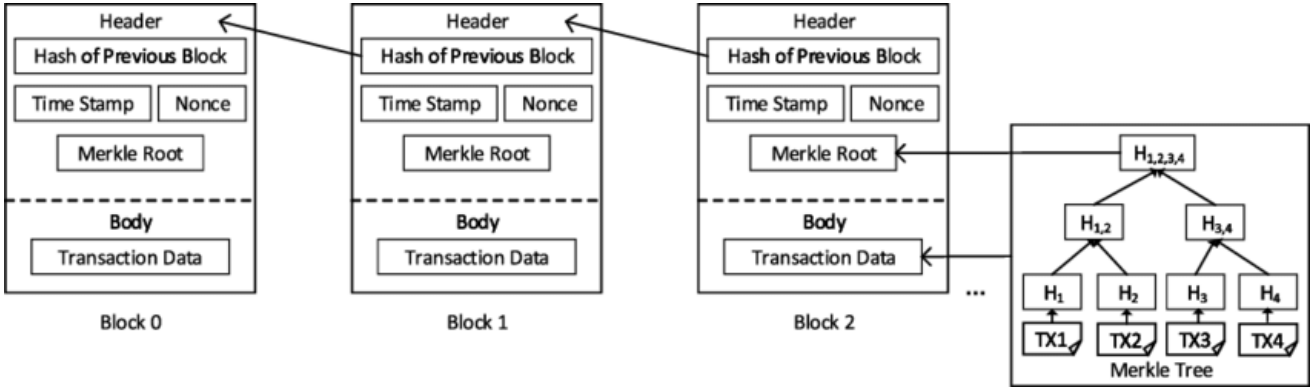

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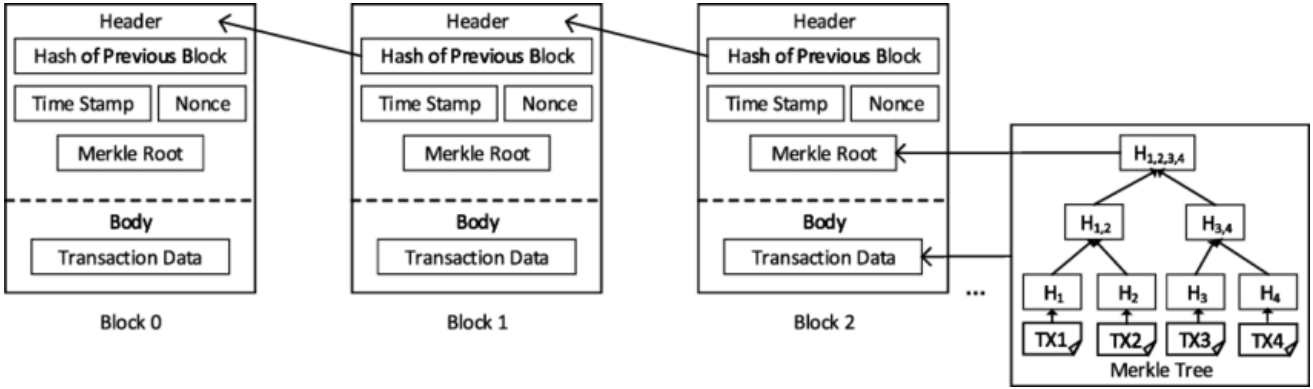

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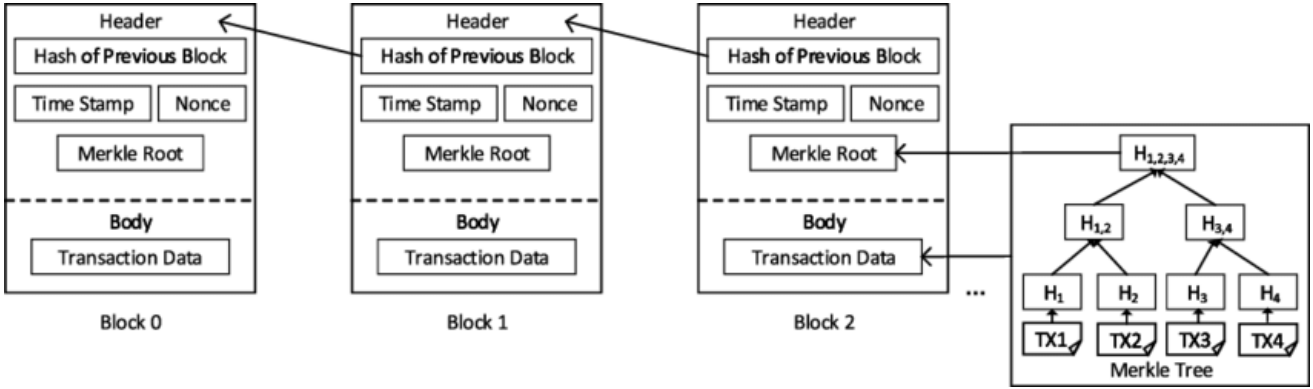

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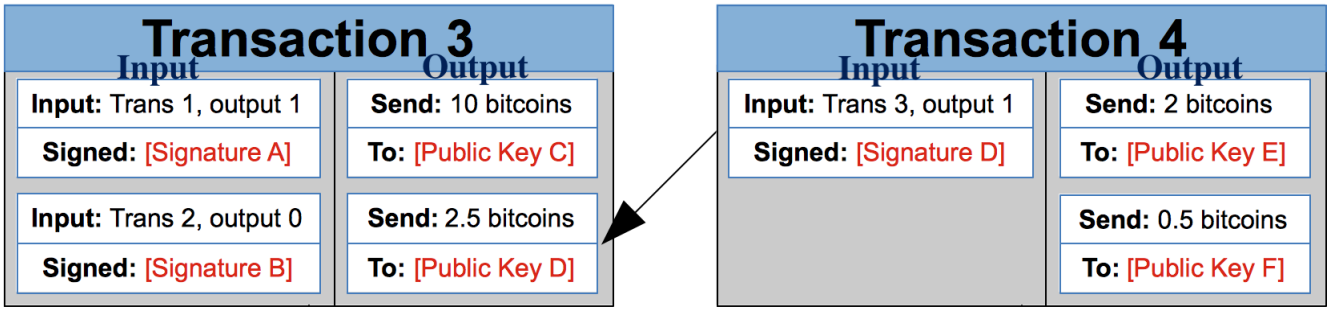
How to Verify if You have the Money You Claimed?



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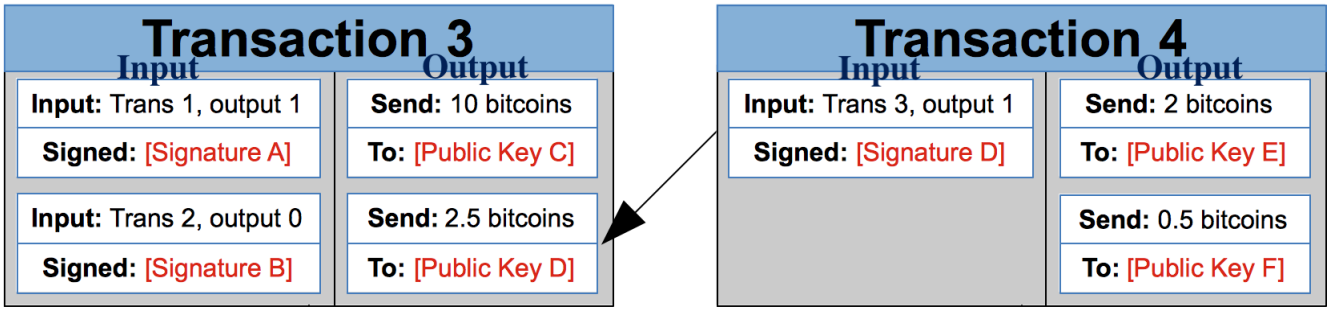

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A digital signature (ECDSA) can be used to unlock outputs, because it shows that we know the private key of an address.

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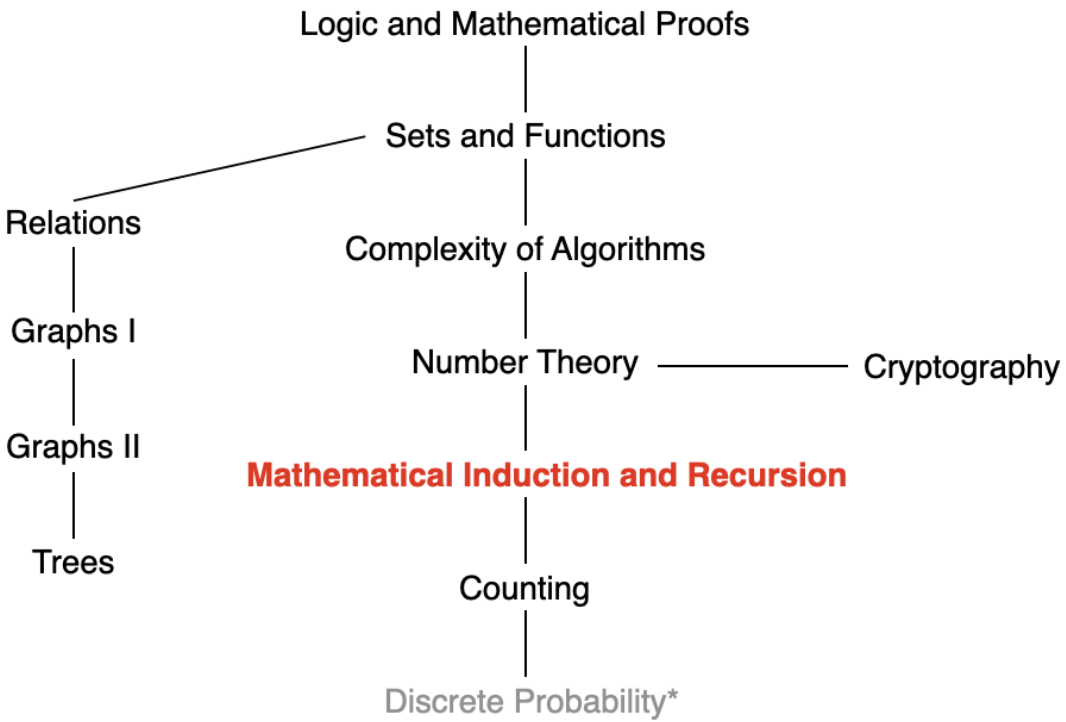
Next Lecture



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