Information

- To have a quantitative measure of information contained in an event, we consider intuitively the following:
  - Information contained in events should be defined in terms of the *uncertainty/probability* of the events.
  - *Monotonous: Less certain (small probability)* events should contain *more information*. 单调性
  - *Additive:* The total information of *unrelated/independent* events should equal the *sum* of the information of each individual event. 可加性

Information Measure of Random Events

A *natural* measure of the *uncertainty* of an event A is the probability $Pr(A)$ of A.

To satisfy the *monotonous* and *additive* properties, the information in the event A could be defined as

$$I(A)_{\text{self-info}} = -\log \Pr(A).$$

If $\Pr(A) > \Pr(B)$, then $I(A) < I(B)$. (monotonous)

If A, B are independent, then $I(A+B) = I(A) + I(B)$. (additive)

$\log_2$	:	bit
$\log_e$	:	nat
$\log_{10}$	:	Hartley

$$\log_a X = \frac{\log_b X}{\log_b a} = \log_a b \cdot \log_b X$$

Average Information Measure of a Discrete R.V.

$x_1, x_2, \cdots, x_q$: *Alphabet $\mathcal{X}$ (realizations) of discrete r.v. X*

$p_1, p_2, \cdots, p_q$: *Probability*

The *average* information of the r.v. X is

$$I(X) = \sum_{i=1}^{q} p_i \log(\frac{1}{p_i}),$$

where $\log \frac{1}{p_i}$ is the *self-information* of event $X = x_i$.

要描述一个R.V.，需要该R.V.的support set及在set上的概率分布. 即对R.V. X.

support set : $\mathcal{X} = \{x_1, \dots, x_n\}$

$\{P_1, \dots, P_n\}$

PMF : $P(x_1) = P_1 \ \dots \ P(x_n) = P_n$ 将support set与概率分布联系起来

那么对R.V. X，其对应的所有随机事件的信息量为

$$I(x_i) = -\log p(x_i)$$

总体的平均信息量即为熵

$$H(X) = \sum_{i=1}^{n} [-\log p(x_i)] \cdot p(x_i)$$

$$= E[\log \frac{1}{p(X)}]$$

(若 $p(x_i) = 0$，则视 $p(x_i) \cdot \log \frac{1}{p(x_i)} = 0$)

Entropy

Definition

The *entropy* of a discrete random variable X is given by

$$\begin{aligned} H(X) &= \sum_{x\in\mathcal{X}} p(x) \log \frac{1}{p(x)} \\ &= -\sum_{x\in\mathcal{X}} p(x) \log p(x) \\ &= E\left[\log \frac{1}{p(X)}\right]. \end{aligned}$$

By convention, let $0 \log 0 = 0$ since $x \log x \to 0$ as $x \to 0$.

Lemma 2.1.1

$$H(X) \geq 0.$$

Proof.

Since $0 \leq p(x) \leq 1$, we have $\log \frac{1}{p(x)} \geq 0$. □

Lemma 2.1.2

$$H_b(X) = (\log_b a) H_a(X).$$

Proof.

Since $\log_b p = \log_b a \log_a p$. □

Example

Example 2.1.1

Let

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p. \end{cases}$$

往熵增最大的方向进行

$$H(X) = -p\log p - (1-p)\log(1-p) = H(p).$$

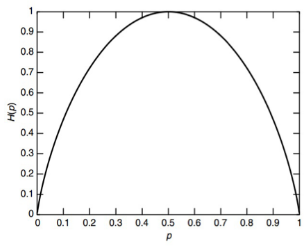


FIGURE 2.1. $H(p)$ vs. p.

Joint Entropy

Definition

The *joint entropy* $H(X,Y)$ of a pair of discrete random variables (X,Y) with a joint distribution $p(x,y)$ is

$$H(X,Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y)$$
$$= -E\log p(X,Y).$$

$$H(X_1,\ldots,X_n) = E\left[\log\frac{1}{p(X_1,\ldots,X_n)}\right]$$

若 X_i 相互独立，则 $H(X_1,\ldots,X_n) = H(X_1)+\cdots+H(X_n)$

If X and Y are *independent*, then

$$\begin{aligned} H(X,Y) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y)\log p(x)p(y) \\ &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y)\log p(x) - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x)p(y)\log p(y) \\ &= \sum_{y\in\mathcal{Y}} p(y)H(X) + \sum_{x\in\mathcal{X}} p(x)H(Y) \\ &= H(X)+H(Y). \end{aligned}$$

Conditional Entropy

- If $(X,Y)\sim p(x,y)$, the *conditional entropy* $H(Y|X)$ is

$$\begin{aligned} H(Y|X) &= \sum_{x\in\mathcal{X}} p(x) H(Y|X=x) \\ &= -\sum_{x\in\mathcal{X}} p(x)\sum_{y\in\mathcal{Y}} p(y|x)\log p(y|x) \\ &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x) \\ &= -E\log p(Y|X). \end{aligned}$$

要注意：$H(Y|X=x) = \sum_y p(y|x)\log\frac{1}{p(y|x)} = E_Y\left[\log\frac{1}{p(Y|x)}\right]$

不是条件熵.

$$H(Y|X) \triangleq \sum_x H(Y|X=x)\cdot p(x)$$

Chain Rule

Theorem 2.2.1 (Chain Rule)

$$H(X,Y) = H(X) + H(Y|X).$$

The *joint entropy* of a pair of random variables = the entropy of one + the *conditional entropy* of the other.

$$\begin{aligned}
H(X,Y) &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y)\\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x)p(y|x)\\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x) - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x)\\
&= -\sum_{x\in\mathcal{X}} p(x)\log p(x) - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x)\\
&= H(X) + H(Y|X)
\end{aligned}$$

Corollary

$$H(X,Y|Z) = H(X|Z) + H(Y|X,Z).$$

Example

Example 2.2.1

Let (X,Y) have the following *joint distribution*:

Y \ X	1	2	3	4
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
4	$\frac{1}{4}$	0	0	0

What are $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, and $H(Y|X)$?

$$H(X) = \frac{7}{4}\text{bits}, H(Y) = 2\text{bits}, H(X|Y) = \frac{11}{8}\text{bits},$$

$$H(Y|X) = \frac{13}{8}\text{bits}, H(X,Y) = \frac{27}{8}\text{bits}.$$

Relative Entropy

- The *entropy* of a random variable is a measure of *the amount of information* required to describe the random variable.

- The *relative entropy* $D(p\|q)$ is a measure of *the distance between two distributions*. We need $H(p)$ bits on average to describe a random variable with distribution p, and need $H(p) + D(p\|q)$ bits on average to describe a random variable with distribution q from the distribution p point of view.

- The *relative entropy* or *Kullback-Leibler distance* between two probability mass functions $p(x)$ and $q(x)$ is defined as

$$\begin{aligned}
D(p\|q) &= \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)}\\
&= E_p\log\frac{p(X)}{q(X)}.
\end{aligned}$$

By convention, $0\log\frac{0}{0} = 0$, $0\log\frac{0}{q} = 0$ and $p\log\frac{p}{0} = \infty$.

两个分布的距离

对R.V. X，有support set $\mathcal{X} = \{x_1, \dots x_n\}$

有两个分布：$p: \{p_1, \cdots, p_n\}$

$q: \{q_1, \dots, q_n\}$

则 $D(p\|q) = \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)}$ 即为这两个分布的距离

对p,q编码

	p	q
Optimal p	$H(p)$	$H(p)+D(p\|q)$
Optimal q	$H(q)+D(q\|p)$	$H(q)$

- $D(p\|q) = D(q\|p)$?

Example 2.3.1

Let $\mathcal{X} = \{0, 1\}$ and consider two distributions p and q on $\mathcal{X}$. Let $p(0) = 1 - r$, $p(1) = r$, and let $q(0) = 1 - s$, $q(1) = s$. Then

$$D(p\|q) = (1 - r)\log\frac{1 - r}{1 - s} + r\log\frac{r}{s},$$
$$D(q\|p) = (1 - s)\log\frac{1 - s}{1 - r} + s\log\frac{s}{r}.$$

In general, $D(p\|q) \neq D(q\|p)$!

Mutual Information

Definition

Consider two random variables X and Y with a joint probability mass function $p(x, y)$ and marginal probability mass functions $p(x)$ and $p(y)$. The *mutual information* $I(X; Y)$ is the relative entropy between the joint distribution and the product distribution $p(x)q(y)$:

$$\begin{aligned} I(X; Y) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x, y)\log\frac{p(x, y)}{p(x)p(y)} \\ &= D(p(x, y)\|p(x)p(y)) \\ &= E_{p(x,y)}\log\frac{p(X, Y)}{p(X)p(Y)} \end{aligned}$$

描述了$p(x,y)$与$p(x)p(y)$两个不同的联合分布的距离

显然，若X,Y独立，则$I(X;Y)=0$.

Relationships

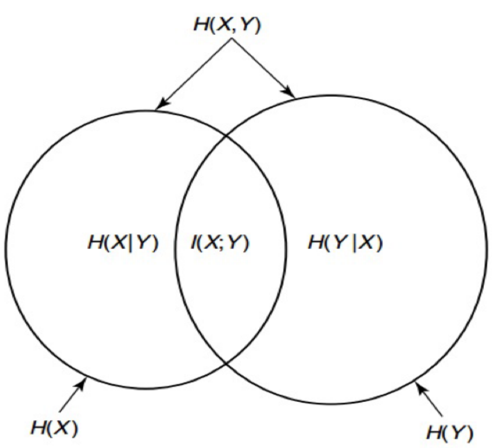
Theorem 2.4.1 (Mutual information and entropy)

$$I(X; Y) = H(X) - H(X|Y)$$
$$I(X; Y) = H(Y) - H(Y|X)$$
$$I(X; Y) = H(X) + H(Y) - H(X, Y)$$
$$I(X; Y) = I(Y; X)$$
$$I(X; X) = H(X)$$

$$I(X_1, X_2; Y) = E_{X_1, X_2, Y}\log\frac{P(X_1, X_2, Y)}{P(X_1, X_2)P(Y)}$$

$$I(X_1, \cdots, X_n; Y_1, \dots, Y_m | Z_1, \dots, Z_k)$$
$$= E_{\substack{X_1, \dots, X_n \\ Y_1, \cdots, Y_m \\ Z_1, \cdots, Z_k}} \log\frac{P(X_1, \dots, X_n, Y_1, \dots, Y_m | Z_1, \dots, Z_k)}{P(X_1, \dots, X_n | Z_1, \dots, Z_k)\cdot P(Y_1, \cdots, Y_m | Z_1, \cdots, Z_k)}$$

- *Mutual information and entropy*



Chain Rules

Theorem 2.5.1 (Chain rule for entropy)

Let $X1, X2, \ldots, X_n$ be drawn according to $p(x_1, x_2, \ldots, x_n)$. Then

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1). = H(X_1) + H(X_2|X_1) + H(X_3|X_2, X_1) + \cdots + H(X_n|X_{n-1}, \ldots, X_1)$$

Definition

The *conditional mutual information* of random variable X and Y given Z is defined by

$$\begin{aligned} I(X;Y|Z) &= H(X|Z) - H(X|Y,Z) \\ &= E_{p(x,y,z)} \log \frac{p(X,Y|Z)}{p(X|Z)p(Y|Z)} \end{aligned}$$

$$I(X;Y,Z) = I(X;Y) + I(X;Z|Y)$$

Theorem 2.5.2 (Chain rule for mutual information)

$$I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_{i-1}, X_{i-2}, \ldots, X_1). = I(X_1;Y) + I(X_2;Y|X_1) + I(X_3;Y|X_2,X_1) + \cdots + I(X_n;Y|X_{n-1}\ldots X_1)$$

Definition

For joint probability mass functions $p(x, y)$ and $q(x, y)$, the *conditional relative entropy* $D(p(y|x)\|q(y|x))$ is the average of the relative entropies between the conditional probability mass functions $p(y|x)$ and $q(y|x)$ averaged over the probability mass function $p(x)$. More precisely,

$$\begin{aligned} D(p(y|x)\|q(y|x)) &= \sum_x p(x) \sum_y p(y|x) \log \frac{p(y|x)}{q(y|x)} \\ &= E_{p(x,y)} \log \frac{p(Y|X)}{q(Y|X)} \end{aligned}$$

Theorem 2.5.3 (Chain rule for relative entropy)

$$D(p(x,y)\|q(x,y)) = D(p(x)\|q(x)) + D(p(y|x)\|q(y|x))$$

Proof.

$$\begin{aligned} & D(p(x,y)\|q(x,y)) \\ &= \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{q(x,y)} \\ &= \sum_x \sum_y p(x,y) \log \frac{p(x)p(y|x)}{q(x)q(y|x)} \\ &= \sum_x \sum_y p(x,y) \log \frac{p(x)}{q(x)} + \sum_x \sum_y p(x,y) \log \frac{p(y|x)}{q(y|x)} \\ &= D(p(x)\|q(x)) + D(p(y|x)\|q(y|x)) \end{aligned}$$