MATLAB and Engineering Application

FEM/FEA

Finite element method

- Finite element method (finite element method) or finite element analysis (finite element analysis) is a very effective tool to obtain approximate solutions of complex differential equations, and is an important basic principle of modern digital technology

- The idea of the finite element method can be traced back to the ancients' practice of "turning the whole into zeros" and "turning the circle into a straight line", such as the allusion of "Cao Chong weighing the elephant", the ancient Chinese mathematician Liu Hui used the method of cutting a circle to determine the circumference of a circle. Carry out calculations; these actually embody the idea of discrete approximation, that is, using a large number of simple small objects to "flush out" complex large objects.

Development of Finite Element Analysis

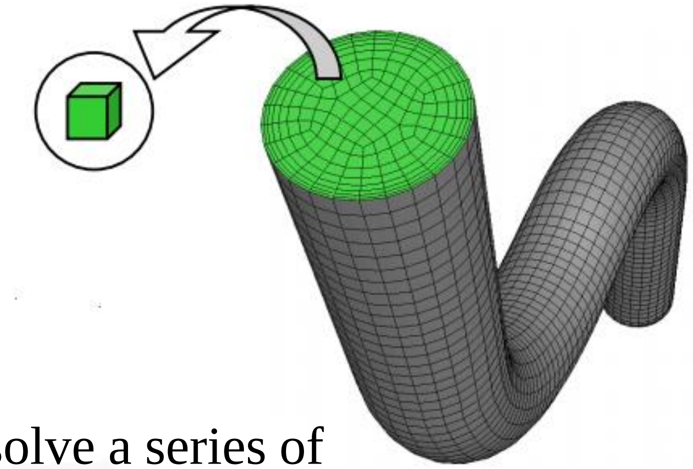
- In 1870, British scientist Rayleigh used an imaginary "trial function" to solve complex differential equations. In 1909, Ritz developed it into a perfect numerical approximation method, laying a solid foundation for modern finite element methods.

- In 1960, Clough first proposed and used the name "finite element method" when dealing with plane elasticity problems; in 1955, Argyris in Germany published the first book on energy principles and matrix methods in structural analysis. , which laid an important foundation for subsequent finite element research. In 1967, Zienkiewicz and Cheung published the first monograph on finite element analysis; after 1970, the finite element method began to be applied to deal with nonlinear and large deformation problems.

FEM

FEM： a numerical calculation method, similar to finite difference method, finite volume method

The algorithm:



Create model → Discretize (mesh) the model to obtain elements → Apply initial and boundary conditions → solve a series of algebraic equations → Obtain other physical quantities(displacement, strain and stress)

FEM

Now, finite element analysis has become the mainstream of numerical calculation, and the steady/transient, linear/nonlinear problems in structure, heat, fluid, and electromagnetic fields can be solved by finite element method. A variety of software that adopte finite element analysis software been developed, such as ANSYS, NASTRAN, ABAQUS, ADINA, LS-DYNA, etc.

The main features of finite element software can be summarized as follows:

depth： Solve a variety of complex problems

breadth： Covers multidisciplinary fields (structural, thermal, fluids, electromagnetics, etc.)

- complex： Multiphysics Coupling
- agility： From simple to complex, from single line to multi-core, parallel processing
- adaptability： compatible to CAD software and data sharing

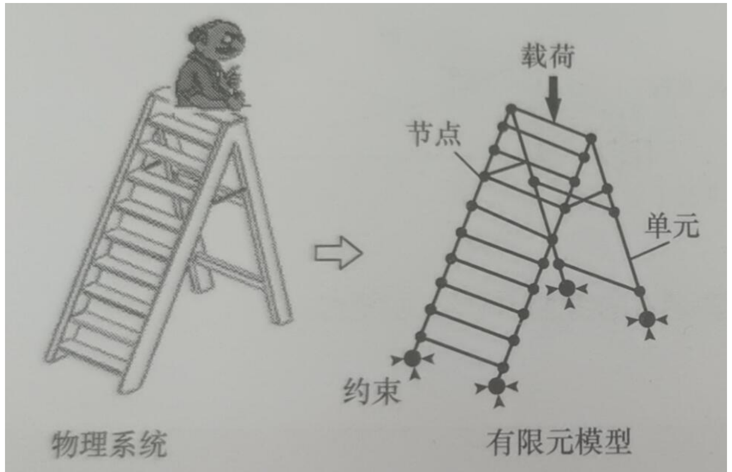
Terminology

- Physical system

All substances in nature do not exist alone in the form of isolated individuals, they interact with the surrounding environment. A physical system is a whole with specific functions which comprised of several interacting elements (geometry and loads) under certain environmental conditions (physical field).

- Finite Element Model

The finite element model is an idealized discrete mathematical abstract model of the real system, which consists of some simple-shaped elements, which are connected by nodes and bear a certain load.



Real physical systems and finite element models

FEA procedure

1. Preprocessing

The solution domain is established and discretized into a finite number of elements.

The element shape function is assumed to represent the physical behavior of the element.

2. Solving

Set up equations for the element.

Construct the global stiffness matrix.

Apply boundary conditions, initial conditions, and loads.

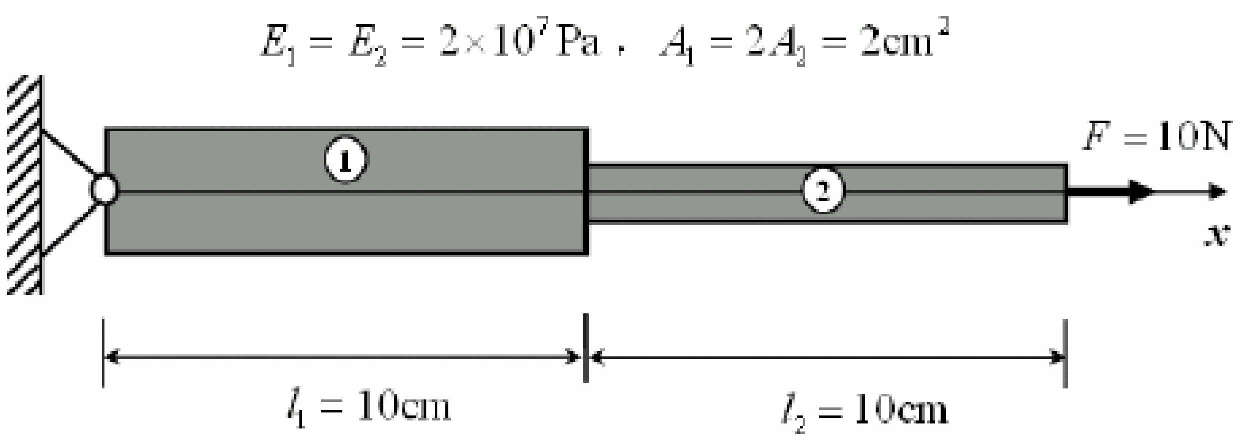
Solve systems of linear or nonlinear differential equations to obtain nodal solutions, such as displacement or temperature values at different nodes.

3. Postprocessing

Obtain other quantities such as stress, strain in the structural field, heat flux in the temperature field, etc.

Verify the results and interpret the solution.

Example—Step bar stretch

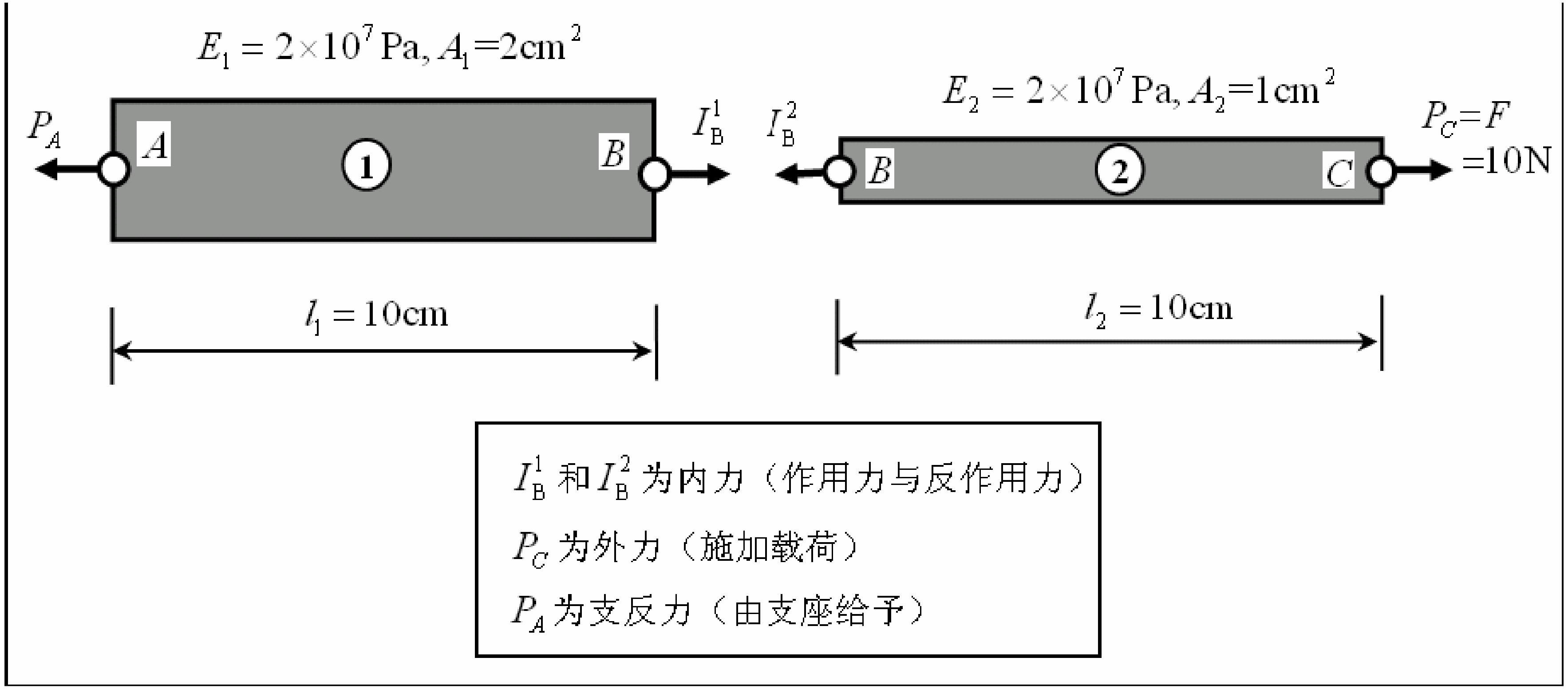


Elastic Modulus and Structural Dimensions are as follows:$E_1=E_2=2\times10^7$Pa，$A_1=2A_2=2cm^2$,$l_1=l_2=10cm$， F=10N

Example—Step bar stretch

Solve the problem analytically

- **perform mechanical analysis on the member ② at the right end**



- The two members are decomposed, and the force balance at each associated node is described under the condition of external force F at point C

$$P_c = F = 10N$$

For rod ②

$$I_B^2 = P_c = F = 10N$$

Example—Step bar stretch

- IB_1 and IB_2 are a pair of internal force

$$P_A = I_B^1 = I_B^2 = P_C = F = 10N$$

- **The stress applied on rod ① is:**

$$\sigma_1 = \frac{P_A}{A_1} = \frac{10N}{2cm^2} = 5\times10^4 \frac{N}{m^2} = 5\times10^4 P_a$$

Example—Step bar stretch

- The stress applied on rod ② is:
- $$\sigma_2 = \frac{P_C}{A_2} = \frac{10N}{1cm^2} = 1\times10^5 \frac{N}{m^2} = 1\times10^5 P_a$$

- For elastic material, the stress follows Hooke law

$$\left.\begin{array}{l} \sigma_1 = E_1\varepsilon_1 \\ \sigma_1 = E_2\varepsilon_2 \end{array}\right\}$$

- The strain on rods ① and ② are described as

$$\left.\begin{array}{l} \varepsilon_1 = \dfrac{\sigma_1}{E_1} = \dfrac{5\times10^4 P_A}{2\times10^7 P_a} = 2.5\times10^{-3} \\ \varepsilon_2 = \dfrac{\sigma_2}{E_2} = \dfrac{1\times10^5 P_A}{2\times10^7 P_a} = 5\times10^{-3} \end{array}\right\}$$

Example—Step bar stretch

- The strain is defined as ε=ΔL/L
- The change of length of rod ① and ② are

$$\left.\begin{aligned}\Delta l_1 &= \varepsilon_1 \cdot l_1 = 2.5\times10^{-3}\times10 = 2.5\times10^{-2}\,\text{cm}\\ \Delta l_2 &= \varepsilon_2 \cdot l_2 = 5\times10^{-3}\times10 = 5\times10^{2}\,\text{cm}\end{aligned}\right\}$$

- As A point is fixed, uA=0， The displacement of point B is described as

$$u_B = \Delta l_1 = 2.5\times10^{-2}\,\text{cm}$$

Example—Step bar stretch

- The displacement of C point is described as

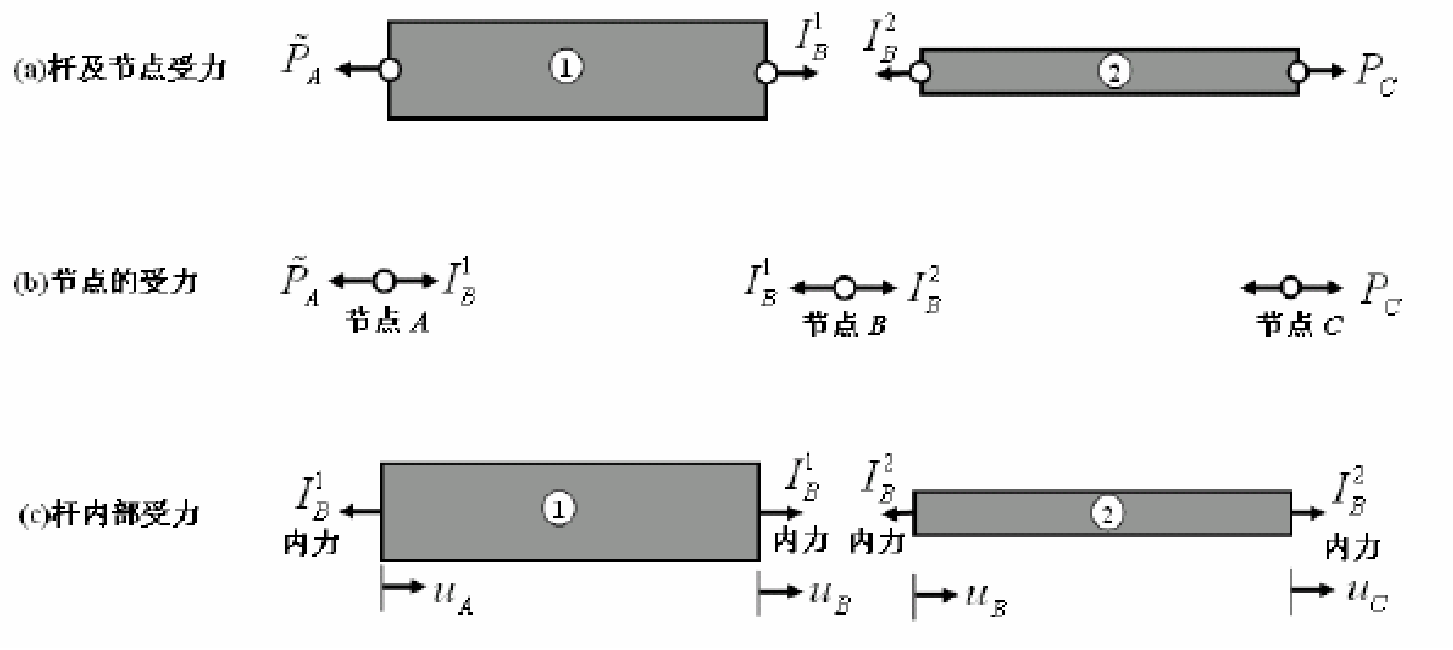
$$u_c = \Delta l_1 + \Delta l_2 = 7.5\times10^{-2}\,\text{cm}\ ,$$

- To sum up, the solution is

$$\left.\begin{array}{l}\sigma_1 = 5\times10^4\,\text{Pa},\ \ \sigma_2 = 1\times10^5\,\text{Pa}\\ \varepsilon_1 = 2.5\times10^{-3},\ \ \varepsilon_2 = 5\times10^{-3}\\ u_A = 0,\ \ u_B = 2.5\times10^{-2}\,\text{cm},\ \ u_C = 7.5\times10^{-2}\,\text{cm}\end{array}\right\}$$

Example—Step bar stretch

- Construct the corresponding equilibrium relationship based on the displacement of the node for each connection node



Example—Step bar stretch

Solve internal force according to rod deformation

- Analyze internal force within rod ①， the strain is written as

$$\varepsilon_1 = \frac{u_B - u_A}{l_1}$$

- For Hook Law， the stress σ_1 is:

$$\sigma_1 = E_1\varepsilon_1 = \frac{E_1}{l_1}(u_B - u_A)$$

- The internal force of rod ①

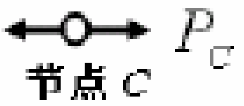
$$I_B^1 = \sigma_1 A_1 = \frac{E_1 A_1}{l_1}(u_B - u_A)$$

- For rod ②, the internal force IB_2 is expressed as：

$$I_B^2 = \sigma_2 A_2 = \frac{E_2 A_2}{l_2}(u_C - u_B)$$

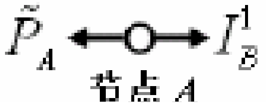
Force balance equation

- **For node C** $$P_C - I_B^2 = 0$$



- **We have** $$P_C - \frac{E_2 A_2}{l_2}\left(u_C - u_B\right) = 0$$

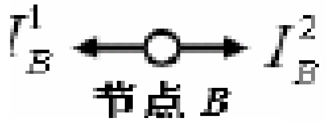
- **For node A** $$-\tilde{P}_A + I_B^1 = 0$$



$$-\tilde{P}_A + \frac{E_1 A_1}{l_1}\left(u_B - u_A\right) = 0$$

- **For node B**

$$-I_B^1 + I_B^2 = 0$$



$$-\frac{E_1 A_1}{l_1}\left(u_B - u_A\right) + \frac{E_2 A_2}{l_2}\left(u_C - u_B\right) = 0$$

Example—Step bar stretch

Write the equilibrium relationship between nodes A, B, and C as a system of equations

$$
\left.\begin{array}{l}
-\tilde{P}_A-\left(\dfrac{E_1A_1}{l_1}\right)u_A+\left(\dfrac{E_1A_1}{l_1}\right)u_B+0=0 \\
0+\left(\dfrac{E_1A_1}{l_1}\right)u_A-\left(\dfrac{E_1A_1}{l_1}+\dfrac{E_2A_2}{l_2}\right)u_B+\left(\dfrac{E_2A_2}{l_2}\right)u_C=0 \\
P_C-0+\left(\dfrac{E_2A_2}{l_2}\right)u_B-\left(\dfrac{E_2A_2}{l_2}\right)u_C=0
\end{array}\right\}
$$

Example—Step bar stretch

- **Write in matrix form**

$$
\begin{bmatrix} -\tilde{P}_A \\ 0 \\ P_C \end{bmatrix} - \begin{bmatrix} \dfrac{E_1 A_1}{l_1} & -\dfrac{E_1 A_1}{l_1} & 0 \\ -\dfrac{E_1 A_1}{l_1} & \dfrac{E_1 A_1}{l_1} + \dfrac{E_2 A_2}{l_2} & -\dfrac{E_2 A_2}{l_2} \\ 0 & -\dfrac{E_2 A_2}{l_2} & \dfrac{E_2 A_2}{l_2} \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

外力
阵列

内力
阵列

Example—Step bar stretch

- **Substitute material elastic modulus and structural dimensions into the equation**

$$\begin{bmatrix} 4\times10^4 & -4\times10^4 & 0 \\ -4\times10^4 & 6\times10^4 & -2\times10^4 \\ 0 & -2\times10^4 & 2\times10^4 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_C \end{bmatrix} = \begin{bmatrix} -\tilde{P}_A \\ 0 \\ 10 \end{bmatrix}$$

- **For fixed point A, u_A=0,the unknowns are** u_B、u_C、$\tilde{P}_A$：

Example—Step bar stretch

- $$\left.\begin{aligned} u_B &= 2.5\times10^{-4}\,\mathrm{m} \\ u_C &= 7.5\times10^{-4}\,\mathrm{m} \\ \tilde{P}_A &= 10\mathrm{N} \end{aligned}\right\}$$

代入:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{u_B - u_A}{l_1} = 2.5\times10^{-3} \\ \varepsilon_2 &= \frac{u_C - u_B}{l_2} = 5\times10^{-3} \\ \sigma_1 &= E_1\varepsilon_1 = 5\times10^{4}\,\mathrm{P_a} \\ \sigma_2 &= E_2\varepsilon_2 = 1\times10^{5}\,\mathrm{P_a} \end{aligned}\right\}$$

Got the same result as the previous method

Example—Step bar stretch

General form of displacement-based solution

- **Rewrite the balance equation as**

$$\begin{bmatrix} \frac{E_1A_1}{l_1} & -\frac{E_1A_1}{l_1} & 0 \\ -\frac{E_1A_1}{l_1} & \frac{E_1A_1}{l_1}+\frac{E_2A_2}{l_2} & -\frac{E_2A_2}{l_2} \\ 0 & -\frac{E_2A_2}{l_2} & \frac{E_2A_2}{l_2} \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_C \end{bmatrix} = \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix}$$

- **Then decompose it into the sum of the equations of two rods**

$$\begin{bmatrix} \frac{E_1A_1}{l_1} & -\frac{E_1A_1}{l_1} & 0 \\ -\frac{E_1A_1}{l_1} & \frac{E_1A_1}{l_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E_2A_2}{l_2} & -\frac{E_2A_2}{l_2} \\ 0 & -\frac{E_2A_2}{l_2} & \frac{E_2A_2}{l_2} \end{bmatrix} \begin{bmatrix} u_A \\ u_B \\ u_C \end{bmatrix} = \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix}$$

Example—Step bar stretch

The first term on the left hand side of the above equation is

$$\begin{bmatrix} \dfrac{E_1 A_1}{l_1} & -\dfrac{E_1 A_1}{l_1} \\ -\dfrac{E_1 A_1}{l_1} & \dfrac{E_1 A_1}{l_1} \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \frac{E_1 A_1}{l_1} \begin{bmatrix} u_A - u_B \\ u_B - u_A \end{bmatrix} = \begin{bmatrix} -I_B^1 \\ I_B^1 \end{bmatrix}$$

The second term on the left hand side of the above equation is

$$\begin{bmatrix} \dfrac{E_2 A_2}{l_2} & -\dfrac{E_2 A_2}{l_2} \\ -\dfrac{E_2 A_2}{l_2} & \dfrac{E_2 A_2}{l_2} \end{bmatrix} \begin{bmatrix} u_B \\ u_C \end{bmatrix} = \frac{E_2 A_2}{l_2} \begin{bmatrix} u_B - u_C \\ u_C - u_B \end{bmatrix} = \begin{bmatrix} -I_B^2 \\ I_B^2 \end{bmatrix}$$

Example—Step bar stretch

- It can be seen that the left end of the equation is the sum of the internal force expression of rod ① and the internal force expression of rod ②, which changes the original node-based balance relationship into a superposition through the balance relationship of each rod.
- The concept of element is naturally introduced here, that is, the original overall structure is "segmented" smaller "components", each "component" has nodes, and the "component" can also be written based on the node displacement. "The internal force expresses the relationship, such a "component" is called an element.
- It means that there is a certain generalization and standardization in the geometric shape and node description, as long as the parameters in the element expression (such as material constants, geometric parameters) are replaced according to the actual situation. , it can be widely used in the description of this type of component (element).

Example—Step bar stretch

The internal force of this element is described as

$$\begin{bmatrix} -I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} \\ -\frac{EA}{l} & \frac{EA}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

as

$$P_1 = -I_1, \quad P_2 = I_2$$

The equation is written as

$$\begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} \\ -\frac{EA}{l} & \frac{EA}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Example—Step bar stretch

The above equation is further expressed as

$$\underset{(2\times2)}{\mathbf{K}^e}\,\underset{(2\times1)}{\mathbf{q}^e} = \underset{(2\times1)}{\mathbf{P}^e}$$

Ke is stiffness matrix of the element，K11、K12、 K21、 K22 are stiffness factor

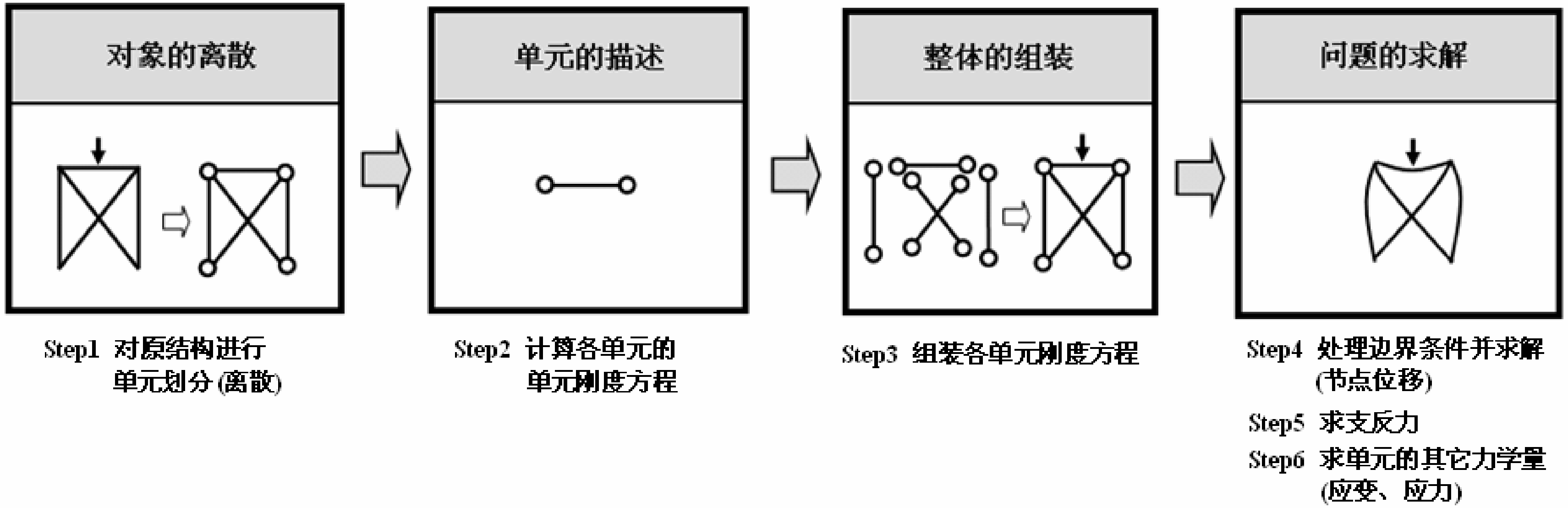
$$\underset{(2\times2)}{\mathbf{K}^e} = \begin{bmatrix} \dfrac{EA}{l} & -\dfrac{EA}{l} \\ -\dfrac{EA}{l} & \dfrac{EA}{l} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

- The so-called element-based analysis method is to divide and number the nodes of the original overall structure according to the changing nature of the geometric shape, and then decompose it into small components (ie: element), and establish the nodes of each element based on the node displacement. The equilibrium relationship (called the element stiffness equation), is

$$\underset{(2\times2)}{\mathbf{K}^e}\,\underset{(2\times1)}{\mathbf{q}^e} = \underset{(2\times1)}{\mathbf{P}^e}$$

Example—Step bar stretch

- The next step is to combine and integrate the individual elements to obtain the overall equilibrium equation (also called the element stiffness equation) for the structure.
- According to the actual situation, some nodal displacements and nodal forces in the equation are given corresponding values (called boundary conditions), and displacements and forces can be solved. After all nodal displacements are obtained, all other secondary variables such as strain, stress can be obtained.



Global Matrix

The global matrix is the overall matrix wave equation.

$$\nabla\times\left(\mu^{-1}\nabla\times\vec{E}\right)-\omega^{2}\varepsilon\vec{E}=b \quad\rightarrow\quad \underbrace{[K]\left[\vec{E}\right]=[b]}_{\text{Global matrix}}$$

$$[K]=\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \qquad [b]=\begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Assembling the Global Matrix

The global matrix is assembled by adding the elements of the element matrices to the corresponding elements of the global matrix.

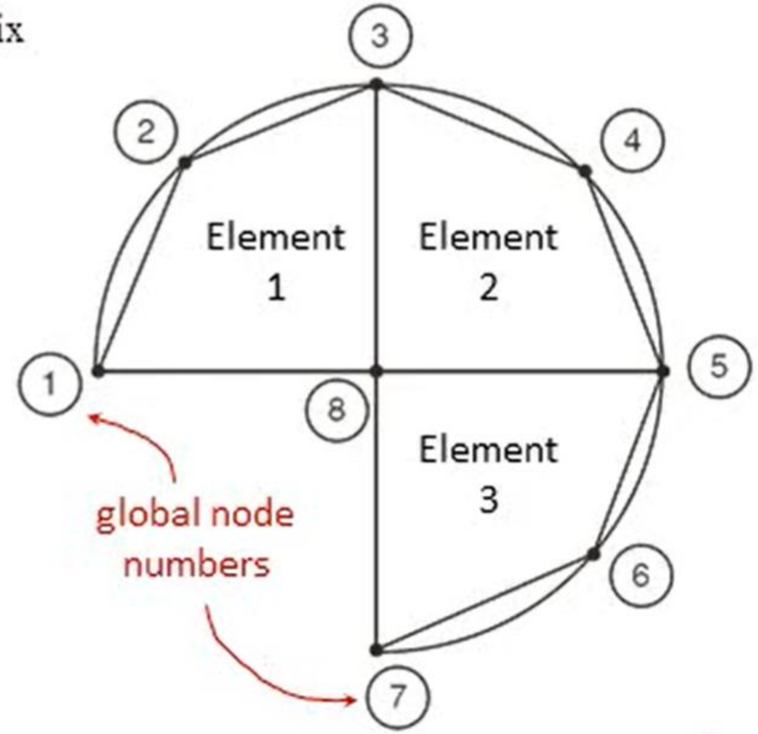
$$[K] = \sum_{e=1}^{N_e} \left[K^{(e)}\right]$$

$[K] \equiv 8\times 8$ global matrix

$\left[K^{(e)}\right] \equiv 4\times 4$ element matrix

The global matrix is initialized as

$$[K] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Assembling the Global Matrix

Add Element 1

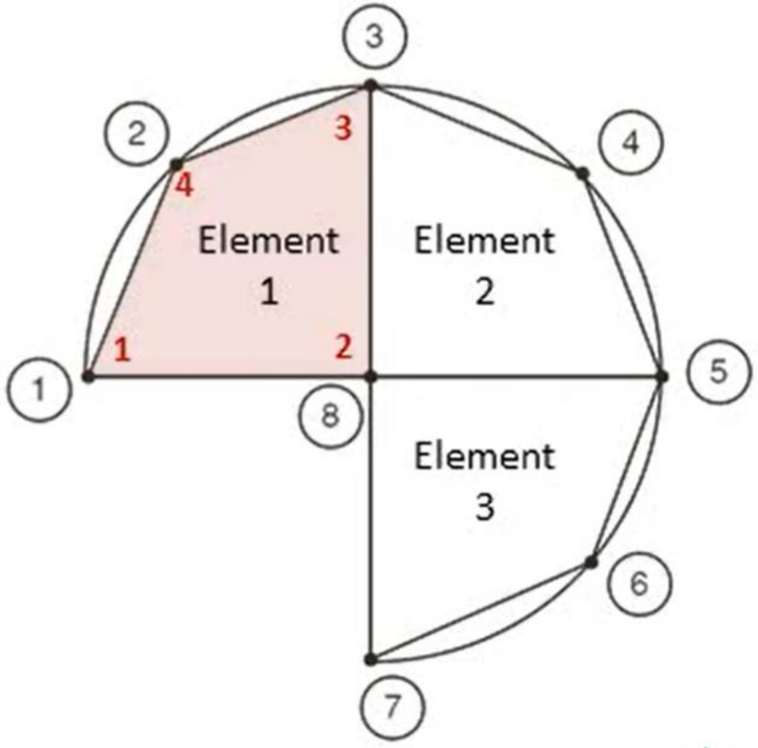
1. We match the local nodes to the global nodes.

Local Node Number	Global Node Number
1	1
2	8
3	3
4	2

$$K_{ij}^{(e)} \rightarrow K_{mn}^{(\text{global})}$$

$$\left.\begin{aligned} i &\rightarrow m \\ j &\rightarrow n \end{aligned}\right\}$$

We use this chart to map local matrix elements to global matrix elements.



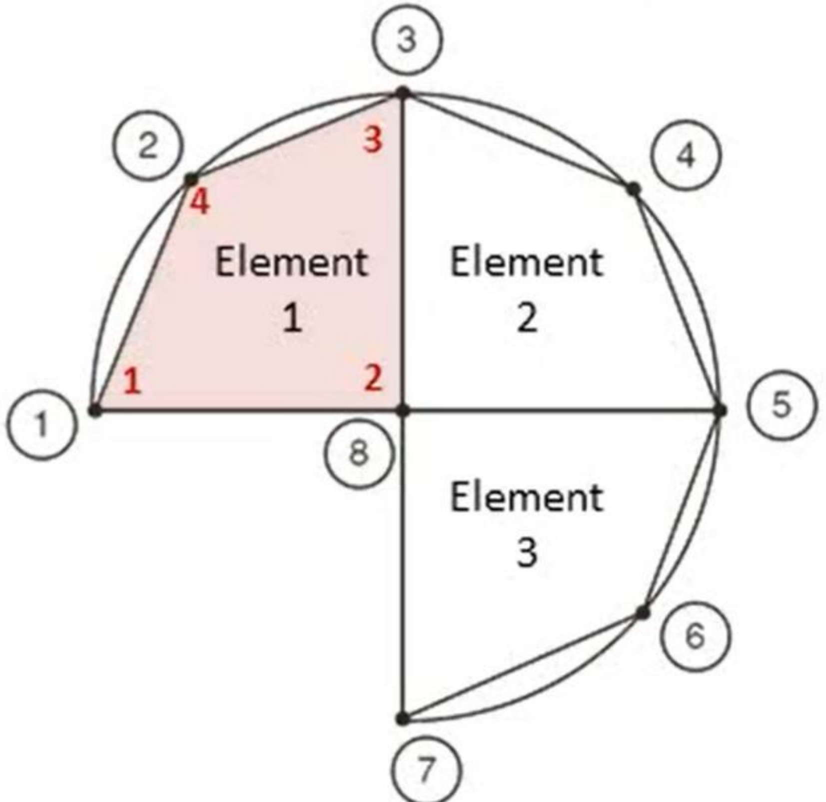
Assembling the Global Matrix

Add Element 1

2. We add $[K^{(1)}]$ to the global matrix.

Local	Global
1	1
2	8
3	3
4	2

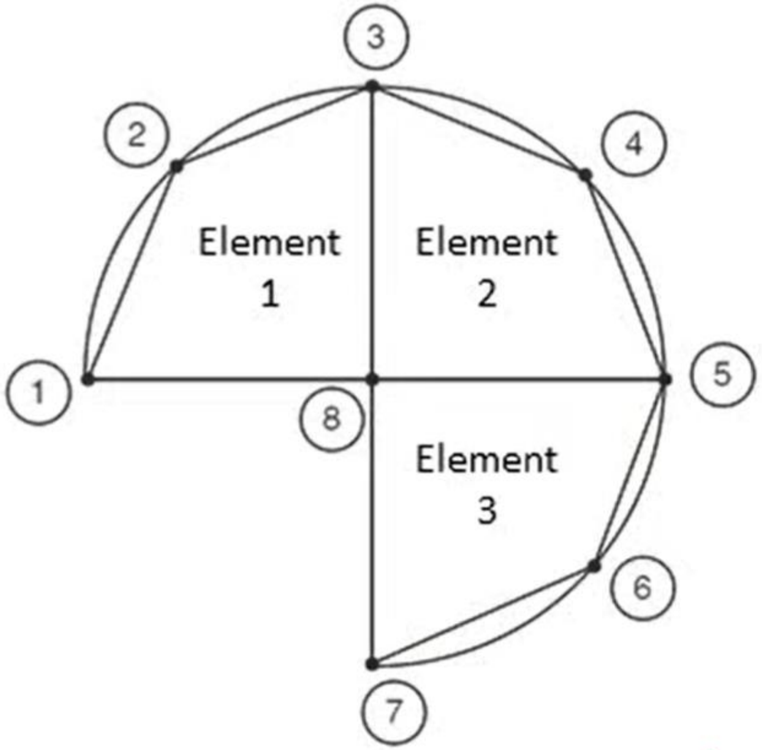
$$K = \begin{bmatrix} K_{11}^{(1)} & K_{14}^{(1)} & K_{13}^{(1)} & 0 & 0 & 0 & 0 & K_{12}^{(1)} \\ K_{41}^{(1)} & K_{44}^{(1)} & K_{43}^{(1)} & 0 & 0 & 0 & 0 & K_{42}^{(1)} \\ K_{31}^{(1)} & K_{34}^{(1)} & K_{33}^{(1)} & 0 & 0 & 0 & 0 & K_{32}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{24}^{(1)} & K_{23}^{(1)} & 0 & 0 & 0 & 0 & K_{22}^{(1)} \end{bmatrix}$$



Don’t forget about [b]

We follow the same procedure for [b] and get

$$\mathbf{b} = \begin{Bmatrix} b_1^{(1)} \\ b_4^{(1)} \\ b_3^{(1)} + b_4^{(2)} \\ b_3^{(2)} \\ b_2^{(2)} + b_3^{(3)} \\ b_2^{(3)} \\ b_1^{(3)} \\ b_2^{(1)} + b_1^{(2)} + b_4^{(3)} \end{Bmatrix}$$



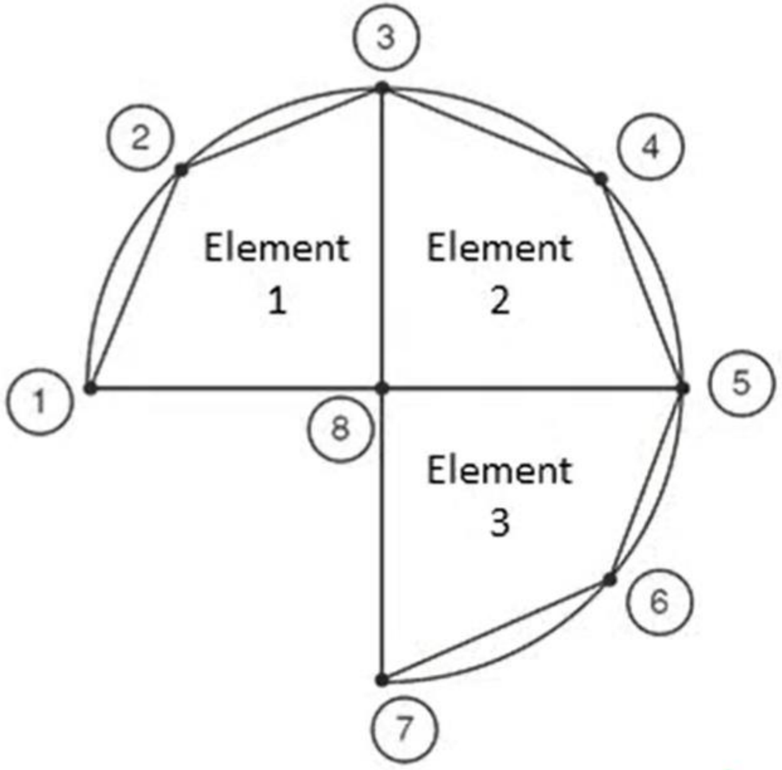
Overall Solution

We solve this problem as

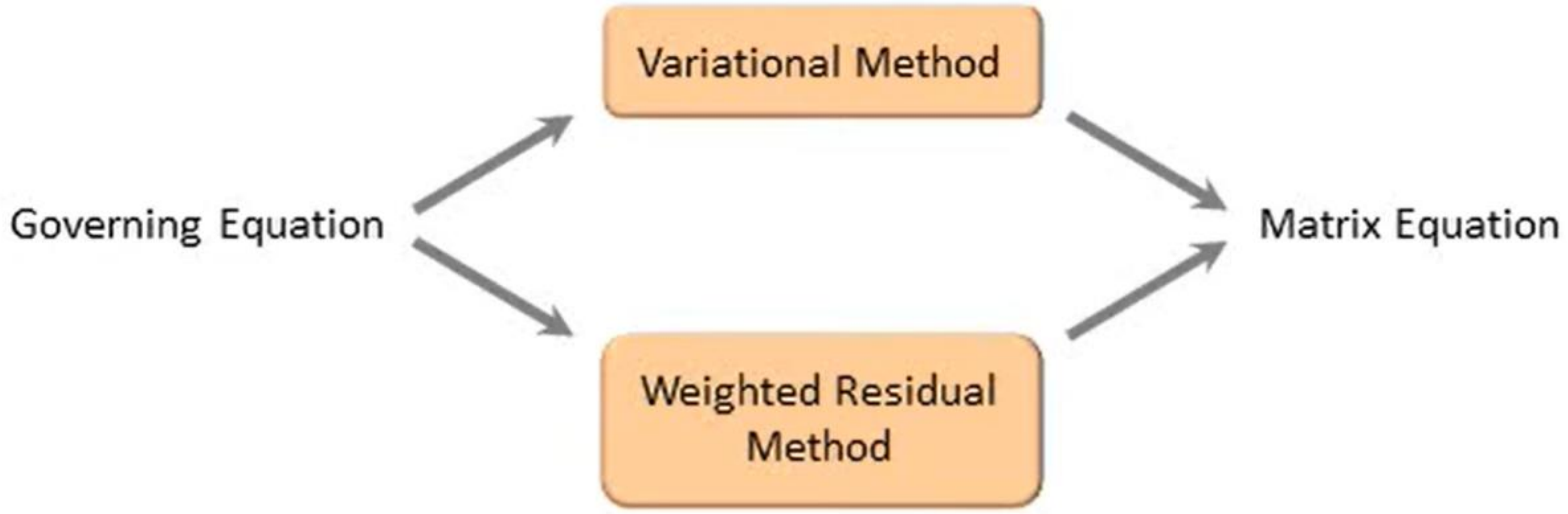
$$[K]\left[\vec{E}\right]=[b]$$

$$\downarrow$$

$$\left[\vec{E}\right]=[K]^{-1}[b]$$



Discretization



Both the variational method and the method of weighted residuals can be used to write a governing equation in matrix form.

Both approaches yield exactly the same matrices.

The Galerkin method is the most popular special case of weighted residual

Linear Equations

Consider the following linear homogeneous equation.

$$L\left[f(x)\right] = g(x)$$

$L[\ ] \equiv$ Linear operation

$f(x) \equiv$ unknown solution

$g(x) \equiv$ known driving function

The linear operator $L[]$ has the following properties

$$L\left[f_1(x) + f_2(x)\right] = L\left[f_1(x)\right] + L\left[f_2(x)\right]$$

$$L\left[af(x)\right] = aL\left[f(x)\right]$$

$$L_1\left[L_2\left[f(x)\right]\right] = L_2\left[L_1\left[f(x)\right]\right]$$

Inner Product

An inner product is a scalar quantity that provides a measure of similarity between two functions.

$$\langle f(x), g(x)\rangle \equiv \text{inner product between } f(x) \text{ and } g(x)$$

$\langle f, g\rangle = 0$	f and g are orthogonal
$\langle f, g\rangle =$ small number	f and g are very different
$\langle f, g\rangle =$ big number	f and g are very similar

It is common to scale functions such that $\langle f, f^*\rangle = 1$

An appropriate inner product must satisfy

$$\langle f, g\rangle = \langle g, f\rangle$$

$$\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

$$\langle f, f^*\rangle \begin{cases} > 0 & f \neq 0 \\ = 0 & f = 0 \end{cases}$$

We will use the following inner product: $\langle f(z), g(z)\rangle = \int_z f(z)\, g(z)\, dz$

Expansion into a a set of basis functons

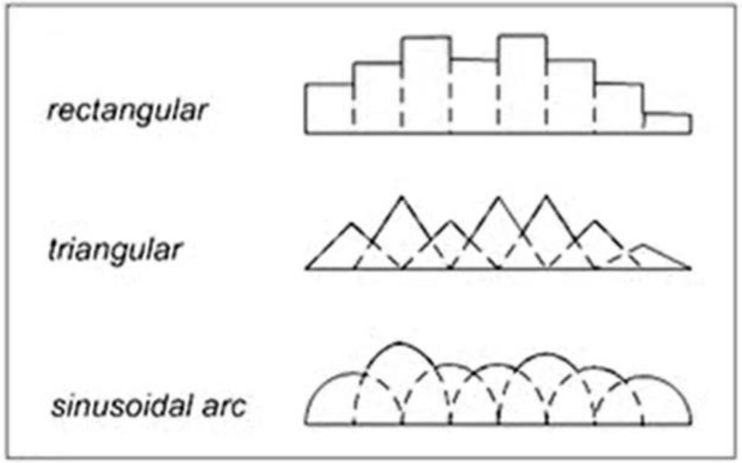
Let $f(x)$ be expanded into a set of basis functions.

$$f(x) = \sum_n a_n v_n(x)$$

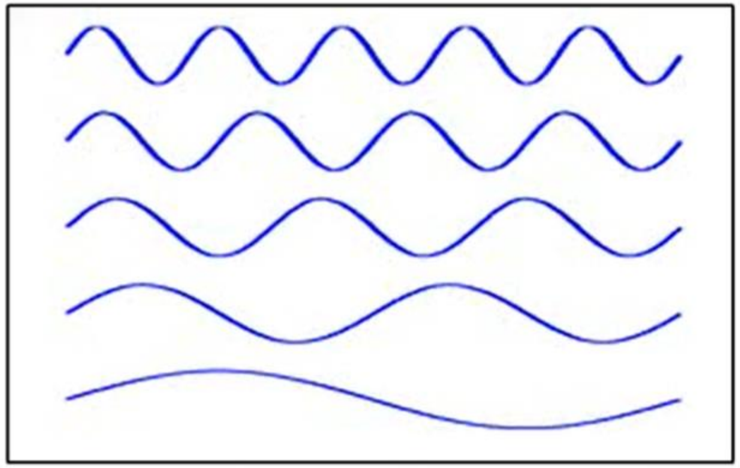
$f(x) \equiv$ unknown function

$a_n \equiv$ coefficient of n^{th} basis function

$v_n(x) \equiv n^{\text{th}}$ basis function



Subdomain Basis Functions



Entire Domain Basis Functions

We choose the basis functions with two considerations: (1) ease of calculations, and (2) minimize how many are needed in the expansion to accurately portray the field.

Linear equation in terms of basis functions

First we substitute the expansion into the original linear equation.

$$f(x) = \sum_n a_n v_n(x)$$

$$L[f(x)] = g(x)$$

Using the properties of linear operations, we get

$$L[f(x)] = g(x)$$

$$L\left[\sum_n a_n v_n(x)\right] = g(x)$$

$$\sum_n L[a_n v_n(x)] = g(x) \longrightarrow \sum_n a_n L[v_n(x)] = g(x)$$

We typically choose $v_n(x)$ so that $L[v_n(x)]$ is easy to calculate.

Method of Weighted Residuals

Similar to how we choose a set of basis functions, we choose another set of weighting functions.

$$w_n(x)$$

We start with our linear inhomogeneous equation and calculate the inner product with $w_m(x)$ of both sides. Here we "test" both sides with $w_m(x)$.

$$\sum_n a_n L[v_n(x)] = g(x)$$

$$\left\langle w_m(x)\,,\, \sum_n a_n L[v_n(x)] \right\rangle = \langle w_m(x)\,,\, g(x)\rangle$$

$$\sum_n a_n \langle w_m(x)\,,\, L[v_n(x)]\rangle = \langle w_m(x)\,,\, g(x)\rangle$$

Method of weighted residuals

This equation can be written in matrix form as

$$\sum_n a_n \left\langle w_m(x)\ ,\ L\left[v_n(x)\right]\right\rangle = \left\langle w_m(x)\ ,\ g(x)\right\rangle \quad \rightarrow \quad \boxed{\left[z_{mn}\right]\left[a_n\right]=\left[g_m\right]}$$

$$\left[z_{mn}\right]=\begin{bmatrix} \langle w_1, L[v_1]\rangle & \langle w_1, L[v_2]\rangle & & \\ \langle w_2, L[v_1]\rangle & \langle w_2, L[v_2]\rangle & & \\ & & \ddots & \\ & & & \langle w_M, L[v_N]\rangle \end{bmatrix}$$

$$\left[a_n\right]=\begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad \left[g_m\right]=\begin{bmatrix} \langle w_1, g\rangle \\ \langle w_2, g\rangle \\ \vdots \\ \langle w_M, g\rangle \end{bmatrix}$$

Galerkin Method

The Galerkin method is the method of weighted residuals, but the weighting functions are made to be the same as the basis functions.

$$w_m(x) = v_m(x)$$

The matrix equation becomes

$$\sum_n a_n \left\langle v_m(x)\ ,\ L\left[v_n(x)\right]\right\rangle = \left\langle v_m(x)\ ,\ g(x)\right\rangle \quad \rightarrow \quad \boxed{[z_{mn}][a_n] = [g_m]}$$

$$[z_{mn}] = \begin{bmatrix} \langle v_1, L[v_1]\rangle & \langle v_1, L[v_2]\rangle & & \\ \langle v_2, L[v_1]\rangle & \langle v_2, L[v_2]\rangle & & \\ & & \ddots & \\ & & & \langle v_M, L[v_N]\rangle \end{bmatrix} \qquad [a_n] = \begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad [g_m] = \begin{bmatrix} \langle v_1, g\rangle \\ \langle v_2, g\rangle \\ \vdots \\ \langle v_M, g\rangle \end{bmatrix}$$

Summary of the Galerkin Method

The Galerkin method can be used to find the solution to any linear inhomogeneous equation.

$$L[f] = g$$

Step 1 – Expand the unknown function into a set of basis functions.

$$f = \sum_n a_n v_n \qquad L\left[\sum_n a_n v_n\right] = g \quad \rightarrow \quad \sum_n a_n L[v_n] = g$$

Step 2 – Test both sides of the equation with the basis functions using an inner product

$$\left\langle v_m , \sum_n a_n L[v_n] \right\rangle = \langle v_m , g \rangle \quad \rightarrow \quad \sum_n a_n \langle v_m , L[v_n] \rangle = \langle v_m , g \rangle$$

Step 3 – Form a matrix equation

$$[z_{mn}][a_n] = [g_m]$$

$$[z_{mn}] = \begin{bmatrix} \langle v_1, L[v_1] \rangle & \langle v_1, L[v_2] \rangle & \\ \langle v_2, L[v_1] \rangle & \langle v_2, L[v_2] \rangle & \\ & & \ddots \end{bmatrix} \qquad [a_n] = \begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad [g_m] = \begin{bmatrix} \langle v_1, g \rangle \\ \langle v_2, g \rangle \\ \vdots \\ \langle v_M, g \rangle \end{bmatrix}$$

Governing equation and its solution

Governing Equation

We will apply the Galerkin method to solve the following differential equation.

$$-\frac{d^2 f}{dx^2} = 1 + 4x^2 \qquad f(0) = f(1) = 0 \qquad 0 \le x \le 1$$

Simple Analytical Solution

This is a simple boundary value problem with the following solution.

$$f(x) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3}$$

We wish to solve this using the Galerkin method.

Choose basis functions

Basis Functions

For this problem, it will be convenient to choose as basis functions:

$$v_n = x - x^{n+1}$$

Expansion of the Function into the Basis

We expand our function into this set of basis functions as

$$f(x) = \sum_{n=1}^{N} a_n \left(x - x^{n+1} \right)$$

Choose Testing Functions

CEM

Testing Functions

The Galerkin method uses the same function for basis and testing.

$$w_n = v_n = x - x^{n+1}$$

Form of Final Solution

Recall that we are converting a linear equation into a matrix equation according to

$$L[f]=g \quad \Longrightarrow \quad [z_{mn}][a_n]=[g_m]$$

$$[z_{mn}]=\begin{bmatrix} \langle v_1, L[v_1]\rangle & \langle v_1, L[v_2]\rangle & \\ \langle v_2, L[v_1]\rangle & \langle v_2, L[v_2]\rangle & \\ & & \ddots \end{bmatrix}$$

$$[a_n]=\begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad [g_m]=\begin{bmatrix} \langle v_1, g\rangle \\ \langle v_2, g\rangle \\ \vdots \\ \langle v_M, g\rangle \end{bmatrix}$$

To do this, we need to evaluate $\langle v_m, L[v_n]\rangle$ and $\langle v_m, g\rangle$.

First Inner Product

$$
\begin{aligned}
\langle v_m, L[v_n] \rangle &= \int_x v_m L[v_n]\, dx \\
&= \int_0^1 \left(x - x^{m+1}\right)\left\{-\frac{d^2}{dx^2}\left[x - x^{n+1}\right]\right\} dx \\
&= -\int_0^1 \left(x - x^{m+1}\right)\left[-n(n+1)x^{n-1}\right] dx \\
&= n(n+1)\int_0^1 \left(x^n - x^{m+n}\right) dx \\
&= n(n+1)\cdot\left[\frac{x^{n+1}}{n+1} - \frac{x^{m+n+1}}{m+n+1}\right]_0^1 \\
&= n(n+1)\cdot\left[\frac{1}{n+1} - \frac{1}{m+n+1}\right] \\
&= n(n+1)\cdot\left[\frac{m+n+1}{(n+1)(m+n+1)} - \frac{n+1}{(n+1)(m+n+1)}\right] \\
&= n(n+1)\cdot\left[\frac{m}{(n+1)(m+n+1)}\right] \\
&= \frac{mn}{(m+n+1)}
\end{aligned}
$$

$$
\boxed{z_{mn} = \langle v_m, L[v_n] \rangle = \frac{mn}{(m+n+1)}}
$$

Second Inner Product

$$
\begin{aligned}
\langle v_m, g \rangle &= \int_x v_m g \, dx \\
&= \int_0^1 \left(x - x^{m+1} \right)\left(1 + 4x^2 \right) dx \\
&= \int_0^1 \left(x + 4x^3 - x^{m+1} - 4x^{m+3} \right) dx \\
&= \left[\frac{x^2}{2} + x^4 - \frac{x^{m+2}}{m+2} - \frac{4x^{m+4}}{m+4} \right]_0^1 \\
&= \frac{1}{2} + 1 - \frac{1}{m+2} - \frac{4}{m+4} \\
&= \frac{3}{2} - \frac{1}{m+2} - \frac{4}{m+4} \\
&= \frac{3(m+2)(m+4)}{2(m+2)(m+4)} - \frac{2(m+4)}{2(m+2)(m+4)} - \frac{8(m+2)}{2(m+2)(m+4)} \\
&= \frac{3(m+2)(m+4) - 2(m+4) - 8(m+2)}{2(m+2)(m+4)} \\
&= \frac{3\left(m^2 + 6m + 8\right) - 10m - 24}{2(m+2)(m+4)} \\
&= \frac{3m^2 + 8m}{2(m+2)(m+4)} \\
&= \frac{m(3m+8)}{2(m+2)(m+4)}
\end{aligned}
$$

$$
\boxed{g_m = \langle v_m, g \rangle = \frac{m(3m+8)}{2(m+2)(m+4)}}
$$

Try N=1

For N=1, our matrix equation is

$$[z_{11}][a_1]=[g_1]$$

$$z_{mn}=\frac{mn}{(m+n+1)}$$

$$g_m=\frac{m(3m+8)}{2(m+2)(m+4)}$$

Applying our equations for z_{mn} and g_m, we get

$$z_{11}=\frac{1\cdot 1}{(1+1+1)}=\frac{1}{3}$$

$$g_1=\frac{1(3\cdot 1+8)}{2(1+2)(1+4)}=\frac{11}{30}$$

The coefficient is

$$a_1=\frac{g_1}{z_{11}}=\frac{11/30}{1/3}=\frac{11}{10}$$

Finally, the solution for N=1 is

$$f(x)=a_1(x-x^2)=\frac{11x}{10}-\frac{11x^2}{10}$$

← Not correct. Need larger N.

Try N=2

For N=2, our matrix equation is

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$z_{mn} = \frac{mn}{(m+n+1)}$$

$$g_m = \frac{m(3m+8)}{2(m+2)(m+4)}$$

Applying our equations for z_{mn} and g_m, we get

$$z_{11} = \frac{1\cdot 1}{(1+1+1)} = \frac{1}{3} \quad z_{12} = \frac{1\cdot 2}{(1+2+1)} = \frac{1}{2} \quad z_{21} = \frac{2\cdot 1}{(2+1+1)} = \frac{1}{2} \quad z_{22} = \frac{2\cdot 2}{(2+2+1)} = \frac{4}{5}$$

$$g_1 = \frac{1(3\cdot 1+8)}{2(1+2)(1+4)} = \frac{11}{30} \quad g_2 = \frac{2(3\cdot 2+8)}{2(2+2)(2+4)} = \frac{7}{12}$$

The coefficients are

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix}^{-1}\begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{2}{3} \end{bmatrix}$$

Finally, the solution for N=2 is

$$f(x) = a_1\left(x - x^2\right) + a_2\left(x - x^3\right) = \frac{23x}{30} - \frac{x^2}{10} - \frac{2x^3}{3}$$

← Still not correct. Need larger N.

Try N=3

For N=3, are matrix equation is

$$z_{mn} = \frac{mn}{(m+n+1)}$$

$$g_m = \frac{m(3m+8)}{2(m+2)(m+4)}$$

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Applying our equations for z_{mn} and g_m, we get

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{5} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \\ \frac{51}{70} \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{5} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{bmatrix}^{-1} \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \\ \frac{51}{70} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

Finally, the solution for N=3 is

$$f(x) = a_1(x - x^2) + a_2(x - x^3) + a_3(x - x^4) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3} \leftarrow \text{Exact solution!}$$

Try N=4

For N=4, we get a larger matrix equation, but it also converges to the exact solution.

In fact, the method converges to the exact solution for all $N \geq 3$.

$$f(x) = \sum_{n=1}^{N} a_n \left(x - x^{n+1} \right) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3} \quad \leftarrow \text{Exact solution}$$