Stock Market

- Initial investment Y_0, daily return ratio r_i, in t-th day, your money is

$$Y_t = Y_0 r_1 \cdot \ldots \cdot r_t.$$

- Now if returns ratio r_i are i.i.d., with

$$r_i = \begin{cases} 4, & \text{w.p. } 1/2 \\ 0, & \text{w.p. } 1/2 \end{cases}$$

- So you think the expected return ratio is $E[r_i] = 2$.
- And then

$$E[Y_t] = E[Y_0 r_1 \cdot \ldots \cdot r_t] = Y_0 (E[r_i])^t = Y_0 2^t???$$

- With $Y_0 = 1$, actual return Y_t goes like

$$1 \quad 4 \quad 16 \quad 0 \quad 0 \quad \cdots$$

- Why?
  - The 'typical' sequences will end up with 0 return.
  - Occasionally, we got high return.
  - The expected return is increasing.
  - Expectation does not show the typical feature of this random sequence. We can turn to typical set.

只看期望的话是纯增的
期望无法体现随机序列的典型特征.
需关注典型集.

Weak Law of Large Numbers

Theorem (Weak Law of Large Numbers)

Suppose that $X_1, X_2, \ldots, X_n$ are n independent, identically distributed (i.i.d.) random variables, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to E[X] \quad \text{in probability},$$

i.e. for every number $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left[|\frac{1}{n}\sum_{i=1}^{n} X_i - E[X]| \le \epsilon \right] = 1.$$

Asymptotic Equipartion Property (AEP)

Definition (Convergence of random variables)

Given a sequence of random variables, $X_1, X_2, \ldots,$ we say that the sequence $X_1, X_2, \ldots$ converges to a random variable X:

1. In probability if for every $\epsilon > 0$, $\Pr[|X_n - X| \ge \epsilon] \to 0$
2. In mean square if $E[(X_n - X)^2] \to 0$
3. With probability 1 (a.k.a. almost surely) if $\Pr[\lim_{n\to\infty} X_n = X] = 1$

Theorem 3.1.1 (AEP)

If $X_1, X_2, \ldots$ are i.i.d. $\sim p(x)$, then

$$-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \to H(X) \qquad \textit{in probability.}$$

Proof.

Since X_i are i.i.d., so are $\log p(X_i)$. Hence, by the weak law of large numbers,

$$\begin{aligned} -\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) &= -\frac{1}{n}\sum_i \log p(X_i) \\ &\to -E[\log p(X)] \qquad \text{in probability} \\ &= H(X) \end{aligned}$$

□

$\forall \varepsilon > 0$

$\lim_{n\to\infty} \Pr[|-\frac{1}{n}\log p(X_1, \ldots X_n) - H(X)| \le \varepsilon] = 1$

$\Rightarrow -\varepsilon \le -\frac{1}{n}\log p(X_1, \ldots, X_n) - H(X) \le \varepsilon$

$-n[H(X)-\varepsilon] \ge \log p(X_1, \ldots, X_n) \ge -n[H(X)+\varepsilon]$

$2^{-n[H(X)-\varepsilon]} \ge p(X_1, \ldots, X_n) \ge 2^{-n[H(X)+\varepsilon]}$

Typical Set

Definition

A *typical set* $A_\epsilon^{(n)}$ contains all sequence realizations $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ with

$$2^{-n(H(X)+\epsilon)} \le p(x_1, x_2, \ldots, x_n) \le 2^{-n(H(X)-\epsilon)}.$$

Consequences of AEP

Theorem 3.1.2

- *If $(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}$, then* $H(X) - \epsilon \le -\frac{1}{n}\log p(x_1, x_2, \ldots, x_n) \le H(X) + \epsilon$.
- $\Pr[(X_1, X_2, ..., X_n) \in A_\epsilon^{(n)}] > 1 - \epsilon$ *for n sufficiently large.*
- $|A_\epsilon^{(n)}| \le 2^{n(H(X)+\epsilon)}$, *where $|A|$ denotes the cardinality of the set A.*
- $|A_\epsilon^{(n)}| \ge (1-\epsilon)2^{n(H(X)-\epsilon)}$ *for n sufficiently large.*

上限是任何情况都成立，下限只在n足够大时成立.

Proof.

1. Immediate from the definition of $A_\epsilon^{(n)}$. □

The number of bits used to describe sequences in typical set is approximately $nH(X)$.

Proof.

2. By Theorem 3.1.1, the probability of the event $(X_1, X_2, \ldots, X_n) \in A_\epsilon^{(n)}$ tends to 1 as $n \to \infty$. Thus, for any $\delta > 0$, there exists an n_0 such that for all $n \ge n_0$, we have

$$\Pr\left\{\left|-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) - H(X)\right| < \epsilon\right\} > 1 - \delta.$$

Setting $\delta = \epsilon$, the conclusion follows. □

Proof.

3.
$$\begin{aligned} 1 = \sum_{\mathbf{x}\in\mathcal{X}^n} p(\mathbf{x}) &\ge \sum_{\mathbf{x}\in A_\epsilon^{(n)}} p(\mathbf{x}) \\ &\ge \sum_{\mathbf{x}\in A_\epsilon^{(n)}} 2^{-n(H(X)+\epsilon)} \\ &= 2^{-n(H(X)+\epsilon)}\left|A_\epsilon^{(n)}\right|. \end{aligned}$$

□

Proof.

4. For sufficiently large n, $\Pr[A_\epsilon^{(n)}] > 1 - \epsilon$, so that

$$\begin{aligned} 1 - \epsilon < \Pr\left[A_\epsilon^{(n)}\right] &\le \sum_{\mathbf{x}\in A_\epsilon^{(n)}} 2^{-n(H(X)-\epsilon)} \\ &= 2^{-n(H(X)-\epsilon)}\left|A_\epsilon^{(n)}\right|. \end{aligned}$$

□

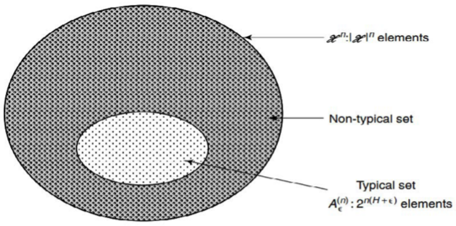
Typical set diagram

This enables us to divide all sequences into two sets

- Typical set: high probability to occur, sample entropy is close to true entropy
  so we will focus on analyzing sequences in typical set
- Non-typical set: small probability, can ignore in general

发生概率高，采样熵接近于真实熵

一般忽略非典型集



Asyptotic Equipartion Property (AEP)

Theorem 3.2.1

Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with distribution $p(x)$, and $X^n = X_1X_2...X_n$. For arbitrarily small $\epsilon > 0$, there exists a code that maps every realization $x^n = x_1x_2...x_n$ of X^n into one binary string, such that the mapping is one-to-one (and therefore invertible) and

$$E\left[\frac{1}{n}\ell(X^n)\right] \leq H(X) + \epsilon$$

for a sufficiently large n.

Proof.

Description in typical set requires no more than $n(H(X)+\epsilon)+1$ bits (correction of 1 bit because of integrality).

Description in atypical set $A_\epsilon^{(n)^C}$ requires no more than $n\log|\mathcal{X}|+1$ bits.

Add another bit to indicate whether in $A_\epsilon^{(n)}$ or not to get whole description. □

Proof.

Let $\ell(x^n)$ be the length of the binary description of x^n. Then, $\forall\epsilon > 0$, there exists n_0 s.t. $\forall n > n_0$,

$$\begin{aligned}
E(\ell(X^n)) &= \sum_{x^n} p(x^n)\,\ell(x^n) \\
&= \sum_{x^n\in A_\epsilon^{(n)}} p(x^n)\,\ell(x^n) + \sum_{x^n\in A_\epsilon^{(n)^C}} p(x^n)\,\ell(x^n) \\
&\leq \sum_{x^n\leq A_\epsilon^{(n)}} p(x^n)\,(n(H+\epsilon)+2) + \sum_{x^n\in A_\epsilon^{(n)^C}} p(x^n)\,(n\log|\mathcal{X}|+2) \\
&= \Pr[A_\epsilon^{(n)}](n(H+\epsilon)+2) + \Pr[A_\epsilon^{(n)^C}](n\log|\mathcal{X}|+2) \\
&\leq n(H+\epsilon) + \epsilon n(\log|\mathcal{X}|) + 2 \\
&= n\,(H+\epsilon')
\end{aligned}$$

where $\epsilon' = \epsilon + \epsilon\log|\mathcal{X}| + \frac{2}{n}$ can be made arbitrarily small by choosing n properly. □