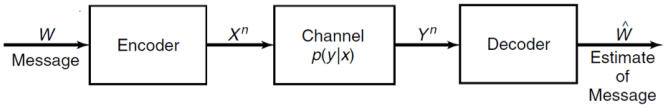
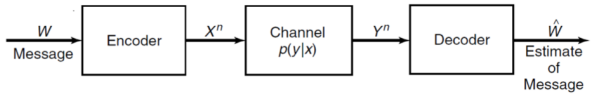
Communication System Model



- $X^n = [X_1, X_2, \ldots, X_n]$
- $Y^n = [Y_1, Y_2, \ldots, Y_n]$
- Channel $p(y^n|x^n)$: probability of observing y^n given input input sequence x^n

Discrete memoryless channel (DMC)



Definition

A discrete channel consists of an input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ and a probability transition matrix $p(y^n|x^n)$ that expresses the probability of observing the output sequence y^n given that we send the sequence x^n.

Definition

The channel is called memoryless if $p(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i)$.

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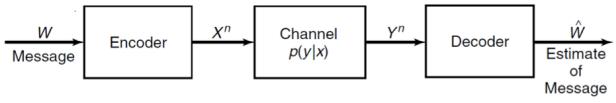
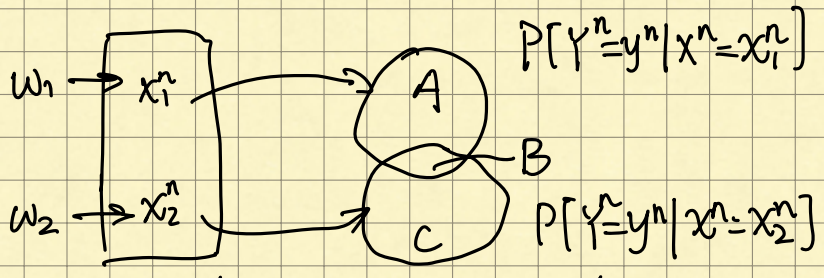
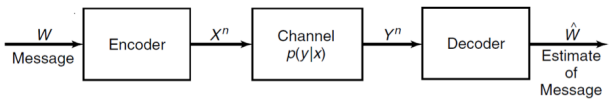


- $X^n = [X_1, X_2, \ldots, X_n] \in \mathcal{X}^n$, $Y^n = [Y_1, Y_2, \ldots, Y_n] \in \mathcal{Y}^n$
  Channel $p(y^n|x^n)$: probability of observing y^n given input symbol x^n
  Memoryless: $p(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i)$
- Messages are mapped into some sequence of the channel symbols. Output sequence is random but has a distribution that depends on the input sequences. Each possible input sequence may induce several possbile outputs, and hence inputs are confusable. Can we choose a *non-confusable* subset of input sequences?



Region A: $\hat{W}=W_1$, Region C: $\hat{W}=W_2$
Region B: $\hat{W}=W_1$ or W_2?
要尽力使region B变小，以致没有error.
n越大，region B越小，但n不能过大.

Duality



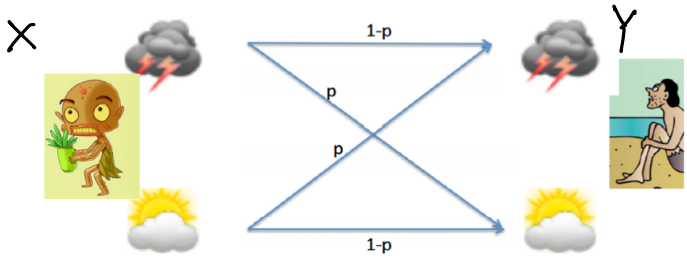
- Data compression: we remove all the redundancy in the data to form the most compressed version possible.
- Data transmission: we add redundancy in a controlled manner to combat errors in the channel.

"Survivor"

- You were deserted on a small island. You met a native and asked about the weather.
- True weather is a random variable X

$$X = \begin{cases} \text{rain} & \text{w.p. } \alpha, \\ \text{sunny} & \text{w.p. } 1-\alpha, \end{cases}$$

- Native knows tomorrow's weather perfectly, but only tells truth with probability $1-p$.
- Native's answer is a random variable $Y \in \{\text{rain, sunny}\}$.
- How informative is the native's answer?



$p(x,y) = p(x)p(y|x)$

↓ source channel

$I(X;Y)$ 表示 X 与 Y 之间传递的信息量. X 是 source 的分布, p 是 channel 的分布.

What is $I(X;Y)$?

- $I(X;Y) = H(X) - H(X|Y)$
- $H(X) = H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$
- $H(X|Y) = H(X|Y = \text{rain})p(\text{rain}) + H(X|Y = \text{sunny})p(\text{sunny})$
- $H(X|Y = \text{rain})$ is equal to $-\sum_{i \in \{\text{rain,sunny}\}} p(X = i|Y = \text{rain}) \log p(X = i|Y = \text{rain})$. Note that

$$p(X = \text{rain}|Y = \text{rain}) = \frac{p(X=\text{rain}|Y=\text{rain})p(X=\text{rain})}{p(Y=\text{rain})} = \frac{(1-p)\alpha}{(1-p)\alpha + p(1-\alpha)}$$

Thus, $H(X|Y) = \alpha H\left(\frac{(1-p)\alpha}{(1-p)\alpha+p(1-\alpha)}\right) + (1-\alpha)H\left(\frac{p\alpha}{p\alpha+(1-p)(1-\alpha)}\right)$

- $I(X;Y) = H(\alpha) - \alpha H\left(\frac{(1-p)\alpha}{(1-p)\alpha+p(1-\alpha)}\right) - (1-\alpha)H\left(\frac{p\alpha}{p\alpha+(1-p)(1-\alpha)}\right)$

Special Cases

- $I(X;Y) = H(\alpha) - \alpha H\left(\frac{(1-p)\alpha}{(1-p)\alpha+p(1-\alpha)}\right) - (1-\alpha)H\left(\frac{p\alpha}{p\alpha+(1-p)(1-\alpha)}\right)$
- Always telling the truth: $p = 0$

$$I(X;Y) = H(\alpha) - \alpha H(1) - (1-\alpha)H(0) = H(\alpha) \le 1 \text{ bit}$$

- Telling truth half of the time: $p = 1/2$

$$I(X;Y) = H(\alpha) - \alpha H(\alpha) - (1-\alpha)H(\alpha) = 0 \text{ bit}$$

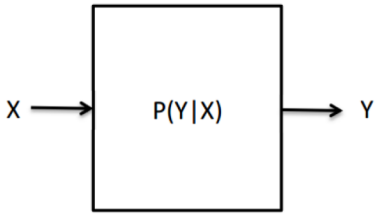
- Fix p, maximize with respect to α, maximum achieved when $\alpha = 1/2$

$$\max_{\alpha} I(X;Y) = H(1/2) - \frac{1}{2}H(1-p) - \frac{1}{2}H(p) = 1 - H(P)$$

Information Channel Capacity

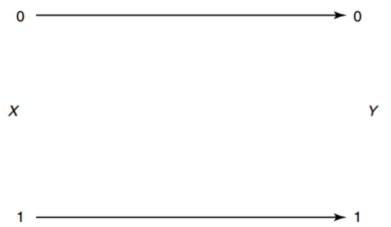
Definition ("Information" Channel Capacity)

$$C = \max_{p(x)} I(X;Y)$$



Examples

- Binary noiseless channel



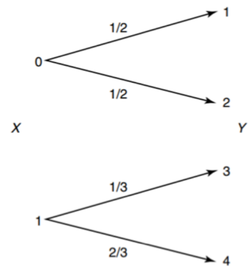
$$C = \max I(X;Y) = \log 2 = 1 \text{ bits } \left(\text{with } p(x) = (\tfrac{1}{2}, \tfrac{1}{2})\right)$$

无随机性, $H(Y|X)=0$.

$I(X;Y) = H(Y) - H(Y|X) = H(Y)$

$C = \max I(X;Y) = \max H(Y) = 1$

- Noisy channel with nonoverlapping outputs



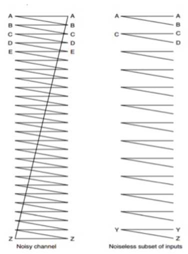
$$C = \max I(X;Y) = \log 2 = 1 \text{ bits } \left(\text{with } p(x) = (\tfrac{1}{2}, \tfrac{1}{2})\right)$$

无重复输出，与上者同.

$I(X;Y) = H(X) - H(X|Y)$, $H(X|Y) = 0$

$C = \max H(X) = 1$

- Noisy typewriter



$$C = \max I(X;Y) = \log \frac{26}{2} = \log 13 \text{ bits } \left(\text{with } p(x) \text{ uniformly distributed}\right)$$

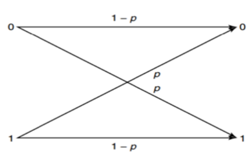
$C = \max H(Y) - H(Y|X)$, 给定X, Y的取值只有两个, 且等可能.

$= \max H(Y) - \log 2$

$= \log 26 - \log 2$

(如X=A时, $Y = \begin{cases} A, & p=\frac{1}{2} \\ B, & p=\frac{1}{2} \end{cases}$)

- Binary symmetric channel



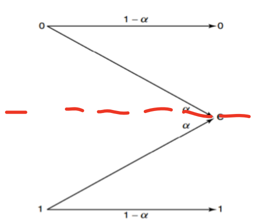
CD-ROM read channel

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - \sum_{x \in \{0,1\}} p(x) H(Y|X=x)$$

$$= H(Y) - \sum_{x \in \{0,1\}} p(x) H(p) = H(Y) - H(p) \le 1 - H(p)$$

$$C = \max I(X;Y) = I - H(p) \text{ bits}$$

- Binary erasure channel



$$C = \max_{p(x)} I(X;Y)$$
$$= \max_{p(x)} \left(H(Y) - H(Y|X)\right)$$
$$= \max_{p(x)} H(Y) - H(\alpha)$$

Let $\Pr[X=1] = \pi$, then

$(1-\pi)\alpha + \pi\alpha = \alpha$

$$H(Y) = H\left((1-\pi)(1-\alpha), \alpha, \pi(1-\alpha)\right) = H(\alpha) + (1-\alpha)H(\pi)$$

Thus, $C = \max_\pi (1-\alpha)H(\pi) = 1 - \alpha$ (with $\pi = \frac{1}{2}$)

Symmetric channel

$$p(y|x) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \end{array}.$$

All the rows of the transition matrix are permutations of each other and so are the columns. Let $\mathbf{r}$ be a row of the transition matrix.

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(\mathbf{r}) \leq \log|\mathcal{Y}| - H(\mathbf{r})$$

with equality if $\mathcal{Y}$ is uniformly distributed. If $p(x) = \frac{1}{|\mathcal{X}|}$, Y is also uniformly distributed:

$$p(y) = \sum_{x\in\mathcal{X}} p(y|x)p(x) = \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} p(y|x) = \frac{c}{|\mathcal{X}|} = \frac{1}{|\mathcal{Y}|},$$

where c is the sum of the entries in one column.

$$C = \max H(Y) - H(Y|X)$$
$$= \max H(Y) - H(0.2, 0.3, 0.5)$$
$$= \log 3 - H(0.2, 0.3, 0.5)$$

Fundamental question

- How fast can we transmit information over a channel?
- Suppose a source sends r messages per second, and the entropy of a message is H bits per message, information rate is $R = rH$ bits/second.
- Intuition: as R increases, error will increase.
- Surprisingly, Shannon showed error can approach to zero, as long as

$$R < C$$

Review

- **Channel capacity.** The logarithm of the number of distinguishable inputs is given by

$$C = \max_{p(x)} I(X;Y).$$

- **Examples**
  - Binary symmetric channel: $C = 1 - H(p)$
  - Binary erasure channel: $C = 1 - \alpha$
  - Symmetric channel: $C = \log|\mathcal{Y}| - H$ (row of trans. matrix)

Channel Code

Definition

An (M, n) code for the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of :
1. An index set $\{1, 2, \ldots, M\}$ representing messages.
2. An encoding function $X^n : \{1, 2, \ldots, M\} \to \mathcal{X}^n$, yielding codewords $x^n(1), x^n(2), \ldots, x^n(M)$. The set of codewords is called codebook.
3. A decoding function $g : \mathcal{Y}^n \to \{1, 2, \ldots, M\}$.

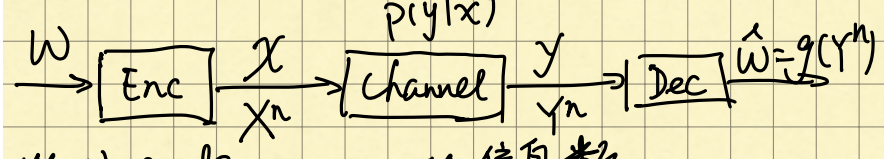
The rate R of an (M, n) code is

$$R = \frac{\log M}{n} \text{ bit per transmission}$$

On the other hand, we usually write

$$M = \lceil 2^{nR} \rceil$$

信息 + 加密 + 解密



(M, n) code

① $W \in \{1, \ldots, M\}$

② $x^n(1), x^n(2), \ldots x^n(M)$

③ $\hat{W} = g(Y^n)$

M: 信息数

n: 编码长度

Data Rate

发送一次能传输的 bit 数

$$R = \frac{\log M}{n}$$

→ M 个可能的信息编码成 D-nary 码需要 $\log_D M$ 个 codeword

→ 一个 codeword 长为 n.

$$\Rightarrow M = \lceil 2^{nR} \rceil$$

Performance Metric

- Conditional probability of error: 给定第i个信息的编码
$$\lambda_i = \Pr[g(Y_n) \neq i | X^n = x^n(i)] = \sum_{y^n} p\left(y^n | x^n(i)\right) I\left(g(y^n) \neq i\right)$$
解码出来不为第i个信息
- Maximal probability of error: $\lambda^{(n)} = \max_{i \in \{1,2,\ldots,M\}} \lambda_i$
- Decoding error probability: $\Pr[W \neq g(Y^n)] = \sum_i \lambda_i \Pr[W = i]$
- Arithmetic average probability of error:

$$P_e^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i, \quad P_e^{(n)} \leq \lambda^{(n)}$$

If W is uniformly distributed:

$$P_e^{(n)} = \Pr[W \neq g(Y^n)]$$ Decoding error probability

λ_i 即为解码错误的概率

$\lambda^{(n)}$ 逼近于0，则出错概率低

Achievable Rate

- A rate R is achievable,

if there exists a sequence of codes with rate R and codeword length n, denoted as $(\lceil 2^{nR} \rceil, n)$, such that the maximal probability of error $\lambda^{(n)} \to 0$ as $n \to \infty$.

Recall that

The rate R of an (M, n) code is

$$R = \frac{\log M}{n}$$ bit per transmission.

允许出错，但 codeword 无限长时，最差出错概率为0.

Joint Typical Set

- Joint typicality. Given two i.i.d. random variable sequences X^n and Y^n, the set of jointly typical sequences is

$$A_\epsilon^{(n)} = \Big\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon \\ \left| -\frac{1}{n} \log p(y^n) - H(Y) \right| < \epsilon \\ \left| -\frac{1}{n} \log p(x^n, y^n) - H(X,Y) \right| < \epsilon \Big\}$$

where $p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$.

typical set: $(X_1 \ldots X_n) \sim p(x) \triangleq X^n$

$$A_\varepsilon^{(n)} = \{ x^n : \left| -\tfrac{1}{n} \log p(x^n) - H(X) \right| < \varepsilon \}$$

joint typical set:

$$[(x_1, y_1), \ldots (x_n, y_n)] \sim p(x,y) \triangleq (X^n, Y^n)$$

$$A_\varepsilon^{(n)} = \{ (x^n, y^n) : \left| -\tfrac{1}{n} \log p(x^n) - H(X) \right| < \varepsilon; \leftarrow X^n \\ \left| -\tfrac{1}{n} \log p(y^n) - H(Y) \right| < \varepsilon; \leftarrow Y^n \\ \left| -\tfrac{1}{n} \log p(x^n, y^n) - H(X,Y) \right| < \varepsilon \} \leftarrow (X^n, Y^n)$$

Joint AEP

- **Joint AEP** Let (X^n, Y^n) be the sequences of length n drawn i.i.d. according to $p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$, then:
  1. $\Pr\left[(X^n, Y^n) \in A_\epsilon^{(n)}\right] \to 1$ as $n \to \infty$.
  2. $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(X,Y)+\epsilon)}$.
  3. If $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$, then
$$\Pr\left[\left(\tilde{X}^n, \tilde{Y}^n\right) \in A_\epsilon^{(n)}\right] \leq 2^{-n(I(X;Y)-3\epsilon)}.$$

Please refer to p196 for the proof (proof of Theorem 7.6.1)

$p(x,y) \sim (X^n, Y^n)$ X与Y有相关性

$p(x)p(y) \sim (\tilde{X}^n, \tilde{Y}^n)$ $\tilde{X}$与$\tilde{Y}$无关，但 X与$\tilde{X}$同分布 Y与$\tilde{Y}$同分布

$n \to \infty$，按$p(x,y)$产生的X与Y一定在typical set上
但按$p(x)p(y)$产生的$\tilde{X}$与$\tilde{Y}$不在typical set上

Channel Coding Theorem

Theorem (Channel coding theorem)

For a discrete memoryless channel, all rates below capacity C are achievable. Specifically, for every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$.

Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \to 0$ must have $R \leq C$.

Achievability: when $R < C$, there exists zero-error code.
Converse: zero-error codes must have $R \leq C$.

① $R<C \Rightarrow R$ is achievable, 即 $\lambda^{(n)} \to 0$
② $\lambda^{(n)} \to 0 \Rightarrow R \leq C$

Random Codebook

- Generate a $(2^{nR}, n)$ code at random according to $p(x)$, where $p(x)$ is the capacity achieving distribution. The 2^{nR} are the rows of a matrix:

$$\mathcal{C} = \begin{bmatrix} x_1(1) & x_2(1) & \dots & x_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \dots & x_n(2^{nR}) \end{bmatrix} \begin{matrix} \to x_{\mathcal{C}}^n(1) \\ \\ \to x_{\mathcal{C}}^n(2^{nR}) \end{matrix}$$

  Each entry is generated i.i.d. according to $p(x)$.
- Encoding: map the message $w = \{1, 2, 3, \dots, 2^{nR}\}$ to codeword $[x_1(w), x_2(w), \dots, x_n(w)]$, i.e.
  $$\mathcal{C} \to [x_1(w), x_2(w), \dots, x_n(w)] = x_{\mathcal{C}}^n(w), w = 1, 2, \dots, 2^{nR}$$
- We shall prove the average detection error probability (over all codebooks) tends to zero as n increase, which implies that there must exists one good codebook whose detection error probability tends to zero

证①$R<C \Rightarrow R$ is achievable, 即 $\lambda^{(n)} \to 0$

C为 $2^{nR} \times n$ 的矩阵，每行都是一个message的codeword

每一项都以 $p(x)$ 独立同分布产生

$W \to X_C^n(W) \to Y^n \to \hat{W} = g(Y^n)$
$p(x)$ $p(y|x)$ $p(x)p(y|x)=p(x,y) \Rightarrow$ 产生 $A_\varepsilon^{(n)}$

因此收到 Y^n 后，g要寻找 $\hat{w}$，使得 $(X_C^n(\hat{w}), Y^n)$ 在 joint typical set 中.

成功的情况：
$(X_C^n(i), Y^n) \in A_\varepsilon^{(n)}$ 且 $i=w$，并且 i 唯一

出错的情况：
① $(X_C^n(w), Y^n) \notin A_\varepsilon^{(n)}$
② $(X_C^n(i), Y^n) \in A_\varepsilon^{(n)}$, $i \neq w$
由joint AEP的第三条性质，②出现的概率 $\to 0$

Jointly Typical Decoding

- Decoding: finds the only $\hat{w}$ such that $(x_{\mathcal{C}}^n(\hat{w}), Y_{\mathcal{C}}^n)$ is jointly typical.
- Decoding error: Suppose message 1 is sent to via codeword $x_{\mathcal{C}}^n(1)$ and $Y_{\mathcal{C}}^n$ is the received signal, the possible decoding error events include:
  - $(x_{\mathcal{C}}^n(1), Y_{\mathcal{C}}^n)$ is not joint typical.
  - $(x_{\mathcal{C}}^n(i), Y_{\mathcal{C}}^n)$ is joint typical $(i = 2, 3, \dots, 2^{nR})$.
- Idea of proof: According to joint AEP, since $x_{\mathcal{C}}^n(1)$ and $Y_{\mathcal{C}}^n$ are generated according to joint distribution $p(x^n, y^n)$, the chance of the first event is small. Moreover, since $Y_{\mathcal{C}}^n$ is generated independently of $x_{\mathcal{C}}^n(i)$, the total chance of the second event is also small.

Proof for achievability

- A message W is chosen according to a uniform distribution
  $$\Pr[W = w] = 2^{-nR},$$
  for $w = 1, 2, \dots, 2^{nR}$. The w-th codeword $x_{\mathcal{C}}^n(w)$, corresponding to the w-th row of $\mathcal{C}$, is sent over the channel.
- The receiver receives a sequence $Y_{\mathcal{C}}^n$ according to the distribution according to the distribution
  $$\Pr\left(y_{\mathcal{C}}^n | x_{\mathcal{C}}^n(w)\right) = \prod_{i=1}^{n} \Pr\left(y_{i,\mathcal{C}} | x_{i,\mathcal{C}}(w)\right),$$
  and guesses which message was sent using jointly typical decoding.

假设有一种传输：virtual transmission
① uniformly pick up one message
② randomly generate a codebook C
③ Joint typical decoding. $A_\varepsilon^{(n)} \sim p(x)p(y|x) = p(x,y)$
若这种传输能有当 $n \to \infty$, $\lambda^{(n)} \to 0$. 则必然存在 codebook 满足 $\lambda^{(n)} \to 0$ 当 $n \to \infty$.

- Let $\varepsilon = \{\hat{W}(Y^n) \neq W\}$ denote the error event, $\lambda_w(\mathcal{C})$ be the error probability of the w-th codeword of code $\mathcal{C}$. The average probability of error, over all codewords and all codebooks, is:

$$\Pr(\varepsilon) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) P_e^{(n)}(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \lambda_w(\mathcal{C})$$
$$= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}),$$

where $\sum_{\mathcal{C}} \Pr(\mathcal{C})\lambda_1(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C})\lambda_w(\mathcal{C})$, $\forall w \neq 1$.

- Let $Y_{\mathcal{C}}^n$ be the received signal for $x_{\mathcal{C}}^n(1)$

限定C $e_i(\mathcal{C}) = \{(x_{\mathcal{C}}^n(i), Y_{\mathcal{C}}^n) \in A_\epsilon^{(n)}\}, i \in \{1, 2, \ldots, 2^{nR}\}$,

and $e_i^c(\mathcal{C}) = !e_i(C)$. Thus,

$$\Pr[\varepsilon] = \sum_{\mathcal{C}} \Pr(\mathcal{C})\lambda_1(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr\left[e_1^c(\mathcal{C}) \cup (\cup_{i=2}^{2^{nR}} e_i(\mathcal{C}))\middle| W = 1\right]$$
$$\leq \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C})|W = 1] + \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{i=2}^{2^{nR}} \Pr[e_i(\mathcal{C})|W = 1]$$
$$= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C})|W = 1] + \sum_{i=2}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_i(\mathcal{C})|W = 1]$$

需要证 $\to 0$ $\to 0$

$$\sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C})|W = 1]$$
$$= \sum_{\mathcal{C}} \left(\prod_{i=1}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i))\right) \Pr[e_1^c(\mathcal{C})|W = 1]$$
$$= \sum_{x_1^n} \sum_{\mathcal{C}: x_{\mathcal{C}}^n(1) = x_1^n} \prod_{i=1}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i)) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1)$$
$$= \sum_{x_1^n} \Pr(x_1^n) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1)$$
$$\times \sum_{\mathcal{C}: x_{\mathcal{C}}^n(1) = x_1^n} \prod_{i=2}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i))$$
$$= \sum_{x_1^n} \Pr(x_1^n) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1)$$
$$= \Pr(X_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1) = \Pr(E_1^c|W = 1)$$

- Similarly,

$$\sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1(\mathcal{C})|W = 1] = \Pr(X_i^n \text{ and } Y^n \text{ are joint typical}|W = 1)$$
$$= \Pr(E_i|W = 1)$$

- As a result,

$$\Pr[\varepsilon] \leq \Pr[E_1^c|W = 1] + \sum_{i=2}^{2^{nR}} \Pr[E_i|W = 1]$$

- By the joint AEP, $\Pr[E_1^c|W = 1] \leq \epsilon$ for n sufficiently large. By the code generation process, $X^n(1)$ and $X^n(i)$ are independent for $i \neq 1$, so are Y^n and $X^n(i)$. Hence the probability that $X^n(i)$ and Y^n are jointly typical is $\leq 2^{-n(I(X;Y)-3\epsilon)}$ by the joint AEP.

$$\Pr[\varepsilon] \leq \epsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(X;Y)-3\epsilon)}$$
$$= \epsilon + (2^{nR} - 1)2^{-n(I(X;Y)-3\epsilon)}$$
$$\leq \epsilon + 2^{3n\epsilon} 2^{-n(I(X;Y)-R)}$$
$$\leq 2\epsilon \quad \text{for } R \leq I(X;Y) - 4\epsilon \text{ and sufficiently large n}$$

Hence, if $R < I(X;Y)$, we can choose ϵ and n so that the average probability of error, over codebooks and codewords, is less than 2ϵ.

- Since $p(x)$ is the capacity achieving distribution, $R < I(X;Y)$ beacomes $R < C$.

$\Pr[\hat{W} \neq W] = \Pr[Er] = \sum_{\forall C} \Pr[C]\lambda_1(C)$

$e_i(C) = \{(X_C^n(i), Y_C^n) \in A_C^{(n)}\}, i = 1, 2, \ldots, 2^{nR}$.

$\lambda_1(C) = \Pr[\bar{e}_1(C) \cup e_2(C) \cup \ldots \cup e_{2^{nR}}(C) | W = 1]$
$\leq \Pr[\bar{e}_1(C)|W=1] + \sum_{i=2}^{2^{nR}} \Pr[e_i(C)|W=1]$

$\Pr[Er] \leq \sum_{\forall C} \Pr[C]\Pr[\bar{e}_1(C)|W=1] + \sum_{i=2}^{2^{nR}} \sum_{\forall C} \Pr[C]\Pr[e_i(C)|W=1]$

$\leq \varepsilon$ $\leq 2^{-n[I(X;Y)-3\varepsilon]}$

由于C中每一行的生成都是 i.i.d 的，故

$\Pr[C] = \Pr[X_C^n(1)]\Pr[X_C^n(2)]\cdots\Pr[X_C^n(2^{nR})]$

$\sum_{\forall C} = \sum_{x_1^n} \sum_{C: X_C^n(1) = x_1^n}$, $\Pr[C] = \prod_{i=1}^{2^{nR}} \Pr[X_C^n(i)]$

$\sum_{\forall C} \Pr[C]\Pr[\bar{e}_1(C)|W=1]$
$= \sum_{\forall C} \prod_{i=1}^{2^{nR}} \Pr[X_C^n(i)]\Pr[\bar{e}_1(C)|W=1]$
$= \sum_{\forall x_1^n} \sum_{C: X_C^n(1) = x_1^n} \Pr[X_C^n(1)] \prod_{i=2}^{2^{nR}} \Pr[X_C^n(i)]\Pr[\bar{e}_1(C)|W=1]$
$= \sum_{\forall x_1^n} \Pr[X_C^n(1)]\Pr[\bar{e}_1(C)|W=1] \sum_{C: X_C^n(1) = x_1^n} \prod_{i=2}^{2^{nR}} \Pr[X_C^n(i)]$

$\sum_{C: X_C^n(1) = x_1^n} = \sum_{\forall X_2^n, X_3^n, \ldots, X_{2^{nR}}^n}$ 只固定了第一行，后面均为随机选取.

$\Rightarrow \sum_{C: X_C^n(1) = x_1^n} \prod_{i=2}^{2^{nR}} \Pr[X_C^n(i)] = \sum_{\forall X_2^n} \Pr[X_C^n(2)] \sum_{\forall X_3^n} \Pr[X_C^n(3)] \cdots \sum_{\forall X_{2^{nR}}^n} \Pr[X_C^n(2^{nR})]$
$= 1$

原式 $= \sum_{\forall X_1^n} \Pr[X_1^n]\Pr[(X_1^n, Y_1^n) \notin A_\varepsilon^{(n)} | W=1]$
$= \Pr[(X_1^n, Y_1^n) \notin A_\varepsilon^{(n)} | W=1] \leq \varepsilon$ for sufficiently large n.

$\Pr[Er] \leq \varepsilon + \sum_{i=2}^{2^{nR}} 2^{-n[I(X;Y)-3\varepsilon]}$
$< \varepsilon + 2^{nR} \cdot 2^{-n[I(X;Y)-3\varepsilon]}$
$= \varepsilon + 2^{-n[I(X;Y)-R-3\varepsilon]}$ $\geq \varepsilon$

若 $R \leq I(X;Y) - 4\varepsilon$, 则有
$\leq \varepsilon + 2^{-n\varepsilon} \leq 2\varepsilon$ for sufficiently large n.

- Get rid of the average over codebooks. Since the average probability of error is $\leq 2\epsilon$, there exists at least one codebook $\mathcal{C}^*$ with a small average probability of error ($\Pr(\varepsilon|\mathcal{C}^*) \leq 2\epsilon$). Since we have chosen $\hat{W}$ according to a uniform distribution, we have

$$\Pr(\varepsilon|\mathcal{C}^*) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(\mathcal{C}^*).$$

- Throw away the worst half of the codewords in the best codebook $\mathcal{C}^*$. We have $\Pr(\varepsilon|\mathcal{C}^*) \leq \frac{1}{2^{nR}} \sum \lambda_i(\mathcal{C}^*) \leq 2\epsilon$. This implies that at least half the indices i and their associated codewords $X^n(I)$ must have conditional probability of error $\lambda_i \leq 4\epsilon$. If we reindex the codewords, we have 2^{nR-1} codewords. The rate now is $R' = R - \frac{1}{n}$ with maximal probability of error $\lambda^{(n)} \leq 4\epsilon$.

所以对足够大的 n，$R < C = \max I(X;Y)$，因有
$P_r[E_r] \leq 2\varepsilon$

也即 $\exists C^*$，$P_r[E_r(C^*)] \leq \varepsilon$，for $R<C$, and sufficiently large n.

$P_r[E_r(C^*)] = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(C^*) \leq 2\varepsilon$

不失一般性，设 $\lambda_1(C_n^*) \leq \lambda_2(C_n^*) \leq \dots \leq \lambda_{2^{nR}}(C^*)$

将 $i > 2^{nR-1}$（也即出错概率较大的那一半）的码去掉，对剩余的 $\tilde{C}_n^*$，有 codeword 2^{nR-1}，此时的 rate 为

$R' = \frac{\log 2^{nR-1}}{n} = R - \frac{1}{n} \to R$，$n \to \infty$

且有 $\lambda^{(n)}(\tilde{C}_n^*) \leq 4\varepsilon$（若不满足，则仍然有 $\frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(C^*) > 2\varepsilon$，不满足前提）

Proof for the converse

- The index W is uniformly distributed on the set $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$, and the sequence Y^n is related to W. From Y^n, we estimate the index W as $\hat{W} = g(Y^n)$. Thus, $W \to X^n(W) \to Y^n \to \hat{W}$ forms a Markov chain.

Data processing inequality: $I(W;\hat{W}) \leq I(X^n(W);Y^n)$

Lemma (Fano's inequality)

For a discrete memoryless channel with a codebook $\mathcal{C}$ and the input message W uniformly distributed over 2^{nR}, we have

$$H(W|\hat{W}) \leq 1 + P_e^{(n)} nR.$$

Lemma

Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C. Then

$$I(X^n;Y^n) \leq nC, \quad \text{for all} \quad p(x^n).$$

Proof.

$$\begin{aligned} I(X^n;Y^n) &= H(Y^n) - H(Y^n|X^n) = H(Y^n) - \sum_{i=1}^{n} H(Y_i|Y_1, \dots, Y_{i-1}, X^n) \\ &= H(Y^n) - \sum_{i=1}^{n} H(Y_i|X_i) \quad \text{memoryless} \\ &\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) \quad \text{independence bound} \\ &= \sum_{i=1}^{n} I(X_i|Y_i) \leq nC \end{aligned}$$

Proof.

Converse to channel coding theorem: Since W has a uniform distribution, we have

$$\begin{aligned} nR = H(W) &= H(W|\hat{W}) + I(W;\hat{W}) \\ &\leq 1 + P_e^{(n)} nR + I(W;\hat{W}) \quad \text{Fano's inequality} \\ &\leq 1 + P_e^{(n)} nR + I(X^n;Y^n) \quad \text{data-processing inequality} \\ &\leq 1 + P_e^{(n)} nR + nC \quad \text{Lemma 7.9.2} \end{aligned}$$

We obtain $R \leq \frac{1}{n(1+P_e^{(n)})} + \frac{C}{1+P_e^{(n)}} \to \frac{1}{n} + C$.
Letting $n \to \infty$, we have $R \leq C$. □

证②：$\lambda^{(n)} \to 0 \Rightarrow R \leq C$

$W \to X_C^n(W) \to Y^n \to \hat{W} = g(Y^n)$ 构成 Markov chain
$(\lceil 2^{nR} \rceil, n)$

令 $P_e^{(n)} = P_r[W \neq \hat{W}]$

对 $W \to Y^n \to \hat{W}$ 使用 Fano 不等式

$H(W|\hat{W}) \leq H(P_e^{(n)}) + P_e^{(n)} \log(2^{nR}-1)$
$< 1 + P_e^{(n)} nR$

又 $H(W|\hat{W}) = H(W) - I(W;\hat{W})$

因为 W 是在 2^{nR} 的均匀分布上取值，故 $H(W) = \log 2^{nR} = nR$

所以 $nR - I(W;\hat{W}) < 1 + P_e^{(n)} nR$

由 data processing inequality 可知：$I(W;\hat{W}) \leq I(X^n;Y^n)$

又 $I(X^n;Y^n) \leq nC$

所以 $nR - nC < 1 + P_e^{(n)} nR$

$\Rightarrow R < \frac{1+nC}{n(1-P_e^{(n)})} = \frac{1}{n(1-P_e^{(n)})} + \frac{C}{1-P_e^{(n)}}$

$n \to \infty$，$\to 0$，$\to C$