CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 5
Due: May 20th, 2022, please submit one pdf file through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Plagiarism in an Assignment or a Quiz:

- **For the first time: the score of the assignment or quiz will be zero**
- **For the second time: the score of the course will be zero**
- **When two assignments are nearly identical, the policy will apply to BOTH students, unless one confesses having copied without the knowledge of the other.**

Any late submission will lead to a zero point with no exception.

Q. 1. (5 points) Show that a subset of an *antisymmetric* relation is also *antisymmetric*.

Solution: Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a, b) \in R_1$ and $(b, a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

□

Q. 2. (5 points) Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution: The result follows from

$$(R^*)^{-1} = (\cup_{n=1}^{\infty} R^n)^{-1} = \cup_{n=1}^{\infty} (R^n)^{-1} = \cup_{n=1}^{\infty} R^n = R^*.$$

□

Q. 3. (5 points) Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Solution: Suppose that $(a, b) \in R$. Because $(b, b) \in R$, it then follows that $(a, b) \in R^2$. Thus, R is a subset of R^2.

$\square$

Q. 4. (5 points) Suppose that R is a *symmetric* relation on a set A. Is $\overline{R}$ also symmetric? Explain your answer.

Solution: Under this hypothesis, $\overline{R}$ must also be symmetric. If $(a,b) \in \overline{R}$, then $(a,b) \notin R$, whence (b,a) cannot be in R since R is symmetric. In other words, (b,a) is also contained in $\overline{R}$. Thus, $\overline{R}$ is symmetric.

$\square$

Q. 5. (10 points) For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3+5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Solution:

(a) Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

(b) No, because $33 \preceq 26$ and $26 \preceq 33$, but $26 \neq 33$.

(c) No, because $33 \preceq 35$ and $35 \preceq 13$, but we do not have $33 \preceq 13$.

$\square$

Q. 6. (10 points) Give an examples of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

Solution: We fix the ground set $S = \{a,b,c,d\}$, and we consider the relation $R = \{(a,b),(b,c),(c,d)\}$. Then the transitive closure of R equals $R^* = \{(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)\}$. On the other hand, $R^2 = \{(a,c),(b,d)\}$, and $R^3 = \{(a,d)\}$. Hence, R^3 is necessary to get R^*.

$\square$

Q. 7. (10 points) Which of the following are equivalence relations on the set of all people?

(1) $\{(x, y) | x$ and y have the same sign of the zodiac$\}$

(2) $\{(x, y) | x$ and y were born in the smae year$\}$

(3) $\{(x, y) | x$ and y have been in the same city$\}$

Solution:

(1) This is an equivalence relation.

(2) This is an equivalence relation.

(3) This is not an equivalence relation, since it is not transitive.

$\square$

Q. 8. (10 points) Show that $\{(x, y) | x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Solution: This relation is reflexive, since $x - x = 0 \in \mathbb{Q}$. To see that it is symmetric, suppose that $x-y \in \mathbb{Q}$. Then $y-x = -(x-y)$ is again a rational number. For transitivity, if $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$, then their sum, namely $x - z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of 1/2 are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, in other words, $\{\pi + r | r \in \mathbb{Q}\}$.

$\square$

Q. 9. (10 points) Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

$\square$

Q. 10. (10 points) Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Solution: The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$.

$\square$

Q. 11. (10 points) We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

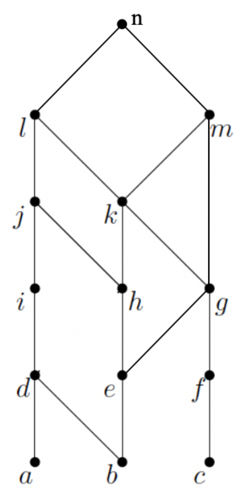
(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \subseteq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \geq i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_{i+1} \subseteq B_i$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k + 1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \subseteq C_{ij} \subseteq C_{i+1,j}$.

$\square$

Q. 12. (10 points) Answer these questions for the partial order represented by this Hasse diagram.



(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) The maximal elements are the ones with no other elements above them, namely n.

(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

(c) There is a greatest element n.

(d) There is no least elements, since neither of a, b, c is less than the others.

(e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that l and n are the elements fitting this description.

(f) Since l is less than n, it is the least upper bound of a, b and c.

(g) No element is less than both f and h, so there are no lower bounds.

(h) Since there is no lower bound, there cannot be greatest lower bound.

□