EE411 · Information Theory and Coding

Assignment 8

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$$Y = X + Z, X \in \{0, 1\}, Z \in \{0, a\}$$

- $a = 0$. $Y = X$ and the channel capacity is $C = \max I(X;Y) = \max H(X) = 1$ bit.
- $a = \pm 1$. It is the same as binary erasure channel with $\alpha = \frac{1}{2}$. So the channel capacity is $C = 1 - \alpha = \frac{1}{2}$ bit.
- $a \neq 0, \pm 1$. $Y \in \{0, 1, a, 1 + a\}$. It is the same as the noisy channel with non-overlapping outputs. So $H(X|Y) = 0$ and the channel capacity is $C = \max I(X;Y) = \max H(X) = 1$ bit.

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$$H(Y|X) = H(X + Z(mod\ 11)|X) = H(Z|X) = H(Z) = \log 3$$

The channel capacity is

$$\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} H(Y) - H(Y|X) \\
&= \max_{p(x)} H(Y) - \log 3 \\
&= \log 11 - \log 3
\end{aligned}$$

when X is uniformly distributed, i.e., $p^*(x) = \frac{1}{11}$.

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To find the capacity of this channel, we must find the distribution $p(x_1, x_2)$ on $\mathscr{X}_1 \times \mathscr{X}_2$ that maximizes $I(X_1, X_2; Y_1, Y_2)$. By the definition of the new channel, $p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)$. Then we have

$$p(x_1, x_2, y_1, y_2) = p(y_1, y_2|x_1, x_2)p(x_1, x_2) = p(x_1, x_2)p(y_1|x_1)p(y_2|x_2)$$

and thus $Y_1 \to X_1 \to X_2 \to Y_2$ forms a Markov chain. So

$$\begin{aligned}
I(X_1, X_2; Y_1, Y_2) &= H(Y_1, Y_2) - H(Y_1, Y_2|X_1, X_2) \\
&= H(Y_1, Y_2) - H(Y_1|X_1, X_2) - H(Y_2|X_1, X_2) \\
&= H(Y_1, Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\
&\leq H(Y_1) - H(Y_1|X_1) + H(Y_2) - H(Y_2|X_2) \\
&= I(X_1; Y_1) + I(X_2; Y_2)
\end{aligned}$$

where the equality holds when X_1 and X_2 are independent. Therefore,

$$\begin{aligned} C &= \max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2) \\ &\leq \max_{p(x_1,x_2)} I(X_1; Y_1) + I(X_2; Y_2) \\ &= \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_2)} I(X_2; Y_2) \\ &= C_1 + C_2 \end{aligned}$$

where the equality holds when $p(x_1, x_2) = p^*(x_1)p^*(x_2)$ and $p^*(x_1)$ and $p^*(x_2)$ are the distributions that maximize C_1 and C_2, respectively.