EE411 · Information Theory and Coding

Assignment 1

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1 2.2

If $Y = f(X)$, then we have

$$p(y|x) = p(Y = f(x)|X = x) = 1$$

1. If $Y = 2^X$, then we have

$$\begin{aligned} H(Y|X) &= \sum_{x\in\mathcal{X}} p(x)H(Y|X = x) \\ &= \sum_{x\in\mathcal{X}} p(x) \sum_{y\in\mathcal{Y}} p(y|x)\log p(y|x) \\ &= 0 \end{aligned}$$

   and thus $H(X,Y) = H(X) + H(Y|X) = H(X)$. Besides, we have $X = \log Y$ and $H(X,Y) = H(Y) + H(Y|X) = H(Y)$ for the same reason. Therefore, $H(X) = H(Y)$.

2. If $Y = \cos X$, then we still have

$$\begin{aligned} H(Y|X) &= \sum_{x\in\mathcal{X}} p(x)H(Y|X = x) \\ &= \sum_{x\in\mathcal{X}} p(x) \sum_{y\in\mathcal{Y}} p(y|x)\log p(y|x) \\ &= 0 \end{aligned}$$

   However, the inverse of $Y = \cos X$ is $X = \cos^{-1} Y$, where X is not determined by given Y, and thus $H(X|Y) \geq 0$. Therefore, $H(X) = H(X,Y) = H(Y) + H(X|Y) \geq H(Y)$

2 2.3

Since

$$H(\mathbf{p}) = E[-\log \Pr[\mathbf{p}]] = -\sum_{i=1}^{n} p_i \log p_i \geq 0$$

and

$$\sum_{i=1}^{n} p_i = 1$$

then the minimum value of $H(\mathbf{p})$ is 0 when $p_i = 1$ and $p_j = 0$ for $j \neq i$

3 2.5

Suppose there exists x_0 and y_1, y_2, where $y_1 \neq y_2$, such that $p(x_0, y_1) > 0$ and $p(x_0, y_2) > 0$. Then we have

$$p(x_0) = \sum_{y\in\mathcal{Y}} p(x_0, y) \geq p(x_0, y_1) + p(x_0, y_2) > 0$$

And we have

$$
\begin{aligned}
H(Y|X) &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x) \\
&= -\sum_{x\in\mathcal{X}} p(x)\sum_{y\in\mathcal{Y}} p(y|x)\log p(y|x) \\
&\geq -p(x_0)(p(y_1|x_0)\log p(y_1|x_0) + p(y_2|x_0)\log p(y_2|x_0))
\end{aligned}
$$

Since $f(t) = t\log t \leq 0$ for $t \in [0,1]$, and $f(t) = t\log t < 0$ for $t \neq 0, 1$, then $H(Y|x) > 0$, which contradicts to $H(Y|X) = 0$.

4 2.6

1. If $X \to Y \to Z$, then $I(X;Y|Z) \leq I(X;Y)$. Following is the proof:
   By the chain rule, we have

   $$
   \begin{aligned}
   I(X;Y,Z) &= I(X;Z) + I(X;Y|Z) \\
   &= I(X;Y) + I(X;Z|Y)
   \end{aligned}
   $$

   Since X and Z are conditionally independent given Y, we have $I(X;Z|Y) = 0$. Since $I(X;Z) \geq 0$, we have

   $$I(X;Y|Z) \leq I(X;Y)$$

   and the equality holds if and only if $I(X;Z) = 0$, i.e., X and Z are independent.
   Therefore, we can construct an example such that X is a fair binary random variable and $Y = X$, $Z = Y$. In this case, we have

   $$
   \begin{aligned}
   I(X;Y) &= H(X) - H(X|Y) = H(X) = 1, \\
   I(X;Y|Z) &= H(X|Z) - H(X|Y,Z) = 0
   \end{aligned}
   $$

   Thus, $I(X;Y|Z) < I(X;Y)$.

2. Let X, Y are fair binary random variables and $Z = X + Y$. In this case, we have

   $$
   \begin{aligned}
   &I(X;Y) = 0 \\
   &I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = H(X|X+Y) - H(X|Y,X+Y) = H(X|X+Y) = \frac{1}{2}
   \end{aligned}
   $$

   Thus, $I(X;Y|Z) > I(X;Y)$.