EE411 · Information Theory and Coding

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Assignment 7

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1. Suppose $\mu = [\mu_0, \mu_1]$ is the stationary distribution vector. By definition, we have

$$\mu P = \mu$$
$$\mu_0 + \mu_1 = 1$$

So we have

$$\mu_0 = \frac{p_{10}}{p_{01} + p_{10}}, \mu_1 = \frac{p_{01}}{p_{01} + p_{10}}$$

By definition, the entropy rate is

$$\begin{aligned} H(\mathcal{X}) &= -\sum_{ij} \mu_i P_{ij} \log P_{ij} = -(\mu_0(1-p_{01})\log(1-p_{01}) + \mu_0 p_{01} \log p_{01} + \mu_1 p_{10} \log p_{10} + \mu_1(1-p_{10})\log(1-p_{10})) \\ &= -(\frac{p_{10}}{p_{01}+p_{10}}((1-p_{01})\log(1-p_{01}) + p_{01}\log p_{01}) + \frac{p_{01}}{p_{01}+p_{10}}(p_{10}\log p_{10} + (1-p_{10})\log(1-p_{10}))) \\ &= \frac{p_{10}}{p_{01}+p_{10}} H(p_{01}) + \frac{p_{01}}{p_{01}+p_{10}} H(p_{10}) \end{aligned}$$

2. By Jensen's Inequality, we have

$$\begin{aligned} H(\mathcal{X}) &= \frac{p_{10}}{p_{01}+p_{10}} H(p_{01}) + \frac{p_{01}}{p_{01}+p_{10}} H(p_{10}) \\ &\le H(2\frac{p_{10}p_{01}}{p_{01}+p_{10}}) \end{aligned}$$

Let $\alpha = 2\frac{p_{10}p_{01}}{p_{01}+p_{10}}$, then we need to find the α that maximizes $H(\alpha)$. Since $H(\alpha)$ increases when $\alpha < \frac{1}{2}$ and decreases when $\alpha > \frac{1}{2}$, then $\alpha = \frac{1}{2}$ maximizes $H(\alpha)$. Since $p_{10} + p_{01} = 1$, we can know that $p_{01} = p_{10} = \frac{1}{2}$ maximize the entropy rate.

3. It is a special case of part (a) when $p_{01} = p$ and $p_{10} = 1$. So the entropy rate is

$$H(\mathcal{X}) = \frac{1}{p+1} H(p) + \frac{p}{p+1} H(1) = \frac{H(p)}{p+1}$$

4. Let $f(p) = \frac{H(p)}{p+1} = \frac{-p\log p - (1-p)\log(1-p)}{p+1}$ with $p \in [0, 1]$. Its deviation is

$$f'(p) = \frac{2\log(1-p) - \log p}{(p+1)^2}$$

Let $f'(p) = 0$, then $p = \frac{3\pm\sqrt{5}}{2}$. And since $p \in [0, 1]$, $p = \frac{3-\sqrt{5}}{2} = 0.382 < \frac{1}{2}$. Since $p \in [0, \frac{3-\sqrt{5}}{2})$, $f'(p) > 0$; $p \in (\frac{3-\sqrt{5}}{2}), 1]$, $f'(p) < 0$, the maximum value of $H(\mathcal{X})$ occurs at $p = \frac{3-\sqrt{5}}{2}$ and the maximum value is $H(p) = 0.694$ bits.

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1. By chain rule, we have

$$H(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, ..., X_1)$$

Since the next position depends only on the previous two for $i > 1$, then we have

$$\begin{aligned} H(X_1, X_2, ..., X_n) &= H(X_1) + \sum_{i=2}^{n} H(X_i|X_{i-1}, X_{i-2}) \\ &= H(X_0) + H(X_1|X_0) + \sum_{i=2}^{n} H(X_i|X_{i-1}, X_{i-2}) \end{aligned}$$

Since $X_0 = 0$ is deterministic, $H(X_0) = 0$. Since the first step is equitable for positive and negative, $H(X_1|X_0) = 1$. Since for $i > 1$, every step can be reversed with possibility 0.1, then

$$H(X_i|X_{i-1}, X_{i-2}) = -0.1 \cdot \log(0.1) - 0.9 \cdot \log(0.9) = \log 2 + \log 5 - \frac{9}{5}\log 3$$

Therefore, we have

$$H(X_1, X_2, ..., X_n) = 1 + (n-1)(\log 2 + \log 5 - \frac{9}{5}\log 3)$$

2. By definition, we have

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, ..., X_n) = \lim_{n\to\infty} \frac{1 + (n-1)(\log 2 + \log 5 - \frac{9}{5}\log 3)}{n} = (\log 2 + \log 5 - \frac{9}{5}\log 3)$$

3. Suppose S be the number of steps taken between reversing direction. Then

$$E(S) = \sum_{s=1}^{\infty} s(0.9)^{s-1}(0.1) = 10$$

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It is a one-state Markov chain so that the choice of next state depends only on the current state.
Since the king can only walk one cell on the row, column and diagonal and cannot stay remained, the state transition matrix is

$$P = \begin{bmatrix} 0 & 1/5 & 0 & 1/5 & 1/8 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/5 & 1/8 & 1/5 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 & 1/8 & 1/5 & 0 & 0 & 0 \\ 1/3 & 1/5 & 0 & 0 & 1/8 & 0 & 1/3 & 1/5 & 0 \\ 1/3 & 1/5 & 1/3 & 1/5 & 0 & 1/5 & 1/3 & 1/5 & 1/3 \\ 0 & 1/5 & 1/3 & 0 & 1/8 & 0 & 0 & 1/5 & 1/3 \\ 0 & 0 & 0 & 1/5 & 1/8 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1/5 & 1/8 & 1/5 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/8 & 1/5 & 0 & 1/5 & 0 \end{bmatrix}$$

Suppose the stationary distribution vector is $\mu = [\mu_1, ..., \mu_9]$. By definition, we can get

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \end{bmatrix}^T = \begin{bmatrix} 3/40 \\ 5/40 \\ 3/40 \\ 5/40 \\ 8/40 \\ 5/40 \\ 3/40 \\ 5/40 \\ 3/40 \end{bmatrix}^T$$

In a random walk, the next state is chosen equitably among possible choices, so we have

$$H(X_2|X_1 = i) = \log 3, i = 1, 3, 7, 9$$
$$H(X_2|X_1 = i) = \log 5, i = 2, 4, 6, 8$$
$$H(X_2|X_1 = i) = \log 8, i = 5$$

Therefore, the entropy rate of the king is

$$H(\mathcal{X}) = \sum_{i=1}^{9} \mu_i H(X_2|X_1 = i) = 0.3\log 3 + 0.5\log 5 + 0.2\log 8 = 2.24$$

Similarly, for the rooks, which can walk any cells on the row and column and cannot stay remained, we have

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \end{bmatrix}^T = \begin{bmatrix} 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \\ 4/36 \end{bmatrix}^T ; H(X_2|X_1 = i) = \log 4, i = 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and thus

$$H(\mathcal{X}) = \sum_{i=1}^{9} \mu_i H(X_2|X_1 = i) = \log 4 = 2$$

For the queens, which can walk any cells on the row, column and diagonal and cannot stay remained, we have

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \end{bmatrix}^T = \begin{bmatrix} 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \\ 6/54 \end{bmatrix}^T ; H(X_2|X_1 = i) = \log 6, i = 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and thus

$$H(\mathcal{X}) = \sum_{i=1}^{9} \mu_i H(X_2|X_1 = i) = \log 6 = 1 + \log 3 = 2.58$$

The bishops, which can walk any cells on the diagonal and cannot stay remained, has two types: one only walks on the even cells and the other only walks on the odd cells. For even-cells bishops, we have

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 2/8 \\ 0 \\ 2/8 \\ 0 \\ 2/8 \\ 0 \\ 2/8 \\ 0 \end{bmatrix}^T ; \begin{array}{c} H(X_2|X_1 = i) = \log 2, i = 2, 4, 6, 8 \\ H(X_2|X_1 = i) = 0, i = 1, 3, 5, 7, 9 \end{array}$$

and thus

$$H(\mathcal{X}) = \sum_{i=1}^{9} \mu_i H(X_2|X_1 = i) = \log 2 = 1$$

For odd-cells bishops, we have

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \end{bmatrix}^T = \begin{bmatrix} 4/20 \\ 0 \\ 4/20 \\ 0 \\ 4/20 \\ 0 \\ 4/20 \\ 0 \\ 4/20 \end{bmatrix}^T ; \quad \begin{array}{c} H(X_2|X_1 = i) = 0, i = 2, 4, 6, 8 \\ H(X_2|X_1 = i) = \log 4, i = 1, 3, 5, 7, 9 \end{array}$$

and thus

$$H(\mathcal{X}) = \sum_{i=1}^{9} \mu_i H(X_2|X_1 = i) = \log 4 = 2$$

The omitted unit of $H(\mathcal{X})$ is $bits$.