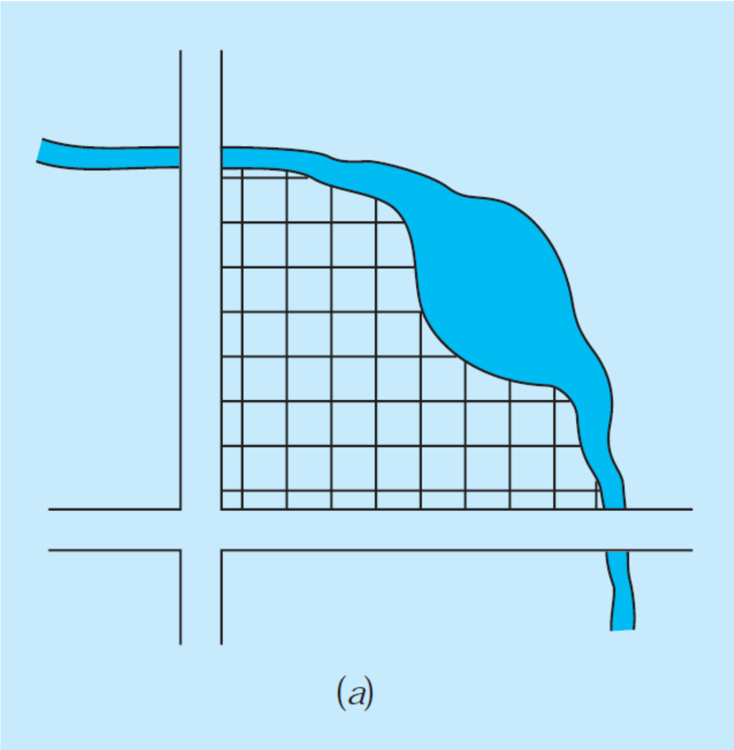
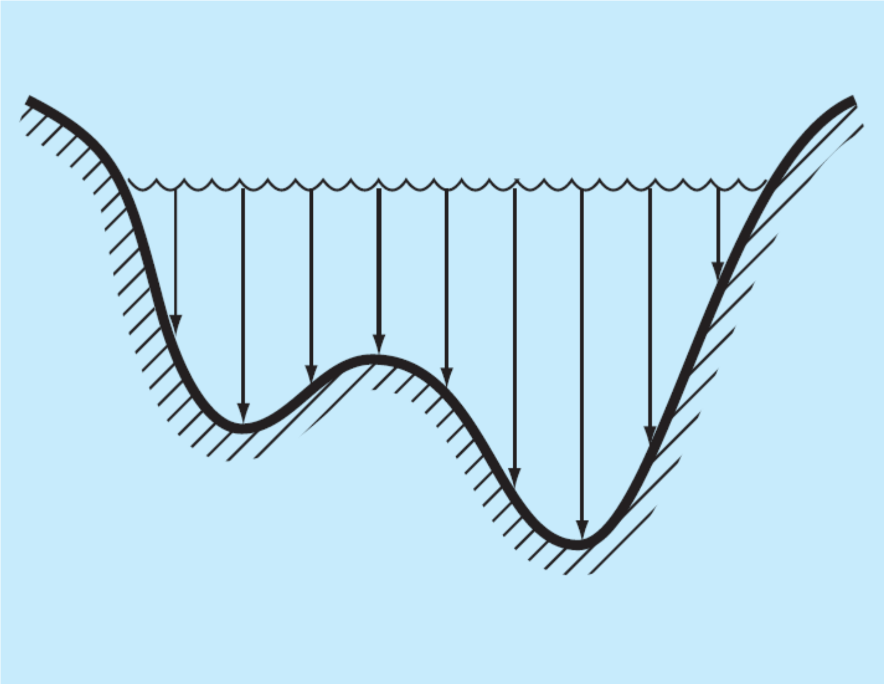
MATLAB 与工程应用

Numerical integration and differentiation

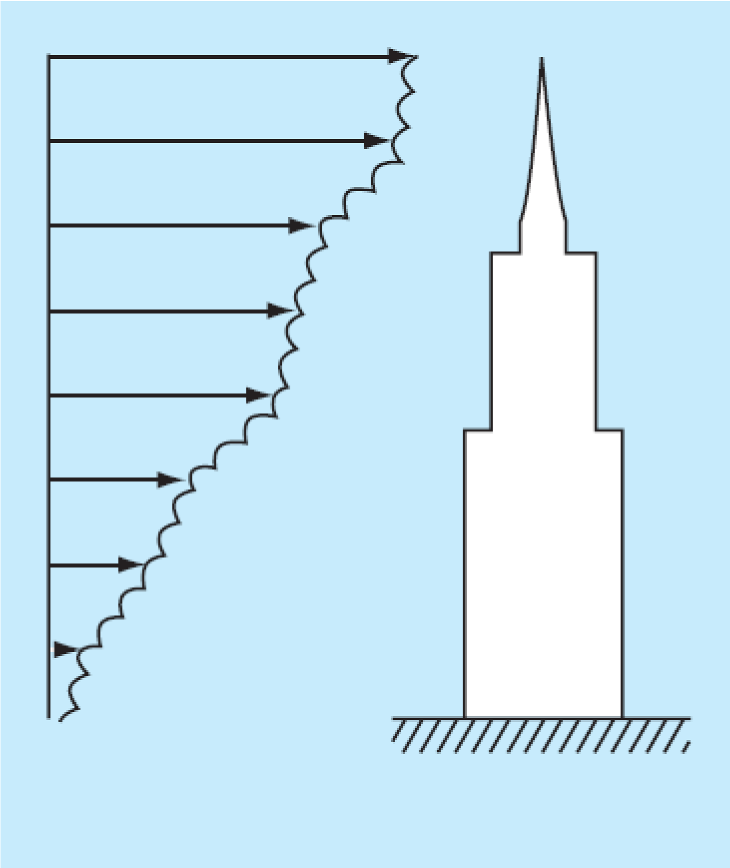
Motivation



Area of a field bounded by a meandering stream and two roads



Cross-sectional area of a river



Net force due to a nonuniform wind blowing against the side of a skyscraper

Motivation

- Integrals are also employed by engineers to evaluate the total amount or quantity of a given physical variable.

$$\text{Heat transfer} = \iint_A \text{flux}\, dA$$

$$\text{Mass} = \iiint c(x, y, z)\, dx\, dy\, dz$$

or

$$\text{Mass} = \iiint_V c(V)\, dV$$

Motivation

- Integrals are used to evaluate differential or rate equations. Suppose the velocity of a particle is known continuous function of time

$$\frac{dy}{dt} = v(t)$$

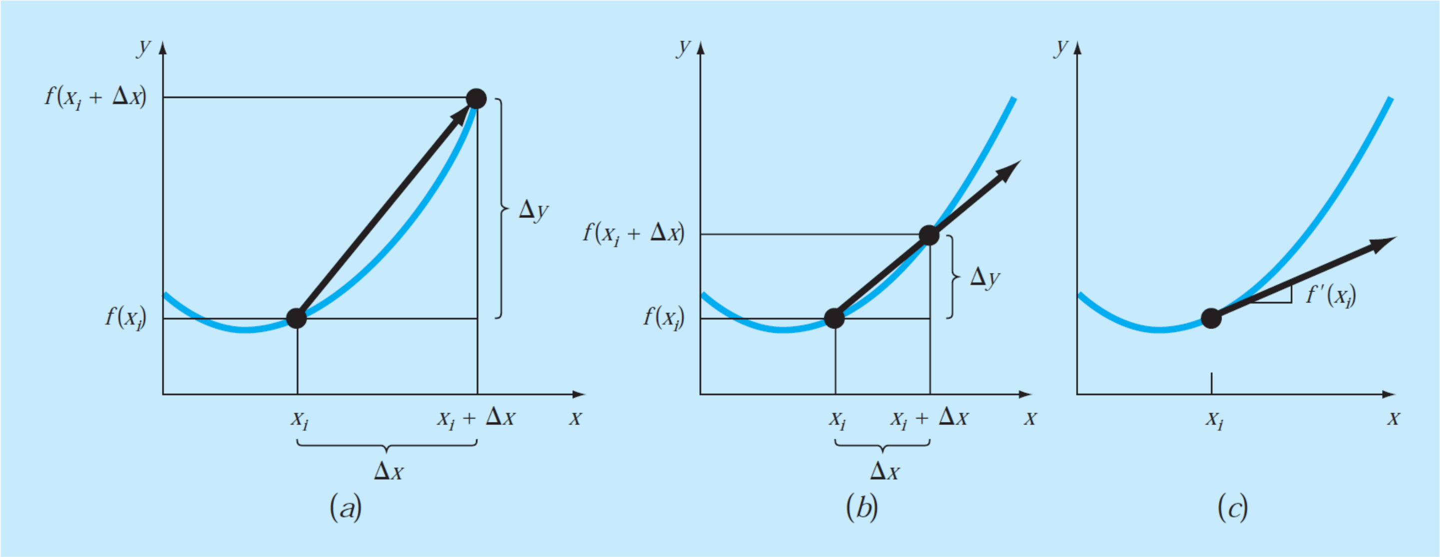
$$y = \int_0^t v(t)\, dt$$

Differentiation

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

The graphical definition of a derivative: as Δx approaches zero in going from (*a*) to (*c*), the difference approximation becomes a derivative.



Partial derivatives

For example, given a function f that depends on both x and y, the partial derivative of f with respect to x at an arbitrary point (x, y) is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

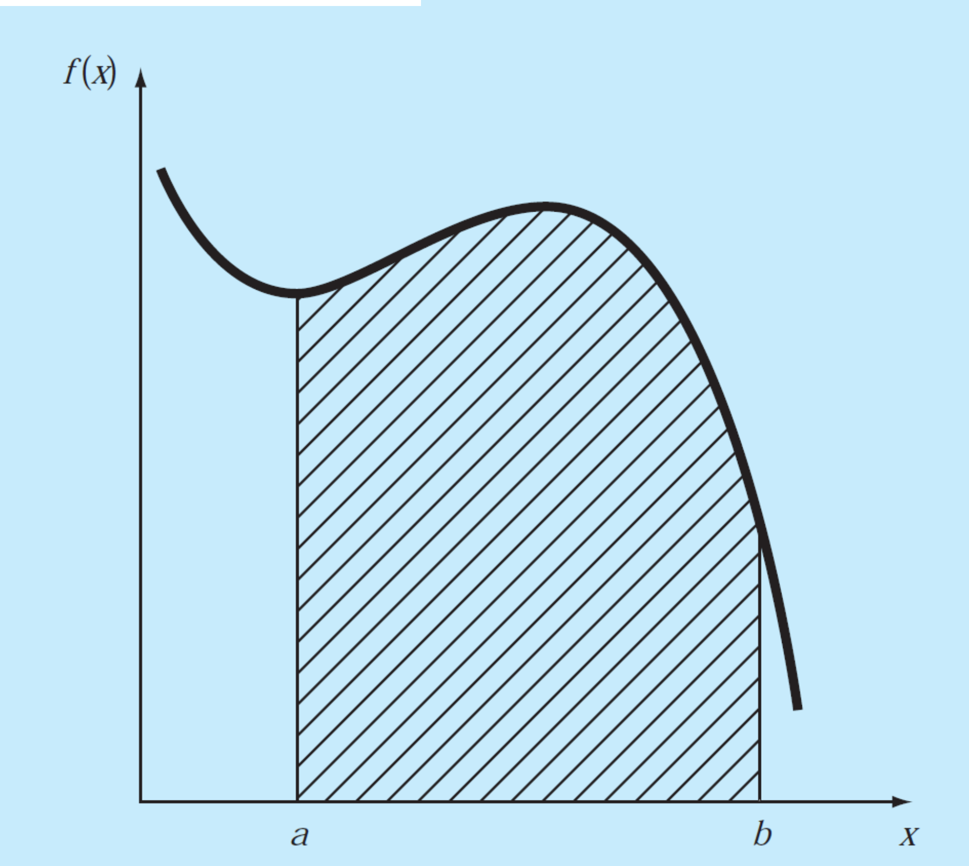
Similarly, the partial derivative of f with respect to y is defined as

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Integration

The inverse process to differentiation in calculus is integration. According to the dictionary definition, to *integrate* means "to bring together, as parts, into a whole; to unite; to indicate the total amount . . . ." Mathematically, integration is represented by

$$I = \int_a^b f(x)\,dx$$



Graphical representation of the integral of $f(x)$ between the limits $x = a$ to b. The integral is equivalent to the area under the curve.

Integration

$$v(t) = \frac{d}{dt} y(t)$$

Conversely, if we are provided with velocity as a function of time, integration can be used to determine its position

$$y(t) = \int_0^t v(t)\, dt$$

Thus, we can make the general claim that the evaluation of the integral

$$I = \int_a^b f(x)\, dx$$

is equivalent to solving the differential equation

$$\frac{dy}{dx} = f(x)$$

for $y(b)$ given the initial condition $y(a) = 0$.

Integral with calculus

$$\int u\,dv = uv - \int v\,du$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b \ln a} + C \qquad a > 0,\ a \neq 1$$

$$\int \frac{dx}{x} = \ln|x| + C \qquad x \neq 0$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \ln|x|\,dx = x\ln|x| - x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{\sqrt{ab}}{a}x + C$$

The strip method

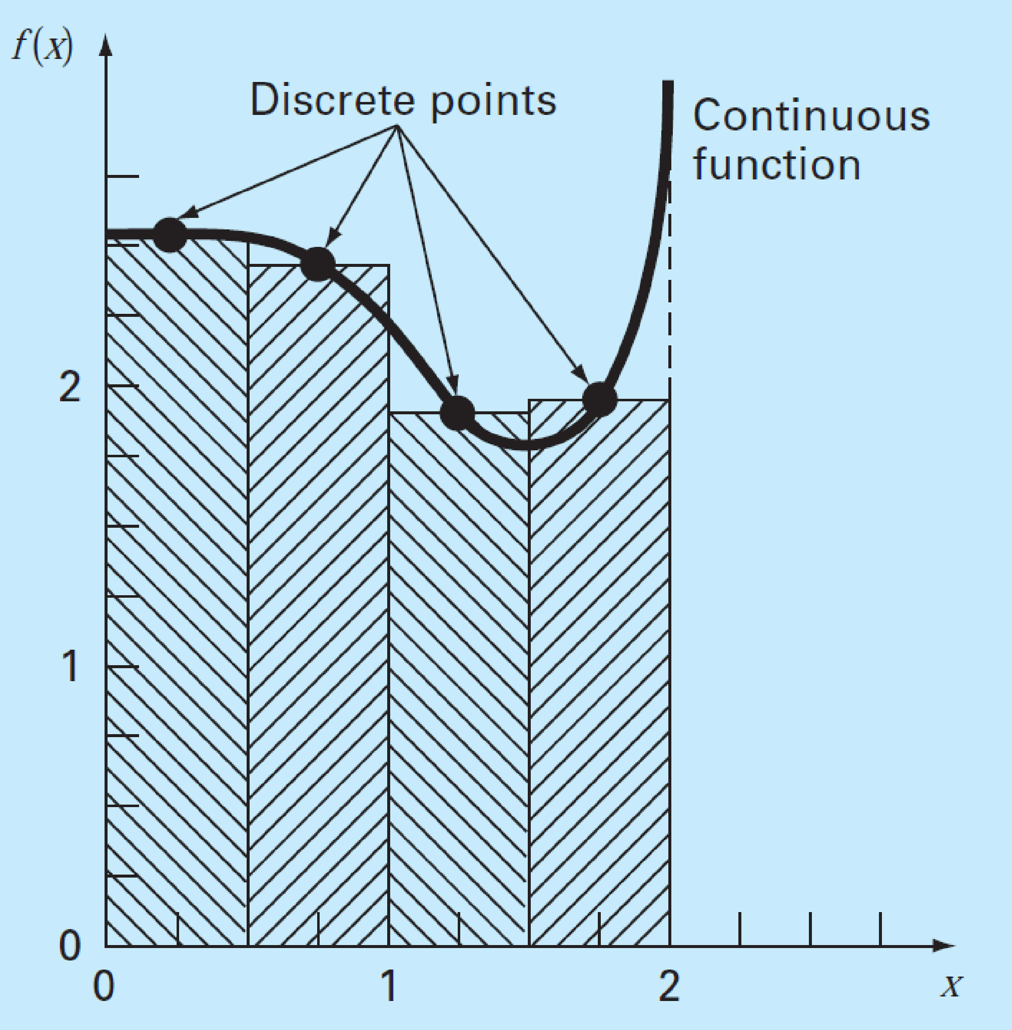
A complicated, continuous function

$$\int_0^2 \frac{2 + \cos(1 + x^{3/2})}{\sqrt{1 + 0.5 \sin x}} e^{0.5x}\, dx$$

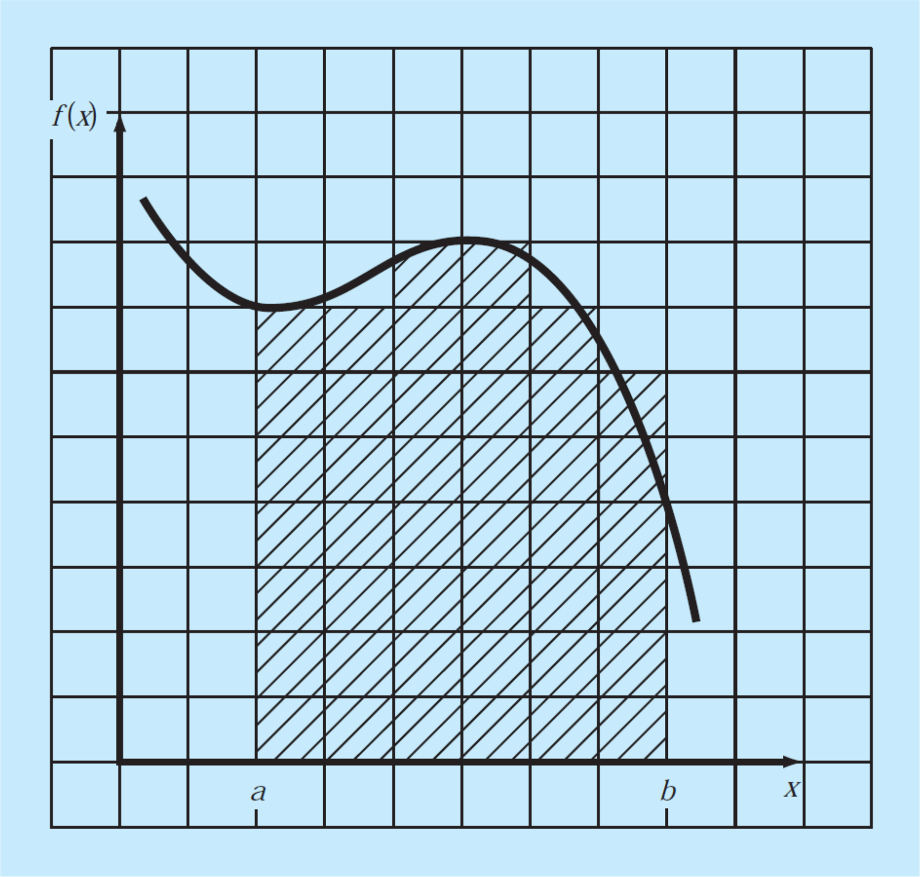
Table of discrete values of f(x) generated from the function

x	$f(x)$
0.25	2.599
0.75	2.414
1.25	1.945
1.75	1.993

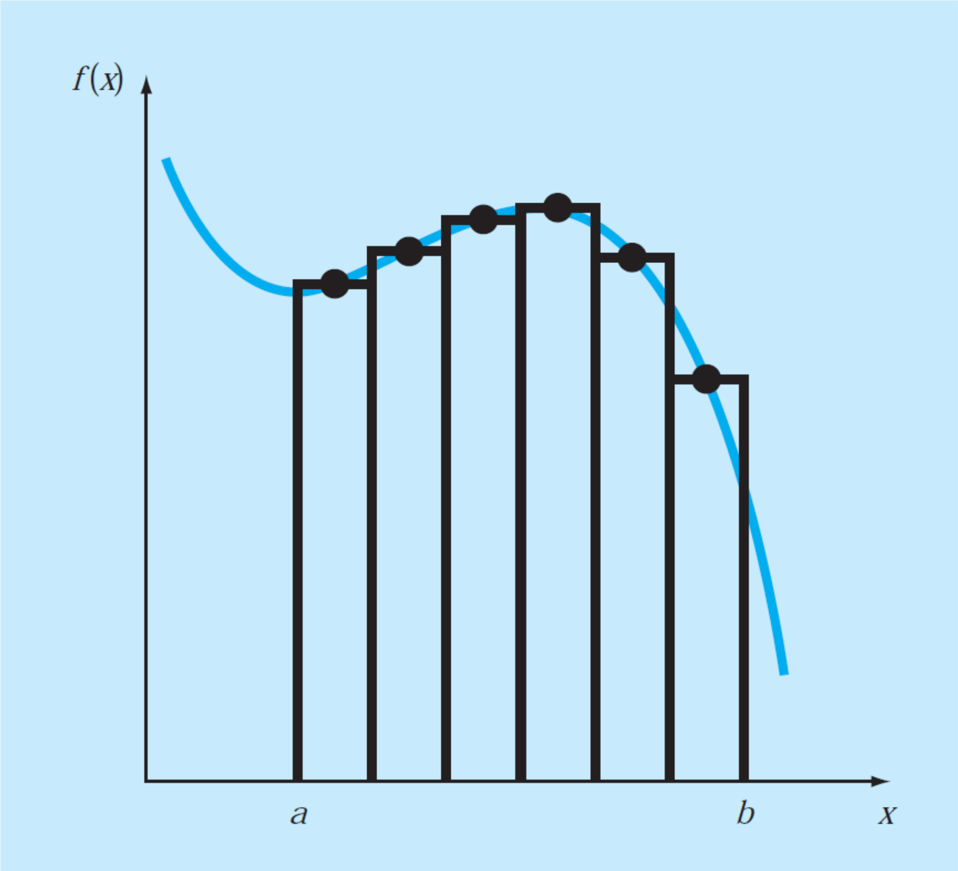
Use of a numerical method(the strip method) to estimate the integral on the basis of the discrete points



Integration



The use of a grid to approximate an integral.



The use of rectangles, or strips, to approximate the integral.

Newton-Cotes Integration Formulas

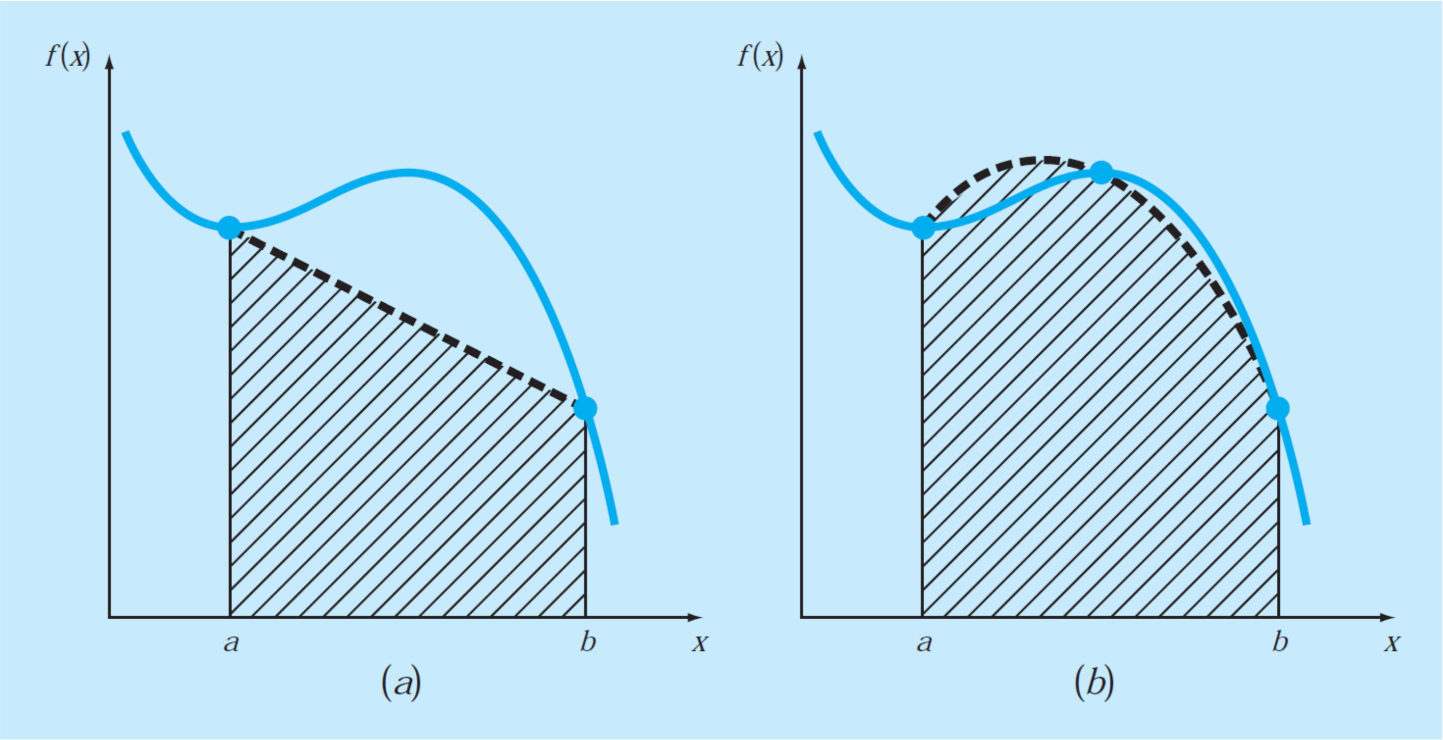
The *Newton-Cotes formulas* are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

where $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

where n is the order of the polynomial.



The approximation of an integral by the area under (*a*) a single straight line and (*b*) a single parabola.

The Trapezoidal rule

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

A straight line can be represented ab

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

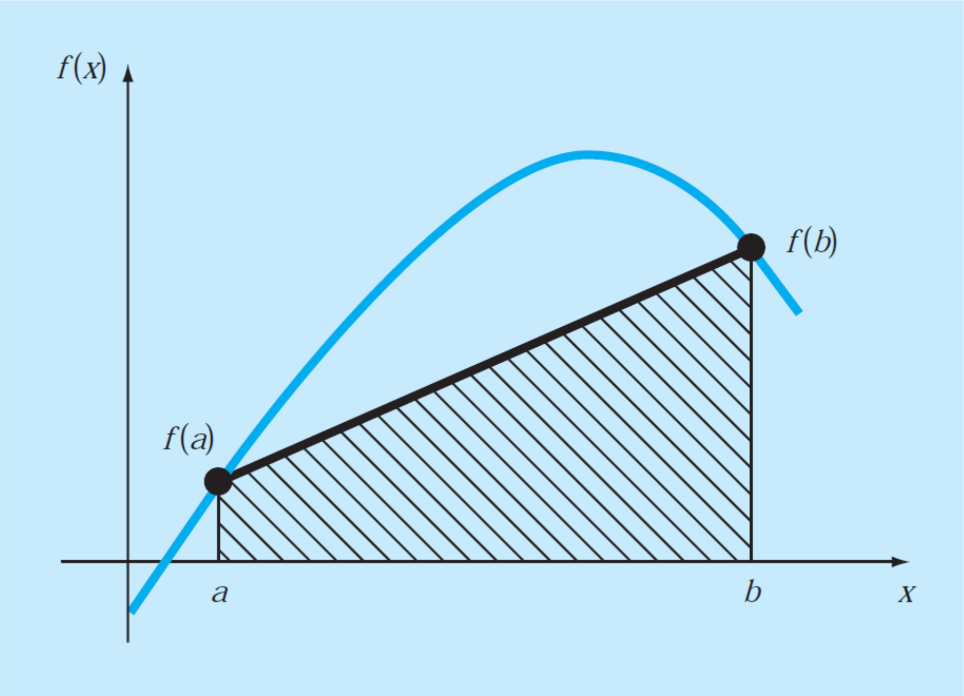
$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

The result of the integration (see Box 21.1 for details) is

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

which is called the *trapezoidal rule.*

$$I \cong \text{width} \times \text{average height}$$



Graphical depiction of the trapezoidal rule.

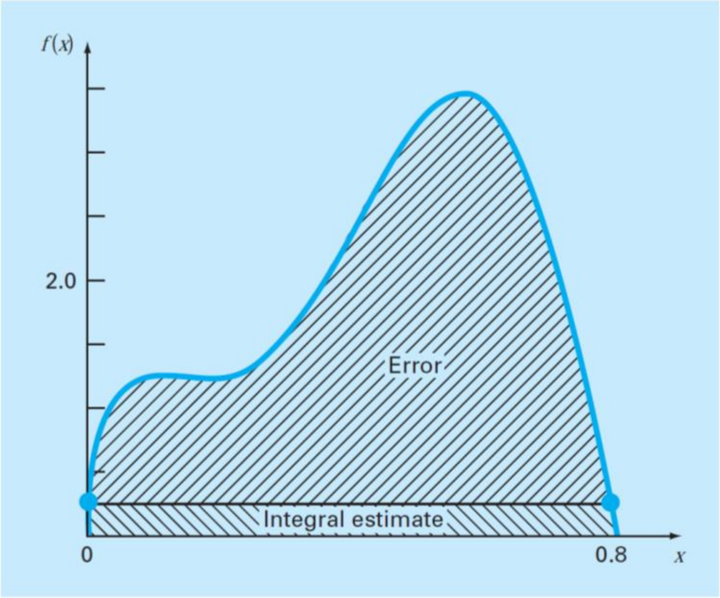
Error of the Trapezoidal Rule

When we employ the integral under a straight-line segment to approximate the integral under a curve, we obviously can incur an error that may be substantial (Fig. 21.6). An estimate for the local truncation error of a single application of the trapezoidal rule is (Box. 21.2)

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 \tag{21.6}$$

where ξ lies somewhere in the interval from a to b. Equation (21.6) indicates that if the function being integrated is linear, the trapezoidal rule will be exact. Otherwise, for functions with second- and higher-order derivatives (that is, with curvature), some error can occur.

Graphical depiction of the use of a single application of the trapezoidal rule to approximate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $x = 0$ to 0.8.



The Multiple-Application Trapezoidal Rule

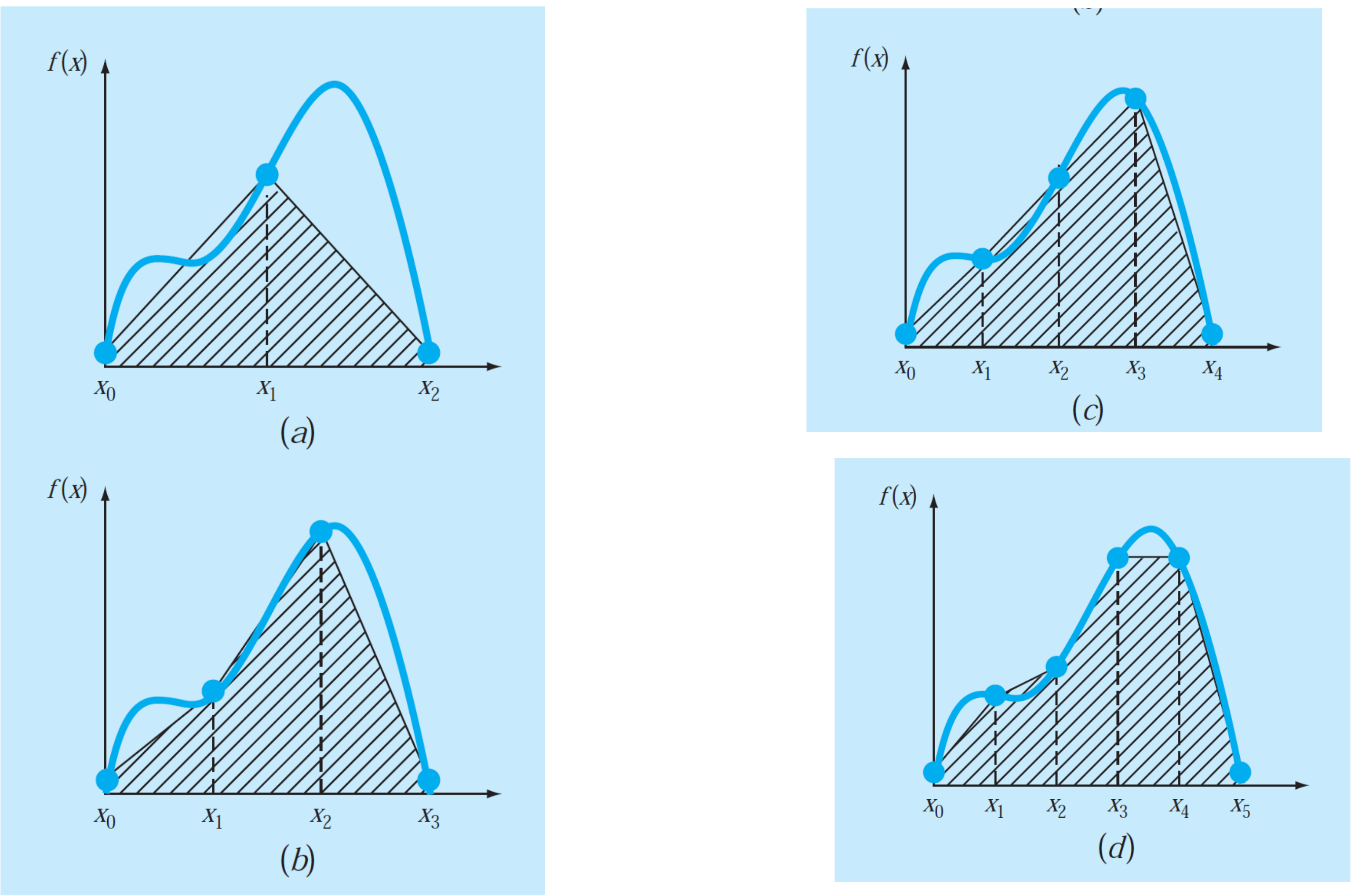


Illustration of the multiple-application trapezoidal rule. (*a*) Two segments, (*b*) three segments, (*c*) four segments, and (*d*) five segments.

The Multiple-Application Trapezoidal Rule

$$h = \frac{b-a}{n} \tag{21.7}$$

If a and b are designated as x_0 and x_n, respectively, the total integral can be represented as

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Substituting the trapezoidal rule for each integral yields

$$I = h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \cdots + h\frac{f(x_{n-1})+f(x_n)}{2} \tag{21.8}$$

or, grouping terms,

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] \tag{21.9}$$

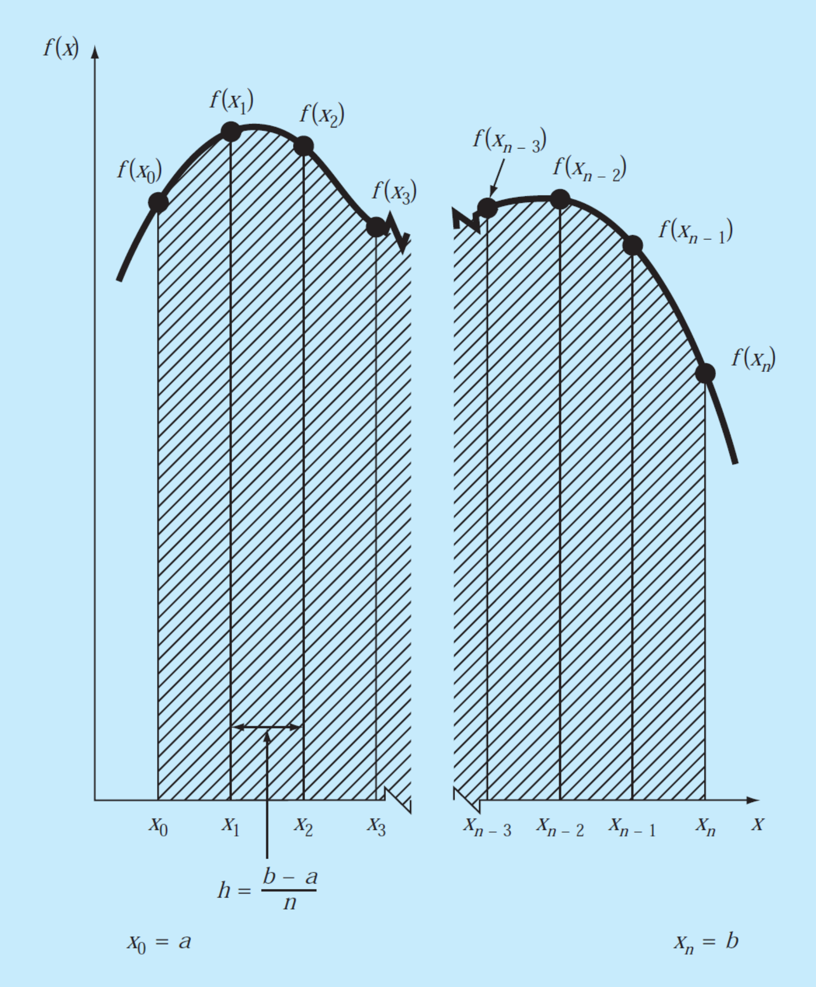
or, using Eq. (21.7) to express Eq. (21.9) in the general form of Eq. (21.5),

$$I = \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}} \tag{21.10}$$

An error for the multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment to give

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$



Example

Problem Statement. Use the two-segment trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Employ Eq. (21.13) to estimate the error. Recall that the correct value for the integral is 1.640533.

Solution. $n = 2$ ($h = 0.4$):

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$$

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$

Results for multiple-application trapezoidal rule to estimate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $x = 0$ to 0.8. The exact value is 1.640533.

n	h	I	ε_t (%)
2	0.4	1.0688	34.9
3	0.2667	1.3695	16.5
4	0.2	1.4848	9.5
5	0.16	1.5399	6.1
6	0.1333	1.5703	4.3
7	0.1143	1.5887	3.2
8	0.1	1.6008	2.4
9	0.0889	1.6091	1.9
10	0.08	1.6150	1.6

Example

The velocity of the parachutist is given as the following function of time

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

where v = velocity (m/s), g = the gravitational constant of 9.8 m/s^2, m = mass of the parachutist equal to 68.1 kg, and c = the drag coefficient of 12.5 kg/s.

How far the parachutist has fallen after a certain time t. The distance is given by

$$d = \int_0^t v(t)\,dt$$

where d is the distance in meters.

$$d = \frac{gm}{c}\int_0^t \left(1 - e^{-(c/m)t}\right) dt$$

Use your software to determine this integral with the multiple-segment trapezoidal rule using different numbers of segments. Note that performing the integration analytically and substituting known parameter values results in an exact value of d = 289.43515 m.

Solution. For the case where $n = 10$ segments, a calculated integral of 288.7491 is obtained. Thus, we have attained the integral to three significant digits of accuracy. Results for other numbers of segments can be readily generated.

Segments	Segment Size	Estimated *d*, m	ε_t (%)
10	1.0	288.7491	0.237
20	0.5	289.2636	0.0593
50	0.2	289.4076	9.5×10^{-3}
100	0.1	289.4282	2.4×10^{-3}
200	0.05	289.4336	5.4×10^{-4}
500	0.02	289.4348	1.2×10^{-4}
1,000	0.01	289.4360	-3.0×10^{-4}
2,000	0.005	289.4369	-5.9×10^{-4}
5,000	0.002	289.4337	5.2×10^{-4}
10,000	0.001	289.4317	1.2×10^{-3}

Up to about 500 segments, the multiple-application trapezoidal rule attains excellent accuracy. However, notice how the error changes sign and begins to increase in absolute value beyond the 500-segment case. The 10,000-segment case actually seems to be diverging from the true value. This is due to the intrusion of round-off error because of the great number of computations for this many segments. Thus, the level of precision is limited, and we would never reach the exact result of 289.4351 obtained analytically.

SIMPSON'S RULES

- For individual applications with nicely behaved functions, the multiple-segment trapezoidal rule is just fine for attaining the type of accuracy required in many engineering applications.
- If high accuracy is required, the multiple-segment trapezoidal rule demands a great deal of computational effort. Although this effort may be negligible for a single application, it could be very important when (*a*) numerous integrals are being evaluated or (*b*) where the function itself is time consuming to evaluate. For such cases, more efficient approaches (like those in the remainder of this chapter and the next) may be necessary.

- Finally, round-off errors can limit our ability to determine integrals. This is due both to the machine precision as well as to the numerous computations involved in simple techniques like the multiple-segment trapezoidal rule.

We now turn to one way in which efficiency is improved. That is, by using higher-order polynomials to approximate the integral.

SIMPSON'S RULES

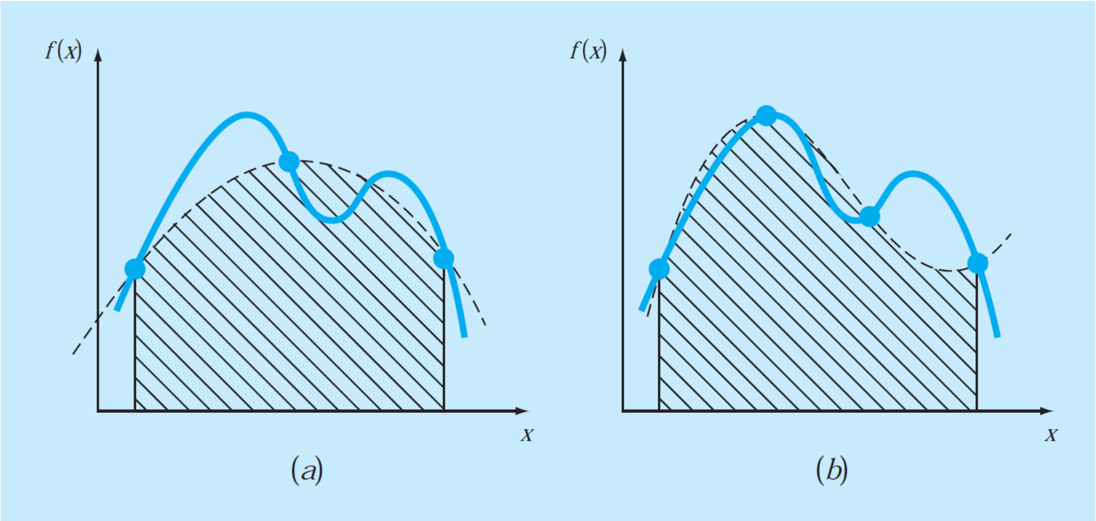
Simpson's 1/3 Rule

Simpson's 1/3 rule results when a second-order interpolating polynomial is substituted into

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial [Eq. (18.23)], the integral becomes

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$



(a) Graphical depiction of Simpson's 1/3 rule: It consists of taking the area under a parabola connecting three points. (b) Graphical depiction of Simpson's 3/8 rule: It consists of taking the area under a cubic equation connecting four points.

After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

$$h = (b-a)/2$$

The label "1/3" stems from the fact that h is divided by 3

Single Application of Simpson's 1/3 Rule

Integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

from $a = 0$ to $b = 0.8$. Recall that the exact integral is 1.640533.

Solution.

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

Therefore, Eq. (21.15) can be used to compute

$$I \cong 0.8 \frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

which represents an exact error of

$$E_t = 1.640533 - 1.367467 = 0.2730667 \qquad \varepsilon_t = 16.6\%$$

which is approximately 5 times more accurate than for a single application of the trapezoidal rule

The estimated error is

$$E_a = -\frac{(0.8)^5}{2880}(-2400) = 0.2730667$$

The Multiple-Application Simpson's 1/3 Rule

Just as with the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width

$$h = \frac{b-a}{n}$$

The total integral can be represented as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

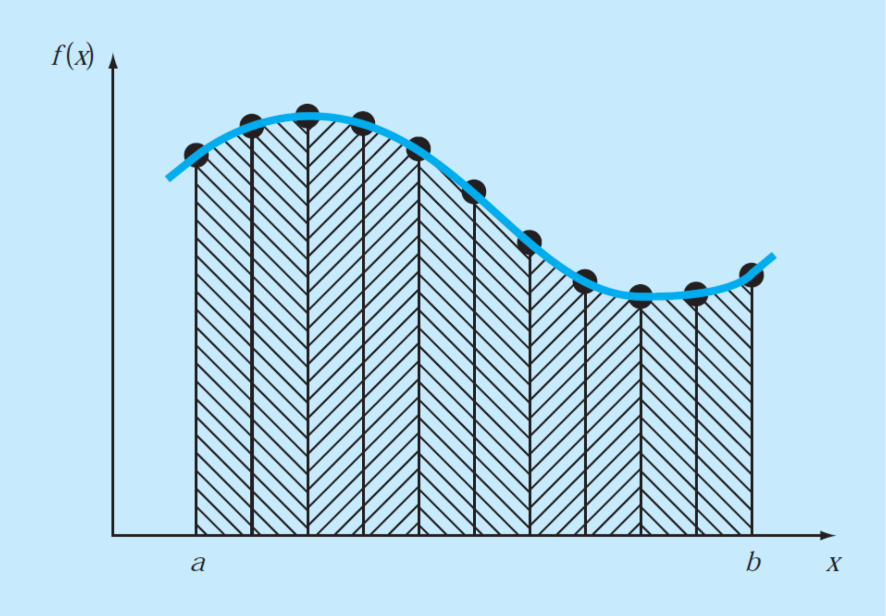
Substituting Simpson's 1/3 rule for the individual integral yields

$$I \cong 2h\frac{f(x_0) + 4\,f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4\,f(x_3) + f(x_4)}{6} + \cdots + 2h\frac{f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)}{6}$$

or,

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$

$$E_a = -\frac{(b-a)^5}{180n^4}\,\bar{f}^{(4)}$$



Graphical representation of the multiple application of Simpson's 1/3 rule. Note that the method can be employed only if the number of segments is even.

Multiple-Application Version of Simpson's 1/3 Rule

Estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Recall that the exact integral is 1.640533.

Solution. $n = 4$ $(h = 0.2)$:

$$f(0) = 0.2 \qquad f(0.2) = 1.288$$
$$f(0.4) = 2.456 \qquad f(0.6) = 3.464$$
$$f(0.8) = 0.232$$

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

The estimated error [Eq. (21.19)] is

$$E_a = -\frac{(0.8)^5}{180(4)^4}(-2400) = 0.017067$$

Simpson's 3/8 Rule

In a similar manner to the derivation of the trapezoidal and Simpson's 1/3 rule, a third-order Lagrange polynomial can be fit to four points and integrated:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

to yield

$$I \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $h = (b - a)/3$. This equation is called *Simpson's 3/8 rule* because h is multiplied by 3/8. It is the third Newton-Cotes closed integration formula. The 3/8 rule can also be expressed in the form of

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}}$$

Thus, the two interior points are given weights of three-eighths, whereas the end points are weighted with one-eighth. Simpson's 3/8 rule has an error of

$$E_t = -\frac{3}{80}h^5 f^{(4)}(\xi)$$

$$E_t = -\frac{(b - a)^5}{6480} f^{(4)}(\xi)$$

(a) Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$.

(b) Use it in conjunction with Simpson's 1/3 rule to integrate the same function for five segments.

Solution.

(a) A single application of Simpson's 3/8 rule requires four equally spaced points:

$$f(0) = 0.2 \qquad f(0.2667) = 1.432724$$
$$f(0.5333) = 3.487177 \qquad f(0.8) = 0.232$$

Using Eq. (21.20),

$$I \cong 0.8\,\frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

$$E_t = 1.640533 - 1.519170 = 0.1213630 \qquad \varepsilon_t = 7.4\%$$

$$E_a = -\frac{(0.8)^5}{6480}(-2400) = 0.1213630$$

(b) The data needed for a five-segment application ($h = 0.16$) is

$$f(0) = 0.2 \qquad f(0.16) = 1.296919$$
$$f(0.32) = 1.743393 \qquad f(0.48) = 3.186015$$
$$f(0.64) = 3.181929 \qquad f(0.80) = 0.232$$

The integral for the first two segments is obtained using Simpson's 1/3 rule:

$$I \cong 0.32\,\frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

For the last three segments, the 3/8 rule can be used to obtain

$$I \cong 0.48\,\frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

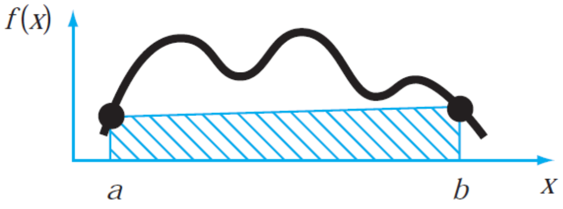
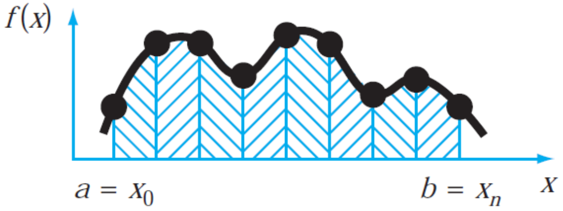
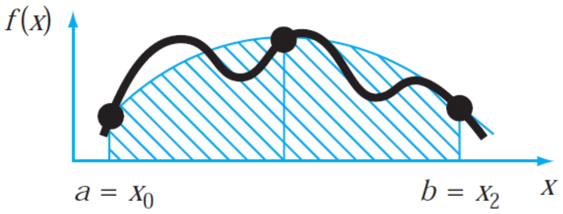
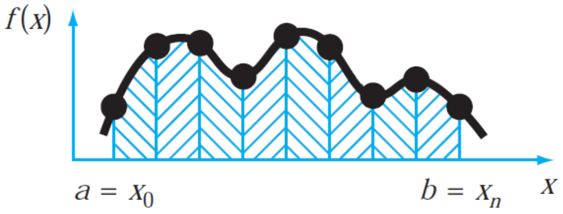
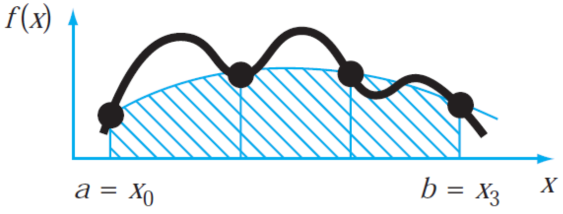
The total integral is computed by summing the two results:

$$I = 0.3803237 + 1.264753 = 1.645077$$
$$E_t = 1.640533 - 1.645077 = -0.00454383 \qquad \varepsilon_t = -0.28\%$$

Newton-Cotes closed integration formulas. The formulas are presented in the format of Eq. (21.5) so that the weighting of the data points to estimate the average height is apparent. The step size is given by $h = (b - a)/n$.

Segments (n)	Points	Name	Formula	Truncation Error
1	2	Trapezoidal rule	$(b-a)\,\dfrac{f(x_0) + f(x_1)}{2}$	$-(1/12)h^3 f''(\xi)$
2	3	Simpson's 1/3 rule	$(b-a)\,\dfrac{f(x_0) + 4f(x_1) + f(x_2)}{6}$	$-(1/90)h^5 f^{(4)}(\xi)$
3	4	Simpson's 3/8 rule	$(b-a)\,\dfrac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$	$-(3/80)h^5 f^{(4)}(\xi)$
4	5	Boole's rule	$(b-a)\,\dfrac{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)}{90}$	$-(8/945)h^7 f^{(6)}(\xi)$
5	6		$(b-a)\,\dfrac{19f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)}{288}$	$-(275/12{,}096)h^7 f^{(6)}(\xi)$

TABLE PT6.5 Summary of important formulas presented in Part Six.

Method	Formulation	Graphic Interpretations	Error
Trapezoidal rule	$I \simeq (b-a)\dfrac{f(a)+f(b)}{2}$		$-\dfrac{(b-a)^3}{12}f''(\xi)$
Multiple-application trapezoidal rule	$I \simeq (b-a)\dfrac{f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)}{2n}$		$-\dfrac{(b-a)^3}{12n^2}\bar{f}''$
Simpson's 1/3 rule	$I \simeq (b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$		$-\dfrac{(b-a)^5}{2880}f^{(4)}(\xi)$
Multiple-application Simpson's 1/3 rule	$I \simeq (b-a)\dfrac{f(x_0)+4\sum_{i=1,3}^{n-1} f(x_i)+2\sum_{j=2,4}^{n-2} f(x_j)+f(x_n)}{3n}$		$-\dfrac{(b-a)^5}{180n^4}f^{(4)}$
Simpson's 3/8 rule	$I \simeq (b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$		$-\dfrac{(b-a)^5}{6480}f^{(4)}(\xi)$

MULTIPLE INTEGRALS

Multiple integrals are widely used in engineering. For example, a general equation to compute the average of a two-dimensional function can be written as

$$\bar{f} = \frac{\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy}{(d - c)(b - a)}$$

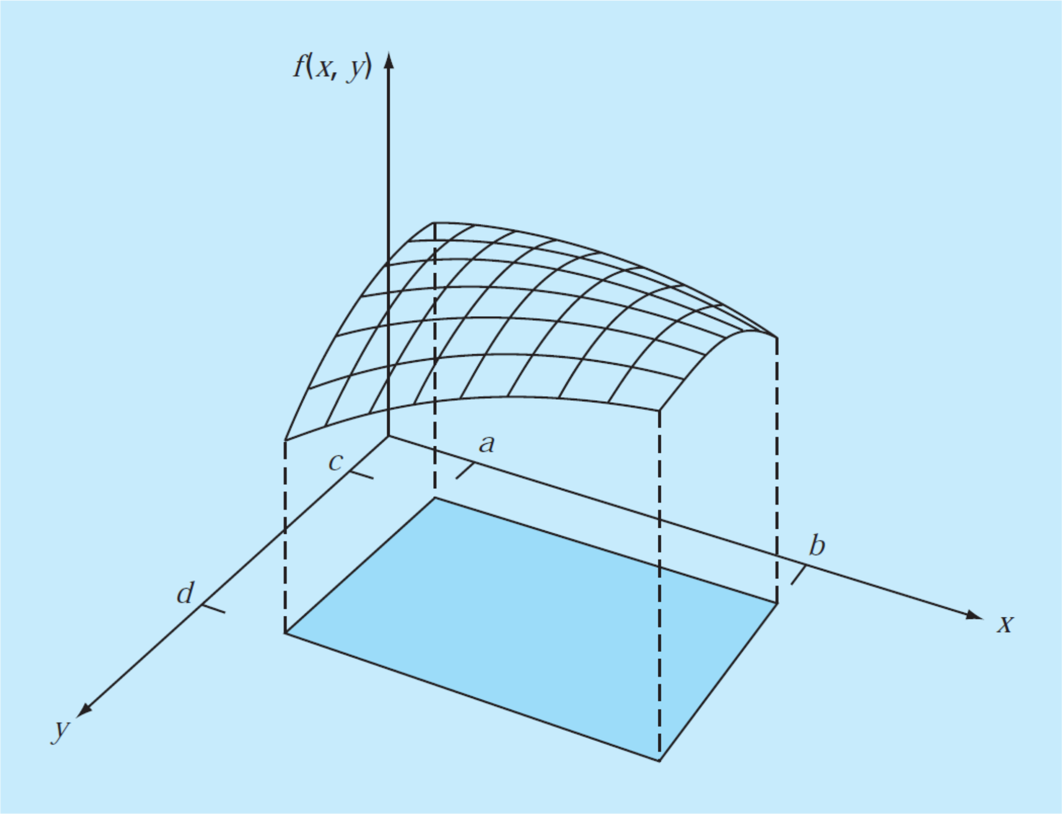
The numerator is called a double integral.

The techniques discussed in this chapter (and the following chapter) can be readily employed to evaluate multiple integrals. A simple example would be to take the double integral of a function over a rectangular area

Recall from calculus that such integrals can be computed as iterated integrals

$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

A numerical double integral would be based on the same idea. First, methods like the multiple-segment trapezoidal or Simpson's rule would be applied in the first dimension with each value of the second dimension held constant. Then the method would be applied to integrate the second dimension. The approach is illustrated in the following example.



Double integral as the area under the function surface.

Using Double Integral to Determine Average Temperature

Problem Statement. Suppose that the temperature of a rectangular heated plate is described by the following function:

$$T(x, y) = 2xy + 2x - x^2 - 2y^2 + 72$$

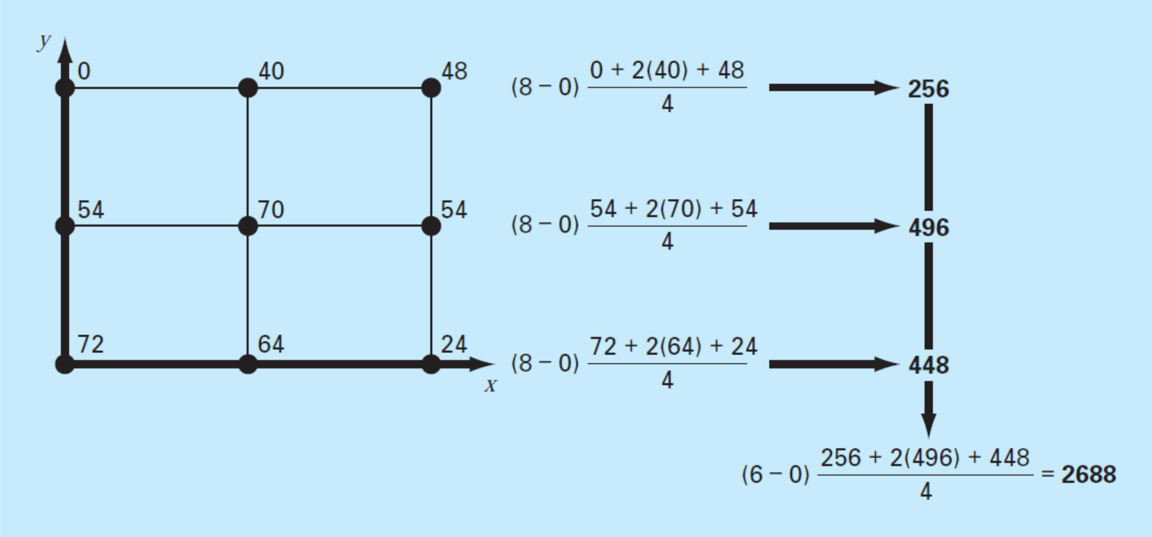
If the plate is 8-m long (x dimension) and 6-m wide (y dimension), compute the average temperature.

Solution. First, let us merely use two-segment applications of the trapezoidal rule in each dimension. The temperatures at the necessary x and y values are depicted in Fig. 21.17. Note that a simple average of these values is 47.33. The function can also be evaluated analytically to yield a result of 58.66667.

To make the same evaluation numerically, the trapezoidal rule is first implemented along the x dimension for each y value. These values are then integrated along the y dimension to give the final result of 2688. Dividing this by the area yields the average temperature as $2688/(6 \times 8) = 56$.

Now we can apply a single-segment Simpson's 1/3 rule in the same fashion. This results in an integral of 2816 and an average of 58.66667, which is exact. Why does this occur? Recall that Simpson's 1/3 rule yielded perfect results for cubic polynomials. Since the highest order term in the function is second order, the same exact result occurs for the present case.

For higher-order algebraic functions as well as transcendental functions, it would be necessary to use multi-segment applications to attain accurate integral estimates. In addition, Chap. 22 introduces techniques that are more efficient than the Newton-Cotes formulas for evaluating integrals of given functions. These often provide a superior means to implement the numerical integrations for multiple integrals.



Numerical evaluation of a double integral using the two-segment trapezoidal rule.

Numerical integration function in MATLAB

integral	Numerical integration
integral2	Numerically evaluate double integral
integral3	Numerically evaluate triple integral
quadgk	Numerically evaluate integral, adaptive Gauss-Kronrod quadrature
quad2d	Numerically evaluate double integral, tiled method
cumtrapz	Cumulative trapezoidal numerical integration
trapz	Trapezoidal numerical integration
polyint	Polynomial integration

TABLE 23.1 MATLAB functions to implement (*a*) integration and (*b*) differentiation.

Function	Description
(*a*) Integration:	
cumtrapz	Cumulative trapezoidal numerical integration
dblquad	Numerically evaluate double integral
polyint	Integrate polynomial analytically
quad	Numerically evaluate integral, adaptive Simpson quadrature
quadgk	Numerically evaluate integral, adaptive Gauss-Kronrod quadrature
quadl	Numerically evaluate integral, adaptive Lobatto quadrature
quadv	Vectorized quadrature
trapz	Trapezoidal numerical integration
triplequad	Numerically evaluate triple integral
(*b*) Differentiation:	
del2	Discrete Laplacian
diff	Differences and approximate derivatives
gradient	Numerical gradient
polyder	Polynomial derivative

Example: Velocity from an accelerometer

An *accelerometer* is used in aircraft, rockets, and other vehicles to estimate the vehicle's velocity and displacement. The accelerometer integrates the acceleration signal to produce an estimate of the velocity, and it integrates the velocity estimate to produce an estimate of displacement. Suppose the vehicle starts from rest at time $t = 0$, and its measured acceleration is given in the following table.

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Acceleration (m/s^2)	0	2	4	7	11	17	24	32	41	48	51

(a) Estimate the velocity v after 10 s.
(b) Estimate the velocity at times $t = 1, 2, \ldots, 10$ s.

■ Solution

(a) The initial velocity is zero, so $v(0) = 0$. The relation between the velocity and acceleration $a(t)$ is

$$v(10) = \int_0^{10} a(t)\,dt + v(0) = \int_0^{10} a(t)\,dt$$

The script le is shown below .

```
t = 0:10;
a = [0,2,4,7,11,17,24,32,41,48,51];
v10 = trapz(t,a)
```

The answer for the velocity after 10 s is `v10`, and it is 211.5 m/s.

(b) The following script le uses the fact that the velocity can be expressed as

$$v(t_{k+1}) = \int_{t_k}^{t_{k+1}} a(t)\,dt + v(t_k) \qquad k = 1, 2, \ldots, 10$$

where $v(t_1) = 0$.

```
t = 0:10;
a = [0,2,4,7,11,17,24,32,41,48,51];

v(1) = 0;
for k = 1:10
   v(k+1) = trapz(t(k:k+1), a(k:k+1))+v(k);
end
disp([t',v'])
```

The answers are given in the following table.

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Velocity (m/s)	0	1	4	9.5	18.5	32.5	53	81	117	162	211.5

Quadrature Functions

Another approach to numerical integration is Simpson’s rule, which divides the integration range b-a into an even number of sections and uses a different quadratic function to represent the integrand for each panel. A quadratic function has three parameters, and Simpson’s rule computes these parameters by requiring that the quadratic pass through the function’s three points corresponding to the two adjacent panels. To obtain greater accuracy, we can use polynomials of degree higher than 2.

The MATLAB function **quad** implements an adaptive version of Simpson’s rule. The **quadl** function is based on an adaptive Lobatto integration method, where the letter “l” in qual stands for Lobatto. The term quad is an abbreviation of quadrature.

The function quad(fun, a, b) computes the integral of the function fun between the limits a and b. The input fun, which represents the integrand f(x), is either a function handle of the integrand function.
A=quad(@sin, 0,pi)
A=quad(‘sin’,0,pi)
A=2.0000

Example: Evaluation for Fresnel's Cosine Integral

Some simple-looking integrals cannot be evaluated in closed form. An example is Fresnel's cosine integral

$$A = \int_0^b \cos x^2 \, dx$$

(a) Demonstrate two ways to compute the integral when the upper limit is $b = \sqrt{2\pi}$

(b) Demonstrate the use of a nested function to compute the more general integral for $n = 2$ and for $n = 3$

$$A = \int_0^b \cos x^n \, dx$$

Solution

(a) The integrand $\cos x^2$ obviously does not contain any singularities that might cause problems for the integration function. We demonstrate two ways to use the quad function.

1. With a function le: De ne the integrand with a user -de ned function as shown by the following function le.

```
function c2 = cossq(x)
c2 = cos(x.^2);
```

   The quad function is called as follows: `A = quad(@cossq,0,sqrt(2*pi))`. The result is $A = 0.6119$.

2. With an anonymous function (anonymous functions are discussed in Section 3.3): The session is

```
>>cossq = @(x)cos(x.^2);
>>A = quad(cossq,0,sqrt(2*pi))
  A =
     0.6119
```

The two lines can be combined into one as follows:

```
A = quad(@(x)cos(x.^2),0,sqrt(2*pi))
```

The advantage of using an anonymous function is that you need not create and save a function le. However, for complicated integrand functions, using a function le is preferable.

(b) Because quad requires that the integrand function have only one argument, the following code will not work.

```
>>cossq = @(x)cos(x.^n);
>>n = 2;
>>A = quad(cossq,0,sqrt(2*pi))
??? Unde ned function or variable 'n'.
```

Instead we will use parameter passing with a nested function (nested functions are discussed in Section 3.3). First create and save the following function.

Example: Evaluation for Fresnel's Cosine Integral

```
function A = integral_n(n)
A = quad(@cossq_n,0,sqrt(2*pi));

% Nested function
   function integrand = cossq_n(x)
      integrand = cos(x.^n);
   end
end
```

The session for $n = 2$ and $n = 3$ is as follows.

```
>>A = integral_n(2)
    A =
       0.6119
>>A = integral_n(3)
    A =
       0.7734
```

The `quad` functions have some optional arguments for analyzing and adjusting the algorithm's ef ciency and accuracy . Type `help quad` for details.

Polynomial Integration

MATLAB proves the **polyint** function to compute the integral of a polynomial. The syntax q=polyint(p,c) return a polynomial q representing the integral of polynomial p with a user-specified scalar constant of integration C. The elements of the vector p are the coefficients of the polynomial, arranged in descending powers.

For example, the integral of $12x^3 + 9x^2 + 8x + 5$ is obtained from `q = polyint([12,9,8,5], 10)`. The answer is `q = [3, 3, 4, 5, 10]`, which corresponds to $3x^4 + 3x^3 + 4x^2 + 5x + 10$. Because polynomial integrals can be obtained from a symbolic formula, the `polyint` function is not a numerical integration operation.

Double Integrals

The function `dblquad` computes double integrals. Consider the integral

$$A = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

The basic syntax is

```
A = dblquad(fun, a, b, c, d)
```

where `fun` is the handle to a user-de ned function that de nes the integrand $f(x, y)$. The function must accept a vector x and a scalar y, and it must return a vector result, so the appropriate array operations must be used. The extended syntax enables the user to adjust the accuracy and to use `quadl` or a user-de ned quadrature routine. See the MATLAB Help for details.

For example, using an anonymous function to compute the integral

$$A = \int_0^1 \int_1^3 xy^2\, dx\, dy$$

you type

```
>>fun = @(x,y)x.*y^2;
>>A = dblquad(fun, 1, 3, 0, 1)
```

The answer is $A = 1.3333$.

Example: Double Integral over a Nonrectangular Region

Compute the integral

$$A = \int\int_R (x - y)^4 (2x + y)^2 \, dx\, dy$$

over the region R bounded by the lines

$$x - y = \pm 1 \qquad 2x + y = \pm 2$$

Solution

We must convert the integral into one that is speci ed over a rectangular region. To do this, let $u = x - y$ and $v = 2x + y$. Thus, using the Jacobian, we obtain

$$dx\,dy = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} du\,dv = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} du\,dv = \frac{1}{3}\,du\,dv$$

Then the region R is speci ed as a rectangular region in terms of u and v. Its boundaries are given by $u = \pm 1$ and $v = \pm 2$, and the integral becomes

$$A = \frac{1}{3}\int_{-2}^{2}\int_{-1}^{1} u^4 v^2 \, du\, dv$$

and the MATLAB session is

```
>>fun = @(u,v)u.^4*v^2;
>>A = (1/3)*dblquad(fun, -1, 1, -2, 2)
```

The answer is $A = 0.7111$.

Triple Integrals

Triple Integrals

The function `triplequad` computes triple integrals. Consider the integral

$$A = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The basic syntax is

```
A = triplequad(fun, a, b, c, d, e, f)
```

where `fun` is the handle to a user-de ned function that de nes the integrand $f(x, y, z)$. The function must accept a vector x, a scalar y, and a scalar z, and it must return a vector result, so the appropriate array operations must be used. The extended syntax enables the user to adjust the accuracy and to use `quadl` or a user-de ned quadrature routine. See the MATLAB Help for details. For example, to compute the integral

$$A = \int_1^2 \int_0^2 \int_1^3 \left(\frac{xy - y^2}{z} \right) dx\, dy\, dz$$

You type

```
>>fun = @(x,y,z)(x*y-y^2)/z;
>>A = triplequad(fun, 1, 3, 0, 2, 1, 2)
```

The answer is $A = 1.8484$.

Practice problems

Evaluate the following integral:

$$\int_0^3 (1 - e^{-x})\,dx$$

(a) analytically; **(b)** single application of the trapezoidal rule; **(c)** multiple-application trapezoidal rule, with $n = 2$ and 4; **(d)** single application of Simpson's 1/3 rule; **(e)** multiple-application Simpson's 1/3 rule, with $n = 4$; **(f)** single application of Simpson's 3/8 rule; and **(g)** multiple-application Simpson's rule, with $n = 5$. For each of the numerical estimates **(b)** through **(g)**, determine the percent relative error based on **(a)**.

Integrate the following function analytically and using the trapezoidal rule, with $n = 1, 2, 3$, and 4:

$$\int_{1}^{2} (x + 1/x)^2 \, dx$$

Use the analytical solution to compute true percent relative errors to evaluate the accuracy of the trapezoidal approximations.

Integrate the following function both analytically and using Simpson's rules, with $n = 4$ and 5. Discuss the results.

$$\int_{-3}^{5} (4x - 3)^3 \, dx$$

Integrate the following function both analytically and numerically. Use both the trapezoidal and Simpson's 1/3 rules to numerically integrate the function. For both cases, use the multiple-application version, with $n = 4$. Compute percent relative errors for the numerical results.

$$\int_0^3 x^2 e^x \, dx$$

The following data was collected for a cross-section of a river (y = distance from bank, H = depth and U = velocity):

y, m	0	1	3	5	7	8	9	10
H, m	0	1	1.5	3	3.5	3.2	2	0
U, m/s	0	0.1	0.12	0.2	0.25	0.3	0.15	0

Use numerical integration to compute the (*a*) average depth, (*b*) cross-sectional area, (*c*) average velocity, and (*d*) the flow rate. Note that the cross-sectional area (A_c) and the flow rate (Q) can be computed as

$$A_c = \int_0^y H(y)\,dy \qquad\qquad Q = \int_0^y H(y)U(y)\,dy$$

The outflow concentration from a reactor is measured at a number of times over a 24-hr period:

t, hr	0	1	5.5	10	12	14	16	18	20	24
c, mg/L	1	1.5	2.3	2.1	4	5	5.5	5	3	1.2

The flow rate for the outflow in m^3/s can be computed with the following equation:

$$Q(t) = 20 + 10 \sin\left(\frac{2\pi}{24}(t - 10)\right)$$

Use the best numerical integration method to determine the flow-weighted average concentration leaving the reactor over the 24-hr period,

$$\bar{c} = \frac{\int_0^t Q(t)c(t)dt}{\int_0^t Q(t)dt}$$

A transportation engineering study requires that you determine the number of cars that pass through an intersection traveling during morning rush hour. You stand at the side of the road and count the number of cars that pass every 4 minutes at several times as tabulated below. Use the best numerical method to determine **(a)** the total number of cars that pass between 7:30 and 9:15, and **(b)** the rate of cars going through the intersection per minute. (*Hint:* Be careful with units.)

Time (hr)	7:30	7:45	8:00	8:15	8:45	9:15
Rate (cars per 4 min)	18	24	26	20	18	9