EE411 · Information Theory and Coding

Assignment 4

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1 3.4

1. By the AEP, we have $Pr\{X^n \in A^n\} \to 1$.
2. By the law of large numbers, we have $Pr\{X^n \in B^n\} \to 1$. So there exists $\epsilon_1 > 0$ and N_1 such that

$$Pr\{X^n \in B^n\} > 1 - \epsilon_1$$

for all $n > N_1$. For the same reason, there exists $\epsilon_2 > 0$ and N_2 such that

$$Pr\{X^n \in A^n\} > 1 - \epsilon_2$$

for all $n > N_2$. Therefore, for all $n > \max(N_1, N_2)$,

$$\begin{aligned} Pr\{X^n \in A^n \cap B^n\} &= Pr\{X^n \in A^n\} + Pr\{X^n \in B^n\} - Pr\{X^n \in A^n \cup B^n\} \\ &> 1 - \epsilon_2 + 1 - \epsilon_1 - 1 \\ &= 1 - \epsilon_1 - \epsilon_2 \end{aligned}$$

since $Pr\{X^n \in A^n \cup B^n\} \le Pr\{X^n \in \mathcal{X}^n\} = 1$. So for any $\epsilon = \epsilon_1 + \epsilon_2 > 0$, there exists $N = \max(N_1, N_2)$ such that $Pr\{X^n \in A^n \cap B^n\} > 1 - \epsilon$ for all $n > N$. Therefore, $Pr\{X^n \in A^n \cap B^n\} \to 1$.

3. We have

$$\sum_{x^n \in A^n \cap B^n} p(x^n) \le \sum_{x^n \in X^n} p(x^n) = 1$$

By the property of typical set, we have for $x^n \in A^n$,

$$p(x^n) \ge 2^{-n(H+\epsilon)}$$

Therefore, we have

$$1 \ge \sum_{x^n \in A^n \cap B^n} p(x^n) \ge \sum_{x^n \in A^n \cap B^n} 2^{-n(H+\epsilon)} = |A^n \cap B^n| 2^{-n(H+\epsilon)}$$

i.e.,

$$|A^n \cap B^n| \le 2^{n(H+\epsilon)}$$

for all n.

4. From (b) we can claim that there exists N such that $Pr\{X^n \in A^n \cap B^n\} \ge \frac{1}{2}$ for all $n > N$. By the property of typical set, we have for $x^n \in A^n$,

$$p(x^n) \le 2^{-n(H-\epsilon)}$$

Therefore, we have

$$\frac{1}{2} \le \sum_{x^n \in A^n \cap B^n} p(x^n) \le \sum_{x^n \in A^n \cap B^n} 2^{-n(H-\epsilon)} = |A^n \cap B^n| 2^{-n(H-\epsilon)}$$

i.e.,

$$|A^n \cap B^n| \ge (\frac{1}{2}) 2^{n(H-\epsilon)}$$

for n sufficiently large ($n > N$).

2 3.8

By the law of large numbers, we have

$$\log(X_1X_2...X_n)^{\frac{1}{n}} = \frac{1}{n}\sum_{i=1}^{n}\log X_i \to E\log X, with\ prob.$$

Thus, $(X_1X_2...X_n)^{\frac{1}{n}} \to 2^{E\log X}$, with prob.. Since $E\log X = \frac{1}{2}\log 1 + \frac{1}{4}\log 2 + \frac{1}{4}\log 3 = \frac{1}{4}\log 6$, we have $(X_1X_2...X_n)^{\frac{1}{n}} \to 2^{\frac{1}{4}\log 6} = \sqrt[4]{6} = 1.565$.

3 3.13

1. $H(X) = -0.6\log 0.6 - 0.4\log 0.4 = 0.971$.
2. By definition, $A_\epsilon^{(n)}$ is the set of sequence such that $-\frac{1}{n}\log p(x^n) \in [H(X)-\epsilon, H(X)+\epsilon]$, i.e., $[0.871, 1.071]$. Checking from the table, we can know that $A_\epsilon^{(n)}$ is the set of all sequences with $k \in [11, 19]$. Thus, the corresponding probability is

$$Pr\{x^n \in A_\epsilon^{(n)}\} = F(19) - F(10) = 0.970638 - 0.034392 = 0.936246$$

where $F(x) = Pr\{X \leq x\} = \sum_{k=0}^{x}\left(\begin{array}{c} n \\ k \end{array}\right)p^k(1-p)^{n-k}$. The number of elements in the typical set is

$$\begin{aligned} |A_\epsilon^{(n)}| = \sum_{k=11}^{19}\left(\begin{array}{c} n \\ k \end{array}\right) &= 4457400 + 5200300 + 5200300 + 4457400 + 3268760 + 2042975 + 1081575 + 480700 + 177100 \\ &= 26366510 \end{aligned}$$

3. To attain the smallest set of probability 0.9, we need to choose the elements with probability as large as possible. From the table, we know that $-\frac{1}{n}\log p(x^n)$ decreases monotonically as k increases, i.e., $p(x^n)$ increases monotonically as k increases. Since $Pr\{k \geq 13\} = 0.846232$ and $Pr\{k \geq 12\} = 0.922199$, $B_\delta^{(n)}$ consists of sequences with $k \geq 13$ and some sequences with $k = 12$. Since the number of sequences with $k \geq 13$ is 16777216 and the number of sequences with $k = 12$ is $\frac{0.9-Pr\{k\geq 13\}}{p_{k=12}(x^n)} = \frac{0.9-0.846232}{2^{-1.041146*25}} = 3680681$. Therefore, $B_\delta^{(n)}$ has $16777216 + 3680681 = 20457897$ elements.
4. The intersection of $A_\epsilon^{(n)}$ and $B_\delta^{(n)}$ consists of the sequences with $k \in [13, 19]$ and 3680681 sequences with $k = 12$, and thus the number of elements are $16708810 + 3680681 = 20389491$. The probability is $0.81687 + 0.053768 = 0.870638$.

The table given on the textbook is wrong. Following is the correct table.

n	25		
p	0.6		
1-p	0.4		
k	n choose k	(n choose k) p^k (1-p)^(n-k)	-1/n log(p^k (1-p)^(n-k))
0	1	0.000000	1.321928
1	25	0.000000	1.298530
2	300	0.000000	1.275131
3	2300	0.000001	1.251733
4	12650	0.000007	1.228334
5	53130	0.000045	1.204936
6	177100	0.000227	1.181537
7	480700	0.000925	1.158139
8	1081575	0.003121	1.134740
9	2042975	0.008843	1.111342
10	3268760	0.021222	1.087943
11	4457400	0.043410	1.064545
12	5200300	0.075967	1.041146
13	5200300	0.113950	1.017748
14	4457400	0.146507	0.994349
15	3268760	0.161158	0.970951
16	2042975	0.151086	0.947552
17	1081575	0.119980	0.924154
18	480700	0.079986	0.900755
19	177100	0.044203	0.877357
20	53130	0.019891	0.853958
21	12650	0.007104	0.830560
22	2300	0.001937	0.807161
23	300	0.000379	0.783763
24	25	0.000047	0.760364
25	1	0.000003	0.736966
	33554432	1.000000	
	equals 2^25		