Kraft inequality for U.D. code.

① ~~ϕ~~ The converse part is ~~its~~ already ~~po~~ proved

② Consider arbitrary source sequence

$x^k = x_1 x_2 \cdots x_k$. With a U.D. code,

$l(x^k) = l(x_1) + l(x_2) + \cdots + l(x_k)$.

$$\left(\sum_{x\in\mathcal{X}} D^{-l(x)}\right)^k = \sum_{x_1\in\mathcal{X}} D^{-l(x_1)} \sum_{x_2\in\mathcal{X}} D^{-l(x_2)} \cdots \sum_{x_k\in\mathcal{X}} D^{-l(x_k)}$$

$$= \sum_{x_1\in\mathcal{X}} \cdots \sum_{x_k\in\mathcal{X}} D^{-[l(x_1)+l(x_2)+\cdots+l(x_k)]}$$

Suppose l_{max} is the maximum codeword ~~len~~ length for one souree symbol.

$\sum_{i=1}^{k} l(x_i) = l(x_1) + l(x_2) + \cdots + l(x_k) \in [1, k l_{max}]$

Let $a(m)$ be # of $(x_1 x_2 \cdots x_k)$ with $\sum_{i=1}^{k} l(x_i) = m$

$$= a(1) D^{-1} + a(2) D^{-2} + \cdots + a(k l_{max}) D^{-k l_{max}}$$

$$= \sum_{m=1}^{k l_{max}} a(m) D^{-m}$$

Notice $a(m) \le D^m$ (Because of U.D. code)

$$\le \sum_{m=1}^{k l_{max}} D^m D^{-m}$$

$$= k\, l_{max}$$

$$\Rightarrow \sum_{x \in \mathcal{X}} D^{-l(x)} \le k^{1/k} \cdot l_{max}^{1/k} \longrightarrow 1 \quad (k \to \infty)$$

proof of Theorem 5.3.1.

$$L - H_D(X) = \sum p_i l_i - \sum p_i \log_D \frac{1}{p_i}$$

$$= -\sum p_i \log_D D^{-l_i} - \sum p_i \log_D \frac{1}{p_i}$$

$$= \sum p_i \log_D \frac{p_i}{D^{-l_i}}$$

Let $r_i = \frac{D^{-l_i}}{\sum_j D^{-l_j}}$ and $c = \sum_j D^{-l_j} \leq 1$

$$= \sum p_i \log_D \frac{p_i}{r_i} + \sum p_i \log_D \frac{r_i}{D^{-l_i}}$$

$$= \sum p_i \log_D \frac{p_i}{r_i} - \log_D c.$$

$$= D(p \| r) - \log_D c.$$

$$\geq 0$$

" = " holds iff $p_i = r_i$ $(\forall i)$ and $c = 1$

$\Leftrightarrow p_i = D^{-l_i}$. D-adic distributi

SUSTech

Proof of Theorem 5.4.1

Suppose the ~~source~~ R.V. PMF is $(p_1, p_2, \cdots p_m)$

Take $l_i = \lceil -\log_D p_i \rceil$.

$$\sum_{i=1}^{m} D^{-l_i} \le \sum_{i=1}^{m} D^{\log_D p_i} = 1$$

$$L^* \le \sum_{i=1}^{m} l_i p_i$$

$$= \sum_{i=1}^{m} p_i \lceil -\log_D p_i \rceil$$

$$< \sum_{i=1}^{m} p_i (-\log_D p_i + 1)$$

$$= H_D(X) + 1$$

Review

$$\underbrace{\text{prefix code} \Leftrightarrow \sum_i D^{-l_i} \le 1}_{\text{code Tree. } D\text{-ary number}} \Leftrightarrow \text{U.D.}$$

$L = \sum_i p_i l_i$

Given a R.V. with $(p_1, p_2, \cdots p_m)$, optimal code

$\min_{\{l_i\}} \sum_i p_i l_i$ s.t. $\sum_i D^{-l_i} \le 1$

ACT TODAY IMPACT TOMORROW

Consider a sequence of i.i.d. R.V.s

$$X^k = X_1 X_2 \cdots X_k$$

~~Coder~~ Coding Scheme 1.

X	codeword
a_1	$c(a_1)$
a_2	$c(a_2)$
$\vdots$	$\vdots$
a_m	$c(a_m)$

optimal ~~pref~~ prefix code L^*

$$H_D(X) \le L^* < H_D(X) + 1$$

$$\underbrace{k H_D(X) \le k L^* < k H_D(X) + k}_{k \text{ bits gap}}$$

Coding Scheme 2:

X^k	codeword
$b_1^1 b_2^1 \cdots b_k^1$	$c(b_1^1 \cdots b_k^1)$
$b_1^2 b_2^2 \cdots b_k^2$	$\vdots$
$\vdots$	$\vdots$
$b_1^{k^m} b_2^{k^m} \cdots b_k^{k^m}$	$c(b_1^{k^m} \cdots b_k^{k^m})$

k^m

optimal prefix code $\tilde{L}^*$

$$\underset{\substack{\| \\ k H_D(X)}}{H_D(X^k)} \le \tilde{L}^* < \underset{\substack{\| \\ k H_D(X)+1}}{H_D(X^k)+1}$$

$$H_D(X) \le \tilde{L}^*/k < H_D(X) + 1/k \Rightarrow \tilde{L}^*/k \to H_D(X)$$