CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 2
Due: Mar. 24th, 2022, please submit through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. (5 points) Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A - B = A) \to (B \subset A)$

(b) $(A \cap B \cap C) \subseteq (A \cup B)$

(c) $\overline{(A - B)} \cap (B - A) = B$

Q. 2. (5 points) The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Q. 3. (5 points) Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Q. 4. (5 points) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Q. 5. (10 points) For each of the following mappings, use the following options to describe them, and explain your answers.
i. Not a function.
ii. A function which is neither one-to-one nor onto.
iii. A function which is onto but not one-to-one.
iv. A function which is one-to-one but not onto.
v. A function which is both one-to-one and onto.

(a) The mapping f from $\mathbf{Z}$ to $\mathbf{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbf{R}$ to $\mathbf{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbf{R}^+$ to $\mathbf{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ defined by $f(x) = x + 1$.

Q. 6. (5 points) Which of the mappings in Q. 5 have an inverse function? What is the inverse function? Please list all such mappings and explain your answer.

Q. 7. (5 points) Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q. 8. (10 points) Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q. 9. (5 points) Derive the formula for $\sum_{k=1}^{n} k^2$.

Q. 10. (5 points) Give an example of two uncountable sets A and B such that the difference $A - B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q. 11. (10 points) For each set defined below, determine whether the set is *countable* or *uncountable*. Explain your answers. Recall that $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b)|a,\ b \in \mathbf{N}\}$

(c) $\{(a, b)|a \in \mathbf{N},\ b \in \mathbf{R}\}$

Q. 12. (5 points) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

Q. 13. (5 points) Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m+n-2)(m+n-1)/2 + m$ is one-to-one and onto.

Q. 14. (5 points) By the Schröder-Bernstein theorem, prove that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Q. 15. (5 points) Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Q. 16. (5 points) Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

Q. 17. (5 points) Consider the following algorithm for evaluating the value of a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ at $x = c$.

(a) How many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).

(b) Under the operations considered in (a), what is the time complexity with respect to n (in Big-Theta Notation)?

Algorithm 1 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
power := 1
y := a_0
for i := 1 to n do
    power := power * c
    y := y + a_i * power
end for
return  y {y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0}
```