EE411 · Information Theory and Coding

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Assignment 2

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1 2.12

From the table, we can get the marginal distribution

$$p[X=0]=\sum_{y=0}^{1}p[X=0,Y=y]=\frac{1}{3}+\frac{1}{3}=\frac{2}{3},\qquad p[X=1]=\sum_{y=0}^{1}p[X=1,Y=y]=0+\frac{1}{3}=\frac{1}{3}$$
$$p[Y=0]=\sum_{x=0}^{1}p[Y=0,X=x]=\frac{1}{3}+0=\frac{1}{3},\qquad p[Y=1]=\sum_{x=0}^{1}p[Y=1,X=x]=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

and the conditional distribution

$$p[X=0|Y=0]=\frac{p[X=0,Y=0]}{p[Y=0]}=1,\qquad p[X=1|Y=0]=\frac{p[X=1,Y=0]}{p[Y=0]}=0$$
$$p[X=0|Y=1]=\frac{p[X=0,Y=1]}{p[Y=1]}=\frac{1}{2},\qquad p[X=1|Y=1]=\frac{p[X=1,Y=1]}{p[Y=1]}=\frac{1}{2}$$
$$p[Y=0|X=0]=\frac{p[X=0,Y=0]}{p[X=0]}=\frac{1}{2},\qquad p[Y=1|X=0]=\frac{p[X=1,Y=0]}{p[X=0]}=\frac{1}{2}$$
$$p[Y=0|X=1]=\frac{p[X=0,Y=1]}{p[X=1]}=1,\qquad p[Y=1|X=1]=\frac{p[X=1,Y=1]}{p[X=1]}=0$$

1.
$$H(X)=-\sum_{x\in\mathcal{X}}p(x)\log p(x)=-(\frac{2}{3}\log\frac{2}{3}+\frac{1}{3}\log\frac{1}{3})=\log 3-\frac{2}{3}\log 2$$
$$H(Y)=-\sum_{y\in\mathcal{Y}}p(y)\log p(y)=-(\frac{1}{3}\log\frac{1}{3}+\frac{2}{3}\log\frac{2}{3})=\log 3-\frac{2}{3}\log 2$$

2.
$$H(X|Y)=-\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}p(x,y)\log p(x|y)=-(0+0+\frac{1}{3}\log\frac{1}{2}+\frac{1}{3}\log\frac{1}{2})=\frac{2}{3}\log 2$$
$$H(Y|X)=-\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}p(x,y)\log p(y|x)=-(\frac{1}{3}\log\frac{1}{2}+\frac{1}{3}\log\frac{1}{2}+0+0)=\frac{2}{3}\log 2$$

3.
$$H(X,Y)=H(X)+H(Y|X)=\log 3-\frac{2}{3}\log 2+\frac{2}{3}\log 2=\log 3$$

4.
$$H(Y)-H(Y|X)=\log 3-\frac{2}{3}\log 2-\frac{2}{3}\log 2=\log 3-\frac{4}{3}\log 2$$

5.
$$I(X;Y)=\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}p(x,y)\log\frac{p(x,y)}{p(x)p(y)}=\frac{1}{3}\log\frac{3}{2}+\frac{1}{3}\log\frac{3}{4}+0+\frac{1}{3}\log\frac{3}{2}=\log 3-\frac{4}{3}\log 2$$

6.

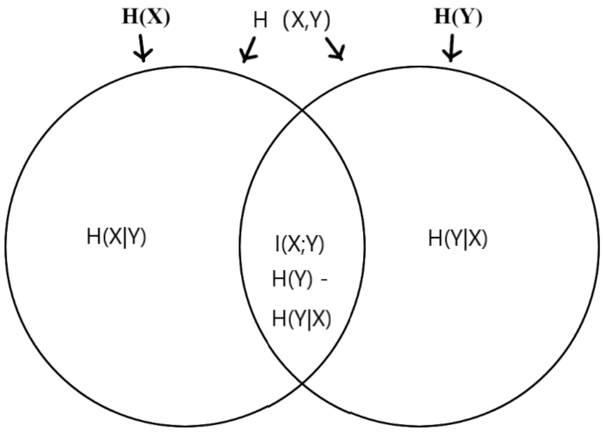

Figure 1: Venn diagram

2 2.28

By the log sum inequality, we have

$$\begin{aligned}
&H((p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)) - H((p_1, \ldots, \frac{p_i + p_j}{2}, \ldots, \frac{p_j + p_i}{2}, \ldots, p_m)) \\
&= (p_i + p_j) \log \frac{p_i + p_j}{2} - (p_i \log p_i + p_j \log p_j) \\
&\leq (p_i \log p_i + p_j \log p_j) - (p_i \log p_i + p_j \log p_j) \\
&= 0
\end{aligned}$$

Therefore, we have

$$H((p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)) \leq H((p_1, \ldots, \frac{p_i + p_j}{2}, \ldots, \frac{p_j + p_i}{2}, \ldots, p_m))$$

3 2.42

1. Since $f(X) = 5X$ is one-to-one, then $H(5X) = H(X)$
2. Since $Y \to$ $X \to g(X)$, by data-processing inequality, we have $I(g(X); Y) = I(Y; g(X)) \leq I(X; Y) = I(Y; X)$. where the equality holds if and only if $I(X; Y|g(X)) = 0$.
3. Since conditioning reduces entropy, i.e., $H(X|Y) \leq H(X)$, then $H(X_0|X_1) \geq H(X_0|X_1, X_1)$, where the equality holds if and only if X_0 and X_1 are independent.
4. Since $H(X, Y) \leq H(X) + H(Y)$, then $\frac{H(X,Y)}{H(X)+H(Y)} \leq 1$, where the equality holds if and only if X and Y are independent.