# Discrete Mathematics for Computer Science

## Lecture 20: Graph

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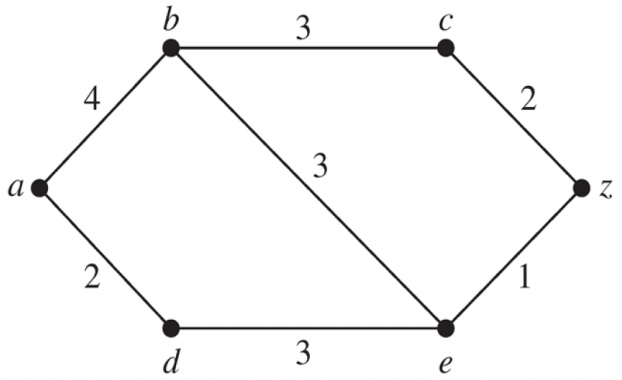
# Shortest Path Problems




What is the length of a shortest path between $a$ and $z$?

# Dijkstra's Algorithm

$S$: a distinguished set of vertices
$L(v)$: the length of a shortest path from $a$ to $v$ that contains vertices only in $S$

(i) Set $L(a) = 0$ and $L(v) = \infty$ for all $v$, $S = \emptyset$

(ii) While $z \notin S$

$u :=$ a vertex not in $S$ with $L(u)$ minimal

$S := S \cup \{u\}$

For all vertices $v$ not in $S$

$L(v) := \min\{L(u) + w(u, v), L(v)\}$

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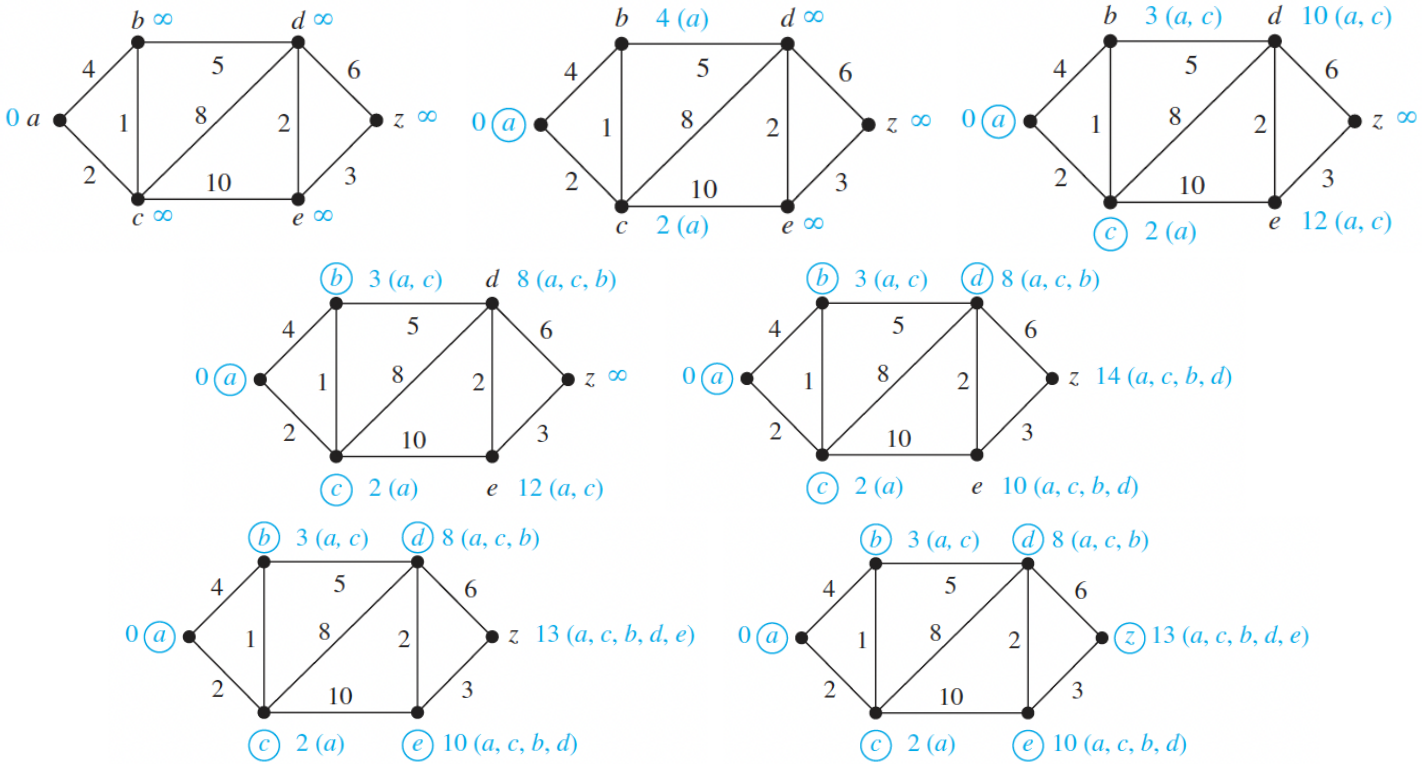
# Dijkstra's Algorithm

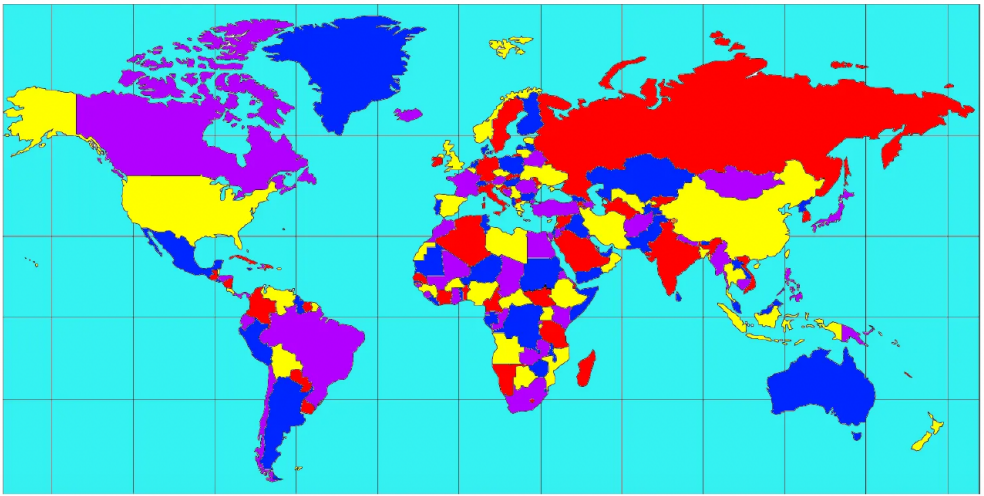



$S = \{a, c, b, d, e, z\}$

$L(a) = 0,\ L(b) = 3,\ L(c) = 2,\ L(d) = 8,\ L(e) = 10,\ L(z) = 13$

# Graph Coloring

**Theorem (Four Color Theorem)**: The chromatic number of a planar graph is no greater than four.



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# Graph Coloring

**Theorem (Six Color Theorem)**: The chromatic number of a planar graph is no greater than six.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

- Basic step: For one single vertex, pick an arbitrary color.
- Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.
- Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

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# Graph Coloring

**Theorem (Five Color Theorem)**: The chromatic number of a planar graph is no greater than five.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

- Basic step: For one single vertex, pick an arbitrary color.
- Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.
- Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in.

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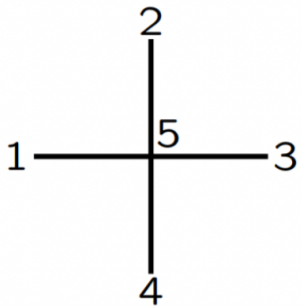
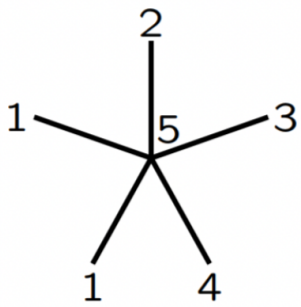
# Graph Coloring

**Theorem (Five Color Theorem)**: The chromatic number of a planar graph is no greater than five.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

Case 1: If the vertex has degree less than 5, or if it has degree 5 and only $\leq 4$ colors are used for vertices connected to it, we can pick an available color for it.

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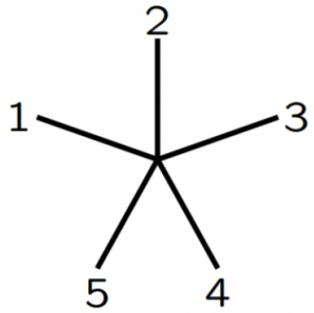
# Graph Coloring

**Theorem (Five Color Theorem)**: The chromatic number of a planar graph is no greater than five.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

Case 2: If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).

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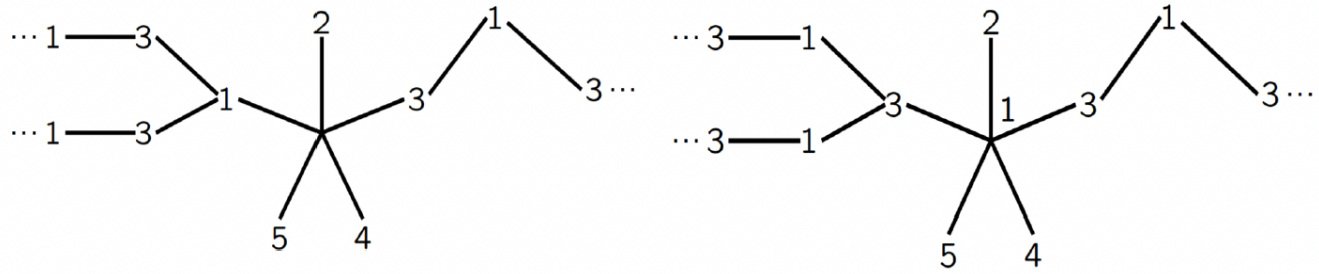
# Graph Coloring

**Theorem (Five Color Theorem)**: The chromatic number of a planar graph is no greater than five.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

Case 2a: The adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

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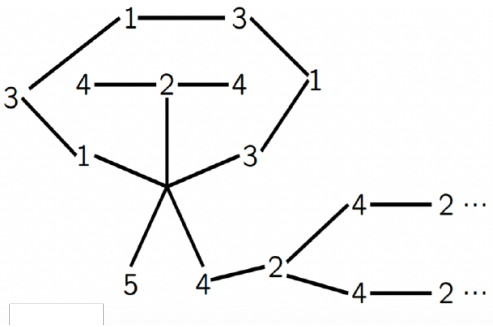
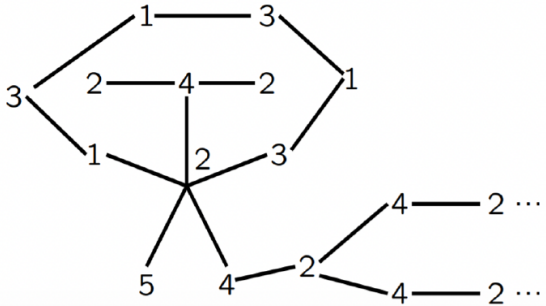
# Graph Coloring

**Theorem (Five Color Theorem)**: The chromatic number of a planar graph is no greater than five.

**Proof** (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

Case 2b: The vertices colored 1 and 3 are connected via a path in the subgraph. We do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3. Why? Planar graph

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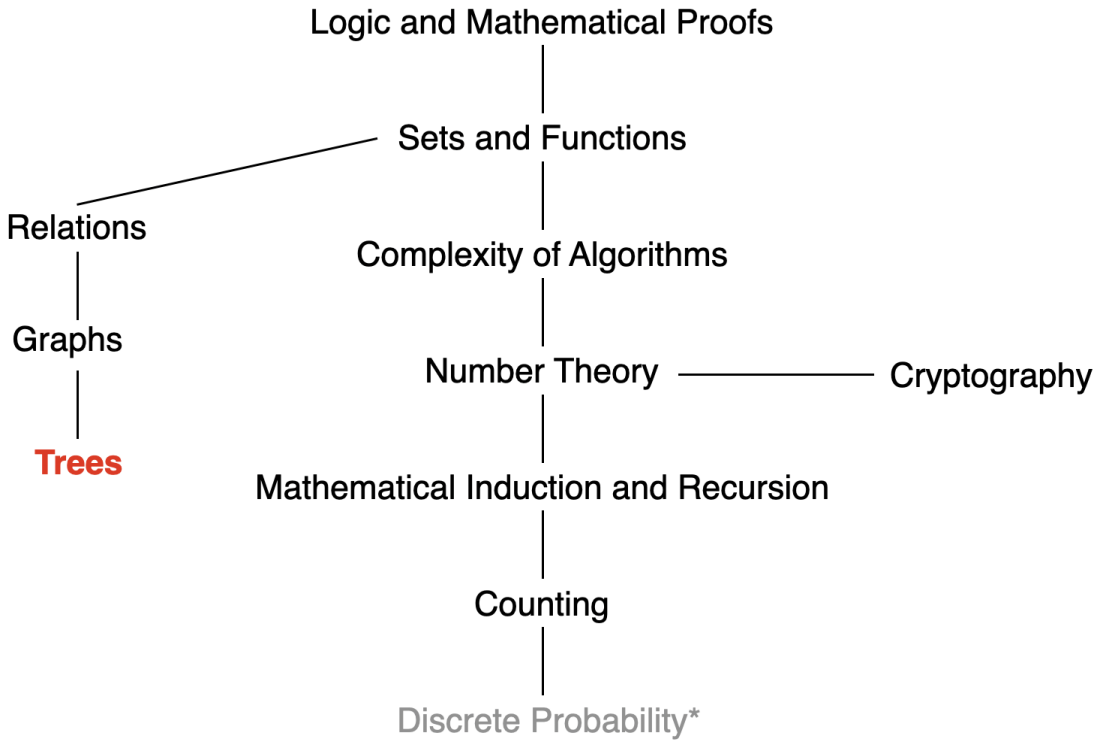
# This Lecture

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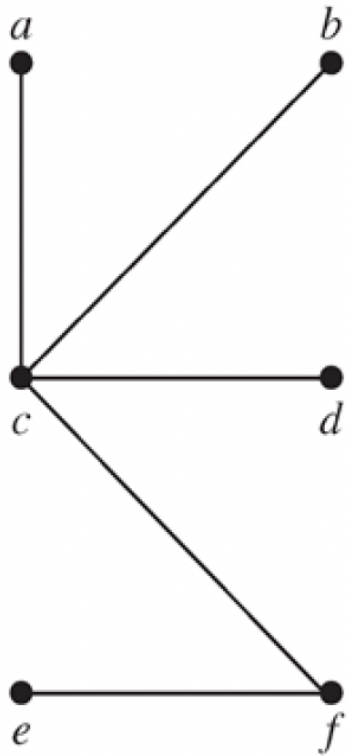
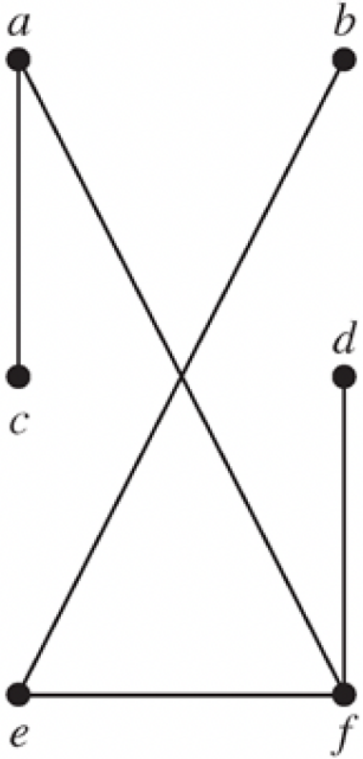
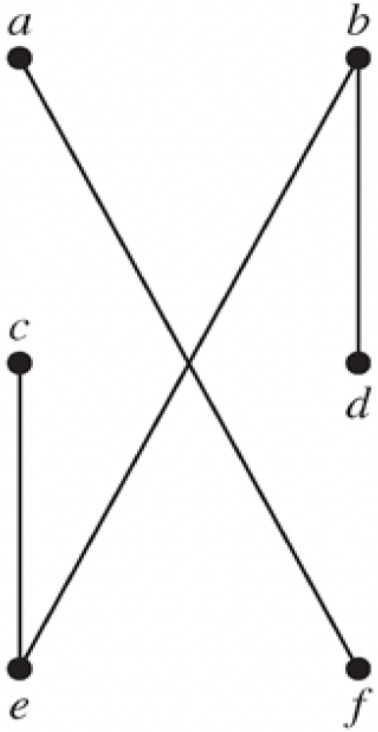
# Trees

**Definition**: A tree is a connected undirected graph with no simple circuits.

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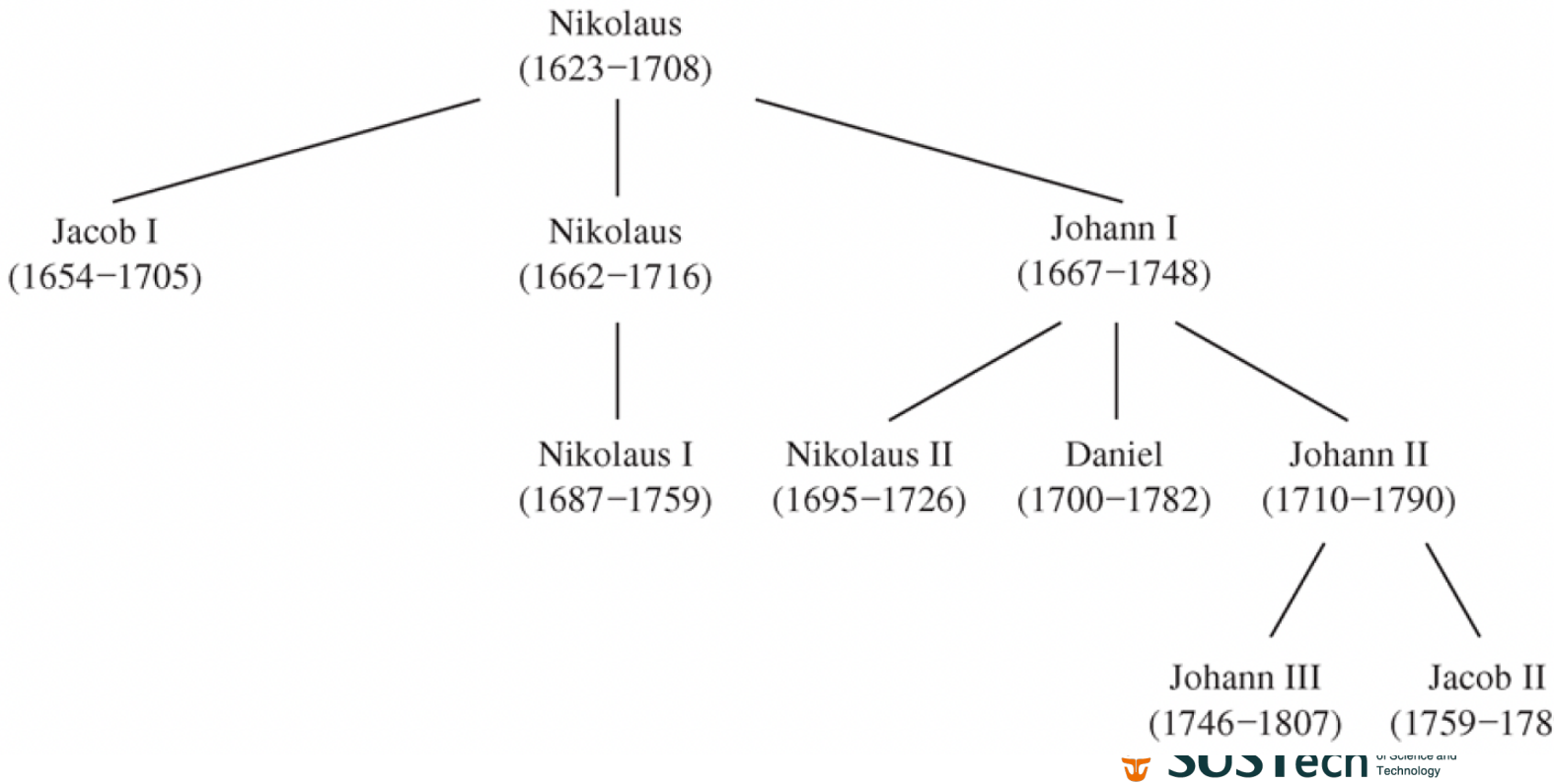
# Trees

**Definition**: A tree is a connected undirected graph with no simple circuits.



# Trees

**Theorem**: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

**Proof**: Two properties of tree: connected, no circuit

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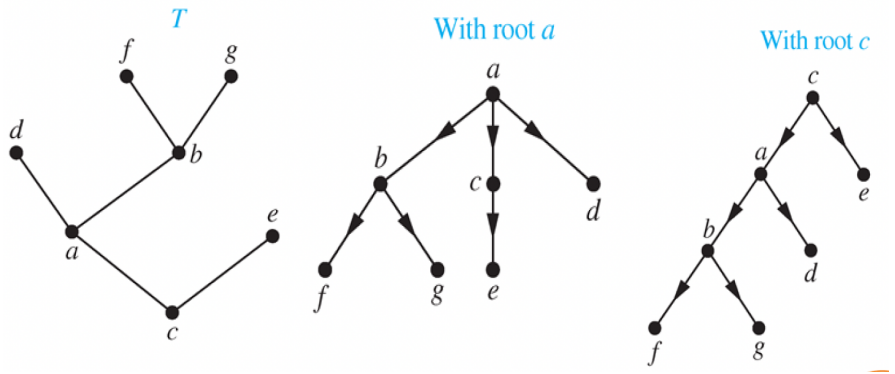
# Rooted Trees

**Definition**: A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.




We can change an unrooted tree into a rooted tree by choosing any vertex as the root.

The arrows indicating the directions of the edges in a rooted tree can be omitted, because the choice of root determines the directions of the edges.

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# Rooted Trees

The parent of $v$ is the unique vertex $u$ such that there is a directed edge from $u$ to $v$.

When $u$ is the parent of $v$, $v$ is called a child of $u$.

Vertices with the same parent are called siblings.

The ancestors of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.

The descendants of a vertex $v$ are those vertices that have $v$ as an ancestor.

A vertex of a rooted tree is called a leaf if it has no children.

Vertices that have children are called internal vertices.

Subtree with $a$ as its root: consists of $a$ and its descendants and all edges incident to these descendants.

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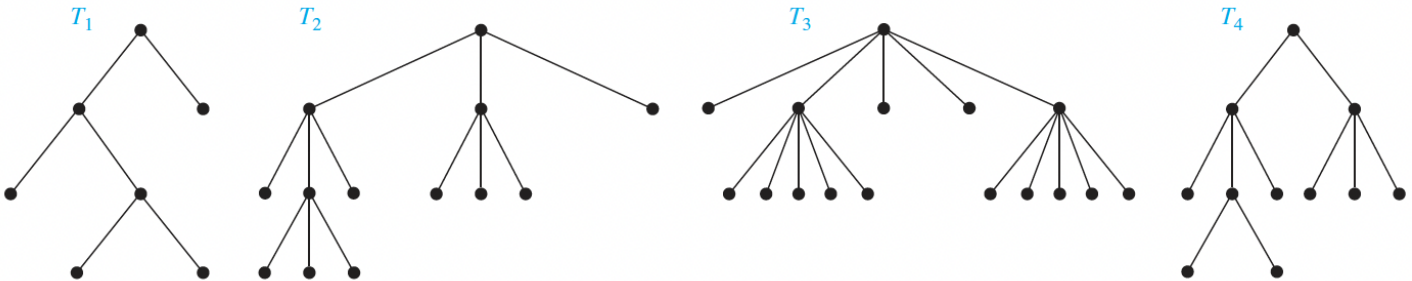
# $m$-Ary Trees

**Definition**: A rooted tree is called an $m$-ary tree if every internal vertex has no more than $m$ children. The tree is called a full $m$-ary tree if every internal vertex has exactly $m$ children. In particular, an $m$-ary tree with $m = 2$ is called a binary tree.

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# Ordered Rooted Tree

**Definition**: An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right.

In an ordered binary tree (usually called just a binary tree), if an internal vertex has two children, the first child is called the left child and the second child is called the right child.

The tree rooted at the left child of a vertex is called the left subtree of this vertex, and the tree rooted at the right child of a vertex is called the right subtree of the vertex.

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# Counting Vertices in a Full $m$-Ary Trees

**Theorem**: A tree with $n$ vertices has $n-1$ edges.

**Theorem**: A full $m$-ary tree with $i$ internal vertices has $n = mi + 1$ vertices.

**Theorem**: A full $m$-ary tree with

(i) $n$ vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n + 1]/m$ leaves,

(ii) $i$ internal vertices has $n = mi + 1$ vertices and $l = (m-1)i + 1$ leaves,

(iii) $l$ leaves has $n = (ml - 1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Using $n = mi + 1$ and $n = i + l$

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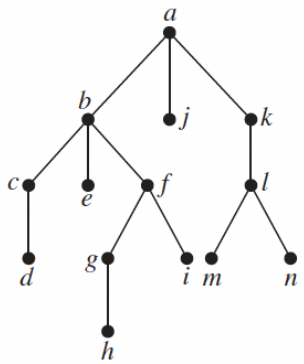
# Level and Height

The level of a vertex $v$ in a rooted tree is the length of the unique path from the root to this vertex.

The height of a rooted tree is the maximum of the levels of the vertices.




The root $a$ is at level 0. Vertices $b$, $j$, and $k$ are at level 1. ... Finally, vertex $h$ is at level 4. Because the largest level of any vertex is 4, this tree has height 4.

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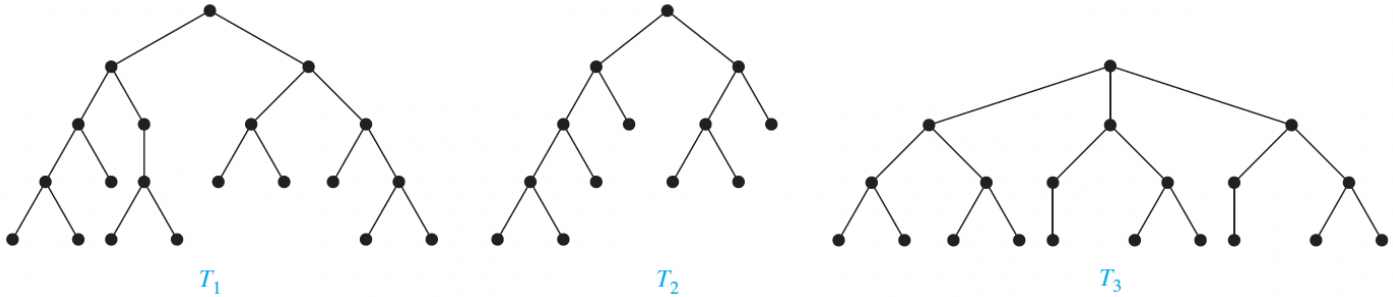
# Level and Height

A rooted $m$-ary tree of height $h$ is balanced if all leaves are at levels $h$ or $h-1$.




$T_1$ is balanced, because all its leaves are at levels 3 and 4.
$T_2$ is not balanced, because it has leaves at levels 2, 3, and 4.
$T_3$ is balanced, because all its leaves are at level 3.

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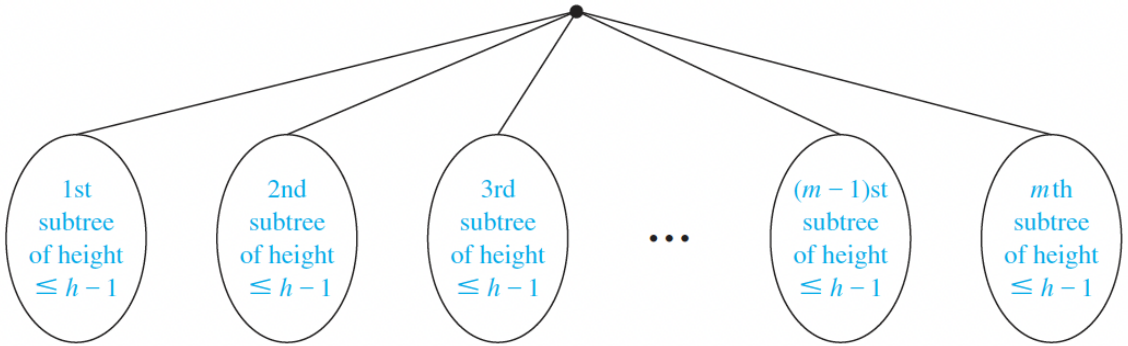
# The Number of Leave

**Theorem**: There are at most $m^h$ leaves in an $m$-ary tree of height $h$.

**Proof** (by induction)

- Basic Step: $h = 1$, at most $m$ leaves
- Inductive Step: Now assume that the result is true for all $m$-ary trees of height less than $h$; Consider an $m$-ary tree of height $h$:

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# The Number of Leave

**Corollary**: If an $m$-ary tree of height $h$ has $l$ leaves, then $h \geq \lceil \log_m l \rceil$. If the $m$-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$.

**Proof**:

- We know that $l \leq m^h$. Taking logarithms to the base $m$ shows that $\log_m l \leq h$. Because $h$ is an integer,we have $h \geq \lceil \log_m l \rceil$.
- Now suppose that the tree is balanced. Then, each leaf is at level $h$ or $h-1$, and because the height is $h$, there is at least one leaf at level $h$. It follows that there must be more than $m^{h-1}$ leaves.
- Because $l \leq m^h$, we have $m^{h-1} < l \leq m^h$. Taking logarithms to the base $m$ in this inequality gives $h - 1 < \log_m l \leq h$. Hence, $h = \lceil \log_m l \rceil$.

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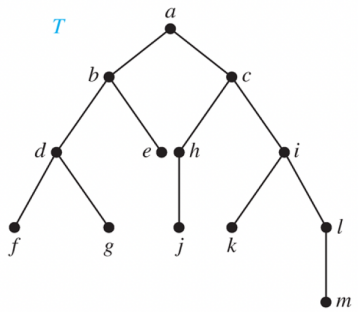
# Binary Trees

**Definition**: A binary tree is an ordered rooted tree where each internal tree has two children, the first is called the left child and the second is the right child. The tree rooted at the left child of a vertex is called the left subtree of this vertex, and the tree rooted at the right child of a vertex is called the right subtree of this vertex.

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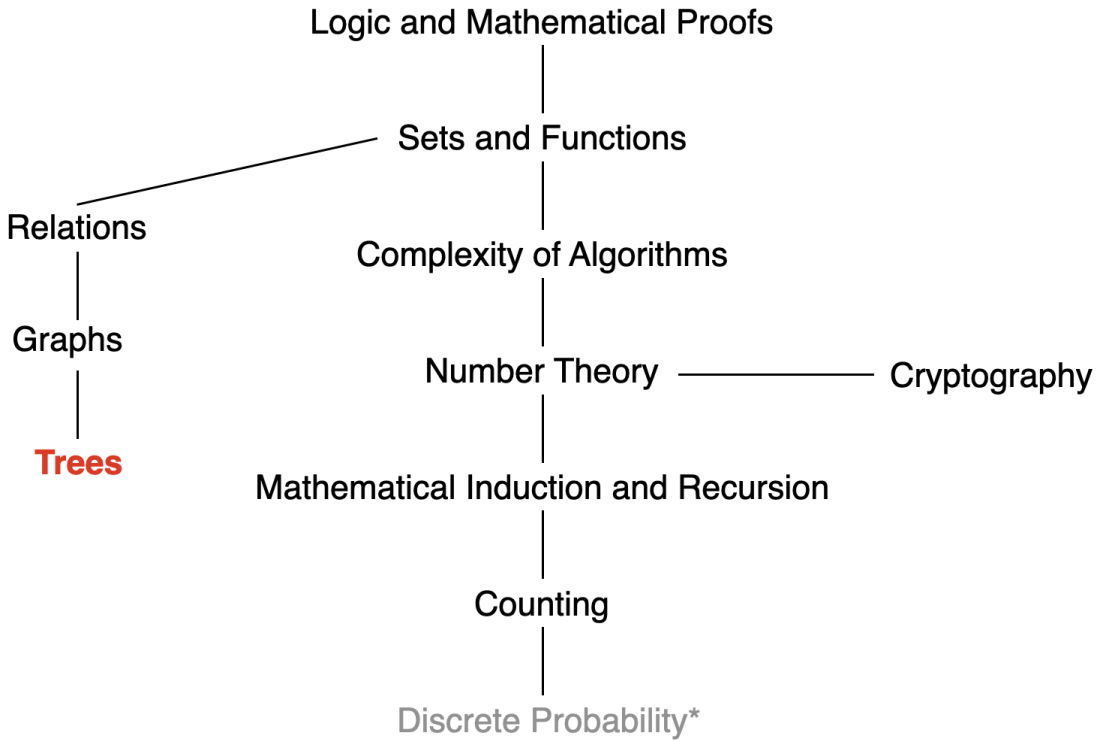
# This Lecture




Tree, Tree Traversal, ...

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# Tree Traversal

The procedures for systematically visiting every vertex of an ordered tree are called traversals.

The three most commonly used traversals are preorder traversal, inorder traversal, postorder traversal.

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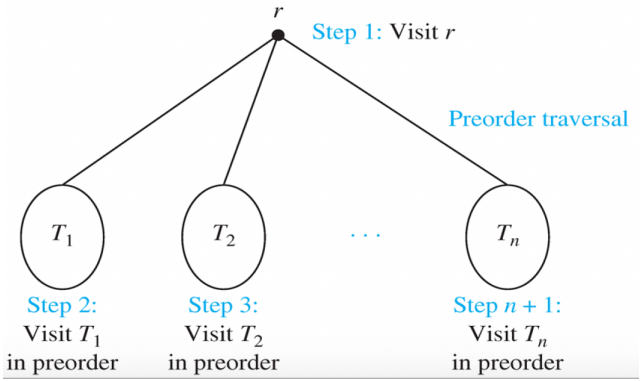
# Preorder Traversal

**Definition** Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the preorder traversal of $T$.

Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of $r$ from left to right in $T$. The preorder traversal begins by visiting $r$, and continues by traversing $T_1$ in preorder, then $T_2$ in preorder, and so on, until $T_n$ is traversed in preorder.

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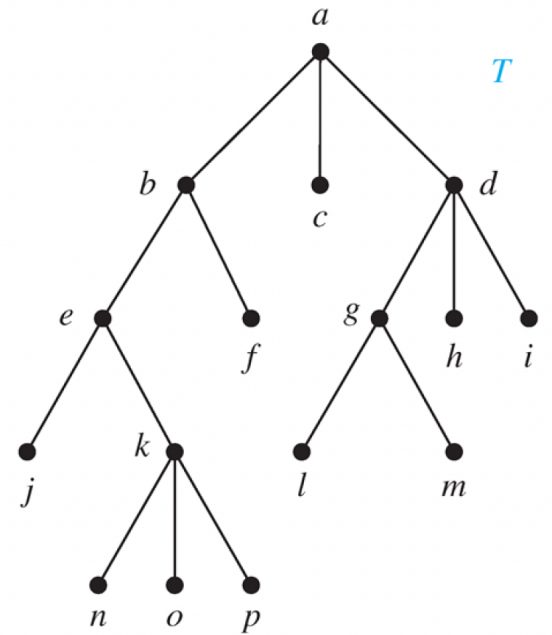
# Preorder Traversal: Example




a, b, e, j, k, n, o, p, f, c, d, g, l, m, h, i

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# Preorder Traversal: Algorithm

**procedure** *preorder* ($T$: ordered rooted tree)
$r$ := root of $T$
list $r$
**for** each child $c$ of $r$ from left to right
    $T(c)$ := subtree with $c$ as root
    *preorder*($T(c)$)

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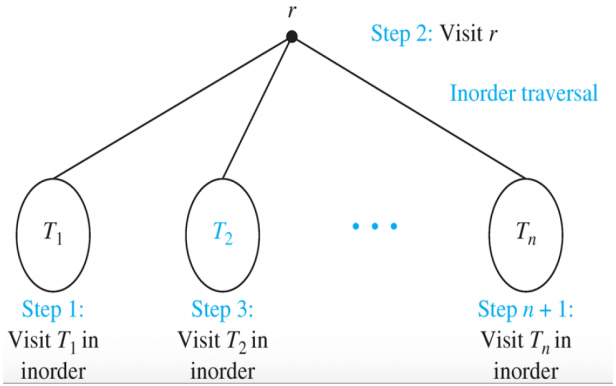
# Inorder Traversal

**Definition**: Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the inorder traversal of T.

Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of $r$ from left to right in $T$. The inorder traversal begins by traversing $T_1$ in inorder, then visiting $r$, and continues by traversing $T_2$ in inorder, and so on, until $T_n$ is traversed in inorder.

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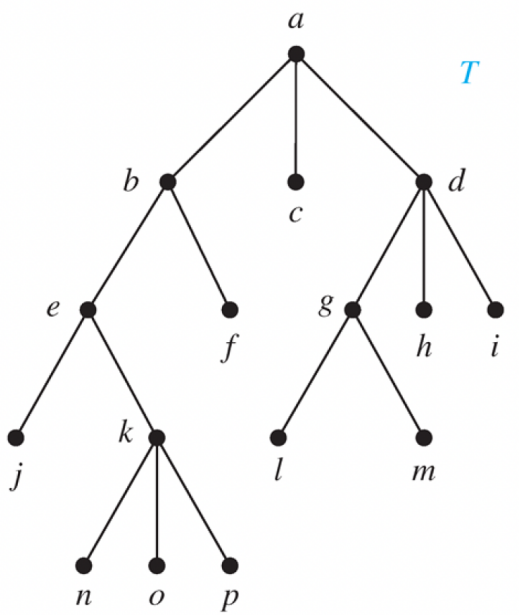
# Inorder Traversal: Example




j, e, n, k, o, p, b, f, a, c, l, g, m, d, h, i

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# Inorder Traversal: Algorithm

```
procedure inorder (T: ordered rooted tree)
r := root of T
if r is a leaf then list r
else
    l := first child of r from left to right
    T(l) := subtree with l as its root
    inorder(T(l))
    list(r)
    for each child c of r from left to right
        T(c) := subtree with c as root
        inorder(T(c))
```

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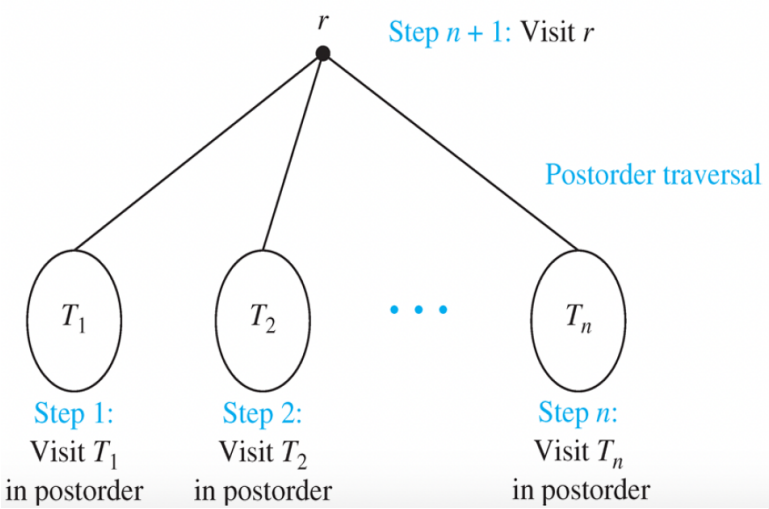
# Postorder Traversal

**Definition**: Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the postorder traversal of $T$.

Otherwise, suppose that $T_1, T_2, ..., T_n$ are the subtrees of $r$ from left to right in $T$. The postorder traversal begins by traversing $T_1$ in postorder, then $T_2$ in postorder, and so on, after $T_n$ is traversed in postorder, $r$ is visited.

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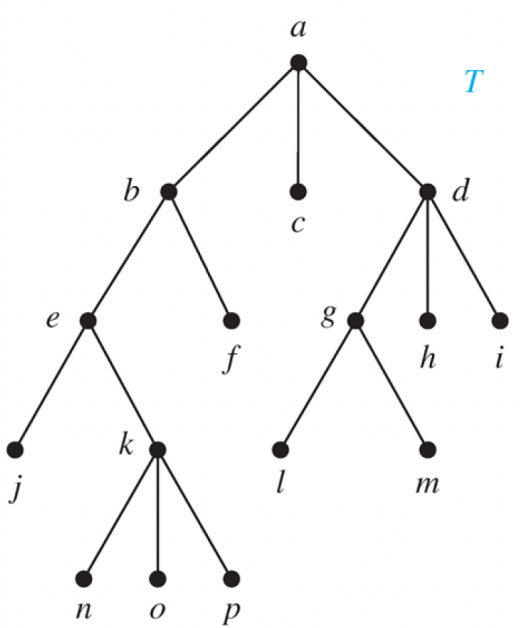
# Postorder Traversal




j, n, o, p, k, e, f, b, c, l, m, g, h, i, d, a

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# Postorder Traversal: Algorithm

**procedure** *postordered* ($T$: ordered rooted tree)
$r$ := root of $T$
**for** each child $c$ of $r$ from left to right
    $T(c)$ := subtree with $c$ as root
    postorder($T(c)$)
list $r$

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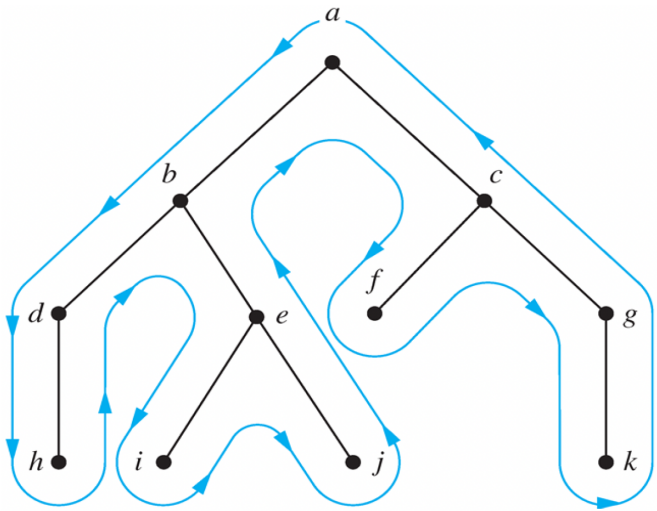
# Preorder, Inorder, Postorder Traversal




- in preorder: listing each vertex the first time this curve passes it.
  a, b, d, h, e, i, j , c, f , g, k
- in inorder: listing a leaf the first time the curve passes it and listing each internal vertex the second time the curve passes it.
  h, d, b, i, e, j, a, f , c, k, g
- in postorder: listing a vertex the last time it is passed on the way back up to its parent.
  h, d, i, j , e, b, f , k, g, c, a

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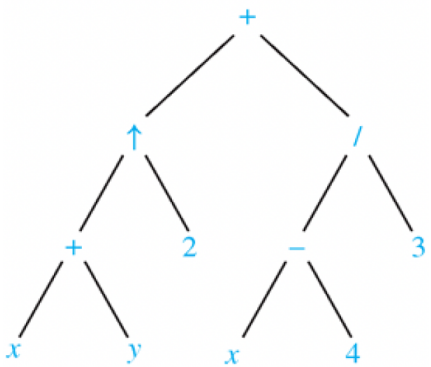
# Expression Trees

Complex expressions can be represented using ordered rooted trees.

- the internal vertices represent operations
- the leaves represent the variables or numbers

**Example**: Consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$

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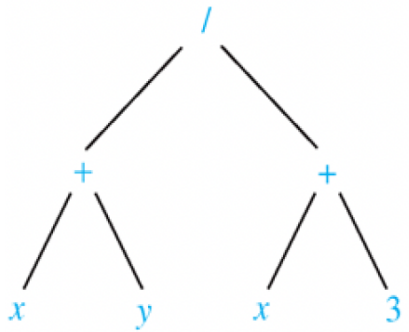
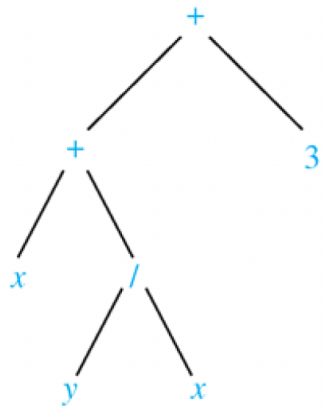
# Infix Notation

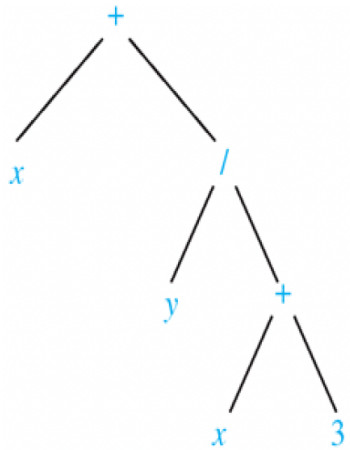
An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operation. The fully parenthesized expression obtained in this way is said to be in infix form.

Why parentheses are needed?

$(x + y)/(x + 3)$, $(x + (y/x)) + 3$, $x + (y/(x + 3))$



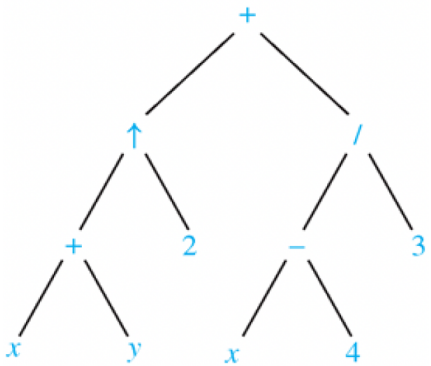

Infix form: $x + y/x + 3$

# Prefix Notation

The preorder traversal of expression trees leads to the prefix form of the expression (Polish notation).

An expression in prefix notation, is unambiguous, so no parentheses are needed in such an expression.

**Example**: What is the prefix form for $((x + y) \uparrow 2) + ((x - 4)/3)$?




Prefix form: $+ \uparrow + x\ y\ 2\ /\ -\ x\ 4\ 3$.

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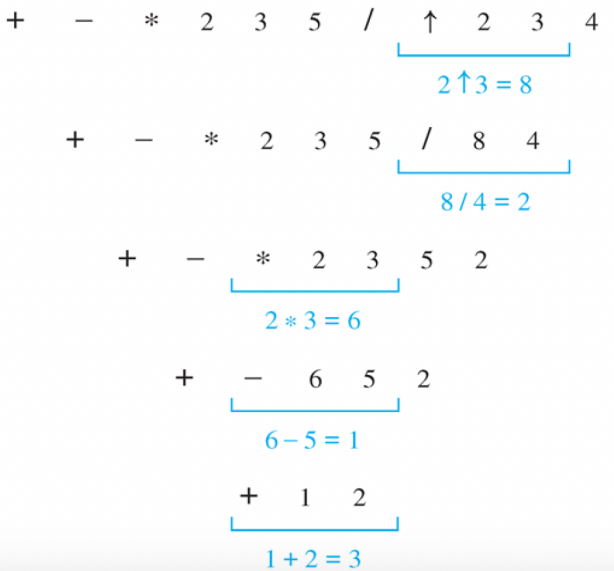
# Prefix Notation

Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

**Example**: $+ \quad - \quad * \quad 2 \; 3 \; 5 \; / \; \uparrow 2 \; 3 \; 4$

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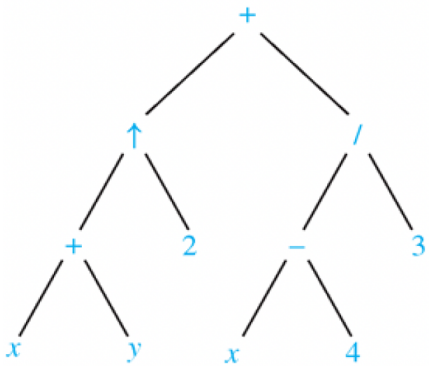
# Postfix Notation

The postorder traversal of expression trees leads to the postfix form of the expression (reverse Polish notation).

Expressions in reverse Polish notation are unambiguous, so parentheses are not needed.

**Example**: What is the postfix form of the expression $((x + y) \uparrow 2) + ((x - 4)/3)$?




Prefix form: $x\ y\ +\ 2\ \uparrow\ x\ 4\ -\ 3\ /\ +$.

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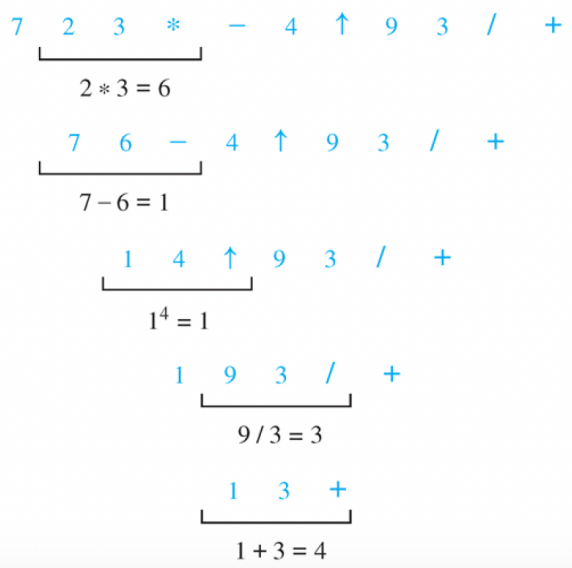
# Postfix Notation

Postfix expressions are evaluated by working from left to right. When we encounter an operator, we perform the operation with the two operands to the left.

**Example**: $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

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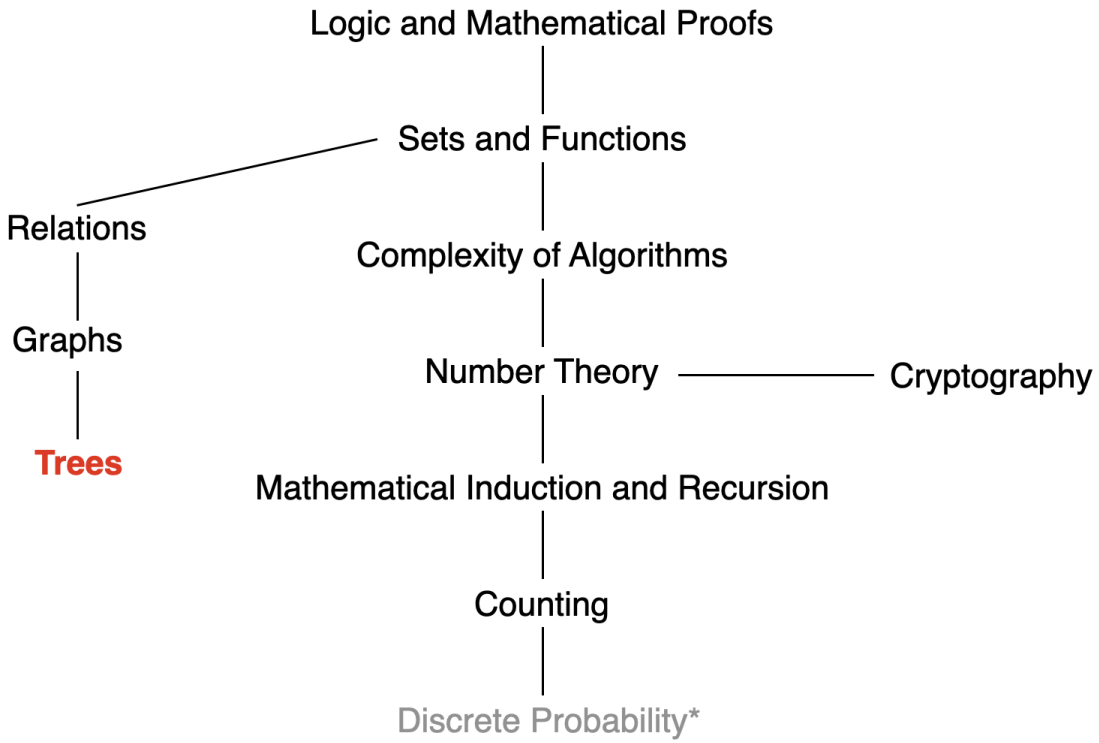
# This Lecture




Tree, Tree Traversal, Spanning Trees ...

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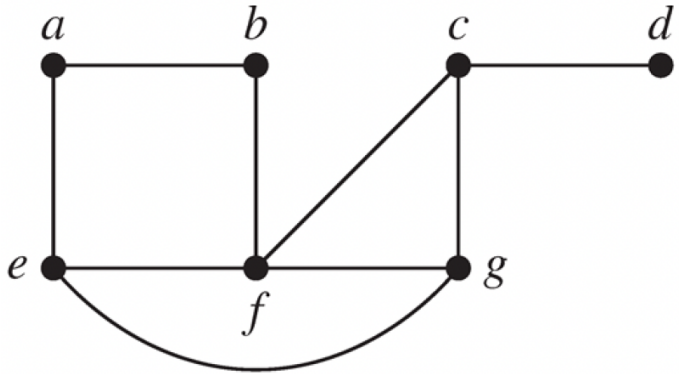
# Spanning Trees

**Definition**: Let $G$ be a simple graph. A spanning tree of $G$ is a subgraph of $G$ that is a tree containing every vertex of $G$.




Remove edges to avoid circuits.

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# Spanning Trees

**Theorem** A simple graph is connected if and only if it has a spanning tree.

**Proof:**

- only if: The spanning tree can be obtained by removing edges from simple circuits.
- if: The spanning tree $T$ contains every vertex of $G$. Furthermore, there is a path in $T$ between any two of its vertices. Because $T$ is a subgraph of $G$, there is a path in $G$ between any two of its vertices. Hence, $G$ is connected.

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# Depth-First Search

We can find spanning trees by removing edges from simple circuits.

But, this is inefficient, since simple circuits should be identified first.

Instead, we build up spanning trees by successively adding edges.

- First, arbitrarily choose a vertex of the graph as the root.
- Form a path by successively adding vertices and edges. Continue adding to this path as long as possible.
- If the path goes through all vertices of the graph, the tree is a spanning tree.
- Otherwise, move back to some vertex to repeat this procedure (backtracking).

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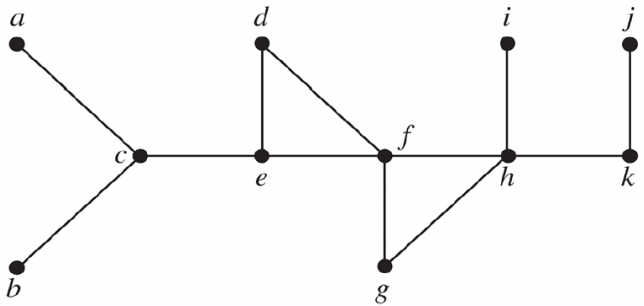
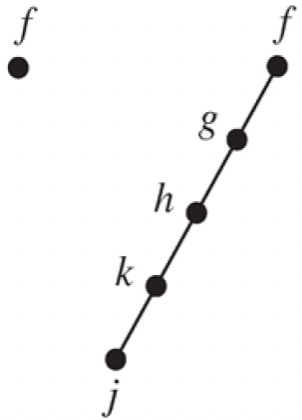
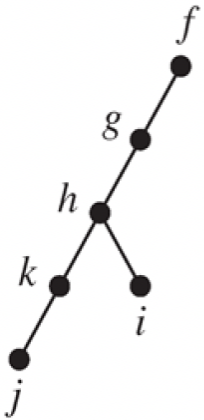
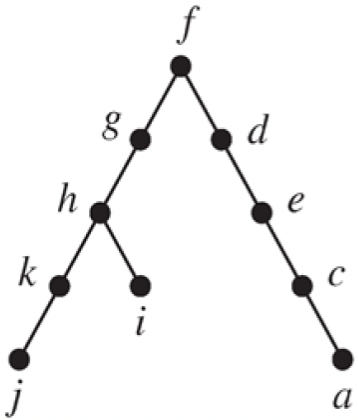
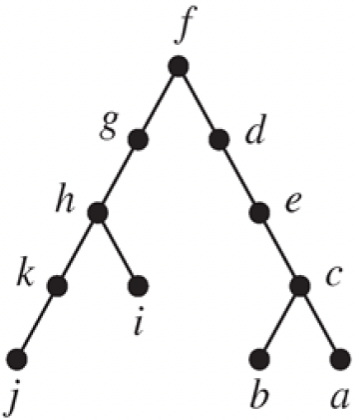
# Depth-First Search: Example

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# Depth-First Search: Algorithm

When we add an edge connecting a vertex $v$ to a vertex $w$, we finish exploring from $w$ before we return to $v$ to complete exploring from $v$.

**procedure** *DFS*($G$: connected graph with vertices $v_1, v_2, ..., v_n$)
$T$ := tree consisting only of the vertex $v_1$
*visit*($v_1$)

**procedure** *visit*($v$: vertex of $G$)
**for** each vertex $w$ adjacent to $v$ and not yet in $T$
  add vertex $w$ and edge $\{v, w\}$ to $T$
  *visit*($w$)

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# Breadth-First Search

This is the second algorithm that we build up spanning trees by successively adding edges.

- First arbitrarily choose a vertex of the graph as the root.
- Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges
- For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.
- Continue in this manner until all vertices have been added.

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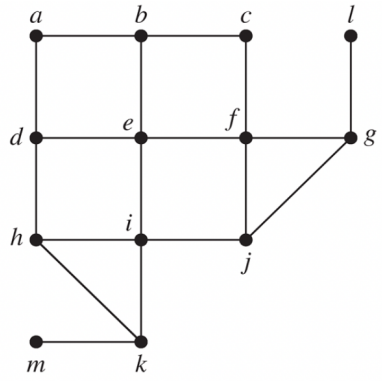
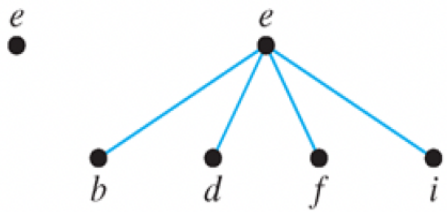
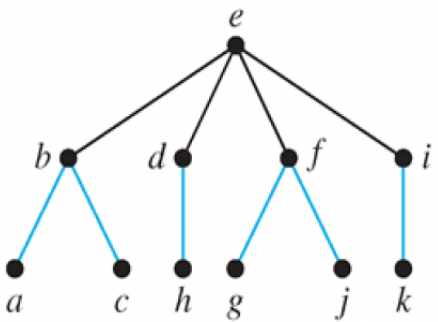
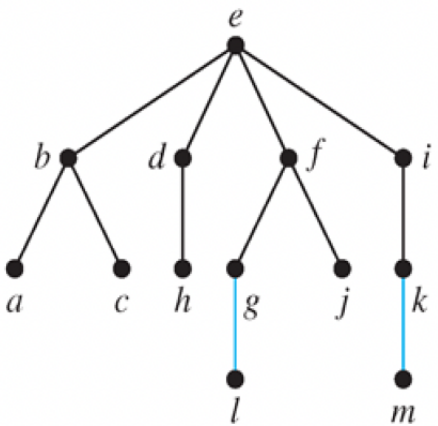
# Breadth-First Search: Example



# Breadth-First Search

**procedure** *BFS*($G$: connected graph with vertices $v_1, v_2, \ldots, v_n$)
$T$ := tree consisting only of the vertex $v_1$
$L$ := empty list *visit*($v_1$)
put $v_1$ in the list $L$ of unprocessed vertices
**while** $L$ is not empty
  remove the first vertex, $v$, from $L$
  **for** each neighbor $w$ of $v$
    **if** $w$ is not in $L$ and not in $T$ **then**
      add $w$ to the end of the list $L$
      add $w$ and edge $\{v,w\}$ to $T$

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# Backtracking Applications

There are problems that can be solved only by performing an exhaustive search of all possible solutions.

One way to search systematically for a solution is to use a decision tree, where each internal vertex represents a decision and each leaf a possible solution.

To find a solution via backtracking

- first make a sequence of decisions in an attempt to reach a solution as long as this is possible.
- Once it is known that no solution can result from any further sequence of decisions, backtrack to the parent of the current vertex and work toward a solution with another series of decisions

The procedure continues until a solution is found, or it is established that no solution exists.

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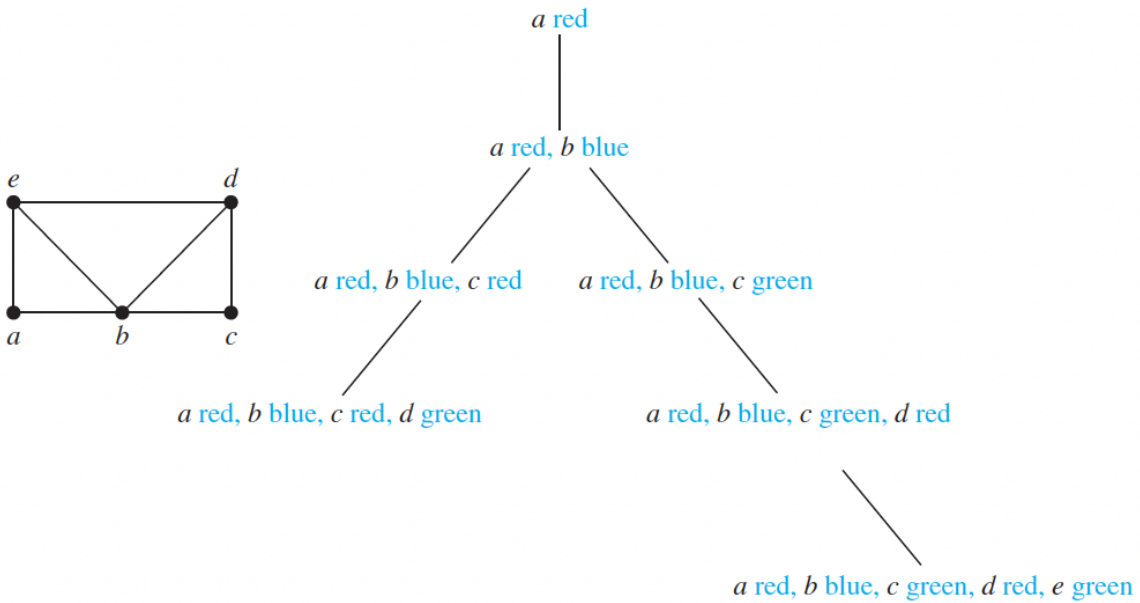
# Backtracking Applications: Graph Colorings

How can backtracking be used to decide whether a graph can be colored using $n$ colors?

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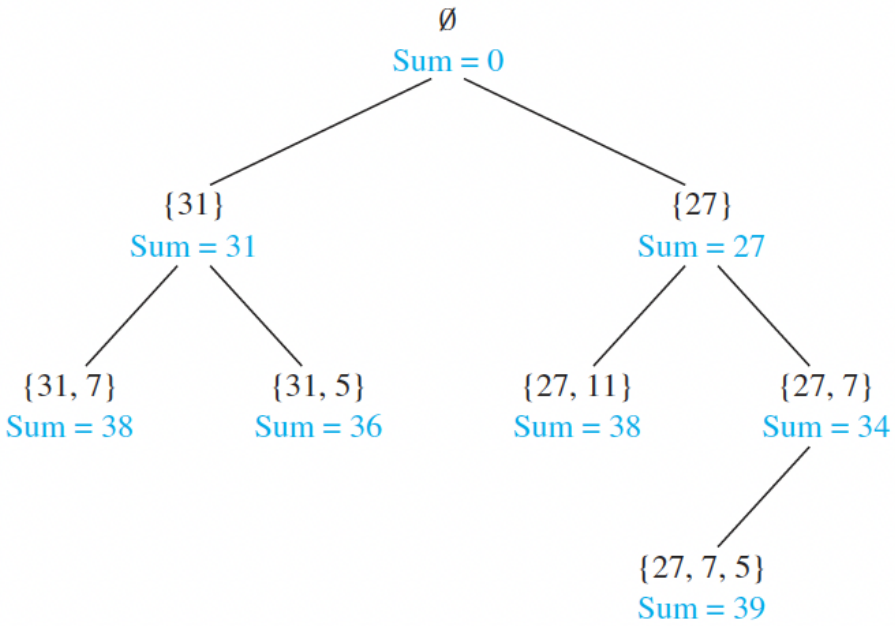
# Backtracking Applications: Sums of Subsets

Consider this problem. Given a set of positive integers $x_1, x_2, \ldots, x_n$, find a subset of this set of integers that has $M$ as its sum. How can backtracking be used to solve this problem?

Finding a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum equal to 39.



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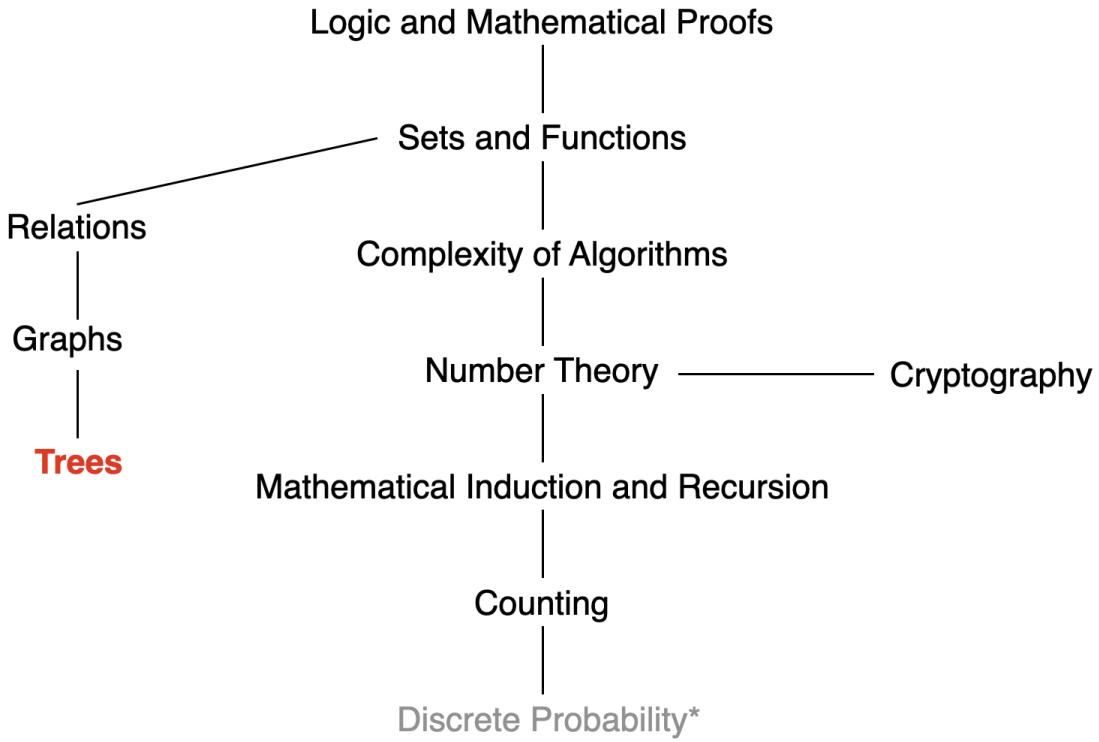


Tree, Tree Traversal, Spanning Trees, Minimum Spanning Trees, ...

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# Minimum Spanning Trees

**Definition**: A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Two greedy algorithms: Prim's Algorithm, Kruscal's Algorithm.

Both algorithms do produce optimal solutions.

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# Prim's Algorithm

**ALGORITHM 1 Prim's Algorithm.**

**procedure** *Prim*($G$: weighted connected undirected graph with $n$ vertices)
$T$ := a minimum-weight edge
**for** $i := 1$ **to** $n - 2$
  $e$ := an edge of minimum weight incident to a vertex in $T$ and not forming a simple circuit in $T$ if added to $T$
  $T := T$ with $e$ added
**return** $T$ {$T$ is a minimum spanning tree of $G$}

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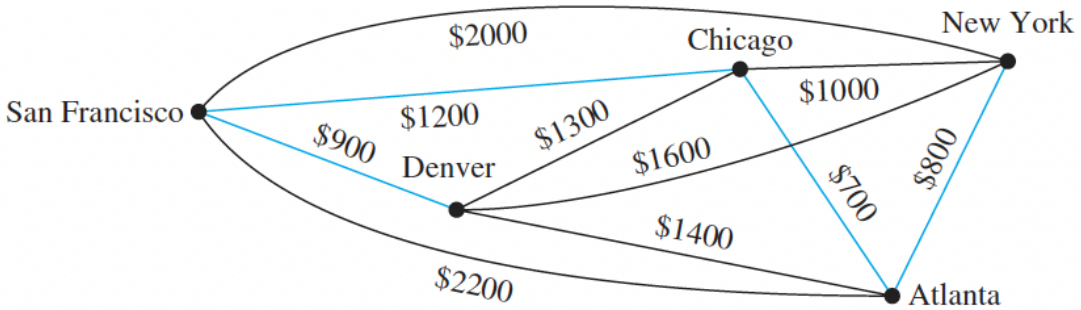
# Prim's Algorithm: Example




| Choice | Edge | Cost |
|---|---|---|
| 1 | {Chicago, Atlanta} | $ 700 |
| 2 | {Atlanta, New York} | $ 800 |
| 3 | {Chicago, San Francisco} | $1200 |
| 4 | {San Francisco, Denver} | $ 900 |
| | Total: | $3600 |

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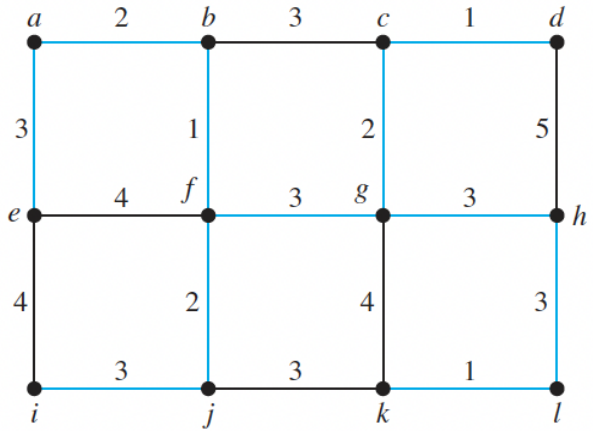
# Prim's Algorithm: Example




| Choice | Edge | Weight |
|---|---|---|
| 1 | $\{b, f\}$ | 1 |
| 2 | $\{a, b\}$ | 2 |
| 3 | $\{f, j\}$ | 2 |
| 4 | $\{a, e\}$ | 3 |
| 5 | $\{i, j\}$ | 3 |
| 6 | $\{f, g\}$ | 3 |
| 7 | $\{c, g\}$ | 2 |
| 8 | $\{c, d\}$ | 1 |
| 9 | $\{g, h\}$ | 3 |
| 10 | $\{h, l\}$ | 3 |
| 11 | $\{k, l\}$ | 1 |
| | Total: | 24 |

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# Kruskal's Algorithm

**ALGORITHM 2 Kruskal's Algorithm.**

**procedure** *Kruskal*($G$: weighted connected undirected graph with $n$ vertices)
$T$ := empty graph
**for** $i := 1$ **to** $n - 1$
  $e$ := any edge in $G$ with smallest weight that does not form a simple circuit
    when added to $T$
  $T := T$ with $e$ added
**return** $T$ {$T$ is a minimum spanning tree of $G$}

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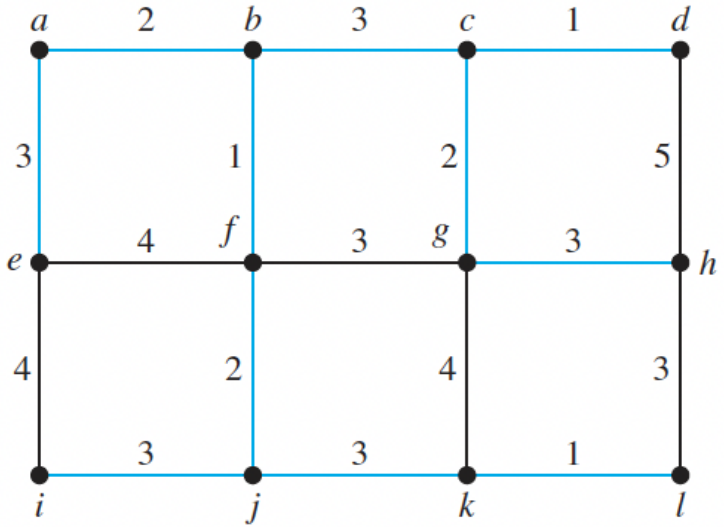
# Kruskal's Algorithm: Example




| Choice | Edge | Weight |
|---|---|---|
| 1 | $\{c, d\}$ | 1 |
| 2 | $\{k, l\}$ | 1 |
| 3 | $\{b, f\}$ | 1 |
| 4 | $\{c, g\}$ | 2 |
| 5 | $\{a, b\}$ | 2 |
| 6 | $\{f, j\}$ | 2 |
| 7 | $\{b, c\}$ | 3 |
| 8 | $\{j, k\}$ | 3 |
| 9 | $\{g, h\}$ | 3 |
| 10 | $\{i, j\}$ | 3 |
| 11 | $\{a, e\}$ | 3 |
| | Total: | 24 |

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