CSE5014 CRYPTOGRAPHY AND NETWORK SECURITY

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Cyclic groups

- Let G be a finite group of order m (written multiplicatively)
- Let g be some element of G
- Consider the set $\langle g \rangle = \{g^0, g^1, \ldots\}$

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  - If the set has m elements, then it is all of G!
    - In this case, we say g is a *generator* of G
    - If G has a generator, we say G is *cyclic*

Examples

- $\mathbb{Z}_N$
  - Cyclic (for any N); 1 is always a generator: $\{0, 1, 2, \ldots, N-1\}$

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- $\mathbb{Z}_8$
  - Is 3 a *generator*?
    $\{0, 3, 6, 1, 4, 7, 2, 5\}$ - Yes!

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  - Is 2 a *generator*?
    $\{0, 2, 4, 6\}$ - No!

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  - Is 2 a *generator*?
    $\{1, 2, 4, 8, 5, 10, 9, 7, 3, 6\}$ - Yes!
  - Is 8 a *generator*?
    $\{1, 8, 9, 6, 4, 10, 3, 2, 5, 7\}$ - Yes!

Important examples

- **Theorem 11.1** Any group of prime order is cyclic, and every non-identity element is a generator.

- **Theorem 11.2** If p is prime, then $\mathbb{Z}_p^*$ is cyclic
  - **Note**: the order is $p-1$, which is not prime for $p > 3$

Uniform sampling

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  - Choose uniform $x \in \{0, \ldots, q-1\}$: set $h := g^x$
- Fix *cyclic* group G of order q, and *generator* g
- We know that $\{g^0, g^1, \ldots, g^{q-1}\} = G$
  - For every $h \in G$, there is a unique $x \in \mathbb{Z}_q$, s.t. $g^x = h$
  - Define $\log_g h$ to be this x – the *discrete logarithm* of h with respect to g (in the group G)

Examples

- In $\mathbb{Z}_{11}^*$
  - What is $\log_2 9$?

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  - What is $\log_8 9$?

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  - What is $\log_2 9$?

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- In $\mathbb{Z}_{13}^*$
  - What is $\log_2 9$?
    - $\langle 2 \rangle = \{1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7\}$, so $\log_2 9 = 8$

Discrete-logarithm problem (informal)

- DLog problem in G:

  Given *generator* g and element h, compute $\log_g h$

- DLog assumption in G:

  Solving the discrete log problem in G is hard

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  Solving the discrete log problem in G is hard

- In $\mathbb{Z}^*_{3092091139}$
  - What is $\log_2 1656755742$?

Discrete-logarithm problem

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- For algorithm A, define $exp't\ Dlog_{A,\mathcal{G}}(n)$:
  - Compute $(G, q, g) \leftarrow \mathcal{G}(1^n)$
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- **Definition 11.3** The *discrete-logarithm problem* is hard relative to $\mathcal{G}$ if for all PPT algorithms A,

$$\Pr[Dlog_{A,\mathcal{G}}(n) = 1] \leq negl(n)$$

Diffie-Hellman problems

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- In $\mathbb{Z}_{11}^*$
  - $\langle 2 \rangle = \{1, 2, 4, 8, 5, 10, 9, 7, 3, 6\}$
  - So $DH_2(7, 5) =?$
- In $\mathbb{Z}_{3092091139}^*$
  - What is $DH_2(1656755742, 938640663) =?$
  - Is 1994993011 the answer, or is it just a random element of $\mathbb{Z}_{3092091139}^*$?

Diffie-Hellman assumptions

- *Computational* Diffie-Hellman (CDH) problem:
  - Given g, h_1, h_2, compute $DH_g(h_1, h_2)$

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Diffie-Hellman assumptions

- *Computational* Diffie-Hellman (CDH) problem:
  - Given g, h_1, h_2, compute $DH_g(h_1, h_2)$

- *Decisional* Diffie-Hellman (DDH) problem:
  - Given g, h_1, h_2, distinguish $DH_g(h_1, h_2)$ from a uniform element of G

DDH problem

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- The DDH problem is hard relative to $\mathcal{G}$ if for all PPT algorithm A:

  $|\Pr[A(G, q, g, g^x, g^y, g^z) = 1] - \Pr[A(G, q, g, g^x, g^y, g^{xy}) = 1]| \leq \epsilon(n)$

Relating the Diffie-Hellman problems

- Relative to $\mathcal{G}$
  - If the discrete-logarithm problem is easy, so is the CDH problem
  - If the CDH problem is easy, so is the DDH problem

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  - If the CDH problem is easy, so is the DDH problem

  - I.e., the DDH assumption is *stronger* than the CDH assumption
  - I.e., the CDH assumption is *stronger* than the dlog assumption

Group selection

- The *discrete logarithm* is not hard in all groups!

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- The *discrete logarithm* is not hard in all groups!
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- For cryptographic applications, best to use *prime-order* groups
  - The *dlog* problem becomes easier if the order of the group has small prime factors
  - Prime-order groups have several nice features: e.g., every element except identity is a generator

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- For cryptographic applications, best to use *prime-order* groups
  - The *dlog* problem becomes easier if the order of the group has small prime factors
  - Prime-order groups have several nice features: e.g., every element except identity is a generator
- Two common choices of groups

Group selection: choice 1

- *Prime-order* subgroup of $\mathbb{Z}_p^*$, p prime
  - E.g., $p = tq + 1$ for q prime
  - Take the subgroup of t^{th} powers, i.e., $G = \{[x^t \bmod p] | x \in \mathbb{Z}_p^*\}$
    - This is a group
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    - This is a group
    - It has order $(p-1)/t = q$
    - Since q is prime, the group must be *cyclic*
  - Generalizations based on finite fields are also used

Group selection: choice 2

- *Prime-order* subgroup of an *elliptic curve* group
  - See book for details

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Group selection: choice 2

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- We will describe algorithm in "abstract" groups
  - Can ignore details of the underlying group in the analysis
  - Can instantiate with any (appropriate) group for an implementation

Concrete parameters

- We have discussed two classes of cryptographic assumptions
  - *Factoring-based* (factoring, RSA assumptions)
  - *Dlog-based* (dlog, CDH, and DDH assumptions)

- All these problems are (believed to be) "hard", i.e., to have no polynomial-time algorithms
  - But how hard are they, concretely?

- The goal here is to give an idea as to how parameters are calculated, and what relevant parameters are

Security

- Recall: For symmetric-key algorithms
  - Block cipher with n-bit key $\approx$ security againt 2^n-time attacks
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- Computing discrete logarithms in a group of order 2^n takes 2^n time
  - Are these the best algorithms possible?

Algorithms for factoring

- There exist algorithms factoring an integer N that run in much less than $2^{\|N\|/2}$ time

- Best known algorithm (asymptotically):
  general number field sieve
  - Running time (heuristic): $2^{O(\|N\|^{1/3} \log^{2/3} \|N\|)}$
  - Makes a huge difference in practice
  - Exact constant term also important!

Algorithms for dlog

- Two classes of algorithms:
  - Ones that work for *arbitrary* ("generic") groups
  - Ones that target specific groups
    - Recall that in some groups the problem is not even hard

- Best "generic" algorithms:
  - Time $2^{n/2}$ in a group of order $\approx 2^n$
  - This is known to be optimal for generic algorithms

Algorithms for dlog

- Best known algorithm for (subgroups of) $\mathbb{Z}_p^*$: *number field sieve*
  - Running time (heuristic): $2^{O(\|p\|^{1/3} \log^{2/3} \|p\|)}$

- For (appropriately chosen) elliptic-curve groups, nothing better than generic algorithms is known!
  - This is why elliptic-curve groups can allow for more-efficient cryptography

Choosing parameters

- As recommended by NIST (112-bit security):
  - *Factoring*: 2048-bit modulus
  - *Dlog*: order-q subgroup of $\mathbb{Z}_p^*$: $\|q\| = 224$, $\|p\| = 2024$
  - *Dlog*, elliptic-curve group of order q: $\|q\| = 224$

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  - *Dlog*, elliptic-curve group of order q: $\|q\| = 224$

- Much longer than for symmetric-key algorithms!
  - Explains in part why public-key crypto is less efficient than symmetric-key crypto

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- Can be difficult or expensive to solve in other settings

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- Imagine an organization with N employees, where each pair of employees might need to communicate securely

- Solution using *private-key cryptography*
  - Each user shares a key with all other users

  $\Rightarrow$ Each user must store/manage $N - 1$ secret keys!

  $\Rightarrow$ $O(N^2)$ keys overall!

Lack of support for "open systems"

- Say two users *who have no prior relationship* want to communicate securely
  - When would they ever have shared a key?

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- This happens all the time!
  - Customer sending credit-card data to merchant
  - Contacting a friend-of-a-friend on social media
  - Emailing a colleague
  - ...

“Classical” cryptography offers no solution

New Directions in Cryptography

Invited Paper

WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

***Abstract*—Two kinds of contemporary developments in cryptography are examined. Widening applications of teleprocessing have given rise to a need for new types of cryptographic systems, which minimize the need for secure key distribution channels and supply the equivalent of a written signature. This paper suggests ways to solve these currently open problems. It also discusses how the theories of communication and computation are beginning to provide the tools to solve cryptographic problems of long standing.**

I. Introduction

WE STAND TODAY on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of mechanical computing and brought the cost of high grade cryptographic devices down to where they can be used in such commercial applications as remote cash dispensers and computer terminals. In turn, such applications create a need for new types of cryptographic systems which minimize the necessity of secure key distribution channels and supply the equivalent of a written signature. At the same time, theoretical developments in information theory and computer science show promise of providing provably secure cryptosystems, changing this ancient art into a science.

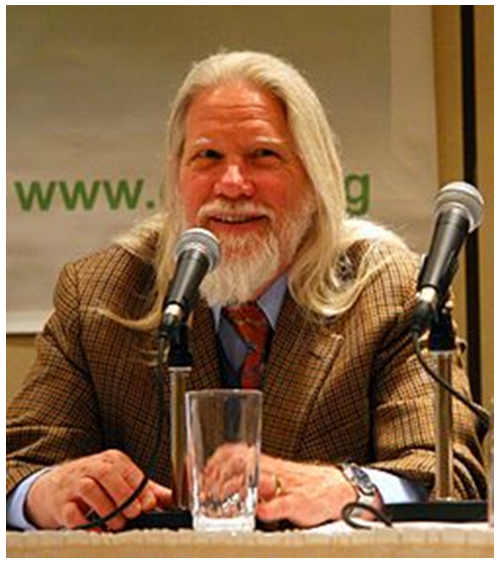
The best known cryptographic problem is that of privacy: preventing the unauthorized extraction of information from communications over an insecure channel. In order to use cryptography to insure privacy, however, it is currently necessary for the communicating parties to share a key which is known to no one else. This is done by sending the key in advance over some secure channel such as private courier or registered mail. A private conversation between two people with no prior acquaintance is a common occurrence in business, however, and it is unrealistic to expect initial business contacts to be postponed long enough for keys to be transmitted by some physical means. The cost and delay imposed by this key distribution problem is a major barrier to the transfer of business communications to large teleprocessing networks.

Section III proposes two approaches to transmitting keying information over public (i.e., insecure) channels without compromising the security of the system. In a *public key cryptosystem* enciphering and deciphering are governed by distinct keys, E and D, such that computing D from E is computationally infeasible (e.g., requiring 10^{100} instructions). The enciphering key E can thus be publicly disclosed without compromising the deciphering key D. Each user of the network can, therefore, place his enciphering key in a public directory. This enables any user of the system to send a message to any other user enci-

Cryptography History

- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

"We stand today on the brink of a revolution in cryptography."



Bailey W. Diffie

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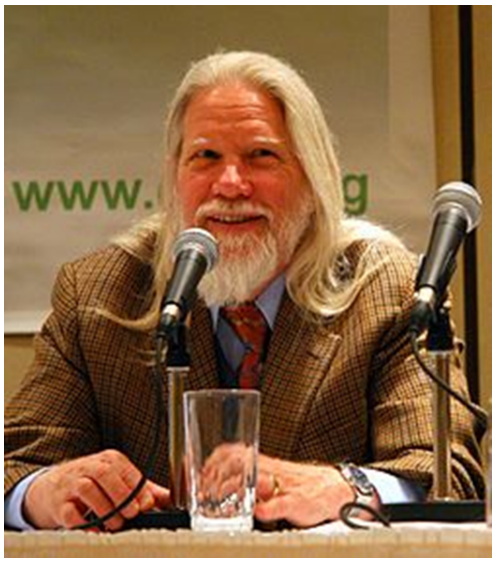

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2015 **Turing Award**

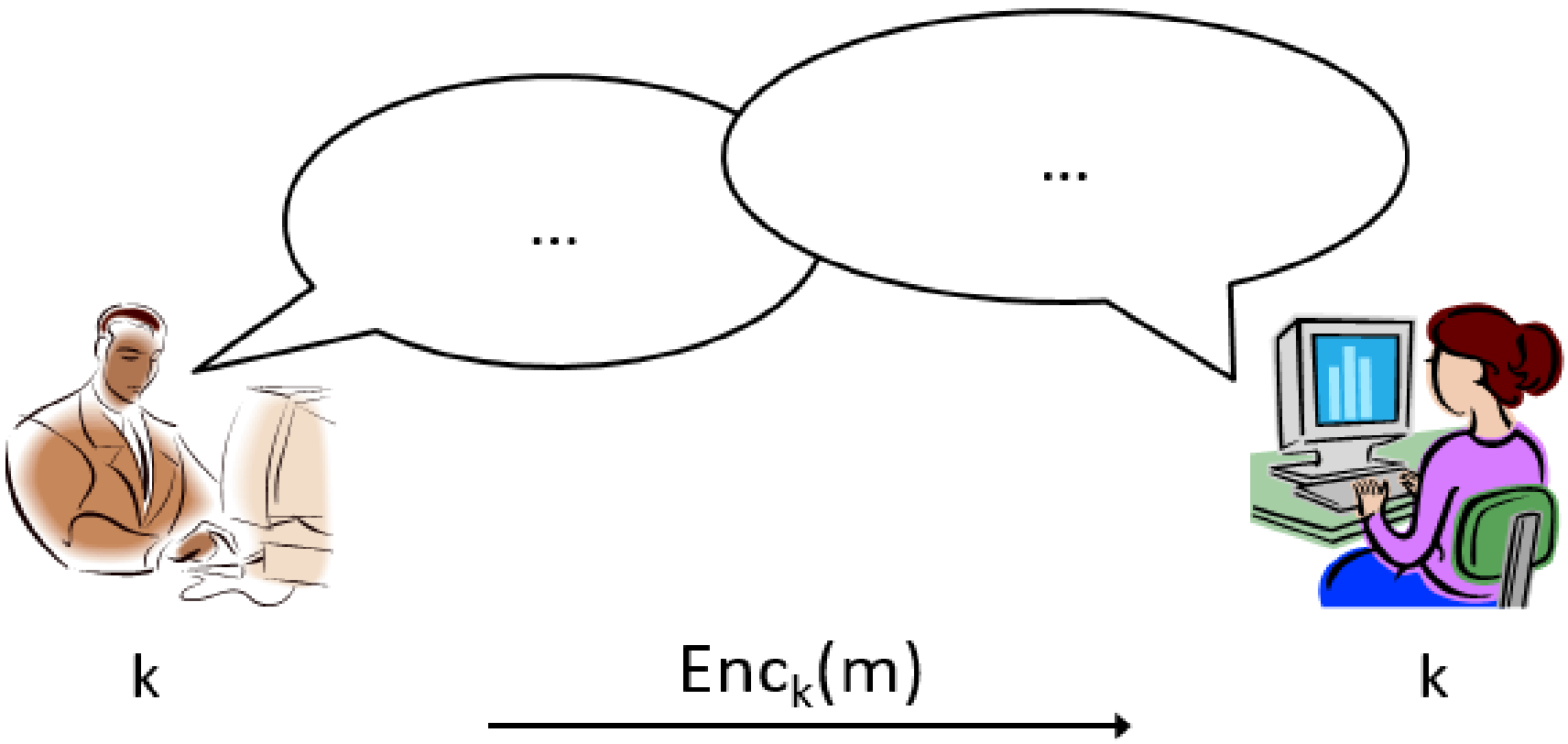
2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

New directions

- Main ideas:
  - Some problems exhibit *asymmetry* - easy to compute, but hard to invert (*factoring*, *RSA*, *group exponentiation*, ...)
  - Use this *asymmetry* to enable two parties to agree on a shared secret key via public discussion
    - *Key exchange*

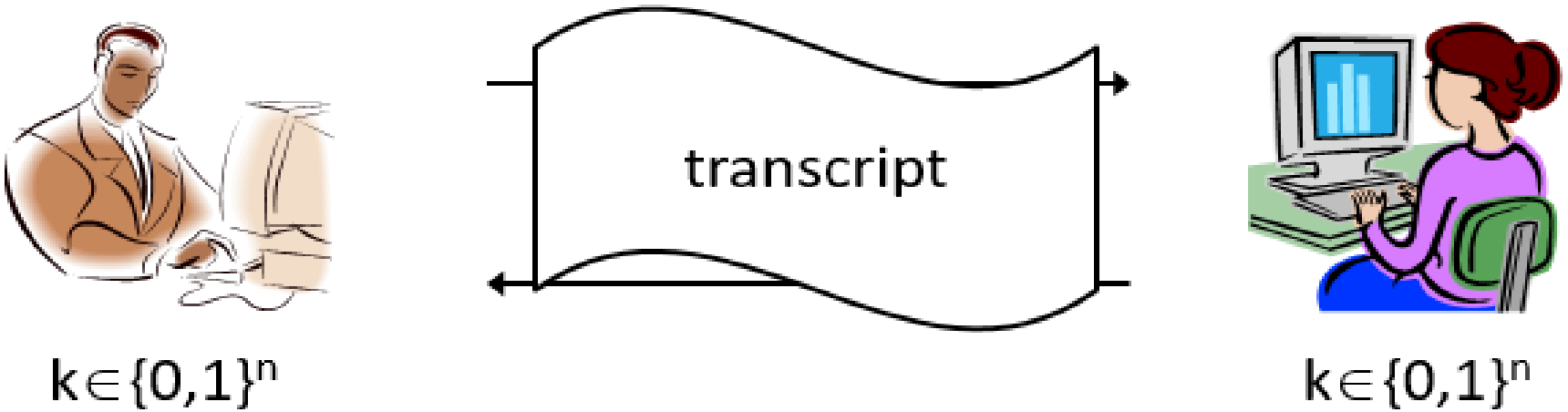
Key exchange

- *Secure* against an eavesdropper who sees everything!



More formally ...

- *Security goal*: even after observing the transcript, the shared key k should be *indistinguishable* from a uniform key



Formally

- Fix a key-exchange protocol Π and an attacker (passive eavesdropper) A

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- Fix a key-exchange protocol Π and an attacker (passive eavesdropper) A
- Define the following experiment $KE_{A,\Pi}(n)$:
  - Honest parties run Π using security parameter n, resulting in a transcript *trans* and (shared) key k
  - Choose uniform bit b. If $b = 0$, then set $k' = k$; if $b = 1$, then choose uniform $k' \in \{0,1\}^n$
  - Give *trans* and k' to A, which outputs a bit b'
  - Exp't evaluates to 1 (A *succeeds*) if $b = b'$

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  - Give *trans* and k' to A, which outputs a bit b'
  - Exp't evaluates to 1 (A *succeeds*) if $b = b'$
- **Definition 12.1** Key-exchange protocol Π is *secure* (against passive eavesdropping) if for *all* PPT A it holds that

$$\Pr[KE_{A,\Pi}(n) = 1] \leq 1/2 + negl(n)$$

Notes

- Being unable to compute the key given the transcript is not a strong enough guarantee

- Indistinguishability of the shared key from uniform is a much stronger guarantee
  - And is necessary if the shared key will subsequently be used for *private-key crypto*!

Diffie-Hellman key exchange

G, q, g, h_1 →

← h_2

$k_1 = (h_2)^x$

$k_2 = (h_1)^y$

$(G, q, g) \leftarrow \mathcal{G}(1^n)$

$x \leftarrow \mathbb{Z}_q$

$h_1 = g^x$

$y \leftarrow \mathbb{Z}_q$

$h_2 = g^y$

In practice

G, q, g

h_1

h_2

$k_1 = (h_2)^x$

$x \leftarrow \mathbb{Z}_q$

$h_1 = g^x$

$k_2 = (h_1)^y$

$y \leftarrow \mathbb{Z}_q$

$h_2 = g^y$

Recall: Diffie-Hellman assumptions

- *Computational* Diffie-Hellman (CDH) problem:
  - Given g, h_1, h_2, compute $DH_g(h_1, h_2)$

- *Decisional* Diffie-Hellman (DDH) problem:
  - Given g, h_1, h_2, distinguish $DH_g(h_1, h_2)$ from a uniform element of G

Security

- Eavesdropper sees G, q, g, g^x, g^y
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- Shared key k is g^{xy}
- Computing k from the transcript is exactly the *computational* Diffie-Hellman problem
- Distinguishing k from uniform group element is exactly the *decisional* Diffie-Hellman problem

  $\Rightarrow$ If the DDH problem is hard relative to $\mathcal{G}$, this is a *secure* key-exchange protocol!

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  - Not clear how to use this as, e.g., an AES key
- Solution: *key derivation*
  - Set $k' = H(k)$ for suitable hash function H

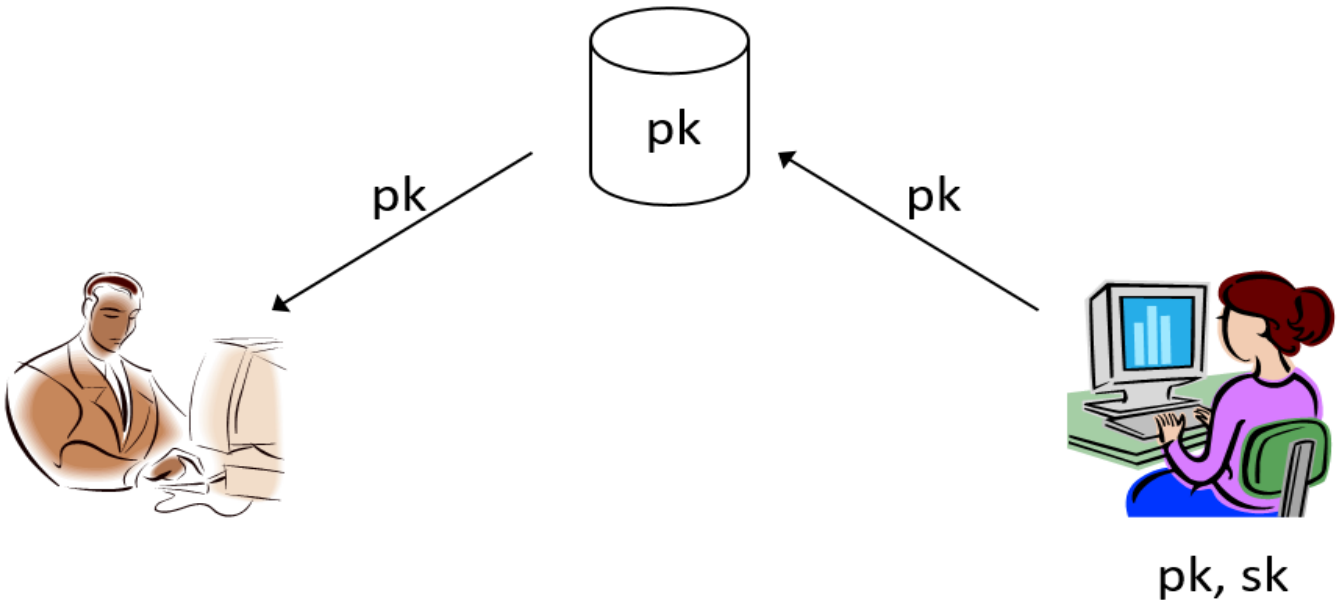
Modern key-exchange protocols

- Security against passive eavesdroppers is insufficient
- Want *authenticated* key exchange
  - This requires some form of setup in advance
- Modern key-exchange protocols provide this
  - We will return to this later

The public-key setting

- A party generates a *pair* of keys: a *public* key *pk* and a *private* key *sk*
  - Public key is widely disseminated
  - Private key is kept secret, and shared with no one
- Private key used by the party who generated it; public key used by everyone else
  - Also called *asymmetric cryptography*
- Security must hold even if the attacker knows *pk*

Public-key distribution I, II



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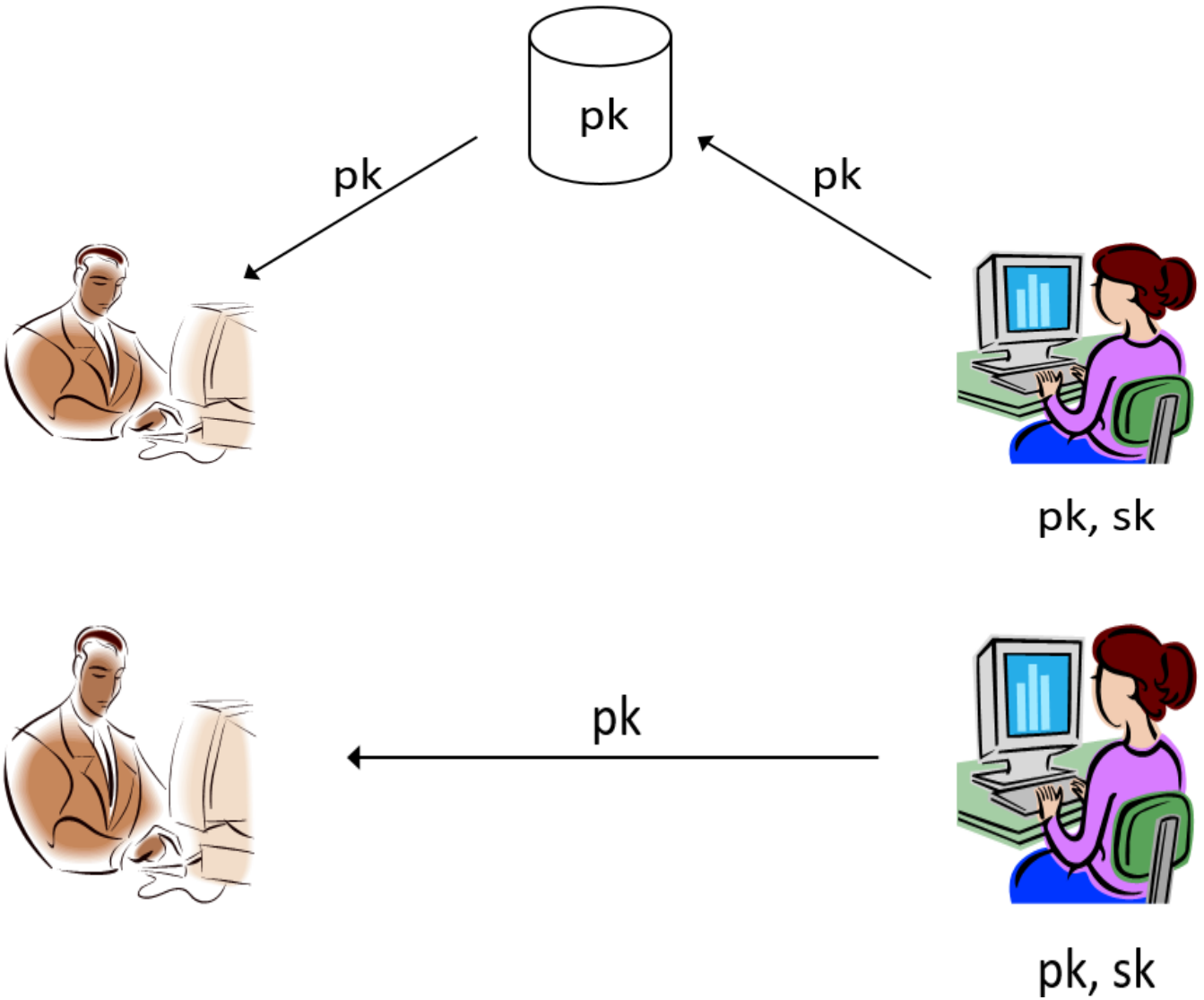

Public-key distribution

- Previous figures (implicitly) assume parties are able to obtain correct cipies of each others' public keys
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	Private-key setting	Public-key setting
Secrecy	Private-key encryption	Public-key encryption
Integrity	Message authentication codes	Digital signature schemes

Addressing drawbacks of private-key crypto

- Key distribution
  - Public keys can be distributed over *public* (but *authenticated*) channels!

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- Key management in large systems of N users
  - Each user stores 1 private key and $N - 1$ public keys; only N keys overall
  - Public keys can be stored in a central directory
- Applicability in "open systems"
  - Even parties who have no prior relationship can find each others' public keys and use them

Why study private-key crypto?

- Private-key cryptography is more suitable for certain applications
  - E.g., disk encryption

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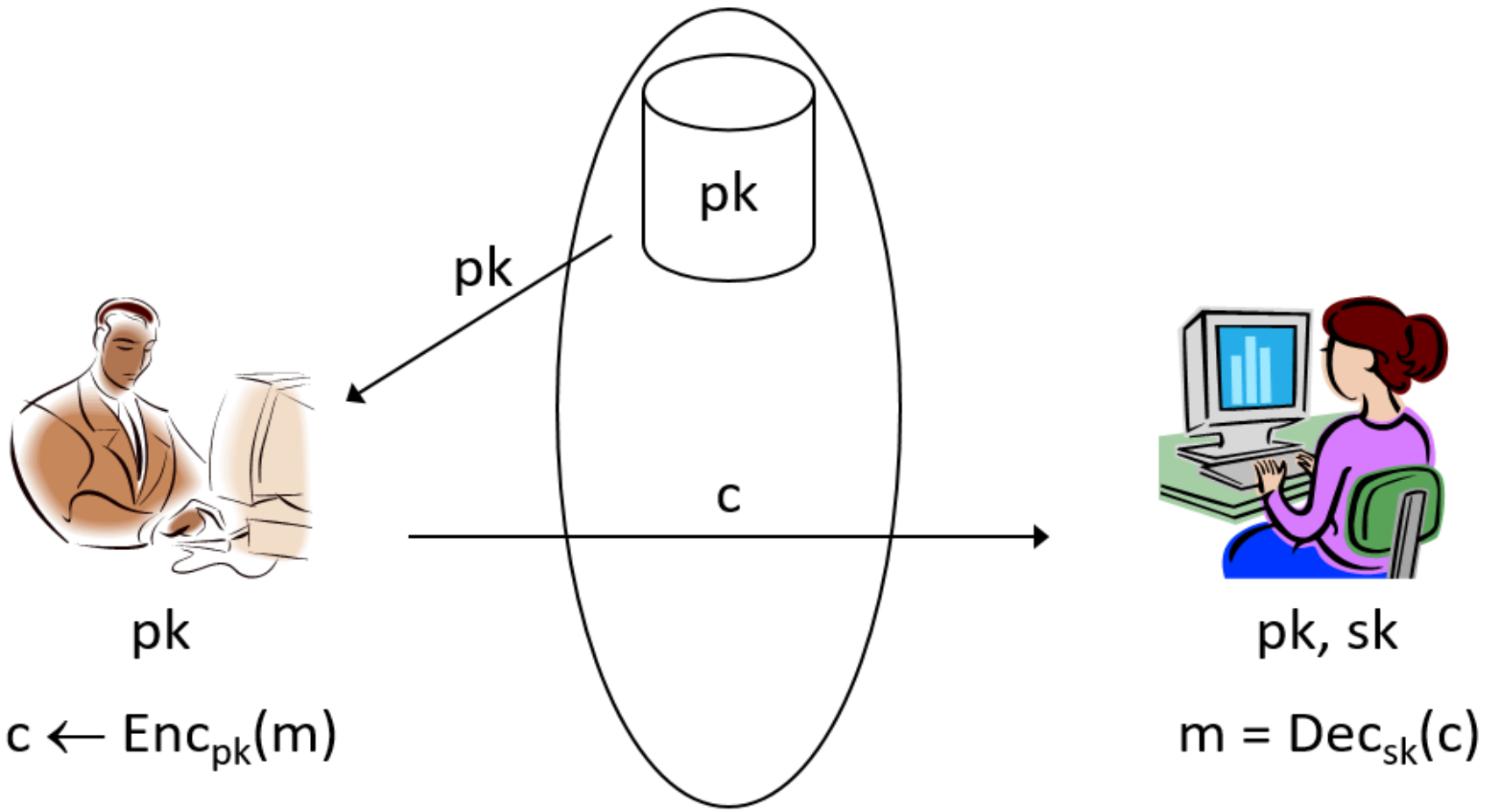
- Public-key crypto is roughly $2 - 3$ orders of magnitude *slower* than private-key crypto
  - If private-key crypto is an option, use it!
  - Private-key crypto is used for *efficiency* even in the public-key setting

Public-key encryption



Public-key encryption

- **Theorem 12.2** A *public-key encryption* scheme is composed of three PPT algorithms:
  - *Gen*: *key-generation algorithm* that on input 1^n outputs pk, sk
  - *Enc*: *encryption algorithm* that on input pk and a message m outputs a ciphertext c
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  For all m and pk, sk output by *Gen*,

$$Dec_{sk}(Enc_{pk}(m)) = m$$

CPA-security

- Fix a public-key encryption scheme Π and an adversary A
- Define experiment $PubK\text{-}CPA_{A,\Pi}(n)$:
  - Run $Gen(1^n)$ to get keys pk, sk
  - Give pk to A, who outputs m_0, m_1 of same length
  - Choose uniform $b \in \{0,1\}$ and compute the ciphertext $c \leftarrow Enc_{pk}(m_b)$; give c to A
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- **Theorem 12.3** Public-key encryption scheme Π is *CPA-secure* if for all PPT adversaries A:

$$\Pr[PubK\text{-}CPA_{A,\Pi}(n) = 1] \leq 1/2 + negl(n)$$

Notes on the definition

- No encryption oracle?!

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$\Rightarrow$ No *perfectly secret* public-key encryption

$\Rightarrow$ No *deterministic* public-key encryption can be CPA-secure

$\Rightarrow$ CPA-security implies security for encryption multiple messages as in the private-key case

Perfectly secret public-key encryption

- **Definition 12.4** A public-key encryption scheme is *perfectly secret* if for all public keys pk, all messages m_0, m_1, all ciphertexts c, and all algorithms A, we have:

$$\Pr[A(pk, c) = 0 | c \leftarrow Enc_{pk}(m_0)] = \Pr[A(pk, c) = 0 | c \leftarrow Enc_{pk}(m_1)]$$

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Theorem 12.5 No public-key encryption scheme is *prefectly secret*.

Proof.

Chosen-ciphertext attacks

pk

$c \leftarrow Enc_{pk}(m)$

c

pk, sk

c'

m'

Chosen-ciphertext attacks

- *Chosen-ciphertext attacks* are arguably even a greater concern in the public-key setting
  - Attacker might be a legitimate sender
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- Related concern: *malleability*
  - I.e., given a ciphertext c that is the encryption of an unknown message m, might be possible to produce ciphertext c' that decrypts to a related message m'
  - This is also undesirable in the public-key setting

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- Compute modulus $N = pq$
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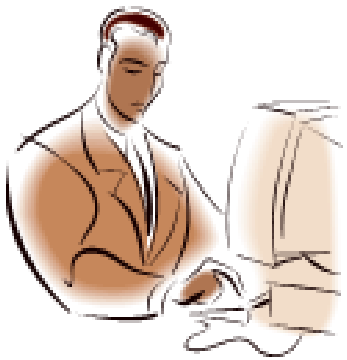
- *RSA assumption*: given N, e only, it is *hard* to compute the e^{th} root of a uniform $c \in \mathbb{Z}_N^*$

“Plain” RSA encryption



N, e

c

$(N, e, d) \leftarrow \text{RSAGen}(1^n)$

$pk = (N, e)$

$sk = d$

$c = [m^e \bmod N]$

$m = [c^d \bmod N]$

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- *RSA assumption* only refers to hardness of computing the e^{th} roots *in its entirety*
  - Partial information about the e^{th} root may be leaked
- Plain RSA should *never* be used!

Next Lecture

- El Gamal encryption, digital signature ...