1. Proof:
$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} = \frac{\sum_{i=1}^{n} y_i (x_i - \bar{x})}{SS_{xx}}, \quad SS_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$
$$= \sum_{i=1}^{n} \underbrace{\frac{(x_i - \bar{x})}{S_{xx}}}_{c_i} \cdot y_i$$
$$= \sum_{i=1}^{n} c_i \cdot y_i, \quad c_i = \frac{x_i - \bar{x}}{S_{xx}}$$

- $SS_{xy} \equiv S_{xy}, \quad SS_{xx} \equiv S_{xx}$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$= \sum_{i=1}^{n} \frac{y_i}{n} - \bar{x} \cdot \sum_{i=1}^{n} c_i y_i$$
$$= \sum_{i=1}^{n} \underbrace{\left(\frac{1}{n} - c_i \cdot \bar{x} \right)}_{d_i} y_i$$
$$= \sum_{i=1}^{n} d_i y_i$$

2. $\mathrm{Var}(\hat{\beta}_1) = \mathrm{Var}\left(\sum_{i=1}^{n} c_i y_i \right)$

Because y_i's are uncorrelated, and c_is are non-random.

$$\boxed{\begin{array}{l} \mathrm{Var}(aX+bY) \\ = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) \end{array}}$$

$$= \sum_{i=1}^{n} c_i^2 \,\mathrm{Var}(y_i)$$
$$= \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{S_{xx}^2} \cdot \sigma^2$$
$$= \frac{\sigma^2 \underbrace{\left(\sum_{i=1}^{n} (x_i - \bar{x})^2 \right)}_{S_{xx}}}{S_{xx}^2}$$
$$= \frac{\sigma^2}{S_{xx}}$$

- $\sum_{i=1}^{n} e_i = 0$

Proof: $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i), \quad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$
$$= \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \right)$$
$$= \sum_{i=1}^{n} \left(y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 \cdot x_i \right)$$
$$= \sum_{i=1}^{n} (y_i - \bar{y}) - \hat{\beta}_1 \sum_{i=1}^{n} (\bar{x} - x_i)$$
$$= \underbrace{\sum_{i=1}^{n} y_i - n \cdot \bar{y}}_{0} - \hat{\beta}_1 \underbrace{\left(n\bar{x} - \sum_{i=1}^{n} x_i \right)}_{0} = 0$$

$$\sum_{i=1}^{n} y_i = n \cdot \bar{y} = n \left(\frac{\sum_{i=1}^{n} y_i}{n} \right)$$

- $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i$

Proof: $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i) = 0$

so $\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{y}_i = 0$

$\Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i$

- $\sum_{i=1}^{n} x_i e_i = 0$

Proof: $\sum_{i=1}^{n} x_i (y_i - \hat{y}_i) = \sum_{i=1}^{n} x_i \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \right), \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$
$$= \sum_{i=1}^{n} x_i \left(y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) - \hat{\beta}_1 x_i \right)$$
$$= \sum_{i=1}^{n} x_i (y_i - \bar{y}) + \hat{\beta}_1 \sum_{i=1}^{n} x_i (\bar{x} - x_i)$$
$$= S_{xy} + \hat{\beta}_1 (-S_{xx})$$
$$= S_{xy} + \frac{S_{xy}}{\cancel{S_{xx}}} (-\cancel{S_{xx}})$$
$$= S_{xy} - S_{xy}$$
$$= 0$$

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- $\sum_{i=1}^{n} \hat{y}_i e_i = 0$

Proof: $\sum_{i=1}^{n} \hat{y}_i e_i = \sum_{i=1}^{n} (\hat{\beta}_0 + \hat{\beta}_1 x_i) e_i$
$$= \hat{\beta}_0 \underbrace{\sum_{i=1}^{n} e_i}_{0} + \hat{\beta}_1 \underbrace{\sum_{i=1}^{n} x_i e_i}_{0}$$
$$= 0$$

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- The Least-square regression line always passes through the centroid $(\bar{x}, \bar{y})$

Proof: fitted regression line
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$
plug-in $\bar{x}$ to the RHS.
$$\hat{\beta}_0 + \hat{\beta}_1 \bar{x} = \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 \bar{x} = \bar{y}$$

$\Rightarrow (\bar{x}, \bar{y})$ is on the line

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